Comparison Of Accuracy Assessment Techniques For Numerical Integration

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Introduction

- Numerical integration of the problem:

$$\dot{\vec{x}} = \vec{f}(t, \vec{x}), \quad \vec{x}(a) = \vec{s}$$

gives some error,

$$\xi_n = \vec{x}(t_n) - \tilde{\vec{x}}$$

- Total error is from truncation error and round-off error.
- We wish to measure the error to choose the best integrator for a given application.

Test Cases

- Two test integrators:
  - $\mathbf{4^{th}}$ order Runge-Kutta (single-step)
  - $\mathbf{8^{th}}$ order Gauss-Jackson (multi-step)
- Three test case orbits:
  - Case 1: Low earth orbit (RK step: 5sec, GJ step: 30sec)
    $\boldsymbol{h_p = 300}$km, $\boldsymbol{e = 0}$, $\boldsymbol{i = 40^\circ}$, $\boldsymbol{B = 0.01\ \mathrm{m^2/kg}}$
  - Case 2: Elliptical orbit (RK step: 5sec, GJ step: 30sec)
    $\boldsymbol{h_p = 200}$km, $\boldsymbol{e = 0.75}$, $\boldsymbol{i = 40^\circ}$, $\boldsymbol{B = 0.01\ \mathrm{m^2/kg}}$
  - Case 3: Geostationary orbit (RK step: 1min, GJ step: 20min)
    $\boldsymbol{h_p = 35800}$km, $\boldsymbol{e = 0}$, $\boldsymbol{i = 0^\circ}$

Error ratio

- Compare computed numerical integration to some reference.
- Define an error ratio:

$$\rho_r = \frac{1}{r_A N_{\text{orbits}}} \sqrt{\frac{1}{n} \sum_{i=1}^{n} (\Delta r_i)^2}$$

where $\Delta r = \left| r_{\text{computed}} - r_{\text{ref}} \right|$.

- Comparisons are over 3 days with and w/o perturbations.
- Perturbations include 36 $\times$ 36 WGS-84 geopotential, Jacchia 70 drag model, and lunar/solar forces.

Two-Body Test

- Integration performed without perturbations, compared to analytic solution.
- Advantage is that the reference is exact.
- Disadvantage is that the effect of perturbations on integration error is not considered.
- Used by Fox (1984) in an accuracy / speed study.
- Used by Montenbruck (1992) to test integrators.

Two Body Test Results

	Error Ratio		Position Error (mm)	
test #	RK	GJ	RK	GJ
1	$\mathbf{2.05 \times 10^{-10}}$	$\mathbf{7.96 \times 10^{-14}}$	133	.0494
2	$\mathbf{2.49 \times 10^{-10}}$	$\mathbf{1.03 \times 10^{-11}}$	286	14.9
3	$\mathbf{3.27 \times 10^{-11}}$	$\mathbf{8.95 \times 10^{-12}}$	7.21	2.60

Step-Size Halving

- Reference is from same integrator, with half the step size.
- Perturbations can be tested.
- Gives a good measure of truncation error, which is related to the step size.
- Similar technique can be used to measure the order of the integrator.
- Does not work well if round-off error is dominant.

Step-Size Halving Results

Two-Body Results

test #	RK	GJ
1	1.96×10^{-10}	2.22×10^{-14} $\downarrow$
2	2.34×10^{-10}	1.03×10^{-11}
3	3.07×10^{-11}	8.94×10^{-12}

Perturbed Results

test #	RK	GJ
1	1.19×10^{-9}	4.63×10^{-9}
2	1.16×10^{-9}	9.93×10^{-9}
3	3.07×10^{-11}	8.95×10^{-12}

High Order Test

- Reference integration is performed with a high-order, high-accuracy integrator.
- Perturbations can be tested.
- Assumes that the reference integrator is much more accurate than the integrator being tested.
- We used a $\mathbf{14^{th}}$ order Gauss-Jackson, with a 15 sec step size for cases 1 & 2, 1 min for case 3.

High Order Test Results

	test #	RK	GJ
Two-Body Results	1	2.05×10^{-10}	$5.34 \times 10^{-14} \downarrow$
	2	2.49×10^{-10}	1.04×10^{-11}
	3	3.28×10^{-11}	9.02×10^{-12}

	test #	RK	GJ
Perturbed Results	1	4.59×10^{-9}	4.62×10^{-9}
	2	7.19×10^{-9}	9.94×10^{-9}
	3	3.27×10^{-11}	9.07×10^{-12}

Reverse Test

- Final state of integration is used as initial conditions in a reverse integration.
- The forward and backward integrations should be the same.
- Used by Hadjifotinou and Gousidou-Koutita (1998) to test accuracy in the N-body problem.
- Does not measure reversible error.
- Zadunaisky (1979) claims that the reverse test is always unreliable.

Reverse Test Results

	test #	RK	GJ
	1	2.27×10^{-10}	$4.55\times10^{-15}\Downarrow$
Two-Body Results	2	$5.13\times10^{-11}\Downarrow$	$2.21\times10^{-11}\uparrow$
	3	$3.53\times10^{-12}\Downarrow$	$2.11\times10^{-11}\Uparrow$

	test #	RK	GJ
	1	2.28×10^{-10}	7.79×10^{-10}
Perturbed Results	2	5.18×10^{-11}	2.46×10^{-11}
	3	3.52×10^{-12}	1.97×10^{-11}

Zadunaisky's Technique

- Zadunaisky (1966) suggests integrating a *pseudo-problem*.

$$\dot{\vec{z}} = \vec{f}(t, \vec{z}) + \dot{\vec{P}}(t) - \vec{f}(t, \vec{P}(t))$$

- $\vec{P}(t)$ is a polynomial constructed to fit the original integration.
- $\vec{P}(t)$ is the exact solution of the pseudo-problem.
- Matches error of the original problem if the $\vec{P}(t)$ is well chosen.
- Problem broken into subintervals to use low-order polynomials.
- Polynomials match actual derivatives at subinterval endpoints.
- Use a $\mathbf{5^{th}}$ order polynomial for RK, $\mathbf{3^{rd}}$ for GJ.

Zadunaisky's Method Results

	test #	RK	GJ
Two-Body Results	1	$3.08 \times 10^{-10} \uparrow$	$3.33 \times 10^{-14} \downarrow$
	2	$3.39 \times 10^{-9} \Uparrow$	$6.83 \times 10^{-14} \Downarrow$
	3	3.87×10^{-11}	$1.86 \times 10^{-14} \Downarrow$

	test #	RK	GJ
Perturbed Results	1	1.81×10^{-9}	8.06×10^{-8}
	2	2.11×10^{-9}	6.55×10^{-8}
	3	3.82×10^{-11}	1.01×10^{-12}

Conclusions

- Reverse test is not reliable.
- Two-body test does not give enough information, but is useful for evaluating other methods.
- Step-size halving and high order test give consistent results.
- Zadunaisky's method gives reasonable results for RK, not for GJ.
- More work needed choosing $\vec{P}(t)$ to improve Zadunaisky results with GJ.