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STOCHASTIC ORDERS ASSOCIATED WITH THE FORWARD RECURRENCE TIME OF A RENEWAL PROCESS

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ABSTRACT

Let $\mathcal{L}$ be a collection of $I\!R$–valued mappings defined on $I\!R_+$, and let $\leq_{\mathcal{L}}$ denote the integral order induced by $\mathcal{L}$ on the class of probability distributions on $I\!R_+$. For integrable distributions F and G with no mass at the origin, we write $F \leq_{FR-\mathcal{L}} G$ provided $\tilde{F} \leq_{\mathcal{L}} \tilde{G}$ where $\tilde{F}$ and $\tilde{G}$ denote the stationary forward recurrence times associated with the renewal processes with interarrival times $\tilde{F}$ and $\tilde{G}$, respectively. Simple analytical characterizations of the stochastic order $\leq_{FR-\mathcal{L}}$ are developed, their relationships with the classical integral orders $\leq_{\mathcal{L}}$ clarified and several of their properties obtained with the view towards building a "calculus." Emphasis is put on conditions which ensure the stability of the orderg $\leq_{FR-\mathcal{L}}$ under various transformations. They include classical operations on distribution functions such as convolution and weak convergence, as well as transformations on renewal (and thus more generally point) processes, e.g., thinning and superposition. Particular attention is given to the case where the order $\leq_{\mathcal{L}}$ is either the standard stochastic order $\leq_{st}$ or the convex increasing order $\leq_{icx}$.

Key Words: Stationary point processes, excess operator, Feller's paradox, thinning, superposition, integral order, stochastic order, convex order.

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I. Introduction

In this paper we focus on a family of stochastic orders for probability distributions on the positive real line which were first introduced by Whitt [11]. These order are naturally associated with the notion of forward recurrence time and allow for a stochastic comparison between renewal processes (and therefore more generally between point processes). To set the stage, let $\mathcal{L}$ be a collection of $I\!R$–valued mappings defined on the positive real line $I\!R_+$, and let $\leq_{\mathcal{L}}$ denote the integral order induced by $\mathcal{L}$ on the probability distributions on $I\!R_+$. For integrable distributions F and G with no mass at the origin, we write $F \leq_{FR-\mathcal{L}} G$ provided $\tilde{F} \leq_{\mathcal{L}} \tilde{G}$ where $\tilde{F}$ and $\tilde{G}$ denote the stationary forward recurrence times associated with the renewal processes with interarrival times $\tilde{F}$ and $\tilde{G}$, respectively.

We develop new analytical and qualitative results on these orders $\leq_{FR-\mathcal{L}}$, thereby expanding on and complementing the work already carried out in [1]. Here, most notably, we put emphasis on three types of properties with the view towards building a "calculus" for these non–integral stochastic orders.

1. Following the developments in Stoyan's monograph [10], we seek conditions which ensure the stability of the order $\leq_{FR-\mathcal{L}}$ under classical operations on distribution functions, as usually done for the integral orders $\leq_{\mathcal{L}}$. Transformations of interest include convolution and limit operation under weak convergence;
2. These orders $\leq_{FR-\mathcal{L}}$ have a natural interpretation within the framework of the theory of point processes, as indicated more fully in [1]; in fact they can be interpreted as orders on the class of renewal processes and more generally on the class of stationary point processes [3,1]. This connection thus suggests a whole new class of transformations on probability distributions which are defined via a corresponding transformation on point processes; for instance the standard thinning and superposition of point processes induce such transformations. Here too, we seek conditions under which the order $\leq_{FR-\mathcal{L}}$ will be preserved when applying such transformations; and
3. We explore the relationship of these non–integral orders to the classical (integral) stochastic orders such as the convex order $\leq_{cx}$, the increasing convex order $\leq_{icx}$ and the usual strong stochastic order $\leq_{st}$ [2,9,10]. In particular we show (Theorem 5.3) that the order $\leq_{FR-st}$ is located between the integral orders $\leq_{cx}$ and $\leq_{st}$.

The beginnings of a rich theory emerges in this paper which is organized as follows: In Section 2, we review various facts form the theory of integral stochastic orders $\leq_{\mathcal{L}}$ as discussed in [9,10]. The orders $\leq_{FR-\mathcal{L}}$ associated with the forward recurrence time are introduced in Section 3, where basic analytical characterizations are developed, and general preservation properties are established in Section 4. When the underlying order $\leq_{\mathcal{L}}$ is the usual strong stochastic order $\leq_{st}$, the corresponding order $\leq_{FR-\mathcal{L}}$ is shown in Section 5 to exhibit rather interesting properties. In Section 6, we introduce the so–called Palm orders $\leq_{P-\mathcal{L}}$ which can be viewed as the "dual" of the orders $\leq_{FR-\mathcal{L}}$; the terminology will become apparent from the definitions.

The relation of the orders $\leq_{FR-\mathcal{L}}$ and $\leq_{P-\mathcal{L}}$ to classical integral orders is taken on in Section 7 where we have also developed various counterexamples to possible implications. Finally, Section 8 is devoted to the preservation of forward recurrence time order by the thinning and superposition (of point processes).

The notation adopted here is the one used in [2]: We find it convenient to define all the random variables (rvs) of interest on some common probability triple $(\Omega, \mathcal{F}, P)$. Let $\mathcal{D}(\mathbb{R}_+)$ denote the collection of all probability distribution functions on $\mathbb{R}_+$, i.e., $\mathcal{D}(\mathbb{R}_+)$ is the collection of all mappings $F : \mathbb{R}_+ \to [0,1]$ which are non–decreasing and right–continuous with $\lim_{x\to\infty} F(x) = 1$ [5]. We identify an element F of $\mathcal{D}(\mathbb{R}_+)$ with an $\mathbb{R}_+$–valued rv X which has distribution F, in which case $F(x) = P[X \leq x]$ for all $x \geq 0$, and we denote the first moment of F either by $m(F)$ or by $m(X)$. Finally two $\mathbb{R}$–valued rvs X and Y are said to be equal in law if they have the same distribution, a fact we denote by $X =_{st} Y$, in agreement with notation introduced below.

II. Preliminaries on integral orders

We begin by briefly reviewing some basic notions from the theory of integral stochastic orders; the reader is referred to the monographs by Ross [9] and Stoyan [10] for additional information on this material.

Let S be an arbitrary set. A binary relation $\mathcal{R}$ on S which is reflexive and transitive is called a partial semi–order on S. If, in addition, $\mathcal{R}$ has the anti–symmetry property (i.e., $x\mathcal{R}y$ and $y\mathcal{R}x$ imply $x = y$), then $\mathcal{R}$ is said to define a partial order on S.

Let $\mathcal{L}$ be any non–empty collection of Borel mappings $\phi : \mathbb{R}_+ \to \mathbb{R}$. Any such collection $\mathcal{L}$ induces a binary relation $\leq_{\mathcal{L}}$ on $\mathcal{D}(\mathbb{R}_+)$ by requiring that

$$F \leq_{\mathcal{L}} G \quad \text{iff} \quad \int_0^\infty \phi(x)F(dx) \leq \int_0^\infty \phi(x)G(dx) \tag{2.1}$$

for every ϕ in $\mathcal{L}$ such that the integrals are well defined. Equivalently, if X and Y are rvs with distribution F and G, respectively, we require that

$$X \leq_{\mathcal{L}} Y \quad \text{iff} \quad E[\phi(X)] \leq E[\phi(Y)] \tag{2.2}$$

for every mapping ϕ in $\mathcal{L}$ such that the expectations in (2.2) exist. Since the binary relation $\leq_{\mathcal{L}}$ is clearly reflexive and transitive on $\mathcal{D}(\mathbb{R}_+)$, it always defines a partial semi–order on $\mathcal{D}(\mathbb{R}_+)$. If the collection $\mathcal{L}$ is large enough to yield the anti–symmetry property, then $\leq_{\mathcal{L}}$ defines a partial order on $\mathcal{D}(\mathbb{R}_+)$.

During the discussion, we shall consider the following collections $\mathcal{L}$:

1. The set $\{st\}$ defined by

$$\{st\} = \{\phi : \mathbb{R}_+ \to \mathbb{R} \mid \phi \text{ increasing } \} \tag{2.3}$$

induces the *strong order* $\leq_{st}$ on $\mathcal{D}(I\!R_+)$. The anti-symmetry property of $\leq_{st}$ immediately follows from the following equivalent definition of $\leq_{st}$ [10, pp. 4–6].

Lemma 2.1. *For distributions F and G in $\mathcal{D}(I\!R_+)$, we have*

$$F \leq_{st} G \quad \text{iff} \quad F(x) \geq G(x), \quad x \geq 0. \tag{2.4}$$

2. The set $\{cx\}$ defined by

$$\{cx\} = \{\phi : I\!R_+ \to I\!R \mid \phi \text{ convex } \} \tag{2.5}$$

induces the *convex order* $\leq_{cx}$ on $\mathcal{D}(I\!R_+)$. For each $s \geq 0$, the function $x \to \exp(-sx)$ is an element of $\{cx\}$. Consequently, $F \leq_{cx} G$ and $G \leq_{cx} F$ imply F and G to have identical Laplace transforms, and the equality $F = G$ follows, i.e., the relation $\leq_{cx}$ is anti-symmetric on $\mathcal{D}(I\!R_+)$.

3. We shall find it useful to consider the following subset $\{icx\}$ of both (2.3) and (2.5) defined by

$$\{icx\} = \{st\} \bigcap \{cx\}. \tag{2.6}$$

The collection $\{icx\}$ induces the *increasing convex* order $\leq_{icx}$ on $\mathcal{D}(I\!R_+)$. The following characterization of $\leq_{icx}$ [10, pp. 8–9] will find use in later sections.

Lemma 2.2. *For distributions F and G in $\mathcal{D}(I\!R_+)$ with finite first moment, we have*

$$F \leq_{icx} G \quad \text{iff} \quad \int_x^\infty (1 - F(t))\, dt \leq \int_x^\infty (1 - G(t))\, dt, \quad x \geq 0. \tag{2.7}$$

From (2.7) (with $x = 0$) we already see that $F \leq_{icx} G$ implies $m(F) \leq m(G)$. Consequently, $F \leq_{icx} G$ and $G \leq_{icx} F$ imply $m(F) = m(G)$. But the comparison $F \leq_{icx} G$ with $m(F) = m(G)$ is equivalent to $F \leq_{cx} G$ [10, Thm. 1.3.1, p. 9]. Therefore, $F \leq_{icx} G$ and $G \leq_{icx} F$ together imply $F = G$ by the antisymmetry property of $\leq_{cx}$, and the antisymmetry property of $\leq_{icx}$ follows.

If we replace convex functions by concave functions in the definitions **2** and **3**, we get the *concave* and *increasing concave* orders on $\mathcal{D}(I\!R_+)$, which are respectively denoted $\leq_{cv}$ and $\leq_{icv}$.

The obvious relations between these partial (semi–)orders are summarized in the implications

$$F \leq_{st} G \quad \Longrightarrow \quad F \leq_{icx} G \tag{2.8a}$$

and

$$F \leq_{cx} G \quad \Longrightarrow \quad F \leq_{icx} G \tag{2.8b}$$

Since $-\phi$ is concave whenever ϕ is convex, we readily conclude that

$$X \leq_{cx} Y \quad \text{if and only if} \quad Y \leq_{cv} X. \tag{2.9}$$

Moreover, each of the inequalities $X \leq_{st} Y$, $X \leq_{cx} Y$ and $\leq_{icx} Y$ implies a corresponding inequality between the moments (of any order) of X and Y. Finally, we have

$$X \leq_{cx} Y \quad \text{if and only if} \quad -X \leq_{cv} -Y \tag{2.10}$$

since for any convex function ϕ, the mapping $x \to \phi(-x)$ is also convex. This immediately yields

$$X \leq_{cx} Y \quad \Longrightarrow \quad E[X] = E[Y]. \tag{2.11}$$

III. Orderings associated with forward recurrence times

We introduce $\mathcal{D}$ as the subset of distribution functions on $I\!R_+$ which are integrable and which have no mass at the origin, i.e., F belongs to $\mathcal{D}$ if F is an element of $\mathcal{D}(I\!R_+)$ such that $m(F) < \infty$ and $F(0) = 0$.

Let X be a non–negative rv with distribution F in $\mathcal{D}$. The (stationary) forward recurrence time of X is any non–negative rv $\tilde{X}$ with distribution $\tilde{F}$ given by

$$\tilde{F}(x) = \frac{\int_0^x (1 - F(t))\,dt}{m(F)}, \quad x \geq 0. \tag{3.1}$$

Since

$$m(F) = \int_0^\infty (1 - F(t))\,dt, \tag{3.2}$$

we can also write

$$1 - \tilde{F}(x) = \frac{\int_x^\infty (1 - F(t))\,dt}{\int_0^\infty (1 - F(t))\,dt}, \quad x \geq 0. \tag{3.3}$$

This formula will find its use in what follows.

The correspondence implicit in (3.1) forms the basis for a new class of stochastic orders defined on $\mathcal{D}$ which we now introduce. Let $\mathcal{L}$ be any non–empty collection of Borel mappings $\phi : I\!R_+ \to I\!R$. Any such collection $\mathcal{L}$ induces a binary relation $\leq_{FR-\mathcal{L}}$ on $\mathcal{D}$ by requiring that

$$F \leq_{FR-\mathcal{L}} G \quad \text{iff} \quad \tilde{F} \leq_{\mathcal{L}} \tilde{G} \tag{3.4}$$

where the distribution $\tilde{F}$ and $\tilde{G}$ are the forward recurrence time distributions associated with F and G, respectively, through (3.1). Equivalently, if X and Y are rvs with distribution F and G in $\mathcal{D}$, respectively, we require

$$X \leq_{FR-\mathcal{L}} Y \quad \text{iff} \quad \tilde{X} \leq_{\mathcal{L}} \tilde{Y} \tag{3.5}$$

where $\tilde{X}$ and $\tilde{Y}$ are rvs with distribution $\tilde{F}$ and $\tilde{G}$, respectively. Using (2.1)–(2.2), the definitions (3.4)–(3.5) become

$$F \leq_{FR-\mathcal{L}} G \quad \text{iff} \quad \int_0^\infty \phi(x)\tilde{F}(dx) \leq \int_0^\infty \phi(x)\tilde{G}(dx) \tag{3.6}$$

and

$$X \leq_{\mathcal{L}} Y \quad \text{iff} \quad E[\phi(\tilde{X})] \leq E[\phi(\tilde{Y})] \tag{3.7}$$

for every ϕ in $\mathcal{L}$ such that the integrals (resp. expectations) in (3.6) (resp. (3.7)) are well defined.

We pause at this point to draw the reader's attention to the following connection which was developed more fully by the first author in [1]: Each distribution F in $\mathcal{D}$ determines a synchronous renewal process with i.i.d. inter–renewal times of common distribution F. Therefore, any order on $\mathcal{D}$ expressed through the forward recurrence time transformation (3.1) – and the orders $\leq_{FR-\mathcal{L}}$ are such orders – can be interpreted as an order on the class of renewal processes. In fact, as shown in [1], since a time–stationary point process is fully determined by its Palm version, this connection can be extended to the class of all time–stationary point processes with inter–event time distributions in $\mathcal{D}$. This will not be used throughout the paper, except in Section 8 for motivating some of the results obtained there.

Lemma 3.1. *Let $\mathcal{L}$ be an non–empty collection of Borel mappings $\phi : I\!R_+ \to I\!R$. If the binary relation $\leq_{\mathcal{L}}$ is a partial order on $\mathcal{D}(I\!R_+)$, so is $\leq_{FR-\mathcal{L}}$ on $\mathcal{D}$.*

Proof. We need only check the antisymmetry property of $\leq_{FR-\mathcal{L}}$: By definition (3.4), if the distributions F and G in $\mathcal{D}$ satisfy $F \leq_{FR-\mathcal{L}} G$ and $G \leq_{FR-\mathcal{L}} F$, then $\tilde{F} \leq_{\mathcal{L}} \tilde{G}$ and $\tilde{G} \leq_{\mathcal{L}} \tilde{F}$. The antisymmetry of $\leq_{\mathcal{L}}$ implies $\tilde{F} = \tilde{G}$, whence

$$\frac{\int_0^x (1 - F(t))\, dt}{m(F)} = \frac{\int_0^x (1 - G(t))\, dt}{m(G)}, \quad x \geq 0 \tag{3.8}$$

upon using (3.1). Differentiating this equality with respect to x, we finally get

$$\frac{1 - F(x)}{m(F)} = \frac{1 - G(x)}{m(G)}, \quad x \geq 0. \tag{3.9}$$

The condition $F(0) = G(0) = 0$, when used in (3.9), yields $m(F) = m(G)$, and this in turn implies $F(x) = G(x)$ for all $x \geq 0$.

∎

In view of the results of Section 2, we observe from Lemma 3.1 that $\leq_{FR-st}$ and $\leq_{FR-icx}$ are both partial orders on $\mathcal{D}$. The remainder of this section is devoted to deriving various properties and analytical characterizations of the order $\leq_{FR-\mathcal{L}}$, when $\mathcal{L} = \{st\}$ and $\mathcal{L} = \{icx\}$.

We begin with a simple result which will allow us to directly relate the order $\leq_{FR-\mathcal{L}}$ to the order $\leq_{\mathcal{L}}$. For any Borel mapping $\phi : I\!R_+ \to I\!R$ which is locally integrable (with respect to Lebesgue measure), we define its antiderivative $\Phi : I\!R_+ \to I\!R$ by

$$\Phi(x) := \int_0^x \phi(t)dt, \quad x \geq 0. \tag{3.10}$$

Lemma 3.2. *Let X be a rv distributed according to F in $\mathcal{D}$. For any Borel mapping $\phi : I\!R_+ \to I\!R$ which is locally integrable, the equality*

$$E[\phi(\tilde{X})] = \frac{E[\Phi(X)]}{m(X)} \tag{3.11}$$

holds whenever these expectations are well defined; in fact both expectations are simultaneously well defined and finite (resp. infinite).

Proof. We start with a locally integrable mapping $\phi : I\!R_+ \to I\!R_+$, whence both expectations in (3.11) are well defined, though possibly infinite. The very definition of $\tilde{X}$ immediately yields

$$\begin{aligned} E[\phi(\tilde{X})] &= \int_0^\infty \phi(x)\tilde{F}(dx) \\ &= \frac{1}{m(X)} \cdot \int_0^\infty \phi(x)(1 - F(x))dx. \end{aligned} \tag{3.12}$$

Integrating by parts, with the notation (3.10), we see that

$$\begin{aligned} \int_0^A \phi(x)(1 - F(x))dx &= \left[\Phi(x)(1 - F(x))\right]_0^A + \int_0^A \Phi(x)F(dx) \\ &= \Phi(A)(1 - F(A)) + \int_0^A \Phi(x)F(dx), \quad A \geq 0. \end{aligned} \tag{3.13}$$

Moreover, when $E[\Phi(X)]$ is finite, we get

$$\lim_{A \to \infty} \Phi(A)(1 - F(A)) = 0. \tag{3.14}$$

It is now immediate to check with the help of (3.12), (3.13) (with A going to infinity) and (3.14) that $E[\phi(\tilde{X})]$ and $E[\Phi(X)]$ are simultaneously finite or infinite.

For an arbitrary mapping $\phi : I\!R_+ \to I\!R$ which is locally integrable, we decompose ϕ into its positive and negative parts, and use the first part of the proof on each component.

■

In order to take advantage of Lemma 3.2, we assume from now on that $\mathcal{L}$ contains only Borel mappings which are locally integrable with respect to Lesbegue measure so that the antiderivative (3.10) is always well defined. This is a rather weak assumption since it holds in all cases discussed in practice, namely $\{st\}$, $\{icx\}$ and

$\{cx\}$. Combining Lemma 3.2 and the definition (3.7), we readily obtain the following characterization.

Theorem 3.3. *Let $\mathcal{L}$ be a non–empty collection of Borel mappings which are locally integrable. For rvs X and Y with distribution F and G in $\mathcal{D}$, respectively, we have $X \leq_{FR-\mathcal{L}} Y$ if and only if*

$$\frac{E[\Phi(X)]}{m(X)} \leq \frac{E[\Phi(Y)]}{m(Y)} \tag{3.15}$$

for all mappings $\phi : I\!\!R_+ \to I\!\!R$ in $\mathcal{L}$ provided the expectations in (3.15) exist.

Since all the rvs considered in this paper are non–negative, it is easy to see for $\mathcal{L} = \{st\}$ and $\mathcal{L} = \{icx\}$ that only $I\!\!R_+$–valued mappings ϕ in $\mathcal{L}$ need to be considered in the characterization of Theorem 3.3.

IV. General properties of the order $\leq_{\mathbf{FR}-\mathcal{L}}$

In this section we discuss several general properties of the orders $\leq_{FR-\mathcal{L}}$. Following Stoyan [10, p. 2], as was done for integral orders, we consider properties **(R)**, **(M)**, **(E)**, **(C)** and **(W)**.

IV.1. Property (R)

For every $a \geq 0$, let U_a denote the probability distribution with all its mass at $x = a$, so that

$$U_a(x) = \begin{cases} 0 & \text{if } 0 \leq x < a \\ \\ 1 & \text{if } a \leq x. \end{cases} \tag{4.1}$$

The order $\leq_{FR-\mathcal{L}}$ is said to have the property **(R)** if the distributions $\{U_a,\ a \geq 0\}$ are monotone increasing in the order $\leq_{FR-\mathcal{L}}$, i.e., $U_a \leq_{FR-\mathcal{L}} U_b$ whenever $0 \leq a < b$.

Theorem 4.1. *The order $\leq_{FR-\mathcal{L}}$ has the property* **(R)** *if and only if for any mapping $\phi : I\!\!R_+ \to I\!\!R$ in $\mathcal{L}$, its antiderivative Φ is star–shaped, i.e., the mapping $x \to \frac{\Phi(x)}{x}$ is non–decreasing on $(0, +\infty)$*

Proof. Fix a and b scalars, with $0 < a < b$. Using Theorem 3.3, we see that $U_a \leq_{FR-\mathcal{L}} U_b$ if and only

$$\frac{\Phi(a)}{a} \leq \frac{\Phi(b)}{b} \tag{4.2}$$

for all mappings $\phi : I\!\!R_+ \to I\!\!R$ in $\mathcal{L}$, and the result follows from [7, Prop. B.9, p. 453].

■

By the remark that follows Theorem 3.3, for $\mathcal{L} = \{st\}$ and $\mathcal{L} = \{icx\}$, we need only consider (4.2) for $I\!\!R_+$–valued mappings ϕ in $\mathcal{L}$, in which case the antiderivative Φ is necessarily convex with $\Phi(0) = 0$, thus star–shaped [7, p. 453] and the orders $\leq_{FR-st}$ and $\leq_{FR-icx}$ both enjoy the property **(R)**. On the other hand, the order

$\leq_{FR-cx}$ does not enjoy the property (**R**) as can be seen by taking the convex mapping $\phi : x \to e^{-x}$, in which case the mapping $x \to \frac{\Phi(x)}{x}$ is readily checked to be decreasing on $(0, +\infty)$.

IV.2. Property (M)

Let X be a rv with distribution F in $\mathcal{D}$. For every $c > 0$, we define the distribution F_c as the distribution of the rv cX, i.e., $F_c(x) = F(x/c)$ for all $x \geq 0$. We shall say that the order $\leq_{FR-\mathcal{L}}$ has the property (**M**) if for rvs X and Y with distribution F and G in $\mathcal{D}$, respectively, the comparison $F \leq_{FR-\mathcal{L}} G$ implies $F_c \leq_{FR-\mathcal{L}} G_c$ for all $c > 0$. In other words, $\leq_{FR-\mathcal{L}}$ has the property (**M**) provided the order $\leq_{FR-\mathcal{L}}$ is stable under multiplication (of the corresponding rvs) by positive scalars.

It is now plain after a simple change of variable that

$$\tilde{F}_c(x) = \frac{\int_0^x (1 - F(t/c))\, dt}{cm(F)} = \tilde{F}(x/c), \quad x \geq 0. \tag{4.3}$$

Equivalently, $\widetilde{cX} =_{st} c\tilde{X}$, a remark which leads to the next result.

Theorem 4.2. *The order $\leq_{FR-\mathcal{L}}$ has the property* (**M**) *provided the order $\leq_{\mathcal{L}}$ itself has the property* (**M**).

IV.5. Property (W)

Let $\{X_n,\ n = 1, 2, \ldots\}$ and $\{Y_n,\ n = 1, 2, \ldots\}$ be sequences of rvs with distributions in $\mathcal{D}$ such that

$$X_n \leq_{FR-\mathcal{L}} Y_n. \qquad n = 1, 2, \ldots \tag{4.4}$$

If the sequences $\{X_n,\ n = 1, 2, \ldots\}$ and $\{Y_n,\ n = 1, 2, \ldots\}$ converge weakly to the rvs X_∞ and Y_∞, respectively, it is natural to ask if and when the comparison (4.4) holds in the limit, in which case the property (**W**) is said to hold for the order $\leq_{FR-\mathcal{L}}$. The next result explores this question.

Theorem 4.3. *Let the rvs $\{X_n,\ n = 1, 2, \ldots\}$ and $\{Y_n,\ n = 1, 2, \ldots\}$ with distributions in $\mathcal{D}$ converge weakly to X_∞ and Y_∞, respectively.*

1. *If the limiting rvs X_∞ and Y_∞ have finite mean, and*

$$\lim_{n\to\infty} m(X_n) = m(X_\infty) \quad \text{and} \quad \lim_{n\to\infty} m(Y_n) = m(Y_\infty), \tag{4.5}$$

   then both X_∞ and Y_∞ have distributions in $\mathcal{D}$;

2. *If in addition, the order $\leq_{\mathcal{L}}$ has the property* (**W**), *then the condition (4.4) implies*

$$X_\infty \leq_{FR-\mathcal{L}} Y_\infty. \tag{4.6}$$

Proof. The weak convergence of the rvs $\{X_n,\ n=1,2,\ldots\}$ implies that X_∞ has no mass at the origin, and X_∞ thus has a distribution in $\mathcal{D}$ since $m(F_\infty)$ is assumed finite. Similar comments hold for the rvs $\{Y_n,\ n=1,2,\ldots\}$, and Claim 1 is established.

To prove Claim 2, we first observe with the help of (3.1) that the forward recurrence times $\{\tilde{X}_n,\ n=1,2,\ldots\}$ converge weakly to the forward recurrence time $\tilde{X}_\infty$ of X_∞. This is an easy consequence of the weak convergence of the rvs $\{X_n,\ n=1,2,\ldots\}$ to X_∞, of the convergence of moments (4.5) and of the finiteness assumption on $m(F_\infty)$. Similar comments hold for the rvs $\{Y_n,\ n=1,2,\ldots\}$. With this in mind, we note from the definition of $\leq_{FR-\mathcal{L}}$, that (4.4) is equivalent to

$$\tilde{X}_n \leq_{\mathcal{L}} \tilde{Y}_n, \qquad n=1,2,\ldots \quad (4.7)$$

and the result (4.6) is now a consequence of (4.7) since the order $\leq_{\mathcal{L}}$ is assumed stable under weak convergence.

■

When $\mathcal{L}=\{st\}$, we observe from (4.4) by Lemma 5.2 that $m(X_n)\leq m(Y_n)$ for all $n=1,2,\ldots$. As a result, the finiteness of $m(Y_\infty)$ implies $m(X_\infty)<\infty$, and the convergence conditions (4.5) are equivalent to the rvs $\{X_n,\ n=1,2,\ldots\}$ and $\{Y_n,\ n=1,2,\ldots\}$ being uniformly integrable [4, Thm. 5.4, p. 32]. Finally, we recall that the order $\leq_{st}$ has the property **(W)** [10, Prop. 1.2.3, p. 6]. For the case $\mathcal{L}=\{icx\}$, conditions for the stability of $\leq_{icx}$ under weak convergence are available in [10, Prop. 1.3.2, p. 10].

IV.3. Property (E)

The order $\leq_{FR-\mathcal{L}}$ is said to have the expectation property **(E)** if for rvs X and Y with distribution F and G in $\mathcal{D}$, respectively, the comparison $X\leq_{FR-\mathcal{L}} Y$ implies $m(X)\leq m(Y)$. There seems to be no general result concerning this property. In Lemma 5.2 we show that the order $\leq_{FR-st}$ has the property **(E)**. On the other hand, for $\mathcal{L}=\{cx\}$, the comparison $X\leq_{FR-\mathcal{L}} Y$ is equivalent to $\tilde{X}\leq_{cx}\tilde{Y}$, whence $E[\tilde{X}]=E[\tilde{X}]$ by (2.11). From standard results on the moments of forward recurrence times [6, p. 173], this last equality becomes

$$\frac{1}{2}\frac{E[X^2]}{E[X]}=\frac{1}{2}\frac{E[Y^2]}{E[Y]} \quad (4.8)$$

under an asssumption of finite second moments. It is easy to infer from (4.8) that property **(E)** cannot hold for $\leq_{FR-cx}$. Similar comments apply for $\leq_{FR-icx}$.

IV.4. Property (C)

The order $\leq_{FR-\mathcal{L}}$ is said to enjoy the convolution property **(C)** if for independent rvs X, Y, U and V with distributions in $\mathcal{D}$, $X\leq_{FR-\mathcal{L}} Y$ and $U\leq_{FR-\mathcal{L}} V$ imply $X+U\leq_{FR-\mathcal{L}} Y+V$. Although this property does not hold in this general form, we shall obtain somewhat weaker versions of it. Before doing so, we observe that

$\mathcal{D}$ is indeed stable under convolution, i.e., if for independent rvs X and U with distributions in $\mathcal{D}$, the rv $X+U$ has distribution in $\mathcal{D}$.

We prepare the discussion with a simple representation result for the forward recurrence time of the sum of two independent rvs. This representation is readily obtained either by direct probabilistic arguments or by straightforward calculations on the Laplace–Stieltjes transforms.

Lemma 4.4. *Let X and U be independent rvs with distributions in $\mathcal{D}$. The forward recurrence time of their sum $X+U$ can be represented as*

$$\widetilde{X+U} =_{st} B(\tilde{X}+U) + (1-B)\tilde{U} \tag{4.9}$$

where the rvs X, $\tilde{X}$, $\tilde{U}$ and B are taken mutually independent, and B is a $\{0,1\}$–valued rv with

$$P[B=1] = \frac{m(X)}{m(X+U)} = 1 - P[B=0]. \tag{4.10}$$

We can express (4.9) in a more suggestive manner as

$$\widetilde{X+U} = \begin{cases} \tilde{X}+U & \text{w.p. } \frac{m(X)}{m(X+U)} \\ \tilde{U} & \text{w.p. } \frac{m(U)}{m(X+U)}. \end{cases} \tag{4.11}$$

Note also that

$$E[\phi(\widetilde{X+U})] = \frac{m(X)}{m(X+U)} E[\phi(\tilde{X}+U)] + \frac{m(U)}{m(X+U)} E[\phi(\tilde{U})] \tag{4.12}$$

for any mapping $\phi : I\!R_+ \to I\!R$ in $\mathcal{L}$ provided the expectations exist.

Lemma 4.4 paves the way to several convolution results for the order $\leq_{FR-\mathcal{L}}$. The most basic one is considered first.

Theorem 4.5. *Assume the order $\leq_{\mathcal{L}}$ itself to have the convolution property* **(C)**, *and let X, Y, U and V be independent rvs with distributions in $\mathcal{D}$ such that*

$$X \leq_{FR-\mathcal{L}} Y \quad \text{and} \quad U \leq_{FR-\mathcal{L}} V. \tag{4.13}$$

In addition, if either

$$U \leq_{\mathcal{L}} V \tag{4.14a}$$

or

$$X \leq_{\mathcal{L}} Y, \tag{4.14b}$$

and if

$$\frac{m(X)}{m(X+U)} = \frac{m(Y)}{m(Y+V)} =: p, \tag{4.15}$$

then

$$X + U \leq_{FR-\mathcal{L}} Y + V. \tag{4.16}$$

Proof. By symmetry, we need only consider the case when (4.14a) is satisfied, as we do from now on. Recall that (4.16) is equivalent to $\widetilde{X+U} \leq_{\mathcal{L}} \widetilde{Y+V}$. In view of (3.7), we see from (4.12) under the condition (4.15) that this last comparison holds true if and only

$$pE[\phi(\tilde{X} + U)] + (1-p)E[\phi(\tilde{U})] \leq pE[\phi(\tilde{Y} + V)] + (1-p)E[\phi(\tilde{V})] \tag{4.17}$$

for any mapping $\phi : I\!R_+ \to I\!R$ in $\mathcal{L}$ provided the expectations exist. Under (4.13)–(4.14a) we have $\tilde{U} \leq_{\mathcal{L}} \tilde{V}$, $\tilde{X} \leq_{\mathcal{L}} \tilde{Y}$ and $U \leq_{\mathcal{L}} V$, whence $\tilde{X} + U \leq_{\mathcal{L}} \tilde{Y} + V$ since the order $\leq_{\mathcal{L}}$ itself has the convolution property **(C)**. These comparisons readily imply that (4.17) holds true for all mappings $\phi : I\!R_+ \to I\!R$ in $\mathcal{L}$ provided the expectations exist, and the result (4.16) follows.

■

This last result has several interesting consequences.

Corollary 4.6. *Assume the order $\leq_{\mathcal{L}}$ itself to have the convolution property* **(C)**, *and let X, Y and U be independent rvs with distributions in $\mathcal{D}$ such that $X \leq_{FR-\mathcal{L}} Y$. If $m(X) = m(Y)$, then*

$$X + U \leq_{FR-\mathcal{L}} Y + U. \tag{4.18}$$

Proof. With the identification $U =_{st} V$, we see that the conditions (4.13), (4.14a) and (4.15) are in place, and the result immediately follows from Theorem 4.5.

■

By a repeated application of Corollary 4.6 we obtain another set of conditions under which (4.18) holds.

Theorem 4.7. *Assume the order $\leq_{\mathcal{L}}$ itself to have the convolution property* **(C)**, *and let X, Y, U and V be independent rvs with distributions in $\mathcal{D}$ such that*

$$X \leq_{FR-\mathcal{L}} Y \quad \text{and} \quad U \leq_{FR-\mathcal{L}} V. \tag{4.19}$$

In addition, if

$$m(X) = m(Y) \quad \text{and} \quad m(U) = m(V), \tag{4.20}$$

then (4.16) holds true.

Note that Theorem 4.7 is neither subsumed by nor implie Theorem 4.5.

To formulate additional consequences of Theorem 4.5, we consider the sequences of i.i.d. rvs $\{X_n,\ n = 1, 2, \ldots\}$ and $\{Y_n,\ n = 1, 2, \ldots\}$ with common distribution F

and G in $\mathcal{D}$, respectively. We also define the corresponding partial sums $\{S_n,\ n = 1,2,\ldots\}$ and $\{T_n,\ n = 1,2,\ldots\}$ by

$$S_n = \sum_{k=1}^{n} X_k \quad \text{and} \quad T_n = \sum_{k=1}^{n} Y_k \qquad n = 1,2,\ldots \tag{4.21}$$

with the usual convention $S_0 = T_0 = 0$.

Corollary 4.8. *Assume the order $\leq_{\mathcal{L}}$ itself to have the convolution property* (C). *If the comparisons*

$$F \leq_{\mathcal{L}} G \quad \text{and} \quad F \leq_{FR-\mathcal{L}} G \tag{4.22}$$

hold true, then

$$S_n \leq_{FR-\mathcal{L}} T_n. \qquad n = 1,2,\ldots \tag{4.23}$$

Proof. The proof proceeds by induction on n: For $n = 1$, the result (4.23) is trivially true by the first part of assumption (4.22), and the basis step is established.

Next, assuming (4.23) for some $n \geq 1$, we show that it also holds for $n+1$. This will follow from Theorem 4.5 with the identification $X = S_n$, $U = X_{n+1}$, $Y = T_n$, $V = Y_{n+1}$, provided the assumptions of Theorem 4.5 are satisfied. To see that it is indeed the case, we observe that assumption (4.22) is equivalent to $U \leq_{\mathcal{L}} V$ and $U \leq_{FR-\mathcal{L}} V$, whereas the condition $X \leq_{FR-\mathcal{L}} Y$ holds by the induction hypothesis. Here, we obviously have

$$\frac{m(X)}{m(X+U)} = \frac{m(Y)}{m(Y+V)} = \frac{n}{n+1} \tag{4.24}$$

and the condition (4.15) on the moments thus holds.

■

In the same way that Theorem 4.5 leads to Corollary 4.8, we can invoke Theorem 4.7 to get another set of conditions for (4.23) to hold. This is the content of Corollary 4.9; its proof proceeds by induction and is omitted in the interest of brevity.

Corollary 4.9. *Assume the order $\leq_{\mathcal{L}}$ itself to have the convolution property* (C). *If the comparison*

$$F \leq_{FR-\mathcal{L}} G \tag{4.25}$$

hold true, with

$$M(F) = M(G), \tag{4.26}$$

then (4.23) holds true.

V. Properties of the order $\leq_{\mathbf{FR-st}}$

This section is devoted to a more detailed discussion of some of the properties of the order $\leq_{FR-st}$. From the previous section, we already know from Theorems 4.1 and 4.2 that properties **(R)** and **(M)** are satisfied. Furthermore, Theorem 4.3 (and the remark that follows its proof) gives simple conditions for the order $\leq_{FR-st}$ to be stable under weak convergence.

We first provide a convenient characterization of the order $\leq_{FR-st}$ by specializing Theorem 3.3 for $\mathcal{L} = \{st\}$.

Theorem 5.1. *For rvs X and Y with distribution F and G in $\mathcal{D}$, respectively, we have $F \leq_{FR-st} G$ if and only if*

$$\frac{\int_x^\infty (1-F(t))\,dt}{m(F)} \leq \frac{\int_x^\infty (1-G(t))\,dt}{m(G)}, \quad x \geq 0. \tag{5.1}$$

Moreover (5.1) holds if and only if

$$\frac{E[\phi(X)]}{m(X)} \leq \frac{E[\phi(Y)]}{m(Y)} \tag{5.2}$$

for every mapping $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ in $\{icx\}$ such that $\phi(0) = 0$, provided the expectations are well defined.

Proof. The first characterization (5.1) follows from the definition (3.4) when specialized to $\mathcal{L} = \{st\}$ with the help of Lemma 2.1 and of the remark (3.3).

By Theorem 3.3 and the remark that follows it, we see that $F \leq_{FR-st} G$ if and only if

$$\frac{E[\Psi(X)]}{m(X)} \leq \frac{E[\Psi(Y)]}{m(Y)} \tag{5.3}$$

for every mapping $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ in $\{st\}$, with Ψ denoting the antiderivative (3.11) of ψ. The second characterization (5.2) is now easily obtained upon observing that any function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ in $\{icx\}$ with $\phi(0) = 0$ is indeed the antiderivative Ψ of an $\mathbb{R}_+$-valued increasing mapping ψ.

■

We then proceed to show that the order $\leq_{FR-st}$ has the expectation property **(E)**.

Lemma 5.2. *The order $\leq_{FR-st}$ has the expectation property* **(E)**, *i.e., for distribution F and G in $\mathcal{D}$ such that $F \leq_{FR-st} G$, we have $m(F) \leq m(G)$.*

Proof. Using (3.2) on (5.1), we readily see that $F \leq_{FR-st} G$ if and only if

$$\frac{1}{t}\frac{\int_0^t (1-F(s))\,ds}{m(F)} \geq \frac{1}{t}\frac{\int_0^t (1-G(s))\,ds}{m(G)}, \quad t > 0. \tag{5.4}$$

Letting t go to zero in (5.4), we get

$$\frac{1-F(0)}{m(F)} \geq \frac{1-G(0)}{m(G)}, \tag{5.5}$$

and the result follows from the fact that $F(0) = G(0) = 0$.

■

With the help of this result, we now establish that $\leq_{FR-st}$ is located between the stochastic orders $\leq_{cx}$ and $\leq_{icx}$.

Theorem 5.3. *For distributions F and G in $\mathcal{D}$, we have the following implications:*

1. *If $F \leq_{cx} G$, then $F \leq_{FR-st} G$;*
2. *If $F \leq_{FR-st} G$, then $F \leq_{icx} G$.*

Moreover, when $m(F) = m(G)$, then

$$F \leq_{cx} G \quad \text{iff} \quad F \leq_{FR-st} G \quad \text{iff} \quad F \leq_{icx} G. \tag{5.6}$$

Proof. Let X and Y be rvs with distribution F and G in $\mathcal{D}$. The comparison $X \leq_{cx} Y$ implies $E[X] = E[Y]$ and $E[\phi(X)] \leq E[\phi(Y)]$ for all mappings ϕ in $\{cx\}$, thus in $\{icx\}$. Claim 1 now follows from Theorem 5.1.

Using Theorem 5.1 and Corollary 5.2, from the comparison $X \leq_{FR-st} Y$, we have

$$E[\phi(X)] \leq \frac{E[X]}{E[Y]} E[\phi(Y)] \leq E[\phi(Y)] \tag{5.7}$$

for all mappings $\phi : I\!R_+ \to I\!R_+$ in $\{icx\}$ such that $\phi(0) = 0$, whence (5.7) holds for all mappings $\phi : I\!R_+ \to I\!R$ in $\{icx\}$ since the rvs are non–negative. This establishes Claim 2. Finally, if $E[X] = E[Y]$, then $X \leq_{icx} Y$ if and only if $X \leq_{cx} Y$ [10, Thm. 1.3.1, p. 9], and the equivalences (5.6) follow from Claims 1 and 2.

■

The implications of Theorem 5.3 hold only in one direction as stated; counterexamples are given in Section 7. Additional properties of the order $\leq_{FR-st}$ are given in Section 9 after connections with point processes will have been presented.

VI. Stochastic orders associated with Palm distributions

Not every distribution function in $\mathcal{D}(I\!R_+)$ is the distribution function of a forward recurrence time associated with a distribution in $\mathcal{D}$. The first result of this section identifies conditions for this to happen.

To set the terminology, observe from (3.1) that a distribution F in $\mathcal{D}(I\!R_+)$ will be the distribution of the forward recurrence time associated with some distribution function F^0 in $\mathcal{D}$ if the representation

$$F(x) = \frac{\int_0^x \left(1 - F^0(t)\right) dt}{m(F^0)}, \quad x \geq 0 \tag{6.1}$$

holds. For reasons that will become apparent in the next section, any distribution F^0 in $\mathcal{D}$ satisfying (6.1) is called a *Palm* distribution induced by F.

Lemma 6.1. *A distribution function F in $\mathcal{D}(\mathbb{R}_+)$ is the distribution of the forward recurrence time associated with a distribution F^0 in $\mathcal{D}$ if and only if F is concave on $\mathbb{R}_+$ with $F(0) = 0$ and has a finite non–vanishing (right) derivative at $x = 0$. Moreover, each one of the distributions F and F^0 is determined uniquely by the other, that is F induces exactly one Palm distribution F^0.*

Proof. To establish the necessity part, let F be an element of $\mathcal{D}(\mathbb{R}_+)$ which is the distribution of the forward recurrence time associated with some distribution function F^0 in $\mathcal{D}$, in which case (6.1) holds. It is plain from (6.1) that F is concave non–decreasing, thus differentiable on $\mathbb{R}_+$, except possibly on a countable set of points. In fact, the right derivative F'_+ exists everywhere and is given by

$$F'_+(x) = \frac{1 - F^0(x)}{m(F^0)}, \quad x \geq 0. \tag{6.2}$$

Therefore, since F^0 is continuous at $x = 0$ in view of the condition $F^0(0) = 0$ and F^0 has finite mean, we see that F is indeed differentiable at $x = 0$ with non–vanishing derivative given by

$$F'_+(0) = \frac{1}{m(F^0)}. \tag{6.3}$$

Combining (6.2) and (6.3), we obtain

$$F'_+(x) = [1 - F^0(x)]F'_+(0), \quad x \geq 0. \tag{6.4}$$

Conversely, let F be a distribution on $\mathbb{R}_+$ which is concave with $F(0) = 0$ and which has a finite non–vanishing (right) derivative at $x = 0$. We need to show that F indeed induces a Palm distribution F^0 in $\mathcal{D}$. Guided by (6.4), we define a mapping $F^0 : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$F^0(x) := 1 - \frac{F'_+(x)}{F'_+(0)}, \quad x \geq 0. \tag{6.5}$$

We first show that F^0 is an element of $\mathcal{D}$: From the concavity of F, we conclude [8] that F'_+ is non–increasing and right–continuous with $F'_+(x) \leq 0$ for all $x \geq 0$. Therefore F^0 is non–decreasing and right–continuous with $0 \leq F^0(x) \leq 1$ for all $x \geq 0$. It is also straighforward to see that $\lim_{x\to\infty} F'_+(x) = 0$, whence $\lim_{x\to\infty} F^0(x) = 1$, and F is indeed an element of $\mathcal{D}(\mathbb{R}_+)$ with no mass at $x = 0$, i.e., $F^0(0) = 0$. Its mean is finite since given by

$$m(F^0) = \int_0^\infty \left(1 - F^0(t)\right) dt = \frac{\int_0^\infty F'_+(t)dt}{F'_+(0)} = \frac{1}{F'_+(0)}, \tag{6.6}$$

and F^0 is therefore an element of $\mathcal{D}$. It is now plain that F and F^0 are indeed related by (6.1). The claim on uniqueness is a straightforward byproduct of the arguments given above.

■

Let $\mathcal{C}$ denote the subset of distribution functions on $\mathbb{R}^+$ which are concave with no mass at the origin and which have a finite non–vanishing (right) derivative at the origin, i.e., F belongs to $\mathcal{C}$ if F is an element of $\mathcal{D}(\mathbb{R}_+)$ such that F is concave with $F(0) = 0$ and $0 < F'_+(0) < \infty$. The proof of Lemma 6.1 shows that the correspondence $F \to F^0$ given either by (6.1) or (6.5) defines a bijection between $\mathcal{D}$ and $\mathcal{C}$. This correspondence will allow us to define another class of stochastic orders, this time on $\mathcal{C}$, in the same way that the correspondence (3.1) formed the basis for defining the stochastic orders of Section 2 on $\mathcal{C}$. We refer to this new class of stochastic orders as Palm orders on $\mathcal{C}$.

Let $\mathcal{L}$ be any non–empty collection of Borel mappings $\phi : \mathbb{R}_+ \to \mathbb{R}$. Any such collection $\mathcal{L}$ induces a binary relation $\leq_{P-\mathcal{L}}$ on $\mathcal{C}$ by requiring that

$$F \leq_{P-\mathcal{L}} G \quad \text{iff} \quad F^0 \leq_{\mathcal{L}} G^0 \tag{6.7}$$

where F^0 and G^0 are the Palm distributions in $\mathcal{D}$ induced by F and G, respectively. Equivalently, if X and Y are rvs with distribution F and G in $\mathcal{C}$, respectively, we require

$$X \leq_{P-\mathcal{L}} Y \quad \text{iff} \quad X^0 \leq_{\mathcal{L}} Y^0 \tag{6.8}$$

where X^0 and Y^0 are rvs with distribution F^0 and G^0, respectively. The next fact parallels Lemma 3.1.

Lemma 6.2. *Let $\mathcal{L}$ be an non–empty collection of Borel mappings $\phi : \mathbb{R}_+ \to \mathbb{R}$. If the binary relation $\leq_{\mathcal{L}}$ is a partial order on $\mathcal{D}$, so is $\leq_{P-\mathcal{L}}$ on $\mathcal{C}$.*

Proof. We need only check the antisymmetry property of $\leq_{P-\mathcal{L}}$ on $\mathcal{C}$: By the definition (6.8), if the distributions F and G in $\mathcal{C}$ satisfy $F \leq_{P-\mathcal{L}} G$ and $G \leq_{P-\mathcal{L}} F$, then $F^0 \leq_{\mathcal{L}} G^0$ and $G^0 \leq_{\mathcal{L}} F^0$. The antisymmetry of $\leq_{\mathcal{L}}$ implies $F^0 = G^0$, whence $F = G$ by the arguments of Lemma 6.1.

■

We conclude this section with some obvious analytical results on these stochastic orders; proofs are straightforward and are omitted in the interest of brevity.

Theorem 6.3. *For distributions F and G in $\mathcal{C}$, we have $F \leq_{P-st} G$ if and only if*

$$\frac{F'_+(x)}{F'_+(0)} \leq \frac{G'_+(x)}{G'_+(0)}, \quad x \geq 0. \tag{6.9}$$

Theorem 6.4. *For distributions F and G in $\mathcal{C}$, we have $F \leq_{P-icx} G$ if and only if*

$$\frac{1 - F(x)}{F'_+(0)} \leq \frac{1 - G(x)}{G'_+(0)}, \quad x \geq 0. \tag{6.10}$$

Theorem 6.5. *For distributions F and G in $\mathcal{C}$, we have $F \leq_{P-cx} G$ if and only if $F \leq_{st} G$ and $F'_+(0) = G'_+(0)$.*

As a direct consequence of Theorem 5.3, we also conclude with

Theorem 6.6. *For distributions F and G in $\mathcal{C}$, we have the implications:*

1. *If $F \leq_{P-cx} G$, then $F \leq_{st} G$;*
2. *If $F \leq_{st} G$, then $F \leq_{P-icx} G$.*

Claims 1 and 2 can also be validated through Theorems 6.4 and 6.5, respectively.

VII. Summary of the relations between the orders

Collecting the results of Sections 5 and 6, we summarize the relationships between these new orders and the integral orders in the following chart.

$$\begin{array}{ccccccc} & & & & FR-cx & \Longrightarrow & FR-FR-st \\ & & & & & & \Downarrow \\ & & cx & \Longrightarrow & FR-st & \Longrightarrow & FR-icx \\ & & & & \Downarrow & & \\ P-cx & \Longrightarrow & st & \Longrightarrow & icx & & \\ & & \Downarrow & & & & \\ P-st & \Longrightarrow & P-icx & & & & \\ \Downarrow & & & & & & \\ P-P-icx & & & & & & \end{array} \qquad (7.1)$$

We conclude this section with several examples which show that some of the implications of the tableau (7.1) cannot be reversed. In each case, the implication which is shown *not* to hold is displayed in square brackets:

1. $[\leq_{\mathbf{FR-st}} \Longrightarrow \leq_{\mathbf{cx}}]$: First we exhibit two distributions F and G in $\mathcal{D}$ such that $F \leq_{FR-st} G$, yet F and G are not comparable for $\leq_{cx}$: With $0 < \mu < \lambda$, take F deterministic with mean $\frac{1}{\lambda}$ and G exponentially distributed with mean $\frac{1}{\mu}$. Since $m(F) < m(G)$, F and G cannot be compared in the order $\leq_{cx}$. On the other hand, $\tilde{F}$ is uniform on $[0, \frac{1}{\lambda}]$ while $\tilde{G}$ is still exponentially distributed with parameter μ. Under the enforced assumption, we readily check

$$\tilde{F}(x) = (\lambda x) \wedge 1 \geq \tilde{G}(x) = 1 - e^{-\mu x}, \quad x \geq 0 \qquad (7.2)$$

so that $\tilde{F} \leq_{st} \tilde{G}$, and $F \leq_{FR-st} G$.

2. $[\leq_{\mathbf{icx}} \not\Longrightarrow \leq_{\mathbf{FR-st}}]$: Next we find F and G in $\mathcal{D}$ such that $F \leq_{icx} G$, yet F and G are not comparable for $\leq_{FR-st}$: Let F be any distribution function in $\mathcal{D}$ with the property that $F(x) > 0$ whenever $0 < x < u$ for some $u > 0$. Define the distribution function G in $\mathcal{D}$ by

$$G(x) = \begin{cases} 0 & \text{if } 0 \leq x < u \\ \\ F(x) & \text{if } u \leq x. \end{cases} \tag{7.3}$$

In other words G is the distribution of a rv $X \vee u$ where X has distribution F. By construction F and G are distinct distributions satisfying $G(x) \leq F(x)$ for all $x \geq 0$ so that $F \leq_{st} G$, whence $F \leq_{icx} G$. However, F and G are not comparable in the order $\leq_{FR-st}$: Indeed, if they were comparable, it would follow from Theorem 5.3 that $F \leq_{FR-st} G$ necessarily, so that

$$\frac{\int_x^\infty (1 - F(t))\, dt}{m(F)} \leq \frac{\int_x^\infty (1 - G(t))\, dt}{m(G)}, \quad x \geq 0 \tag{7.4}$$

by using the characterization (5.1). From the definition (7.3), we obtain

$$\int_x^\infty (1 - F(t))\, dt = \int_x^\infty (1 - G(t))\, dt, \quad x \geq u \tag{7.5}$$

and the strict inequality $m(F) < m(G)$ precludes (7.4) to hold, at least for $x > u$.

3. $[\leq_{\mathbf{st}} \not\Longrightarrow \leq_{\mathbf{FR-st}}]$: The previous counterexample applies here since the distributions F and G constructed in **2.** have the property that $F \leq_{st} G$, yet F and G are not comparable under the order $\leq_{FR-st}$.

4. $[\leq_{\mathbf{FR-st}} \not\Longrightarrow \leq_{\mathbf{st}}]$: Let X be a non–constant rv, so that $X \neq m(X)$ with probability one. By Jensen's inequality, $m(X) \leq_{cx} X$, thus $m(X) \leq_{FR-st} X$ by Theorem 5.3, yet $m(X)$ and X are clearly not comparable under the order $\leq_{st}$.

Counterexample **3** shows that increasing F for $\leq_{st}$ does not automatically result in a increase of $\tilde{F}$ for $\leq_{st}$, that is an increase of F for $\leq_{FR-st}$. We expand on this point by showing on a more concrete parametric example derived from the construction (7.3) that a strict increase of F in $\leq_{st}$ gives rise to a decrease of the mean value $m(\tilde{F})$: Take X to be an exponentially distributed rv with parameter 1, and define the delayed rvs $X_u = X \vee u$ with $u \geq 0$. It is plain that $X \leq_{st} X_u$ for all $u \geq 0$. By straightforward computations we find

$$E[X_u] = u + e^{-u} \quad \text{and} \quad E[X_u^2] = u^2 + 2(1+u)e^{-u}, \tag{7.6}$$

so that

$$E[\tilde{X}_u] = \frac{u^2 + 2(1+u)e^{-u}}{2(u + e^{-u})}, \quad u \geq 0. \tag{7.7}$$

Elementary considerations show that

$$E[\tilde{X}_u] \leq E[\tilde{X}], \quad 0 \leq u \leq u^* \tag{7.8}$$

where u^* is the root of the transcendental equation

$$u + 2e^{-u} - 2 = 0, \quad u > 0. \tag{7.9}$$

Direct numerical computations yield $u^* \simeq 1.5936$. In particular, it is easily shown that for $u = u^*$, the mean forward recurrence time is 1, namely the same as for $u = 0$, while the mean value of X_{u^*} is approximately 1.797. The value of u that minimizes $E[\tilde{X}_u]$ is $u^0 = \ln 2$, in which case we get

$$E[\tilde{X}_{u^0}] \simeq .9109. \tag{7.10}$$

In other words, Feller's paradox [6] can be continued by the following comments: By delaying the interarrival times of a Poisson point process of intensity 1 as it was done above, a point process is constructed with both a smaller intensity and a smaller mean forward recurrence time. In particular, the delay u can be selected in such a way that the intensity of the delayed process is .56, while keeping the same mean forward recurrence time as the initial Poisson process, namely 1. The smallest possible mean forward recurrence time obtained by this technique is roughly .91.

VIII. Preservation of $\leq_{\mathbf{FR-st}}$ under thinning and superposition

As should be apparent from the results obtained so far, the orders $\leq_{FR-\mathcal{L}}$ do not behave quite like the standard integral orders, although they are obtained from them by a simple transformation, namely (3.1). In particular, we see that these orders $\leq_{FR-\mathcal{L}}$ are not naturally compatible with basic operations, e.g., convolution and expectation. At first glance, this may seem rather unfortunate since this compatibility with elementary operations provided the basis for a "calculus" which has made integral orders so useful in many applications [2]. However, in light of the correspondence with orders on point processes developed in [1], we argue that the basic and natural operations of interest for the orders $\leq_{FR-\mathcal{L}}$ should correspond to basic and natural operations on point processes. Below we take this viewpoint one step further and consider two operations which have simple interpretations in the theory of (renewal) point processes, namely thinning and superposition. We have elected to carry out the discussion only for the order $\leq_{FR-st}$ for which these two operations are shown to be preserved.

VIII.1. Thinning

The setup is as follows: Consider a sequence of i.i.d. $I\!R_+$–valued rvs $\{X_n,\ n = 1, 2, \ldots\}$ with common distribution F in $\mathcal{D}$. We define the corresponding partial sums $\{S_n,\ n = 0, 1, \ldots\}$ by (4.21), and denote by N_X the *synchronous* renewal process with event times $\{S_n,\ n = 0, 1, \ldots\}$ [3]. Let also $\{\alpha_n,\ n = 1, 2, \ldots\}$ be another sequence of i.i.d. $\{1, 2, \ldots\}$–valued rvs which are independent of the rvs $\{X_n,\ n = 1, 2, \ldots\}$, hereafter called a thinning sequence. The α_n–thinning of the renewal process N_X is the synchronous renewal process N_X^α with inter–event times $\{X_n^\alpha,\ n = 1, 2, \ldots\}$ given by

$$X_{n+1}^\alpha = \sum_{k=a_n+1}^{a_{n+1}} X_k, \qquad n = 0, 1, \ldots \quad (8.1)$$

where we have defined

$$a_0 = 0, \quad a_{n+1} = a_n + \alpha_{n+1}. \qquad n = 0, 1, \ldots \quad (8.2)$$

The rvs $\{X_n^\alpha,\ n = 1, 2, \ldots\}$ are of course i.i.d. with some common distribution F^α which is easily determined from F and from the common distribution of the rvs $\{\alpha_n,\ n = 1, 2, \ldots\}$. To simplify the notation, let X, α and X^α denote generic rvs for the i.i.d. rvs $\{X_n,\ n = 1, 2, \ldots\}$, $\{\alpha_n,\ n = 1, 2, \ldots\}$ and $\{X_n^\alpha,\ n = 1, 2, \ldots\}$, respectively.

Following the viewpoint developed in [1], and briefly indicated in Section 3, it is quite natural to investigate in what sense should two thinning sequences $\{\alpha_n,\ n = 1, 2, \ldots\}$ and $\{\beta_n,\ n = 1, 2, \ldots\}$ be comparable in order to guarantee that the α_n–thinning N_X^α and the β_n–thinning N_X^β of the same synchronous renewal process N_X be comparable in the order $\leq_{FR-st}$. As in [1] this is equivalent to the rvs X^α and X^β be comparable in the order $\leq_{FR-st}$, and requires that of the order $\leq_{FR-st}$ under random summation be investigated. We take this on in the following proposition.

Theorem 8.1. *Let α and β be two integrable $\{1, 2, \ldots\}$–valued rvs, each independent of the i.i.d rvs $\{X_n,\ n = 1, 2, \ldots\}$. If $\alpha \leq_{FR-st} \beta$, then*

$$S_\alpha \leq_{FR-st} S_\beta \quad (8.3)$$

or equivalently,

$$X^\alpha \leq_{FR-st} X^\beta. \quad (8.4)$$

Proof. By Theorem 5.1 and Wald's equation [5, Thm. 5.5.3, p.137], it suffices to show that

$$\frac{E[\phi(S_\alpha)]}{E[X]E[\alpha]} \leq \frac{E[\phi(S_\beta)]}{E[X]E[\beta]} \quad (8.5)$$

for every mapping $\phi : I\!R_+ \to I\!R_+$ in $\{icx\}$ with $\phi(0) = 0$. Introducing the notation

$$\hat{\phi}(n) := E[\phi(S_n)], \qquad n = 0, 1, \ldots \quad (8.6)$$

we observe that

$$E[\phi(S_\alpha)] = E[\hat{\phi}(\alpha)] \quad \text{and} \quad E[\phi(S_\beta)] = E[\hat{\phi}(\beta)] \quad (8.7)$$

since the rvs α and β are assumed independent of the rvs $\{X_n,\ n = 1, 2, \ldots\}$.

Using the fact that the i.i.d. rvs $\{X_n,\ n = 1, 2, \ldots\}$ are non–negative, we see that the mapping $n \to \hat{\phi}(n)$ is increasing and integer–convex with $\hat{\phi}(0) = 0$ since the mapping ϕ is also increasing convex with $\phi(0) = 0$; details are available in [9, Lemma 8.6.7, p. 278]. The interpolated mapping $\hat{\phi}_e : I\!R_+ \to I\!R$ defined by

$$\hat{\phi}_e(t) := \hat{\phi}(n), \quad n \leq t < n+1 \qquad\qquad n = 0, 1, \ldots \tag{8.8}$$

is thus in $\{icx\}$ with $\hat{\phi}_e(0) = 0$. Now, as we combine (8.7) and (8.8), we get (8.5) in the equivalent form

$$\frac{E[\hat{\phi}_e(\alpha)]}{E[\alpha]} \leq \frac{E[\hat{\phi}_e(\beta)]}{E[\beta]}. \tag{8.9}$$

From the necessary part of Theorem 5.1 we see that (8.9), thus (8.5), holds under the assumption $\alpha \leq_{FR-st} \beta$.

■

We extend readily this last result to the situation where two different synchronous renewal processes N_X and N_Y are each thinned by its own thinning sequence, say $\{\alpha_n,\ n = 1, 2, \ldots\}$ and $\{\beta_n,\ n = 1, 2, \ldots\}$. Here, the renewal process N_Y is generated by a sequence of i.i.d. rvs $\{Y_n,\ n = 1, 2, \ldots\}$ which is independent of the thinning sequence $\{\beta_n,\ n = 1, 2, \ldots\}$. The sequence of partial sums $\{T_n,\ n = 1, 2, \ldots\}$ associated with N_Y is defined by (4.21). Using the generic notation introduced earlier in this section, we have

Corollary 8.2. *Let α and β be two integrable $\{1, 2, \ldots\}$–valued rvs, each independent of the sequences of i.i.d. rvs $\{X_n,\ n = 1, 2, \ldots\}$ and $\{Y_n,\ n = 1, 2, \ldots\}$. If $\alpha \leq_{FR-st} \beta$ and if both conditions $X \leq_{st} Y$ and $X \leq_{FR-st} Y$ hold, then*

$$S_\alpha \leq_{FR-st} T_\beta, \tag{8.10}$$

or equivalently

$$X^\alpha \leq_{FR-st} Y^\beta. \tag{8.11}$$

Proof. In view of Theorem 8.1, it is enough to establish

$$S_\alpha \leq_{FR-st} T_\alpha. \tag{8.12}$$

By Theorem 5.1, as in the proof of Theorem 8.1, this is equivalent to showing

$$\frac{E[\phi(S_\alpha)]}{E[X]E[\alpha]} \leq \frac{E[\phi(T_\alpha)]}{E[Y]E[\alpha]} \tag{8.13}$$

for every mapping $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ in $\{icx\}$ with $\phi(0) = 0$. From the independence assumption we get

$$\frac{E[\phi(S_\alpha)]}{E[X]} = \sum_{n=1}^{\infty} P[\alpha = n] \frac{E[\phi(S_n)]}{E[X]} \tag{8.14}$$

with a similar decomposition for $E[\phi(T_\alpha)]$. By Corollary 4.8, the comparison assumptions on X and Y now imply $S_n \leq_{FR-st} T_n$ for all $n = 1, 2, \ldots$. Invoking again the characterization (5.2) we see that these comparisons are equivalent to

$$\frac{E[\phi(S_n)]}{nE[X]} \leq \frac{E[\phi(T_n)]}{nE[Y]}, \qquad n = 1, 2, \ldots \tag{8.15}$$

and the proof of (8.13) is concluded by using (8.15) on the representation (8.14).

■

By Theorem 5.3 we see that in Theorem 8.1. and its corollary, the condition $\alpha \leq_{FR-st} \beta$ can be replaced by $\alpha \leq_{cx} \beta$. In particular, this will be the case when $\alpha = K$ for some positive integer K, and β is the geometric rv with parameter $\frac{1}{K}$, i.e.,

$$P[\beta = n + 1] = \frac{1}{K}\left(1 - \frac{1}{K}\right)^n. \qquad n = 0, 1, \ldots \tag{8.16}$$

Finally, in the same way that Corollary 4.8 yielded (8.10)-(8.11), we can invoke Corollary 4.9 to get another set of conditions under which this comparison will hold.

Corollary 8.3. *Let α and β be two integrable $\{1, 2, \ldots\}$–valued rvs, each independent of the sequences of i.i.d. rvs $\{X_n,\ n = 1, 2, \ldots\}$ and $\{Y_n,\ n = 1, 2, \ldots\}$. If $\alpha \leq_{FR-st} \beta$ and if both conditions $X \leq_{FR-st} Y$ and $m(X) = m(Y)$ hold, then (8.10)–(8.11) hold true.*

Proof. The proof is identical with the proof of Corollary 8.2 up to (8.14), but this time (8.15) follows from Corollary 4.9.

■

VIII.2. Superposition

Consider now two independent sequences of i.i.d. rvs $\{X_n,\ n = 1, 2, \ldots\}$ and $\{Y_n,\ n = 1, 2, \ldots\}$ with common distribution F and G in $\mathcal{D}$, respectively. The corresponding synchronous renewal processes N_X and N_Y are thus independent. The superposition of the two renewal processes N_X and N_Y is the synchronous point process $N_X + N_Y$ defined by

$$N_X + N_Y := \sum_{n=0}^{\infty} (\delta_{S_n} + \delta_{T_n}). \tag{8.17}$$

The inter-event distribution $F \oplus G$ of the stationary version of the point process $N_X + N_Y$ is given [3, pp. 18–19] by

$$F \oplus G(x) = \frac{m(F)}{m(F)+m(G)}(1-(1-F(x))(1-\tilde{G}(x))) \\ + \frac{m(G)}{m(F)+m(G)}(1-(1-G(x))(1-\tilde{F}(x))), \quad x \geq 0. \tag{8.18}$$

The following result paves the way to a stability result of the order $\leq_{FR-st}$ under superposition operation (8.18).

Lemma 8.3 *For distributions F and G in $\mathcal{D}$, we have the representation*

$$\widetilde{F \oplus G}(x) = 1-(1-\tilde{F}(x))(1-\tilde{G}(x)), \quad x \geq 0 \tag{8.19}$$

Proof. The result is an immediate consequence of the expression (8.18) of the forward recurrence time distribution associated with $F \oplus G$.

■

Now consider another pair of independent synchronous renewal processes N_U and N_V with inter–renewal time distributions H and K in $\mathcal{D}$. As before $H \oplus K$ is the inter–event time distribution of the staionary version of the superposition $N_U + N_V$ of the renewal processes N_U and N_V.

Theorem 8.4. *For distributions F, G, H and K in $\mathcal{D}$, if $F \leq_{FR-st} H$ and $G \leq_{FR-st} K$, then $F \oplus G \leq_{FR-st} H \oplus K$.*

Proof. Under the enforced assumptions, we have $1-\tilde{F}(x) \leq 1-\tilde{H}(x)$ and $1-\tilde{G}(x) \leq 1-\tilde{K}(x)$ for all $x \geq 0$, and the result now follows from Lemma 8.3.

■

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