GRAVITATIONAL RADIATION IN THE LIMIT OF HIGH FREQUENCY

by

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ABSTRACT

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This dissertation deals with a technique for obtaining approximate radiative solutions to the Einstein equations of general relativity in situations where the gravitational fields of interest are quite strong. In the first chapter, we review the history of the problem and discuss previous work along related lines. In the second chapter, we assume the radiation to be of high frequency and expand the field equations in powers of the small wavelength this supplies. This assumption provides an approximation scheme valid for all orders of 1/r, for arbitrary velocities up to that of light, and for arbitrary intensities of the gravitational field. To lowest order we obtain a gauge invariant linear wave equation for gravitational radiation, which is a covariant generalization of that for massless spin-two fields in flat space. This wave equation is then solved by the W.K.B. approximation to show that gravitational waves travel on null geodesics with amplitude and frequency modified by gravitational fields in exactly the same way as are those of light waves, and with their polarization parallel transported along the geodesics, again as is the case for light. The metric containing high frequency gravitational waves is shown to be type N to lowest order, and some limits to the methods used are discussed. In the third chapter we go beyond the linear terms in the high frequency expansion, and consider the lowest order non-linear terms. They are shown to provide a natural, gauge invariant, averaged effective stress tensor for the energy localized in the high frequency radiation. By assuming the W.K.B. form for the field, this tensor is found to have the same structure as that for an electromagnetic null field. A Poynting vector is used to investigate the flow of energy and momentum in the gravitational wave field,

and it is seen that high frequency waves propagate along null hypersurfaces and are not backscattered off by the curvature of space. Expressions for the total energy and momentum carried by the field to flat null infinity are given in terms of coordinate independent integrals valid within regions of strong field strength. The formalism is applied to the case of spherical gravitational waves where a news function is obtained, and where the source is found to lose exactly the energy and momentum contained in the radiation field.

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CHAPTER I

HISTORICAL REVIEW

1. Introductory Remarks

From an experimentalist's point of view, it was perhaps quite surprising to see the Newtonian theory of gravitation, a theory whose accuracy had been painstakingly verified by centuries of astronomical observation, so suddenly replaced by the much more complex and mathematical generalization of Einstein. While the experimental justification was meager indeed, due to the extremely weak coupling constant of the gravitational field, the theoretical arguments were overwhelming. Newton's theory suffered from the incurable and eventually fatal disease of incompatibility with special relativity. This is perhaps most readily seen by the fact that changes in a mass distribution are instantly propagated, via Poisson's differential equation, as changes in the gravitational field throughout space, providing a mechanism to violate causality by sending information to arbitrary distances faster than the speed of light. Thus, Newtonian theory did not admit the possibility of gravitational waves and this was one of the causes of its rejection. The logical question to ask next is whether general relativity admits waves as solutions. The answer to this is that we do not yet have a completely rigorous treatment showing how waves are produced by the internal motion of sources, propagate out away from the scene of their creation and finally arrive in a radiation zone. When we consider the complexity of the Einstein field equations, the lack of such a rigorous proof is not surprising, for the analogous problem in electromagnetic theory was not completely solved until comparatively recent times. If, however, we wish to settle for a lesser degree of rigor, then we have many extremely convincing arguments for the reality of gravitational waves within Einstein's theory. Some of the variety of means used in the past to substantiate the existence of gravity waves are

a) exact solutions representing waves propagating through space, b) perturbation on known geometries, c) multipole or asymptotic analysis of fields due to localized sources, d) algebraic consideration of the field and its derivatives, e) study of the characteristic surfaces for the Einstein equations and f) the high frequency limit of the radiation field.

Before we embark on a more detailed discussion, it is well to ask a more fundamental, and surprisingly non-trivial question: What is a gravitational wave? It is not just a solution of Einstein's differential equations which contains lots of wiggles, for ripples in coordinates or motion of the observer may easily give rise to spurious coordinate waves in a static situation. We must reject such coordinate juggling since the resultant waves lack the physical characteristics of real waves. In the words of Eddington (1960), "we can 'propagate' coordinate changes with the speed of thought" and not be bound by causal restrictions, since coordinate waves carry no information, while it is precisely this feature which is essential to our notion of any physical wave. Free gravitational radiation must contain provision for impressing arbitrary, non-analytic functions into its structure (say via AM or FM modulation by the source) which may later be extracted out with a receiver. The transfer of news is made possible by the closely related physical fact that waves also transfer energy. A precise local definition of energy is only possible within the context of special relativity, and so when we deal with gravitational waves the concept of energy transfer is rather difficult to apply quantitatively except in rather specialized situations. For space-times with asymptotically flat Euclidean topology surrounding localized sources of outgoing radiation, the energy carried by the wave should cause a corresponding decrease in the energy of the source, or equivalently, the source should loose mass. Besides serving as a criterion for the elimination of coordinate waves, the loss of mass by a radiating source tells us not to expect true gravitational waves to be of infinite extent in sensible geometries. Rather, finite wave packets containing a spread of frequencies are characteristic of the true physical wave.

Now that we have some idea as to what gravitational radiation is, it is perhaps valid to ask why one should spend time studying it. Of course, as in any field of pure research, curiosity is regarded as a legitimate enough purpose for the investigation of any phenomenon. Certainly in relativity, where Einstein's theory has stood for fifty years without much experimental validation, this is especially true. There are other reasons however, since gravitation must be united to the rest of what we understand about physics. Eventually, a quantum theory of gravitation will be necessary, and it is clear from the quantization of electromagnetism that waves will play an essential role in any attempts to understand quantum effects. Electrodynamics seems to justify our study of waves in many other ways as well, and indeed, it is almost entirely in electromagnetic wave theory that the concepts and tools used in gravitation originate. Unfortunately, the weakness of gravity seems to preclude the development of a useful technology based upon its radiative features; yet it may be possible that this weakness will itself be an asset for investigation of effects otherwise inaccessible, such as the large scale structure of the universe, or the internal structure of the sun.

Finally, it is well to ask what sort of difficulties are encountered in the study of gravitational radiation. As has already been indicated, the fact that general relativity is a covariant theory gives rise to coordinate problems. Without the existence of preferred frames, we are faced with the task of extracting out the essentials of a wave, and stating them in a covariant fashion. All physical observables must therefore be expressed in a coordinate independent way, and the masks they wear upon changing from one set of coordinates to another must be well understood. An additional problem lies in the extreme complexity of the non-linear Einstein equations. Any exact solution is necessarily very simple and specialized or it could not have been found in the first place, and, unlike Maxwell's theory, there is no possibility of building additional exact solutions out of a given one. The only hope of getting to some of the complexities of a physically realistic case is to use some approximation scheme, but then it is hard to do this in a covariant way as well as to demonstrate the convergence of the method used. Finally, the lack of a tensorial description for gravitational fields is built in with the equivalence principle. Since it is possible to locally transform away a field by simply choosing coordinates correctly, we must use some non-local scheme such as averaging or equivalently we must go to derivatives of the fields themselves or multipole expansion techniques before essential features can be observed. In any practical problem it is the simultaneous combination of approximations, covariance, and non-locality which must be overcome before sensible results emerge. Let us now see how some selected methods have been developed to do this.

2. Exact Wave Solutions

Although there is much information known about exact solutions, most of this data is necessarily about rather unrealistic, highly symmetric situations. While real wave solutions may tend toward these at great distances from localized sources, exact solutions presently known seem to be too specialized to reveal the essential characteristics of physical waves.

Birkhoff (1927) showed that any spherically symmetric solution to the vacuum field equations is necessarily static, so no spherically symmetric gravitational waves can exist.

Cylindrical waves were discovered by Einstein and Rosen (1937), while Weber and Wheeler (1957) have treated cylindrical pulse solutions. However, the infinite dimensions of the source is clearly unattainable in practice and leads to great difficulty in interpretation.

Plane waves have been considered by Rosen (1937); Bondi, Pirani, and Robinson (1959); and Takeno (1961). They have been shown to be transverse, and generalized to so-called "plane-fronted" waves (see review article by Ehlers and Kundt in Witten 1962.)

Other interesting exact solutions have been found by Robinson and Trautman (1962), but these have singularities stretching along lines to infinity.

So far, no one has found solutions which can be interpreted as representing waves from an isolated source, with singularities only within the region of the source itself. Such solutions must necessarily have little symmetry (say axial at best) and so involve three coordinates, which presents enormous mathematical difficulties.

3. Invariant Methods

In order to insure that the study of gravitational waves is free of coordinate dependence, elegant methods have been developed based on geometric properties of space-time or on the algebra of tensors which characterize this geometry. Much of this work is naturally centered about the Riemann tensor since it is the only new tensor which may be constructed out of the metric and its first two derivatives, and therefore describes those variations of the gravitational field (through the equation of geodesic deviation) which cannot be annihilated by a coordinate transformation.

The study of electromagnetic radiation fields showed their close relation to the geometric null cone structure. This has been extended by analogy to gravitational radiation, where the Riemann tensor may be classified by its symmetries with vectors from the null cone. This depends on the result of Debever (1958) that in any empty space-time there is at least one and at most four null directions k^α satisfying

$$(3.1)\quad k_{[\alpha}R_{\beta]\sigma\tau[\gamma}k_{\delta]}k^\sigma k^\tau = 0.$$

The null directions k^α are called <u>rays</u> and in general are not null geodesics. Vacuum fields may be classified into the following types (see Sachs 1961; Appendix A)

Type	Condition	Coincident Rays
N	$R_{\alpha\beta\gamma\delta}k^\delta = 0$	4
III	$R_{\alpha\beta\gamma[\delta}k^\delta k_{\varepsilon]} = 0$	3, 1
D	$\begin{cases} R_{\alpha\beta\gamma[\delta}k_{\varepsilon]}k^\alpha k^\gamma = 0 \\ R_{\alpha\beta\gamma[\delta}m_{\varepsilon]}m^\alpha m^\gamma = 0 \end{cases}$	2, 2
II	$R_{\alpha\beta\gamma[\delta}k_{\varepsilon]}k^\alpha k^\gamma = 0$	2, 1 1
I	$k_{[\alpha}R_{\beta]\sigma\tau[\gamma}k_{\delta]}k^\sigma k^\tau = 0$	1 1 1 1.

Sachs (1961) gives a description of the geometry of the rays themselves, and shows that physically realistic sources produce fields which are generally of type I but which tend to type N at large distances (in the radiation zone) from the source in an asymptotically flat space.

Another geometrical approach to the problem of gravitational waves is based upon the fact that the transfer of information necessitates construction of non-analytic solutions to the hyperbolic system of second order partial differential equations for the gravitational potentials $g_{\mu\nu}$. Except along certain surfaces, the Cauchy problem for the Einstein equations can be set correctly and the metric components have unique, analytic solutions (at least in a neighborhood of a given surface). The exception to this is along null or characteristic surfaces specified by the equations

(3.2) $\phi(x^\alpha) = \text{constant}$

(3.3) $g^{\alpha\beta} \frac{\partial\phi}{\partial x^\alpha} \frac{\partial\phi}{\partial x^\beta} = 0.$

Along these characteristic surfaces, given $g_{\alpha\beta}$ and its first derivatives, the second derivatives are left indeterminate by the field equations. Consequently it is possible for the potentials $g_{\alpha\beta}$ to suffer discontinuities in their second derivatives across such a surface. Null surfaces represent wave fronts across which a disturbance may change its magnitude from zero to a finite value. From this, we can conclude that gravitational waves travel with the velocity of light and that discontinuities in the metric and Riemann tensors across a wave front have the form (see Trautman 1958)

(3.4) $\Delta g_{\alpha\beta,\gamma\delta} = h_{\alpha\beta} k_\gamma k_\delta \ , \ \Delta R_{\alpha\beta\gamma\delta} = 2k_{[\delta} h_{\gamma]\ [\alpha} k_{\beta]}$

where

(3.5) $k_\alpha = \phi_{,\alpha} \qquad k_\alpha k^\alpha = 0$

and $h_{\alpha\beta}$ is subject to

(3.6) $(h_\alpha{}^\beta - \frac{1}{2} h_\rho{}^\rho \delta_\alpha^\beta)\ k_\beta = 0.$

If we assign the discontinuity to the presence of a gravitational wavefront, these algebraic relations should hold true for the wave itself.

4. Weak Field Approximation

In order to obtain approximate solutions to the field equations, we may expand the metric in a power series about a flat background

(4.1) $g_{\alpha\beta} = \eta_{\alpha\beta} + \varepsilon h^{(1)}_{\alpha\beta} + \varepsilon^2 h^{(2)}_{\alpha\beta} + \ldots$

where ε is assumed small. If we normalize our scale by the choice of a coordinate condition which agrees with harmonic coordinates to first order, then

$$(4.2)\quad (h_{\alpha}^{(1)\beta} - \frac{1}{2} h_{\alpha}^{(1)\rho}\delta_{\alpha}^{\beta})_{,\beta} = 0$$

where all indice are raised and lowered using the background metric $\eta_{\alpha\beta}$. Then the linearized vacuum field equations assume the form

$$(4.3)\ \Box h_{\alpha\beta}^{(1)} \equiv \eta^{\mu\nu} h_{\alpha\beta,\mu\nu}^{(1)} = 0$$

familiar to us from electrodynamics, which admits wave-like solutions. This weak field linearization destroys many essential features of the exact theory, e.g., (4.3) admits periodic wave solutions and so must neglect the loss of mass of a source. In order to overcome this difficulty, we naturally try to go to the second order terms in this approximation scheme. Here grave difficulties emerge (Fock 1957; Trautman 1958). In the words of Trautman (1965) "it is far from clear whether the method converges or even whether it can be continued, in the general case beyond the first step." Trautman argues that if we assume the coordinate condition to second order given by $h_{\alpha\beta}^{(2)}$ satisfying the same equation (4.2) as $h_{\alpha\beta}^{(1)}$ does, then the field equations take the symbolic second order form

$$(4.4)\ \Box h^{(2)} = Q(h^{(1)})$$

where $Q(h^{(1)})$ represents the effective source created by the first order terms and which is a quadratic function in $\partial h^{(1)}$. Looking at the solution of (4.4) for large r and along the flat space coordinate light-cone $t - r = \text{constant}$, Trautman (1965) concludes $h^{(2)}$ behaves as $(\log r)/r$. Since this contradicts the Sommerfeld radiation condition and gives an infinite total energy in the radiation field, we see that the linear approximation has broken down. The reason seems to be due to a poor coordinate choice. Physically, this may be understood as the result of the following process (Misner, private communication 1966). The linear wave $h^{(1)}$ is a pulse moving along coordinate light cones given, say, by the equation $t = r$, but since it carries energy it will slightly distort the geometry. (In the Schwarzschild geometry the radial light cones are given by $t = r + 2m \log[r - 2m]$.) In the second order, the wave energy is recognized by equation (4.4) and $h^{(2)}$ must therefore try to move the wave pulse $h^{(1)} \sim r^{-1}f(t-r)$ off the coordinate light cone and on to the actual light cone given by the exact wave $h \sim r^{-1}f(t-r-2m \log[r-2m])$. Evidently $h^{(2)}$ which is approximately $h - h^{(1)}$ will, to first order in $2m$, have a $(\log r)/r$ term, and the needed correction $h - h^{(1)}$ will be large (Fig. 1). In the next section, we will see how Bondi and his co-workers used the true physical light cones as coordinates and were thus able to circumvent this problem. Also, near the end of this thesis, we will show how a perturbation expansion about the metric already curved by $h^{(1)}$ instead of about the flat geometry also cures the $(\log r)/r$ difficulties.

5. Asymptotic Methods Using Multipole Expansions

When we look for the total energy, momentum, and angular momentum radiated by a physical system, we are after the properties of gravitation which extend to infinite distances from its source. It is then advantageous to step back and look at the field in the asymptotically flat background geometry, where the significant properties of the field manifest themselves, and all the irrelevent details of intermediate regions have been stripped away. Several authors have therefore decided to investigate the features of the gravitational field which emerge asymptotically at large distances r from an isolated source by expanding the metric as a multipole series in r^{-1}, in analogy with electromagnetic theory.

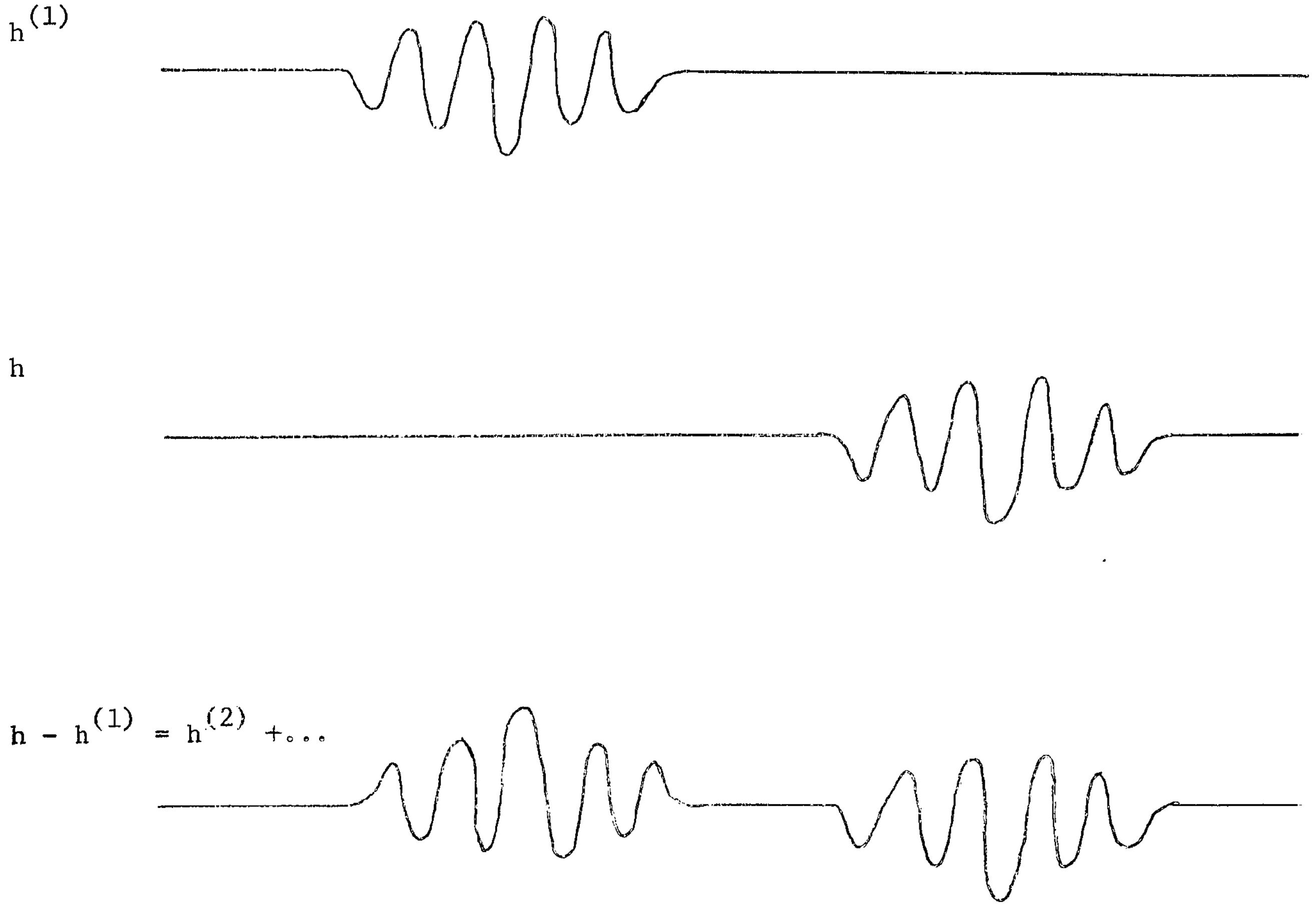

Fig. 1 A linear wave pulse $h^{(1)}$ may asymptotically agree with an exact solution h in shape, amplitude, and velocity, but if it needs a displacement in position ℓ (with $\ell/r \ll 1$), then this inherently minor correction will appear to be as large as the solution h itself when we attempt to represent it as an additive correction $h - h^{(1)}$.

In order to insure that only outgoing waves arrive at infinity, Trautman (1958, 1965) studied the gravitational analogue of the Sommerfeld radiation conditions. With an eye towards the usual electrodynamical analogy, he considered the class of space-times which admit a null vector field k^α, a parameter r (thought of as roughly a luminosity distance), and a set of coordinates x^μ in which the metric for the coordinate patch near $r \to \infty$ takes the form (Trautman 1965)

(5.1) $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad h_{\mu\nu} = O(r^{-1})$

(5.2) $g_{\mu\nu,\alpha} = i_{\mu\nu}k_\alpha + O(r^{-2}), \; i_{\mu\nu} = O(r^{-1})$

(5.3) $(i_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\sigma\tau}i_{\sigma\tau})k^\nu = O(r^{-2})$

(5.4) $g_{\mu\nu,\alpha\beta} = j_{\mu\nu}k_\alpha k_\beta + O(r^{-2}), \qquad j_{\mu\nu} = O(r^{-1})$

(5.5) $(j_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\sigma\tau}j_{\sigma\tau})k^\nu = O(r^{-2})$

(5.6) $k_\mu = O(1), \qquad k_{\mu,\nu} = O(r^{-1}) \; , \; k_\mu k^\mu = 0$

The Riemann tensor is easily seen to be

(5.7) $R_{\alpha\beta\gamma\delta} = 2\, k_{[\alpha}j_{\beta][\gamma}k_{\delta]} + O(r^{-2}),$

with the leading r^{-1} part having Petrov type N. The energy-momentum pseudotensor becomes

(5.8) $t_\mu{}^\nu = \rho k_\mu k^\nu + O(r^{-3})$

with

(5.9) $32\pi\rho \equiv i^{\alpha\beta}(i_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}\eta^{\sigma\tau}i_{\sigma\tau}).$

Under a coordinate transformation we may mix coordinate waves with the $h_{\mu\nu}$ metric. To study this, we consider the change of coordinates

(5.10) $x^\mu \to \bar{x}^\mu = x^\mu + a^\mu$

where a^μ satisfies

(5.11) $a_{\mu,\nu} = b_\mu k_\nu + O(r^{-2})$, $b_\mu = O(r^{-1})$

(5.12) $a_{\mu,\nu\tau} = c_\mu k_\nu k_\tau + O(r^{-2})$, $c_\mu = O(r^{-1})$

This induces the "gauge transformation"

(5.13) $i_{\mu\nu} \rightarrow \bar{i}_{\mu\nu} = i_{\mu\nu} + c_\mu k_\nu + c_\nu k_\mu$

which leaves (5.1)-(5.9) entirely unchanged. By means of such gauge transformations, we may insure (Komar 1962)

(5.14) $i^{\mu\nu} k_\nu = O(r^{-2})$

(5.15) $\eta^{\sigma\tau} i_{\sigma\tau} = O(r^{-2})$

which is slightly stronger than (5.3). We should note the close similarity of (3.4)-(3.6) to the appropriate equations in (5.1)-(5.15). This also will be seen to hold for radiation in the high frequency limit which will be the subject of future discussion.

Trautman's pioneering steps in the development of multipole techniques have been extended beyond the linear approximation by several other authors. Bondi and his co-workers (Bondi 1960; Bondi, van der Burg and Metzner 1962) look at the outgoing radiation from an isolated source with axial symmetry which is invariant under reflections about the equatorial plane. The key to their analysis is the use of coordinates which are closely related to the physical processes which they study. To circumvent the spurious (log r)/r divergences in the weak field approximation, Bondi chooses as coordinate surfaces the true light cones of the metric by introducing the retarded time $x^0 = u$ as the time coordinate, and uses the luminosity distance by choosing it as the radial coordinate $x^1 = r$. Angular variables $x^2 = \theta$ and $x^3 = \phi$ are set up so that a light ray has u, θ, and ϕ constant along it, and the area of a surface element of constant u and r is given by $r^2 \sin\theta d\theta d\phi$. The most general metric subject to these restrictions can be written in the form

$$(5.16)\quad ds^2 = (Vr^{-1}e^{2\beta} - r^2U^2e^{2\gamma})du^2 + 2e^{2\beta}dudr + 2r^2Ue^{2\gamma}dud\theta - r^2(e^{2\gamma}d\theta^2 + e^{-2\gamma}\sin^2\theta d\phi^2)$$

with β, γ, U and V functions of u, r, and θ. Of the ten field equations, axial symmetry causes R_{03}, R_{13} and R_{23} to vanish, leaving seven non-trivial equations. If we satisfy the so-called main equations

$$R_{11} = R_{12} = R_{22} = R_{33} = 0$$

then the Bianchi identities imply that $R_{01} = 0$, while R_{02} and R_{00} vanish everywhere if they do so at one point. Three of the "main equations" are constraint conditions within a hypersurface of constant u, while the remaining one gives the time development off of the hypersurface. Bondi assumes the "outgoing wave condition" that it is possible to express the dependent variables as power series in r^{-1}, e.g.,

$$\gamma = c(u,\theta)r^{-1} + d(u,\theta)r^{-2} + \ldots$$

and similarly for β, U, and V. When this is substituted into the field equations, all the unknown expansion coefficients except $c(u,\theta)$ are determined if their initial values on a given null hypersurface are known. The "news function" $c(u,\theta)$ has arbitrary time development, and describes source data flowing out along other light cones. This information is laid down on all null cones by the source and therefore could not be obtained by looking at just one such initial surface. Given $c_o \equiv c,_u$ our information about the gravitational radiation from a source is fully determined; for static systems c_o vanishes, while for systems with news, the source mass decreases according to

$$(5.17)\quad \frac{dm}{du} = -\frac{1}{2}\int_o^\pi c_o^2 \sin^2\theta d\theta .$$

Sach (1962) has relaxed the requirements which Bondi imposed and generalized the results to asymptotically flat spaces without symmetry.

He finds that there are two news functions corresponding to the two independent polarization modes in the general case. He studies the behavior of the Riemann tensor to find that it peels off in powers of r^{-1} as (indices suppressed)

$$(5.18) \quad R = Nr^{-1} + IIIr^{-2} + IIr^{-3} + Ir^{-4} + I'r^{-5} + \ldots$$

where N, III, D, II are as classified in §3, while types I and I' have respectively geodesic and non-geodesic rays and are algebraically general.

While the methods of Bandi and Sachs work well at null infinity where they were intended to, if we try to extend their techniques back to finite distances from a source into regions of strong fields, the methods eventually break down as light rays form caustic surfaces and intersect resulting in coordinate singularities.

6. A.D.M. Analysis of the Wave Zone

Working with the canonical (Hamiltonian) form of the field equations of general relativity, Arnowitt, Deser, and Misner (A.D.M.) analyse gravitational radiation in a fashion which simultaneously clarifies both the weak field and multipole approximations (Arnowitt, Deser, and Misner 1961; Misner 1964). They separate the metric into a) dynamical variables ($g_{ij}{}^{TT}$, π^{ij} where $i,j = 1,2,3$), b) Newtonian like parts (g^{T}, π^{i}) and c) coordinate dependent parts (g_{i}, π^{T}, $g_{o\mu}$). For asymptotically flat geometries they show that we may go to sufficiently large distance r from sources to insure the existence of a "wave zone" in which the canonical variables obey the weak field linearized equations

$$(6.1) \quad \Box g_{ij}{}^{TT} = 0, \quad \Box \pi^{ijTT} = 0$$

neglecting terms of order r^{-2}. These dynamic modes travel independent of their origin, without any self-interaction or dependence on the Newtonian -like parts, and without any coordinate dependent effects (to order r^{-1}).

The wave zone is not equivalent to the weak field region, since the Newtonian-like parts determined by the interior region are not small compared to the radiation modes, and in fact these parts (which are quadratic in the metric) are asymptotically of the form $(g^T, \pi^i) \to (E/r, P^i/r)$.

The decomposition of the space metric g^{ij} into the various modes requires the free use of the projection operator $\nabla^{-2}\partial_i\partial_j$. This becomes a local operation for radiation of short wavelength, i.e., for

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

with $h_{\mu\nu} \ll 1$ in a region of length L, the short wavelength part of h_{ij}^{TT} for $|x| < L$ is independent of the behavior of h_{ij} for $|x| > L$ (Misner 1964). Therefore (6.1) is really only meaningful for the high frequency components of the radiation field.

To measure the flux of energy in the canonical modes A.D.M. use the Poynting vector

$$S^i = \pi^{\ell m} \left(2g^{TT}_{\ell i,m} - g^{TT}_{\ell m,i}\right).$$

When this is averaged over oscillations as in electrodynamics, it is found to be coordinate independent.

For the remainder of this thesis, we will repeatedly come back to some of the lessons learned in the A.D.M. analysis. Specifically, the restriction to high frequencies, and the use of averaging to insure coordinate invariance will be essential to our discussion.

7. Asymptotic Methods for High Frequency

Gravitation differs uniquely from other classical fields in its ability to generate itself in the absence of all other material sources. This self-interaction inevitably leads to a non-linear set of field equations to describe gravity, and it is possible to use the self-interaction to "bootstrap" up from a flat space linear spin-two field and reach the non-linear Einstein field (Feynman 1962; Gupta 1957; Thirring 1961).

We may always rewrite the equations of general relativity in such a way as to exhibit an effective contribution to the total stress-energy which comes from gravity itself. This shifting of terms from the left to the right side of the field equations is trivial from a mathematical viewpoint, but can contribute to our physical insight about gravitation in certain classes of geometries. These are geometries which consist of a smoothly changing "background" metric which has been altered by perturbations of small amplitude but of high frequency. For example, Wheeler (1962) has used this viewpoint in estimating the possible energy density present in gravitational waves (perturbations) moving through the large scale structure of the universe (background geometry). Here, energy is regarded as localized in the high frequency waves and when averaged over many wavelengths serves as a source for the curvature of the cosmos.

Following Wheeler, Brill and Hartle (1964) published a study of spherical gravitational geons, i.e., localized bound concentrations of high frequency gravitational waves. In it, they present an exciting new approximation method for treating gravitational waves in a highly curved space, and they again emphasize the reality of the effective stress-energy for gravitational radiation of high frequency. For their geon, they assume a metric of the form

$$g_{\mu\nu} = \gamma_{\mu\nu} + h_{\mu\nu} \tag{7.1}$$

where $\gamma_{\mu\nu}$ is regarded as the highly curved metric for the long range, time-averaged gravitational field of the geon, and $h_{\mu\nu}$ is a small perturbation which describes the local effect of the gravitational waves in the geon.

The Einstein equations are then expanded as

$$(7.2)\quad G_{\mu\nu}(g_{\alpha\beta}) \equiv G_{\mu\nu}(\gamma_{\alpha\beta}) + \Delta G(\gamma_{\alpha\beta}, h_{\alpha\beta}) = 0$$

where $G_{\mu\nu}(m_{\alpha\beta})$ is the Einstein tensor for the metric $m_{\alpha\beta}$. Since the background is regarded as being produced by the time-averaged effects of the perturbation, (7.2) is replaced by the equivalent set of two equations

$$(7.3)\quad \text{a)}\quad \Delta G_{\mu\nu}(\gamma_{\alpha\beta}, h_{\alpha\beta}) = \langle \Delta G_{\mu\nu}(\gamma_{\alpha\beta}, h_{\alpha\beta}) \rangle$$

$$\text{b)}\quad G_{\mu\nu}(\gamma_{\alpha\beta}) = -\langle \Delta G_{\mu\nu}(\gamma_{\alpha\beta}, h_{\alpha\beta}) \rangle$$

where $\langle \ldots \rangle$ denotes an average over a time long compared to the period of $h_{\mu\nu}$. The exact equations (7.3) are now made into an approximation scheme by assuming $h_{\alpha\beta}$ small and expanding $\Delta G_{\mu\nu}$ in powers of $h_{\alpha\beta}$ as

$$(7.4)\quad \Delta G_{\mu\nu} = \Delta_1 G_{\mu\nu} + \Delta_2 G_{\mu\nu} + \ldots$$

which yields the approximate equations

$$(7.5)\quad \text{a)}\quad \gamma^{\alpha\beta}[h_{\mu\nu;\alpha\beta} + h_{\alpha\beta;\mu\nu} - h_{\mu\alpha;\nu\beta} - h_{\nu\alpha;\mu\beta}] = 0$$

(where the semicolon denotes covariant derivatives using the background metric)

$$\text{b)}\quad G_{\mu\nu}(\gamma_{\alpha\beta}) = -\langle \Delta_2 G_{\mu\nu}(\gamma_{\alpha\beta}, h_{\alpha\beta}) \rangle$$

Here, Brill and Hartle have arrived at the gravitational analogue of the Hartree-Fock self-consistent field approach for atomic wavefunctions and potentials. That is, the perturbation $h_{\mu\nu}$ satisfies the wave equation (7.5a) but this can not be solved until the background geometry $\gamma_{\alpha\beta}$ is determined from (7.5b). The latter involves $h_{\alpha\beta}$, so we have come full circle to find that (7.5a) and (7.5b) must be solved simultaneously. Brill and Hartle proceed to do this for a spherical background and find that the geon traps gravitational waves for long periods, but small amounts of energy continually diffuse away as the waves tunnel through the gravitational potential barrier which serves to bind them.

Brill and Hartle have neatly demonstrated that localized regions of space-time may be strongly curved up by the energy in high frequency gravitational waves. At the opposite extreme, Brill (1964) has also shown that the entire universe may be curved up and in fact closed if it is filled only with high frequency gravitational radiation carrying large amounts of effective energy. Here the metric is again assumed of the form (7.1), with background geometry

$$(7.6) \quad \gamma_{\alpha\beta}dx^{\alpha}dx^{\beta} = dt^2 - \frac{R^2(t)}{(1 + r^2/4R_o^2)}(dr^2 + r^2d\Omega^2)$$

Using the fact that $<h_{\alpha\beta}> = 0$, he shows that the averaged background curvature scalar vanishes to second order, i.e.,

$$(7.7) \quad <R(\gamma_{\alpha\beta})> = 0$$

from which the time development of the radius of the universe is given by

$$(7.8) \quad R(t) = R_o(1-t^2/R_o^2)^{1/2}.$$

Therefore, in the limit of high frequency, the time development of Brill's gravitational radiation filled universe is identical to that of Tolman's (1958) universe filled with electromagnetic radiation.

In summary, the work of Brill and Hartle has provided us with a powerful method for uncovering many features of gravitation which emerge clearly in the high frequency limit. Here it seems that gravitational energy may be localized (at least to a region on the scale of a wavelength) and that gravitational radiation fields exhibit properties similar to electromagnetic radiation fields. It is with this as motivation that we will go on to explore the consequences of the high frequency assumption in the next two chapters of this thesis. We will study the Brill and Hartle approximation method for gravitational waves in a self-consistent background field, but with a more detailed discussion of the basis they gave, paying attention to gauge (coordinate) invariance, and developing some new applications.

CHAPTER II

THE HIGH FREQUENCY EXPANSION AND THE LINEAR APPROXIMATION

1. Introductory Remarks

Objects such as neutron stars, collapsing supernovae, and quasars may endow regions of space with gravitational fields which seem enormous by terrestrial standards and may provide us with natural sources of intense gravitational radiation. In order to mathematically describe the waves from such objects, we must use the full apparatus of general relativity. This, however, runs us up against the notorious complexity of the nonlinear Einstein field equations. If we do not wish to be limited to unphysically overspecialized models with high symmetry, we must abandon any practical hope of getting exact solutions to these equations, and must content ourselves with just finding good approximations to the true radiative solutions. Here other problems arise, for the conventional technique of obtaining approximate wave solutions via linearization of the field equations (Einstein 1916) has its own drawbacks. First of all, linearization is accomplished by assuming that field strengths are weak and that space-time is essentially flat to lowest order. From the very start then, this procedure is manifestly inapplicable to the really interesting strong field problems, where gravitational waves can be expected to impart huge curvature to the fabric of space-time. Secondly, linearization of the field equations inherently destroys the possibility of describing the interaction of the gravitational field with itself. Therefore, within any linear approximation we cannot explore the resulting secular changes in geometry as energy is transported away from regions in space containing purely gravitational fields. We seem to be challenged to find a new method of obtaining approximate solutions representing strong gravitational radiation propagating through a highly curved space, and capable of including at least some nonlinear features of the theory as well.

Approximation procedures overcoming some of these limitations have already been developed (Bondi et al 1960, 1962; Sachs 1962). These depend on expanding the metric in powers of the luminosity distance from an isolated source which is embedded in an asymptotically flat space and emitting outgoing radiation. Even with these hypotheses, much algebra must be done before basic results are uncovered. In this paper we will follow the approach of Brill and Hartle (I §7). The essential assumption we will make is that the gravitational radiation is to have a high frequency, and our plan will be to expand the Einstein equations in powers of the wavelength of the radiation, the small parameter this assumption supplies. We find an approximation scheme correct to all orders of 1/r, all magnitudes of the field strength and valid for arbitrary velocities up to that of light. To lowest order we will have a linear wave equation for the gravitational radiation which will tell how the curvature of space interacts with and modifies the wave. Later, we will incorporate higher order terms in the expansion to see how the wave reacts back on the geometry of space in a nonlinear feedback process. The higher order nonlinearities will be left as the subject of Chapter III, while the present chapter will treat the basic expanision procedure, its gauge invariance, and the linear approximation in great detail. We will analyse the linear equations in such a way that the strong analogy between gravitation and electromagnetism in the geometrical optics limit clearly emerges.

What exactly do we mean when we say the radiation is of "high frequency", and under what circumstances can we expect such fields to be important? We will call the frequency "high" whenever the wavelength of the gravitational field is small compared to the radius of curvature of the background geometry. This assumption already is seen to hold in the conventional weak field linearization where the background is flat and thus has infinite radius of curvature, and so all weak field results will just be a special case of our general theory. Besides the binary star systems, fissioning stars, oscillating and rotating spheroids, and other conventional weak field sources (Vishveshwara 1964; Chin 1965; Chau 1966), there exist sources of gravitational radiation at optical frequencies which may be among the more important sources which persist for long periods of time. For example, the predominant source of gravitational waves from the sun is in the thermal motion of matter causing gravitational bremsstrahlung (Mironovskii 1965; Weinberg 1965; Carmeli 1966). Also, gravitational waves of optical frequency should arise as photons are converted into gravitons of the same frequency in the presence of a constant electromagnetic field (Gertsenshtein 1962). Moreover, all gravitational radiation from isolated systems is of high frequency when it gets far enough away from its source, for assuming that the wavelength λ remains approximately constant, as we increase the distance r from the source of mass m, the ratio of wavelength to radius of curvature of space is of order $(\lambda^2 m/r^3)^{1/2}$ which becomes negligible for large r. In fact, even wavelengths on the scale of galactic diameters or intergalactic distances are seen to be short when compared to the average background cosmological curvature of the universe. These examples give some idea of the scope to which the high frequency approximation can be applied.

While the hypothesis of high frequency will hold throughout this entire thesis, we will sometimes combine it with an additional hypothesis. This will be to assume the existence of a simple asymptotic expansion of the exact solution, valid when the frequency becomes very large, and wavefront curvatures are negligible, in which the leading term is locally a single plane wave. Asymptotic methods are often applied to physical problems containing a parameter in order to give results where exact solutions are difficult to obtain.

Moreover, even if such exact solutions are available, it is invariably simpler to obtain the asymptotic expansion directly than to first find the exact solution and then to extract out its asymptotic behavior. We shall call the existence of such an expansion the "W.K.B. assumption", after its best known application.

2. Expansion of the Ricci Tensor

We picture gravitational radiation as a small ripple in the geometry of space time, running through a highly curved, slowly changing background. The frequency of the ripple is high, but its amplitude quite small since we do not want pathological situations where physicists struck by radiation change size even faster than Alice in Wonderland. However, this by no means implies that the energy content of the wave is small. Quite the contrary, the energy carried along by the gravitational wave is pictured as a major (if not the only) cause for the background geometry to curve up. Following Brill and Hartle we assume the total metric $g_{\mu\nu}$ takes the form

$$g_{\mu\nu} = \gamma_{\mu\nu} + \varepsilon\, h_{\mu\nu}$$

where $\gamma_{\mu\nu}$ represents the background metric which is a slowly varying function of space-time, $h_{\mu\nu}$ is the high frequency ripple which changes significantly over a much smaller distance, and ε is a smallness parameter which insures that laboratory geometry has only microscopic fluctuations.

We introduce estimates of how fast the metric components vary by saying that typically their derivatives are of order

$$\partial\gamma \sim \gamma/L \qquad \partial h \sim h/\lambda$$

where L and λ are characteristic lengths over which the background and ripple change significantly, and where L is assumed very much larger than λ. The effective energy density contained in the wave is of order $(c^4/G)(\varepsilon/\lambda)^2$, while the curvature of the background is of order L^{-2}. The Einstein equations then tell us that the background curvature is equal to G/c^4 times the total energy density curving the background, or $L^{-2} \geq (G/c^4)(c^4/G)(\varepsilon/\lambda)^2$, i.e., $\varepsilon \leq \lambda/L$. The most interesting case occurs when no other sources of energy besides gravitational waves are present, and the two dimensionless numbers are equal, $\varepsilon = \frac{\lambda}{L} << 1$. We will make this assumption in order to simplify matters, and will only need to concern ourselves with the one small parameter ε. Once this is done the total metric remains slowly changing on a macroscopic scale, and the total curvature will be entirely due to the microscopic wave. We are now in a position to formalize our order of magnitude arguments and give an axiomatic characterization to the types of metrics of interest. All we need do is regard L as a constant (say of order one) and λ as a parameter which is to be replaced by $\varepsilon = \lambda/L$ since $O(\lambda) = O(\varepsilon)$ [$f(x) = O(\varepsilon^n)$ is defined to mean that there exists a constant M such that $|f(x)| < M\varepsilon^n$ as $\varepsilon \to 0$]. Then we see that we are studying the one parameter class of geometries differing infinitesimally by a high frequency radiation field which serves as a source for the background metric common to all. We will say that <u>a metric contains a high frequency radiation field</u> if and only if there exist a family of coordinate systems (called steady coordinates), related by infinitesimal coordinate transformations, in which the total metric takes the form

(2.1) $$g_{\mu\nu}(x) = \gamma_{\mu\nu}(x) + \varepsilon\, h_{\mu\nu}(x,\varepsilon)$$

(2.2) $$\varepsilon << 1, \qquad \gamma_{\mu\nu} = O(1), \qquad h_{\mu\nu} = O(1)$$

(2.3) $\gamma_{\mu\nu,\alpha} = O(1) \quad h_{\mu\nu,\alpha} = O(\epsilon^{-1})$

(2.4) $\gamma_{\mu\nu,\alpha\beta} = O(1) \quad h_{\mu\nu,\alpha\beta} = O(\epsilon^{-2})$

It should be noted that (2.1)-(2.4) imply a highly curved space, since in steady coordinates the Riemann tensor is $R_{\alpha\beta\gamma\delta} = O(\epsilon^{-1})$. Letting $R_{\alpha\beta}(m_{\alpha\beta})$ denote the Ricci tensor formed out of the metric $m_{\alpha\beta}$ (for sign conventions see Appendix A), we may expand the Ricci tensor for the total metric in powers of ϵ to obtain (Brill and Hartle 1964)

(2.5) $$R_{\alpha\beta}(\gamma_{\mu\nu}+\epsilon h_{\mu\nu}) \equiv R^{(0)}_{\alpha\beta} + \epsilon\, R^{(1)}_{\alpha\beta} + \epsilon^2\, R^{(2)}_{\alpha\beta} + \epsilon^3\, R^{(3+)}_{\alpha\beta}$$

where

(2.6) $$R^{(0)}_{\alpha\beta} = R_{\alpha\beta}(\gamma_{\mu\nu})$$

(2.7) $$R^{(1)}_{\alpha\beta} = \frac{1}{2}\gamma^{\rho\tau}(h_{\rho\tau;\alpha\beta} + h_{\alpha\beta;\rho\tau} - h_{\tau\alpha;\beta\rho} - h_{\tau\beta;\alpha\rho})$$

(2.8) $$R^{(2)}_{\alpha\beta} = -\frac{1}{2}[\frac{1}{2}h^{\rho\tau}{}_{;\beta}h_{\rho\tau;\alpha} + h^{\rho\tau}(h_{\tau\rho;\alpha\beta} + h_{\alpha\beta;\tau\rho} - h_{\tau\alpha;\beta\rho} - h_{\tau\beta;\alpha\rho}) + h_{\beta}{}^{\tau;\rho}(h_{\tau\alpha;\rho} - h_{\rho\alpha;\tau}) - (h^{\rho\tau}{}_{;\rho} - \frac{1}{2}h^{;\tau})(h_{\tau\alpha;\beta} + h_{\tau\beta;\alpha} - h_{\alpha\beta;\tau})].$$

Here, the semicolons denote covariant differentiation with respect to the background metric which is also used to raise or lower all indices. The remainder term $R^{(3+)}_{\alpha\beta}$ is now fully defined by (2.5)-(2.8). Even if we allow for the manifest powers of ϵ in (2.5), the quantities defined by (2.6, 7, and 8) are not intrinsically of the same magnitude. Symbolically, the Ricci tensor is

$$R(g) = g^{-1}\partial^2 g$$

and the metric and its inverse are

$$g = \gamma + \varepsilon h$$

$$g^{-1} = \gamma^{-1} + \varepsilon h\gamma^{-2} + \varepsilon^2 h^2 \gamma^{-3} + \ldots$$

Although γ and h are the same order, by (2.3) and (2.4) their derivatives are very different. Therefore when second derivatives are applied to γ, the result is very much smaller than when applied to h. The results are summarized in Table I. We see that the dominant term is $\varepsilon R^{(1)}_{\alpha\beta} = O(\varepsilon^{-1})$. Smaller than this by a factor ε are both $R^{(0)}_{\alpha\beta}$, and $\varepsilon^2 R^{(2)}_{\alpha\beta} = O(1)$. Smallest of all is the remainder term $\varepsilon^3 R^{(3+)}_{\alpha\beta} = O(\varepsilon)$, down by ε^2 from the dominant term.

TABLE I. MAGNITUDE OF TERMS IN EXPANSION OF RICCI TENSOR

Term	Symbolic Form	Order of Magnitude
$R^{(0)}_{\alpha\beta}$	$\gamma^{-1}\ \partial^2\gamma$	$L^{-2} \cong \varepsilon^2\lambda^{-2} = O(1)$
$\varepsilon\ R^{(1)}_{\alpha\beta}$	$\gamma^{-1}\ \partial^2(\varepsilon h)$	$\varepsilon\ \lambda^{-2} = O(\varepsilon^{-1})$
$\varepsilon^2\ R^{(2)}_{\alpha\beta}$	$\varepsilon h\gamma^{-2}\ \partial^2(\varepsilon h)$	$\varepsilon^2\ \lambda^{-2} = O(1)$
$\varepsilon^3\ R^{(3+)}_{\alpha\beta}$	$\varepsilon^2 h^2\gamma^{-3}\ \partial^2(\varepsilon h)$	$\varepsilon^3\lambda^{-2} = O(\varepsilon)$

3. Expansion of the Riemann Tensor

Just as we did for the Ricci tensor, we may expand the Riemann tensor $R_{\alpha\beta\gamma\delta}(g_{\mu\nu})$ in powers of ε to obtain

$$R_{\alpha\beta\gamma\delta}(\gamma_{\mu\nu}+\varepsilon\ h_{\mu\nu}) = R^{(0)}_{\alpha\beta\gamma\delta} + \varepsilon R^{(1)}_{\alpha\beta\gamma\delta} + \varepsilon^2\ R^{(2)}_{\alpha\beta\gamma\delta} + \varepsilon^3\ R^{(3+)}_{\alpha\beta\gamma\delta} \quad (3.1)$$

where

$$(3.2) \qquad R^{(0)}_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta}(\gamma_{\mu\nu})$$

$$(3.3) \qquad R^{(1)}_{\alpha\beta\gamma\delta} = \frac{1}{2}\,(h_{\alpha\gamma;\beta\delta} + h_{\beta\delta;\alpha\gamma} - h_{\beta\gamma;\alpha\delta} - h_{\alpha\delta;\beta\gamma} + R^{(0)}_{\alpha\sigma\gamma\delta}\, h^{\sigma}{}_{\beta} - R^{(0)}_{\beta\sigma\gamma\delta}\, h^{\sigma}{}_{\alpha})$$

As before $\varepsilon\, R^{(1)}_{\alpha\beta\gamma\delta} = O(\varepsilon^{-1})$ is dominant in magnitude, $R^{(0)}_{\alpha\beta\gamma\delta}$ and $\varepsilon^2\, R^{(2)}_{\alpha\beta\gamma\delta} = O(1)$ are smaller by a factor of ε, and $\varepsilon^3\, R^{(3+)}_{\alpha\beta\gamma\delta} = O(\varepsilon)$ is smaller by ε^2.

The expressions defined in (2.7) and (3.3) will be shown to be gauge invariant in the high frequency limit. Because of this $R^{(1)}_{\alpha\beta\gamma\delta}$ will play a central role in distinguishing the presence of coordinate waves from true gravitational effects. In the absence of radiation (i.e., $h_{\mu\nu} = 0$), the Riemann tensor reduces in value to $R^{(0)}_{\alpha\beta\gamma\delta}$. However, if gravitational waves are present, the total curvature of space-time grows enormously in magnitude to the dominant $\varepsilon\, R^{(1)}_{\alpha\beta\gamma\delta}$ term. If space is empty of waves, but a gauge transformation has mixed in coordinate effects, the total curvature is still only of order $R^{(0)}_{\alpha\beta\gamma\delta}$, and so, easy to distinguish.

$R^{(1)}_{\alpha\beta\gamma\delta}$ satisfies the same symmetries as the total Riemann tensor $R_{\alpha\beta\gamma\delta}(g_{\mu\nu})$, but the "Bianchi identities" hold only in the limit of zero wavelength. That is, we find

$$R^{(1)}_{\alpha\beta\gamma\delta} = -R^{(1)}_{\alpha\beta\delta\gamma}$$

$$R^{(1)}_{\alpha\beta\gamma\delta} = R^{(1)}_{\gamma\delta\alpha\beta}$$

$$R^{(1)}_{\alpha\beta\gamma\delta} + R^{(1)}_{\alpha\gamma\delta\beta} + R^{(1)}_{\alpha\delta\beta\gamma} = 0$$

$$R^{(1)}_{\mu\nu\alpha\beta;\gamma} + R^{(1)}_{\mu\nu\gamma\alpha;\beta} + R^{(1)}_{\mu\nu\beta\gamma;\alpha} \doteq \varepsilon^2 \;.$$

We have introduced a new symbol " $\doteq$ " which can be read "is down by a factor ϵ^2 (from a priori expectations)" whose use and definition can be seen by the following. The terms on the left hand side of the last equation each involve three derivatives of h. Thus, if we multiply this side by successive powers of ϵ, we would expect that a factor of ϵ^3 is necessary to yield a finite limit for the left side as $\epsilon \to 0$. In reality, we find that the last expression is of the form $R^{(1)}_{\mu\nu\alpha\beta;\gamma} + \ldots = \Sigma\, R^{(0)}\, \partial h$. The right hand side is seen to remain finite if we multiply through by a factor of ϵ, two orders less than we would expect. This is the meaning of $\doteq \epsilon^2$. In a general case, $f \doteq \epsilon^p$ means that while $f = O(\epsilon^n)$ is expected by counting the number of derivatives of h in f, actually, due to some internal cancellation $f = O(\epsilon^{n+p})$. This new notation saves us from having to write confusing equations like $\epsilon^{-n} f = O(\epsilon^p)$.

4. Gauge Transformations and Invariance

We now wish to study infinitesimal coordinate transformations and the gauge transformations they induce, in order to establish the gauge invariance of individual terms in the expansions of the total Riemann and Ricci tensors.

Consider an infinitesimal coordinate transformation

$$(4.1) \qquad x^\alpha \to \bar{x}^\alpha = x^\alpha + \epsilon\, \xi^\alpha$$

In the new coordinate system, we find (Landau and Lifshitz 1962a) neglecting terms of order ϵ^2,

$$(4.2) \qquad g_{\alpha\beta} = \gamma_{\alpha\beta} + \epsilon(h_{\alpha\beta} - \xi_{\alpha;\beta} - \xi_{\beta;\alpha}) \quad .$$

Since $\epsilon\, h_{\alpha\beta}$ is defined as the difference between the background and total perturbed metric, we have

$$(4.3) \qquad \bar{h}_{\alpha\beta} = h_{\alpha\beta} - \xi_{\alpha;\beta} - \xi_{\beta;\alpha}$$

for the perturbation in the new coordinate system.

We interpret this to mean that the infinitesimal coordinate change induces a "gauge transformation" similar to those found in flat space spin-one and spin-two fields. Under this change in gauge, quantities dependent upon $h_{\mu\nu}$ also undergo transformations. Either by direct calculation or from the definition of the Lie derivative, we find that under a gauge transformation the dominant parts of the Ricci and Riemann tensors become

$$(4.4) \qquad R^{(1)}_{\alpha\beta} \rightarrow \bar{R}^{(1)}_{\alpha\beta} = R^{(1)}_{\alpha\beta} - \underset{\xi}{\mathcal{L}} R^{(0)}_{\alpha\beta}$$

$$R^{(1)}_{\alpha\beta\gamma\delta} \rightarrow \bar{R}^{(1)}_{\alpha\beta\gamma\delta} = R^{(1)}_{\alpha\beta\gamma\delta} - \underset{\xi}{\mathcal{L}} R^{(0)}_{\alpha\beta\gamma\delta}$$

where the Lie derivatives are explicitly given by

$$(4.5) \qquad \underset{\xi}{\mathcal{L}} R^{(0)}_{\alpha\beta} = R^{(0)}_{\alpha\beta;\sigma} \xi^{\sigma} + R^{(0)}_{\sigma\beta} \xi^{\sigma}_{;\alpha} + R^{(0)}_{\alpha\sigma} \xi^{\sigma}_{;\beta}$$

$$\underset{\xi}{\mathcal{L}} R^{(0)}_{\alpha\beta\gamma\delta} = R^{(0)}_{\alpha\beta\gamma\delta;\sigma} \xi^{\sigma} + R^{(0)}_{\sigma\beta\gamma\delta} \xi^{\sigma}_{;\alpha} + R^{(0)}_{\alpha\sigma\gamma\delta} \xi^{\sigma}_{;\beta} + R^{(0)}_{\alpha\beta\sigma\delta} \xi^{\sigma}_{;\gamma} + R^{(0)}_{\alpha\beta\gamma\delta} \xi^{\sigma}_{;\delta}$$

In (4.2) the additional terms $\varepsilon\, \xi_{(\alpha;\beta)}$ can only be regarded as resulting from an infinitesimal coordinate transformation if they allow us to still unambiguously call $\gamma_{\alpha\beta}$ the background metric. This implies that $\varepsilon\, \xi_{(\alpha;\beta)}$ is truly a small quantity compared to $\gamma_{\alpha\beta}$. If ξ^{α} is to be the generator of an infinitesimal coordinate transformation, it may be assumed to satisfy

$$(4.6) \qquad \xi_{\mu} = O(1)$$

$$\xi_{\mu;\nu} = O(1) \quad ,$$

a form sufficiently general to admit both high and low frequency coordinate waves.

If we insert (4.6) and (2.1 - 2.4) into (4.4) and (4.5) we then see that

$$(4.7) \qquad R^{(1)}_{\alpha\beta} - \bar{R}^{(1)}_{\alpha\beta} \doteq \varepsilon^2$$

$$R^{(1)}_{\alpha\beta\gamma\delta} - \bar{R}^{(1)}_{\alpha\beta\gamma\delta} \doteq \varepsilon^2 \quad .$$

In the high frequency limit $\varepsilon \to 0$, we see that the "perturbations" of the Riemann and Ricci tensor are gauge invariant to an extremely good approximation and therefore meaningful entities, capable of physical measurement. The basic reason behind this is that on a scale of distance of order λ, space appears locally flat, and curvature is locally gauge invariant as in weak field linear theory. As long as $\lambda << L$, perturbations do not have any long wavelength components, and this local behavior carries over to curved backgrounds to give a global gauge invariance. The fact that our expansion is gauge invariant is extremely important since any physically observable effects cannot be coordinate dependent. Finally, it should be emphasized that our gauge invariance has resulted only from the assumption of high frequency.

5. The Linear Approximation

We have just found a remarkable degree of freedom in the choice of a gauge and so it is natural to exploit this in order to either simplify the labor involved in future calculation, or to exhibit interesting results most effectively. To decide on a convenient gauge, let us briefly review the theory of massless spin-two fields in flat space. It is well-known (Wentzel 1949) that such a field may be described by a real symmetric tensor field $\psi^{\mu\nu}$ satisfying

$$(5.1) \quad \text{a)} \quad \psi^{\mu\nu,\beta}{}_{,\beta} = 0$$

$$\text{b)} \quad \psi^{\mu\nu}{}_{,\nu} = 0$$

$$\text{c)} \quad \psi \equiv \eta_{\mu\nu}\,\psi^{\mu\nu} = 0$$

where $\eta_{\mu\nu} \equiv \mathrm{diag}\,(1,-1,-1,-1)$. The supplementary conditions (5.1 b,c) insure that the field energy is positive definite and the field pure spin-two without spin-zero or spin-one components. The number of degrees of freedom of $\psi^{\mu\nu}$ is reduced from the five implied by (5.1) to just two because (5.1) is left unaltered by the gauge transformation

$$(5.2) \quad \psi^{\mu\nu} \rightarrow \bar{\psi}^{\mu\nu} = \psi^{\mu\nu} - \xi^{\mu,\nu} - \xi^{\nu,\mu}$$

where ξ^{μ} is a vector field satisfying

$$(5.3) \quad \text{a)} \quad \xi^{\mu,\alpha}{}_{,\alpha} = 0$$

$$\text{b)} \quad \xi^{\mu}{}_{,\mu} = 0$$

The standard weak field linearization of gravity can be put into the form described in (5.1) by means of infinitesimal coordinate transformations, and it retains this form under the class of coordinate transformations $x^{\mu} \rightarrow \bar{x}^{\mu} = x^{\mu} + \xi^{\mu}$ satisfying (5.3) (Wentzel 1949; Pauli 1958).

Now we are in position to proceed to examine the strong field case, using the "correspondence principle" that the theory we derive must reduce back to equations (5.1) when field strengths become negligible. Applying the decomposition of the Ricci tensor to the Einstein equations _in vacuuo,_ we find to lowest order

$$R^{(1)}_{\alpha\beta} = 0 \tag{5.4}$$

and to the next order

$$R^{(0)}_{\alpha\beta} = -\varepsilon^2 R^{(2)}_{\alpha\beta} \quad . \tag{5.5}$$

The remainder of this chapter will be devoted to the analysis of the linear equation (5.4). This was derived by Lanczos (1925), and used by Regge and Wheeler (1957) to study the stability of the Schwarzschild solution. The nonlinear equation (5.5) (which we will treat in Chapter III later) was first written by Brill and Hartle (1964) to show that gravitational radiation can be thought to have an effective stress-energy which can produce the background curvature.

Let us define

$$\psi_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}\, h \tag{5.6}$$

$$\psi \equiv \gamma^{\alpha\beta}\, \psi_{\alpha\beta}$$

where $h \equiv \gamma^{\alpha\beta}\, h_{\alpha\beta}$. Regarding the $\psi_{\mu\nu}$ as our basic field quantities, we may rewrite (5.4) as

$$\psi_{\mu\nu}{}^{;\beta}{}_{;\beta} - \frac{1}{2}\gamma_{\mu\nu}\,\psi^{;\beta}{}_{;\beta} - \psi_{\mu\beta}{}^{;\beta}{}_{;\nu} - \psi_{\nu\beta}{}^{;\beta}{}_{;\mu} \tag{5.7}$$

$$+ 2R^{(0)}_{\sigma\nu\mu\beta}\,\psi^{\beta\sigma} + R^{(0)}_{\mu\sigma}\,\psi^{\sigma}{}_{\nu} + R^{(0)}_{\nu\sigma}\,\psi^{\sigma}{}_{\mu} = 0$$

Comparing this to the flat space equations (5.1), we have the strong temptation to impose as our choice of gauge that

$$\psi^{\mu\nu}{}_{;\nu} = 0 \tag{5.8}$$

$$\psi = 0 \quad . \tag{5.9}$$

Let us investigate whether it is possible to make an infinitesimal coordinate transformation $x^\alpha \rightarrow \bar{x}^\alpha = x^\alpha + \varepsilon\, \xi^\alpha$ to bring about the desired conditions (5.8) and (5.9). Under this coordinate transformation we have $\psi_{\mu\nu} \rightarrow \bar{\psi}_{\mu\nu}$ and

$$(5.10) \qquad \psi_{\mu\nu}{}^{;\nu} \rightarrow \bar{\psi}_{\mu\nu}{}^{;\nu} = \psi_{\mu\nu}{}^{;\nu} - \xi_\mu{}^{;\nu}{}_{;\nu} + R^{(0)}_{\mu\nu}\, \xi^\nu$$

$$(5.11) \qquad \psi \rightarrow \bar{\psi} = \psi + 2\xi^\mu{}_{;\mu} \quad .$$

If we choose ξ^μ to be the solution to the inhomogeneous equation

$$\xi_\mu{}^{;\nu}{}_{;\nu} - R^{(0)}_{\mu\nu}\, \xi^\nu = \psi_{\mu\nu}{}^{;\nu}$$

we have that in our new gauge (dropping the bar) $\psi^{\mu\nu}{}_{;\nu} = 0$. From (4.7) we see that the wave equation in the new coordinate system retains the form (5.7) when we drop the negligible Lie derivative terms. Upon contracting (5.7) and using (5.8) we obtain $\psi^{;\beta}{}_{;\beta} = 0$, and so the wave equation becomes in the new coordinates

$$(5.12) \qquad -\Delta\psi_{\mu\nu} \equiv \psi_{\mu\nu}{}^{;\beta}{}_{;\beta} + 2R^{(0)}_{\sigma\nu\mu\beta}\, \psi^{\beta\sigma}\, R^{(0)}_{\mu\sigma}\, \psi^\sigma{}_\nu + R^{(0)}_{\nu\sigma}\, \psi^\sigma{}_\mu = 0 \quad .$$

The operator Δ is precisely the generalization of the flat space d'Alembertian which Lichnerowicz introduced for symmetric tensors following de Rham's definition of a similar operator for antisymmetric ones (Lichnerowicz 1964; de Rham 1965) and (5.12) clearly reduces to (5.1a) for weak fields. The gauge condition (5.9) is consistent with (5.12), but if we differentiate (5.12) we obtain

$$(5.13) \qquad (-\Delta\psi_{\mu\nu})^{;\nu} = \psi_{\mu\nu}{}^{;\nu\sigma}{}_{;\sigma} + R^{(0)}_{\mu\sigma}\, \psi^{\sigma\nu}{}_{;\nu} + \psi^{\sigma\nu}(2R^{(0)}_{\mu\sigma;\nu} - R^{(0)}_{\sigma\nu;\mu}) = 0$$

which shows that (5.12) and (5.8) are in general inconsistent. There are, however, important advantages to be derived from using (5.12) instead of other equations (such as $\psi_{\mu\nu}{}^{;\beta}{}_{;\beta} = 0$) which differ from it by terms down by a factor ε^2, since Δ is a mathematically well-behaved operator. Thus in (5.13) the terms involving the derivatives of the Ricci tensor vanish in spaces with backgrounds of constant curvature, such as the Schwarzschild metric, and then (5.12), (5.8), and (5.9) will have a consistent solution. We thus regard (5.8), (5.9), and (5.12) as our fundamental equations for gravitational waves in the first (linear) approximation.

We may write (5.13) as

$$(5.14) \qquad \psi_{\mu\nu}{}^{;\nu\sigma}{}_{;\sigma} \doteq \varepsilon^2 \ .$$

Then, if the gauge conditions (5.8) and (5.9) hold on some initial hypersurface, the wave equation (5.12) guarantees that

$$(5.15) \qquad \psi_{\mu\nu}{}^{;\nu} \doteq \varepsilon^2 \ , \qquad \psi \doteq \varepsilon^2$$

everywhere else in space. It is always possible to find an infinitesimal coordinate transformation to bring about equations (5.15), after which the wave equation (5.12) and gauge conditions (5.15) are left invariant by the class of gauge transformations generated by ξ^μ satisfying

$$(5.16) \qquad \xi_\mu{}^{;\nu}{}_{;\nu} - R^{(0)}_{\mu\nu}\xi^\nu = 0$$

$$\xi^\mu{}_{;\mu} = 0 \ .$$

This final gauge freedom can be pinned down if desired by requiring additional conditions, for example $\psi_{\mu o} = 0$.

The wave equation (5.7) without a gauge specialization is derivable from a variational principle with Lagrangian density.

$$(5.17)\quad \mathcal{L} = -(c^4/32\pi G)\left(\tfrac{1}{2}\, h^{\alpha\beta;\rho}\, h_{\alpha\beta;\rho} - \tfrac{1}{2}\, h^{;\rho}\, h_{;\rho} + h_{;\alpha}\, h^{\alpha\beta}{}_{;\beta} - h_{\rho\beta;\alpha}\, h^{\alpha\beta;\rho}\right)(-\gamma)^{1/2}$$

However $\mathcal{L}$ is not gauge invariant and contains explicit x^{μ} dependence. Therefore, the canonical energy-momentum tensor

$$t_{\alpha}^{\beta} = \mathcal{L}\,\delta_{\alpha}^{\beta} - \frac{\partial \mathcal{L}}{\partial h_{\gamma\delta,\beta}}\, h_{\gamma\delta,\alpha}$$

is neither symmetric, conserved, nor gauge invariant, and t_{o}^{o} is not necessarily positive definite. A much better tensor than t_{α}^{β} will be shown to describe the energy content of our gravitational radiation when we consider the higher order nonlinearities in the approximation. (See III.)

6. W.K.B. Analysis of the Wave Equation

In the limit of flat space, the wave equation reduces to (5.1a) and it is sufficient to consider only one Fourier component of the solution, which may be written as

$$(6.1)\qquad \psi_{\mu\nu} = A_{\mu\nu}\, e^{ik_{\alpha}x^{\alpha}}$$

where $A_{\mu\nu}$ and k_{μ} are constants and the exponential is a rapidly fluctuating function of position. When we deal with a space containing gravitational fields, we know that geometry may be considered locally flat over distances of order L (Remember $R^{(0)}_{\alpha\beta\gamma\delta} \sim L^{-2}$) as can be seen by introducing normal Riemannian coordinates. On this scale of distance (6.1) should remain a fairly decent solution to the wave equation, however, due to the slowly varying geometry, the previously constant $A_{\mu\nu}$ and k_{μ} can be expected to slowly change in value over a characteristic distance of order L. Thus, we may expect to try for a solution to (5.12) of the form

$$(6.2) \qquad \psi_{\mu\nu} = A_{\mu\nu} e^{i\phi}$$

(actually only the real part of (6.2) is to be used) where $A_{\mu\nu}$ is a slowly changing real function of position, and ϕ is a real function with a large first derivative but no larger derivatives beyond this to correspond to a slowly changing k_α. Solutions of this form are frequently assumed in mathematical physics for both theoretical insight and computational ease. Perhaps the most familiar example for modern physicists is the W.K.B. approximation for one-dimensional problems in quantum mechanics where a solution in the form (6.2) is sought for an ordinary differential equation. For this reason we will call (6.2) the W.K.B. approximation, although its application to partial differential equations predates quantum mechanics. This method seems to have been first used by Sommerfeld and Runge (Sommerfeld and Runge 1911) to establish the transition from Maxwell's equations to classical geometrical optics as the wavelength of light approaches zero. Since that time the W.K.B. approximation has been used in many fields besides electromagnetism and quantum mechanics, such as acoustics, plasma physics, elasticity and hydrodynamics (see bibliography in Keller, Lewis, and Seckler 1956; see also Appendix B). In the varied literature this approximation is sometimes also known as the eikonel approximation or the method of stationary phase; however, it is really just the first term in an asymptotic expansion of the exact solution in the limit of the wavelength going to zero (see Kline 1954; Lewis 1958).

Returning to (6.2), let us now introduce ray vectors (normals to surfaces of constant phase ϕ) by

(6.3) $$k_{\alpha} \equiv \phi_{,\alpha}$$

We may estimate the order of various derivatives needed in calculations as

(6.4) $$R^{(0)}_{\alpha\beta\gamma\delta} \sim O(1)$$

$$A^{\mu\nu}{}_{;\tau} \sim O(1)$$

$$k_{\mu} \sim O(\varepsilon^{-1})$$

$$k_{\mu,\nu} \sim O(\varepsilon^{-1})$$

Note that (6.2) - (6.4) are compatible with (2.1) - (2.4). Now, substitute (6.2) into (5.8) - (5.9) to get

(6.5) $$\gamma^{\mu\nu} A_{\mu\nu} = 0$$

(6.6) $$i\, k_{\beta} A^{\alpha\beta} + A^{\alpha\beta}{}_{;\beta} = 0,$$

In this last equation, the first term is of order ε^{-1} while the second is of order 1, and must be neglected for a consistent approximation. This then gives

(6.7) $$k_{\beta} A^{\alpha\beta} = 0 \quad .$$

Similarly substituting (6.2) into (5.12) and grouping terms of the same size we find

(6.8) $$[-k_{\beta}k^{\beta}A_{\mu\nu}] + i[2k_{\beta}A_{\mu\nu}{}^{;\beta} + k^{\beta}{}_{;\beta} A_{\mu\nu}]$$

$$+ [A_{\mu\nu}{}^{;\beta}{}_{;\beta} + 2R^{(0)}_{\sigma\nu\mu\beta} A^{\beta\sigma} + R^{(0)}_{\mu\sigma} A^{\sigma}{}_{\nu} + R^{(0)}_{\nu\sigma} A^{\sigma}{}_{\mu}] = 0$$

The terms in the first, second, and third brackets are of order ε^{-2}, ε^{-1}, and 1 respectively. To lowest order, (6.8) is

(6.9) $$k_\beta k^\beta = 0 \ .$$

This gives us the important result that gravitational wave rays are null vectors. Alternately, we may write (6.9) as an eikonel equation

(6.10) $$\gamma^{\mu\nu} \phi_{,\mu} \phi_{,\nu} = 0 \ .$$

We may introduce a congruence of curves with rays as tangents by

(6.11) $$\frac{dx^\mu}{d\ell} = k^\mu \ .$$

The solution curves $x^\mu(\ell)$ are null geodesics and ℓ is a preferred affine parameter, since by differentiating (6.9) and remembering that k_α is a gradient we get

(6.12) $$0 = k_{\alpha;\beta} k^\alpha = k_{\beta;\alpha} k^\alpha$$

Thus the rays of the gravitational radiation field are parallel propagated tangentially along null geodesics, just as are the rays for electromagnetic waves (Kristian and Sachs 1966; Zipoy 1966; Trautman 1964; Appendix B).

Consequently, high frequency gravitational radiation is red shifted exactly the same way as light when, for example, passing through localized strong gravitational fields or while traveling across the universe.

Now, we may proceed to the second order terms in (6.8) which give us

(6.13) $$A_{\mu\nu}{}^{;\beta} k_\beta + \frac{1}{2} A_{\mu\nu} k^\beta{}_{;\beta} = 0 \ .$$

It is convenient to separate the behavior of the amplitude from that of the polarization of the gravitational radiation. At each point in space where there is radiation, we define a polarization tensor field $e_{\mu\nu}$ proportional to

$A_{\mu\nu}$ and whose arbitrary magnitude is fixed by normalizing so that $e_{\mu\nu} e^{\mu\nu} = 1$. This is given by $e_{\mu\nu} = (A^{\mu\nu}A_{\mu\nu})^{-1/2} A_{\mu\nu}$. As a shorthand, define the amplitude $\mathcal{A}$ of the radiation by $\mathcal{A} = (A_{\mu\nu}A^{\mu\nu})^{1/2}$. This is a real, positive scalar measure of the intensity of the radiation and vanishes only when no radiation is present. Now, substitute $A_{\mu\nu} = \mathcal{A} e_{\mu\nu}$ into (6.13) to obtain

$$(6.14) \qquad (\mathcal{A}_{,\beta}k^{\beta} + \tfrac{1}{2}\mathcal{A}\, k^{\beta}_{;\beta})e_{\mu\nu} + \mathcal{A} e_{\mu\nu;\beta}\, k^{\beta} = 0 \; .$$

Multiplying by $e_{\mu\nu}$, we have

$$(6.15a) \qquad (\log\mathcal{A})_{,\beta}k^{\beta} + \tfrac{1}{2}\, k^{\beta}_{;\beta} = 0 \; .$$

This is just an ordinary differential equation along the null ray, and gives the amplitude once the geometry of the ray congruence is known. This is easily seen if we rewrite (6.15a) using (6.11) to get

$$(6.15b) \qquad \frac{d}{d\ell} \log\mathcal{A} = -\frac{1}{2}\, k^{\beta}_{;\beta}$$

showing how the radiation decreases as the rays diverge.

Now from (6.14) and (6.15a) we see that

$$(6.16) \qquad e_{\mu\nu;\beta}k^{\beta} = \frac{\delta}{\delta\ell}\, e_{\mu\nu} = 0 \; .$$

This gives the important physical result that the polarization tensor is parallel transported along the null geodesic $x^{\mu}(\ell)$. At a fixed point along $x^{\mu}(\ell)$ we may impose the initial conditions

$$(6.17) \qquad \text{a)} \quad e_{\mu\nu}k^{\nu} = 0$$

$$\text{b)} \quad e_{\mu\nu}\gamma^{\mu\nu} = 0$$

Since both $e_{\mu\nu}$ and k_μ are parallel transported, these conditions consequently hold everywhere along $x^\mu(\ell)$, guaranteeing consistency with (6.5) and (6.7).

We see the remarkable similarity between light and gravitation, since the geometrical optics of light tells us its amplitude satisfies exactly the same transport equation (5.15a), and its polarization is also parallel propagated along the null geodesics with ray vectors k_μ as tangents (Kristian and Sachs 1966; Zipoy 1966; Appendix B).

We may rewrite (6.15a) in still another form

$$(6.15c) \qquad (\mathcal{A}^2 k^\beta)_{;\beta} = 0$$

which may be interpreted as a conservation law for the total number of gravitons present in our radiation, since if we let

$$N = \int_S k^0 \mathcal{A}^2 (-\gamma)^{1/2} \, d^3x$$

where S is a space-like hypersurface, x^0 a time-like coordinate and (x^1, x^2, x^3) three space-like coordinates, then $\frac{dN}{dx^0} = 0$ for localized pulses of radiation. Alternately (6.15c) shows that $\mathcal{A}^{-1}$ is a luminosity distance.

Additional insight into (5.14c) follows from work by Kristian and Sachs (1966) intended for light but which holds equally well for gravitational radiation. They consider two observers located at different points along a given ray and moving in such a fashion that the frequency ω they measure is the same. Each observer measures intensity $I = \mathcal{A}^2 \omega^2$ and cross-sectional area dA for the same bundle of rays. Then (6.15c) implies that the measured energy flux through dA is the same for either observer

$$I_1 \, dA_1 = I_2 \, dA_2 \; .$$

This shows that energy is transported with the rays serving as guidelines.

We may also find the dominant part of the total Riemann tensor using the W.K.B. approximation. Inserting (6.2) into (3.3), we find

$$R^{(1)}_{\alpha\beta\gamma\delta} - 2k_{[\alpha}h_{\beta][\gamma}k_{\delta]} \doteq \varepsilon$$

(where square brackets denote the antisymmetric part). Then $k^{\delta}R_{\alpha\beta\gamma\delta}(g_{\mu\nu}) \doteq \varepsilon$. This tells us that the ray vectors serve as principle null vectors of multiplicity four for the Riemann tensor of the total metric (at least to order ε), and in the high frequency limit the total metric in Petrov type N to lowest order in ε. The invariant classification of our metric agrees with the beautiful results of Sachs for radiation at large distances from bounded sources (see I (5.18)) and so this serves as additional confirmation that our original decomposition of the metric into a background plus small high frequency ripple does indeed correspond to the presence of gravitational radiation. It should be noted that the high frequency assumption can be applied to spaces which are not asymptotically flat.

7. Limits of Validity

While we have seen some of the power of the W.K.B. and high frequency approaches, they do have their limitations. Essentially, these are the same limits one finds in the geometrical optics approach to light (Kline and Kay 1965; Born and Wolf 1959). Thus, we may show that if a beam of gravitational waves is initially convergent, it will collapse, causing the energy density to become infinite at a finite parameter distance along the ray. Before this singularity arises, our W.K.B. approximation must break down, since the radius of curvature of wavefronts no longer is large, and $A_{\mu\nu}$ becomes a rapidly varying function of position. In general, whenever the ray congruence has a caustic (i.e., a manifold of dimension equal to or less than three on which any neighborhood contains a point with more than one ray of the congruence passing through it) we expect this to happen.

CHAPTER III

THE EFFECTIVE STRESS TENSOR FOR GRAVITATIONAL WAVES OF HIGH FREQUENCY

1. Introductory Remarks

The principle of equivalence tells us that it is possible to transform away any uniform gravitational fields by simply changing coordinates, but is this to hold for gravitational waves as well? The energy and momentum carried by an electromagnetic radiation field is measured by its stress tensor, but a gravitational field is usually described by a pseudotensor which can be locally annihilated by a suitable coordinate transformation. While the pseudotensor is satisfactory enough for calculating the total energy or momentum of an isolated system, there is no way of localizing the distribution of these quantities if we keep using it.

In the preceeding chapter, we showed how the assumption that the gravitational field is of high frequency led to a gauge invariant approximation procedure to first order. When this was combined with the assumption of a W.K.B. form $A_{\mu\nu}e^{i\phi}$ for the radiation, we found that the gravitational field was remarkably similar to the electromagnetic field in the behavior of its amplitude, frequency, and polarization. In this present chapter, we will extend the results of the linear approximation in II to incorporate some of the essential nonlinear features of the Einstein equations and in so doing also extend the analogy between light and gravitation. We shall find that in the high frequency limit, the gravitational radiation field has a natural gauge invariant stress tensor which does not involve a choice of special coordinates or arbitrary vector fields. Since we now have a true tensor, it cannot be made to vanish by a simple coordinate transformation. Like the Maxwell stress tensor, the effective stress tensor for gravitational waves involves only first derivatives of the radiation field, allows us to introduce a Poynting vector to describe the flow of energy and momentum, and acts as a source generating curvature of space-time.

While the hypothesis of high frequency (i.e., the wavelength of radiation is much smaller than the radius of curvature of the background geometry) will be used throughout this entire chapter, we will sometimes combine it with either or both of two additional and logically independent assumptions. The first of these is to assume the W.K.B. form for the radiation field (see II for details). The second of our working tools will be an averaging procedure whereby the fine details of some property of gravitational radiation is replaced by its space-time average over a region containing many wavelengths. This method is familiar from electromagnetism or statistical mechanics, and we will call it the "B.H. assumption" after Brill and Hartle (I §7) who applied this technique to the analysis of gravitational geons. When either of these two assumptions is used to derive important results, we will indicate it in the resultant formula by placing the letters W.K.B. or B.H. after the appropriate equations.

2. The Effective Stress Tensor for Gravitational Radiation

In II we expanded the vacuum field equations in powers of the wavelength of the gravitational radiation. To lowest order, the field equations become $R^{(1)}_{\mu\nu} = 0$, or choosing our gauge with the reservations discussed in II, this was shown to reduce to

$$\text{(2.1)} \quad \text{a)} \quad h_{\mu\nu}{}^{;\beta}{}_{;\beta} + 2R^{(0)}_{\sigma\nu\mu\beta}\, h^{\beta\sigma} + R^{(0)}_{\sigma\mu}\, h_{\nu}{}^{\sigma} + R^{(0)}_{\sigma\nu}\, h_{\mu}{}^{\sigma} = 0$$

$$\text{b)} \quad h^{\mu\nu}{}_{;\nu} = 0$$

$$\text{c)} \quad h \equiv \gamma^{\alpha\beta} h_{\alpha\beta} = 0 .$$

These equations determine the gravitational wave $h_{\mu\nu}$ once the background geometry $\gamma_{\mu\nu}$ is given. The second order terms in the Einstein equations can be written

$$\text{(2.2)} \quad R^{(0)}_{\mu\nu} = - \varepsilon^2 R^{(2)}_{\mu\nu} .$$

(See II eqns. (2.5) - (2.8) for explicit expressions.)

Equation (2.2) shows us how the radiation field apparently acts as a source for the curvature of the background. Notice that (2.1) and (2.2) cannot be solved individually, but rather only in a self-consistent field scheme as Brill and Hartle (1964) have emphasized. We now observe that the Einstein field equations may be solved to an error of order ε, by simultaneously solving

$$(2.3) \quad \text{a)} \quad R^{(1)}_{\mu\nu} = 0$$

$$\text{b)} \quad R^{(0)}_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu} R^{(0)} = -8\pi\, T^{\text{eff.}}_{\mu\nu}$$

where the effective stress tensor for the high frequency radiation field is given (in a completely general choice of gauge) by

$$(2.4) \quad T^{\text{eff.}}_{\mu\nu} \equiv (\varepsilon^2/8\pi)\,(R^{(2)}_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}R^{(2)}) \quad , \quad R^{(2)} \equiv \gamma^{\alpha\beta}R^{(2)}_{\alpha\beta}$$

$$= (\varepsilon^2/16\pi)\,(Q_{\mu\nu} - S_{\mu\nu}{}^{\rho}{}_{;\rho})$$

where

$$(2.5) \quad Q_{\mu\nu} \equiv \frac{1}{2} h^{\rho\tau}{}_{;\mu} h_{\rho\tau;\nu} - h_{\nu}{}^{\tau;\rho}(h_{\tau\mu;\rho} - h_{\rho\mu;\tau})$$

$$- \frac{1}{2} h^{;\tau}(h_{\tau\mu;\nu} + h_{\tau\nu;\mu} - h_{\mu\nu;\tau})$$

$$+ \frac{1}{2}\gamma_{\mu\nu}(\frac{1}{2} h^{\rho\tau;\alpha} h_{\rho\tau;\alpha} - h^{\alpha\tau;\rho} h_{\alpha\rho;\tau}$$

$$+ h^{;\tau} [h_{\tau\alpha}{}^{;\alpha} - \frac{1}{2} h_{;\tau}])$$

and

$$(2.6) \quad S_{\mu\nu}{}^{\rho} \equiv \delta^{\rho}_{\nu} h^{\alpha\tau} h_{\alpha\tau;\mu} + h^{\rho\tau}(h_{\mu\nu;\tau} - h_{\tau\mu;\nu} - h_{\tau\nu;\mu})$$

$$+ \gamma_{\mu\nu}(h^{\rho\tau} [h_{\tau\alpha}{}^{;\alpha} - \frac{1}{2} h_{;\tau}] - \frac{1}{2} h_{\alpha\tau} h^{\alpha\tau;\rho}).$$

Here $Q_{\mu\nu}$ is a tensor quadratic in ∂h whereas $S_{\mu\nu}{}^{\rho}$ is of the form $h\partial h$. Since only the divergence of $S_{\mu\nu}{}^{\rho}$ appears in (2.4), this piece of the effective stress-tensor does not contribute under integral averages (see Appendix C).

From (2.3b) we find that, in the high frequency approximation, gravitational radiation fields are uncoupled from their sources and are endowed with a vitality and independent existence of their own. They are just as good as any other source of energy when it comes to curving space, and later on we will see that they behave like any other conceivable field as far as energy and momentum transport and conservation are concerned.

3. The Brill-Hartle Averaging Scheme

The high frequency oscillations of the gravitational waves are seen to produce the background curvature, but we are not really interested in all the fine details of the latter's fluctuations. The situation is somewhat analogous to the problem of finding electric fields in macroscopic dielectrics. While it is in principle possible to take into account all the atomic charge distributions in a dielectric to find the local electric field at any interior point, it is scarcely interesting to arrive at electric fields which fluctuate over a huge range as we move the observation point by 10^{-13} cm, and which require an exact description of the precise location of 10^{23} atoms. This sort of detail is totally irrelavent to answer any reasonable question about bulk matter. Rather, we take the field equation $\nabla \cdot \underset{\sim}{E} = 4\pi\rho$ and average it over a region of space which is large compared to the scale of charge fluctuation, but small compared to the dimensions of the material of interest. Then, we say that the average field is given as a solution to $\nabla \cdot \underset{\sim}{E}_{av.} = 4\pi\langle\rho\rangle$ where $\langle\rho\rangle$ denotes the space-averaged charge distribution.

Returning to the problem of gravitation, whenever regions of interest are large enough to contain many wavelengths, it is natural and advantageous to introduce a similar averaging process. Time averaging has been done in the past by Tolman (1958) for a radiation filled universe, by Wheeler (1955) for electromagnetic geons, and by Brill and Hartle (1964) for gravitational geons. Here we wish to average over space-time as Arnowitt, Deser and Misner (I §6) have done, and so we let the symbol $\langle\cdots\rangle$ denote an average over a region whose characteristic dimension is small compared to the scale over which the background changes, but independent of ε (i.e., $O(1)$) and therefore large compared to the wavelength of the radiation in the limit $\varepsilon \to 0$. Then, the averaged approximate field equations can be cast into the final form as given by Brill and Hartle (see I (6.5)):

$$(3.1) \quad \text{a)} \quad R^{(1)}_{\mu\nu} = 0$$

$$\text{b)} \quad R^{(0)}_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu} R^{(0)} = -8\pi T^{BH}_{\mu\nu}$$

where the B.H. averaged effective stress tensor is

$$(3.2) \quad T^{BH}_{\mu\nu} = (\varepsilon^2/16\pi) \langle Q_{\mu\nu} - S_{\mu\nu}{}^{\rho}{}_{;\rho} \rangle .$$

4. <u>B.H. and W.K.B. Simplification of the Effective Stress Tensor</u>

The general expressions (2.4)-(2.6) defining the effective stress tensor are rather unwieldy, and even if we specialize to the "Lorentz gauge" (i.e., the class of gauges satisfying (2.1)), $T^{eff.}_{\mu\nu}$ is still rather clumsy. When we perform the B.H. averaging indicated in (3.2) we obtain a neat result. The rules we follow to do this are (see Appendix C for justification):

1) Under integrals, divergences become reduced by a factor of ε. We may therefore drop $S_{\mu\nu}{}^{\rho}{}_{;\rho}$ and similar terms.

2) Under integrals we may "integrate by parts", e.g.,

$$\langle h_{\nu}{}^{\tau;\rho} h_{\rho\mu;\tau} \rangle = - \langle h_{\nu}{}^{\tau;\rho}{}_{;\tau} h_{\rho\mu} \rangle$$

if we ignore terms down by a factor ε.

3) Covariant derivatives commute on high frequency waves as $\varepsilon \to 0$, <u>e.g.</u>

$$h_{\mu\nu;[\rho\tau]} = \frac{1}{2} R^{(0)}_{\mu\sigma\rho\tau} h^{\sigma}{}_{\nu} + \frac{1}{2} R^{(0)}_{\nu\sigma\rho\tau} h_{\mu}{}^{\sigma}$$

or

$$h_{\mu\nu;[\rho\tau]} \doteq \varepsilon^2 .$$

Using these rules, we find that in the Lorentz gauge $T^{BH}_{\mu\nu}$ is simply

$$(4.1) \qquad T^{BH}_{\mu\nu} = (\epsilon^2/32\pi) \langle h^{\rho\tau}{}_{;\mu}\, h_{\rho\tau;\nu} \rangle + O(\epsilon). \qquad \text{(B.H.)}$$

If we fix the gauge so that $h_{0\mu} = 0$, we see that T^{BH}_{00} is positive definite.

To see if (4.1) has any invariant significance we must investigate how $T^{BH}_{\mu\nu}$ behaves under a change of gauge (see II §4). Since we have from (2.5) $Q_{\mu\nu} \sim (\partial h)(\partial h)$, under the change of gauge $h_{\mu\nu} \to \bar{h}_{\mu\nu} \sim h + \partial\xi$ we obtain $Q_{\mu\nu} \to \bar{Q}_{\mu\nu} \sim (\partial h)(\partial h) + (\partial h)(\partial^2\xi) + (\partial^2\xi)(\partial^2\xi)$.

The first group of terms of the form $(\partial h)(\partial h)$ are just the old $Q_{\mu\nu}$. The additional terms induced by the gauge transformation can be roughly divided up into either high or low frequency. For low frequency waves we have

$$\xi_\mu = O(1) \quad , \quad \xi_{\mu;\nu} = O(1) \quad , \quad \xi_{\mu;\nu\tau} = O(1)$$

and therefore the second and third group of terms in $\bar{Q}$ are negligible compared to the first group. On the other hand, for high frequency waves we assume

$$\xi_\mu = O(\epsilon) \quad , \quad \xi_{\mu;\nu} = O(1) \quad , \quad \xi_{\mu;\nu\tau} = O(\epsilon^{-1}).$$

In this case $\xi_{\mu;[\nu\tau]} \doteq \epsilon^2$ and so covariant derivatives on high frequency coordinate waves commute. We find that the terms in $\bar{Q}$ of the form $(\partial h)(\partial^2\xi)$ can be converted to a divergence by integrating by parts, commuting derivatives and using the general wave equation II (5.7) for h. Similarly, by integrating by parts and commuting derivatives we may reduce the terms like $(\partial^2\xi)(\partial^2\xi)$ to a divergence. Putting this all together, we obtain the behavior of the effective stress tensor under a general gauge transformation

$$(4.2) \qquad T^{\text{eff.}}_{\mu\nu} \to \bar{T}^{\text{eff.}}_{\mu\nu} = T^{\text{eff.}}_{\mu\nu} + U_{\mu\nu}{}^{\rho}{}_{;\rho} + O(\epsilon).$$

While $U_{\mu\nu}{}^{\rho}{}_{;\rho} = O(\epsilon)$ for low frequency coordinate waves, it is of order unity for high frequency coordinate transformations, and so $T^{\text{eff.}}_{\mu\nu}$ is not gauge invariant in general, and hence not a physical observable. However since $U_{\mu\nu}{}^{\rho}{}_{;\rho}$

is reduced under integrals,

$$(4.3) \quad T^{BH}_{\mu\nu} \to \bar{T}^{BH}_{\mu\nu} + O(\varepsilon)$$

and the B.H. averaged stress tensor is gauge invariant and this given by (4.1) for all choices of gauge.

A corresponding simplification of the stress-tensor occurs if we assume the W.K.B. form for $h_{\mu\nu}$, but not the B.H. assumption. Thus we let (see II §6)

$$(4.4) \quad \text{a)} \quad h_{\mu\nu} = \mathrm{Re}\{\mathcal{A} e_{\mu\nu} \, e^{i\phi}\}$$

$$\text{b)} \quad k_\nu = \phi_{,\nu} \qquad k^\nu k_\nu = 0$$

$$\text{c)} \quad e_{\mu\nu;\alpha} k^\alpha = 0$$

$$\text{d)} \quad (\mathcal{A}^2 k^\beta)_{;\beta} = 0$$

$$\text{e)} \quad e_{\mu\nu} e^{\mu\nu} = 1$$

$$\text{f)} \quad k_\mu e^{\mu\nu} = 0$$

$$\text{g)} \quad \gamma_{\mu\nu} e^{\mu\nu} = 0 .$$

We find that the effective W.K.B. Stress tensor is

$$(4.5) \quad T^{WKB}_{\mu\nu} = (\varepsilon^2/32\pi)\mathcal{A}^2 \sin^2\phi \; k_\mu k_\nu + O(\varepsilon) \qquad \text{(W.K.B)}$$

where T^{WKB}_{00} is positive definite as expected.

Finally, we combine the B.H. and W.K.B. approximations to obtain the effective averaged radiation stress tensor in the geometrical optics form (to lowest order)

$$(4.6) \quad T^{rad.}_{\mu\nu} = q^2 k_\mu k_\nu , \qquad q^2 = \varepsilon^2 \mathcal{A}^2/64\pi \qquad \text{(B.H. - W.K.B)}$$

Equations (4.1), (4.5) and (4.6) may also be derived independently using the Einstein equations rewritten in the Landau and Lifshitz (1962b) form:

$$H^{\mu\alpha\nu\beta}{}_{,\alpha\beta} = 16\pi \, (-g) \, (T^{\mu\nu} + t^{\mu\nu})$$

where $T^{\mu\nu}$ is the stress tensor of material sources which are present in the general case, $t^{\mu\nu}$ is quadratic in the first derivatives of the metric, and

$$H^{\mu\nu\alpha\beta} = (-g)\ (g^{\mu\nu}g^{\alpha\beta} - g^{\mu\beta}g^{\alpha\nu}).$$

If we insert $g_{\mu\nu} = \gamma_{\mu\nu} + \varepsilon h_{\mu\nu}$ into the field equations and either average them, use the W.K.B. assumption, or do both, we find

$$t^{\mu\nu} \rightarrow T^{\mu\nu}_{BH}\ ,\quad T^{\mu\nu}_{WKB}\ ,\quad T^{\mu\nu}_{rad.}$$

respectively.

From now on, we will use (4.6) as the final form for the stress tensor for gravitational radiation. This is precisely the form for electromagnetic null radiation fields, further extending the analogy between light and gravitation. Several other authors have obtained results similar to this in different contexts. Thus, Trautman (I §5) has shown that (4.6) arises from weak field linearized theory with Sommerfeld radiation conditions, while Brill (I §6) finds that in the limit of short wavelength, the development of a universe closed by the presence of gravitational waves is the same as for the Tolman electromagnetically filled universe.

From (4.6) we see that the background geometry is required to have the form

$$(4.7) \qquad R^{(0)} = -\sigma^2 k_\mu k_\nu \quad , \quad \sigma^2 = \varepsilon^2 \mathcal{A}^2/8 \qquad \text{(B.H. - W.K.B)}$$

while from (4.4b,d) we obtain

$$(4.8) \qquad T^{\mu\nu}_{rad.\ ;\nu} = 0. \qquad \text{(B.H. - W.K.B)}$$

These conservation laws tell us that the effective stress tensor for the gravitational radiation field is conserved on an equal footing with other stress tensors, and will later allow us to calculate the energy and momentum carried by the radiation field. Analogously to electromagnetic theory, we may define a space-like gravitational Poynting vector S^α describing the flow of gravitational energy measured by an observer with time-like four-velocity v^μ. This is given by

$$S^\alpha = (\delta^\alpha_\mu - v^\alpha v_\mu)\, T^{\mu\nu}_{rad.}\, v_\nu = \omega q^2 (k^\alpha - \omega v^\alpha) \qquad (4.9) \quad \text{(B.H. - W.K.B.)}$$

where $\omega \equiv k^\alpha v_\alpha$ is the frequency of the radiation as measured in the rest frame of the observer. We see that $S^\alpha v_\alpha = 0$, $S^\alpha S_\alpha = -\omega^4 q^4$, and that S^α only vanishes in the limit that the observer moves with the speed of light. In the rest frame of the observer $k^\alpha = (\omega, \underset{\sim}{k})$ and $S^\alpha = (0, \omega q^2 \underset{\sim}{k})$, and we find that gravitational energy flows along the rays of the radiation field, and is not scattered off of null hypersurfaces (in this approximation) by the background curvature of space-time.

5. Energy and Momentum of the Gravitational Radiation Field

In special relativity, the homogeneity of space-time leads us to the laws of conservation of energy and momentum, however, in general relativity, space-time is curved and does not in general have any symmetries. Without some method of introducing the special relativistic reference for comparison it seems hopeless to try to extend these concepts to curved space. We must therefore content ourselves with being restricted to asymptotically flat spaces where we require energy and momentum to behave as the components of a four-vector under Lorentz transformations. For the usual problem of outgoing radiation emitted by bounded sources in a vacuum, it is reasonable to expect asymptotically flat (but radiative) space at large distances from the source, and we shall later show this to be the case for a specific example. For the rest of this section we will assume that there exists a coordinate system $\tilde{x}^\mu$ which asymptotically becomes lorentzian as space becomes flat, and we will use (4.8) along with Stokes' theorem to obtain formulas for computing the energy and momentum which arrives at flat infinity after being radiated from an isolated source. These expressions will be in the form of integrals over general hypersurfaces where gravitational fields may be large, and in which arbitrary coordinate systems may be used.

By Stokes' law, we may write

$$(5.1) \quad \int_V P^\mu{}_{;\mu} (-\gamma)^{1/2} d^4x = \int_{\partial V} P^\mu dS_\mu$$

where, in the notation of exterior forms

$$(5.2) \quad dS_\mu = \frac{1}{3!} (-\gamma)^{1/2} \epsilon_{\mu\alpha\beta\gamma} dx^\alpha \wedge dx^\beta \wedge dx^\gamma$$

V is a four-dimensional region, ∂V its boundary. For the following, we choose V to be the region bounded by two non-intersecting null cones Σ_1 and Σ_2, and two three-dimensional hypersurfaces σ_1 and σ_2 which cut both null cones. The hypersurface σ_2 will always be assumed space-like, but σ_1 may be either space-like or time-like as shown in Fig. 2 and Fig. 3. Σ_1 and Σ_2 should be throught of as future light cones emanating from the world line z^μ of a gravitational wave source, while σ_1 and σ_2 are choosen to make V lie entirely in a region where the high frequency, W.K.B., and B.H. assumptions are simultaneously valid. Let us define

$$I(\sigma) \equiv \int_\sigma P^\mu dS_\mu$$

where σ is an open three-surface. If P^μ has vanishing divergence ($P^\mu{}_{;\mu} = 0$) then (5.1) becomes

$$(5.3) \quad 0 = I(\Sigma_1) + I(\Sigma_2) + I(\sigma_1) + I(\sigma_2).$$

If an addition $I(\Sigma_1) = I(\Sigma_2) = 0$ (as is usually the case since radiation travels outward along null cones) we have an integral conservation law $I(\sigma_2) = - I(\sigma_1)$, where the minus sign just reflects the face that the outward normals to V at σ_1 and σ_2 are oppositely directed.

For our first example let us choose $P^\mu = \mathcal{A}^2 k^\mu$. Then by (4.4d) $P^\mu{}_{;\mu} = 0$. To evaluate $I(\Sigma)$ where $\Sigma = \Sigma_1$ or Σ_2 choose retarded time

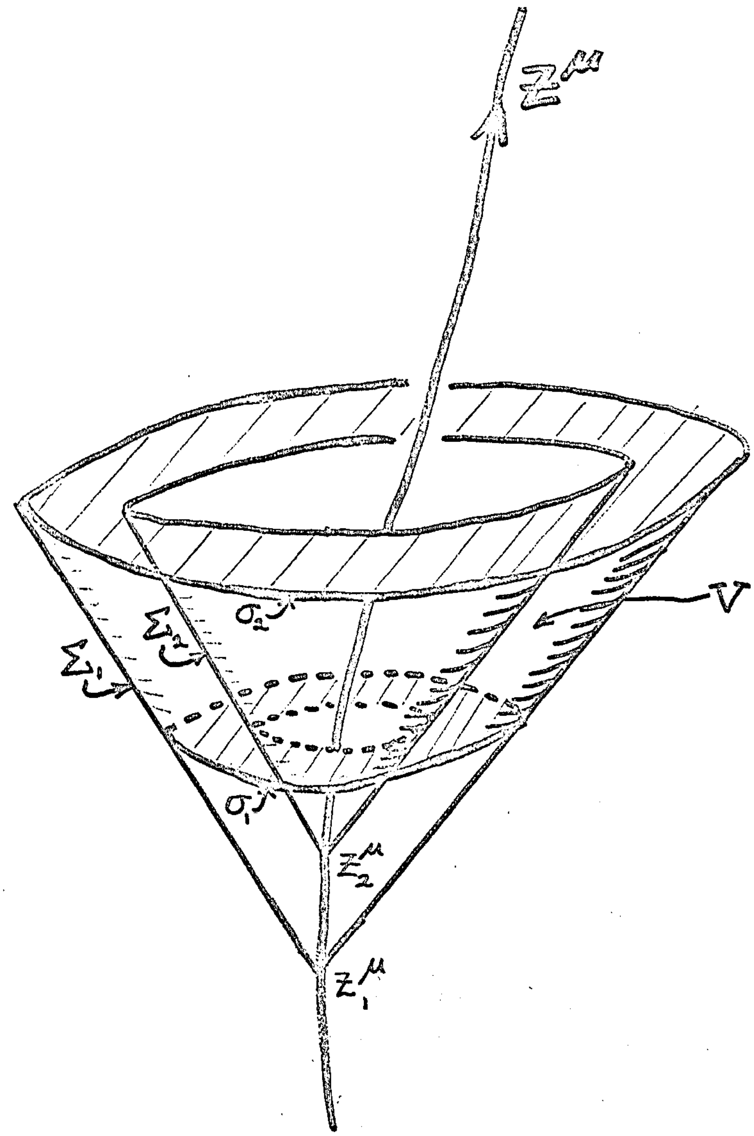

Fig. 2 The light cones Σ_1 and Σ_2 enclose an outgoing pulse of radiation from a source traveling along world-line z^μ . They are cut by the three-dimensional space-like hypersurfaces σ_1 and σ_2.

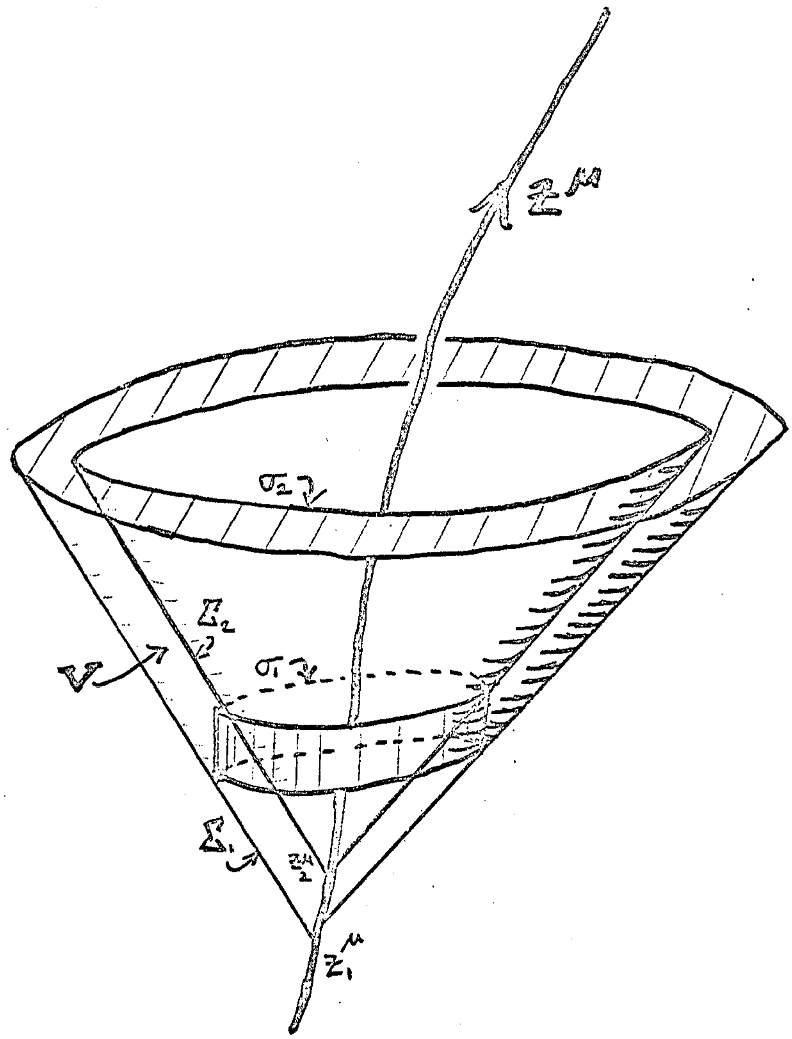

Fig. 3 The light cones Σ_1 and Σ_2 enclose an outgoing pulse of radiation from a source traveling along world-line z^μ. They are cut by the space-like hypersurface σ_2 and the time-like hypersurface σ_1.

coordinates $x^\mu = (x^0 = u, x^1, x^3, x^3)$. Then $du = 0$ on the light cones Σ, and

$$I(\Sigma) = \int_\Sigma P^0 \ (-\gamma)^{1/2} \ dx^1 \wedge dx^2 \wedge dx^3$$

The phase ϕ in (4.4a) must be a function of retarded time only, and since k^α is a null vector this in turn implies $P^0 = 0$ and $I(\Sigma) = 0$. Then $N \equiv I(\sigma_2) = - I(\sigma_1)$ is a conserved quantity, independent of the (space-like) hypersurface used to evaluate it. If a source radiates only while it is between points $z_1{}^\mu$ and $z_2{}^\mu$ on its worldline, P^μ vanishes outside the region included between light cones Σ_1 and Σ_2, since radiation travels outward only along null surfaces and is not scattered by background curvature to lowest order in ε. We may therefore extend σ_2 to include an entire space-like surface S oriented the same as σ_2, to obtain

$$(5.4) \qquad N = \int_S c\mathcal{A}^2 k^\mu \ (-\gamma)^{1/2} \ dS_\mu$$

and N is our conserved graviton number.

In order to develop expressions for the total energy and momentum arriving at null infinity, we need to first define a tetrad $e^{(\alpha)}_\nu$ (α selects one vector out of the family of four, ν gives its vector components). We do this by requiring the tetrad vectors to be parallel propagated along the rays k^α and to point along the asymptotically lorentzian coordinates $\tilde{x}^\mu$ at null infinity. Then the tetrad is the unique solution to the differential equation

$$(5.5) \qquad e^{(\alpha)}{}_{\nu;\mu} k^\mu = 0$$

with the boundary condition $e^{(\alpha)}_\nu \rightarrow \delta^\alpha_\nu$ in $\tilde{x}^\mu$ coordinates at null infinity. We define a four-momentum density for the gravitational radiation as

$$P^{(\alpha)\mu} = e^{(\alpha)}_\nu \ T^{\nu\mu}_{\text{rad.}} .$$

Then, by (4.6) and (5.5) we find $P^{(\alpha)\mu}{}_{;\mu} = 0$ and we may use (5.3). As before, introducing retarded time coordinates on Σ_1 and Σ_2, we find $I(\Sigma_1) = I(\Sigma_2) = 0$, and the total four-momentum arriving at infinity is

$$\tilde{P}^{(\alpha)} \equiv I(\sigma_2) = - I(\sigma_1).$$

To show that this agrees with the usual special relativistic result, we evaluate $I(\sigma_2)$ with coordinates $\tilde{x}^\mu$ and choose σ_2 as the flat region at null infinity given by $\tilde{x}^0$ = constant. Then we find

$$\tilde{P}^{(\alpha)} = \int_{\sigma_2} T^{\alpha 0}_{\text{rad.}} \, d^3\tilde{x}$$

which is the expected result. However, the utility of all this lies in the fact that we may also evaluate $\tilde{P}^{(\alpha)}$ on the general hypersurface σ_1 where

$$(5.6) \quad \tilde{P}^{(\alpha)} = -\frac{1}{3!} \int_{\sigma_1} q^2 e^{(\alpha)}_{\nu} k^\nu k^\mu \, \varepsilon_{\mu\alpha\beta\gamma} \, (-\gamma)^{1/2} \, dx^\alpha \wedge dx^\beta \wedge dx^\gamma \, .$$

Here, σ_1 may be taken inside a region of strong gravitational fields, so the tetrad field $e^{(\alpha)}_{\nu}$ has explored space to the extent that (5.6) automatically separates off the part of the radiation which will escape the binding of the source and incorporates all red shifts which this radiation suffers as it climbs out of the strong gravitational potential near its point of creation. The four integrals (5.6) are all scalars under coordinate transformations on σ_1 once the asymptotic coordinates $\tilde{x}^\mu$ are specified. This means (5.6) may be evaluated in any coordinate system on σ_1, not necessarily in lorentzian coordinates as is required for expressions dependent on the Landau and Lifschitz pseudotensor (1962b). It should be remembered, however, that σ_1 may not be taken arbitrarily close to an intense source, since if space becomes highly curved, the ray congruence k^α develops caustics as light rays intersect and the geometrical optics limit breaks down. As we saw in (I §5), the same limitation arose in the Bondi-Sachs multipole analysis.

Now let us investigate the effect of changing our choice of asymptotic coordinates $\tilde{x}^\mu$ by a Lorentz transformation corresponding to a rotation of four-space at infinity. Then, new asymptotically lorentzian coordinates are given by

$$(5.7) \quad \hat{x}^\mu = L^\mu{}_\nu \, \tilde{x}^\nu .$$

We require a new tetrad of vectors denoted by $f^{(\alpha)}_{\nu}$ which point along the $\hat{x}^{\nu}$ axis at null infinity. This is given by

$$(5.8) \qquad f^{(\alpha)}_{\nu} = L^{\alpha}_{\ \mu}\, e^{(\mu)}_{\nu} \ .$$

In (5.6) the net result of changing our asymptotic lorentzian coordinates is the effect induced by the change of tetrad. Then (5.4) and (5.6) transform as a scalar and four-vector, namely

$$(5.9) \qquad \hat{N} = \tilde{N} \qquad \hat{P}^{(\alpha)} = L^{\alpha}_{\ \mu}\, \tilde{P}^{(\mu)}$$

confirming their special relativistic character.

6. Generalization to Several Wave Fronts

So far, we have been concerned with the presence of only one monochromatic wave front at a point, as is implicit in the W.K.B. form (4.4a). We may extend this to the case where several sources are present, or where a source (or sources) produces a polychromatic spectral distribution. In either case, we assume the existence of an asymptotic expansion as ε approaches zero in the form

$$(6.1) \qquad h_{\mu\nu} = \sum_{m} h_{\mu\nu}(m) \qquad \text{(W.K.B.)}$$

where each component $h_{\mu\nu}(m) = A_{\mu\nu}(m)\, e^{i\phi(m)}$ is the (approximate) solution to the wave equation (2.1). Since the wave equation is linear, (6.1) is also a solution. Combining the W.K.B. and B.H. approximations we substitute (6.1) in (4.1) and drop corrections of order ε. Then since k_{μ} and $A^{\rho\tau}$ are slowly varying functions over the region of integration, we have

$$\langle h^{\rho\tau}_{\ \ ,\mu}\, h^{\rho\tau}_{\ \ ,\nu} \rangle = \sum_{m,n} k_{\mu}(m)\, k_{\mu}(n)\, A^{\rho\tau}(m)\, A_{\rho\tau}(n)\, \langle \sin\phi(m)\, \sin\phi(n) \rangle$$

For the usual case of incoherent sources of radiation, we may drop all terms where $m \neq n$ and use $\langle \sin^2\phi(m) \rangle = \frac{1}{2}$. In this manner, we find that for incoherent sources

$$(6.2) \qquad T^{\text{rad.}}_{\mu\nu} = \sum_{m} T^{\text{rad.}}_{\mu\nu}(m) \quad , \quad T^{\text{rad.}}_{\mu\nu}(m) = q^2(m)\, k_{\mu}(m)\, k_{\nu}(m)$$

and so we have a superposition principle for the stress tensors as well as for the amplitudes of the various components of the radiation field.

7. Spherical Gravitational Waves

To give concreteness to the formalism which has been developed, in this section we apply both the B.H. and W.K.B. approximations to a specific example. In order to simplify the algebra a highly symmetric situation is needed, so we will investigate radiation from a spherical body of mass m. This may be interpreted as a star or gravitational geon leaking away high frequency gravitational waves. Of course, if the star were truly spherically symmetric in its motion no radiation could be emitted, so symmetry is to be interpreted as holding after some sort of average in time or over many independent modes of oscillation. Radiation emitted by some perturbation on the average symmetry will propagate into space, still feeling the influence of the gravitational field of the star. The result is a spherical shell of radiation expanding in a spherically symmetric background geometry determined from (4.7). Vaidya has found an exact solution to these Einstein equations for the background in the form (Vaidya, 1951, 1953)

$$(7.1) \qquad dS^2 = (1 - 2m(u)/r)du^2 + 2dudr - r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad .$$

Here m(u) is a non-increasing but otherwise arbitrary function of the retarded time u, and may be interpreted as the mass of the star measured by an observer at infinity. If m(u) is constant then the substitution

$$u = t - r - 2m \log(r-2m)$$

brings us to the Schwarzschild form of the metric.

Now that we have found the background geometry created by and consistent with the radiation field, we may introduce observers and see what properties of the gravitational waves they may measure. We assume following Lindquist, Schwartz, and Misner (1965), that our observer has four-velocity v^μ but only moves radially. Then defining

$$U = \frac{dr}{d\tau} \equiv r_{;\mu}v^\mu$$

and using $v^\mu v_\mu = 1$, we find

$$(7.2) \qquad v^\mu = ((\gamma+U)^{-1},U,0,0)$$

where

$$\gamma \equiv (1 + U^2 - 2m/r)^{1/2} \quad .$$

Any observable generator of gravitational radiation must have its boundary radius outside the physically unaccessible region bounded by $r = 2m(u)$ and must emit waves whose phase can only depend upon the retarded time u. Hence, in (4.4) $\phi = \phi(u)$ and so $k_\mu = (\dot{\phi},0,0,0)$ where $\dot{\phi} = \phi_{,u}$. The moving observer measures a frequency in his rest frame given by

$$(7.3) \qquad \omega(u) = k_\mu U^\mu = (\gamma+U)^{-1}\,\dot{\phi} \quad .$$

For an observer at rest at infinity, the measured frequency is $\omega_\infty(u) = \dot{\phi}(u)$. Thus in general we have the frequency shift formula

$$(7.4) \qquad \omega(\gamma+U) = \omega_\infty$$

relating the frequency ω measured (or emitted) by an observer with radial velocity U to that measured by an observer at rest at infinity, including all gravitational and Doppler effects.

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relating the frequency ω measured (or emitted) by an observer with radial velocity U to that measured by an observer at rest at infinity, including all gravitational and Doppler effects.

The modifications which the background impose on the radiation are given by (4.4) in the Vaidya geometry. From these we find

$$(7.5) \qquad h_{\mu\nu} = r^{-1} A(u)\, e_{\mu\nu}\, e^{i\phi(u)}$$

allowing for AM or FM transmission of information via the functions $A(u)$ and $\phi(u)$, which must satisfy II (6.4) but which are otherwise arbitrary. There are precisely two polarizations representing true gravitational effects rather than just coordinate waves. These are the only two modes which give a nonvanishing contribution to the dominant part of the Riemann tensor $R^{(1)}_{\alpha\beta\gamma\delta} \simeq 2k_{[\alpha}h_{\beta][\gamma}k_{\delta]}$ (see II). They are transverse traceless modes with explicit form

$$(7.6) \qquad e^{(1)}_{\mu\nu} = \frac{r^2 \sin^2\theta}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and

$$(7.7) \qquad e^{(2)}_{\mu\nu} = \frac{r^2}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\sin^2\theta \end{pmatrix}$$

where $\mu,\nu = u,r,\theta,\phi$. The only nonvanishing component of the Ricci tensor of the background is

$$R^{(0)}_{00} = \frac{2}{r^2}\frac{dm}{du}$$

which gives from (4.7) that

$$(7.8) \qquad \frac{dm(u)}{du} = -(\varepsilon^2/16)\ [A(u)\ \omega_\infty(u)]^2$$

$$= -(\varepsilon^2/16\ [A(u)\ \omega_o(u)]^2[1 - 2m(u)/r_o(u)]$$

where ω_o is the frequency of the radiation measured at the surface of the star where the radius is $r = r_o$. If the star continuously emits radiation with frequency $\omega_o(u)$ and specific amplitude $A(u)$ as it collapses toward $r_o = 2m(u)$, then the observed mass at infinity tends toward a constant value and ω_∞ tends toward zero, i.e., all the radiation is red shifted away as r_o approaches $2m(u)$.

From (7.8) we see that there is no tail to the radiation--A and ω are non-zero only when the driving mass of the source is changing. The radiation travels outward along the lightcones u = constant without being scattered off these null hypersurfaces. Conversely, from (7.8) we see that the $A^2\ \omega_\infty^2$ is a news function of the kind found by Bondi (I §5) and a radiating star must loose mass whenever information is carried away by radiation.

The existence of radiation is compatible with having space asymptotically flat, for we may introduce coordinates

$$(7.9) \qquad \tilde{x}^0 = u + r$$

$$\tilde{x}^1 = r \sin\theta \cos\phi$$

$$\tilde{x}^2 = r \sin\theta \sin\phi$$

$$\tilde{x}^3 = r \cos\theta$$

in terms of which the metric becomes

$$(7.10) \qquad g^{00} = 1 + 2m/r$$

$$g^{0i} = 2mx^i/r^2$$

$$g^{ij} = 2mx^ix^j/r^3 - \delta^i_j$$

which tends to the Lorentz form as r approaches infinity.

If we evaluate the conserved graviton number N by (5.4) over a general hypersurface r = constant, we find that it satisfies the differential law

$$(7.11) \qquad \frac{dN}{du} = -\frac{64\pi}{\epsilon^2}\frac{1}{\omega_\infty}\frac{dm}{du} \quad .$$

In order to calculate the energy and momentum in the radiation fields, we express the tetrad components in the (u,r,θ,ϕ) coordinate system which displays the symmetry of the problem. Then, using (5.6) on the r = constant hypersurface, we find

$$P^{(\mu)} = (\Delta m,0,0,0)$$

where Δm is the change in mass of the source during the time it radiates gravitationally, as measured by an observer at infinity. Due to spherical symmetry the source cannot loose momentum, so we see that the energy and momentum carried by the radiation is just balanced by the energy and momentum lost by the source, and overall total conservation is maintained.

8. Higher Order Corrections to the Metric

Having explored many features of the high frequency approximation, we now will see if it is possible to extend it to give a systematic method for generating an exact solution. In (I, §4) we saw how the weak field approximation already ran into divergences when pushed to the second order. We will now check the analogous case for the high frequency approximation. We consequently assume that to second order the exact metric is of the form

$$(8.1) \qquad g_{\mu\nu}(x) = \gamma_{\mu\nu}(x) + \varepsilon h^{(1)}_{\mu\nu}(x,\varepsilon) + \varepsilon^2 h^{(2)}_{\mu\nu}(x,\varepsilon)$$

where γ is the slowly changing background and $h^{(1)}$, $h^{(2)}$ are high frequency wave components. The Ricci tensor may now be expanded to give

$$(8.2) \qquad R_{\mu\nu}(g) = R^{(0)}_{\mu\nu}(\gamma) + \varepsilon R^{(1)}_{\mu\nu}(h^{(1)}) + \varepsilon^2[R^{(2)}_{\mu\nu}(h^{(1)}) + R^{(1)}_{\mu\nu}(h^{(2)})] + \varepsilon^3 R^{(3+)}_{\mu\nu}$$

where $R^{(0)}$, $R^{(1)}$, and $R^{(2)}$ are defined as in II(2.6) - (2.8) with either $h^{(1)}$ or $h^{(2)}$ replacing the argument h given there, as indicated in (8.2), and $R^{(3+)}$ is again a remainder term. We group terms of the same magnitude together, and solve the vacuum Einstein equations by setting terms which are the same size equal to zero. This gives the equations

$$(8.3) \qquad \text{a)} \quad R^{(1)}_{\mu\nu}(h^{(1)}) = 0$$

$$\text{b)} \quad R^{(0)}_{\mu\nu} + \varepsilon^2 R^{(2)}_{\mu\nu}(h^{(1)}) + \varepsilon^2 R^{(1)}_{\mu\nu}(h^{(2)}) = 0 \quad .$$

Now since $R^{(2)}$ contains terms typically of the form $\partial h^{(1)}\partial h^{(1)}$, if $h^{(1)}$ is a wave of frequency ω, then $R^{(2)}$ will have terms of frequency 0 and 2ω. Under an average, the zero frequency terms remain and the double frequency terms average to zero. The B.H. approximation of (III, §3) amounts to only choosing the zero frequency terms as a source for curving the background, and we may retain it and still solve (8.3b) if we use the 2ω terms as a source for $h^{(2)}$.

This motivates the scheme of solving the Einstein equations approximately via

$$(8.4) \qquad \text{a)} \quad R^{(1)}_{\mu\nu}(h^{(1)}) = 0$$

$$\text{b)} \quad R^{(0)}_{\mu\nu} = -\varepsilon^2 \langle R^{(2)}_{\mu\nu} \rangle$$

$$\text{c)} \quad R^{(1)}_{\mu\nu}(h^{(2)}) = J_{\mu\nu} \equiv \langle R^{(2)}_{\mu\nu} \rangle - R^{(2)}_{\mu\nu} \quad .$$

Thus (8.4a) and (8.4b) are precisely the old B.H. approximation to find the metric to order ε. They may be solved to obtain the self-consistent solution for γ and $h^{(1)}$. Having this, we proceed to the next order and may find the metric to order ε^2 by solving the wave equation (8.4c), with a source term $J_{\mu\nu}$ of frequency 2ω acting to drive the wave $h^{(2)}$.

As an example of how this may be done, we now add the W.K.B. assumption and calculate according to (8.4). Choosing $h^{(1)}$ to have the W.K.B. form

$$(8.5) \qquad h^{(1)}_{\mu\nu} = A_{\mu\nu} \cos\phi \text{ where } \phi_{,\nu} \equiv k_\nu \text{ and } A_{\mu\nu} = \mathcal{A} e_{\mu\nu}$$

we find (see III (4.5)) that to lowest order

$$R^{(2)}_{\mu\nu}(h^{(1)}) = (1/4)\mathcal{A}^2 \sin^2\phi = (1/8)\mathcal{A}^2(1 - \cos 2\phi)$$

(Here the 1 represents the zero frequency part while the 2ϕ gives the double frequency component.) Now γ and $h^{(1)}$ are found from (4.7) and (8.5), while $h^{(2)}$ is determined from

$$(8.6) \qquad \frac{1}{2}\,[h^{(2);\beta}_{\mu\nu\ \ ;\beta} + h^{(2)}_{\ \ ;\mu\nu} - h^{(2)\ \beta}_{\ \mu\ ;\nu\beta} - h^{(2)\ \beta}_{\ \nu\ ;\mu\beta}]$$

$$= (1/8)\mathcal{A}^2 \cos 2\phi\, k_\mu k_\nu$$

We now assume $h^{(2)}$ has the W.K.B. form

$$h^{(2)}_{\mu\nu} = B_{\mu\nu} \cos\psi , \qquad \psi_{,\nu} \equiv q_\nu ,$$

substitute this into (8.6), retain only the dominant terms for a consistent approximation and thereby get

$$(8.7) \qquad \frac{1}{2} [-B_{\mu\nu} \cos\psi\, q_\beta q^\beta - B^\alpha{}_\alpha \cos\psi\, q_\mu q_\nu + B_\mu{}^\beta \cos\psi\, q_\nu q_\beta + B_\nu{}^\beta \cos\psi\, q_\mu q_\beta] = (1/8) \cos 2\phi\, k_\mu k_\nu .$$

Let $\psi = 2\phi$, $q_\nu = 2k_\mu$ and (8.7) becomes

$$(8.8) \qquad -B^\alpha{}_\alpha k_\mu k_\nu + B_\mu{}^\beta k_\beta k_\nu + B_\nu{}^\beta k_\beta k_\mu = (1/16)\mathcal{A}^2 k_\mu k_\nu .$$

Then to solve (8.8) we need only require

$$(8.9) \qquad \text{a)} \quad B^\alpha{}_\alpha = -(1/16)\mathcal{A}^2$$

$$\text{b)} \quad B^{\mu\nu} k_\nu = 0$$

and with these algebraic conditions we have found the metric to second order

$$(8.10) \qquad g_{\mu\nu} = \gamma_{\mu\nu} + \varepsilon A_{\mu\nu} e^{i\phi} + \varepsilon^2 B_{\mu\nu} e^{2i\phi} .$$

Note that no divergences arise to second order, and the second order terms are really smaller than the first order ones by a factor ε. Note also that the coordinate condition (8.9) in second order is neither that used by Trautman (1965) nor harmonic coordinates to second order.

The success uncovered so far makes it seem likely that the high frequency expansion (8.10) may be pushed to still higher orders, and the results to be uncovered thereby should be an interesting subject for future research.

APPENDIX A

NOTATION AND CONVENTIONS

We assume space-time to be described by a four-dimensional normal-hyperbolic Riemannian manifold with first fundamental form

$$\text{(A.1)} \qquad dS^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

and signature (+ − − −). We denote the determinant of $g_{\mu\nu}$ by g. Greek indices take on the values $\mu,\nu,\ldots = 0,1,2,3$, Latin indices take the values $i,j\ldots = 1,2,3$, and summation over repeated indices is implied.

Partial derivatives are indicated by a comma, e.g.,

$$f_{,\nu} = \frac{\partial f}{\partial x^\nu} .$$

Covariant derivatives with respect to the total metric $g_{\mu\nu}$ are <u>never</u> used, however, covariant derivatives with respect to the background metric $\gamma_{\mu\nu}$ will be indicated by a semicolon (as in $T_{\mu\nu;\alpha}$).

Christoffel symbols for a metric $m_{\mu\nu}$ are defined by

$$\text{(A.2)} \qquad \left\{{\alpha \atop \beta\ \gamma}\right\} \equiv \frac{1}{2} m^{\alpha\tau}(m_{\tau\beta,\gamma} + m_{\tau\gamma,\beta} - m_{\beta\gamma,\tau})$$

and the Riemann tensor for this metric is given by

$$\text{(A.3)} \qquad R^\alpha{}_{\beta\gamma\delta}(m_{\mu\nu}) \equiv \left\{{\alpha \atop \beta\ \gamma}\right\}_{,\delta} - \left\{{\alpha \atop \beta\ \delta}\right\}_{,\gamma}$$

$$+ \left\{{\alpha \atop \tau\ \delta}\right\}\left\{{\tau \atop \gamma\ \beta}\right\} - \left\{{\alpha \atop \tau\ \gamma}\right\}\left\{{\tau \atop \delta\ \beta}\right\}$$

The Ricci tensor is $R_{\mu\nu}(m_{\alpha\beta}) = m^{\tau\sigma} R_{\tau\mu\sigma\nu}(m_{\alpha\beta})$.

Pairs of indices respectively symmetrized or antisymmetrized are denoted as

$$T_{(\mu\nu)} = \frac{1}{2}(T_{\mu\nu} + T_{\nu\mu})$$

$$T_{[\mu\nu]} = \frac{1}{2}(T_{\mu\nu} - T_{\nu\mu}) \quad .$$

The alternating symbol is defined by

$$\varepsilon_{\mu\nu\sigma\tau} = \varepsilon_{[\mu\nu\sigma\tau]}$$

$$\varepsilon_{0123} = 1 \quad .$$

Units are chosen so that the speed of light c and the gravitational constant G both equal unity, and the flat space metric assumes the lorentzian form

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} \quad .$$

APPENDIX B

THE GEOMETRICAL OPTICS LIMIT FOR LIGHT

In this appendix, we present a brief treatment of the geometrical optics of light moving through a curved background geometry. The analysis is based upon class lectures given by Prof. C. W. Misner at the University of Maryland (1965). It should be read in conjunction with (II §6) and so a similar notation has been adapted to emphasize the close relation between light and gravity.

In vacuum, the Maxwell equations have the form

$$\text{(B.1)} \qquad \text{a)} \quad F_{\mu\nu} = \psi_{\nu;\mu} - \psi_{\mu;\nu}$$

$$\text{b)} \quad F_{\mu\nu}{}^{;\nu} = 0 \quad .$$

We may combine these and specialize to the Lorentz gauge where

$$\text{(B.2)} \qquad \text{a)} \quad \psi_{\mu}{}^{;\alpha}{}_{;\alpha} + R^{(0)}_{\mu\sigma}\,\psi^{\sigma} = 0$$

$$\text{b)} \quad \psi^{\mu}{}_{;\mu} = 0$$

(compare II (5.8, 5.9, 5.12)).

In flat space, (B.2a) has solution

$$\text{(B.3)} \qquad \psi_{\mu} = A_{\mu}\, e^{ik_{\alpha}x^{\alpha}}$$

where A_μ, k_μ are constants, however in a curved space this is no longer true. If we consider only high frequency light waves with wavelength λ and the geometry varies over a characteristic distance $L >> \lambda$, then locally the wave finds itself moving in a reasonably flat domain. The flat space solution (B.3) should then be good if A_μ and k_α are assumed to vary slowly over a distance L.

We may therefore assume a trial solution of the W.K.B. form (see II §6)

$$(B.4) \qquad \psi_\mu = A_\mu \, e^{i\phi} \quad .$$

We define

$$\epsilon \equiv \lambda/L \qquad k_\mu \equiv \phi_{,\mu}$$

and assume

$$(B.5) \qquad R^{(0)}{}_{\alpha\beta} = O(1)$$

$$A^\mu = O(1), \qquad A^\mu{}_{;\tau} = O(1)$$

$$k_\mu = O(\epsilon^{-1}), \qquad k_{\mu;\tau} = O(\epsilon^{-1}) \qquad .$$

When these are substituted into (B.2), we find

$$(B.6) \qquad a) \quad [-k_\alpha k^\alpha A^\mu] + i[2k_\alpha A^{\mu;\alpha} + k_\alpha{}^{;\alpha} A^\mu]$$

$$+ [A^\mu{}_{;\alpha}{}^{;\alpha} + R^\mu{}_\alpha A^\alpha] = 0$$

$$b) \quad ik_\mu A^\mu + A^\mu{}_\mu = 0 \quad .$$

In (B.6a) the terms in the various brackets are of order ϵ^{-2}, ϵ^{-1}, and 1 respectively. Setting terms of the same size equal to zero, to lowest order we have

$$(B.7) \qquad k_\beta \, k^\beta = 0$$

which implies the eikonel equation

$$\gamma^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} = 0$$

and that k^{α} is tangent to null geodesics since $k_{\alpha;\beta} k^{\alpha} = k_{\beta;\alpha} k^{\alpha} = 0$.

To lowest order (B.6b) yields

$$\text{(B.8)} \qquad k_{\mu} A^{\mu} = 0$$

or that the vector potential ψ_{μ} is orthogonal to the direction of propagation of the wave.

The second order terms in (B.6a) imply

$$\text{(B.9)} \qquad A_{\mu;\beta} k^{\beta} + \frac{1}{2} A_{\mu} k_{\beta}{}^{;\beta} = 0 \quad .$$

We may introduce an amplitude $\mathcal{A}$ and polarization vector e_{μ} by the decomposition

$$A_{\mu} = \mathcal{A} e_{\mu}$$

where $e_{\mu} e^{\mu} = 1$, $\mathcal{A} = (A_{\mu} A^{\mu})^{1/2}$. (B.9) may be solved to give

$$\text{(B.10)} \qquad \text{a)} \quad e^{\mu}{}_{;\beta} k^{\beta} = 0$$

$$\text{b)} \quad (\mathcal{A}^2 k^{\beta})_{;\beta} = 0 \quad .$$

Since the polarization is parallel propagated along null geodesics with k^{α} as tangent, we may impose $k_{\alpha} e^{\alpha} = 0$ as an initial condition at one point and it will be true along the entire curve, guaranteeing consistency with (B.8).

Electromagnetic radiation in the geometrical optics limit travels along null geodesics with its polarization parallel propagated and with its amplitude satisfying (B.10b), just as does gravitational radiation (see II (6.9, 16, 15c))

APPENDIX C

COMPUTATION OF AVERAGES

In this appendix, we will show how to construct the average of a rapidly oscillating tensor field, and will justify integration by parts within such an average.

The result of integrating a tensor field does not give a tensor in a curved space, as tensors at different points have different transformation properties. Since it is permissible to add tensors at the same point, we must go about constructing an average by somehow carrying the tensors back to a common point, and adding them there. To do this in a unique manner, we introduce the bi-vector of geodesic parallel displacement, denoted by $g_{\alpha}{}^{\beta'}(x,x')$ (DeWitt and Brehme 1960; Synge 1960). This transforms as a vector with respect to coordinate transformations at either x or x' and, assuming x and x' are sufficiently close together to insure the existence of a unique geodesic of the metric $\gamma_{\alpha\beta}$ between them, given the vector $A_{\beta'}$ at x', then $A_{\alpha} = g_{\alpha}{}^{\beta'} A_{\beta'}$ is the unique vector at x which can be obtained by parallel transporting $A_{\beta'}$ from x' back to x along the geodesic.

Given a tensor $T_{\mu\nu}$ which is assumed to have high frequency components of wavelength λ and a background geometry $\gamma_{\mu\nu}$ containing only low frequency componen of wavelength $L >> \lambda$, then we define the average of $T_{\mu\nu}$ to be the tensor

$$\text{(C.1)} \qquad \langle T_{\mu\nu}(x)\rangle \equiv \int_{\substack{\text{all}\\ \text{space}}} g_{\mu}{}^{\alpha'}(x,x')\, g_{\nu}{}^{\beta'}(x,x')\, T_{\alpha'\beta'}(x')\, f(x,x')\, d^4x'$$

where f(x,x') is a weighting function which falls smoothly to zero when x and x' differ by a distance d, ($\lambda << d << L$) and where

$$\int_{\substack{\text{all}\\ \text{space}}} f(x,x')\, d^4x = 1 .$$

Since f vanishes well within the region where the background remains flat, then there is no problem about the global existence of unique geodesics from x needed in the definition of $g_\mu^{\ \alpha'}$. Also, $\partial f \sim f/d = O(1)$.

Since $g_\mu^{\ \alpha'}$ depends on the background geometry, it clearly changes only over a distance L and $\partial g_\mu^{\ \alpha'} \sim g_\mu^{\ \alpha'}/L = O(1)$.

The only rapidly varying element in the construction is $T_{\mu\nu}$, since $\partial T \sim T/\lambda = O(\varepsilon^{-1})$.

Let us now ask what happens to tensors of the form $T_{\mu\nu} = S_{\mu\nu\ ;\rho}^{\ \ \ \rho}$ under averages. Inserting this into (C.1), we obtain

$$\text{(C.2)} \qquad \langle S_{\mu\nu\ ;\rho}^{\ \ \ \rho} \rangle = \int g_\mu^{\ \alpha'} g_\nu^{\ \beta'} S_{\alpha'\beta'\ \ ;\rho'}^{\ \ \ \ \rho'} f\, d^4x'$$

$$= \int \{(g\, g\, Sf)_{;\rho} - (g_{;\rho}\, g\, Sf) - (g\, g_{;\rho}\, Sf)$$

$$- (g\, g\, Sf_{;\rho})\}\, d^4x'$$

Since the first term may be converted to a surface integral taken in the region where $f \to 0$, we see that the right hand side of (C.2) contains no contributions of the form ∂S. Since we assume $\partial S = O(\varepsilon^{-1})$, this implies

$$\langle S_{\mu\nu\ ;\rho}^{\ \ \ \rho} \rangle \doteq \varepsilon$$

proving our earlier assertion that divergence may be neglected in averages as $\varepsilon \to 0$.

As for justifying integration by parts, this is a trivial corollary, since

$$\langle h_\nu^{\ \tau;\rho}\, h_{\rho\mu;\tau} \rangle = -\langle h_{\nu\ \ ;\tau}^{\ \tau;\rho}\, h_{\rho\mu} \rangle + \langle S_{\mu\nu\ ;\tau}^{\ \ \ \tau} \rangle$$

where

$$S_{\mu\nu}{}^{\tau} = h_{\nu}{}^{\tau;\rho} h_{\rho\mu}$$

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