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Large Deviations for Partial Sum Processes Over Finite Intervals

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Abstract

With any sequence $\{x_n,\ n = 0, \pm 1, \pm 2, \ldots\}$ of $\mathbb{R}^p$-valued random variables, we associate the partial sum processes $\{X_n^T(\cdot),\ n = 1, 2, \ldots\}$ which take value in the space $(D[0,T]^p, \tau_0)$ of right-continuous functions $[0,T] \to \mathbb{R}^p$ with left-hand limits equipped with Skorohod's J_1 topology. Furthermore, in an attempt to capture the past of the sequence, we introduce the negative partial sum processes $\{X_n^{T,-}(\cdot),\ n = 1, 2, \ldots\}$ defined by

$$X_n^{T,-}(t)(\omega) \equiv \begin{cases} \frac{1}{n} \sum_{i=1-\lceil nt \rceil}^{0} x_i(\omega) & \text{if} \quad \lceil nt \rceil \geq 1 \\ 0 & \text{otherwise} \end{cases}, \quad t \in [0,T], \quad \omega \in \Omega.$$

These processes take value in the space $(D_l[0,T]^p, \tau_0)$ of left-continuous functions $[0,T] \to \mathbb{R}^p$ with right-hand limits also equipped with the Skorohod's J_1 topology.

This paper explores some of the issues associated with transfering the LDP for the family $\{X_n^1(\cdot),\ n = 1, 2, \ldots\}$ in $(D[0,1]^p, \tau_0)$ to the families $\{X_n^T(\cdot),\ n = 1, 2, \ldots\}$ in $(D[0,T]^p, \tau_0)$, $\{X_n^{T,-}(\cdot),\ n = 1, 2, \ldots\}$ in $(D_l[0,T]^p, \tau_0)$ and $\{(X_n^T(\cdot), X_n^{T,-}(\cdot)),\ n = 1, 2, \ldots\}$ in $(D[0,T]^p, \tau_0) \times (D_l[0,T]^p, \tau_0)$ for arbitrary $T > 0$; the last two types of transfers require stationarity of the underlying sequence $\{x_n,\ n = 0, \pm 1, \pm 2, \ldots\}$.

The motivation for this work can be found in the study of large deviations properties for general single server queues, and more specifically, in the derivation of the effective bandwidth of its output process, all discussed in a companion paper.

In a significant departure from the situation under the uniform topology, such transfers are not automatic under the Skorohod topology, as additional continuity properties are required on the elements of the effective domain of the rate function I_X of the LDP for $\{X_n^1(\cdot),\ n = 1, 2, \ldots\}$ in $(D[0,1]^p, \tau_0)$. However, when the rate function I_X is of the usual integral form, the transfers are automatic, and the new rate functions assume very simple forms suggesting that from the perspective of large deviations, the past of the underlying stationary process behaves *as if* it were independent of its future.

Key words: Large deviations, partial sum processes

1 Introduction

Consider a bi-infinite sequence of $\mathbb{R}^p$-valued random variables $\{x_n,\ n = 0, \pm 1, \pm 2, \ldots\}$ defined on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$. For each $T > 0$, let $D[0,T]^p$ denote the space of $\mathbb{R}^p$-valued right-continuous functions with left-hand limits defined on $[0,T]$. We construct a family $\{X_n^T(\cdot),\ n = 1, 2, \ldots\}$ of mappings $X_n^T(\cdot) : \Omega \to D[0,T]^p$ by setting for each $n = 1, 2, \ldots$

$$X_n^T(t)(\omega) \equiv \begin{cases} \frac{1}{n}\sum_{i=1}^{\lfloor nt \rfloor} x_i(\omega) & \text{if} \quad \lfloor nt \rfloor \geq 1 \\ 0 & \text{otherwise} \end{cases}, \quad t \in [0,T], \quad \omega \in \Omega,$$

and write $X_n(\cdot)$ for $X_n^1(\cdot)$.

Much effort has been expanded on identifying conditions on the sequence $\{x_n,\ n = 0, \pm 1, \pm 2, \ldots\}$ under which the partial sum process $\{X_n(\cdot),\ n = 1, 2, \ldots\}$ satisfies a Large Deviations Principle (LDP) [6, 10, 14, 17, 19, 21], [11, Section 5.1 p. 152]. These sample path LDP results are typically given on the space $(D[0,1]^p, \tau_\infty)$ (where τ_∞ denotes the uniform topology), and it is often noted that the method used to obtain this LDP would also yield the LDP for $\{X_n^T(\cdot),\ n = 1, 2, \ldots\}$ in $(D[0,T]^p, \tau_\infty)$ for arbitrary $T > 0$; this can indeed be done by appropriately modifying the arguments of Section 6.

Here instead we are interested in LDPs for the partial sum processes $\{X_n^T(\cdot),\ n = 1, 2, \ldots\}$ but in $(D[0,T]^p, \tau_0)$ where τ_0 denotes Skorohod's J_1 topology [20]. As we shall see shortly, this change in the topological structure of the spaces has profound implications for the transfer of the LDP satisfied by $\{X_n(\cdot),\ n = 1, 2, \ldots\}$ in $(D[0,1]^p, \tau_0)$ to the family $\{X_n^T(\cdot),\ n = 1, 2, \ldots\}$ in $(D[0,T]^p, \tau_0)$ for arbitrary $T > 0$, as well as for other large deviations issues addressed in this paper.

Our focus on the Skorohod topology is rooted in technical considerations that surface naturally when considering various large deviations properties of the output process from a single server queue. Such questions have recently been investigated by several authors [4, 8, 9, 15] in an effort to ascertain the usefulness of the notion of effective bandwidth in networks. In that context, as discussed in [2], it is necessary to consider large deviations properties associated with the *past* of the underlying sequence $\{x_n,\ n = 0, \pm 1, \pm 2, \ldots\}$. More precisely, for each $T > 0$, we introduce $D_l[0,T]^p$ as the space of $\mathbb{R}^p$-valued left-continuous functions with right-hand limits defined on $[0,T]$. The space $D_l[0,T]^p$ is endowed with the analog of the Skorohod topology, which we also denote by τ_0; details are presented in Section 4. We then define the family $\{X_n^{T,-}(\cdot),\ n=1,2,\ldots\}$ of mappings $X_n^{T,-}(\cdot):\Omega\to D_l[0,T]^p$ by setting

$$X_n^{T,-}(t)(\omega) \;\equiv\; \begin{cases} \dfrac{1}{n}\displaystyle\sum_{i=1-\lceil nt\rceil}^{0} x_i(\omega) & \text{if} \quad \lceil nt\rceil \geq 1 \\ 0 & \text{otherwise} \end{cases}, \quad t\in[0,T], \quad \omega\in\Omega.$$

The arguments of [2] make use of the joint LDP for the family $\{(X_n^T(\cdot), X_n^{T,-}(\cdot)),\ =1,2,\ldots\}$ in order to characterize the large deviations of queueing systems in *equilibrium*, e.g., the large deviations of the stationary output of a general single-server queue. As briefly discussed in Section 3, the necessity of considering the joint LDP of the family $\{(X_n^T(\cdot), X_n^{T,-}(\cdot)),\ =1,2,\ldots\}$ leads very naturally to working with the (separable) Skorohod topology (rather than with the *non*-separable uniform topology) on $D[0,1]^p$, as this ensures that the joint partial sum processes are *random elements* in the product space $(D[0,T]^p,\tau_0)\times(D_l[0,T]^p,\tau_0)$. However, this use of the Skorohod topology introduces technicalities which render the proofs much more complicated than if the uniform topology were used.

This paper explores some of the issues associated with transfering the LDP for the family $\{X_n(\cdot),\ n=1,2,\ldots\}$ in $(D[0,1]^p,\tau_0)$ to the families $\{X_n^T(\cdot),\ n=1,2,\ldots\}$ in $(D[0,T]^p,\tau_0)$ [Theorem 2.1], $\{X_n^{T,-}(\cdot),\ n=1,2,\ldots\}$ in $(D[0,T]^p,\tau_0)$ [Theorem 2.2], and $\{(X_n^T(\cdot), X_n^{T,-}(\cdot)),\ =1,2,\ldots\}$ in $(D[0,T]^p,\tau_0)\times(D_l[0,T]^p,\tau_0)$ [Theorem 2.3] for arbitrary $T>0$. The latter two results assume stationarity of the sequence $\{x_n,\ n=0,\pm1,\pm2,\ldots\}$. In a significant departure from the situation under the uniform topology, this transfer is *not* automatic and requires additional *continuity* properties on the elements of the effective domain of the rate function I_X of the LDP for $\{X_n(\cdot),\ n=1,2,\ldots\}$ in $(D[0,1]^p,\tau_0)$. This can be traced back to the method of proof, and to peculiarities of the Skorohod topology. Indeed, our main tool for deriving the desired LDPs is a slight extension of the standard Contraction Principle [Theorem 3.3] which demands continuity of the transformation only on the effective domain of I_X. Unfortunately, the relevant functionals are *not* continuous on the entire space under the Skorohod topologies (while trivially continuous under the uniform topologies), and the enforced assumptions are needed to fill this gap.

These continuity assumptions are automatically satisfied in the important special case when I_X has the *integral* form (2.8), and the various rate functions of the derived LDPs can then be computed explicitly [Corollaries 2.4–2.6]. As a result of these computations, we note an interesting byproduct which can be suggestively rephrased as follows: From the perspective of large deviations, a strictly stationary process satisfying a sample path LDP in $D[0,1]$ with good rate function of the integral form has the property that its past behaves *as if* it were independent of its future! Thus far, to the best of the authors' knowledge, rate functions of the type (2.8) have only been derived for i.i.d. or stationary hyper-mixing random sequences, so that this last result is perhaps not too surprising. Our result then seems to imply a converse in the sense that if a stationary sequence satisfies a sample path LDP with good rate function of the type (2.8), then the process necessarily exhibits some form of asymptotic independence or mixing property.

The paper is organized as follows: In Section 2 we present the main results of the paper. The requisite background in the theory of large deviations is given in Section 3, while in Section 4 we discuss the spaces $D[a,b]$ and $D_l[a,b]$ endowed with Skorohod's topology. Topological properties of some functionals on the space $D[a,b]$ and $D_l[a,b]$ are developed in Section 5. Finally the proofs of Theorems 2.1, 2.2 and 2.3 are given in Sections 6, 7 and 8, respectively. Section 9 contains the discussion of the special case when the rate function is of integral form. The proofs of various technical preliminaries are relegated to the Appendix.

2 Summary of results and comments thereon

Throughout, we carry out the discussion for a *given* sequence of $\mathbb{R}^p$-valued random variables $\{x_n,\ n = 0, \pm 1, \pm 2, \ldots\}$ under various assumptions which are now listed for easy reference.

Assumption (L) — *The family of partial sum processes* $\{X_n(\cdot),\ n = 1, 2, \ldots\}$ *satisfies the LDP in* $(D[0,1]^p, \tau_0)$ *with good rate function* $I_X : D[0,1]^p \to [0,\infty]$;

Assumption (S) — *The sequence* $\{x_n,\ n = 0, \pm 1, \pm 2, \ldots\}$ *is strictly stationary.*

In stating the results, we often use the following notation: For each $T > 0$ and each mapping $\varphi : [0,T] \to \mathbb{R}^p$, we define the mapping $\varphi_T : [0,1] \to \mathbb{R}^p$ by

$$\varphi_T(t) \ = \ \frac{1}{T}\varphi(T\,t), \quad t \in [0,1]. \tag{2.1}$$

First, the transfer to $\{X_n^T(\cdot),\ n = 1, 2, \ldots\}$:

Theorem 2.1 *Assume* **(L)**, *and let* $T > 0$. *If* T *is non-integer, assume further that every element of the effective domain of* I_X *is continuous at* $t = \frac{T}{\lceil T \rceil}$.

Then, the family of partial sum processes $\{X_n^T(\cdot),\ n = 1, 2, \ldots\}$ *satisfies the LDP* *in* $(D[0,T]^p, \tau_0)$ *with good rate function* $I_X^T : D[0,T]^p \to [0,\infty]$ *given by*

$$I_X^T(\varphi) \;\equiv\; \inf_{\psi \in D[0,\lceil T\rceil]^p} \left\{ \lceil T\rceil I_X(\psi_{\lceil T\rceil}) : \psi = \varphi \quad on \ [0,T] \right\}, \quad \varphi \in D[0,T]^p. \tag{2.2}$$

The transfer to $\{X_n^{T,-}(\cdot),\ n = 1,2,\ldots\}$ is presented next:

Theorem 2.2 *Assume* **(L)** *and* **(S)**, *and let* $T > 0$. *For* T *non-integer, assume further that every element of the effective domain of* I_X *is continuous at* $t = 1 - \frac{T}{\lceil T\rceil}$.

Then, the family of partial sum processes $\{X_n^{T,-}(\cdot),\ n = 1,2,\ldots\}$ *satisfies the LDP in* $(D_l[0,T]^p, \tau_0)$ *with good rate function* $I_{X^-}^T : D_l[0,T]^p \to [0,\infty]$ *given by*

$$I_{X^-}^T(\varphi) \;\equiv\; \inf_{\psi \in D_l[0,\lceil T\rceil]^p} \left\{ I_{X^-}^{\lceil T\rceil}(\psi) : \varphi = \psi \quad on \ [0,T] \right\}, \quad \varphi \in D_l[0,T]^p \tag{2.3}$$

where $I_{X^-}^{\lceil T\rceil} : D_l[0,\lceil T\rceil]^p \to [0,\infty]$ *is a good rate function given by*

$$I_{X^-}^{\lceil T\rceil}(\psi) \;\equiv\; \begin{cases} \lceil T\rceil\, I_X(\psi_{\lceil T\rceil}(1) - \psi_{\lceil T\rceil}(1-\cdot)) & if \quad \psi(0) = 0 \\ \infty & otherwise \end{cases} \tag{2.4}$$

for ψ *in* $D[0,\lceil T\rceil]^p$ *and* $\psi_{\lceil T\rceil}$ *associated to it through (2.1).*

We now give conditions to validate the transfer to the joint family $\{(X_n^T(\cdot), X_n^{T,-}(\cdot)),\ = 1,2,\ldots\}$.

Theorem 2.3 *Assume* **(L)** *and* **(S)**, *and let* $T > 0$. *Assume further that every element of the effective domain* $\mathcal{D}(I_X)$ *of* I_X *is continuous at* $t = \frac{1}{2}$, *as well as at* $t = \frac{1}{2} \pm \frac{T}{2\lceil T\rceil}$ *for* T *non-integer.*

Then, the family $\{(X_n^T(\cdot), X_n^{T,-}(\cdot)),\ = 1,2,\ldots\}$ *satisfies the LDP in the product space* $(D[0,T]^p, \tau_0) \times (D_l[0,T]^p, \tau_0)$ *with good rate function* $I_{X,X^-}^T : D[0,T]^p \times D_l[0,T]^p \to [0,\infty]$ *given by*

$$I_{X,X^-}^T(\varphi_1, \varphi_2) \;=\; \inf_{\substack{\psi_1 \in D[0,\lceil T\rceil]^p \\ \psi_2 \in D_l[0,\lceil T\rceil]^p}} \left\{ I_{X,X^-}^{\lceil T\rceil}(\psi_1, \psi_2) : \begin{array}{l} \varphi_1 = \psi_1 \\ \varphi_2 = \psi_2 \end{array} \quad on \ [0,T] \right\} \tag{2.5}$$

for φ_1 *in* $D[0,T]^p$ *and* φ_2 *in* $D_l[0,T]^p$ *where* $I_{X,X^-}^{\lceil T\rceil} : D[0,\lceil T\rceil]^p \times D_l[0,\lceil T\rceil]^p \to [0,\infty]$ *is a good rate function given by*

$$I_{X,X^-}^{\lceil T\rceil}(\psi_1, \psi_2) \;\equiv\; \begin{cases} 2\lceil T\rceil\, I_X(\frac{1}{2\lceil T\rceil}\psi) & if \quad \psi_1(0) = \psi_2(0) = 0 \\ \infty & otherwise \end{cases} \tag{2.6}$$

for ψ_1 in $D[0,\lceil T\rceil]^p$, ψ_2 in $D_l[0,\lceil T\rceil]^p$ and $\psi:[0,1]\to\mathbb{R}$ defined through

$$\psi(t) = \begin{cases} \psi_2(\lceil T\rceil) - \psi_2(\lceil T\rceil - 2\lceil T\rceil t), & t\in[0,\frac{1}{2}] \\ \psi_2(\lceil T\rceil) + \psi_1(2\lceil T\rceil t - \lceil T\rceil), & t\in[\frac{1}{2},1] \end{cases} . \tag{2.7}$$

In their general form these results already display the influence of the topology on large deviations properties of partial sum processes. The continuity assumptions on the elements of the effective domain of I_X are required in order to overcome the fact that various natural projection mappings are not necessarily continuous (in the Skorohod topology) on their entire domain of definition. Note that for each one of these results to be true for *all* $T>0$, it is necessary for the effective domain of I_X to be contained in the space $C_0[0,1]^p$ of continuous functions $\varphi:[0,1]\to\mathbb{R}^p$ with $\varphi(0)=0$, this latter requirement following from a standard argument given in the proof of Proposition 7.1.

The results given thus far take a simpler form in one special case which is often encountered in applications. To characterize this situation, for each $T>0$, let $AC_0[0,T]^p$ denote the space of functions $\varphi:[0,T]\to\mathbb{R}^p$ which are absolutely continuous with $\varphi(0)=0$.

Assumption (I) — *The family of partial sum processes $\{X_n(\cdot),\ n=1,2,\ldots\}$ satisfies the LDP in $(D[0,1]^p,\tau_0)$ with good rate function $I_X: D[0,1]^p\to[0,\infty]$ of the integral form*

$$I_X(\varphi) = \begin{cases} \int_0^1 r(\dot\varphi(t))\,dt & \text{if } \varphi\in AC_0[0,1]^p \\ \infty & \text{otherwise} \end{cases} \tag{2.8}$$

for some Borel-measurable mapping $r:\mathbb{R}^p\to[0,\infty]$ which satisfies $\inf_{x\in\mathbf{R}^p} r(x)=0$.

A rate function I_X of the form (2.8) is said to be of the *usual integral* form. In most situations, the integrand r is in fact the rate function governing the LDP for the sample mean sequence $\{X_n(1),\ n=1,2,\ldots\}$, and as such, is Borel-measurable with $\inf_{x\in\mathbf{R}^p} r(x)=0$.

Under **(I)**, the continuity assumptions of Theorems 2.1, 2.2 and 2.3 are automatically satisfied, and the new rate functions can be explicitly computed as the next corollaries indicate.

Corollary 2.4 *Assume* **(I)**. *Then, for each $T>0$ the conclusion of Theorem 2.1* *holds with good rate function* $I_X^T : D[0,T]^p \to [0,\infty]$ *given by*

$$I_X^T(\varphi) = \begin{cases} \int_0^T r(\dot\varphi(t))\,dt & \text{if} \quad \varphi \in AC_0[0,T]^p \\ \infty & \text{otherwise} \end{cases} . \tag{2.9}$$

Corollary 2.5 *Assume* **(I)** *and* **(S)**. *Then, for each* $T > 0$ *the conclusion of Theorem 2.2 holds with good rate function* $I_{X^-}^T : D_l[0,T]^p \to [0,\infty]$ *given by*

$$I_{X^-}^T(\varphi) = \begin{cases} \int_0^T r(\dot\varphi(t))\,dt & \text{if} \quad \varphi \in AC_0[0,T]^p \\ \infty & \text{otherwise} \end{cases} . \tag{2.10}$$

More surprising perhaps is the following corollary to Theorem 2.3.

Corollary 2.6 *Assume* **(I)** *and* **(S)**. *Then, for each* $T > 0$ *the conclusion of Theorem 2.3 holds with good rate function* $I_{X,X^-}^T : D[0,T]^p \times D_l[0,T]^p \to [0,\infty]$ *given by*

$$I_{X,X^-}^T(\varphi_1,\varphi_2) = \begin{cases} \int_0^T r(\dot\varphi_1(t))\,dt + \int_0^T r(\dot\varphi_2(t))\,dt & \text{if} \quad \varphi_1,\varphi_2 \in AC_0[0,T]^p \\ \infty & \text{otherwise} \end{cases} . \tag{2.11}$$

Under assumptions **(I)** and **(S)**, Corollaries 2.4, 2.5 and 2.6 together imply

$$I_{X,X^-}^T(\varphi_1,\varphi_2) = I_X^T(\varphi_1) + I_{X^-}^T(\varphi_2), \quad \varphi_1 \in D[0,T]^p,\ \varphi_2 \in D_l[0,T]^p. \tag{2.12}$$

Hence, in view of the result contained in [11, Exercise 4.2.7 p. 113] on the joint LDP of two independent sequences, each satisfying the LDP, this last relation implies that the joint process of the past and future has the same large deviations behavior *as* two *independent* processes with same marginal distributions. In other words, from the perspective of large deviations, $\{X_n^T(\cdot),\ = 1,2,\ldots\}$ and $\{X_n^{T,-}(\cdot),\ = 1,2,\ldots\}$ behave *as if* they were independent. We close this section by wondering whether the validity of (2.12) for all $T > 0$ under the assumptions of Theorems 2.1, 2.2 and 2.3 necessarily implies the usual integral form for the rate function I_X.

3 Background on large deviations

We highlight some of the key ideas from the theory of large deviations which are used in the paper; most of this material is available in [11].

Let $(\mathcal{X},\tau)$ be a topological space, and let $\mathcal{B}(\mathcal{X})$ denote its Borel σ-field , i.e., the σ-field generated by the open sets of τ.

Definition 3.1 *A rate function I is a lower semi-continuous mapping $I: \mathcal{X} \to [0, \infty]$. A rate function I is said to be a good rate function if it is level compact, i.e., all its level sets $\Psi_I(\alpha) \equiv \{x : I(x) \leq \alpha\}$ are compact subsets of $\mathcal{X}$. The effective domain $\mathcal{D}(I)$ of I is the set of points in $\mathcal{X}$ of finite rate, namely $\mathcal{D}(I) \equiv \{x \in \mathcal{X} : I(x) < \infty\}$.*

Let $\{\mu_n,\ n = 1, 2, \ldots\}$ be a collection of probability measures defined on some σ-field $\mathcal{B}$ of $\mathcal{X}$.

Definition 3.2 *The family $\{\mu_n,\ n = 1, 2, \ldots\}$ satisfies the Large Deviations Principle (LDP) in $(\mathcal{X}, \tau)$ with rate function $I : \mathcal{X} \to [0, \infty]$ if*

$$\liminf_{n\to\infty} \frac{1}{n} \ln \mu_n(\Gamma) \ \geq \ - \inf_{x\in\Gamma^\circ} I(x) \tag{3.13}$$

and

$$\limsup_{n\to\infty} \frac{1}{n} \ln \mu_n(\Gamma) \ \leq \ - \inf_{x\in\overline{\Gamma}} I(x) \tag{3.14}$$

for every Γ in $\mathcal{B}$.

Throughout we shall always assume $\mathcal{B}(\mathcal{X}) \subseteq \mathcal{B}$ as most of the developments in [11] require this inclusion. As will be explained shortly in this section, this assumption is automatically satisfied when considering random elements.

The Contraction Principle constitutes our main tool for establishing the LDPs of this paper, and we we shall use a version of it mentioned in a remark following the proof of Theorem 4.2.1 of [11, p. 110].

Theorem 3.3 (Contraction Principle) *Let $\mathcal{X}$ and $\mathcal{Y}$ be two regular topological spaces and let $\{\mu_n,\ n = 1, 2, \ldots\}$ be a family of probability measures on the Borel σ-field $\mathcal{B}(\mathcal{X})$ of $\mathcal{X}$. Assume the family $\{\mu_n,\ n = 1, 2, \ldots\}$ satisfies the LDP in $\mathcal{X}$ with good rate function $I : \mathcal{X} \to [0, \infty]$, and let the Borel-measurable mapping $f : \mathcal{X} \to \mathcal{Y}$ be continuous on the effective domain $\mathcal{D}(I)$ of I.*

Then, the family $\{\mu_n \circ f^{-1},\ n = 1, 2, \ldots\}$ of probability measures on $\mathcal{B}(\mathcal{Y})$ satisfies the LDP in $\mathcal{Y}$ with good rate function I' given by

$$I'(y) \ \equiv \ \inf_{x\in\mathcal{X}} \{I(x) : y = f(x)\}, \quad y \in \mathcal{Y}. \tag{3.15}$$

In [11, p. 110] the Contraction Principle is established only for *continuous* $f : \mathcal{X} \to \mathcal{Y}$; a proof of Theorem 3.3 in its general form is given in Appendix A.1.

In this paper we use the technical machinery of [11] in the somewhat more concrete setup where probability measures are induced by random elements: A *random element* ξ defined on some probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and taking values in a topological space $(\mathcal{X}, \tau)$ — in short, a random element in $(\mathcal{X}, \tau)$ — is understood as a measurable mapping $\xi : (\Omega, \mathcal{F}) \to (\mathcal{X}, \mathcal{B}(\mathcal{X}))$. The distribution law of the random element ξ is the probability measure μ on $\mathcal{B}(\mathcal{X})$ defined by

$$\mu(B) \equiv \mathbf{P}[\xi \in B], \quad B \in \mathcal{B}(\mathcal{X}).$$

Given a collection of random elements $\{\xi_n,\ n = 1, 2, \ldots\}$ in $(\mathcal{X}, \tau)$, the LDP for $\{\xi_n,\ n = 1, 2, \ldots\}$ is then defined as the LDP for the collection of induced probability measures $\{\mu_n,\ n = 1, 2, \ldots\}$; for each $n = 1, 2, \ldots$, the distribution law μ_n of ξ_n is automatically defined on the Borel σ-field $\mathcal{B}(\mathcal{X})$ of $(\mathcal{X}, \tau)$.

At this level of generality, the notion of random element suffers from limitations which are tied to the interplay between measurable and topological structures: To see this, start with random elements $X_1, \ldots, X_m$, i.e., X_i is a random element on some topological space $(\mathcal{X}_i, \tau_i)$, or equivalently, $X_i : (\Omega, \mathcal{F}) \to (\mathcal{X}_i, \mathcal{B}(\mathcal{X}_i))$ is a measurable mapping, $i = 1, 2, \ldots, m$. Consider the product topological space $(\mathcal{X}, \tau)$ where $\mathcal{X} = \prod_{i=1}^m \mathcal{X}_i$ and τ denotes the product topology on $\mathcal{X}$. In general, the mapping $X : \Omega \to \mathcal{X}$ defined by

$$X(\omega) = (X_1(\omega), \ldots, X_m(\omega)), \quad \omega \in \Omega$$

may not be $\mathcal{F}/\mathcal{B}(\mathcal{X})$-measurable, and is thus *not* necessarily a random element in $(\mathcal{X}, \tau)$. Indeed, with $\mathcal{B} = \otimes_{i=1}^m \mathcal{B}(\mathcal{X}_i)$, it is well known [12, p. 55] that X is $\mathcal{F}/\mathcal{B}$-measurable and that the inclusion $\mathcal{B} \subseteq \mathcal{B}(\mathcal{X})$ always holds [16, p. 6]. However, the reverse inclusion $\mathcal{B}(\mathcal{X}) \subseteq \mathcal{B}$ requires additional properties on the topologies. In particular it holds for *separable* Hausdorff spaces [16, Theorem 1.10 p. 6], in which case $\mathcal{B} = \mathcal{B}(\mathcal{X})$ and the m-variate X is now a random element in $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$.

Therefore, in this paper and its sequel [2], we consider only *separable* Hausdorff spaces, so as to ensure that m-uples of random elements are themselves random elements. The distribution law of the m-uple of random elements is now defined on the Borel σ-field of the product space (and the the blanket assumption made earlier automatically holds).

4 Skorohod topologies on the spaces $D[a,b]^p$ and $D_l[a,b]^p$

For each $T > 0$ and each $n = 1,2,\ldots$, the mappings $\omega \to X_n^T(\cdot,\omega)$ and $\omega \to X_n^{T,-}(\cdot,\omega)$ take values in $D[0,T]^p$ and $D_l[0,T]^p$, respectively. As pointed out in Section 3, it is crucial to identify the topological structures on these spaces under which the families $\{X_n^T(\cdot),\ n=1,2,\ldots\}$ and $\{X_n^{T,-}(\cdot),\ n=1,2,\ldots\}$ satisfy LDPs.

Generalizing somewhat the set-up, we consider the spaces $D[a,b]^p$ and $D_l[a,b]^p$, $0 \le a < b$. Two topologies have traditionally been put on these spaces, namely the uniform topology τ_∞, induced by the uniform metric, and Skorohod's J_1 topology [20].

The uniform metric is defined by

$$d_\infty(x,y) \equiv \sup_{t\in[a,b]} |x(t)-y(t)|, \quad x,y \in D[a,b]^p \ \ (\text{or } D_l[a,b]^p) \tag{4.16}$$

with $|\cdot|$ denoting Euclidean norm in $\mathbb{R}^p$.

Unfortunately, the complete metric spaces $(D[a,b]^p,\tau_\infty)$ and $(D_l[a,b]^p,\tau_\infty)$ are *not* separable [5, p. 150]. As a consequence, although $X_n^T(\cdot)$ and $X_n^{T,-}(\cdot)$ are random elements in $(D[0,T]^p,\tau_\infty)$ and $(D_l[0,T]^p,\tau_\infty)$, respectively, the pair $(X_n^T(\cdot),X_n^{T,-}(\cdot))$ is not necessarily a random element in the product space $(D[0,T]^p,\tau_\infty)\times(D_l[0,T]^p,\tau_\infty)$. To avoid this problem we seek instead to equip the spaces $D[0,T]^p$ and $D_l[0,T]^p$ with (metrizable) topologies which are coarser than the uniform topology *and* separable.

A natural candidate topology is the (separable) Skorohod J_1 topology. Following [20], we introduce the set Λ_{ab} consisting of continuous bijections $\lambda : [a,b] \to [a,b]$ which are strictly increasing; we necessarily have $\lambda(a) = a$ and $\lambda(b) = b$. A sequence $\{x_n,\ n=1,2,\ldots\}$ in $D[a,b]^p$ is said to be J_1-convergent to the element x of $D[a,b]^p$ if there exists a sequence of mappings $\{\lambda_n,\ n=1,2,\ldots\}$ in Λ_{ab} such that

$$\lim_{n\to\infty} d_\infty(x_n\circ\lambda_n, x) \ = \ \lim_{n\to\infty} d_\infty(\lambda_n, e) \ = \ 0 \tag{4.17}$$

where e denotes the identity mapping $[a,b]\to[a,b]$. The Skorohod J_1 topology on $D[a,b]^p$ is induced by a metric d_0 which makes it into a Polish space [5]. This metric d_0 is defined by

$$d_0(x,y) \equiv \inf_{\lambda\in\Lambda_{ab}} \left(\sup_{t\in[a,b]} |x(t) - y\circ\lambda(t)| \ \vee \ \sup_{s\neq t} \left| \ln \frac{\lambda(s)-\lambda(t)}{s-t} \right| \right) \tag{4.18}$$

for x and y in $D[a,b]^p$.

The Skorohod topology has been extensively studied since its introduction in the context of weak convergence [20]; comprehensive treatments are available for instance in the texts [5, 13, 18]. However, its transposition to the space $D_l[a,b]^p$ appears not to have attracted much attention; for the sake of completeness, we briefly indicate how to define the Skorohod topology on $D_l[a,b]^p$.

To this end, consider the mapping $\varphi_{ab} : [a,b] \to [a,b]$ defined by

$$\varphi_{ab}(t) = a + b - t, \quad t \in [a,b],$$

and define the mapping $\Phi_{ab} : D_l[a,b]^p \to D[a,b]^p$ by

$$\Phi_{ab}(x) \equiv x \circ \varphi_{ab}, \quad x \in D_l[a,b]^p. \tag{4.19}$$

It is plain that φ_{ab} and Φ_{ab} are both bijections with $\varphi_{ab}^{-1} = \varphi_{ab}$ and $\Phi_{ab}^{-1}(x) = x \circ \varphi_{ab}$ for all x in $D[a,b]^p$.

Identically to the definition (4.18) of d_0 on $D[a,b]^p$, we define the mapping $d_0^l : D_l[a,b]^p \times D_l[a,b]^p \to \mathbb{R}_+$ by setting

$$\begin{aligned} d_0^l(x,y) &\equiv \inf_{\lambda \in \Lambda_{ab}} \left(\sup_{t \in [a,b]} |x(t) - y \circ \lambda(t)| \vee \sup_{s \neq t} |\ln \frac{\lambda(s) - \lambda(t)}{s-t}| \right) \\ &= d_0(\Phi_{ab}(x), \Phi_{ab}(y)), \quad x, y \in D_l[a,b]^p \end{aligned}$$

where the last equality is validated with the help of Lemma A.2 (with $\alpha = -1$ and $\beta = a+b$). Therefore, Φ_{ab} being one-to-one, it follows that d_0^l is a metric on $D_l[a,b]^p$; in fact, Φ_{ab} establishes an *isometry* between $(D_l[a,b]^p, d_0^l)$ and $(D[a,b]^p, d_0)$. All topological and metric properties of $(D[a,b]^p, d_0)$ then translate to $(D_l[a,b]^p, d_0^l)$; in particular, $(D_l[a,b]^p, d_0^l)$ is also a Polish space.

In [1, Lemma 2.3] Skorohod's J_1-convergence on the space $D_l[a,b]^p$ (as defined by (4.17)) is shown to be equivalent to d_0^l-convergence. For this reason, we can safely refer to the topology induced by the metric d_0^l as the Skorohod topology on $D_l[a,b]^p$, and in the sequel we shall use either one of the characterizations of Skorohod convergence in this space. We abuse the notation somewhat by writing d_0 for d_0^l, as well as τ_∞ and τ_0 for the uniform and Skorohod topologies on $D_l[a,b]^p$ induced respectively by the metrics d_∞ and d_0^l. From now on, unless otherwise mentioned, the topologies considered on $D[a,b]^p$ and $D_l[a,b]^p$, as well as on any of their subspaces are the Skorohod topologies induced by the metric d_0. In particular, S-continuity and S-convergence refer to continuity and convergence in these Skorohod topologies.

5 S-continuity considerations on $D[a,b]^p$ and $D_l[a,b]^p$

The results obtained here on the S-continuity (or lack thereof) of certain functionals may at first appear counter-intuitive as many of these functionals are continuous in the uniform topology but not in the Skorohod topology. Unless stated otherwise, measurability is understood with respect to the Borel σ-fields induced by the Skorohod topologies.

Lemma 5.1 *For $\alpha > 0$ (resp. $\alpha < 0$), β in $\mathbb{R}$ and $\gamma \neq 0$, the mapping $F : D[a,b]^p \to D[\alpha a+\beta, \alpha b+\beta]^p$ (resp. $F : D[a,b]^p \to D_l[\alpha b+\beta, \alpha a+\beta]^p$) defined by*

$$F(x)(t) \;=\; \gamma\, x\left(\frac{1}{\alpha}(t-\beta)\right), \quad t \in [\alpha a+\beta, \alpha b+\beta] \text{ (resp. } [\alpha b+\beta, \alpha a+\beta]), \quad x \in D[a,b]^p$$

is d_0-Lipschitz, hence S-continuous on $D[a,b]^p$.

Proof: Fix $\alpha > 0$, β in $\mathbb{R}$ and $\gamma \neq 0$, and define the mapping $f : [\alpha a+\beta\,,\, \alpha b+\beta] \to [a,b]$ by $f(t) = \frac{1}{\alpha}(t-\beta)$ for all t in $[\alpha a+\beta, \alpha b+\beta]$. Then, for all x in $D[a,b]^p$, $F(x) = \gamma\, x \circ f$, and from Lemma A.2 we get

$$\begin{aligned} d_0(F(x), F(y)) &= d_0(\gamma\, x \circ f, \gamma\, y \circ f) \\ &= d_0(\gamma\, x, \gamma\, y) \\ &\leq \max\{1, |\gamma|\}\, d_0(x,y), \quad x,y \in D[a,b]^p. \end{aligned}$$

The case $\alpha < 0$ is handled in a similar way. ∎

Lemma 5.2 *Let t_0 in $[a,b]$ and define the projection mapping $\pi_{t_0} : D[a,b]^p \to \mathbb{R}^p$ given by*

$$\pi_{t_0}(x) \equiv x(t_0), \quad x \in D[a,b]^p.$$

Then, the mappings π_a and π_b are always S-continuous, while for $a < t_0 < b$, the Borel-measurable mapping π_{t_0} is S-continuous at x if and only if t_0 is a continuity point of x.

The same properties hold true on $D_l[a,b]^p$ for the projection mapping $\pi^l_{t_0} : D_l[a,b]^p \to \mathbb{R}^p : x \to x(t_0)$.

Proof: A proof of the Borel-measurability of π_{t_0} can be found in [3, Theorem 1 p. 170] together with the S-continuity results.

We check that the same properties hold true for $\pi^l_{t_0}$ for each t_0 in $[a,b]$ upon noting that

$$\pi^l_{t_0}(x) = x(t_0) = x \circ \varphi_{ab}(\varphi_{ab}^{-1}(t_0)) = \pi_{\varphi_{ab}^{-1}(t_0)}(x \circ \varphi_{ab}), \quad x \in D_l[a,b]^p,$$

and that continuity of x at t_0 is equivalent to continuity of $\Phi_{ab}(x) = x \circ \varphi_{ab}$ at $\varphi_{ab}^{-1}(t_0)$. ■

The Borel-measurability and S-continuity of the restriction mappings are based on the following preliminary fact [22].

Lemma 5.3 *Let $\{x_n,\ n = 1, 2, \ldots\}$ be a sequence which is S-converging to x in $D[a,b]^p$, and let c in (a,b) be a continuity point of x. Then, the restrictions of $\{x_n,\ n = 1, 2, \ldots\}$ in $D[a,c]^p$ (resp. $D[c,b]^p$) form a sequence which is S-converging to the restriction of x in $D[a,c]^p$ (resp. $D[c,b]^p$).*

As the details of its proof are not available in [22], we give a complete proof in Appendix A.3.

Lemma 5.4 *Let $[c,d] \subseteq [a,b]$, and define the restriction $r_{cd} : D[a,b]^p \to D[c,d]^p$ by*

$$r_{cd}(x)(t) \; = \; x(t), \quad t \in [c,d], \quad x \in D[a,b]^p.$$

Then, the Borel-measurable mapping r_{cd} is S-continuous at x in $D[a,b]^p$ if and only if c and d are continuity points of x or endpoints of $[a,b]$.

The same properties hold true on $D_l[a,b]^p$ for the restriction $r^l_{cd} : D_l[a,b]^p \to D_l[c,d]^p$.

Proof: Borel-measurability of r_{cd} is shown as Lemma 2.3 in [22].

In view of Lemma 5.3, we need only consider the case $a < c$, $b < d$ when showing the S-continuity of r_{cd}. In that case, let x in $D[a,b]^p$ be continuous at both interior points c and d, and let a sequence $\{x_n,\ n = 1, 2, \ldots\}$ in $D[a,b]^p$ be S-converging to x. Upon applying Lemma 5.3 twice, first to $D[c,b]^p$ and then to $D[c,d]^p$, we conclude that the restrictions $\{x_n,\ n = 1, 2, \ldots\}$ on $[c,d]$ are S-converging to the restriction of x on $[c,d]$, whence the sequence $\{r_{cd}(x_n),\ n = 1, 2, \ldots\}$ is S-converging to $r_{cd}(x)$ in $D[c,d]^p$, and r_{cd} is thus S-continuous at x.

We now establish the "only if" part by proving its contraposition: To this end, let x be an element of $D[a,b]^p$ with a discontinuity at c, and suppose now that r_{cd} were S-continuous at x. Let $\pi_c^{ab} : D[a,b]^p \to \mathbb{R}^p$ and $\pi_c^{cd} : D[c,d]^p \to \mathbb{R}^p$ denote the natural projections at c. Because c is an endpoint for $[c,d]$, π_c^{cd} is S-continuous on $D[c,d]^p$ by Lemma 5.2, so in particular at $r_{cd}(x)$. The equality $\pi_c^{ab} = \pi_c^{cd} \circ r_{cd}$ then yields S-continuity of π_c^{ab} at x, a contradiction with Lemma 5.2 since c is a discontinuity point of x. A similar argument holds if d is a discontinuity point, and we conclude that r_{cd} is not S-continuous at x whenever c or d are discontinuity point of x.

In order to establish the Borel-measurability and S-continuity properties of r^l_{cd}, we start with the observation

$$r^l_{cd}(x) = \left(F \circ r_{\varphi_{ab}^{-1}(d)\varphi_{ab}^{-1}(c)} \circ \Phi_{ab}\right)(x), \quad x \in D_l[a,b]^p,$$

with $F : D[\varphi_{ab}^{-1}(d), \varphi_{ab}^{-1}(c)] \to D_l[c,d]$ as defined in Lemma 5.1 with $\alpha = -1$, $\beta = a + b$, and $\gamma = 1$. The properties of r^l_{cd} then follow from that of $r_{\varphi_{ab}^{-1}(d)\varphi_{ab}^{-1}(c)}$ and from the S-continuity of Φ_{ab} and F because c, d are either continuity points of x or endpoints of $[a,b]$ if and only if $\varphi_{ab}^{-1}(d), \varphi_{ab}^{-1}(c)$ are either continuity points of $\Phi_{ab}(x)$ or endpoints of $[a,b]$. ∎

Lemma 5.5 *The addition mapping $S : D[a,b]^p \times D[a,b]^p \to D[a,b]^p$ (resp. $S^l : D_l[a,b]^p \times D_l[a,b]^p \to D_l[a,b]^p$) is Borel-measurable, and S-continuous at those pairs of points x and y which do not have common discontinuity points.*

Proof: A proof of the results for the addition on $D[a,b]^p \times D[a,b]^p$ can be found in [22, Theorem 4.1].

We readily check that the same result holds on $D_l[a,b]^p \times D_l[a,b]^p$ once we observe that

$$x + y = \Phi_{ab}^{-1}\left(\Phi_{ab}(x) + \Phi_{ab}(y)\right), \quad x, y \in D_l[a,b]^p,$$

or equivalently,

$$S^l(x,y) = \Phi_{ab}^{-1}\left(S\left(\Phi_{ab}(x), \Phi_{ab}(y)\right)\right), \quad x, y \in D_l[a,b]^p$$

and that x and y have no common discontinuity points if and only if $\Phi_{ab}(x)$ and $\Phi_{ab}(y)$ have no common discontinuity points. ∎

6 LDP for $X_n^T(\cdot)$ in the space $D[0,T]^p$

The proof of Theorem 2.1, presented in this section, begins with a simple preliminary lemma.

Lemma 6.1 *Assume* **(L)**. *Then, for each $K = 1, 2, \ldots$, the subsequence $\{X_{nK}(\cdot),\ n = 1, 2, \ldots\}$ in $D[0,1]^p$ satisfies the LDP in $(D[0,1]^p, \tau_0)$ with good rate function $K\, I_X$.*

Proof: Fix $K = 1, 2, \ldots$. For any subset B in the Borel σ-field of $(D[0,1]^p, \tau_0)$, we have

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{n} \ln \mathbf{P}\left[X_{nK}(\cdot) \in B \right] &= K \limsup_{n\to\infty} \frac{1}{nK} \ln \mathbf{P}\left[X_{nK}(\cdot) \in B \right] \\
&\leq K \limsup_{n\to\infty} \frac{1}{n} \ln \mathbf{P}\left[X_{n}(\cdot) \in B \right]
\end{aligned}$$

and

$$\liminf_{n\to\infty} \frac{1}{n} \ln \mathbf{P}\left[X_{nK}(\cdot) \in B \right] \geq K \liminf_{n\to\infty} \frac{1}{n} \ln \mathbf{P}\left[X_{n}(\cdot) \in B \right].$$

From these inequalities, it is then plain that the existence of a LDP for the sequence $\{X_n(\cdot),\ n = 1, 2, \ldots\}$ in $D[0,1]^p$ with good rate function I_X translates into one for the subsequence $\{X_{nK}(\cdot),\ n = 1, 2, \ldots\}$ in $D[0,1]^p$ with good rate function $K\, I_X$. ■

The simpler case where T is integer is discussed first.

Proposition 6.2 *Assume* **(L)**. *Then, for each $K = 1, 2, \ldots$, the family $\{X_n^K(\cdot),\ n = 1, 2, \ldots\}$ satisfies the LDP in $(D[0,K]^p, \tau_0)$ with good rate function $I_X^K : D[0,K]^p \to [0, \infty]$ given by*

$$I_X^K(\varphi) \equiv K I_X(\varphi_K), \quad \varphi \in D[0,K]^p \tag{6.20}$$

where φ_K is the element of $D[0,1]^p$ associated with φ through (2.1) (with $T = K$).

Proof: Fix $K = 1, 2, \ldots$ and $n = 1, 2, \ldots$; we have

$$X_n^K(t) = \frac{1}{n} \sum_{i=1}^{\lfloor nt \rfloor} x_i = K \frac{1}{nK} \sum_{i=1}^{\lfloor nK \frac{t}{K} \rfloor} x_i, \quad t \in [0, K].$$

Therefore,

$$X_n^K(\cdot) = F(X_{nK}(\cdot)) \tag{6.21}$$

where the mapping $F : D[0,1]^p \to D[0,K]^p$ is defined by

$$F(z)(t) = K\, z(\frac{t}{K}), \quad t \in [0, K], \quad z \in D[0,1]^p .$$

The S-continuity of F is ensured by Lemma 5.1, and the LDP for the family $\{X_n^K(\cdot),\ n = 1, 2, \ldots\}$ in $D[0,K]^p$ follows from that for the family $\{X_{nK}(\cdot),\ n =$

$1, 2, \ldots\}$ in $D[0,1]^p$ (Lemma 6.1) and the Contraction Principle (via (6.21)). The corresponding rate function $I_X^K : D[0,K]^p \to [0,\infty]$ is given by

$$\begin{aligned} I_X^K(\varphi) &= \inf\left\{ K\, I_X(\psi) : \varphi(t) = K\psi(\frac{t}{K}),\ t \in [0,K],\ \psi \in D[0,1]^p \right\} \\ &= \inf\left\{ K\, I_X(\psi) : \psi(u) = \frac{1}{K}\varphi(Ku),\ u \in [0,1],\ \psi \in D[0,1]^p \right\} \\ &= K\, I_X(\varphi_K), \quad \varphi \in D[0,K]^p, \end{aligned}$$

and the proof is completed. ■

We now turn to the proof of Theorem 2.1.

A proof of Theorem 2.1: Fix $T > 0$. In view of Proposition 6.2 we need only consider the case when T is non-integer. In fact, Proposition 6.2 already yields the LDP for the family $\{X_n^{\lceil T\rceil}(\cdot),\ n = 1, 2, \ldots\}$ with good rate function $I_X^{\lceil T\rceil} : D[0, \lceil T\rceil]^p \to [0,\infty]$ given by

$$I_X^{\lceil T\rceil}(\psi) = \lceil T\rceil\, I_X(\psi_{\lceil T\rceil}), \quad \psi \in D[0, \lceil T\rceil]^p, \tag{6.22}$$

where the element $\psi_{\lceil T\rceil}$ is associated with ψ through (2.1).

Next, we note that

$$X_n^T(\cdot) = r_T(X_n^{\lceil T\rceil}),\ n = 1, 2, \ldots .$$

with the restriction $r_T : D[0, \lceil T\rceil]^p \to D[0,T]^p$ being Borel-measurable by Lemma 5.4. The Contraction Principle will thus yield the LDP for $\{X_n^T(\cdot),\ n = 1, 2, \ldots\}$ from that for the family $\{X_n^{\lceil T\rceil}(\cdot),\ n = 1, 2, \ldots\}$ *provided* we can establish the S-continuity of $r_T : D[0, \lceil T\rceil]^p \to D[0,T]^p$ on the effective domain of $I_X^{\lceil T\rceil}$. In that case, the good rate function $I_X^T : D[0,T]^p \to [0,\infty]$ is given by (2.2). From (2.1) and (6.22), we see that $\mathcal{D}(I_X^{\lceil T\rceil})$ is in one-to-one correspondence with $\mathcal{D}(I_X)$; in fact ψ belongs to $\mathcal{D}(I_X^{\lceil T\rceil})$ if and only if $\psi_{\lceil T\rceil}$ belongs to $\mathcal{D}(I_X)$. Hence, the assumption on the continuity at $t = \frac{T}{\lceil T\rceil}$ of the elements in $\mathcal{D}(I_X)$ translates into the continuity at $t = T$ for the elements of $\mathcal{D}(I_X^{\lceil T\rceil})$, and by Lemma 5.4, the restriction r_T is thus S-continuous on the effective domain of $I_X^{\lceil T\rceil}$. ■

7 LDP for $X_n^{T,-}(\cdot)$ in $D_l[0,T]^p$

The proof of Theorem 2.2 passes through the intermediate Proposition 7.1.

Proposition 7.1 *Assume* **(L)** *and* **(S)**. *Then, for each* $K = 1, 2, \ldots,$ *the family* $\{X_n^{K,-}(\cdot),\ n = 1, 2, \ldots\}$ *satisfies the LDP in* $(D_l[0, K]^p, \tau_0)$ *with good rate function* $I_X^{K,-} : D_l[0, K]^p \to [0, \infty]$ *given by*

$$I_X^{K,-}(\varphi) \equiv \begin{cases} K\, I_X(\varphi_K(1) - \varphi_K(1 - \cdot)) & \text{if } \varphi(0) = 0 \\ \infty & \text{otherwise} \end{cases}, \quad \varphi \in D_l[0, K]^p \tag{7.23}$$

where φ_K *is the element of* $D_l[0, 1]^p$ *associated with* φ *through (2.1) (with* $T = K$*).*

Proof: Fix $K = 1, 2, \ldots$. For each $n = 1, 2, \ldots,$ we have

$$X_n^{K,-}(\cdot) = \frac{1}{n} \sum_{i=1-\lceil n\cdot\rceil}^{0} x_i =_{st} \frac{1}{n} \sum_{i=nK+1-\lceil n\cdot\rceil}^{nK} x_i \tag{7.24}$$

where the last step follows from the stationarity of $\{x_n,\ n = 0, \pm 1, \pm 2, \ldots\}$ and the fact that the shift (nK) does not depend on t.

Next, for each t in $[0, K]$, we find after some simple algebra that

$$nK - \lceil nt \rceil = \lfloor nK - nt \rfloor = \lfloor n(K - t) \rfloor .$$

Therefore, from (7.24) we get that

$$\begin{aligned} X_n^{K,-}(\cdot) &=_{st} \frac{1}{n} \sum_{i=1+\lfloor n(K-\cdot)\rfloor}^{nK} x_i \\ &= \frac{1}{n}\sum_{i=1}^{nK} x_i - \frac{1}{n} \sum_{i=1}^{\lfloor n(K-\cdot)\rfloor} x_i \\ &= X_n^K(K) - X_n^K(K - \cdot) \\ &= F(X_n^K(\cdot)), \end{aligned} \tag{7.25}$$

where the mapping $F : D[0, K]^p \to D_l[0, K]^p$ is defined by

$$F(z)(t) \equiv z(K) - z(K - t), \quad t \in [0, K], \quad z \in D[0, K]^p.$$

By Lemma 5.2, the projection mapping $z \to z(K)$ is S-continuous on $D[0, K]^p$, while by Lemma 5.1 the mapping $z \to z(K - \cdot)$ is S-continuous on $D[0, K]^p$. Constant mappings having no discontinuity points, we can now invoke Lemma 5.5 to conclude that F is S-continuous on $D[0, K]^p$.

By Proposition 6.2 we already know that the family $\{X_n^K(\cdot),\ n = 1, 2, \ldots\}$ satisfies the LDP in $(D[0, K]^p, \tau_0)$ with good rate function I_X^K given by (6.20). Thus, in view of (7.25), and of the S-continuity of F on $D[0,K]^p$, the Contraction Principle yields the LDP for the family $\left\{F(X_n^K(\cdot)),\ \ n=1,2,\ldots\right\}$ in $(D_l[0,K]^p,\tau_0)$, whence for the (stochastically equivalent) processes $\{X_n^{K,-}(\cdot),\ n=1,2,\ldots\}$. The corresponding good rate function $I_{X^-}^K : D_l[0,K]^p \to [0,\infty]$ is given by

$$I_{X^-}^K(\varphi) \ = \inf_{\psi\in D[0,K]^p}\left\{I_X^K(\psi) : \varphi = F(\psi)\right\}, \quad \varphi\in D_l[0,K]^p. \tag{7.26}$$

Fix φ in $D_l[0,K]^p$. The constraint $\varphi = F(\psi)$ obviously requires that φ vanishes at $t=0$, whence $I_{X^-}^K(\varphi)=\infty$ whenever $\varphi(0)\neq 0$. On the other hand, $X_n^K(0)=0$ for all $n=1,2,\ldots$ and the set $\{\psi\in D[0,K]^p : \psi(0)=0\}$ being closed in $D[0,K]^p$, we conclude from Lemma 4.1.5 in [11, p. 104] that $I_X^K(\psi)=\infty$ for $\psi(0)\neq 0$. Therefore, with $\varphi(0)=0$, the constraint on ψ entering (7.26) can be further sharpened to

$$\psi(0)=0, \quad \varphi(t)=\psi(K)-\psi(K-t), \quad t\in[0,K].$$

Expressing ψ in terms of φ, we find

$$\psi(K)=\varphi(K), \quad \psi(t)=\psi(K)-\varphi(K-t), \quad t\in[0,K]$$

and the desired expression (7.23) is now easily deduced from (6.20) and (7.26). ■

A proof of Theorem 2.2: Fix $T>0$. In view of Proposition 7.1 we need only consider the case when T is non-integer. In fact, by Proposition 7.1 the family $\{X_n^{\lceil T\rceil,-}(\cdot),\ n=1,2,\ldots\}$ satisfies the LDP with good rate function $I_X^{\lceil T\rceil,-}$: $D_l[0,\lceil T\rceil]^p\to[0,\infty]$ given by (2.4).

Next, we note that

$$X_n^{T,-}(\cdot) \ = \ r_T(X_n^{\lceil T\rceil,-}),\ n=1,2,\ldots$$

with the restriction $r_T : D_l[0,\lceil T\rceil]^p\to D_l[0,T]^p$ being Borel-measurable by Lemma 5.4. The Contraction Principle will thus yield the LDP for $\{X_n^{T,-}(\cdot),\ n=1,2,\ldots\}$ from that for the family $\{X_n^{\lceil T\rceil,-}(\cdot),\ n=1,2,\ldots\}$ *provided* we can establish that $r_T : D_l[0,\lceil T\rceil]^p\to D_l[0,T]^p$ is S-continuous on the effective domain of $I_X^{\lceil T\rceil,-}$, in which case the good rate function $I_{X^-}^T : D_l[0,T]^p\to[0,\infty]$ will be given by (2.3). But, in view of (2.4) if ψ belongs to the effective domain of $I_{X^-}^{\lceil T\rceil}$, then $\psi_{\lceil T\rceil}(1)-\psi_{\lceil T\rceil}(1-\cdot)$ is necessarily an element of $\mathcal{D}(I_X)$, and the continuity at $t=1-\frac{T}{\lceil T\rceil}$ of the elements in $\mathcal{D}(I_X)$ now translates into the continuity at $t=T$ for ψ. By Lemma 5.4, the restriction r_T is thus S-continuous on the effective domain of $I_X^{\lceil T\rceil}$. ■

8 LDP for $\left(X_n^T(\cdot), X_n^{T,-}(\cdot)\right)$ in $D[0,T]^p \times D_l[0,T]^p$

In order to prove the main result of the paper (Theorem 2.3), we begin by deriving the result for the particular situation where T is integer.

Proposition 8.1 *Assume* **(L)** *and* **(S)**. *Assume further that every element of the effective domain of I_X is continuous at $t = \frac{1}{2}$.*

Then, for each $K = 1, 2, \ldots$, the family $\left\{\left(X_n^K(\cdot), X_n^{K,-}(\cdot)\right),\ n = 1, 2, \ldots\right\}$ satisfies the LDP in the product space $(D[0,K]^p, \tau_0) \times (D_l[0,K]^p, \tau_0)$ with good rate function $I_{X,X^-}^K : D[0,K]^p \times D_l[0,K]^p \to [0,\infty]$ given by

$$I_{X,X^-}^K(\varphi_1,\varphi_2) \equiv \begin{cases} 2K\, I_X(\frac{1}{2K}\varphi) & \text{if } \varphi_1(0) = \varphi_2(0) = 0 \\ \infty & \text{otherwise} \end{cases}, \tag{8.27}$$

for φ_1 in $D[0,K]^p$, φ_2 in $D_l[0,K]^p$ and $\varphi : [0,1] \to \mathbb{R}^p$ given by

$$\varphi(t) = \begin{cases} \varphi_2(K) - \varphi_2(K - 2Kt), & t \in [0, \frac{1}{2}] \\ \varphi_2(K) + \varphi_1(2Kt - K), & t \in [\frac{1}{2}, 1] \end{cases}. \tag{8.28}$$

For φ_1 in $D[0,K]^p$ and φ_2 in $D_l[0,K]^p$, the mapping φ defined by (8.28) is indeed right-continuous with left-hand limits, and is therefore an element of $D[0,1]^p$.

Proof: Fix $K = 1, 2, \ldots$; for $n = 1, 2, \ldots$, we have

$$\begin{aligned}\left(X_n^K(\cdot), X_n^{K,-}(\cdot)\right) &= \left(\frac{1}{n}\sum_{i=1}^{\lfloor n\cdot\rfloor} x_i\,,\ \frac{1}{n}\sum_{i=1-\lceil n\cdot\rceil}^{0} x_i\right) \\ &=_{st} \left(\frac{1}{n}\sum_{i=nK+1}^{\lfloor n\cdot\rfloor+nK} x_i\,,\ \frac{1}{n}\sum_{i=nK+1-\lceil n\cdot\rceil}^{nK} x_i\right)\end{aligned} \tag{8.29}$$

where the last step follows from the stationarity of $\{x_n,\ n = 0, \pm1, \pm2, \ldots\}$ and the fact that the shift (nK) does not depend on t.

Next, for each t in $[0,K]$, we find after some simple algebra that

$$nK + \lfloor nt\rfloor = \lfloor nK + nt\rfloor = \lfloor n(t+K)\rfloor$$

and

$$nK - \lceil nt\rceil = \lfloor nK - nt\rfloor = \lfloor n(K-t)\rfloor .$$

Therefore, from (8.29) we get that

$$
\begin{aligned}
\left(X_n^K(\cdot), X_n^{K,-}(\cdot)\right) &=_{st} \left(\frac{1}{n}\sum_{i=nK+1}^{\lfloor n(K+\cdot)\rfloor} x_i\ ,\ \frac{1}{n}\sum_{i=1+\lfloor n(K-\cdot)\rfloor}^{nK} x_i\right)\\
&= \left(\frac{1}{n}\sum_{i=1}^{\lfloor n(K+\cdot)\rfloor} x_i - \frac{1}{n}\sum_{i=1}^{nK} x_i\ ,\ \frac{1}{n}\sum_{i=1}^{nK} x_i - \frac{1}{n}\sum_{i=1}^{\lfloor n(K-\cdot)\rfloor} x_i\right).
\end{aligned}
\tag{8.30}
$$

Finally, upon noting that

$$
\begin{aligned}
\frac{1}{n}\sum_{i=1}^{\lfloor n(K\pm s)\rfloor} x_i &= 2K\frac{1}{2nK}\sum_{i=1}^{\lfloor 2nK\frac{K\pm s}{2K}\rfloor} x_i\\
&= 2K\, X_{2nK}(\frac{K\pm s}{2K}), \quad s\in[0,K],
\end{aligned}
$$

we can rewrite (8.30) as

$$
\begin{aligned}
&\left(X_n^K(\cdot), X_n^{K,-}(\cdot)\right)\\
&=_{st} \left(2K\, X_{2nK}(\frac{K+\cdot}{2K}) - 2K\, X_{2nK}(\frac{1}{2})\ ,\ 2K\, X_{2nK}(\frac{1}{2}) - 2K\, X_{2nK}(\frac{K-\cdot}{2K})\right)\\
&= G(X_{2nK}(\cdot))
\end{aligned}
\tag{8.31}
$$

where the mapping $G : D[0,1]^p \to D[0,K]^p \times D_l[0,K]^p$ is defined by

$$
G(z)(t) \equiv \begin{pmatrix} 2K\, z(\frac{K+t}{2K}) - 2K\, z(\frac{1}{2}) \\ 2K\, z(\frac{1}{2}) - 2K\, z(\frac{K-t}{2K}) \end{pmatrix}, \quad t\in[0,K], \quad z\in D[0,1]^p\ .
$$

We now argue that G is Borel-measurable and S-continuous on $\mathcal{D}(I_X)$. By [12, p. 55] and Proposition I in [7, p. 44], it suffices to show these properties for each of the coordinate mappings G_1 and G_2 of G. Because of the similarity between the two coordinate mappings, we only consider $G_1 : z \to 2K\, z(\frac{K+\cdot}{2K}) - 2K\, z(\frac{1}{2})$.

By Lemma 5.2, the projection mapping $z \to z(\frac{1}{2})$ is Borel-measurable, and S-continuous on $\mathcal{D}(I_X)$ under the assumption that the elements of $\mathcal{D}(I_X)$ are continuous at $t = \frac{1}{2}$. By Lemma 5.1, the mapping $z \to 2K\, z(\frac{K+\cdot}{2K})$ is S-continuous (hence Borel-measurable), and it follows easily by [12, p. 55] and Proposition I in [7, p. 44] that the mapping $z \to \left(2K\, z(\frac{K+\cdot}{2K}), z(\frac{1}{2})\right)$ is Borel-measurable and S-continuous on $\mathcal{D}(I_X)$. Finally, upon noting that constant mappings have no discontinuity point, we again conclude by Lemma 5.5 to the Borel-measurability and S-continuity of G_1 on $\mathcal{D}(I_X)$.

Next, by Lemma 6.1 the family $\{X_{2nK}(\cdot),\ n = 1,2,\ldots\}$ satisfies the LDP in $(D[0,1]^p, \tau_0)$ with good rate function $2K\, I_X$. Thus, in view of (8.31), and of the Borel-measurability and S-continuity of G on $\mathcal{D}(I_X) = \mathcal{D}(2KI_X)$, the Contraction Principle readily yields the LDP for $\{G(X_{2nK}(\cdot)),\ n = 1,2,\ldots\}$ in $(D[0,K]^p, \tau_0) \times (D_l[0,K]^p, \tau_0)$, whence for the (stochastically equivalent) processes $\{(X_n^K(\cdot), X_n^{K,-}(\cdot)),\ n = 1,2,\ldots\}$. The corresponding rate function $I^K_{X,X^-} : D[0,K]^p \times D_l[0,K]^p \to [0,\infty]$ is good and given by

$$\begin{aligned} & I^K_{X,X^-}(\varphi_1,\varphi_2) && (8.32)\\ &= \inf_{\psi\in D[0,1]^p} \{2K\, I_X(\psi) : (\varphi_1,\varphi_2) = G(\psi)\}, \quad \varphi_1 \in D[0,K]^p, \quad \varphi_2 \in D_l[0,K]^p. \end{aligned}$$

Fix φ_1 in $D[0,K]^p$ and φ_2 in $D_l[0,K]^p$. Here, the constraint $(\varphi_1,\varphi_2) = G(\psi)$ requires $\varphi_1(0) = \varphi_2(0) = 0$, so that $I^K_{X,X^-}(\varphi_1,\varphi_2) = \infty$ if either $\varphi_1(0) \neq 0$ or $\varphi_2(0) \neq 0$. Moreover, as pointed out in the proof of Proposition 7.1, $I_X(\psi) = \infty$ if $\psi(0) \neq 0$, and in the case $\varphi_1(0) = \varphi_2(0) = 0$, the non-vacuous constraint entering the optimization problem (8.32) thus reduces to

$$\psi(0) = 0; \quad \left\{ \begin{array}{lcl} \varphi_1(t) & = & 2K\psi(\frac{K+t}{2K}) - 2K\psi(\frac{1}{2}) \\ \varphi_2(t) & = & 2K\psi(\frac{1}{2}) - 2K\psi(\frac{K-t}{2K}) \end{array} \right. , \quad t \in [0,K].$$

Solving for ψ in terms of φ_1 and φ_2, we find

$$\psi(\frac{1}{2}) = \frac{1}{2K}\varphi_2(K); \quad \left\{ \begin{array}{lcll} \psi(t) & = & \psi(\frac{1}{2}) + \frac{1}{2K}\varphi_1(2Kt-K), & t \in [\frac{1}{2},1] \\ \psi(t) & = & \psi(\frac{1}{2}) - \frac{1}{2K}\varphi_2(K-2Kt), & t \in [0,\frac{1}{2}] \end{array} \right.$$

and (8.32) yields the desired expression (8.27) for I^K_{X,X^-}. ■

A proof of Theorem 2.3: Fix $T > 0$. In view of Proposition 8.1 we need only consider T non-integer. In fact, under the assumption that $\{X_n(\cdot),\ n = 1,2,\ldots\}$ satisfy the LDP with each of the elements of $D(I_X)$ being continuous at $t = \frac{1}{2}$, Proposition 8.1 (with $K = \lceil T\rceil$) already yields the LDP for $\left\{\left(X_n^{\lceil T\rceil}(\cdot), X_n^{\lceil T\rceil,-}(\cdot)\right),\ n = 1,2,\ldots\right\}$ in $D[0,T]^p \times D_l[0,T]^p$ with good rate function $I^{\lceil T\rceil}_{X,X^-} : D[0,\lceil T\rceil]^p \times D_l[0,\lceil T\rceil]^p \to [0,\infty]$ given by (2.6)–(2.7).

Here we have

$$\tilde{r}_T\left(X_n^{\lceil T\rceil}(\cdot), X_n^{\lceil T\rceil,-}(\cdot)\right) = \left(X_n^T(\cdot), X_n^{T,-}(\cdot)\right), \quad n = 1,2,\ldots,$$

with the restriction mapping $\widetilde{r}_T : D[0,\lceil T\rceil]^p \times D_l[0,\lceil T\rceil]^p \to D[0,T]^p \times D_l[0,T]^p$. By [12, p. 55] and Lemma 5.4, the restriction mapping $\widetilde{r}_T$ is Borel-measurable. Hence, the Contraction Principle yields the LDP for $\{(X_n^T(\cdot), X_n^{T,-}(\cdot)),\ n = 1,2,\ldots\}$ in $(D[0,T]^p, \tau_0) \times (D_l[0,T]^p, \tau_0)$ with good rate function $I^T_{X,X^-} : D[0,T]^p \times D_l[0,T]^p \to [0,\infty]$ given by (2.5) *provided* $\widetilde{r}_T$ is continuous on the effective domain of $I^{\lceil T\rceil}_{X,X^-}$. In view of Lemma 5.4 and Proposition I in [7, p. 49], this latter requirement is implied by the continuity at $t = T$ of each element in the effective domain of $I^{\lceil T\rceil}_{X,X^-}$, which is easily seen via (8.28) to be a consequence of the assumption on the continuity of the elements of $\mathcal{D}(I_X)$ at $t = \frac{1}{2} \pm \frac{T}{2\lceil T\rceil}$. ■

9 LDP for $X_n^T(\cdot)$ with integral rate function

Under Assumption **(I)**, the continuity assumptions required in Theorems 2.1, 2.2 and 2.3 are automatically satisfied and the rate functions can be explicitly computed; the next technical lemma is key to the calculations. Let $AC[a,b]^p$ denote the space of absolutely continuous functions $\varphi : [a,b] \to \mathbb{R}^p$.

Lemma 9.1 *Consider a Borel-measurable mapping $r : \mathbb{R}^p \to [0,\infty]$ such that $\inf_{x\in\mathbb{R}^p} r(x) = 0$. Then,*

$$\inf\left\{\int_a^b r(\dot{\psi}(t))\,dt : \psi \in AC[a,b]^p\right\} = 0, \quad 0 \le a \le b. \tag{9.33}$$

Proof: By considering the family of functions $\{\psi_x,\ x \in \mathbb{R}^p\}$ in $AC[a,b]^p$ defined by $\psi_x(t) = x\,t$, for t in $[a,b]$, we get the bounds

$$\begin{aligned} 0 \le \inf\left\{\int_a^b r(\dot{\psi}(t))\,dt : \psi \in AC[a,b]^p\right\} &\le \inf_{x\in\mathbb{R}^p} \int_a^b r(\dot{\psi}_x(t))\,dt \\ &= (b-a) \inf_{x\in\mathbb{R}^p} r(x), \end{aligned}$$

and (9.33) follows easily from the assumptions on r. ■

A proof of Corollary 2.4: Fix $T > 0$. Because $\mathcal{D}(I_X) \subseteq AC_0[0,1]^p$, each element of $\mathcal{D}(I_X)$ is continuous at $t = \frac{T}{\lceil T\rceil}$, and by Theorem 2.1, the family $\{X_n^T(\cdot),\ n = 1,2,\ldots\}$ satisfies the LDP with good rate function I_X^T given by (2.2). In order to simplify this expression, we substitute the form (2.8) for I_X into (2.2). We note that for φ *not* element of $AC_0[0,T]^p$, any ψ in $D[0,\lceil T\rceil]^p$ such that $\psi = \varphi$ on $[0,T]$

will not belong to $AC_0[0, \lceil T \rceil]^p$, and the associated mapping $\psi_{\lceil T \rceil}$ will not belong to $AC_0[0,1]^p$. Combining these remarks, we find that $I_X^T(\varphi) = \infty$ for φ not in $AC_0[0,T]^p$. On the other hand, for φ in $AC_0[0,T]^p$, we get

$$\begin{aligned} I_X^T(\varphi) &= \inf_{\psi \in AC_0[0,\lceil T \rceil]^p} \left\{ \lceil T \rceil \int_0^1 r(\dot{\psi}_{\lceil T \rceil}(t))\, dt : \varphi = \psi \ \text{ on } [0,T] \right\} \\ &= \inf_{\psi \in AC_0[0,\lceil T \rceil]^p} \left\{ \lceil T \rceil \int_0^1 r(\dot{\psi}(\lceil T \rceil t))\, dt : \varphi = \psi \ \text{ on } [0,T] \right\} \\ &= \inf_{\psi \in AC_0[0,\lceil T \rceil]^p} \left\{ \int_0^{\lceil T \rceil} r(\dot{\psi}(t))\, dt : \varphi = \psi \ \text{ on } [0,T] \right\} \\ &= \int_0^T r(\dot{\varphi}(t))\, dt \ + \inf_{\psi \in AC_0[0,\lceil T \rceil]^p} \left(\int_T^{\lceil T \rceil} r(\dot{\psi}(t))\, dt \right) \end{aligned}$$

and the desired conclusion follows by a direct application of Lemma 9.1. ■

The proof of Corollary 2.5 can be handled similarly, with the details left to the interested reader.

A proof of Corollary 2.6: Fix $T > 0$. With $\mathcal{D}(I_X) \subseteq AC_0[0,1]^p$, the elements of $\mathcal{D}(I_X)$ are automatically continuous at $t = \frac{1}{2}$ and $t = \frac{1}{2} \pm \frac{T}{2\lceil T \rceil}$, and by Theorem 2.3 the LDP for $\{(X_n^T(\cdot), X_n^{T,-}(\cdot)),\ n = 1,2,\ldots\}$ holds with good rate function given by (2.5). We claim that under the assumed expression (2.8), the rate function $I_{X,X^-}^{\lceil T \rceil}$ given by (8.27) takes the form

$$I_{X,X^-}^{\lceil T \rceil}(\varphi_1, \varphi_2) = \begin{cases} \int_0^{\lceil T \rceil} r(\dot{\varphi}_1(t))\, dt + \int_0^{\lceil T \rceil} r(\dot{\varphi}_2(t))\, dt & \text{if } \ \varphi_1, \varphi_2 \in AC_0[0, \lceil T \rceil]^p \\ \infty & \text{otherwise} \end{cases},$$

and the desired expression (2.11) follows upon applying Lemma 9.1 once again, as in the proof of Corollary 2.4.

To that end, we turn to the expression (8.27) for $I_{X,X^-}^{\lceil T \rceil}$: It is plain from the definition (8.28) of φ that φ_1 and φ_2 belong to $AC_0[0,K]^p$ if and only if φ belongs to $AC_0[0,1]^p$. Hence, under (2.8), $I_X(\frac{1}{2\lceil T \rceil}\varphi) = \infty$ for φ not in $AC_0[0,1]^p$, and $I_{X,X^-}^{\lceil T \rceil}(\varphi_1, \varphi_2) = \infty$ for either φ_1 or φ_2 not in $AC_0[0, \lceil T \rceil]$. Finally, for φ_1 and φ_2 in $AC_0[0, \lceil T \rceil]^p$, we see that

$$\begin{aligned} I_X(\frac{1}{2\lceil T \rceil}\varphi) &= \int_0^1 r(\frac{1}{2\lceil T \rceil}\dot{\varphi}(t))\, dt \\ &= \int_0^{\frac{1}{2}} r(\dot{\varphi}_2(\lceil T \rceil - 2\lceil T \rceil t))\, dt \ + \int_{\frac{1}{2}}^1 r(\dot{\varphi}_1(2\lceil T \rceil t - \lceil T \rceil))\, dt \end{aligned}$$

and a change of variable yields the desired expression. ■

A Appendix

A.1 A proof of Theorem 3.3

We begin with some elementary topological facts whose proof can be found in [1, Lemma A.2].

Lemma A.1 *Let $(\mathcal{X}, \tau_{\mathcal{X}})$ and $(\mathcal{Y}, \tau_{\mathcal{Y}})$ be two topological spaces, and assume the mapping $f : \mathcal{X} \to \mathcal{Y}$ to be continuous on the subset D of $\mathcal{X}$. Then, for any subset Γ of $\mathcal{Y}$ we have the inclusions*

$$\overline{f^{-1}(\Gamma)} \cap D \subset f^{-1}(\overline{\Gamma}) \quad \text{and} \quad f^{-1}(\Gamma^\circ) \cap D \subset \left(f^{-1}(\Gamma)\right)^\circ.$$

To show that I' is indeed a good rate function, we first note that y belongs to $\mathcal{D}(I')$ if and only if it belongs to $f(\mathcal{D}(I))$, so that $\mathcal{D}(I') = f(\mathcal{D}(I))$ and $\mathcal{D}(I) \subset f^{-1}(\mathcal{D}(I'))$.

Next, fix $\alpha \geq 0$ and consider the level sets $\Psi_{I'}(\alpha)$ and $\Psi_I(\alpha)$: If y belongs to $f(\Psi_I(\alpha))$, then $y = f(x)$ for some x in $\Psi_I(\alpha)$ and by the definition of I', we get $I'(y) \leq I(x) \leq \alpha$. Hence, y belongs to $\Psi_{I'}(\alpha)$ and the inclusion $f(\Psi_I(\alpha)) \subset \Psi_{I'}(\alpha)$ follows.

On the other hand, if y belongs to $\Psi_{I'}(\alpha)$, then by definition of the infimum, for every any $n = 1, 2, \ldots$, there exists x_n in $\mathcal{X}$ such that $f(x_n) = y$ and

$$I'(y) \leq I(x_n) < I'(y) + \frac{1}{n}. \tag{A.34}$$

The sequence $\{x_n,\ n = 1, 2, \ldots\}$ belongs to the compact set $\Psi_I(\alpha+1)$, thus contains a subsequence $\{x_{n_k},\ k = 1, 2, \ldots\}$ converging to some x^* in $\Psi_I(\alpha+1)$. Upon letting n go to ∞ in (A.34), we readily obtain $\lim_{n\to\infty} I(x_n) = I'(y)$, and the lower semi-continuity of I then yields

$$I(x^*) \leq \liminf_{n\to\infty} I(x_n) = I'(y).$$

Hence, x^* belongs in fact to $\Psi_I(\alpha)$; by continuity of f at x^*, we also have

$$f(x^*) = \lim_{k\to\infty} f(x_{n_k}) = y$$

and the conclusion $I'(y) \leq I(x^*)$ follows. From the reverse inequality obtained earlier, we finally get $I'(y) = I(x^*)$, and y thus belongs to $f(\Psi_I(\alpha))$. The equality $\Psi_{I'}(\alpha) = f(\Psi_I(\alpha))$ is then immediate.

Next, to show compactness of the level set $\Psi_{I'}(\alpha)$, we consider an open covering $\cup_\alpha O_\alpha$ (in $\mathcal{Y}$) of $\Psi_{I'}(\alpha) = f(\Psi(\alpha))$. Using the continuity of f on $\mathcal{D}(I)$, we conclude from Lemma A.1 that

$$\begin{aligned}
\Psi_I(\alpha) &= \Psi_I(\alpha) \cap \mathcal{D}(I) \\
&\subset \bigcup_\alpha \left(f^{-1}(O_\alpha) \cap \mathcal{D}(I)\right) \\
&\subset \bigcup_\alpha \left(f^{-1}(O_\alpha)\right)^\circ,
\end{aligned}$$

and $\cup_\alpha (f^{-1}(O_\alpha))^\circ$ is indeed an open covering of $\Psi_I(\alpha)$. The rate function I being good, $\Psi_I(\alpha)$ is compact, and there exists $\alpha_1, \ldots, \alpha_n$ such that

$$\Psi_I(\alpha) \subset \bigcup_{i=1}^n \left(f^{-1}(O_{\alpha_i})\right)^\circ \subset \bigcup_{i=1}^n f^{-1}(O_{\alpha_i}),$$

whence

$$\Psi_{I'}(\alpha) = f(\Psi_I(\alpha)) \subset \bigcup_{i=1}^n O_{\alpha_i}.$$

Therefore, for each $\alpha \geq 0$, $\Psi_{I'}(\alpha)$ is compact and I' is a good rate function.

Before establishing the LDP bounds, we note from (3.15) that

$$\inf_{y\in\Gamma} I'(y) = \inf_{x\in f^{-1}(\Gamma)} I(x), \quad \Gamma \subset \mathcal{Y}. \tag{A.35}$$

Now, let Γ be a Borel set in $\mathcal{Y}$. By Borel-measurability of f, $f^{-1}(\Gamma)$ is a Borel set in $\mathcal{X}$, and we get from the upper bound in the LDP for $\{\mu_n,\ n = 1, 2, \ldots\}$

$$\limsup_{n\to\infty} \frac{1}{n} \ln \mu_n \circ f^{-1}(\Gamma) \leq - \inf_{x\in\overline{f^{-1}(\Gamma)}} I(x). \tag{A.36}$$

From the definition of the effective domain, Lemma A.1 (f is continuous on $\mathcal{D}(I)$) and (A.35), we obtain

$$\begin{aligned}
\inf\left\{I(x) : x \in \overline{f^{-1}(\Gamma)}\right\} &= \inf\left\{I(x) : x \in \overline{f^{-1}(\Gamma)} \cap \mathcal{D}(I)\right\} \\
&\geq \inf\left\{I(x) : x \in f^{-1}\left(\overline{\Gamma}\right)\right\} \\
&= \inf\left\{I'(y) : y \in \overline{\Gamma}\right\}
\end{aligned} \tag{A.37}$$

and the LDP upper bound for $\{\mu_n \circ f^{-1},\ n = 1,2,\ldots\}$ follows from (A.36) and (A.37).

Similarly, from the lower bound in the LDP for $\{\mu_n,\ n = 1,2,\ldots\}$, we get

$$\liminf_{n\to\infty} \frac{1}{n} \ln \mu_n \circ f^{-1}(\Gamma) \ \geq\ - \inf_{x\in(f^{-1}(\Gamma))^\circ} I(x), \tag{A.38}$$

and by Lemma A.1 and (A.35) we see that

$$\begin{aligned}
\inf\left\{I(x) : x \in \left(f^{-1}(\Gamma)\right)^\circ\right\} &\leq \inf\left\{I(x) : x \in f^{-1}(\Gamma^\circ) \cap \mathcal{D}(I)\right\} \\
&= \inf\left\{I(x) : x \in f^{-1}(\Gamma^\circ)\right\} \\
&= \inf\left\{I'(y) : y \in \Gamma^\circ\right\}. \qquad (A.39)
\end{aligned}$$

The LDP lower bound for $\{\mu_n \circ f^{-1},\ n = 1,2,\ldots\}$ then becomes an immediate consequence of (A.38) and (A.39). ■

A.2 A technical Lemma

Lemma A.2 *For $\alpha \neq 0$ and β in $\mathbb{R}$, let $c \equiv \min\{\alpha a + \beta,\, \alpha b + \beta\}$ and $d \equiv \max\{\alpha a + \beta,\, \alpha b + \beta\}$. Define the mapping $f : [c,d] \to [a,b]$ by*

$$f(t) = \frac{1}{\alpha}(t - \beta), \quad t \in [c,d].$$

Then, for each pair x and y in $D[a,b]^p$ (resp. in $D_l[a,b]^p$), we have

$$\begin{aligned}
&\left(\sup_{t\in[c,d]} |x \circ f(t) - y \circ f \circ \lambda(t)|\right) \vee \left(\sup_{t\in[c,d]} |\lambda(t) - t|\right) \qquad (A.40) \\
&= \left(\sup_{t\in[a,b]} |x(t) - y \circ f \circ \lambda \circ f^{-1}(t)|\right) \vee \left(|\alpha| \sup_{t\in[a,b]} |f \circ \lambda \circ f^{-1}(t) - t|\right),
\end{aligned}$$

for all λ in Λ_{cd}, and

$$\begin{aligned}
&\inf_{\lambda'\in\Lambda_{ab}} \left(\sup_{t\in[a,b]} |x(t) - y \circ \lambda'(t)| \vee \sup_{t\neq s\in[a,b]} |\ln \frac{\lambda'(t) - \lambda'(s)}{t-s}|\right) \qquad (A.41) \\
&= \inf_{\lambda\in\Lambda_{cd}} \left(\sup_{t\in[c,d]} |x \circ f(t) - y \circ f \circ \lambda(t)| \vee \sup_{t\neq s\in[c,d]} |\ln \frac{\lambda(t) - \lambda(s)}{t-s}|\right).
\end{aligned}$$

Proof: Let x and y in $D[a,b]^p$ (or $D_l[a,b]^p$). First note that for any λ in Λ_{cd},

$$x \circ f(t) - y \circ f \circ \lambda(t) \ =\ x(f(t)) - y \circ f \circ \lambda\left(f^{-1}(f(t))\right), \quad t \in [c,d],$$

whence, f being a bijection from $[c,d]$ onto $[a,b]$,

$$\sup_{t\in[c,d]} |x\circ f(t) - y\circ f\circ\lambda(t)| = \sup_{t\in[a,b]} |x(t) - y\circ f\circ\lambda\circ f^{-1}(t)|. \tag{A.42}$$

On the other hand, upon using the equality

$$f(t) - f(s) = \frac{1}{\alpha}(t-s), \quad s,t\in[c,d].$$

derived from the definition of f, we see for each λ in Λ_{cd} that

$$\begin{aligned}
\sup_{t\in[c,d]} |\lambda(t) - t| &= \sup_{t\in[a,b]} |\lambda(f^{-1}(t)) - f^{-1}(t)| \\
&= \sup_{t\in[a,b]} |\alpha| \left| f\left(\lambda(f^{-1}(t)\right) - f\left(f^{-1}(t)\right) \right| \\
&= \sup_{t\in[a,b]} |\alpha| |f\circ\lambda\circ f^{-1}(t) - t|.
\end{aligned} \tag{A.43}$$

Equality (A.40) then becomes an easy consequence of (A.42) and (A.43).

Similarly, for all λ in Λ_{cd} we have

$$\begin{aligned}
\sup_{s\neq t\in[c,d]} |\ln\frac{\lambda(t)-\lambda(s)}{t-s}| &= \sup_{s\neq t\in[a,b]} |\ln\frac{\lambda\circ f^{-1}(t) - \lambda\circ f^{-1}(s)}{f^{-1}(t) - f^{-1}(s)}| \\
&= \sup_{s\neq t\in[a,b]} |\ln\frac{f\left(\lambda\circ f^{-1}(t)\right) - f\left(\lambda\circ f^{-1}(s)\right)}{f(f^{-1}(t)) - f(f^{-1}(s))}| \\
&= \sup_{s\neq t\in[a,b]} |\ln\frac{f\circ\lambda\circ f^{-1}(t) - f\circ\lambda\circ f^{-1}(s)}{t-s}|.
\end{aligned} \tag{A.44}$$

Finally, f and f^{-1} are either both strictly increasing ($\alpha > 0$) or both strictly decreasing ($\alpha < 0$). Hence, $f\circ\lambda\circ f^{-1}$ spans Λ_{ab} as λ spans Λ_{cd}, and the desired equality (A.41) follows from (A.42) and (A.44). ■

A.3 A proof of Lemma 5.3

Let $\{x_n,\ n=1,2,\dots\}$ be a sequence in $D[a,b]^p$ which is S-converging to x, and let c in (a,b) be a continuity point of x. By definition of S-convergence, $\{x_n,\ n=1,2,\dots\}$ is J_1-convergent to x, and there exists a sequence of mapping $\{\lambda_n,\ n=1,2,\dots\}$ in Λ_{ab} such that

$$\lim_{n\to\infty}\sup_{t\in[a,b]} |x_n(t) - x\circ\lambda_n(t)| = \lim_{n\to\infty}\sup_{t\in[a,b]} |\lambda_n(t) - t| = \lim_{n\to\infty}\sup_{t\in[a,b]} |\lambda_n^{-1}(t) - t| = 0. \tag{A.45}$$

Because we may not have $\lambda_n(c) = c$, the restriction of λ_n to $[a, c]$ (resp. $[c, b]$) may not be in Λ_{ac} (resp. Λ_{cb}), and this prevents us from obtaining the convergence of the restrictions of $\{x_n,\ n = 1, 2, \ldots\}$ on $D[a, c]^p$ and $D[c, b]^p$ directly via (A.45). However, it is possible to construct from $\{\lambda_n,\ n = 1, 2, \ldots\}$ a sequence $\{\sigma_n,\ n = 1, 2, \ldots\}$ in Λ_{ab} with $\sigma_n(c) = c$ for all $n = 1, 2, \ldots$, such that

$$\lim_{n\to\infty} \sup_{t\in[a,b]} |x_n(t) - x \circ \sigma_n(t)| \;=\; \lim_{n\to\infty} \sup_{t\in[a,b]} |\sigma_n(t) - t| \;=\; 0.$$

To that end, for each $n = 1, 2, \ldots$, we define c_n and C_n by

$$c_n \;\equiv\; \min\{c - \frac{1}{n}, \lambda_n^{-1}(c - \frac{1}{n})\} \quad \text{and} \quad C_n \;\equiv\; \max\{c + \frac{1}{n}, \lambda_n^{-1}(c + \frac{1}{n})\}. \tag{A.46}$$

Fix $n = 1, 2, \ldots$ large enough so that $[c_n, C_n] \subset [a, b]$. We construct the mapping $\sigma_n : [a, b] \to [a, b]$ as follows: If $\lambda_n(c) < c$, we set $\sigma_n = \lambda_n$ on $[a, c - \frac{1}{n}]$ and on $[\lambda_n^{-1}(c + \frac{1}{n}), b]$, and complete σ_n on $[c - \frac{1}{n}, c]$ and $[c, \lambda_n^{-1}(c + \frac{1}{n})]$ by a piecewise linear interpolation passing through the point (c, c). On the other hand, if $\lambda_n(c) > c$, we set $\sigma_n = \lambda_n$ on $[a, \lambda_n^{-1}(c - \frac{1}{n})]$ and $[c + \frac{1}{n}, b]$, and complete σ_n similarly by a linear interpolation passing trough the point (c, c). By taking into account both constructions, we can readily check that

$$\begin{aligned} \sup_{t\in[a,b]} |\lambda_n(t) - \sigma_n(t)| \;&\le\; \sup_{t\in[c_n,C_n]} |\lambda_n(t) - \sigma_n(t)| \\ &\le\; |\lambda_n(C_n) - \lambda_n(c_n)|. \end{aligned} \tag{A.47}$$

Furthermore, because

$$\lambda_n(C_n) = \max\{\lambda_n(c + \frac{1}{n}), c + \frac{1}{n}\}, \tag{A.48}$$

we see that $\lim_{n\to\infty} \lambda_n(C_n) = c$ by the uniform convergence (A.45); similarly, we find that $\lim_{n\to\infty} \lambda_n(c_n) = c$. It is now plain from (A.47) that

$$\lim_{n\to\infty} \sup_{t\in[a,b]} |\lambda_n(t) - \sigma_n(t)| \;=\; 0,$$

whence

$$\lim_{n\to\infty} \sup_{t\in[a,b]} |\sigma_n(t) - t| \;=\; 0 \tag{A.49}$$

by virtue of the triangle inequality and of (A.45).

For large enough $n = 1, 2, \ldots$, the definition of the supremum yields the existence of some t_n in $[c_n, C_n]$ such that

$$\sup_{t\in[c_n,C_n]} |x \circ \lambda_n(t) - x \circ \sigma_n(t)| \;\le\; \frac{1}{n} + |x \circ \lambda_n(t_n) - x \circ \sigma_n(t_n)|. \tag{A.50}$$

The functions λ_n and σ_n being monotone increasing, we get $\lim_{n\to\infty} \lambda_n(t_n) = \lim_{n\to\infty} \sigma_n(t_n) = c$, and by the continuity of x at c we finally see that

$$0 \leq \lim_{n\to\infty} \sup_{t\in[c_n,C_n]} |x \circ \lambda_n(t) - x \circ \sigma_n(t)| \leq |x(c) - x(c)| = 0 \tag{A.51}$$

upon letting n go to infinity in (A.50).

Consequently, as we have

$$\begin{aligned}\sup_{t\in[a,b]} |x_n(t) - x \circ \sigma_n(t)| &\leq \sup_{t\in[a,b]} |x_n(t) - x \circ \lambda_n(t)| + \sup_{t\in[a,b]} |x \circ \lambda_n(t) - x \circ \sigma_n(t)| \\ &\leq \sup_{t\in[a,b]} |x_n(t) - x \circ \lambda_n(t)| + \sup_{t\in[c_n,C_n]} |x \circ \lambda_n(t) - x \circ \sigma_n(t)|,\end{aligned}$$

for large enough n for which σ_n is defined, we get from (A.45) and (A.51) that

$$\lim_{n\to\infty} \sup_{t\in[a,b]} |x_n(t) - x \circ \sigma_n(t)| = 0. \tag{A.52}$$

The desired result easily follows (after renumbering of σ_n) from (A.49) and (A.52) by considering the restrictions of $\{\sigma_n,\ n = 1,2,\ldots\}$ on $[a,c]$ and $[c,b]$, once it is recalled that for each $n = 1,2,\ldots$, σ_n belongs to Λ_{ab} with $\sigma_n(c) = c$. ■

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