ABSTRACT

Title of dissertation: POSITIVE TUPLES OF FLAGS, PIECEWISE CIRCULAR WAVEFRONTS, AND THE 3-DIMENSIONAL EINSTEIN UNIVERSE

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Fock and Goncharov defined the notion of positive subsets of a complete flag manifold G/B in order to study higher Teichmüller spaces. In this dissertation, we study the finite positive subsets when $G = \mathsf{PSp}(4,\mathbb{R}) \cong \mathsf{SO}^0(3,2)$. The main tool is the fact that the 3-dimensional Einstein universe, or Lie quadric, is one of the parabolic homogeneous spaces of G and it parametrizes oriented circles in the 2-sphere. We interpret complete flags in this setting as pointed oriented circles in the 2-sphere and the action of G as contactomorphisms of the unit tangent bundle of S^2. This leads to an interpretation of positive subsets in G/B in terms of oriented piecewise circular curves in the 2-sphere, or equivalently piecewise linear Legendrian curves in $\mathbb{RP}^3$. We parametrize positive triples of flags by a pair of real-valued cross ratios. We explicitly describe a homeomorphism between the configurations space of positive triples of flags and the moduli space of 6-sided, labeled, positive, oriented piecewise circular wavefronts in S^2.

POSITIVE TUPLES OF FLAGS, PIECEWISE CIRCULAR WAVEFRONTS, AND THE 3-DIMENSIONAL EINSTEIN UNIVERSE

by

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Dedication

To my daughter Nora,

without whom this would have been much easier but far less enjoyable.

To my wife Megan,

who has always been my biggest supporter, motivator, and believer.

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Table of Contents

List of Tables

List of Figures

Chapter 1: Introduction

Let S be a closed connected oriented topological surface of negative Euler characteristic $\chi(S)$. The Teichmüller space $\mathcal{T}(S)$ of S is the space of marked conformal classes of Riemannian metrics on S. By the uniformization theorem, there is a unique hyperbolic metric in each conformal class, therefore $\mathcal{T}(S)$ can be characterized as the space of marked hyperbolic structures on S ([Wie18]). There is a further, well-known characterization of the Teichmüller space. While this characterization of Teichmüller space, stated below, was known prior, one can find a proof of the results in Goldman's thesis ([Gol80]).

Theorem 1.0.1. *The holonomy representation of a closed connected oriented surface S of negative Euler characteristic $\chi(S)$ gives a homeomorphism between the Teichmüller space of S and a particular connected component of*

$$\mathsf{Hom}(\pi_1(S), \mathsf{PSL}(2, \mathbb{R}))/\mathsf{PSL}(2, \mathbb{R}) \tag{1.1}$$

which consists entirely of discrete and faithful representations.

In higher Teichmüller theory, $\mathrm{PSL}(2, \mathbb{R})$ is replaced by a semisimple Lie group G of higher rank. A higher Teichmüller space is a subset of the character variety $\mathrm{Hom}(\pi_1(S), G)/G$ which is a union of connected components consisting entirely of discrete and faithful representations. There are several instances where higher Teichmüller spaces are known to parametrize geometric objects. Goldman [Gol90] and Choi and Goldman ([CG97], [CG05]) showed for $G = \mathsf{PGL}(3, \mathbb{R})$ that the higher Teichmüller space parametrizes convex projective structures on surfaces. Equivalently, the higher Teichmüller space for $G = \mathsf{PGL}(3, \mathbb{R})$ parametrizes closed convex curves in the projective plane which are equivariant with respect to the representation ([FG07]). By Guichard and Wienhard [GW08], the higher Teichmüller space for $G = \mathsf{PSL}(4, \mathbb{R})$ parametrizes convex foliated projective structures on the unit tangent bundle of the surface and for $G = \mathsf{PSp}(4, \mathbb{R})$ parametrizes convex foliated contact projective structures. Collier, Tholozan, and Toulisse [CTT17] show that the higher Teichmüller space for $G = \mathsf{SO}^0(3, 2) \cong \mathsf{PSp}(4, \mathbb{R})$ also parametrizes a class of conformally flat Lorentzian structures on the unit tangent bundle of the surface.

In their highly influential paper [FG06], Fock and Goncharov propose the following paradigm for studying higher Teichmüller spaces of punctured surfaces. Let S be a finite area hyperbolic surface with at least one cusp. The collection of lifts of the cusps gives a dense subset of the boundary of $\mathbb{H}^2$. As a subset of $S^1 \cong \partial\mathbb{H}^2$, the set of lifted cusps admits a natural cyclic ordering. This cyclic ordering is independent of the choice of finite area hyperbolic structure. The study of the Teichmüller space of S then translates to the study of certain cyclic subsets of the circle, up to the action of $\mathsf{PSL}(2, \mathbb{R})$. The complete flag manifold G/B of a real split semisimple Lie group G admits a positive (cyclic) structure which generalizes the cyclic structure of S^1. The positive structure on the complete flag manifold is defined using Lusztig's notion of total positivity for reductive groups [Lus94]. In this way, Fock and Goncharov are able to translate the study of higher Teichmüller spaces to the study of positive subsets of complete flag manifolds.

The study of finite positive subsets of the complete flag manifold is already interesting. Fock and Goncharov [FG07] consider this finite analog for $G = \mathsf{PSL}(3, \mathbb{R})$. They show that positive subsets in G/B of size k parametrize pairs of convex k-gons, where one of the convex k-gons is inscribed inside the other in the projective plane, up to projective transformations. They then apply their general theory to define coordinates on this space and show that it is homeomorphic to a ball of dimension $3k - 8$.

In this dissertation, we study the finite positive subset problem for the next simplest rank two Lie group, $\mathsf{PSp}(4, \mathbb{R}) \cong \mathsf{SO}^0(3, 2)$. In this case, the analog of inscribed convex polygons in the projective plane will be positive, oriented piecewise circular curves in the 2-sphere. The unit tangent bundle of S^2 identifies with $\mathbb{RP}^3$ and has a natural contact structure. The lifts of oriented piecewise circular curves under this identification are piecewise linear Legendrian curves in this contact $\mathbb{RP}^3$. We classify equivalence classes of 6-sided positive, oriented piecewise circular curves up to projective contactomorphisms of $\mathbb{RP}^3$.

The geometry of piecewise circular curves is of independent interest ([Arn95], [BG93], [BG94]). In particular, piecewise circular curves have been used in C^1-approximation. Here one approximates smooth curves in such a way that the tangent lines of the piecewise circular curve approximate the tangent lines at points of the smooth curve while also having the piecewise circular curve pointwise approximate the smooth curve ([NM88], [MP84], [RR87]).

The interpretation of positive subsets of G/B, for $G = \mathsf{PSp}(4,\mathbb{R}) \cong \mathsf{SO}^0(3,2)$, in terms of oriented piecewise circular curves in the 2-sphere comes from the fact that the 3-dimensional Einstein universe, or Lie quadric, is one of the parabolic homogeneous spaces of $\mathsf{PSp}(4,\mathbb{R}) \cong \mathsf{SO}^0(3,2)$ and it parametrizes oriented circles in the 2-sphere. With this in mind, we parametrize the totally positive part of the configuration space of triples of flags in the 3-dimensional Einstein universe. We then explicitly describe the correspondence between this space and the space of labeled, positive, oriented piecewise circular hexagons in S^2. We conclude that one can parametrize the space of labeled, positive, oriented piecewise circular hexagons by $\mathbb{R}^2$.

The second chapter describes several models of the 3-dimensional Einstein universe Ein. The first model, the projective model, states that Ein can be thought of as the projectivization of the nullcone of a 5-dimensional vector space $V^{3,2}$ with respect to a symmetric bilinear form of signature $(3,2)$. By choosing a particular diagonal basis for $V^{3,2}$, we give an explicit description of Ein as the moduli space of oriented circles and points in the 2-sphere. This description of the 3-dimensional Einstein universe is referred to as the Lie circles model. See [Cec08] for a generalization to the $(n+1)$-dimensional Einstein universe. A change of basis in the projective model to a particular anti-diagonal basis is used in the third chapter to compute coordinates for the totally positive part of the configuration space of triples of flags. The next model of Ein is the Lagrangian Grassmannian model. Here we use the isomorphism $\mathsf{SO}^0(V^{3,2}) \cong \mathsf{PSp}(4,\mathbb{R})$ to show that points in Ein correspond to Lagrangian planes in a 4-dimensional real symplectic vector space. We refer to [Bur17] for an exposition of Ein as the moduli space of oriented circles and points in the 2-sphere directly from the Lagrangian Grassmannian model. The final model of the 3-dimensional Einstein universe we give is the quotient model which describes Ein as a quotient of $S^2 \times S^1$ by the simultaneous antipodal map.

The $(n+1)$-dimensional Einstein universe, where $n \geq 2$ is defined to be the quotient

$$S^n \times S^1 \big/ \{\pm 1\} \tag{1.2}$$

where scalar multiplication by -1 is the simultaneous antipodal map on both the S^n and the S^1 factors. This quotient is endowed with the metric induced by $ds^2 - d\theta^2$ where ds^2, $d\theta^2$ are the usual spherical Riemannian metrics on S^n and S^1, respectively. This metric descends to a metric on the quotient since the antipodal map is an isometry. The conformal class of that metric defines a conformal Lorentzian structure for the $(n+1)$-dimensional Einstein universe. One reason to consider the $(n+1)$-dimensional Einstein universe is that it is a compactification of Minkowski space respecting the conformal Lorentzian structure ([CK83], [Fra02]). In particular, the referenced papers prove the following Lorentzian analog of Louiville's theorem.

Theorem 1.0.2. *Every conformal diffeomorphism on open sets of the* $(n+1)$*-dimensional Einstein universe is the restriction of a unique element of* $\mathsf{PO}(n+1,2)$.

The terminology "Einstein universe" is due to a relationship with the first cosmological model for the universe, the Einstein Static Universe, that A. Einstein considered soon after the birth of general relativity. The Einstein Static Universe is the universal cover $S^3 \times \mathbb{R}$ of the 4-dimensional Einstein universe we defined endowed with the Lorentz metric $ds^2 - dt^2$ where ds^2 is the usual spherical Riemannian metric ([Bar18], [BEE96], [Ein97]).

A central theme of this dissertation is positivity notions. A matrix in $\mathsf{GL}(n, \mathbb{R})$ is said to be totally positive if each of its minors are positive. The notion of totally positive matrices first appeared in the works of Schoenberg [Sch30] and then Gantmacher and Krein [GK37]. Lusztig ([Lus94]) then generalized the notion of total positivity to real split semisimple Lie groups. Total positivity in this context plays an integral role in representation theory with many noteworthy connections to other mathematical fields such as the study of cluster algebras, linear algebra, and stochastic processes as well as problems in theoretical physics ([And87], [Fom10], [Kar68]). Fock and Goncharov [FG06] use Lusztig's positivity to define a notion of positivity applicable to the study of higher Teichmüller theory. A further generalization of Lusztig's total positivity has been described by Guichard and Wienhard ([GW18]) for other semisimple Lie groups which are not necessarily split. Fock and Goncharov's definition of positivity for tuples of flags in such a Lie group is of particular interest in this dissertation.

In Chapter 3, we begin by describing the complete isotropic flag manifold $\mathscr{F}$ for $\mathsf{SO}(V^{3,2})$. We show that an isotropic flag can be seen as a pointed oriented circle in the Lie circles model of the 3-dimensional Einstein universe. Here the oriented circle corresponds to the 1-dimensional subspace and the unit tangent vector to the oriented circle corresponds to the 2-dimensional subspace. We then explicitly describe a certain positive sub-semigroup $U_+^{>0}$ defined in [Lus94] for the Lie group

$\mathsf{SO}^0(V^{3,2})$. We show that for the choice of anti-diagonal basis in $V^{3,2}$, the positive sub-semigroup is exactly the unipotent upper triangular matrices whose non-trivial minors are strictly positive. We then consider the Fock and Goncharov [FG06] notion of positivity for a tuple of flags in $\mathsf{SO}(V^{3,2})$. For the fixed anti-diagonal basis of $V^{3,2}$, we define coordinates (x, y) for a given element in the totally positive part of the configuration space for triple of flags $\mathsf{Conf}^{(3)}(\mathscr{F})$. We proceed to describe a second type of coordinates (c_1, c_2) of the totally positive part of $\mathsf{Conf}^{(3)}(\mathscr{F})$ in a basis-independent fashion. This basis-independent definition is motivated by the results of Fock and Goncharov [FG07] for $\mathbb{RP}^2$. These coordinates are computed in a way which is similar to their method of taking the cross ratio of four points on a projective line. We then explicitly describe the change of coordinates from one type to the other. The main result in this chapter is a parametrization of the totally positive part of $\mathsf{Conf}^{(3)}(\mathscr{F})$ in the following way.

Theorem 1.0.3. *a) In terms of the coordinates (x, y), the totally positive part of $\mathsf{Conf}^{(3)}(\mathscr{F})$ is given by the quadrant $\{(x, y) \in \mathbb{R}^2 \mid x > 1$ and $y > \sqrt{2}x\}$.*

b) In terms of the coordinates (c_1, c_2), the totally positive part of $\mathsf{Conf}^{(3)}(\mathscr{F})$ is given by the quadrant $\{(c_1, c_2) \in \mathbb{R}^2 \mid 1 < c_1 < c_2\}$.

We conclude the third chapter by analogously describing positivity in $\mathsf{PSp}(4, \mathbb{R})$. The complete isotropic flag manifold for $\mathsf{PSp}(4, \mathbb{R})$ is isomorphic to $\mathscr{F}$. In this context where we use the Lagrangian Grassmanian model of Ein, an isotropic flag can be understood as a pointed oriented circle where the oriented circle corresponds to the 2-dimensional subspace while the unit tangent vector corresponds to the 1-dimensional subspace. Through the isomorphism $\mathsf{PSp}(4,\mathbb{R}) \cong \mathsf{SO}^0(V^{3,2})$, we are able to describe the totally positive part of the configuration space of flags in $\mathsf{PSp}(4,\mathbb{R})$. Understanding this relationship is important as we use the symplectic form of isotropic flags to prove the relationship between positive triples of flags and labeled, positive, oriented piecewise circular hexagons in the 2-sphere which we describe in the final chapter.

In the case of $\mathbb{RP}^2$, Fock and Goncharov [FG07] show the following relationship between tuples of flags in $\mathbb{RP}^2$ and polygons in the projective plane.

Theorem 1.0.4. *For $G = \mathsf{PGL}(3,\mathbb{R})$, the totally positive part of $\mathsf{Conf}^{(k)}(G/B)$ parametrizes pairs of convex k-gons, with one inscribed inside the other in the projective plane, up to the action of $\mathsf{PGL}(3,\mathbb{R})$.*

To do this they use the fact that such flags can be seen as pointed lines in the projective plane. Similarly, we recall that the 3-dimensional Einstein universe Ein, or Lie quadric, parametrizes the moduli space of oriented circles and points in the 2-sphere. This, along with the description of isotropic flags as pointed oriented circles, leads us to define a class of curves in the 2-sphere.

Definition 1.0.5. A *piecewise circular curve* γ in S^2 is a closed curve consisting of finitely many circular arcs with matching tangent lines at the intersections of adjacent arcs. We will refer to the circular pieces as *edges* or *arcs* of the curve and the junctions between adjacent arcs as the *vertices*.

An *orientation* on a piecewise circular curve is a continuous choice of unit vector tangent to γ a each point of γ. Note that if a piecewise circular curve is orientable, then it will necessarily have exactly two orientations.

We show that the Lie group $\mathsf{SO}(V^{3,2})$ acts on the set of oriented piecewise circular curves. It is generated by Möbius transformations, equidistant transformations, and the orientation reversing transformation. These equidistant transformations can be understood as a uniform growth or decrease of the signed radii of all oriented circles. This picture is reminiscent of wavefront propagation in physics. As described by Huygens' principle for wave propagation ([BC39]), the equidistant transformations can be used to express the wavefront propagation both forwards and backwards. For this reason, when we consider oriented piecewise circular curves up to the action of $\mathsf{SO}(V^{3,2})$, we refer to them as oriented *piecewise circular wavefronts*.

As discussed, the unit tangent bundle of the 2-sphere identifies with $\mathbb{RP}^3$ and has a natural contact structure. The lifts of oriented piecewise circular curves in S^2 under this identification are piecewise linear Legendrian curves in this contact $\mathbb{RP}^3$. The action of $\mathsf{SO}(V^{3,2})$ on this class of curves respects the contact structure and therefore acts by contactomorphisms.

By showing that $\mathsf{PSp}^{\pm}(4,\mathbb{R}) \cong \mathsf{SO}(V^{3,2})$ where

$$\mathsf{PSp}^{\pm}(4,\mathbb{R}) = {\{A \in \mathsf{GL}(4,\mathbb{R}) \mid A^t\Omega A = \Omega\} \cup \{A \in \mathsf{GL}(4,\mathbb{R}) \mid A^t\Omega A = -\Omega\}}\big/{\{\pm \mathrm{I}\}}, \tag{1.3}$$

we are able to think of flags in $\mathsf{SO}(V^{3,2})$ as isotropic flags in a 4-dimensional real symplectic vector space.

Given a labeled, oriented piecewise circular curve γ with $2k$ vertices $p_1, \ldots, p_{2k}$, we will associate to γ a k-tuple of flags in $\mathsf{PSp}^{\pm}(4,\mathbb{R}) \cong \mathsf{SO}(V^{3,2})$ as follows. Since

γ is oriented, we have a unit tangent vector for the curve γ based at each vertex p_i. Denote by p_i the points in $\mathbb{RP}^3$ corresponding to the unit tangent vectors at p_i by $\mathbb{RP}^3 \cong T^1(S^2)$. Let P_i be the 1-dimensional subspaces of $V \cong \mathbb{R}^4$ corresponding to $p_i \in \mathbb{RP}^3$.

Definition 1.0.6. For a labeled, oriented piecewise circular curve γ with $2k$ vertices $p_1, \ldots, p_{2k}$, we define the *associated k-tuple of flags* in $\mathsf{PSp}^{\pm}(4, \mathbb{R}) \cong \mathsf{SO}(V^{3,2})$ to be $F_1 = (P_1, \mathrm{span}(P_1, P_2)), F_2 = (P_3, \mathrm{span}(P_3, P_4)), \ldots, F_k = (P_{2k-1}, \mathrm{span}(P_{2k-1}, P_{2k}))$.

We define a notion of positivity of oriented piecewise circular curves which, in the case of a 6-sided curve, will relate to positivity of the corresponding triple of flags. Positivity of oriented piecewise circular curves can be understood as a generalization of convexity of polygons in the projective plane.

Definition 1.0.7. An oriented piecewise circular curve γ in the 2-sphere is *positive* if the set of tangency points between any oriented circle and γ is either a single point or an arc of γ. Here, tangency is understood to imply matching orientations, and we allow zero radius circles as arcs of the curve γ.

With this notion of positivity of oriented piecewise circular curves, we prove the following result.

Theorem 1.0.8. *The induced map*

$$\begin{aligned} \left\{ \begin{matrix} \textit{labeled, positive, oriented} \\ \textit{piecewise circular hexagons} \end{matrix} \right\} \Big/ \mathsf{PSp}^{\pm}(4, \mathbb{R}) &\longrightarrow \mathsf{Conf}^{(3)}_{+}(\mathscr{F}) \\ [\gamma] &\longmapsto [(F_1, F_2, F_3)] \end{aligned} \tag{1.4}$$

where $\mathsf{Conf}^{(3)}_{+}(\mathscr{F})$ *is the totally positive part of* $\mathsf{Conf}^{(3)}(\mathscr{F})$ *is a bijection.*

Combined with Theorem 1.0.3, we have the following corollary.

Corollary 1.0.9. *The space of labeled, positive, oriented piecewise circular hexagons in the 2-sphere up to Möbius transformations, equidistant transformations, and the orientation reversing transformation is parametrized by* $\mathbb{R}^2$.

We believe that it is possible to develop equations to describe gluing positive oriented piecewise circular hexagons together to get a $2k$-gon which will correspond to gluing equations for k-tuples of flags. With this in mind, we make the following conjecture which will be the topic of future work.

Conjecture 1.0.10. *a) There are geometrically defined invariants giving coordinates on the totally positive part of* $\mathsf{Conf}^{(k)}(\mathscr{F})$ *which parametrizes the totally positive part of* $\mathsf{Conf}^{(k)}(\mathscr{F})$ *by* $\mathbb{R}^{4k-10}$.

b) The space of labeled, positive, oriented piecewise circular $2k$*-gons in the 2-sphere up to Möbius transformations, equidistant transformations, and the orientation reversing transformation is parametrized by* $\mathbb{R}^{4k-10}$.

Additionally, we wish to study the case of infinite positive sets of isotropic flags. In order to do this we will generalize the notion of positivity to piecewise C^2 curves in the 2-sphere. These curves should be the limit curves of positive representations into $\mathsf{SO}(3,2)$. Here the limit curve of a positive representation $\rho : \pi_1(S) \to \mathsf{SO}(3,2)$ can be understood as a map $f : S^1 \cong \partial\mathbb{H}^2 \to \mathscr{F}$ which is ρ-equivariant, i.e. for every $g \in \pi_1(S)$, we have $f(gp) = \rho(g)f(p)$. In this way we wish to provide a characterization of the higher Teichmüller space for $\mathsf{SO}(3,2)$.

Chapter 2: The Einstein universe

In this chapter, we describe several models of the 3-dimensional Einstein universe. First, we describe the projective model of the Einstein universe. The projective model will be used in Chapter 3 when defining each of our coordinates for positive triples of flags in $\mathsf{SO}(V^{3,2})$.

Next we describe the Lie circles model of the Einstein universe. We first relate the Lie circles model to the projective model in a coordinate-independent fashion. We then fix a basis in order to more fully describe the Lie circles model. The Lie circles model gives an accessible way to visualize the Einstein universe. We will use the Lie circles model in Chapter 4 to relate positive triples of isotropic flags to labeled, positive, oriented piecewise circular hexagons.

We then describe the Lagrangian Grassmannian model of the Einstein universe. The Lagrangian Grassmannian model is used to describe positive triples of flags in $\mathsf{PSp}(4, \mathbb{R})$ and relate these to positive triples of flags in $\mathsf{SO}(V^{3,2})$. The Lagrangian Grassmannian model is then used again in Chapter 4 to relate positive triples of isotropic flags to 6-sided labeled, positive, oriented piecewise circular wavefronts.

We finish this chapter by describing the quotient model of the Einstein universe. Although we don't utilize the quotient model in later chapters, we provide it as a further model of the Einstein universe.

2.1 The projective model

In this section, we describe the projective model of the 3-dimensional Einstein universe.

Let $V^{3,2}$ be a 5-dimensional real vector space with a given symmetric bilinear form $\langle , \rangle$ of signature $(3,2)$. The bilinear form $\langle , \rangle$ separates vectors in $V^{3,2}$ into three classes. Namely, a vector $\mathbf{v} \in V^{3,2}$ is said to be *lightlike* or *null* whenever $\langle \mathbf{v}, \mathbf{v} \rangle = 0$, *timelike* whenever $\langle \mathbf{v}, \mathbf{v} \rangle < 0$, and *spacelike* whenever $\langle \mathbf{v}, \mathbf{v} \rangle > 0$.

We call the set of all lightlike vectors the *nullcone* of $V^{3,2}$, denoted $\mathscr{C}$. For any subset $U \subseteq V^{3,2}$, the intersection of U with the nullcone will be denoted $\mathscr{C}(U) := \mathscr{C} \cap U$.

For any subset S of the vector space $V^{3,2}$, we define the corresponding subset of the projectivization $\mathbb{P}(V^{3,2})$ of $V^{3,2}$ to be

$$\mathbb{P}(S) := \{[s] \in \mathbb{P}(V^{3,2}) \mid s \in S \backslash \{\mathbf{0}\}\}. \tag{2.1}$$

We refer to this set as the *projectivization* of S.

Definition 2.1.1. The *Einstein universe* is defined by

$$\mathsf{Ein} := \left\{ [\mathbf{v}] \in \mathbb{P}(V^{3,2}) \mid \langle \mathbf{v}, \mathbf{v} \rangle = 0 \right\}. \tag{2.2}$$

In other words, $\mathsf{Ein} = \mathbb{P}\mathscr{C}$, the projective nullcone. Note that Ein as defined is a 3-dimensional submanifold of $\mathbb{P}(V^{3,2})$.

The special orthogonal group $\mathsf{SO}(V^{3,2})$ is the group of automorphisms of Ein.

Definition 2.1.2. For a linear subspace $U \subseteq V^{3,2}$ the *orthogonal complement* $U^{\perp}$ of U is the defined to be the set

$$U^{\perp} := \{\mathbf{v} \in V^{3,2} \mid \langle \mathbf{v}, \mathbf{u} \rangle = 0 \text{ for all } \mathbf{u} \in U\}. \tag{2.3}$$

A linear subspace $U \subseteq V^{3,2}$ is said to be *isotropic* if $U \subseteq U^{\perp}$.

Note that the existence of isotropic subspaces in guaranteed as our symmetric bilinear form is neither positive definite nor negative definite.

Lemma 2.1.3. *If $U \subseteq V^{3,2}$ is isotropic, then $\mathsf{dim}(U) \leq 2$.*

Proof. Suppose instead that $\mathsf{dim}(U) > 2$. By definition of $V^{3,2}$, there exists a subspace $V^{3,0}$ such that $\langle \mathbf{v}, \mathbf{v} \rangle > 0$ for all nonzero $\mathbf{v} \in V^{3,0}$ and $\mathsf{dim}(V^{3,0}) = 3$. Since $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ for all $\mathbf{u} \in U$, it must follow that $V^{3,0} \cap U = \{\mathbf{0}\}$. This would imply that

$$\mathsf{dim}(V^{3,0} + U) = \mathsf{dim}(V^{3,0}) + \mathsf{dim}(U) = 3 + \mathsf{dim}(U) > 5, \tag{2.4}$$

giving a contradiction. $\square$

Definition 2.1.4. A *photon* in Ein is the image under projectivization of an isotropic 2-plane in $V^{3,2}$.

Definition 2.1.5. The group $\mathsf{SO}(V^{3,2})$ acts transitively on photons. Thus, we can regard the space of all photons as a homogeneous space of $\mathsf{SO}(V^{3,2})$. We will refer to this homogeneous space as the *photon space* and denote it by Pho.

There is a natural incidence relation on photons and points in Ein defined by the following. Two points $p, q \in \mathsf{Ein}$ are said to be *incident* if there exists a photon

$\varphi \in \mathsf{Pho}$ such that $p, q \in \varphi$. Equivalently, the isotropic lines in $V^{3,2}$ corresponding to p and q are orthogonal with respect to $\langle , \rangle$. Next, we say that a point is *incident* to a photon if the photon contains the point. Lastly, two photons are said to be *incident* if they intersect in a point.

Lemma 2.1.6. *For $p \in \mathsf{Ein}$ and $\varphi \in \mathsf{Pho}$ such that p is not incident to φ, there exists a unique point q that is incident to both p and φ. Furthermore, there exists a unique photon ψ that is incident to both p and φ.*

Proof. Suppose that $p = [\mathbf{u}]$ and $\varphi = \mathbb{P}(\mathrm{span}(\mathbf{v}, \mathbf{w}))$. Consider the point $q = [\langle \mathbf{w}, \mathbf{u}\rangle \mathbf{v} - \langle \mathbf{v}, \mathbf{u}\rangle \mathbf{w}] \in \varphi$. Since

$$\langle \langle \mathbf{w}, \mathbf{u}\rangle \mathbf{v} - \langle \mathbf{v}, \mathbf{u}\rangle \mathbf{w}, \mathbf{u}\rangle = 0, \tag{2.5}$$

it follows that q is incident to p, and their common photon is

$$\psi = \mathbb{P}(\mathrm{span}(\mathbf{u}, \langle \mathbf{w}, \mathbf{u}\rangle \mathbf{v} - \langle \mathbf{v}, \mathbf{u}\rangle \mathbf{w})). \tag{2.6}$$

Suppose there exists another point q' which is also incident to both p and φ. Since this would imply that p, q, and q' are distinct and pairwise incident, these points would define an isotropic 3-dimensional subspace in $V^{3,2}$, a contradiction.

To finish the proof of the lemma, consider the photon defined by p and q, $\psi = \mathbb{P}(\mathrm{span}(\mathbf{u}, \langle \mathbf{w}, \mathbf{u}\rangle \mathbf{v} - \langle \mathbf{v}, \mathbf{u}\rangle \mathbf{w}))$. It is clear that ψ is incident to both p and φ. Moreover, the uniqueness of q guarantees the uniqueness of ψ. $\square$

Definition 2.1.7. The *lightcone*, $\mathcal{L}([\mathbf{u}])$, of a point $p = [\mathbf{u}] \in \mathsf{Ein}$ is the set

$$\mathcal{L}([\mathbf{u}]) = \left\{ [\mathbf{v}] \in \mathsf{Ein} \mid \langle \mathbf{v}, \mathbf{u}\rangle = 0 \right\} \tag{2.7}$$

The lightcone $\mathcal{L}([\mathbf{u}])$ is the set of all points in Ein incident to $[\mathbf{u}]$. Equivalently, it is the union of all photons containing $[\mathbf{u}]$. Also note that $\mathcal{L}([\mathbf{u}]) = \mathbb{P}\mathscr{C}(\mathbf{u}^\perp)$.

Definition 2.1.8. A *sphere in* $\mathbb{P}(V^{3,2})$ is the projectivization of the nullcone of a signature $(3,1)$ subspace in $V^{3,2}$.

Definition 2.1.9. A *circle* of nonzero radius in a sphere in $\mathbb{P}(V^{3,2})$ is the projectivization of the nullcone of a signature $(2,1)$ subspace of the signature $(3,1)$ subspace defining the sphere.

We will see in Lemmas 2.2.12 and 2.2.13 that these definitions correspond topologically to a sphere and a circle in that sphere, respectively.

2.2 The Lie circles model

In this section, we describe the Einstein universe as the moduli space of oriented circles in the 2-sphere. See for instance [Cec08] for further details and a generalization to higher dimensions. We will first approach this model in a basis-independent fashion.

Many of our arguments will rely on discussions of signatures of orthogonal subspaces so we will first prove two results that will be helpful. The convention for notation that we use when discussing signature is $(+,-,0)$. In other words, for signature (p,q,r), we can find an orthogonal basis consisting of p spacelike vectors, q timelike vectors, and r lightlike vectors. When $r=0$, we will just write (p,q) for the signature.

Lemma 2.2.1. *Let V be a vector space with a symmetric bilinear form $\langle , \rangle$ of signature (p, q). If $U \subseteq V$ is a subspace such that the restriction of $\langle , \rangle$ to U is nondegenerate of signature (p', q'), then $\langle , \rangle$ restricted to $U^\perp$ is nondegenerate of signature $(p - p', q - q')$.*

Proof. First we show that for any subspace W,

$$\dim(W) + \dim(W^\perp) = \dim(V). \tag{2.8}$$

Consider the linear map

$$T : V \longrightarrow V^* \longrightarrow W^*, \qquad \mathbf{v} \longmapsto \langle -, \mathbf{v} \rangle \longmapsto \langle -, \mathbf{v} \rangle \tag{2.9}$$

where the first map is the isomorphism between a vector space and its dual space given by $\langle , \rangle$ and the second map is the surjection induced by the inclusion of W into V. Now the linear function in W^* is the zero map if and only if $\mathbf{v} \in W^\perp$, i.e. Ker $T = W^\perp$. By the rank-nullity theorem, we get

$$\begin{aligned} \dim(V) &= \dim(\text{Ker } T) + \dim(\text{Im } T) \\ &= \dim(W^\perp) + \dim(W^*) \\ &= \dim(W^\perp) + \dim(W). \end{aligned} \tag{2.10}$$

Now since the restriction of $\langle , \rangle$ to U is nondegenerate, we must get that $U \cap U^\perp = \{\mathbf{0}\}$. With this and the fact that $\dim(V) = \dim(U) + \dim(U^\perp)$, we get that

$$V = U \oplus U^\perp. \tag{2.11}$$

As signature is naturally additive over direct sums, it must follow that the restriction of $\langle , \rangle$ to $U^\perp$ must be nondegenerate of signature $(p - p', q - q')$. $\square$

Lemma 2.2.2. *Let V be a vector space with a symmetric bilinear form $\langle , \rangle$ of signature (p, q). If $U \subseteq V$ is a subspace such that the restriction of $\langle , \rangle$ to U has signature (p', q', r'), then the restriction of $\langle , \rangle$ to $U^{\perp}$ has signature*

$(p - p' - r', q - q' - r', r')$.

Proof. Take $\mathbf{e}_1, \ldots, \mathbf{e}_{p'}, \mathbf{f}_1, \ldots, \mathbf{f}_{q'}, \mathbf{g}_1, \ldots, \mathbf{g}_{r'}$ to be a basis of U which diagonalizes the restriction of $\langle , \rangle$ to U. So

$$\begin{gathered} \langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij}, \qquad \langle \mathbf{f}_i, \mathbf{f}_j \rangle = -\delta_{ij}, \qquad \langle \mathbf{g}_i, \mathbf{g}_j \rangle = 0, \\ \langle \mathbf{e}_i, \mathbf{f}_j \rangle = \langle \mathbf{e}_i, \mathbf{g}_j \rangle = \langle \mathbf{f}_i, \mathbf{g}_j \rangle = 0 \qquad \text{for all } i, j. \end{gathered} \tag{2.12}$$

Now we will complete this basis to a basis of V. First, for each $\mathbf{g}_i$, find a vector ${\mathbf{g}'}_i$ such that $\langle \mathbf{g}_i, {\mathbf{g}'}_i \rangle = 1$ and ${\mathbf{g}'}_i$ is orthogonal to each of the other basis vectors. At this point we have defined a new subspace $U' = \operatorname{span}(U, {\mathbf{g}'}_1, \ldots, {\mathbf{g}'}_r) \supseteq U$. By construction $\dim(U') = p' + q' + 2r'$ and $\langle , \rangle$ restricted to U' will be nondegenerate of signature $(p' + r', q' + r')$. Since the restriction of $\langle , \rangle$ to U' is nondegenerate, it follows that the signature of $\langle , \rangle$ restricted to $(U')^{\perp}$ is $(p - p' - r', q - q' - r')$ by Lemma 2.2.1.

Complete the extension of the basis to a basis of V with a diagonal orthogonal basis. In this basis, it is clear that $U^{\perp} \supseteq \operatorname{span}(\mathbf{g}_1, \ldots, \mathbf{g}_{r'}) \oplus (U')^{\perp}$. Note that

$$\dim(U^{\perp}) = p + q - \dim(U) = p + q - p' - q' - r', \tag{2.13}$$

and similarly we get that

$$\begin{aligned} \dim(\operatorname{span}(\mathbf{g}_1, \ldots, \mathbf{g}_{r'}) \oplus (U')^{\perp}) &= \dim(\operatorname{span}(\mathbf{g}_1, \ldots, \mathbf{g}_{r'})) + \dim((U')^{\perp}) \\ &= r + p + q - \dim(U') \\ &= p + q - p' - q' - r'. \end{aligned} \tag{2.14}$$

So we necessarily get that

$$U^{\perp} = \mathrm{span}(\mathbf{g}_1, \ldots, \mathbf{g}_{r'}) \oplus (U')^{\perp} \tag{2.15}$$

which has signature $(p - p' - r', q - q' - r', r')$. $\square$

Lemma 2.2.3. *For a fixed timelike vector* $\mathbf{v} \in V^{3,2}$, $\mathbb{P}\mathscr{C}(\mathbf{v}^{\perp})$ *is a sphere in* $\mathbb{P}(V^{3,2})$.

Proof. Let $\mathbf{v} \in V^{3,2}$ such that $\langle \mathbf{v}, \mathbf{v} \rangle = -1$. The restriction of the bilinear form $\langle , \rangle$ to $\mathbf{v}^{\perp}$ must have signature $(3,1)$. Therefore, the projectivized nullcone $\mathbb{P}\mathscr{C}(\mathbf{v}^{\perp})$ is naturally a sphere in $\mathbb{P}(V^{3,2})$. $\square$

Let us now choose $\mathbf{v}_0$ to be a fixed vector in $V^{3,2}$ such that $\langle \mathbf{v}_0, \mathbf{v}_0 \rangle = -1$. This fixes a copy of a sphere in $\mathbb{P}(V^{3,2})$ which is contained in Ein, namely $\mathbb{P}\mathscr{C}(\mathbf{v}_0^{\perp})$.

Lemma 2.2.4. *For distinct points* $[\mathbf{u}], [\mathbf{v}] \in \mathsf{Ein}$, *the corresponding lightcones are distinct, i.e.* $\mathcal{L}([\mathbf{u}]) \neq \mathcal{L}([\mathbf{v}])$.

Proof. Suppose instead that $[\mathbf{u}], [\mathbf{v}] \in \mathsf{Ein}$ are distinct, but $\mathcal{L}([\mathbf{u}]) = \mathcal{L}([\mathbf{v}])$. Note that this is equivalent to saying $\mathscr{C}(\mathbf{u}^{\perp}) = \mathscr{C}(\mathbf{v}^{\perp})$. Therefore, we know that $\mathbf{u}, \mathbf{v} \in \mathbf{u}^{\perp}$. By Lemma 2.2.2, we have that $\langle , \rangle$ restricted to $\mathbf{u}^{\perp}$ has signature $(2,1,1)$. We can choose a basis $\mathbf{f}_1, \ldots, \mathbf{f}_4$ of $\mathbf{u}^{\perp}$ with Gram matrix C given by

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & -1 & 0 & -1 \\ 0 & -1 & -1 & 0 \end{pmatrix} \tag{2.16}$$

where $\langle \mathbf{f}_i, \mathbf{f}_j \rangle = C_{ij}$ since this has signature $(2,1,1)$. Therefore, one can find a basis of $\mathbf{u}^{\perp}$ consisting of four lightlike vectors. Since $\mathbf{u}, \mathbf{v} \in \mathscr{C}(\mathbf{u}^{\perp})$, complete a basis $\mathbf{u}, \mathbf{v}, \mathbf{w}_1, \mathbf{w}_2$ of $\mathbf{u}^\perp$ which are all lightlike vectors and linearly independent. Then we necessarily have that $\mathbf{u}, \mathbf{v}, \mathbf{w}_1, \mathbf{w}_2 \in \mathscr{C}(\mathbf{u}^\perp) = \mathscr{C}(\mathbf{v}^\perp)$. Then we have that $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{w}_1 \rangle = \langle \mathbf{v}, \mathbf{w}_1 \rangle = 0$ and therefore $\operatorname{span}(\mathbf{u}, \mathbf{v}, \mathbf{w}_1)$ is isotropic. However, $\mathbf{u}, \mathbf{v}, \mathbf{w}_1$ are linearly independent so we have that $\dim(\operatorname{span}(\mathbf{u}, \mathbf{v}, \mathbf{w}_1)) = 3 > 2$, a contradiction since the maximal dimension of an isotropic subspace is 2. □

For our construction of this model of Ein, we will consider the corresponding lightcone $\mathcal{L}([\mathbf{u}])$ for each $[\mathbf{u}] \in \mathsf{Ein}$ rather than the point itself, as the lightcone $\mathcal{L}([\mathbf{u}])$ is unique to the point $[\mathbf{u}]$ by Lemma 2.2.4. In particular, we will observe how lightcones intersect with $\mathbb{P}\mathscr{C}(\mathbf{v}_0^\perp)$ for the fixed timelike vector $\mathbf{v}_0$.

Lemma 2.2.5. *For any point $[\mathbf{u}] \in \mathbb{P}\mathscr{C}(\mathbf{v}_0^\perp)$, $\mathcal{L}([\mathbf{u}]) \cap \mathbb{P}\mathscr{C}(\mathbf{v}_0^\perp)$ corresponds to a single point in S^2.*

Proof. Note that $[\mathbf{u}] \in \mathcal{L}([\mathbf{u}]) \cap \mathbb{P}\mathscr{C}(\mathbf{v}_0^\perp)$ therefore the intersection is non-empty. Now suppose there was another vector $\mathbf{w}$ such that $[\mathbf{w}] \in \mathcal{L}([\mathbf{u}]) \cap \mathbb{P}\mathscr{C}(\mathbf{v}_0^\perp)$. Since $\mathbf{u}, \mathbf{w} \in \mathbf{v}_0^\perp$, we get that $\operatorname{span}(\mathbf{u}, \mathbf{w}) \subseteq \mathbf{v}_0^\perp$. Furthermore, the subspace $\operatorname{span}(\mathbf{u}, \mathbf{w})$ is isotropic because $[\mathbf{w}] \in \mathcal{L}([\mathbf{u}])$. Now, the bilinear form $\langle , \rangle$ restricted to $\mathbf{v}_0^\perp$ must have signature $(3, 1)$ so any subspace of $\mathbf{v}_0^\perp$ must have dimension at most 1. It follows that $\mathbf{w}$ must be a multiple of $\mathbf{u}$ so $[\mathbf{u}] = [\mathbf{w}] \in \mathsf{Ein}$. We conclude that $\{[\mathbf{u}]\} = \mathcal{L}([\mathbf{u}]) \cap \mathbb{P}\mathscr{C}(\mathbf{v}_0^\perp) \subseteq \mathbb{P}\mathscr{C}(\mathbf{v}_0^\perp) \cong S^2$, a single point in S^2. □

Lemma 2.2.6. *For any point $[\mathbf{u}] \in \mathsf{Ein}$, $\mathcal{L}([\mathbf{u}]) \cap \mathbb{P}\mathscr{C}(\mathbf{v}_0^\perp)$ corresponds to a circle, possibly of zero radius, in S^2.*

Proof. Recall that in the 2-sphere given by $\mathbb{P}\mathscr{C}(\mathbf{v}_0^\perp)$, a circle of nonzero radius will be given by the projectivization of the nullcone of a subspace of $\mathbf{v}_0^\perp$ of signature $(2,1)$.

For $[\mathbf{u}] \in \mathsf{Ein}$, we are interested in describing

$$\mathcal{L}([\mathbf{u}]) \cap \mathbb{P}\mathscr{C}(\mathbf{v}_0^\perp) = \mathbb{P}\mathscr{C}(\mathbf{u}^\perp \cap \mathbf{v}_0^\perp) = \mathbb{P}\mathscr{C}(\mathrm{span}(\mathbf{u}, \mathbf{v}_0)^\perp). \tag{2.17}$$

Since $\mathbf{u}$ is lightlike and $\mathbf{v}_0$ is timelike, the bilinear form $\langle , \rangle$ restricted to $\mathrm{span}(\mathbf{u}, \mathbf{v}_0)$ will either be nondegenerate of signature $(1,1)$ or degenerate of signature $(0,1,1)$.

Let us first consider the case where $\langle , \rangle$ restricted to $\mathrm{span}(\mathbf{u}, \mathbf{v}_0)$ is nondegenerate of signature $(1,1)$. In this case, $\mathbf{u} \notin \mathrm{span}(\mathbf{u}, \mathbf{v}_0)^\perp$ and $\mathbf{v}_0 \notin \mathrm{span}(\mathbf{u}, \mathbf{v}_0)^\perp$. Also if there exists $\alpha, \beta \in \mathbb{R}$ such that $\alpha\mathbf{u} + \beta\mathbf{v}_0 \in \mathrm{span}(\mathbf{u}, \mathbf{v}_0)^\perp$, then we get that $0 = \langle \alpha\mathbf{u} + \beta\mathbf{v}_0, \mathbf{u}\rangle = \langle \beta\mathbf{v}_0, \mathbf{u}\rangle$. This implies that $\beta = 0$ since $\langle \mathbf{u}, \mathbf{v}_0\rangle \neq 0$. Similarly, $0 = \langle \alpha\mathbf{u}, \mathbf{v}_0\rangle$ implies that $\alpha = 0$. So we see that

$$\mathrm{span}(\mathbf{u}, \mathbf{v}_0) \cap \mathrm{span}(\mathbf{u}, \mathbf{v}_0)^\perp = \{\mathbf{0}\} \tag{2.18}$$

while $\dim(\mathrm{span}(\mathbf{u}, \mathbf{v}_0)) = 2$ and $\dim(\mathrm{span}(\mathbf{u}, \mathbf{v}_0)^\perp) = 3$. Therefore, we get that

$$V^{3,2} = \mathrm{span}(\mathbf{u}, \mathbf{v}_0) \oplus \mathrm{span}(\mathbf{u}, \mathbf{v}_0)^\perp. \tag{2.19}$$

Since signature is naturally additive over orthogonal direct sums, we get that the restriction of $\langle , \rangle$ to $\mathrm{span}(\mathbf{u}, \mathbf{v}_0)^\perp$ must be of signature $(2,1)$. This means that $\mathcal{L}([\mathbf{u}]) \cap \mathbb{P}\mathscr{C}(\mathbf{v}_0^\perp)$ corresponds to a circle $C([\mathbf{u}])$.

Next we will consider the case where $\langle , \rangle$ restricted to $\mathrm{span}(\mathbf{u}, \mathbf{v}_0)$ is degenerate of signature $(0,1,1)$. Here we necessarily have that $\mathbf{u} \in \mathbf{v}_0^\perp$. Thus, $[\mathbf{u}] \in \mathbb{P}\mathscr{C}(\mathbf{v}_0^\perp)$. By Lemma 2.2.5, we get that $\mathcal{L}([\mathbf{u}]) \cap \mathbb{P}\mathscr{C}(\mathbf{v}_0^\perp)$ corresponds to a single point in S^2, i.e. a circle of zero radius $C([\mathbf{u}])$. □

For the case above where the restriction of $\langle , \rangle$ to $\text{span}(\mathbf{u}, \mathbf{v}_0)$ is of signature $(1, 1)$, there will be exactly one other point $q \in \mathsf{Ein}$ such that $C([\mathbf{u}]) = C(q)$. Suppose $[\mathbf{w}] \in \mathsf{Ein}$ is another such point such that $C([\mathbf{u}]) = C([\mathbf{w}])$. First we fix the representative $\mathbf{u}$ for the point $[\mathbf{u}]$. Then $\text{span}(\mathbf{u}, \mathbf{v}_0)^{\perp} = \text{span}(\mathbf{w}, \mathbf{v}_0)^{\perp}$. By the arguments in the proof, we must also have that

$$V^{3,2} = \text{span}(\mathbf{u}, \mathbf{v}_0) \oplus \text{span}(\mathbf{u}, \mathbf{v}_0)^{\perp} = \text{span}(\mathbf{w}, \mathbf{v}_0) \oplus \text{span}(\mathbf{w}, \mathbf{v}_0)^{\perp}. \tag{2.20}$$

This tells us that $\text{span}(\mathbf{u}, \mathbf{v}_0) = \text{span}(\mathbf{w}, \mathbf{v}_0)$. So there must exist $\alpha, \beta \in \mathbb{R}$ such that $\mathbf{w} = \alpha\mathbf{u} + \beta\mathbf{v}_0$. Since $\mathbf{w}$ is lightlike, we get

$$\begin{aligned} 0 &= \langle \alpha\mathbf{u} + \beta\mathbf{v}_0, \alpha\mathbf{u} + \beta\mathbf{v}_0 \rangle \\ &= \alpha^2 \langle \mathbf{u}, \mathbf{u} \rangle + 2\alpha\beta \langle \mathbf{u}, \mathbf{v}_0 \rangle + \beta^2 \langle \mathbf{v}_0, \mathbf{v}_0 \rangle \\ &= 2\alpha\beta \langle \mathbf{u}, \mathbf{v}_0 \rangle - \beta^2. \end{aligned} \tag{2.21}$$

Now $\beta \neq 0$ otherwise we would have $[\mathbf{u}] = [\mathbf{w}]$. Therefore this simplifies to $\beta = 2\alpha \langle \mathbf{u}, \mathbf{v}_0 \rangle$ giving that $\mathbf{w} = \alpha\mathbf{u} + 2\alpha \langle \mathbf{u}, \mathbf{v}_0 \rangle \mathbf{v}_0$. Since $\mathbf{u}$ and $\mathbf{v}_0$ have been fixed, this gives a 1-dimensional subspace in $V^{3,2}$, hence a single point in Ein.

Remark. This leads us to define an orientation for $C([\mathbf{u}])$.

First we will normalize $\mathbf{u}$ such that $\langle \mathbf{u}, \mathbf{v}_0 \rangle = 1$. Note that this normalization is unique since we have fixed $\mathbf{v}_0$. Next we fix a timelike vector $\mathbf{v}_1 \in \mathbf{v}_0^{\perp} \cong V^{3,1}$ such that $\langle \mathbf{v}_1, \mathbf{v}_1 \rangle = -1$.

Definition 2.2.7. We define a *time orientation* on $\mathbf{v}_0^{\perp}$ by defining timelike and lightlike vectors $\mathbf{v} \in \mathbf{v}_0^{\perp}$ to be *future-pointing* if $\langle \mathbf{v}_1, \mathbf{v} \rangle < 0$.

Definition 2.2.8. We define an *orientation* on $C([\mathbf{u}])$ by associating to this circle the projectivization of the set of lightlike, future-pointing vectors whose product with $\mathbf{u}$ is positive. Equivalently, we are defining an orientation on $C([\mathbf{u}])$ by associating to this circle the "inside" portion of the 2-sphere

$$D([\mathbf{u}]) = \mathbb{P}(\{\mathbf{v} \in \mathscr{C}(\mathbf{v}_0^{\perp}) \mid \langle \mathbf{v}_1, \mathbf{v} \rangle < 0 \text{ and } \langle \mathbf{u}, \mathbf{v} \rangle \geq 0\}).$$

After normalization, the lightlike vector giving the same circle without orientation considered will be $\mathbf{w} = -\mathbf{u} - 2\mathbf{v}_0$. It is clear from the definition of orientation that $C([-\mathbf{u} - 2\mathbf{v}_0])$ will give the same circle as $C([\mathbf{u}])$ with the opposite orientation.

For the second case, where $\langle , \rangle$ restricted to $\mathrm{span}(\mathbf{u}, \mathbf{v}_0)$ is degenerate of signature $(0, 1, 1)$, $C([\mathbf{u}])$ will be unique to $[\mathbf{u}]$ by Lemma 2.2.5.

The set of oriented circles in S^2 has the topology of a 3-dimensional manifold where the local charts are as follows. For any point $p_0 \in S^2$, the stereographic projection based at p_0 identifies $S^2 \backslash \{p_0\}$ with $\mathbb{R}^2$. Every circle in the 2-sphere which does not go through p_0 will be a euclidean circle in $\mathbb{R}^2$ under stereographic projection. Associate to each such circle the local coordinates (x_0, y_0, r) where (x_0, y_0) is the center of the circle in the plane and r is the signed radius. We choose that a positive radius is equivalent to p_0 being "outside" the circle and a negative radius is equivalent to p_0 being "inside" the circle.

Definition 2.2.9. The *moduli space of oriented circles in* S^2 is the set of oriented circles in S^2 endowed with this topology.

Proposition 2.2.10. *There is a bijection between* Ein *and the moduli space of oriented circles, of possibly zero radius, in* S^2 *given by* $[\mathbf{u}] \mapsto C([\mathbf{u}])$. *In the case where* *the radius of a circle is zero, there is no orientation.*

Proof. From Lemma 2.2.6 and the discussion on orientation that followed, we know that there is a copy of Ein that lies in the set of oriented circles in the 2-sphere. What remains to be shown is that all such circles in S^2 can arise from points in Ein.

First, an unoriented circle with nonzero radius corresponds to a subspace of $U \subseteq \mathbf{v}_0^\perp \cong V^{3,1}$ with the restriction of $\langle , \rangle$ to U having signature $(2,1)$. As discussed in the proof of Lemma 2.2.1, $\mathbf{v}_0^\perp = U \oplus U^\perp$ and the signature of $\langle , \rangle$ restricted to $U^\perp$ will have signature $(1,0)$. This tells us that there is a one-to-one correspondence between unoriented circles in S^2 and spacelike unit vectors in $V^{3,1}$. Recall that $\mathsf{SO}(V^{3,2})$ is the group of automorphisms of Ein. The subgroup $\mathsf{SO}(V^{3,1})$ of $\mathsf{SO}(V^{3,2})$ acting on $\mathbf{v}_0^\perp \cong V^{3,1}$ acts transitively on spacelike unit vectors and thus acts transitively on unoriented circles in S^2. From our discussion on orientation, we know that if we have an orientation for a circle realized from a point in Ein, we can always find a point in Ein giving the opposite orientation. Therefore, we can conclude that all oriented circles of nonzero radius in S^2 correspond to points in Ein.

Furthermore, $\mathsf{SO}(V^{3,1})$ acts transitively on the points in the projectivized null-cone of $V^{3,1} \cong \mathbf{v}_0^\perp$. These points are exactly the zero radius circles in S^2. Therefore, we get that $\mathsf{SO}(V^{3,1})$ acts transitively on zero radius circles. Thus we conclude that all zero radius circles can be realized as points in Ein. Again, there is no orientation given to circles of zero radius.

Finally, we can conclude there is a bijection between Ein and the moduli space of oriented circles, possibly with zero radius, in S^2. $\square$

In Definition 2.2.9, we define a natural topology on the set of oriented circles in the 2-sphere. In Proposition 2.2.16 we will show that the bijection described above is, in fact, a homeomorphism.

Lemma 2.2.11. *Let* $[\mathbf{u}_1], [\mathbf{u}_2] \in \mathsf{Ein}$*. If* $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = 0$*, then* $C([\mathbf{u}_1])$ *and* $C([\mathbf{u}_2])$ *must be tangent as unoriented circles.*

Proof. We ignore the case where two circles of nonzero radius coincide as the statement for $[\mathbf{u}_1] = [\mathbf{u}_2]$ is trivial, and after normalizing $\mathbf{u}_1$ and $\mathbf{u}_2$ where $C([\mathbf{u}_1])$ coincides with $C([\mathbf{u}_2])$ with opposite orientation, we see that

$$\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = \langle \mathbf{u}_1, -\mathbf{u}_1 - 2\mathbf{v}_0 \rangle = -2 \neq 0 \tag{2.22}$$

so the hypothesis cannot hold. Therefore, any two circles we would consider would intersect at 0, 1, or 2 points. Two circles would then be tangent if and only if they intersect at exactly one point. This still holds for when one of the circles has zero radius, as we say that it is tangent to a circle if the point it defines lies on the circle, i.e. they intersect at one point.

If $\mathbf{u}_1, \mathbf{u}_2$ are linearly independent lightlike vectors such that $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = 0$, then $\operatorname{span}(\mathbf{u}_1, \mathbf{u}_2)$ is an isotropic plane in $V^{3,2}$, i.e. a subspace where the restriction of $\langle , \rangle$ is of signature $(0, 0, 2)$. Now $C([\mathbf{u}_i])$ as an unoriented circle is

$$\mathbb{P}\mathscr{C}(\mathbf{u}_i^{\perp} \cap \mathbf{v}_0^{\perp}). \tag{2.23}$$

Therefore, the intersection of the two circles is

$$\mathbb{P}\mathscr{C}(\mathbf{u}_1^{\perp} \cap \mathbf{u}_2^{\perp} \cap \mathbf{v}_0^{\perp}) = \mathbb{P}\mathscr{C}(\operatorname{span}(\mathbf{u}_1, \mathbf{u}_2)^{\perp} \cap \mathbf{v}_0^{\perp}). \tag{2.24}$$

Since $\langle , \rangle$ restricted to $\mathrm{span}(\mathbf{u}_1, \mathbf{u}_2)$ has signature $(0, 0, 2)$, the restriction to $\mathrm{span}(\mathbf{u}_1, \mathbf{u}_2)^\perp$ must have signature $(1, 0, 2)$. Recall that the restriction of $\langle , \rangle$ to $\mathbf{v}_0^\perp$ has signature $(3, 1)$. Since $\dim(\mathrm{span}(\mathbf{u}_1, \mathbf{u}_2)^\perp) + \dim(\mathbf{v}_0^\perp) = 3 + 4$ and $\dim(V^{3,2}) = 5$, it must follow that $\dim(\mathrm{span}(\mathbf{u}_1, \mathbf{u}_2)^\perp \cap \mathbf{v}_0^\perp) \geq 2$. In the intersection, maximal positive and negative subspaces have dimensions at most 1 and 0 respectively since the restriction of $\langle , \rangle$ to $\mathrm{span}(\mathbf{u}_1, \mathbf{u}_2)^\perp$ has signature $(1, 0, 2)$. Also, a maximal isotropic subspace has dimension at most 1 because $\langle , \rangle$ restricted to $\mathbf{v}_0^\perp$ has signature $(3, 1)$. Combined, these observations show that the signature of $\langle , \rangle$ restricted to $\mathrm{span}(\mathbf{u}_1, \mathbf{u}_2)^\perp \cap \mathbf{v}_0^\perp$ is exactly $(1, 0, 1)$. Therefore, the intersection has a single lightlike direction which corresponds to a unique point of intersection between the two circles. $\square$

This suggests that an element of Pho in the Lie circles model for Ein is realized as a set of circles in the 2-sphere all tangent to the same point. In the next section, we will develop a complete description of photons in the Lie circles model.

2.2.1 Fixing a diagonal basis

In this section, we discuss and improve on the results for the Lie circles model of Ein by fixing a basis $\mathbf{e}_1, \ldots, \mathbf{e}_5$ of $V^{3,2}$ such that the bilinear form $\langle , \rangle$ is given by the Gram matrix

$$J = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.25}$$

where $J_{ij} = \langle \mathbf{e}_i, \mathbf{e}_j \rangle$.

This choice gives an isomorphism with $\mathbb{R}^{3,2}$, that is, $\mathbb{R}^5$ with the bilinear form given by the matrix J. We will refer to this choice of basis as our *diagonal basis*. In the next section, we will discuss the Lie circles model of Ein for a different choice of basis, referred to as the anti-diagonal choice. The basis we fix in this section is well-suited to the description in terms of circles, their centers, and their radii. The choice of basis in the next section will be used in our development of coordinates for triples of flags. We will make it clear which choice of basis is being used in each section of this paper.

As we did in the previous section, we will fix a timelike vector $\mathbf{v}_0 = (0\ 0\ 0\ 1\ 0)^t$. Here it is clear that $\mathbf{v}_0^\perp = \{(x_1 \quad x_2 \quad x_3 \quad 0 \quad x_5)^t \in \mathbb{R}^{3,2}\} \cong \mathbb{R}^{3,1}$.

For our choice of timelike vector, we will now consider the result of Lemma 2.2.3 and see that a sphere in $\mathbb{P}(V^{3,2})$ is in fact homeomorphic to S^2. Although we have fixed a timelike vector, the following result generalizes to all timelike vectors.

Lemma 2.2.12. *For our fixed timelike vector $\mathbf{v}_0 \in V^{3,2}$, the map*

$$\mathbb{P}\mathscr{C}(\mathbf{v}_0^\perp) \longrightarrow S^2, \qquad [(x_1 \quad x_2 \quad x_3 \quad 0 \quad x_5)] \longmapsto \left(\frac{x_1}{x_5}, \frac{x_2}{x_5}, \frac{x_3}{x_5}\right). \tag{2.26}$$

is well-defined and, in fact, a homeomorphism.

Proof. First, we have that the nullcone of $\mathbf{v}_0^\perp$ is given by

$$\mathscr{C}(\mathbf{v}_0^\perp) = \{(x_1 \quad x_2 \quad x_3 \quad 0 \quad x_5)^t \in \mathbb{R}^{3,2} \mid x_1^2 + x_2^2 + x_3^2 = x_5^2\}. \tag{2.27}$$

Note that $x_5 = 0$ if and only if we have the zero vector. Since we ignore the zero vector under projectivization, every projective equivalence class has a unique representative with $x_5 = 1$. Therefore, we have

$$\mathbb{P}\mathscr{C}(\mathbf{v}_0^\perp) = \{[(x_1 \quad x_2 \quad x_3 \quad 0 \quad 1)^t] \in \mathsf{Ein} \mid x_1^2 + x_2^2 + x_3^2 = 1\}. \tag{2.28}$$

This shows that $\mathbb{P}\mathscr{C}(\mathbf{v}_0^\perp)$ is clearly homeomorphic to the 2-sphere by the map

$$[(x_1 \quad x_2 \quad x_3 \quad 0 \quad x_5)] \longmapsto \left(\frac{x_1}{x_5}, \frac{x_2}{x_5}, \frac{x_3}{x_5}\right). \tag{2.29}$$

$\square$

Just as we did in the previous section, we will be modeling Ein by considering the intersection of the lightcone $\mathcal{L}([\mathbf{u}])$ with $\mathbb{P}\mathscr{C}(\mathbf{v}_0^\perp) \cong S^2$ for each $[\mathbf{u}] \in \mathsf{Ein}$ rather than the point itself.

We will now make results of Lemma 2.2.6 explicit using our choice of basis and timelike vector.

Lemma 2.2.13. *For any point $[\mathbf{u}] \in \mathsf{Ein}$, $\mathcal{L}([\mathbf{u}]) \cap \mathbb{P}\mathscr{C}(\mathbf{v}_0^\perp)$ corresponds to a circle, possibly of zero radius, in S^2.*

Proof. We proved this result in a coordinate-independent fashion in Lemma 2.2.6. We wish to give an alternate proof for our choice of basis and fixed timelike vector $\mathbf{v}_0$.

Stereographic projection maps circles on the sphere to circles and lines in the plane. We will use this correspondence to show our desired result.

Define $\mathbf{p} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}^t \in \mathbf{v}_0^\perp$ and consider $U = \mathbf{p}^\perp \cap \mathbf{v}_0^\perp$ as a subspace of $\mathbf{v}_0^\perp \cong \mathbb{R}^{3,1}$. Lemma 2.2.2 tells us that $\langle , \rangle$ restricted to U will have signature $(2, 0, 1)$. This intersection will be a 3-dimensional subspace which contains $\mathbf{p}$ as $\mathbf{p}$ is lightlike. Therefore, we get that ${}^{U}\!/\!_{\mathbf{p}} \cong \mathbb{R}^2$. The vectors

$$\mathbf{x} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \end{pmatrix}^t \qquad \mathbf{y} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \end{pmatrix}^t \tag{2.30}$$

are a basis for the plane $\mathbb{R}^2 \cong {}^{U}\!/\!_{\mathbf{p}}$. The inverse of stereographic projection, with north and south poles $[\mathbf{q}]$ and $[\mathbf{p}]$ respectively, will be given by the injective map

$$\begin{aligned} \Pi : \mathbb{R}^2 &\longrightarrow \mathbb{P}\mathscr{C}(\mathbf{v}_0^\perp), \\ (x, y) &\longmapsto \mathbf{p} + (x\mathbf{x} + y\mathbf{y}) - \langle x\mathbf{x} + y\mathbf{y}, x\mathbf{x} + y\mathbf{y} \rangle \mathbf{q} \end{aligned} \tag{2.31}$$

where $\mathbf{q} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix}^t$ is the north pole where stereographic projection is not defined.

We wish to see that this is indeed the inverse of stereographic projection with north and south poles $[\mathbf{q}]$ and $[\mathbf{p}]$ respectively. Here $\mathbb{P}\mathscr{C}(\mathbf{v}_0^\perp)$ is the 2-sphere in 3-dimensional projective space $\mathbb{P}(\mathbf{v}_0^\perp) \cong \mathbb{P}(\mathbb{R}^{3,1})$. The plane ${}^{U}\!/\!_{\mathbf{p}}$ we are projecting the sphere onto corresponds to the plane tangent to the 2-sphere at $[\mathbf{p}]$ defined by

$$\{[\mathbf{p} + x\mathbf{x} + y\mathbf{y}] \in \mathbb{P}(\mathbf{v}_0^\perp) \mid x, y \in \mathbb{R}\} \tag{2.32}$$

where $(x, y) \in \mathbb{R}^2$ corresponds to $[\mathbf{p} + x\mathbf{x} + y\mathbf{y}]$. By definition of $\Pi(x, y)$, it lies on the line containing $[\mathbf{q}]$ and $[\mathbf{p} + x\mathbf{x} + y\mathbf{y}]$ and is a point on the 2-sphere. Therefore, our definition of Π as the inverse of stereographic projection is valid.

Note that $\Pi(x,y)$ simplifies to give

$$\Pi(x,y) = \begin{pmatrix} \frac{1-x^2-y^2}{2} \\ x \\ y \\ 0 \\ \frac{1+x^2+y^2}{2} \end{pmatrix}. \tag{2.33}$$

Now for a given $[\mathbf{u}] \in \mathsf{Ein}$, the points in the intersection $\mathcal{L}([\mathbf{u}]) \cap \mathbb{P}\mathscr{C}(\mathbf{v}_0^{\perp})$, except for possibly $[\mathbf{q}]$, will correspond exactly to points $(x,y) \in \mathbb{R}^2$ such that $\langle \Pi(x,y), \mathbf{u} \rangle = 0$. To show our desired result, we will show that $\langle \Pi(x,y), \mathbf{u} \rangle = 0$ gives the appropriate equations for lines and circles in the plane. Note that this condition on the bilinear form will be independent of the choice of representative in $\mathbb{R}^{3,2}$ for the point $[\mathbf{u}] \in \mathsf{Ein}$.

For a vector $\mathbf{u} = (u_1 \quad u_2 \quad u_3 \quad u_4 \quad u_5)^t$, we know that $u_1 + u_5$ is either zero or nonzero. Therefore, each $[\mathbf{u}] \in \mathsf{Ein}$ is one of the following two standard forms.

First we consider when $u_1 + u_5 \neq 0$. In this case, we have

$$\mathbf{u} = \begin{pmatrix} \frac{1+r^2-x_0^2-y_0^2}{2} \\ x_0 \\ y_0 \\ \pm r \\ \frac{1-r^2+x_0^2+y_0^2}{2} \end{pmatrix} \tag{2.34}$$

where $r \geq 0$. Note that the sum of the first and fourth entries must always be 1 in this standard form. Again consider $(x,y) \in \mathbb{R}^2$ such that $\langle \Pi(x,y), \mathbf{u} \rangle = 0$. We have that

$$
\begin{aligned}
0 =& \langle \Pi(x,y), \mathbf{u} \rangle \\
=& \frac{1}{4} + \frac{r^2}{4} - \frac{x_0^2 + y_0^2}{4} - \frac{x^2+y^2}{4} - r^2 \frac{x^2+y^2}{4} \\
&+ \frac{(x^2+y^2)(x_0^2+y_0^2)}{4} + xx_0 + yy_0 - \frac{1}{4} + \frac{r^2}{4} - \frac{x_0^2+y_0^2}{4} \\
&- \frac{x^2+y^2}{4} + r^2\frac{x^2+y^2}{4} - \frac{(x^2+y^2)(x_0^2+y_0^2)}{4} \\
=& \frac{1}{2}\left(r^2 - x^2 + 2xx_0 - x_0^2 - y^2 + 2yy_0 - y_0^2\right) \\
=& \frac{1}{2}\left(r^2 - (x-x_0)^2 - (y-y_0)^2\right)
\end{aligned}
\tag{2.35}
$$

This gives the equation for the circle, possibly of zero radius, $(x-x_0)^2+(y-y_0)^2 = r^2$. Unlike before, $\langle \mathbf{q}, \mathbf{u} \rangle \neq 0$. So we get that $\mathcal{L}([\mathbf{u}]) \cap \mathbb{P}\mathscr{C}(\mathbf{v}_0^\perp)$ is a circle in the 2-sphere that does not go through the north pole, $[\mathbf{q}]$. The $\pm$ in the fourth entry gives us that each such circle with nonzero radius will correspond to two points in Ein. We will show later that the choice of sign for the fourth entry (when the radius is nonzero) is equivalent to the choice of orientation we described in the previous section.

Now we consider when $u_1+u_5 = 0$. Since our vector we consider is lightlike, we have that $u_2^2 + u_3^2 = u_4^2$. Therefore, we can consider two possibilities, either $u_4 = 0$ or $u_4 \neq 0$.

If $u_1 + u_5 = 0$ and $u_4 = 0$, then, up to projectivization, we have $\mathbf{u} = (-1 \quad 0 \quad 0 \quad 0 \quad 1)^t$. Note that $[\mathbf{u}] = [\mathbf{q}]$. We see that

$$
\langle \Pi(x,y), \mathbf{u} \rangle = -\frac{1}{2} + \frac{x^2+y^2}{2} - \frac{1}{2} - \frac{x^2+y^2}{2} = -1 \neq 0. \tag{2.36}
$$

This tells us that $\mathcal{L}([\mathbf{u}]) \cap \mathbb{P}\mathscr{C}(\mathbf{v}_0^\perp)$ will not correspond to any points in the plane. Since we know that $\mathbf{u} \in \mathscr{C}(\mathbf{v}_0^\perp)$ and is a lightlike vector, it must follow that the intersection is nonempty. Therefore, $\mathcal{L}([\mathbf{u}]) \cap \mathbb{P}\mathscr{C}(\mathbf{v}_0^{\perp}) = [\mathbf{q}]$, a zero radius circle at the north pole.

Now suppose that $u_1 + u_5 = 0$ and $u_4 \neq 0$. Then we have

$$\mathbf{u} = \begin{pmatrix} -l \\ m \\ n \\ \pm\sqrt{m^2+n^2} \\ l \end{pmatrix} \tag{2.37}$$

with m and n not both zero. To fix a standard form for this case, we suppose that $m > 0$ or $m = 0$ and $n > 0$. Consider $(x, y) \in \mathbb{R}^2$ such that $\langle \Pi(x, y), \mathbf{u} \rangle = 0$. This gives us

$$\begin{aligned} 0 &= \langle \Pi(x, y), \mathbf{u} \rangle \\ &= -\frac{l}{2} + l\frac{x^2+y^2}{2} + mx + ny - \frac{l}{2} - l\frac{x^2+y^2}{2} \\ &= -l + mx + ny. \end{aligned} \tag{2.38}$$

Simplifying, we get the equation for the line $mx + ny = l$. Note also that $\langle \mathbf{q}, \mathbf{u} \rangle = 0$. Therefore, in the 2-sphere, $\mathcal{L}([\mathbf{u}]) \cap \mathbb{P}\mathscr{C}(\mathbf{v}_0^{\perp})$ is circle through the north pole, $[\mathbf{q}]$. Again, the $\pm$ in the fourth entry of $\mathbf{u}$ shows that each such circle in the 2-sphere will correspond to two points in Ein. Similarly, we will show later that the choice of sign for the fourth entry is equivalent to our description of orientation in the previous section. $\square$

In the previous proof, we showed that points in the Einstein universe have a unique standard form. We summarize this in the following lemma.

Lemma 2.2.14. *Any point* $[\mathbf{u}] \in \mathsf{Ein}$ *will have a unique representative vector given by exactly one of the following forms:*

(a) $\mathbf{u} = \begin{pmatrix} \frac{1+r^2-x_0^2-y_0^2}{2} \\ x_0 \\ y_0 \\ \pm r \\ \frac{1-r^2+x_0^2+y_0^2}{2} \end{pmatrix}$ *where* $r \geq 0$

(b) $\mathbf{u} = (-1 \quad 0 \quad 0 \quad 0 \quad 1)^t$

(c) $\mathbf{u} = \begin{pmatrix} -l \\ m \\ n \\ \pm\sqrt{m^2+n^2} \\ l \end{pmatrix}$ *where either* $m > 0$ *or* $m = 0$ *and* $n > 0$.

We refer to this form for a point in Ein *as the* standard form*.*

Remark. Unlike in our proof of Lemma 2.2.6, this proof allows for explicit description of the equation of the circle corresponding to a point in Ein. This proof also clearly shows that every oriented circle in the 2-sphere is realized by a point in Ein without utilizing the group action of $\mathsf{SO}(3,1)$ on $\mathbf{v}_0^\perp$ as we did in Proposition 2.2.10.

We now wish to understand how the notion of orientation of a circle $C([\mathbf{u}])$ with nonzero radius discussed in the previous section is reflected in the coordinates for $\mathbf{u}$ with respect to the fixed diagonal basis. The only cases described in the previous proof where we have nonzero radius are when $u_1 + u_5 \neq 0$ and $r \neq 0$ or when $u_1 + u_5 = 0$ and $u_4 \neq 0$.

Let us define a time orientation on $\mathbf{v}_0^\perp \cong \mathbb{R}^{3,1}$ by fixing the timelike $\mathbf{v}_1 = (0 \quad 0 \quad 0 \quad 0 \quad 1)^t$. This gives the description being future-pointing for timelike and lightlike vectors in $\mathbf{v}_0^\perp$ as simply those such vectors whose fifth entry is positive. For determining the orientation, we only consider lightlike vectors in $\mathbf{v}_0^\perp$ that are future-pointing. As in the previous proof, we can see that such vectors are either of the form

$$\mathbf{v_q} = \alpha(-1 \quad 0 \quad 0 \quad 0 \quad 1)^t \text{ where } \alpha > 0, \tag{2.39}$$

denoted as such because $[\mathbf{v_q}] = [\mathbf{q}]$ or they are of the form

$$\mathbf{w} = \beta \begin{pmatrix} \frac{1-x^2-y^2}{2} \\ x \\ y \\ 0 \\ \frac{1+x^2+y^2}{2} \end{pmatrix} \text{ where } \beta > 0. \tag{2.40}$$

Recall that the orientation for $C([\mathbf{u}])$ will be determined by the "inside" portion of the 2-sphere

$$D([\mathbf{u}]) = \mathbb{P}(\{\mathbf{v} \in \mathscr{C}(\mathbf{v}_0^\perp) \mid \langle \mathbf{v}_1, \mathbf{v} \rangle < 0 \text{ and } \langle \mathbf{u}, \mathbf{v} \rangle \geq 0\}) \tag{2.41}$$

Orientation for when $u_1 + u_5 \neq 0$ and $r \neq 0$: The normalized representative vector for such a point in Ein is of the form

$$\mathbf{u} = \pm\frac{1}{r}\begin{pmatrix} \frac{1+r^2-x_0^2-y_0^2}{2} \\ x_0 \\ y_0 \\ \pm r \\ \frac{1-r^2+x_0^2+y_0^2}{2} \end{pmatrix}. \tag{2.42}$$

In this case, we get that $\langle \mathbf{u}, \mathbf{v_q} \rangle = \mp\frac{\alpha}{r}$. Therefore, we get that $[\mathbf{q}]$ will lie "inside" $C([\mathbf{u}])$ if and only if the fourth entry in the standard form for $\mathbf{u}$ is negative. Next we get that

$$\langle \mathbf{u}, \mathbf{w} \rangle = \pm\frac{\beta}{2r}(r^2 - (x-x_0)^2 - (y-y_0)^2). \tag{2.43}$$

Thus $\langle \mathbf{u}, \mathbf{w} \rangle \geq 0$ is equivalent to $(x-x_0)^2 + (y-y_0)^2 \leq r^2$ when the fourth entry of the standard form for $\mathbf{u}$ is positive and $(x-x_0)^2 + (y-y_0)^2 \geq r^2$ when the fourth entry of the standard form for $\mathbf{u}$ is negative. We conclude that the orientation of $C([\mathbf{u}])$ is determined by the sign of the fourth entry of the standard form of $\mathbf{u}$. Furthermore, the "inside" of the $C([\mathbf{u}])$ is the region bounded by the circle in our choice of stereographic projection if and only if the fourth entry is positive.

Orientation when $u_1 + u_5 = 0$ and $u_4 \neq 0$: The normalized representative vector for such a point in Ein is of the form

$$\mathbf{u} = \pm\frac{1}{\sqrt{m^2+n^2}}\begin{pmatrix} -l \\ m \\ n \\ \pm\sqrt{m^2+n^2} \\ l \end{pmatrix}. \tag{2.44}$$

Now $\langle \mathbf{u}, \mathbf{v_q} \rangle = 0$ so $[\mathbf{q}]$ will always lie on $C([\mathbf{u}])$ for all such $\mathbf{u}$ which we already knew from the previous proof. Also we get that

$$\langle \mathbf{u}, \mathbf{w} \rangle = \pm\frac{\beta}{\sqrt{m^2+n^2}}(-l+mx+ny) \tag{2.45}$$

Therefore, the condition that $\langle \mathbf{u}, \mathbf{w} \rangle \geq 0$ is equivalent to $-l+mx+ny \geq 0$ when $\mathbf{u}$ is of the form described and the fourth entry is positive and $-l+mx+ny \leq 0$ when $\mathbf{u}$ is of the form described and the fourth entry is negative. This describes the "inside" of the circle in the 2-sphere. We conclude that the orientation of $C([\mathbf{u}])$ is determined by the sign of the fourth entry when $\mathbf{u}$ is in the standard form.

From the above discussion, we see that the following definition of orientation agrees with Definition 2.2.8. Refer to Lemma 2.2.14 to recall the types of standard forms of representative vectors of points in Ein.

Definition 2.2.15. For $[\mathbf{u}] \in \mathsf{Ein}$, we define *orientation* for a non-zero radius circle $C([\mathbf{u}])$ to be the "inside" of the circle in the 2-sphere corresponding to the following regions in the plane. When the standard form of $\mathbf{u}$ is of type (a) with $r > 0$ the "inside" of $C([\mathbf{u}])$ is

- $(x-x_0)^2+(y-y_0)^2 \leq r^2$ if the sign is $+$

- $(x-x_0)^2+(y-y_0)^2 \geq r^2$ if the sign is $-$.

When the standard form of $\mathbf{u}$ is of type (c) the "inside" of $C([\mathbf{u}])$ is

- $-l+mx+ny \geq 0$ if the sign is $+$

- $-l+mx+ny \leq 0$ if the sign is $-$.

Proposition 2.2.16. Ein *is homeomorphic to the moduli space of oriented circles, of possibly zero radius, in S^2. In the case where the radius of a circle is zero, there is no orientation.*

Proof. Recall that the moduli space of oriented circles in S^2 has the topology of a 3-dimensional manifold where the local charts are as follows. For any point $p_0 \in S^2$, the stereographic projection based at p_0 identifies $S^2\backslash\{p_0\}$ with $\mathbb{R}^2$. Every circle in the 2-sphere which does not go through p_0 will be a euclidean circle in $\mathbb{R}^2$ under stereographic projection. Associate to each such circle the local coordinates (x_0, y_0, r) where (x_0, y_0) is the center of the circle in the plane and r is the signed radius. We choose that a positive radius is equivalent to p_0 being "outside" the circle and a negative radius is equivalent to p_0 being "inside" the circle. For $p_0 = [\mathbf{q}]$, it is clear that this is equivalent to the sign of the fourth entry of $\mathbf{u}$ when given in standard form. From our computations in the proof of Lemma 2.2.13 and similar computations for a different choice of p_0, it is clear that the bijection between Ein and the moduli space of oriented circles in the 2-sphere induces a homeomorphism between the two spaces. $\square$

Now that we have described Ein as the moduli space of oriented circles in the 2-sphere with our choice of diagonal basis, we wish to improve the results of Lemma 2.2.11.

Remark. Along with our definition of orientation defined earlier, we also will use the convention that a circle $C([\mathbf{u}])$ has orientation corresponding to traveling around the circle with the "inside" being on the left. This is equivalent to a continuous of unit tangent vector at each point on $C([\mathbf{u}])$.

Definition 2.2.17. We say that two distinct oriented circles of nonzero radius $C([\mathbf{u}_1])$ and $C([\mathbf{u}_2])$ are *tangent* if they intersect in one point and the unit tangent vectors at this tangency point are identical. Equivalently, they are tangent if they intersect in one point and either $D([\mathbf{u}_1]) \subseteq D([\mathbf{u}_2])$ or $D([\mathbf{u}_2]) \subseteq D([\mathbf{u}_1])$. We say that a zero radius circle is *tangent* to any oriented circle which passes through it.

Proposition 2.2.18. *Let* $[\mathbf{u}_1], [\mathbf{u}_2] \in \mathsf{Ein}$. *Then,* $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = 0$ *if and only if the oriented circles* $C([\mathbf{u}_1])$ *and* $C([\mathbf{u}_2])$ *are tangent. A zero radius circle is considered to be tangent to a circle through that point.*

Proof. We will first consider the case when one of the circles, without loss of generality $C([\mathbf{u}_1])$, is a zero radius circle. Here we have no tangency to consider. Now either $[\mathbf{u}_1] = [\mathbf{q}]$ or $[\mathbf{u}_1] = [\Pi(x, y)]$ for some $(x, y) \in \mathbb{R}^2$. We have already established that $\langle \mathbf{q}, \mathbf{u}_2 \rangle = 0$ if and only if $\mathbf{u}_2$ has first and fifth entries summing to 0. These give rise to the lines in the plane model which correspond to circles going through $[\mathbf{q}]$ in the 2-sphere. Therefore, we get that $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = 0$ if and only if $C([\mathbf{u}_1])$ and $C([\mathbf{u}_2])$ are tangent.

Now consider when $[\mathbf{u}_1] = [\Pi(x, y)]$. If $[\mathbf{u}_2]$ has the standard form with the first and fifth entries summing to 0 and the fourth entry is nonzero, then we get that

$$\langle \Pi(x, y), \mathbf{u}_2 \rangle = \alpha(-l + mx + ny) \tag{2.46}$$

So we get that $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = 0$ if and only if (x, y) lies on the line corresponding to $\mathbf{u}_2$ in the plane which is equivalent to $C([\mathbf{u}_1])$ lying on $C([\mathbf{u}_2])$ in the 2-sphere.

Similarly, if $[\mathbf{u}_2]$ is such that the sum of the first and fifth entries is nonzero, we see that

$$\langle \Pi(x, y), \mathbf{u}_2 \rangle = \frac{\alpha}{2}(r^2 - (x - x_0)^2 - (y - y_0)^2) \tag{2.47}$$

Just as above, we see that $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = 0$ if and only if (x, y) lies on the circle corresponding to $\mathbf{u}_2$ in the plane which is equivalent to $C([\mathbf{u}_1])$ lying on $C([\mathbf{u}_2])$ in S^2.

Now we consider the case when both $C([\mathbf{u}_1])$ and $C([\mathbf{u}_2])$ have nonzero radius. There will be three cases to consider: when both correspond to circles in the plane, when both correspond to lines in the plane, and when we have one line and one circle in the plane.

Both circles: Suppose $C([\mathbf{u}_1])$ and $C([\mathbf{u}_2])$ projected into the plane are both circles of nonzero radius. We consider the representative vectors

$$\mathbf{u}_i = \begin{pmatrix} \frac{1+r_i^2-x_i^2-y_i^2}{2} \\ x_i \\ y_i \\ \sigma_i r_i \\ \frac{1-r_i^2+x_i^2+y_i^2}{2} \end{pmatrix} \tag{2.48}$$

where σ_i denotes a sign.

Here we see that

$$\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = \frac{1}{2}((r_1 - \sigma_1\sigma_2 r_2)^2 - (x_1 - x_2)^2 - (y_1 - y_2)^2 \tag{2.49}$$

This gives that $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = 0$ if and only if

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = (r_1 - \sigma_1\sigma_2 r_2)^2. \tag{2.50}$$

Note that if $\sigma_1\sigma_2 = 1$ this equation is equivalent to $C([\mathbf{u}_1])$ and $C([\mathbf{u}_2])$ being internally tangent as unoriented circles. Similarly, if $\sigma_1\sigma_2 = -1$ this equation is equivalent to $C([\mathbf{u}_1])$ and $C([\mathbf{u}_2])$ being externally tangent as unoriented circles.

Suppose $\sigma_1 = 1$ and $\sigma_2 = 1$. In this case $D([\mathbf{u}_1])$ and $D([\mathbf{u}_2])$ are the regions bounded by the respective circles in the plane. Thus, $C([\mathbf{u}_1])$ and $C([\mathbf{u}_2])$ are tangent if and only if they are internally tangent. We have shown that this is equivalent to $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = 0$ as desired.

Similarly, if $\sigma_1 = -1$ and $\sigma_2 = -1$, $D([\mathbf{u}_1])$ and $D([\mathbf{u}_2])$ are the regions unbounded by the respective circles in the plane. In this case, $C([\mathbf{u}_1])$ and $C([\mathbf{u}_2])$ are tangent if and only if they are internally tangent. We have shown this to be equivalent to $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = 0$.

Now if $\sigma_1\sigma_2 = -1$ then, without loss of generality, $D([\mathbf{u}_1])$ is the region bound by its corresponding circle and $D([\mathbf{u}_2])$ is the region unbounded by its corresponding circle. Therefore, $C([\mathbf{u}_1])$ and $C([\mathbf{u}_2])$ are tangent if and only if they are externally tangent. We have shown this is equivalent to $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = 0$.

Both lines: Recall that the circles in S^2 giving lines in $\mathbb{R}^2$ after stereographic projection are exactly those circles going through the north pole. Furthermore, such circles are tangent as unoriented circles if and only if the lines in the plane are parallel. Suppose $C([\mathbf{u}_1])$ and $C([\mathbf{u}_2])$ are both lines in the plane. We consider the representative vectors

$$\mathbf{u}_i = \begin{pmatrix} -l_i \\ m_i \\ n_i \\ \sigma_i\sqrt{m_i^2 + n_i^2} \\ l_i \end{pmatrix} \tag{2.51}$$

where σ_i denotes a sign.

Recall that

$$D([\mathbf{u}_i]) = \{[\mathbf{q}]\} \cup \{[\Pi(x, y)] | \sigma_i(-l_i + m_i x + n_i y) \geq 0\}. \tag{2.52}$$

Without loss of generality, $D([\mathbf{u}_1]) \subseteq D([\mathbf{u}_2])$ if and only if the lines are parallel and $\sigma_1\sigma_2 = 1$. Thus, $C([\mathbf{u}_1])$ and $C([\mathbf{u}_2])$ are tangent if and only if their lines are parallel and $\sigma_1\sigma_2 = 1$.

Now note that

$$\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = m_1 m_2 + n_1 n_2 - \sigma_1\sigma_2\sqrt{(m_1^2 + n_1^2)(m_2^2 + n_2^2)}. \tag{2.53}$$

If the lines corresponding to $C([\mathbf{u}_1])$ and $C([\mathbf{u}_2])$ are parallel and $\sigma_1\sigma_2 = 1$, then after scaling so that $m_1 = m_2 = m$ and $n_1 = n_2 = n$ we have that

$$\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = m^2 + n^2 - \sqrt{(m^2 + n^2)^2} = 0. \tag{2.54}$$

Therefore, $C([\mathbf{u}_1])$ and $C([\mathbf{u}_2])$ tangent implies that $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = 0$.

Now if $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = 0$, then the above gives

$$m_1 m_2 + n_1 n_2 = \sigma_1 \sigma_2 \sqrt{(m_1^2 + n_1^2)(m_2^2 + n_2^2)} \tag{2.55}$$

Squaring both sides and simplifying, we see that

$$(m_2 n_1 - m_1 n_2)^2 = 0 \tag{2.56}$$

Since $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = 0$ and m_i and n_i cannot simultaneous be zero, we must have that it is impossible to have $m_i = 0$, $n_j = 0$ for any i, j. Thus, (2.56) implies that the two lines must be parallel. With scaling we can let $m_1 = m_2 = m$ and $n_1 = n_2 = n$. So the left side of the equation in (2.55) becomes $m^2 + n^2 \geq 0$. Therefore, it must be true that $\sigma_1\sigma_2 = 1$. We have shown that $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = 0$ implies that the lines are parallel and $\sigma_1\sigma_2 = 1$. Finally, we can conclude that, in fact, $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = 0$ if and only if $C([\mathbf{u}_1])$ and $C([\mathbf{u}_2])$ tangent.

One line and one circle: Suppose, without loss of generality, that $C([\mathbf{u}_1])$ gives a line under stereographic projection and $C([\mathbf{u}_2])$ gives a circle with nonzero radius.

Let their reprsentative vectors be

$$\mathbf{u}_1 = \begin{pmatrix} -l \\ m \\ n \\ \sigma_1\sqrt{m^2+n^2} \\ l \end{pmatrix}, \qquad \mathbf{u}_2 = \begin{pmatrix} \frac{1+r^2-x_0^2-y_0^2}{2} \\ x_0 \\ y_0 \\ \sigma_2 r \\ \frac{1-r^2+x_0^2+y_0^2}{2} \end{pmatrix} \tag{2.57}$$

where σ_i denotes a sign.

Taking their product gives us

$$\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = -l + mx_0 + ny_0 - \sigma_1\sigma_2(r\sqrt{m^2+n^2}). \tag{2.58}$$

Note that the line $mx + ny - l$ is tangent, without considering orientation, to the circle $(x-x_0)^2 + (y-y_0)^2 = r^2$ exactly when

$$|mx_0 + ny_0 - l| = r\sqrt{m^2+n^2}. \tag{2.59}$$

First suppose that $\sigma_1 = 1$ and $\sigma_2 = 1$. For this case, $D([\mathbf{u}_1])$ is the region of the plane such that $-l + mx + ny \geq 0$ and $D([\mathbf{u}_2])$ is the region bounded by the corresponding circle. Therefore, $C([\mathbf{u}_1])$ and $C([\mathbf{u}_2])$ are tangent if and only if they are tangent without considering orientation and $D([\mathbf{u}_2]) \subseteq D([\mathbf{u}_1])$. Equivalently, the circle and line are tangent without considering orientation and the center of the circle $C([\mathbf{u}_2])$, (x_0, y_0), satisfies the inequality $-l + mx_0 + ny_0 \geq 0$. This is true if and only if $mx_0 + ny_0 - l = r\sqrt{m^2+n^2}$ by (2.59). With our assumptions on σ_i and using (2.58), this completes our chain of equivalences and gives that $C([\mathbf{u}_1])$ and $C([\mathbf{u}_2])$ are tangent if and only if $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = 0$.

Now consider when $\sigma_1 = 1$ and $\sigma_2 = -1$. Here we have that $D([\mathbf{u}_1])$ is the region of the plane such that $-l + mx + ny \geq 0$ and $D([\mathbf{u}_2])$ is the region unbounded by the corresponding circle. Similar to before, this tells us that $C([\mathbf{u}_1])$ and $C([\mathbf{u}_2])$ are tangent if and only if the circle and line are tangent without considering orientation and the center of the circle $C([\mathbf{u}_2])$, (x_0, y_0), satisfies the inequality $-l + mx_0 + ny_0 \leq 0$. This is equivalent to $mx_0 + ny_0 - l = -r\sqrt{m^2 + n^2}$ by (2.59). With $\sigma_1 = 1$ and $\sigma_2 = -1$ and recalling (2.58), we can conclude that $C([\mathbf{u}_1])$ and $C([\mathbf{u}_2])$ are tangent if and only if $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = 0$.

Next, suppose that $\sigma_1 = -1$ and $\sigma_2 = 1$. This gives us that $D([\mathbf{u}_1])$ is the region of the plane such that $-l + mx + ny \leq 0$ and $D([\mathbf{u}_2])$ is the region bounded by the corresponding circle. We recognize that $C([\mathbf{u}_1])$ and $C([\mathbf{u}_2])$ are tangent if and only if the circle and line are tangent without considering orientation and the center of the circle $C([\mathbf{u}_2])$, (x_0, y_0), satisfies the inequality $-l + mx_0 + ny_0 \leq 0$. At this point, the argument follows exactly as the previous case and we can conclude that $C([\mathbf{u}_1])$ and $C([\mathbf{u}_2])$ are tangent if and only if $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = 0$.

Finally, take $\sigma_1 = -1$ and $\sigma_2 = -1$. Now we get that $D([\mathbf{u}_1])$ is the region of the plane such that $-l + mx + ny \leq 0$ and $D([\mathbf{u}_2])$ is the region unbounded by the corresponding circle. In this case, we note that $C([\mathbf{u}_1])$ and $C([\mathbf{u}_2])$ are tangent if and only if the circle and line are tangent without considering orientation and the center of the circle $C([\mathbf{u}_2])$, (x_0, y_0), satisfies the inequality $-l + mx_0 + ny_0 \geq 0$. The rest of the proof for this case follows exactly the first case where $\sigma_1 = \sigma_2 = 1$. Therefore, we have that $C([\mathbf{u}_1])$ and $C([\mathbf{u}_2])$ are tangent if and only if $\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = 0$.

We have now considered all cases and subcases and can conclude that the result holds in each case. $\square$

Corollary 2.2.19. *A photon* $\varphi = \mathbb{P}(\mathrm{span}(\mathbf{u}_1, \mathbf{u}_2)) \in \mathsf{Pho}$ *in the Lie circles model of* Ein *is the set of all oriented circles tangent to both* $C([\mathbf{u}_1])$ *and* $C([\mathbf{u}_2])$. *An example of a photon is pictured below in Figure 2.1.*

We have already established that a photon in Ein corresponds to a family of pairwise-tangent oriented circles parametrized by S^1. All the circles in this family will be tangent to the same point. We associate to each photon this point together with the unit tangent vector which is tangent to every circle in the family.

Corollary 2.2.20. *By associating to each photon this point together with the unit tangent vector which is tangent to every circle in the family, we define a bijection*

$$\{\textit{Photons in } \mathsf{Ein}\} \longleftrightarrow \ T^1(S^2) \tag{2.60}$$

where $T^1(S^2)$ *is the unit tangent bundle of the 2-sphere. For a photon* $\varphi \in \mathsf{Pho}$, *we denote its associated unit tangent vector by* $\mathbf{u}_\varphi$.

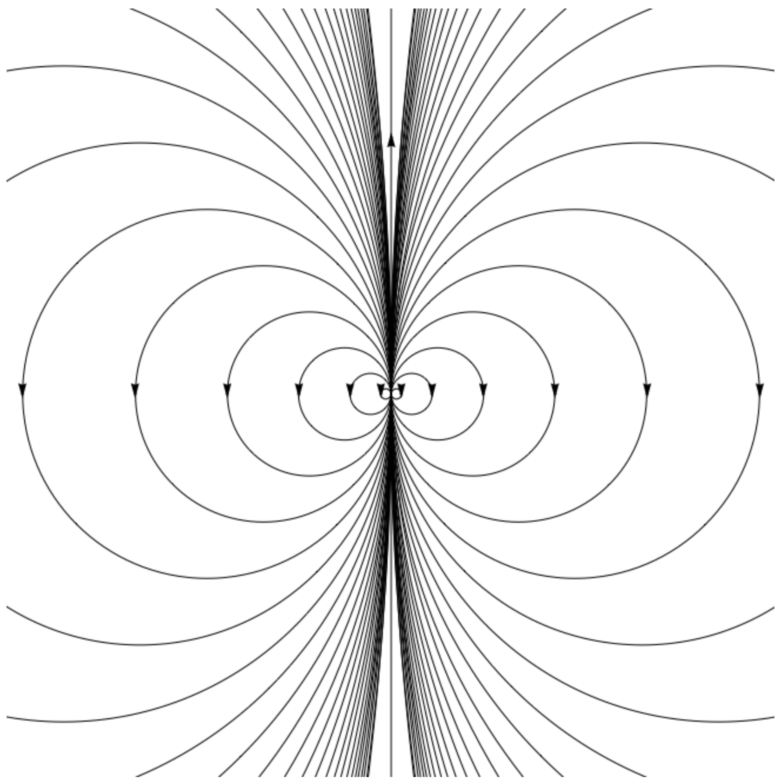


Figure 2.1: A photon in the Lie circles model of the Einstein universe.

2.2.2 Fixing an anti-diagonal basis

In this section, we consider the Lie circles model of Ein by fixing a basis $\mathbf{f}_1, \ldots, \mathbf{f}_5$ of $V^{3,2}$ such that the bilinear form $\langle , \rangle$ is given by the Gram matrix

$$\mathscr{J} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.61}$$

where $\mathscr{J}_{ij} = \langle \mathbf{f}_i, \mathbf{f}_j \rangle$. We refer to this choice of basis as the *anti-diagonal choice*. We will primarily fix this basis when describing coordinates for triples of flags later. Our primary focus in this section will be to establish the conversion from the anti-diagonal choice of basis to the diagonal choice and vice versa. The diagonal choice of basis considered in the previous section gives an easy way of graphing elements in the Lie circles model of Ein since the center and radius can be immediately read off from the standard form. When considering coordinates of triples of flags, it will be beneficial to be able visualize the process. Therefore, it will be important be well-versed in the conversions between our two choices of basis.

The change of basis is defined as follows

$$\begin{gathered} \mathbf{f}_1 = \tfrac{\mathbf{e}_1+\mathbf{e}_5}{\sqrt{2}}, \qquad \mathbf{f}_2 = \tfrac{\mathbf{e}_2+\mathbf{e}_4}{\sqrt{2}}, \qquad \mathbf{f}_3 = \mathbf{e}_3 \\ \mathbf{f}_4 = \tfrac{\mathbf{e}_4-\mathbf{e}_2}{\sqrt{2}}, \qquad \mathbf{f}_5 = \tfrac{\mathbf{e}_1-\mathbf{e}_5}{\sqrt{2}}. \end{gathered} \tag{2.62}$$

Similarly, the reverse change of basis is given as

$$\begin{aligned} &\mathbf{e}_1 = \tfrac{\mathbf{f}_1+\mathbf{f}_5}{\sqrt{2}}, \qquad \mathbf{e}_2 = \tfrac{\mathbf{f}_2-\mathbf{f}_4}{\sqrt{2}}, \qquad \mathbf{e}_3 = \mathbf{f}_3 \\ &\mathbf{e}_4 = \tfrac{\mathbf{f}_2+\mathbf{f}_4}{\sqrt{2}}, \qquad \mathbf{e}_5 = \tfrac{\mathbf{f}_1-\mathbf{f}_5}{\sqrt{2}}. \end{aligned} \tag{2.63}$$

The change of basis matrix from $\mathbf{e}_1, \ldots, \mathbf{e}_5$ to $\mathbf{f}_1, \ldots, \mathbf{f}_5$ is defined as

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}. \tag{2.64}$$

Note that $J = P^t \mathscr{J} P$ as desired for a change of basis matrix from J to $\mathscr{J}$.

We will conclude this section by describing the standard form, in the anti-diagonal basis, for representative vectors of points in Ein. We do this by applying the matrix P to the standard forms.

Circles in the plane: For this case we get that the standard form under the new basis is

$$P\mathbf{u} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{x_0 \pm r}{\sqrt{2}} \\ y_0 \\ \frac{-x_0 \pm r}{\sqrt{2}} \\ \frac{r^2 - x_0^2 - y_0^2}{\sqrt{2}} \end{pmatrix}. \tag{2.65}$$

The north pole: Here $P\mathbf{u} = (0 \quad 0 \quad 0 \quad 0 \quad \sqrt{2})^t$.

Lines in the plane: For the final case we see the standard form in terms of the new basis is

$$P\mathbf{u} = \begin{pmatrix} 0 \\ \frac{m\pm\sqrt{m^2+n^2}}{\sqrt{2}} \\ n \\ \frac{-m\pm\sqrt{m^2+n^2}}{\sqrt{2}} \\ -\sqrt{2}l \end{pmatrix}. \tag{2.66}$$

2.3 The Lagrangian Grassmannian model

In this section, we refer to [BCD+08] as we describe an alternate model for the Einstein universe as the *Lagrangian Grassmannian*. We will see that Ein can be thought of as the manifold of Lagrangian 2-planes in a real symplectic 4-dimensional vector space V. While points in the projective model of Ein will correspond to Lagrangian 2-planes, photons will be given by families of Lagrangian 2-planes passing through a common line. These correspondences will come from the isomorphism of Lie groups $\mathsf{SO}^0(3,2) \cong \mathsf{PSp}(4,\mathbb{R})$ where $\mathsf{SO}^0(3,2)$ denotes the identity component of $\mathsf{SO}(3,2)$. This model of Ein will be of particular importance in relating positive triples of flags to labeled, positive, oriented piecewise circular curves in the last chapter.

We take V to be a 4-dimensional real vector space. Let $\mathbf{e}_1,\ldots,\mathbf{e}_4$ be our choice of basis for V. Fix a generator of the fourth exterior power of V

$$\mathrm{vol} = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4 \in \Lambda^4(V). \tag{2.67}$$

The special linear group $\mathsf{SL}(V)$ is the group of automorphisms of (V, vol).

Now consider the second exterior power $\Lambda^2(V)$. The dimension of $\Lambda^2(V)$ is $\binom{4}{2} = 6$. Let's fix the basis $\mathbf{e}_1 \wedge \mathbf{e}_2, \mathbf{e}_1 \wedge \mathbf{e}_3, \mathbf{e}_1 \wedge \mathbf{e}_4, \mathbf{e}_2 \wedge \mathbf{e}_3, \mathbf{e}_2 \wedge \mathbf{e}_4, \mathbf{e}_3 \wedge \mathbf{e}_4$ of $\Lambda^2(V)$. By considering the action of $\mathsf{SL}(V)$ on V, we can induce an action on $\Lambda^2(V)$. This action will preserve the bilinear form

$$B: \ \Lambda^2(V) \times \Lambda^2(V) \longrightarrow \mathbb{R} \tag{2.68}$$

which we define by

$$\alpha_1 \wedge \alpha_2 = B(\alpha_1, \alpha_2)\mathrm{vol}. \tag{2.69}$$

It is clear from the properties of the exterior product that B is both symmetric and nondegenerate. Furthermore, the Gram matrix for $B(,)$ is given by

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.70}$$

By diagonalizing this S, we see that $B(,)$ has signature $(3,3)$.

The induced action of $\mathsf{SL}(4, \mathbb{R})$ gives a homomorphism

$$\mathsf{SL}(4, \mathbb{R}) \longrightarrow \mathsf{SO}(3,3). \tag{2.71}$$

One can check that this is a local isomorphism of Lie groups whose kernel is $\{\pm \mathrm{I}\}$ and whose image is the identity component of $\mathsf{SO}(3,3)$.

Let ω be a symplectic form on V, i.e. a skew-symmetric nondegenerate bilinear form on V. We will say that V is a *symplectic vector space*. Since $B(,)$ is nondegenerate, we have that ω defines a dual exterior bivector $\omega^* \in \Lambda^2(V)$ given by

$$\omega(\mathbf{v}_1, \mathbf{v}_2) = B(\mathbf{v}_1 \wedge \mathbf{v}_2, \omega^*). \tag{2.72}$$

Without loss of generality, we assume that

$$\omega^* \wedge \omega^* = -2\text{vol}. \tag{2.73}$$

We see that $B(\omega^*, \omega^*) = -2 < 0$. As $B(,)$ is nondegenerate of signature $(3,3)$ on $\Lambda^2(V)$ and its restriction to $\text{span}(\omega^*)$ is nondegenerate of signature $(0,1)$, we see that the restriction of $B(,)$ on its symplectic complement

$$W_0 := (\omega^*)^\perp \subset \Lambda^2(V) \tag{2.74}$$

has signature $(3,2)$. This allows us to restrict the local isomorphism (2.71) to a local isomorphism

$$\mathsf{Sp}(4, \mathbb{R}) \longrightarrow \mathsf{SO}(3,2) \tag{2.75}$$

whose kernel is $\{\pm \mathrm{I}\}$ and whose image is the identity component of $\mathsf{SO}(3,2)$. Thus, we have an isomorphism

$$\mathsf{PSp}(4, \mathbb{R}) \cong \mathsf{SO}^0(3,2). \tag{2.76}$$

2.3.1 The Einstein universe in terms of Lagrangian planes

In this section, we use the isomorphism $\mathsf{PSp}(4, \mathbb{R}) \cong \mathsf{SO}^0(3,2)$ to describe the Einstein universe in terms of Lagrangian planes in a 4-dimensional real symplectic vector space. We conclude this section with Table 2.1 giving a dictionary between the projective, Lie circles, and Lagrangian Grassmannian models of the Einstein universe.

Let V be a 4-dimensional real symplectic vector space with symplectic form ω. Define $B(,)$, ω^*, and W_0 as we did above.

A 2-dimensional subspace $P \subset V$ is referred to as a *Lagrangian plane* if the restriction of ω to P is identically zero. The space of all 2-dimensional subspaces of V is called the *Grassmannian* of 2-planes in V. Finally, the *Lagrangian Grassmannian* is the subspace consisting of all Lagrangian planes in V. We wish to establish the equivalence between Ein and the Lagrangian Grassmannian.

We first note that $W_0 \cong \mathbb{R}^{3,2}$. Therefore, the projectivization of the nullcone in W_0 is equivalent to Ein. Suppose that $P \subset V$ is a Lagrangian plane and that $\mathbf{v}_1, \mathbf{v}_2$ is a basis for P. The line generated by the bivector

$$w = \mathbf{v}_1 \wedge \mathbf{v}_2 \in \Lambda^2(V) \tag{2.77}$$

is necessarily independent of choice of basis for P. Furthermore,

$$B(w, \omega^*) = \omega(\mathbf{v}_1, \mathbf{v}_2) = 0 \tag{2.78}$$

so we see that w generates a 1-dimensional isotropic subspace of $W_0 \cong \mathbb{R}^{3,2}$. Thus, a Lagrangian plane P corresponds to a point in Ein.

Now we wish to show the converse correspondence. Since $\mathsf{Ein} \cong \mathbb{P}\mathscr{C}(W_0)$, we know that any point in Ein can be represented by a bivector $a \in W_0$ such that $B(a, a) = 0$, but by definition of $B(,)$ this is equivalent to stating that $a \wedge a = 0$. One can show (See Lemma 2.3.2 later) that the bivectors $a \in \Lambda^2(V)$ such that $a \wedge a = 0$

are exactly those which are decomposable, i.e. there exists vectors $\mathbf{v}_1, \mathbf{v}_2 \in V$ such that $a = \mathbf{v}_1 \wedge \mathbf{v}_2$. Since $a \in W_0 = (\omega^*)^\perp$, we have

$$0 = B(a, \omega^*) = B(\mathbf{v}_1 \wedge \mathbf{v}_2, \omega^*) = \omega(\mathbf{v}_1, \mathbf{v}_2). \tag{2.79}$$

So we get that a corresponds to the Lagrangian plane $P = \mathrm{span}(\mathbf{v}_1, \mathbf{v}_2)$ in V. Therefore, we have completed our identification of Ein with the Lagrangian Grassmannian.

Notation. For a point $p \in \mathsf{Ein}$, we will denote the Lagrangian plane in V corresponding to p by L_p.

Remark. A photon $\varphi \in \mathsf{Pho}$ corresponds to the line ℓ_φ in V defined by

$$\ell_\varphi = \bigcap_{p \in \varphi} L_p. \tag{2.80}$$

To show this correspondence, suppose we have any two points $p, q \in \varphi$ with representative bivectors u_p and u_q in $\Lambda^2(V)$. Furthermore, by the discussion above, we can write $u_p = \mathbf{v}_1 \wedge \mathbf{v}_2$ and $u_q = \mathbf{w}_1 \wedge \mathbf{w}_2$. Then $L_p = \mathrm{span}(\mathbf{v}_1, \mathbf{v}_2)$ and $L_q = \mathrm{span}(\mathbf{w}_1, \mathbf{w}_2)$. By the assumption that p and q are incident, we must have that $B(u_p, u_q) = 0$. By definition of $B(,)$, we have that

$$(\mathbf{v}_1 \wedge \mathbf{v}_2) \wedge (\mathbf{w}_1 \wedge \mathbf{w}_2) = 0. \tag{2.81}$$

Hence, $\mathbf{v}_1, \mathbf{v}_2, \mathbf{w}_1, \mathbf{w}_2$ must be linearly dependent. Since $L_p = \mathrm{span}(\mathbf{v}_1, \mathbf{v}_2)$ and $L_q = \mathrm{span}(\mathbf{w}_1, \mathbf{w}_2)$ are distinct 2-dimensional subspaces of V, we see they must intersect in a 1-dimensional subspace. This 1-dimensional subspace, ℓ_φ, is independent of choice of p and q.

To see this suppose we take three points $p, q, r \in \varphi$. Pick vectors $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$

that span the intersections as follows

$$L_p \cap L_q = \operatorname{span}(\mathbf{x}), \qquad L_p \cap L_r = \operatorname{span}(\mathbf{y}), \qquad L_q \cap L_r = \operatorname{span}(\mathbf{z}). \tag{2.82}$$

Since we are dealing with Lagrangian planes, we have that

$$\omega(\mathbf{x}, \mathbf{y}) = \omega(\mathbf{x}, \mathbf{z}) = \omega(\mathbf{y}, \mathbf{z}) = 0. \tag{2.83}$$

The maximal dimension of an isotropic subspace in a 4-dimensional symplectic vector space is 2. Thus, without loss of generality, $\mathbf{z} = a\mathbf{x} + b\mathbf{y}$. With the previous description of the intersections, we must have that $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ all lie in the same 1-dimensional subspace. We conclude that, while φ could be considered as the collection of Lagrangian planes L_p for all $p \in \varphi$, there is a natural correspondence between the photon φ and the common 1-dimensional subspace for all the Lagrangian planes coming from points in φ.

The incidence relations discussed in the projective model can be translated to the Lagrangian Grassmannian model as follows. A point $p \in \mathsf{Ein}$ is *incident* to a photon $\varphi \in \mathsf{Pho}$ if $\ell_\varphi \subset L_p$. Two points $p, q \in \mathsf{Ein}$ are *incident* when $L_p \cap L_q \neq \emptyset$. And finally, two photons $\varphi, \psi \in \mathsf{Pho}$ are *incident* if the plane spanned by ℓ_φ and ℓ_ψ is Lagrangian.

We summarize the relations between the projective model, the Lie circles model, and the Lagrangian Grassmannian with the dictionary given in Table 2.1.

Projective model	Lie circles model	Lagrangian Grassmannian model
The point p	The oriented circle $C(p)$	The Lagrangian plane L_p
The photon φ	The unit tangent vector $\mathbf{u}_\varphi$	The line ℓ_φ
The point p is incident to the photon φ	The vector $\mathbf{u}_\varphi$ is tangent to the oriented circle $C(p)$	$\ell_\varphi \subset L_p$
The points p and q are incident	The oriented circles $C(p)$ and $C(q)$ are tangent	$L_p \cap L_q \neq \emptyset$
The photons φ and ψ are incident	There exists an oriented circle C such that $\mathbf{u}_\varphi$ and $\mathbf{u}_\psi$ are both tangent to C	The plane spanned by ℓ_φ and ℓ_ψ is Lagrangian

Table 2.1: The relations between the projective, Lie circles, and Lagrangian Grassmannian models of the Einstein universe.

Condition for bivectors to be decomposable

In this section, we provide the proof of a claim used in Section 2.3.1. The claim was that for $a \in \Lambda^2(V)$ $a \wedge a = 0$ is equivalent to a being decomposable. We first prove a useful lemma.

Lemma 2.3.1. *Any bivector $\alpha \in \Lambda^2(V)$ can be written as the sum of two decomposable bivectors, i.e. $\alpha = \mathbf{v}_1 \wedge \mathbf{v}_2 + \mathbf{v}_3 \wedge \mathbf{v}_4$ for some $\mathbf{v}_1, \ldots, \mathbf{v}_4 \in V$.*

Proof. We will explicitly give such a sum for each case described below. We decompose α in terms of the basis we chose for $\Lambda^2(V)$ so that

$$\alpha = f_1\mathbf{e}_1 \wedge \mathbf{e}_2 + f_2\mathbf{e}_1 \wedge \mathbf{e}_3 + f_3\mathbf{e}_1 \wedge \mathbf{e}_4 + f_4\mathbf{e}_2 \wedge \mathbf{e}_3 + f_5\mathbf{e}_2 \wedge \mathbf{e}_4 + f_6\mathbf{e}_3 \wedge \mathbf{e}_4. \tag{2.84}$$

First, we suppose that $f_5 - f_3 \neq 0$. In this case, we can verify that

$$\begin{aligned} \alpha &= (\mathbf{e}_1 + \mathbf{e}_2) \wedge (f_1\mathbf{e}_2 + f_2\mathbf{e}_3 + f_3\mathbf{e}_4) \\ &\quad + \left(\mathbf{e}_2 + \frac{f_6}{f_5 - f_3}\mathbf{e}_4\right) \wedge ((f_4 - f_2)\mathbf{e}_3 + (f_5 - f_3)\mathbf{e}_4). \end{aligned} \tag{2.85}$$

Now if $f_3 = f_5 = a$, then we can show that

$$\alpha = (\mathbf{e}_1 + \mathbf{e}_2) \wedge (f_1\mathbf{e}_2 + f_2\mathbf{e}_3 + a\mathbf{e}_4) + \mathbf{e}_3 \wedge ((f_2 - f_4)\mathbf{e}_2 + f_6\mathbf{e}_4). \tag{2.86}$$

This concludes our proof. □

Lemma 2.3.2. *A bivector $\alpha \in \Lambda^2(V)$ is decomposable, i.e. there exist vectors $\mathbf{v}, \mathbf{w} \in V$ such that $\alpha = \mathbf{v} \wedge \mathbf{w}$, if and only if $\alpha \wedge \alpha = 0$.*

Proof. The forwards direction is immediate as

$$\alpha \wedge \alpha = (\mathbf{v} \wedge \mathbf{w}) \wedge (\mathbf{v} \wedge \mathbf{w}) = -(\mathbf{v} \wedge \mathbf{v}) \wedge (\mathbf{w} \wedge \mathbf{w}) = 0. \tag{2.87}$$

Now suppose we have that $\alpha \wedge \alpha = 0$. By the previous lemma, we can write α as the sum of two decomposable bivectors $\alpha = \mathbf{v}_1 \wedge \mathbf{v}_2 + \mathbf{v}_3 \wedge \mathbf{v}_4$. This gives us that

$$0 = \alpha \wedge \alpha = (\mathbf{v}_1 \wedge \mathbf{v}_2 + \mathbf{v}_3 \wedge \mathbf{v}_4) \wedge (\mathbf{v}_1 \wedge \mathbf{v}_2 + \mathbf{v}_3 \wedge \mathbf{v}_4). \tag{2.88}$$

This simplifies to

$$0 = 2\mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \mathbf{v}_3 \wedge \mathbf{v}_4 \tag{2.89}$$

which can only be true if $\mathbf{v}_1, \dots, \mathbf{v}_4$ are linearly dependent. Without loss of generality, suppose that $\mathbf{v}_4 = a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3$.

If $a = 0$, then we see that

$$\begin{aligned}\alpha &= \mathbf{v}_1 \wedge \mathbf{v}_2 + \mathbf{v}_3 \wedge (b\mathbf{v}_2 + c\mathbf{v}_3) \\ &= \mathbf{v}_1 \wedge \mathbf{v}_2 + b\mathbf{v}_3 \wedge \mathbf{v}_2 \\ &= (\mathbf{v}_1 + b\mathbf{v}_3) \wedge \mathbf{v}_2\end{aligned} \tag{2.90}$$

Therefore, α is decomposable.

Similarly, if $a \neq 0$, then we see that

$$\begin{aligned}\alpha &= \mathbf{v}_1 \wedge \mathbf{v}_2 + \mathbf{v}_3 \wedge (a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3) \\ &= \mathbf{v}_1 \wedge \mathbf{v}_2 + \mathbf{v}_3 \wedge (a\mathbf{v}_1 + b\mathbf{v}_2) \\ &= \left(\mathbf{v}_1 + \frac{b}{a}\mathbf{v}_2\right) \wedge (\mathbf{v}_2 - a\mathbf{v}_3).\end{aligned} \tag{2.91}$$

Again, we get that α is decomposable. □

2.4 The quotient model

In this section, we will describe the Einstein universe as a quotient of the space $S^2 \times S^1$. This model was given for the Einstein universe of arbitrary dimension in [BCD+08]. For the remainder of this section, we will consider $\mathbb{R}^{3,2}$, the 5-dimension vector space with a bilinear form of signature $(3, 2)$ with the fixed diagonal basis described in Section 2.2.1.

Here the nullcone, $\mathscr{C}$, is clearly

$$\mathscr{C} = \left\{ \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{pmatrix}^t \mid x_1^2 + x_2^2 + x_3^2 = x_4^2 + x_5^2 \right\}. \tag{2.92}$$

We can define the *double covering* of Ein, denoted $\widehat{\mathsf{Ein}}$, as the quotient of $\mathscr{C}$ by the action of multiplying by positive scalars, after removing $\mathbf{0}$ which does not correspond to a point in Ein. Again, we know that for any vector in the nullcone satisfies

$$x_1^2 + x_2^2 + x_3^2 = x_4^2 + x_5^2. \tag{2.93}$$

This value is always nonnegative and only zero when the vector is the zero vector, and hence does not correspond to a point in Ein. So for all vectors we are considering, $\sqrt{x_4^2 + x_5^2}$ is positive. Then by scaling the vector by dividing by $\sqrt{x_4^2 + x_5^2}$, we can assume that

$$x_1^2 + x_2^2 + x_3^2 = x_4^2 + x_5^2 = 1. \tag{2.94}$$

Therefore, we can think of a point in $\widehat{\mathsf{Ein}}$ as a point in $S^2 \times S^1$. This gives us that

$$\widehat{\mathsf{Ein}} \cong S^2 \times S^1. \tag{2.95}$$

Now multiplying by -1 in $\widehat{\mathsf{Ein}}$ acts by the simultaneous antipodal map on $S^2 \times S^1$. We conclude that

$$\mathsf{Ein} = {}^{\widehat{\mathsf{Ein}}}\!\big/_{\{\pm 1\}} \cong {}^{S^2 \times S^1}\!\big/_{\sim} \tag{2.96}$$

where $\sim$ is the equivalence relation identifying a point with its image under the simultaneous antipodal map.

Chapter 3: Coordinates on positive configurations of triples of flags in $\mathsf{SO}(V^{3,2})$

In this chapter, we define two types of coordinates on the totally positive part of the space of configurations of triples of flags in $\mathsf{SO}(V^{3,2})$. First, we define the complete isotropic flag manifold $\mathscr{F}$. Isotropic flags can be thought of as pointed photons and pointed circles in the projective and Lie circle models of Ein, respectively. We then define positivity in $\mathsf{SO}(V^{3,2})$ by explicitly describing a certain positive sub-semigroup $U_+^{>0}$ for the semisimple Lie group $\mathsf{SO}(V^{3,2})$. Using the definition of positivity in $\mathsf{SO}(V^{3,2})$, we define positivity for tuples of isotropic flags. We define two types of coordinates for positive triples of isotropic flags, one type dependent on a choice of basis and the other basis-independent. The main results of this chapter are that, for each type of coordinates, we show that a particular quadrant of the plane parametrizes the totally positive part of the space of configurations of triples of isotropic flags.

We finish this chapter by describing flags and positivity in $\mathsf{PSp}(4,\mathbb{R})$. We do this in a way that is analogous to what was done to describe the same for $\mathsf{SO}(V^{3,2})$. Using the isomorphism $\mathsf{PSp}(4,\mathbb{R}) \cong \mathsf{SO}^0(V^{3,2})$ given in Section 2.3, we show that the positivity in $\mathsf{PSp}(4,\mathbb{R})$ is equivalent to positivity in $\mathsf{SO}(V^{3,2})$.

3.1 Complete isotropic flags in $\mathsf{SO}(V^{3,2})$

In this section, we describe complete isotropic flags in $V^{3,2}$. We will observe that a complete isotropic flag can be expressed as a pointed oriented circle in the Lie circles model of the Einstein universe.

We fix the anti-diagonal basis defined in Section 2.2.2 where the symmetric bilinear form $\langle , \rangle$ is given by the Gram matrix $\mathscr{J}$ where

$$\mathscr{J} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.1}$$

For a refresher on Cartan subgroups, simple roots, and other Lie theory ideas used here, we refer the reader to [Kna02].

Consider the Cartan subgroup $A \subset \mathsf{SO}(V^{3,2})$ given by the set of diagonal matrices in the group

$$A = \left\{ \begin{pmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2^{-1} & 0 \\ 0 & 0 & 0 & 0 & \lambda_1^{-1} \end{pmatrix} \,\middle|\, \lambda_1, \lambda_2 \in \mathbb{R}^* \right\}. \tag{3.2}$$

The Cartan subalgebra $\mathfrak{a}$ corresponding to A is given by

$$\mathfrak{a} = \left\{ \begin{pmatrix} a & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -b & 0 \\ 0 & 0 & 0 & 0 & -a \end{pmatrix} \;\middle|\; a, b \in \mathbb{R} \right\}. \tag{3.3}$$

As our Lie group is rank two, we know that we will have two simple roots. With respect to this Cartan subalgebra, we choose our simple roots $\alpha_i : \mathfrak{a} \to \mathbb{R}$ to be

$$\alpha_1(a, b) = a - b, \qquad \alpha_2(a, b) = b. \tag{3.4}$$

For α_1 and α_2, there are associated maps $a_i : \mathbb{R} \to \mathsf{SO}(V^{3,2})$ given by

$$a_1(t) = \begin{pmatrix} 1 & t & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad a_2(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & \sqrt{2}t & t^2 & 0 \\ 0 & 0 & 1 & \sqrt{2}t & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.5}$$

These a_1, a_2 generate the unipotent subgroup $U_+ \subset \mathsf{SO}(V^{3,2})$ consisting of unipotent upper triangular matrices. Now this unipotent subgroup U_+ together with the Cartan subgroup A span a *Borel subgroup* $B_+ = AU_+$ of $\mathsf{SO}(V^{3,2})$. This Borel subgroup B_+ is the set of all upper triangular matrices in $\mathsf{SO}(V^{3,2})$.

Definition 3.1.1. The *complete isotropic flag manifold* is the homogeneous space

$$\mathscr{F} := \mathsf{SO}(V^{3,2}) / B_+. \tag{3.6}$$

We refer to an element $F \in \mathscr{F}$ as an *isotropic flag.*

We now wish to understand the elements of $\mathscr{F}$. First, note that B_+ stabilizes the following flag in $V^{3,2}$:

$$0 \subset \mathrm{span}(\mathbf{f}_1) \subset \mathrm{span}(\mathbf{f}_1, \mathbf{f}_2) \subset \mathrm{span}(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3) \subset \mathrm{span}(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \mathbf{f}_4) \subset V^{3,2}. \tag{3.7}$$

With respect to the symmetric bilinear form $\langle , \rangle$, we can rewrite this flag as

$$0 \subset \mathrm{span}(\mathbf{f}_1) \subset \mathrm{span}(\mathbf{f}_1, \mathbf{f}_2) \subset \mathrm{span}(\mathbf{f}_1, \mathbf{f}_2)^{\perp} \subset \mathrm{span}(\mathbf{f}_1)^{\perp} \subset V^{3,2}. \tag{3.8}$$

Observe that $\mathrm{span}(\mathbf{f}_1)$ and $\mathrm{span}(\mathbf{f}_1, \mathbf{f}_2)$ are isotropic subspaces of $V^{3,2}$. Since $\mathsf{SO}(V^{3,2})$ preserves the bilinear form, the 1-dimensional and 2-dimensional subspaces of any flag in $\mathscr{F}$ are necessarily isotropic. Furthermore, the 3-dimensional and 4-dimensional subspaces of any flag $F \in \mathscr{F}$ must be the orthogonal subspaces associated to the 2-dimensional and 1-dimensional subspaces, respectively. Therefore, any flag $F \in \mathscr{F}$ is defined entirely by its 1-dimensional and 2-dimensional isotropic subspaces. Note that this is why we refer to $\mathscr{F}$ as the complete isotropic flag manifold. Since these defining subspaces are each isotropic, they correspond to elements of Ein and Pho, respectively. We conclude that any flag in $\mathscr{F}$ can be considered as a pointed photon as defined below.

Definition 3.1.2. A *pointed photon* is a pair

$$(p, \varphi) \in \mathsf{Ein} \times \mathsf{Pho} \tag{3.9}$$

such that $p \in \varphi$.

Recall that $\mathsf{SO}(V^{3,2})$ both acts transitively on points in Ein and acts transitively on photons. We conclude that the identification of a flag with a pointed photon is in fact a one-to-one correspondence between flags in $\mathscr{F}$ and pointed photons. For this reason, we sometimes refer to elements of $\mathscr{F}$ as flags in Ein.

In the Lie circles model of the Einstein universe, we saw that a point $p \in \mathsf{Ein}$ corresponds to an oriented circle $C(p)$ in the 2-sphere. Also, Corollary 2.2.20 gave an equivalence between Pho and $T^1(S^2)$. For a photon $\varphi \in \mathsf{Pho}$, the corresponding unit tangent vector $\mathbf{u}_\varphi$ is based at the common tangency point for oriented circles in φ and is tangent to all such oriented circles. Note that if we fix an oriented circle $C(p)$ with nonzero radius and wish to describe a photon φ incident to p, we only need to determine the basepoint of $\mathbf{u}_\varphi$, denoted by x_φ, as the direction is already established by $C(p)$.

This gives the following correspondence between pointed photons and pointed oriented circles in the 2-sphere:

$$(p, \varphi) \longleftrightarrow (C(p), \mathbf{u}_\varphi) \tag{3.10}$$

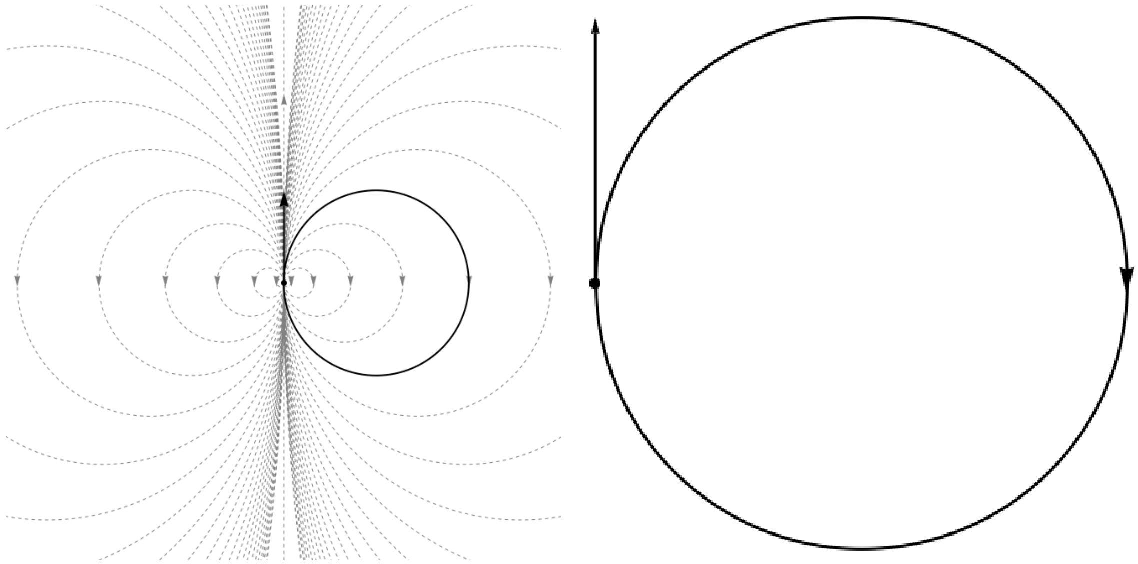
where $p \in \varphi$ and $\mathbf{u}_\varphi \in T^1(C(p))$. Thus we have shown the following result.

Proposition 3.1.3. *There exists a bijection*

$$\mathscr{F} \longleftrightarrow \{\textit{Pointed oriented circles in } S^2\} \tag{3.11}$$

given by $F \mapsto (C(\mathbb{P}(F^1)), \mathbf{u}_{\mathbb{P}(F^2)})$. *Here* F^i *denotes the* i*-dimensional subspace in* F.

Examples of an isotropic flag realized as a pointed photon and as a pointed circle are pictured in Figure 3.1(a) and Figure 3.1(b), respectively.



(a) An isotropic flag as a pointed photon. (b) An isotropic flag as a pointed oriented circle.

Figure 3.1: Isotropic flags in the Lie Circles model of the Einstein universe.

3.2 Positivity in $\mathsf{SO}(V^{3,2})$

In this section, we recall the definition given by Lusztig in [Lus94] of the positive sub-semigroup $U_+^{>0}$ for a semisimple Lie group G. We then explicitly describe this sub-semigroup when $G = \mathsf{SO}^0(V^{3,2})$. Further, we will show that $U_+^{>0}$ consists of all matrices in U_+ whose non-trivial minors are strictly positive.

For the entirety of this section, we will fix the anti-diagonal basis defined in Section 2.2.2 where the symmetric bilinear form $\langle , \rangle$ is given by the Gram matrix $\mathscr{J}$.

As defined in Section 3.1, we have the Cartan subgroup A, its associated Cartan subalgebra $\mathfrak{a}$, the simple roots α_1 and α_2, their corresponding maps a_1 and a_2, the unipotent subgroup U_+, and the Borel subgroup $B_+ = AU_+$. We fix the opposite Borel subgroup $B_- = U_- A$ where $U_- = \mathscr{J} U_+ \mathscr{J}$.

The Borel subgroup B_+, Cartan subgroup A, and the maps a_1 and a_2 determine a pinning for $\mathsf{SO}(V^{3,2})$. For the general definition of a pinning see [Bou05]. Here a pinning can be thought of as a pair of representations $\rho_1, \rho_2 : \mathsf{SL}(2,\mathbb{R}) \to \mathsf{SO}(V^{3,2})$ such that the image by ρ_i of the one-parameter upper triangular subgroup $\left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \middle| \ t \in \mathbb{R} \right\}$ is exactly $\{a_i(t) \mid t \in \mathbb{R}\}$. We can explicitly define these representations as

$$\rho_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b & 0 & 0 & 0 \\ c & d & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & c & d \end{pmatrix} \tag{3.12}$$

$$\rho_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & a^2 & \sqrt{2}ab & b^2 & 0 \\ 0 & \sqrt{2}ac & bc+ad & \sqrt{2}bd & 0 \\ 0 & c^2 & \sqrt{2}cd & d^2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.13}$$

Now we wish to describe the *Weyl group* $W = {}^{N_G(A)}\!/\!_{Z_G(A)}$ of $G = \mathsf{SO}(V^{3,2})$. Note that we have a set of generators $s_i = \rho_i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ associated to the pinning described above. Now each of these generators

$$s_1 = \rho_1 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix} \tag{3.14}$$

$$s_2 = \rho_2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.15}$$

are of order 2 when considered as acting by conjugation on $\mathfrak{a}$, the Lie subalgebra associated to A. Furthermore, their product

$$s_1 s_2 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix} \tag{3.16}$$

is of order 4 when considered as acting by conjugation on $\mathfrak{a}$. Therefore, we conclude that W is isomorphic to the dihedral group D_4. As the Weyl group W is a finite Coxeter group, there is a well defined notion of longest word, i.e. the element with the longest length for a reduced expression in terms of the generators (See [Hum90]).

The *longest word* of the Weyl group W has two reduced expression in terms of our generators: $s_1s_2s_1s_2$ and $s_2s_1s_2s_1$.

Definition 3.2.1 ([Lus94])**.** Let w_0 denote the longest element of W and let $j_1,\dots,j_l$ be a sequence of indices such that $s_{j_1}\dots s_{j_l}$ is a reduced expression for w_0. The *positive sub-semigroup* $U_+^{>0}$ is defined to be

$$U_+^{>0} := \{a_{j_1}(t_1)\dots a_{j_l}(t_l) \mid t_i > 0\} \subset U_+. \tag{3.17}$$

This is independent of the choice of reduced expression for w_0.

For $G = \mathsf{SO}^0(V^{3,2})$ and with the choice of reduced expression $s_1s_2s_1s_2$, we have that

$$\begin{aligned} &a_1(t_1)a_2(t_2)a_1(t_3)a_2(t_4) \\ &= \begin{pmatrix} 1 & t_1+t_3 & \sqrt{2}(t_3t_4+t_1(t_2+t_4)) & t_3t_4^2+t_1(t_2+t_4)^2 & t_1t_2^2t_3 \\ 0 & 1 & \sqrt{2}(t_2+t_4) & (t_2+t_4)^2 & t_2^2t_3 \\ 0 & 0 & 1 & \sqrt{2}(t_2+t_4) & \sqrt{2}t_2t_3 \\ 0 & 0 & 0 & 1 & t_1+t_3 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{3.18}$$

Definition 3.2.2. An $n\times n$ upper (lower) triangular matrix is said to be *upper (lower) strictly totally positive* if each minor is strictly positive unless it is zero due to upper (lower) triangularity.

Theorem 3.2.3 (Theorem 2.8 in [Pin10])**.** *Let A be an $n\times n$ upper triangular matrix such that every connected minor containing an element of the first row of A is strictly positive. Then A is upper strictly totally positive. Similarly, if A is* *an $n \times n$ lower triangular matrix such that every connected minor containing an element of the first column of A is strictly positive. Then A is lower strictly totally positive.*

Lemma 3.2.4. *The positive sub-semigroup $U_+^{>0}$ is the subset of all matrices in U_+ such that all their minors which are not zero by upper triangularity are strictly positive.*

Proof. We first wish to show that all non-trivial minors of $a_1(t_1)a_2(t_2)a_1(t_3)a_2(t_4)$ are strictly positive. By Theorem 3.2.3, it is sufficient to check that all connected minors that contain the first row are strictly positive. Any such minor that also contains the first column is necessarily strictly positive as they are the determinants of upper triangular matrices with 1's along the diagonal. Also, all non-trivial entries in the matrix are obviously positive so we only need to consider minors given by submatrices size 2×2 or larger. Thus, we only need to confirm that the following minors are strictly positive.

$$\det \begin{pmatrix} t_1 + t_3 & \sqrt{2}(t_3t_4 + t_1(t_2 + t_4)) \\ 1 & \sqrt{2}(t_2 + t_4) \end{pmatrix} = \sqrt{2}t_2t_3 > 0 \tag{3.19}$$

$$\det \begin{pmatrix} t_1 + t_3 & \sqrt{2}(t_3t_4 + t_1(t_2 + t_4)) & t_3t_4^2 + t_1(t_2 + t_4)^2 \\ 1 & \sqrt{2}(t_2 + t_4) & (t_2 + t_4)^2 \\ 0 & 1 & \sqrt{2}(t_2 + t_4) \end{pmatrix} = t_2^2t_3 > 0 \tag{3.20}$$

$$\det\begin{pmatrix} t_1+t_3 & \sqrt{2}(t_3t_4+t_1(t_2+t_4)) & t_3t_4^2+t_1(t_2+t_4)^2 & t_1t_2^2t_3 \\ 1 & \sqrt{2}(t_2+t_4) & (t_2+t_4)^2 & t_2^2t_3 \\ 0 & 1 & \sqrt{2}(t_2+t_4) & \sqrt{2}t_2t_3 \\ 0 & 0 & 1 & t_1+t_3 \end{pmatrix} \tag{3.21}$$

$$= t_1t_2^2t_3 > 0$$

$$\det\begin{pmatrix} \sqrt{2}(t_3t_4+t_1(t_2+t_4)) & t_3t_4^2+t_1(t_2+t_4)^2 \\ \sqrt{2}(t_2+t_4) & (t_2+t_4)^2 \end{pmatrix} = \sqrt{2}t_2t_3t_4(t_2+t_4) > 0 \tag{3.22}$$

$$\det\begin{pmatrix} \sqrt{2}(t_3t_4+t_1(t_2+t_4)) & t_3t_4^2+t_1(t_2+t_4)^2 & t_1t_2^2t_3 \\ \sqrt{2}(t_2+t_4) & (t_2+t_4)^2 & t_2^2t_3 \\ 1 & \sqrt{2}(t_2+t_4) & \sqrt{2}t_2t_3 \end{pmatrix} = t_2^2t_3^2t_4^2 > 0 \tag{3.23}$$

$$\det\begin{pmatrix} t_3t_4^2+t_1(t_2+t_4)^2 & t_1t_2^2t_3 \\ (t_2+t_4)^2 & t_2^2t_3 \end{pmatrix} = t_2^2t_3^2t_4^2 > 0 \tag{3.24}$$

Therefore all elements of the positive sub-semigroup $U_+^{>0}$ have strictly positive non-trivial minors.

Conversely, any element of U_+ whose non-trivial minors are strictly positive can be written as

$$M = \begin{pmatrix} 1 & a & ad-b & -bd+\frac{ad^2}{2}+c & ac-\frac{b^2}{2} \\ 0 & 1 & d & \frac{d^2}{2} & c \\ 0 & 0 & 1 & d & b \\ 0 & 0 & 0 & 1 & a \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.25}$$

where all entries above the diagonal are strictly positive as well as $4c^2 - 4bcd + b^2d^2 > 0$ and $-2cd + bd^2 > 0$. The relations above the diagonal come from the fact that we are working in $\mathsf{SO}(V^{3,2})$ with bilinear form given by the anti-diagonal Gram matrix $\mathscr{J}$. The 1's along the diagonal come from the fact that all elements are unipotent and all non-trivial entries of our matrix must be positive.

By setting

$$t_1 = \frac{2ac - b^2}{2c}, \qquad t_2 = \frac{\sqrt{2}c}{b}, \qquad t_3 = \frac{b^2}{2c}, \qquad t_4 = \frac{bd - 2c}{\sqrt{2}b}, \tag{3.26}$$

we see that $a_1(t_1)a_2(t_2)a_1(t_3)a_2(t_4) = M$. Now t_1, t_2, t_3 are clearly strictly positive by the positivity of non-trivial entries of M. Furthermore, we can rewrite $t_4 = \frac{1}{\sqrt{2}bd}(-2cd + bd^2)$ and see that it is positive by the conditions on M. We conclude that M is in the positive sub-semigroup $U_+^{>0}$. $\square$

3.3 Coordinates of the totally positive part of $\mathsf{Conf}^{(3)}(\mathscr{F})$

In this section, we recall the definition of a positive k-tuple of isotropic flags as defined by Fock and Goncharov in [FG06]. This notion generalizes the cyclic ordering on points in $\mathbb{RP}^1 \cong S^1$. We proceed to describe some coordinates on the configuration space of generic triples of isotropic flags. We single out a quadrant of the plane for these coordinates and note that it describes exactly the totally positive part of the configuration space of generic triples of isotropic flags.

For the entirety of this section, we will we fix the anti-diagonal basis defined in Section 2.2.2 where the symmetric bilinear form $\langle , \rangle$ is given by the Gram matrix $\mathscr{J}$.

Definition 3.3.1. A pair of flags (F_1, F_2) in $\mathscr{F}$ is said to be *transverse* if $\mathsf{dim}(\mathrm{span}(F_1^i, F_2^j)) = \mathsf{min}(i + j, 5)$.

Definition 3.3.2. The *space of configurations of k-tuples of isotropic flags*, denoted $\mathsf{Conf}^{(k)}(\mathscr{F})$, is the space of generic k-tuples of isotropic flags in Ein. More precisely, it is the quotient of the set of k-tuples of isotropic flags $\mathscr{F}^{(k)}$ which are pairwise transverse by the diagonal action of $\mathsf{SO}(V^{3,2})$.

Lemma 3.3.3. *The group* $\mathsf{SO}(V^{3,2})$ *acts transitively on pairs of transverse isotropic flags. The stabilizer of such a pair is a Cartan subgroup H of* $\mathsf{SO}(3, 2)$ *isomorphic to* $(\mathbb{R}^*)^2$.

Proof. We first wish to show that there is a bijection between pairs of transverse isotropic flags and orthonormal bases, with respect to the Gram matrix $\mathscr{J}$, up to scaling of the basis vectors.

First consider the pair of flags $(F_0, F_\infty) = (B_+, \mathscr{J} B_+)$. This pair is associated to the standard basis $\mathbf{f}_1, \ldots \mathbf{f}_5$ as F_0 is given by

$$\mathrm{span}(\mathbf{f}_1) \subset \mathrm{span}(\mathbf{f}_1, \mathbf{f}_2) \subset \mathrm{span}(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3) \subset \mathrm{span}(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \mathbf{f}_4) \subset V^{3,2} \tag{3.27}$$

and F_∞ is given by

$$\mathrm{span}(\mathbf{f}_5) \subset \mathrm{span}(\mathbf{f}_5, \mathbf{f}_4) \subset \mathrm{span}(\mathbf{f}_5, \mathbf{f}_4, \mathbf{f}_3) \subset \mathrm{span}(\mathbf{f}_5, \mathbf{f}_4, \mathbf{f}_3, \mathbf{f}_2) \subset V^{3,2}. \tag{3.28}$$

For any other orthonormal basis $\mathbf{g}_1, \ldots, \mathbf{g}_5$, we can get a pair of transverse flags (F, G) in a similar fashion, i.e. define F to be

$$\mathrm{span}(\mathbf{g}_1) \subset \mathrm{span}(\mathbf{g}_1, \mathbf{g}_2) \subset \mathrm{span}(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3) \subset \mathrm{span}(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3, \mathbf{g}_4) \subset V^{3,2} \tag{3.29}$$

and and define G to be

$$\mathrm{span}(\mathbf{g}_5) \subset \mathrm{span}(\mathbf{g}_5, \mathbf{g}_4) \subset \mathrm{span}(\mathbf{g}_5, \mathbf{g}_4, \mathbf{g}_3) \subset \mathrm{span}(\mathbf{g}_5, \mathbf{g}_4, \mathbf{g}_3, \mathbf{g}_2) \subset V^{3,2}. \tag{3.30}$$

For the opposite direction, suppose you have a pair of transverse flags (F, G). Let F^j and G^j denote the j-dimensional subspace associated to F and G, respectively. Then we can get an orthogonal bases by picking a vector in each of the following 1-dimensional subspaces:

$$\begin{gathered} \mathbf{v}_1 \in F^1, \qquad \mathbf{v}_2 \in F^2 \cap G^4, \qquad \mathbf{v}_3 \in F^3 \cap G^3 \\ \mathbf{v}_4 \in F^4 \cap G^2, \qquad \mathbf{v}_5 \in G^1. \end{gathered} \tag{3.31}$$

These vectors will be linearly independent due to the transversality condition on (F, G). To see that this basis is orthogonal, recall that F^1, F^2, G^1, and G^2 are all isotropic and that we have $F^4 = (F^1)^\perp$, $F^3 = (F^2)^\perp$, $G^4 = (G^1)^\perp$, and $G^3 = (G^2)^\perp$. Therefore,

$$\begin{gathered} \langle \mathbf{v}_1, \mathbf{v}_1 \rangle = \langle \mathbf{v}_1, \mathbf{v}_2 \rangle = \langle \mathbf{v}_1, \mathbf{v}_3 \rangle = \langle \mathbf{v}_1, \mathbf{v}_4 \rangle = 0 \\ \langle \mathbf{v}_2, \mathbf{v}_2 \rangle = \langle \mathbf{v}_2, \mathbf{v}_3 \rangle = \langle \mathbf{v}_2, \mathbf{v}_5 \rangle = 0 \\ \langle \mathbf{v}_3, \mathbf{v}_4 \rangle = \langle \mathbf{v}_3, \mathbf{v}_5 \rangle = 0 \\ \langle \mathbf{v}_4, \mathbf{v}_4 \rangle = \langle \mathbf{v}_4, \mathbf{v}_5 \rangle = 0 \\ \langle \mathbf{v}_5, \mathbf{v}_5 \rangle = 0. \end{gathered} \tag{3.32}$$

By the fact that $\langle , \rangle$ is nondegenerate, we must have that

$$\langle \mathbf{v}_1, \mathbf{v}_5 \rangle \neq 0, \qquad \langle \mathbf{v}_2, \mathbf{v}_4 \rangle \neq 0, \qquad \langle \mathbf{v}_3, \mathbf{v}_3 \rangle \neq 0. \tag{3.33}$$

If we scale $\mathbf{v}_1, \ldots, \mathbf{v}_5$ appropriately, we can get an orthonormal basis $\mathbf{u}_1, \ldots, \mathbf{u}_5$ where

$$\langle \mathbf{u}_1, \mathbf{u}_5 \rangle = 1, \qquad \langle \mathbf{u}_2, \mathbf{u}_4 \rangle = -1, \qquad \langle \mathbf{u}_3, \mathbf{u}_3 \rangle = 1. \tag{3.34}$$

Thus, we have shown the desired bijection.

We now wish to show that $\mathsf{SO}(V^{3,2})$ acts transitively on pairs of transverse isotropic flags. To do this, we will define, for an arbitrary pair of transverse isotropic flags (F, G), an element $g \in \mathsf{SO}(V^{3,2})$ such that $(F, G) = (gF_0, gF_\infty)$. It should be clear from construction of the orthonormal basis $\mathbf{u}_1, \ldots, \mathbf{u}_5$ associated to (F, G) that the matrix $g = (\mathbf{u}_1 | \ldots | \mathbf{u}_5)$ is certainly in $\mathsf{SO}(V^{3,2})$ and $(F, G) = (gF_0, gF_\infty)$.

To show that the stabilizer of a transverse pair of isotropic flags is a Cartan subgroup H of $\mathsf{SO}(V^{3,2})$ isomorphic to $(\mathbb{R}^*)^2$, it is sufficient to show that this is the case for (F_0, F_∞).

By definition of F_0, it is clear that $\mathsf{Stab}(F_0) = B_+$. Now recall the definition of the opposite Borel subgroup $B_- = U_- A$ where $U_- = \mathscr{J} U_+ \mathscr{J}$ and note that this is the group consisting of all lower triangular matrices in $\mathsf{SO}(V^{3,2})$. Observe that $\mathscr{J} A \mathscr{J} \subset B_+$ and $U_+ \subset B_+$. With this, we get that $B_- \subseteq \mathsf{Stab}(F_\infty)$. By nature of the maximality of a Borel subgroup and that the stabilizer of a complete flag will always be a Borel subgroup, we conclude that $B_- = \mathsf{Stab}(F_\infty)$. Now we see that

$$\mathsf{Stab}((F_0, F_\infty)) = \mathsf{Stab}(F_0) \cap \mathsf{Stab}(F_\infty) \tag{3.35}$$

where $\mathsf{Stab}(F_0)$ is all upper triangular matrices in $\mathsf{SO}(V^{3,2})$ and $\mathsf{Stab}(F_\infty)$ is all lower triangular matrices in $\mathsf{SO}(V^{3,2})$. Therefore, $\mathsf{Stab}((F_0, F_\infty))$ must be the set of all diagonal matrices in $\mathsf{SO}(V^{3,2})$. A simple computation shows that

$$\mathsf{Stab}((F_0, F_\infty)) = \left\{ \begin{pmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2^{-1} & 0 \\ 0 & 0 & 0 & 0 & \lambda_1^{-1} \end{pmatrix} \middle| \ \lambda_1, \lambda_2 \in \mathbb{R}^* \right\} \cong (\mathbb{R}^*)^2 \tag{3.36}$$

as desired. □

Definition 3.3.4. The *standard pair of transverse flags* we will use is the pair $(F_0, F_\infty) = (B_+, \mathscr{J} B_+)$.

In [FG06], Fock and Goncharov defined the notion of a *positive tuple of flags* which generalizes the cyclic ordering on points of $\mathbb{RP}^1 \cong S^1$. For this definition, we refer back to Section 3.2 for the definition of the positive sub-semigroup $U_+^{>0}$.

Definition 3.3.5 ([FG06])**.** A k-tuple of isotropic flags $(F_1, F_2, F_3, \ldots, F_k)$ with $F_i \in \mathscr{F}$ is *positive* if and only if it is in the $\mathsf{SO}(V^{3,2})$ orbit of a k-tuple of the form

$$(F_0, F_\infty, u_1 F_\infty, u_1 \cdot u_2 F_\infty, \ldots, (u_1 \cdot \cdots \cdot u_{k-2}) F_\infty) \tag{3.37}$$

with $u_i \in U_+^{>0}$.

Lemma 3.3.6 ([FG06])**.** *The positive structure on* $\mathsf{Conf}^{(3)}(\mathscr{F})$ *is invariant under the action of of the symmetric group* S_3*. Equivalently, positivity for triples is independent of the ordering of the triple.*

Notation. We have discussed that an isotropic flag is completely determined by its 1-dimensional and 2-dimensional subspaces. With this is mind and when we have fixed a basis, we will denote a flag F by a 5×2 matrix such that the span of the first column is F^1 and the span of the first two columns is F^2.

Given a triple (F_1, F_2, F_3) of isotropic flags which are in pairwise transverse, we can use Lemma 3.3.3 to find an element of $\mathsf{SO}(V^{3,2})$ which maps F_1 to F_0 and F_2 to F_∞. As shown above, the stabilizer of this pair is given by the subgroup

$$A = \left\{ \begin{pmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2^{-1} & 0 \\ 0 & 0 & 0 & 0 & \lambda_1^{-1} \end{pmatrix} \;\middle|\; \lambda_1, \lambda_2 \in \mathbb{R}^* \right\}. \tag{3.38}$$

Due to the assumed transversality along with the requirement that F_3^1 and F_3^2 are isotropic, we can always write F_3 as a 5×2 matrix of the form

$$F_3 = \begin{pmatrix} 1 & 0 \\ x_1 & 1 \\ x_2 & y_1 \\ x_3 & \frac{y_1^2}{2} \\ x_1 x_3 - \frac{x_2^2}{2} & \frac{x_1 y_1^2}{2} + x_3 - x_2 y_1 \end{pmatrix}. \tag{3.39}$$

Note that x_1, x_2, and x_3 must all be nonzero by transversality.

Now consider the element $a \in A$ with $\lambda_1 = \frac{x_2}{\sqrt{2}}$ and $\lambda_2 = \sqrt{\frac{x_3}{x_1}}$. By having a act on our triple (F_0, F_∞, F_3), we get a new triple (F_0, F_∞, F) which is equivalent in

$\mathsf{Conf}^{(3)}(\mathscr{F})$ where F is given by

$$F = \begin{pmatrix} \frac{x_2}{\sqrt{2}} & 0 \\ \sqrt{x_1 x_3} & \sqrt{\frac{x_3}{x_1}} \\ x_2 & y_1 \\ \sqrt{x_1 x_3} & \frac{y_1^2\sqrt{x_1}}{2\sqrt{x_3}} \\ \frac{2x_1x_3 - x_2^2}{\sqrt{2}x_2} & \frac{x_1y_1^2 + 2x_3 - 2x_2y_1}{\sqrt{2}x_2} \end{pmatrix}. \tag{3.40}$$

Note that we have assumed $\frac{x_3}{x_1} > 0$. If instead $\frac{x_3}{x_1} < 0$, then our triple would not possibly be positive as one can check.

As scaling columns of the 5×2 matrix does not change the associated flag, we can multiply the first column by $\frac{\sqrt{2}}{x_2}$ and multiply the second column by $\sqrt{\frac{x_1}{x_3}}$. If we do this and set $x = \frac{\sqrt{2x_1x_3}}{x_2}$ and $y = y_1\sqrt{\frac{x_1}{x_3}}$, we get a normalized form for F given by

$$F = \begin{pmatrix} 1 & 0 \\ x & 1 \\ \sqrt{2} & y \\ x & \frac{y^2}{2} \\ x^2 - 1 & \frac{xy^2}{2} + x - \sqrt{2}y \end{pmatrix}. \tag{3.41}$$

Definition 3.3.7. For a positive element of $\mathsf{Conf}^{(3)}(\mathscr{F})$, the *normalized form* is the representative triple (F_0, F_∞, F) which is of the form described above. We associate to this positive element of $\mathsf{Conf}^{(3)}(\mathscr{F})$ the *coordinates* (x, y).

Theorem 3.3.8. *In terms of the coordinates (x,y), the totally positive part of $\mathsf{Conf}^{(3)}(\mathscr{F})$ is given by the quadrant $\{(x,y) \in \mathbb{R}^2 \mid x > 1$ and $y > \sqrt{2}x\}$.*

Proof. Suppose we have an element in $\mathsf{Conf}^{(3)}(\mathscr{F})$ with normalized form (F_0, F_∞, F) such that $x > 1$ and $y > \sqrt{2}x$. Observe that, as cosets of B_+, $F_0 = IB_+$, $F_\infty = \mathscr{J}'B_+$, and $F = u_-B_+$ where

$$\mathscr{J}' = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.42}$$

$$u_- = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ x & 1 & 0 & 0 & 0 \\ \sqrt{2} & y & 1 & 0 & 0 \\ x & \frac{y^2}{2} & y & 1 & 0 \\ x^2-1 & \frac{xy^2}{2}+x-\sqrt{2}y & xy-\sqrt{2} & x & 1 \end{pmatrix}. \tag{3.43}$$

Define a new matrix

$$u_+ = \begin{pmatrix} 1 & x & xy-\sqrt{2} & \frac{xy^2}{2}+x-\sqrt{2}y & x^2-1 \\ 0 & 1 & y & \frac{y^2}{2} & x \\ 0 & 0 & 1 & y & \sqrt{2} \\ 0 & 0 & 0 & 1 & x \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.44}$$

and note that $u_- = \mathscr{J}'u_+\mathscr{J}'$. By having $\mathscr{J}'$ act on our triple, we observe that

$$\begin{aligned}(F_0, F_\infty, F) &= (IB_+, \mathscr{J}'B_+, u_-B_+) = (IB_+, \mathscr{J}'B_+, \mathscr{J}'u_+\mathscr{J}'B_+) \\ &= (\mathscr{J}'B_+, IB_+, u_+\mathscr{J}'B_+) = (F_\infty, F_0, u_+F_\infty)\end{aligned} \tag{3.45}$$

as an element of $\mathsf{Conf}^{(3)}(\mathscr{F})$. As positivity is invariant under the action of the symmetric group S_3, we see that (F_0, F_∞, F) will be positive if $u_+ \in U_+^{>0}$. By Lemma 3.2.4, we wish to show that all non-trivial minors of u_+ are strictly positive. By the discussion in the proof of the same lemma, we wish to show that u_+ can be written as M for some a, b, c, d. Take $a = x$, $b = \sqrt{2}$, $c = x$, and $d = y$. As we have assumed $x > 1$ and $y > \sqrt{2}x$, we see that the conditions for positivity are satisfied as follows:

$$\begin{gathered}
a = c = x > 1 > 0, \qquad b = \sqrt{2} > 0, \qquad d = y > \sqrt{2}x > \sqrt{2} > 0, \\
ad - b = xy - \sqrt{2} > \sqrt{2}x^2 - \sqrt{2} > 0, \\
-bd + \tfrac{ad^2}{2} + c = -\sqrt{2}y + \tfrac{xy^2}{2} + x > -\sqrt{2}y + y^2 > -\sqrt{2} + y > 0, \\
ac - \tfrac{b^2}{2} = x^2 - 1 > 0, \\
4c^2 - 4bcd + b^2d^2 = 4x^2 - 4\sqrt{2}xy + 2y^2 = 2(\sqrt{2}x - y)^2 > 0, \\
-2cd + bd^2 = -2xy + \sqrt{2}y^2 = \sqrt{2}y(y - \sqrt{2}x) > 0.
\end{gathered} \tag{3.46}$$

Conversely, suppose that (F_0, F_∞, F) is positive. By construction of u_- and it is clear that this is the unique element in U_- such that $F = u_- B_+$. The uniqueness of u_- gives the uniqueness of $u_+ \in U_+$ such that $(F_0, F_\infty, F) = (F_\infty, F_0, u_+ F_\infty)$. Therefore, $u_+ \in U_+^{>0}$. Therefore, we need only show that the conditions on M must imply that $x > 1$ and $y > \sqrt{2}x$. By upper strict positivity of u_+, we have that $x > 0$ and $x^2 - 1 > 0$, and therefore $x > 1$. Now the minor

$$\mathsf{det}\begin{pmatrix} \frac{y^2}{2} & x \\ y & \sqrt{2} \end{pmatrix} = \frac{y^2}{\sqrt{2}} - xy > 0. \tag{3.47}$$

As $y > 0$ by positivity, this minor yields $y > \sqrt{2}x$. Thus we have shown that the totally positive part of $\mathsf{Conf}^{(3)}(\mathscr{F})$ is exactly the quadrant $\{(x,y) \in \mathbb{R}^2 \mid x > 1 \text{ and } y > \sqrt{2}x\}$. $\square$

3.4 Basis-invariant coordinates of the totally positive part of $\mathsf{Conf}^{(3)}(\mathscr{F})$

In this section, we define alternate coordinates for the totally positive part of the configuration space of triples of isotropic flags which are independent of the choice of basis for $V^{3,2}$. They will also not require the normalization of our triples of flags. The definition of these coordinates is motivated by [FG07] in which Fock and Goncharov define coordinates for positive triples of flags in $\mathbb{RP}^2$. We refer the reader to [CTT18] for a nice exposition and expansion of results by Fock and Goncharov for $\mathbb{RP}^2$.

While the coordinates defined in this section are defined independent of the choice of basis, we will need to fix a basis for computations. Our computations in this section will use both the diagonal and anti-diagonal bases for which the symmetric bilinear form $\langle , \rangle$ is given by J and $\mathscr{J}$, respectively. We will clearly indicate which basis we are using at any given point.

Let (F_1, F_2, F_3) be a triple of isotropic flags which are pairwise transverse. As before, the k-dimensional part of F_i will be denoted by F_i^k. Consider the five of isotropic lines, all contained in F_1^2 given by

$$\ell_1 = F_1^1,$$
$$\ell_2 = F_1^2 \cap (F_2^1)^\perp, \qquad \ell_3 = F_1^2 \cap (F_3^1)^\perp \tag{3.48}$$
$$\ell_4 = F_1^2 \cap (F_2^2 \cap (F_3^1)^\perp)^\perp, \qquad \ell_5 = F_1^2 \cap (F_3^2 \cap (F_2^1)^\perp)^\perp.$$

The fact that the first three subspaces are 1-dimensional is immediate from transversality.

To see that ℓ_4 is 1-dimensional, we first note by transversality that $\mathsf{dim}(F_2^2 \cap (F_3^1)^\perp) = 1$ and thus $\mathsf{dim}((F_2^2 \cap (F_3^1)^\perp)^\perp) = 4$. This tells us that $\mathsf{dim}(\ell_4) \geq 1$. As $\ell_4 \subseteq F_1^2$, we know that $\mathsf{dim}(\ell_4)$ is either 1 or 2.

Suppose that $\mathsf{dim}(\ell_4) = 2$. Choose a $\mathbf{v}$ such that $F_2^2 \cap (F_3^1)^\perp = \mathrm{span}(\mathbf{v})$. Since $\mathbf{v} \in F_2^2$, we know that $\mathbf{v}$ is lightlike. By assumption, $F_1^2 \subset \mathbf{v}^\perp$. Therefore, $\mathrm{span}(F_1^2, \mathbf{v})$ must be isotropic. As the dimension of an isotropic subspace is at most 2, we get that $\mathbf{v} \in F_1^2$, a contradiction on transversality as $\mathbf{v} \in F_2^2$. We conclude that $\mathsf{dim}(\ell_4) = 1$. Similarly, $\mathsf{dim}(\ell_5) = 1$.

Now choose an isomorphism $T : F_1^2 \to \mathbb{R}^2$. For each i, fix a vector $\mathbf{u}_i \in \ell_i$ and set $\hat{\ell}_i = T(\mathbf{u}_i)$. We now define our coordinates of (F_1, F_2, F_3).

Definition 3.4.1. Let (F_1, F_2, F_3) be a triple of isotropic flags which are pairwise transverse. We define the coordinates (c_1, c_2) to be the invariants given by the cross-ratios:

$$c_1 = [\ell_1, \ell_2; \ell_3, \ell_4] = \frac{\mathsf{det}(\hat{\ell}_1\hat{\ell}_4)\mathsf{det}(\hat{\ell}_3\hat{\ell}_2)}{\mathsf{det}(\hat{\ell}_3\hat{\ell}_4)\mathsf{det}(\hat{\ell}_1\hat{\ell}_2)}$$
$$c_2 = [\ell_1, \ell_2; \ell_3, \ell_5] = \frac{\mathsf{det}(\hat{\ell}_1\hat{\ell}_5)\mathsf{det}(\hat{\ell}_3\hat{\ell}_2)}{\mathsf{det}(\hat{\ell}_3\hat{\ell}_5)\mathsf{det}(\hat{\ell}_1\hat{\ell}_2)}. \tag{3.49}$$

The coordinates (c_1, c_2) are clearly independent of the choice of isomorphism T and choice of vectors $\mathbf{u}_i$. Similarly, (c_1, c_2) are invariant under the diagonal action of $\mathsf{SO}(V^{3,2})$ on the triple. This is because the diagonal action of $\mathsf{SO}(V^{3,2})$ induces an isomorphism on $F_1^2 \cong \mathbb{R}^2$ and therefore leaves the cross-ratios unchanged. Hence, we also refer to (c_1, c_2) as coordinates for the element of $\mathsf{Conf}^{(3)}(\mathscr{F})$.

We will show that a particular quadrant of (c_1, c_2) parametrizes the totally positive part of $\mathsf{Conf}^{(3)}(\mathscr{F})$. First we work out an example.

Example 3.4.2. Take $V^{3,2}$ to have the anti-diagonal basis such that the symmetric bilinear form $\langle , \rangle$ is given by the Gram matrix $\mathscr{J}$.

Consider the triple of flags $(F_1, F_2, F_3) = (F_0, F_\infty, F)$ given in the normalized form with $(x, y) = (2, 3\sqrt{2})$ as described in Section 3.3. We wish to compute the coordinates (c_1, c_2) and visualize the lines $\ell_1, \dots, \ell_5$ as points $\mathbb{P}(\ell_1), \dots, \mathbb{P}(\ell_5)$ in Ein.

Our triple of flags can be written as a triple of 5×2 matrices as follows:

$$(F_1, F_2, F_3) = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \\ \sqrt{2} & 3\sqrt{2} \\ 2 & 9 \\ 3 & 14 \end{pmatrix} \right). \tag{3.50}$$

It is clear that we immediately have the following forms for ℓ_1 and ℓ_2:

$$\ell_1 = \operatorname{span}((1 \quad 0 \quad 0 \quad 0 \quad 0)^t), \qquad \ell_2 = \operatorname{span}((0 \quad 1 \quad 0 \quad 0 \quad 0)^t). \tag{3.51}$$

Now we compute ℓ_3. Suppose we have a vector $(a \quad b \quad 0 \quad 0 \quad 0)^t \in F_1^2$ that is orthogonal to F_3^1. By taking the product of this vector and the first column of F_3^1, we get $3a - 2b = 0$. Therefore, we have that

$$\ell_3 = \operatorname{span}\left(\begin{pmatrix}1 & \frac{3}{2} & 0 & 0 & 0\end{pmatrix}^t\right). \tag{3.52}$$

Next we will compute ℓ_4. We begin by computing $F_2^2 \cap (F_3^1)^\perp$. Suppose $(0 \quad 0 \quad 0 \quad a \quad b)^t \in F_2^2$ is orthogonal to F_3^1. Take the product of this vector and the first column of F_3 to get $b - 2a = 0$. This gives us

$$F_2^2 \cap (F_3^1)^\perp = \operatorname{span}((0 \quad 0 \quad 0 \quad 1 \quad 2)^t). \tag{3.53}$$

Now consider any vector $(c \quad d \quad 0 \quad 0 \quad 0)^t \in F_1^2$ that is orthogonal to $F_2^2 \cap (F_3^1)^\perp$. By taking the product of this vector and the representative vector above, we see that $2c - d = 0$. This gives us

$$\ell_4 = \operatorname{span}((1 \quad 2 \quad 0 \quad 0 \quad 0)^t). \tag{3.54}$$

We compute ℓ_5 in a similar way as we did for ℓ_4. Suppose we have a vector

$$a\begin{pmatrix}1\\2\\\sqrt{2}\\2\\3\end{pmatrix} + b\begin{pmatrix}0\\1\\3\sqrt{2}\\9\\14\end{pmatrix} \in F_3^2 \tag{3.55}$$

which is orthogonal to F_2^1. We take the product of this vector with the first column of F_2 to get $a = 0$. Hence,

$$F_3^2 \cap (F_2^1)^\perp = \operatorname{span}((0 \quad 1 \quad 3\sqrt{2} \quad 9 \quad 14)^t). \tag{3.56}$$

Consider a vector $(c \quad d \quad 0 \quad 0 \quad 0)^t \in F_1^2$ which is orthogonal to $F_3^2 \cap (F_2^1)^\perp$. By taking the product of this vector with the representative vector above, we have that $14c - 9d = 0$. Thus, we conclude that

$$\ell_5 = \operatorname{span}\left(\begin{pmatrix}1 & \frac{14}{9} & 0 & 0 & 0\end{pmatrix}^t\right). \tag{3.57}$$

Pick $\mathbf{u}_1, \ldots, \mathbf{u}_5$ to be the representative vectors of $\ell_1, \ldots, \ell_5$ given above. And define an isomorphism

$$T : F_1^2 \longrightarrow \mathbb{R}^2, \qquad (a \quad b \quad 0 \quad 0 \quad 0)^t \longmapsto (a \quad b)^t. \tag{3.58}$$

This gives us

$$\begin{gathered}\hat{\ell}_1 = \begin{pmatrix}1\\0\end{pmatrix}, \qquad \hat{\ell}_2 = \begin{pmatrix}0\\1\end{pmatrix}, \qquad \hat{\ell}_3 = \begin{pmatrix}1\\\frac{3}{2}\end{pmatrix}, \\ \hat{\ell}_4 = \begin{pmatrix}1\\2\end{pmatrix}, \qquad \hat{\ell}_5 = \begin{pmatrix}1\\\frac{14}{9}\end{pmatrix}.\end{gathered} \tag{3.59}$$

We can now compute our coordinates

$$c_1 = \frac{\mathsf{det}\begin{pmatrix}1 & 1\\0 & 2\end{pmatrix}\mathsf{det}\begin{pmatrix}1 & 0\\\frac{3}{2} & 1\end{pmatrix}}{\mathsf{det}\begin{pmatrix}1 & 1\\\frac{3}{2} & 2\end{pmatrix}\mathsf{det}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}} = 4 \tag{3.60}$$

$$c_2 = \frac{\det\begin{pmatrix}1 & 1\\ 0 & \frac{14}{9}\end{pmatrix}\det\begin{pmatrix}1 & 0\\ \frac{3}{2} & 1\end{pmatrix}}{\det\begin{pmatrix}1 & 1\\ \frac{3}{2} & \frac{14}{9}\end{pmatrix}\det\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}} = 28. \tag{3.61}$$

Now that we have computed the coordinates (c_1, c_2), we wish to visualize $\ell_1, \ldots, \ell_5$. As these are isotropic 1-dimensional subspace of $V^{3,2}$, we will consider the corresponding oriented circles $C(\mathbb{P}(\ell_1)), \ldots, C(\mathbb{P}(\ell_5))$ under stereographic projection.

Note that (F_1, F_2, F_3) as expressed above are in terms of the anti-diagonal basis. We wish to express these flags in terms of the diagonal basis to determine the corresponding pointed oriented circles. To do this, we left multiply by the inverse of the change of basis matrix given in (2.64). After scaling to be in standard form for each column, we get

$$(\tilde{F}_1, \tilde{F}_2, \tilde{F}_3) = P^{-1}(F_1, F_2, F_3) = \left(\begin{pmatrix}\frac{1}{2} & 0\\ 0 & 1\\ 0 & 0\\ 0 & 1\\ \frac{1}{2} & 0\end{pmatrix}, \begin{pmatrix}-1 & 0\\ 0 & 1\\ 0 & 0\\ 0 & -1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}2 & -7\\ 0 & 4\\ 1 & -3\\ 2 & -5\\ -1 & 7\end{pmatrix}\right). \tag{3.62}$$

From this, we conclude the following:

- F_1^1 corresponds to the zero radius circle at the origin while F_1^2 corresponds to the photon given by the unit vector based at the origin and pointing downwards.

- F_2^1 corresponds to the point at infinity and F_2^2 corresponds to the photon consisting of all vertical lines with upward orientation.

- F_3^1 corresponds to the circle center at $(0, 1)$ with radius 2 and counterclockwise orientation. F_3^2 will correspond to the unit vector tangent to this circle based at the point where this circle intersects with the line $4x - 3y = 7$.

Now $\ell_1 = F_1^1$ clearly corresponds to the zero radius circle at the origin. Since $\ell_2 = F_1^2 \cap (F_2^1)^\perp$ must go through the north pole and have F_1^2 as a tangent vector, it is the line $x = 0$ with downwards orientation. $\ell_3 = F_1^2 \cap (F_3^1)^\perp$ will correspond to the oriented circle with F_1^2 as a tangent vector which is also tangent to F_3^1.

To determine the circles corresponding to ℓ_4 and ℓ_5, we will first consider the circles corresponding to $F_2^2 \cap (F_3^1)^\perp$ and $F_3^2 \cap (F_2^1)^\perp$, respectively. $F_2^2 \cap (F_3^1)^\perp$ is represented by the oriented vertical line which is tangent to F_3^1. From here, $\ell_4 = F_1^2 \cap (F_2^2 \cap (F_3^1)^\perp)^\perp$ will be the oriented circle with F_1^2 as a tangent vector which is also tangent to this oriented vertical line. Similarly, $F_3^2 \cap (F_2^1)^\perp$ is a line as it goes through the point at infinity. It must also have F_3^2 as a tangent vector. This results in the line $4x - 3y = 7$ with upward orientation. Now we can conclude that $\ell_5 = F_1^2 \cap (F_3^2 \cap (F_2^1)^\perp)^\perp$ is the oriented circle with F_1^2 as a tangent vector which is also tangent to the line $4x - 3y = 7$ with upward orientation.

We have shown how to get $C(\mathbb{P}(\ell_1)), \ldots, C(\mathbb{P}(\ell_5))$ visually from the pointed oriented circles corresponding to F_1, F_2, F_3. This process is pictured in Figure 3.2.

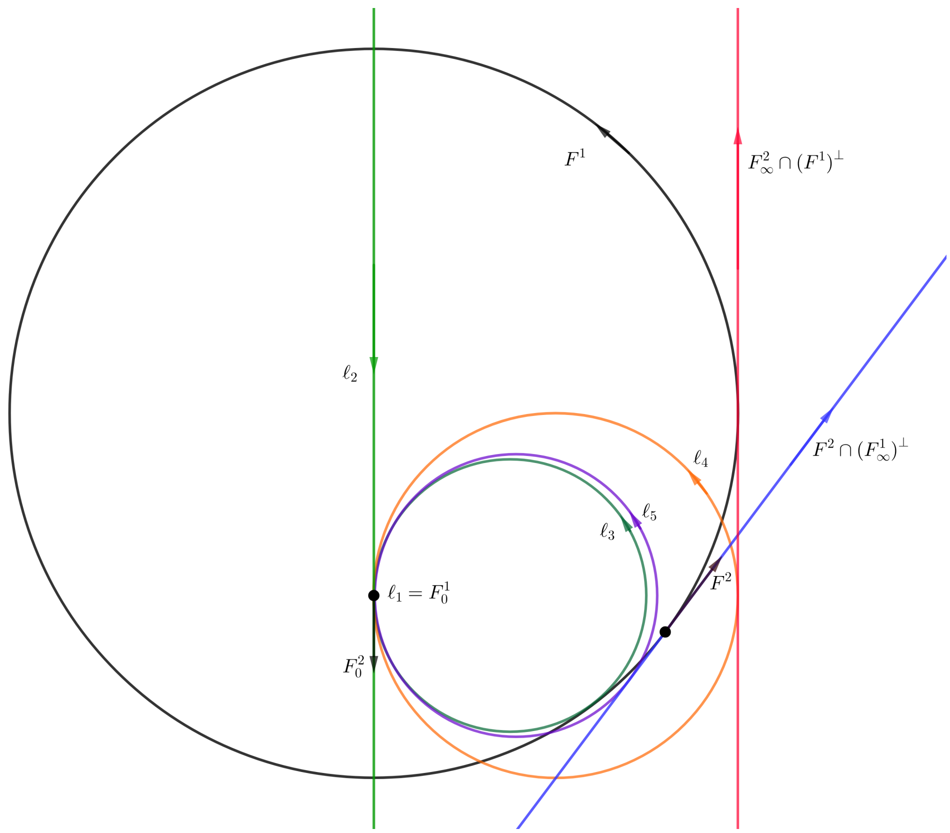


Figure 3.2: The elements $\mathbb{P}(\ell_1), \mathbb{P}(\ell_2), \mathbb{P}(\ell_3), \mathbb{P}(\ell_4), \mathbb{P}(\ell_5) \in \mathsf{Ein}$ used to define our coordinates as seen in the Lie circles model of the Einstein universe.

We now wish to describe the totally positive part $\mathsf{Conf}^{(3)}(\mathscr{F})$ in terms of the coordinates (c_1, c_2). We begin with a lemma relating our two types of coordinates (x, y) and (c_1, c_2).

Lemma 3.4.3. *For a positive element in* $\mathsf{Conf}^{(3)}(\mathscr{F})$*, the change of coordinates from* (x, y) *to* (c_1, c_2) *is given by*

$$(c_1, c_2) = \left(x^2, \frac{x^2y^2 + 2x^2 - 2\sqrt{2}xy}{2x^2 - 2\sqrt{2}xy + y^2} \right). \tag{3.63}$$

As we know that $x > 1$*,* $y > \sqrt{2}x$*, this change of coordinates is well-defined and implies that* $1 < c_1 < c_2$*. Furthermore, the change of coordinates from* (c_1, c_2) *to* (x, y) *is given by*

$$(x, y) = \left(\sqrt{c_1}, \sqrt{2c_1} \left(\frac{1 - c_2 - \sqrt{(c_1 - 1)(c_2 - 1)}}{c_1 - c_2} \right) \right). \tag{3.64}$$

Proof. Take $V^{3,2}$ to have the anti-diagonal basis such that the symmetric bilinear form $\langle , \rangle$ is given by the Gram matrix $\mathscr{J}$.

Consider the normalized form (F_0, F_∞, F) of our positive element of $\mathsf{Conf}^{(3)}(\mathscr{F})$. So we will be working with the triple

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ x & 1 \\ \sqrt{2} & y \\ x & \frac{y^2}{2} \\ x^2 - 1 & \frac{xy^2}{2} + x - \sqrt{2}y \end{pmatrix} \right). \tag{3.65}$$

We will first compute $\ell_1, \dots, \ell_5$. Just as in the previous example, we have that

$$\begin{aligned} \ell_1 &= F_0^1 = \operatorname{span}\left((1 \quad 0 \quad 0 \quad 0 \quad 0)^t\right) \\ \ell_2 &= F_0^2 \cap (F_\infty^1)^\perp = \operatorname{span}\left((0 \quad 1 \quad 0 \quad 0 \quad 0)^t\right). \end{aligned} \tag{3.66}$$

Now suppose we have some vector $(a \quad b \quad 0 \quad 0 \quad 0)^t \in F_0^2$ which is also orthogonal to F^1. This means that

$$0 = \left\langle \begin{pmatrix} a \\ b \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ x \\ \sqrt{2} \\ x \\ x^2 - 1 \end{pmatrix} \right\rangle = a(x^2 - 1) - bx. \tag{3.67}$$

By scaling, we can take $a = 1$ and get that $b = \frac{x^2-1}{x}$. Thus

$$\ell_3 = F_0^2 \cap (F^1)^\perp = \operatorname{span}\left(\begin{pmatrix} 1 & \dfrac{x^2-1}{x} & 0 & 0 & 0 \end{pmatrix}^t\right). \tag{3.68}$$

To compute ℓ_4 we first find a representative of $F_\infty^2 \cap (F^1)^\perp$. To do this, suppose we have a vector $(0 \quad 0 \quad 0 \quad a \quad b)^t \in F_\infty^2$ which is orthogonal to F^1. Then we see that

$$0 = \left\langle \begin{pmatrix} 0 \\ 0 \\ 0 \\ a \\ b \end{pmatrix}, \begin{pmatrix} 1 \\ x \\ \sqrt{2} \\ x \\ x^2 - 1 \end{pmatrix} \right\rangle = b - ax. \tag{3.69}$$

With scaling, we can set $a = 1$ which gives that $b = x$. Therefore

$$F_\infty^2 \cap (F^1)^\perp = \operatorname{span}\left((0 \quad 0 \quad 0 \quad 1 \quad x)^t\right). \tag{3.70}$$

Now if we have a vector $(c \quad d \quad 0 \quad 0 \quad 0)^t \in F_0^2$ which is orthogonal to $F_\infty^2 \cap (F^1)^\perp$, then we must have that

$$0 = \left\langle (c \quad d \quad 0 \quad 0 \quad 0)^t, (0 \quad 0 \quad 0 \quad 1 \quad x)^t \right\rangle = cx - d. \tag{3.71}$$

We can then conclude that

$$\ell_4 = F_0^2 \cap (F_\infty^2 \cap (F^1)^\perp)^\perp = \operatorname{span}\left((1 \quad x \quad 0 \quad 0 \quad 0)^t\right). \tag{3.72}$$

Similarly, to compute ℓ_5, we suppose a vector $\mathbf{v} \in F^2$ is orthogonal to F_∞^1. This means that

$$0 = \left\langle \mathbf{v}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle = \left\langle a \begin{pmatrix} 1 \\ x \\ \sqrt{2} \\ x \\ x^2 - 1 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ y \\ \frac{y^2}{2} \\ \frac{xy^2}{2} + x - \sqrt{2}y \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle = a. \tag{3.73}$$

Now if we have a vector $(c \quad d \quad 0 \quad 0 \quad 0)^t \in F_0^2$ which is orthogonal to $F^2 \cap (F_\infty^1)^\perp$, then we get

$$0 = \left\langle \begin{pmatrix} c \\ d \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ y \\ \frac{y^2}{2} \\ \frac{xy^2}{2} + x - \sqrt{2}y \end{pmatrix} \right\rangle = c\left(\frac{xy^2}{2} + x - \sqrt{2}y\right) - d\frac{y^2}{2}. \tag{3.74}$$

By scaling, we can take $a = 1$ which will gives us that $b = x + \frac{2x}{y^2} - \frac{2\sqrt{2}}{y}$. We then conclude that

$$\ell_5 = F_0^2 \cap (F^2 \cap (F_\infty^1)^\perp)^\perp = \operatorname{span}\left(\begin{pmatrix} 1 \\ x + \frac{2x}{y^2} - \frac{2\sqrt{2}}{y} \\ 0 \\ 0 \\ 0 \end{pmatrix}\right). \tag{3.75}$$

Now we define an isomorphism $T : F_0^2 \to \mathbb{R}^2$ by $T\left((a \quad b \quad 0 \quad 0 \quad 0)^t\right) = (a \quad b)^t$.

With this choice of isomorphism, we have that

$$\begin{gathered} \hat{\ell}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \hat{\ell}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \hat{\ell}_3 = \begin{pmatrix} 1 \\ \frac{x^2-1}{x} \end{pmatrix} \\ \hat{\ell}_4 = \begin{pmatrix} 1 \\ x \end{pmatrix}, \qquad \hat{\ell}_5 = \begin{pmatrix} 1 \\ x + \frac{2x}{y^2} - \frac{2\sqrt{2}}{y} \end{pmatrix}. \end{gathered} \tag{3.76}$$

From here we can compute our change of coordinates from (x, y) to (c_1, c_2) and get

$$c_1 = \frac{\det\begin{pmatrix} 1 & 1 \\ 0 & x \end{pmatrix} \det\begin{pmatrix} 1 & 0 \\ \frac{x^2-1}{x} & 1 \end{pmatrix}}{\det\begin{pmatrix} 1 & 1 \\ \frac{x^2-1}{x} & x \end{pmatrix} \det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} = x^2 \tag{3.77}$$

$$c_2 = \frac{\det\begin{pmatrix} 1 & 1 \\ 0 & x + \frac{2x}{y^2} - \frac{2\sqrt{2}}{y} \end{pmatrix} \det\begin{pmatrix} 1 & 0 \\ \frac{x^2-1}{x} & 1 \end{pmatrix}}{\det\begin{pmatrix} 1 & 1 \\ \frac{x^2-1}{x} & x + \frac{2x}{y^2} - \frac{2\sqrt{2}}{y} \end{pmatrix} \det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} = \frac{x^2y^2 + 2x^2 - 2\sqrt{2}xy}{2x^2 - 2\sqrt{2}xy + y^2}. \tag{3.78}$$

For the change of coordinates from (c_1, c_2) to (x, y), it is clear that $x = \sqrt{c_1}$ since $c_1 = x^2$ and we know that $x > 1$ as our triple is positive. To find an equation for y in terms of c_1 and c_2 we plug $x = \sqrt{c_1}$ into the above equation for c_2. After simplifying, we get

$$(c_1 - c_2)y^2 + 2\sqrt{2c_1}(c_2 - 1) - 2c_1(c_2 - 1) = 0. \tag{3.79}$$

With the quadratic formula, we see that

$$\begin{aligned} y &= \frac{-2\sqrt{2c_1}(c_2 - 1) \pm \sqrt{8c_1(c_2 - 1)^2 + 8c_1(c_1 - c_2)(c_2 - 1)}}{2(c_1 - c_2)} \\ &= \sqrt{2c_1}\left(\frac{1 - c_2 \pm \sqrt{(c_1 - 1)(c_2 - 1)}}{c_1 - c_2}\right). \end{aligned} \tag{3.80}$$

We now wish to determine the sign in the equation above. We note that since our triple is positive, we have $y > \sqrt{2}x$ by Theorem 3.3.8 and hence the equation above tells us that

$$\frac{1 - c_2 \pm \sqrt{(c_1 - 1)(c_2 - 1)}}{c_1 - c_2} > 1. \tag{3.81}$$

Suppose that $c_1 > c_2$, then we can simplify the inequality above to give

$$1 - c_2 \pm \sqrt{(c_1 - 1)(c_2 - 1)} > c_1 - c_2 \tag{3.82}$$

which implies that

$$c_1 - 1 < \pm\sqrt{(c_1 - 1)(c_2 - 1)}. \tag{3.83}$$

However, since our triple is positive, we have that $c_1 = x^2 > 1$ by Theorem 3.3.8. Therefore, we would be forced to have the sign above be $+$. Since we assumed $c_1 > c_2$, we would get

$$\sqrt{(c_1 - 1)(c_2 - 1)} < \sqrt{(c_1 - 1)^2} = c_1 - 1, \tag{3.84}$$

a contradiction. We can therefore conclude that $c_1 < c_2$. Similar to above, since we have $c_1 < c_2$, we can determine that

$$c_1 - 1 > \pm\sqrt{(c_1-1)(c_2-1)}. \tag{3.85}$$

Suppose that the sign above is $+$, then using $c_1 < c_2$, we would have

$$\sqrt{(c_1-1)(c_2-1)} > \sqrt{(c_1-1)^2} = c_1 - 1, \tag{3.86}$$

a contradiction. We can therefore conclude that the choice of sign must be $-$ and therefore

$$y = \sqrt{2c_1}\left(\frac{1 - c_2 - \sqrt{(c_1-1)(c_2-1)}}{c_1 - c_2}\right), \tag{3.87}$$

completing our proof. $\square$

Theorem 3.4.4. *In terms of the coordinates (c_1, c_2), the totally positive part of $\mathsf{Conf}^{(3)}(\mathscr{F})$ is given by the quadrant $\{(c_1, c_2) \in \mathbb{R}^2 \mid 1 < c_1 < c_2\}$.*

Proof. In Theorem 3.3.8, we showed that, in terms of the coordinates (x, y), the totally positive part of $\mathsf{Conf}^{(3)}(\mathscr{F})$ was given by $x > 1$, $y > \sqrt{2}x$. In the proof of Lemma 3.4.3, we used this quadrant to establish that $c_1 > 1$ and $c_1 < c_2$. What remains to be shown is that $1 < c_1 < c_2$ implies that $x > 1$ and $y > \sqrt{2}x$.

Suppose that $1 < c_1 < c_2$. Then it is clear that $x = \sqrt{c_1} > 1$. Now the change of coordinates in Lemma 3.4.3 gives that

$$\begin{aligned} y &= \sqrt{2c_1}\left(\frac{1 - c_2 - \sqrt{(c_1-1)(c_2-1)}}{c_1 - c_2}\right) \\ &= \sqrt{2}x\left(\frac{1 - c_2 - \sqrt{(c_1-1)(c_2-1)}}{c_1 - c_2}\right). \end{aligned} \tag{3.88}$$

Therefore, we need to show that the conditions on (c_1, c_2) imply that

$$1 < \frac{1 - c_2 - \sqrt{(c_1-1)(c_2-1)}}{c_1 - c_2} = 1 + \frac{1 - c_1 - \sqrt{(c_1-1)(c_2-1)}}{c_1 - c_2}. \tag{3.89}$$

This is equivalent to showing

$$\frac{1 - c_1 - \sqrt{(c_1-1)(c_2-1)}}{c_1 - c_2} > 0. \tag{3.90}$$

Since $c_1 < c_2$, we can simplify the above inequality and determine that we need only to show that

$$1 - c_1 - \sqrt{(c_1-1)(c_2-1)} < 0. \tag{3.91}$$

Using that $1 < c_1 < c_2$, we can conclude that this inequality indeed holds since

$$\begin{aligned} 1 - c_1 - \sqrt{(c_1-1)(c_2-1)} &< 1 - c_1 - \sqrt{(c_1-1)^2} \\ &= 1 - c_1 - (c_1 - 1) \\ &= 2(1 - c_1) < 0. \end{aligned} \tag{3.92}$$

Therefore, we get that $x > 1$, $y > \sqrt{2}x$ if and only if $1 < c_1 < c_2$. Hence, the totally positive part of $\mathsf{Conf}^{(3)}(\mathscr{F})$ is given by the quadrant $1 < c_1 < c_2$. $\square$

3.5 Positive triples of flags in $\mathsf{PSp}(4, \mathbb{R})$

In this section, we describe positive triples of flags in $\mathsf{PSp}(4, \mathbb{R})$. First, we describe the complete flag manifold for $\mathsf{PSp}(4, \mathbb{R})$. We then explicitly describe the sub-semigroup $U_+^{>0}$ defined in [Lus94] for the semisimple Lie group $\mathsf{PSp}(4, \mathbb{R})$. Next we recall the definition of a positive tuple of flags in $\mathsf{PSp}(4, \mathbb{R})$ as defined by Fock and Goncharov in [FG06]. We then use the isomorphism $\mathsf{SO}^0(V^{3,2}) \cong \mathsf{PSp}(4, \mathbb{R})$ to show that notions of positivity for $\mathsf{SO}^0(V^{3,2})$ and $\mathsf{PSp}(4,\mathbb{R})$ are analogous. Finally, we translate the notion of the normalized form of a positive triple of isotropic flags in $\mathsf{SO}(V^{3,2})$ and the corresponding coordinates (x, y) in terms of $\mathsf{PSp}(4,\mathbb{R})$ using the isomorphism.

Take V to be a 4-dimensional real symplectic vector space with symplectic form ω. Fix a basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ of V such that ω is given by the matrix

$$\Omega = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{3.93}$$

where $\Omega_{ij} = \omega(\mathbf{e}_i, \mathbf{e}_j)$.

Again, we refer to [Kna02] for a refresher on Cartan subgroups, simple roots, and other Lie theory ideas used here.

Consider the Cartan subgroup $\tilde{A} \subset \mathsf{PSp}(4,\mathbb{R})$ given by the diagonal matrices in the group

$$\tilde{A} = \left\{ \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_2^{-1} & 0 \\ 0 & 0 & 0 & \lambda_1^{-1} \end{pmatrix} \middle| \, \lambda_1, \lambda_2 \in \mathbb{R}^* \right\}. \tag{3.94}$$

The Cartan subalgebra $\tilde{\mathfrak{a}}$ corresponding to $\tilde{A}$ is given by

$$\tilde{\mathfrak{a}} = \left\{ \left. \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & -b & 0 \\ 0 & 0 & 0 & -a \end{pmatrix} \right| \lambda_1, \lambda_2 \in \mathbb{R}^* \right\}. \tag{3.95}$$

As our Lie group is rank two, we know that we will have two simple roots. With respect to this Cartan subalgebra, we choose our simple roots $\alpha_i : \tilde{\mathfrak{a}} \to \mathbb{R}$ to be

$$\tilde{\alpha}_1(a,b) = 2b, \qquad \tilde{\alpha}_2(a,b) = a - b. \tag{3.96}$$

For $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$, there are associated maps $\tilde{a}_i : \mathbb{R} \to \mathsf{PSp}(4,\mathbb{R})$ given by

$$\tilde{a}_1(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \tilde{a}_2(t) = \begin{pmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.97}$$

$\tilde{a}_1$ together with $\tilde{a}_2$ generate the unipotent subgroup $\tilde{U}_+ \subset \mathsf{PSp}(4,\mathbb{R})$ which consists of the unipotent upper triangular matrices. The Cartan subgroup $\tilde{A}$ and the unipotent subgroup $\tilde{U}_+$ together span a Borel subgroup $\tilde{B}_+ = \tilde{A}\tilde{U}_+$ of $\mathsf{PSp}(4,\mathbb{R})$. This Borel subgroup $\tilde{B}_+$ is the set of al upper triangular matrices in $\mathsf{PSp}(4,\mathbb{R})$.

We use this setup in the following sections.

3.5.1 Complete isotropic flags in $\mathsf{PSp}(4,\mathbb{R})$

In this section, we describe complete isotropic flags in a 4-dimensional real symplectic vector space V.

Definition 3.5.1. The *complete isotropic flag manifold* is the homogeneous space

$$\tilde{\mathscr{F}} := \mathsf{PSp}(4,\mathbb{R}) \diagup \tilde{B}_+. \tag{3.98}$$

We refer to an element $F \in \tilde{\mathscr{F}}$ as an *isotropic flag*.

Similar to the case for $\mathsf{SO}(V^{3,2})$, we note that $\tilde{B}_+$ stabilizes the following flag in V:

$$0 \subset \operatorname{span}(\mathbf{e}_1) \subset \operatorname{span}(\mathbf{e}_1, \mathbf{e}_2) \subset \operatorname{span}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \subset V. \tag{3.99}$$

And with respect to the symplectic form ω, we can rewrite this flag as

$$0 \subset \operatorname{span}(\mathbf{e}_1) \subset \operatorname{span}(\mathbf{e}_1, \mathbf{e}_2) \subset \operatorname{span}(\mathbf{e}_1)^{\perp} \subset V. \tag{3.100}$$

We observe that both $\operatorname{span}(\mathbf{e}_1)$ and $\operatorname{span}(\mathbf{e}_1, \mathbf{e}_2)$ are both isotropic subspaces of V. Since $\mathsf{PSp}(4,\mathbb{R})$ preserves the bilinear form, the 1-dimensional and 2-dimensional subspaces of any flag in $\tilde{\mathscr{F}}$ are necessarily isotropic. This also tells us that the 3-dimensional subspace will always be the orthogonal subspace associated to the 1-dimensional subspace. Thus any flag $F \in \tilde{\mathscr{F}}$ is entirely determined by its 1-dimensional and 2-dimensional isotropic subspaces. This is the reason we refer to $\tilde{\mathscr{F}}$ as the complete isotropic flag manifold.

Just as was the case for $\mathsf{SO}(V^{3,2})$, we note that 1-dimensional and 2-dimensional subspaces in V correspond to elements of Pho and Ein, respectively. Thus any flag in $\tilde{\mathscr{F}}$ can be thought of as a pointed photon.

Note $B_+ \subset \mathsf{SO}(V^{3,2})$ contains elements not in the identity component of $\mathsf{SO}(V^{3,2})$. One can check that diagonal matrix $\mathsf{diag}(-1,1,1,1,-1)$ is such an element. These elements necessarily switch the component of $\mathsf{SO}(V^{3,2})$. Therefore, every flag in ${}^{\mathsf{SO}(V^{3,2})}\!\big/\!{}_{B_+}$ has a representative in $\mathsf{SO}^0(V^{3,2})$. This observation along with the isomorphism $\mathsf{PSp}(4,\mathbb{R}) \cong \mathsf{SO}^0(V^{3,2})$ leads us to conclude that $\tilde{\mathscr{F}} \cong \mathscr{F}$. As they are the same flag manifold, we refer to elements of both $\mathscr{F}$ and $\tilde{\mathscr{F}}$ as isotropic flags.

3.5.2 Positivity in $\mathsf{PSp}(4,\mathbb{R})$

In this section, we explicitly describe the positive sub-semigroup $\tilde{U}_+^{>0}$, as defined by Lusztig in [Lus94], for the semisimple Lie group $G = \mathsf{PSp}(4,\mathbb{R})$. Through the isomorphism $\mathsf{PSp}(4,\mathbb{R}) \cong \mathsf{SO}^0(V^{3,2})$, we confirm that $\tilde{U}_+^{>0} \cong U_+^{>0}$ where $U_+^{>0}$ is the sub-semigroup described in Section 3.2. We will show that, for our choice of basis, $\tilde{U}_+^{>0}$ can be described as the subset of all matrices in $\tilde{U}_+$ whose non-trivial minors are strictly positive. Hence, the notion of positivity is the same in both $\mathsf{SO}(V^{3,2})$ and $\mathsf{PSp}(4,\mathbb{R})$.

The Borel subgroup $\tilde{B}_+$, Cartan subgroup $\tilde{A}$, and the maps $\tilde{a}_1$ and $\tilde{a}_2$ determine a pinning for $\mathsf{PSp}(4,\mathbb{R})$. We again refer to [Bou05] for the general definition of a pinning. Just as for $\mathsf{SO}(V^{3,2})$, we think of a pinning as a pair of representations $\tilde{\rho}_1, \tilde{\rho}_2 : \mathsf{SL}(2,\mathbb{R}) \to \mathsf{PSp}(4,\mathbb{R})$ such that the image by $\tilde{\rho}_i$ of the one-parameter upper triangular subgroup $\left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \middle| t \in \mathbb{R} \right\}$ is exactly $\{\tilde{a}_i(t) | t \in \mathbb{R}\}$. We can explicitly define these representations as

$$\tilde{\rho}_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & c & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.101}$$

$$\tilde{\rho}_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix}. \tag{3.102}$$

Next we wish to describe the *Weyl group* $\tilde{W} = {N_G(\tilde{A})}/{Z_G(\tilde{A})}$ of $G = \mathsf{PSp}(4, \mathbb{R})$. For $\tilde{W}$, we have a set of generators $\tilde{s}_i = \tilde{\rho}_i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ associated to the pinning. We note that each of these generators

$$\tilde{s}_1 = \tilde{\rho}_1 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.103}$$

$$\tilde{s}_2 = \tilde{\rho}_2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{3.104}$$

are of order 2 when considered as acting by conjugation on $\tilde{\mathfrak{a}}$, the Lie subalgebra associated to $\tilde{A}$. Furthermore, their product

$$\tilde{s}_1\tilde{s}_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{3.105}$$

is of order 4 when considered as acting by conjugation on $\tilde{\mathfrak{a}}$. We conclude that $\tilde{W}$ is isomorphic to the dihedral group D_4. Just as for $\mathsf{SO}(V^{3,2})$, the Weyl group W is a finite Coxeter group so there is a well defined notion of longest word, i.e. the element with the longest length for a reduced expression in terms of the generators (See [Hum90]). The *longest word* of the Weyl group $\tilde{W}$ has two reduced expressions in terms of our choice of generators $\tilde{s}_1\tilde{s}_2\tilde{s}_1\tilde{s}_2$ and $\tilde{s}_2\tilde{s}_1\tilde{s}_2\tilde{s}_1$.

Definition 3.5.2 ([Lus94]). Let $\tilde{w}_0$ denote the longest element of $\tilde{W}$ and let $j_1, \dots, j_l$ be a sequence of indices such that $\tilde{s}_{j_1} \dots \tilde{s}_{j_l}$ is a reduced expression for $\tilde{w}_0$. The *positive sub-semigroup* $\tilde{U}_+^{>0}$ is defined to be

$$\tilde{U}_+^{>0} := \{\tilde{a}_{j_1}(t_1) \dots \tilde{a}_{j_l}(t_l) | t_i > 0\} \subset \tilde{U}_+. \tag{3.106}$$

This is independent of the choice of reduced expression for $\tilde{w}_0$.

For $G = \mathsf{PSp}(4,\mathbb{R})$ and with choice of reduced expression $\tilde{s}_2\tilde{s}_1\tilde{s}_2\tilde{s}_1$, we get that

$$\tilde{a}_2(t_1)\tilde{a}_1(t_2)\tilde{a}_2(t_3)\tilde{a}_1(t_4) = \begin{pmatrix} 1 & t_1+t_3 & t_3t_4+t_1(t_2+t_4) & t_1t_2t_3 \\ 0 & 1 & t_2+t_4 & t_2t_3 \\ 0 & 0 & 1 & t_1+t_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.107}$$

Now that we have explicitly described the positive sub-semigroup $\tilde{U}_+^{>0}$ for $\mathsf{PSp}(4,\mathbb{R})$, we wish to show that $\tilde{U}_+^{>0} \cong U_+^{>0}$ under the isomorphism $\mathsf{PSp}(4,\mathbb{R}) \cong \mathsf{SO}^0(V^{3,2})$. We refer back to Section 2.3 and briefly recall some of the setup in defining the isomorphism $\Phi : \mathsf{PSp}(4,\mathbb{R}) \xrightarrow{\sim} \mathsf{SO}^0(V^{3,2})$.

We take V to be a 4-dimensional real vector space with basis $\mathbf{e}_1, \ldots, \mathbf{e}_4$. For the 6-dimensional real vector space $\Lambda^2(V)$, we fix a basis $\mathbf{e}_1 \wedge \mathbf{e}_2, \mathbf{e}_1 \wedge \mathbf{e}_3, \mathbf{e}_1 \wedge \mathbf{e}_4, \mathbf{e}_2 \wedge \mathbf{e}_3, \mathbf{e}_2 \wedge \mathbf{e}_4, \mathbf{e}_3 \wedge \mathbf{e}_4$. We define a symmetric nondegenerate bilinear form $B(,)$ on $\Lambda^2(V)$ by

$$\alpha_1 \wedge \alpha_2 = B(\alpha_1, \alpha_2)\mathrm{vol} \tag{3.108}$$

where $\mathrm{vol} = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4$. We let ω be a symplectic form on V. The symplectic form ω defines a dual exterior bivector $\omega^* \in \Lambda^2(V)$ given by

$$\omega(\mathbf{v}_1, \mathbf{v}_2) = B(\mathbf{v}_1 \wedge \mathbf{v}_2, \omega^*). \tag{3.109}$$

Throughout this section, we will fix the basis of V to be $\mathbf{e}_1, \ldots, \mathbf{e}_4$ and its symplectic form to be ω. Now without loss of generality, we assume that

$$\omega^* = \mathbf{e}_1 \wedge \mathbf{e}_4 - \mathbf{e}_2 \wedge \mathbf{e}_3. \tag{3.110}$$

We note that $B(\omega^*, \omega^*) = -2 < 0$ and therefore the restriction of $B(,)$ on

$$W_0 := (\omega^*)^\perp \subset \Lambda^2(V) \tag{3.111}$$

has signature $(3, 2)$.

With this setup in mind, we define a basis for W_0 as follows

$$\begin{gathered} \mathbf{f_1} = \mathbf{e}_1 \wedge \mathbf{e}_2, \qquad \mathbf{f_2} = \mathbf{e}_1 \wedge \mathbf{e}_3, \qquad \mathbf{f_3} = \tfrac{1}{\sqrt{2}}(\mathbf{e}_1 \wedge \mathbf{e}_4 + \mathbf{e}_2 \wedge \mathbf{e}_3) \\ \mathbf{f_4} = \mathbf{e}_2 \wedge \mathbf{e}_4, \qquad \mathbf{f_5} = \mathbf{e}_3 \wedge \mathbf{e}_4. \end{gathered} \tag{3.112}$$

Observe that the Gram matrix for our symmetric nondegenerate bilinear form $B(,)$ restricted to W_0 is given by the anti-diagonal matrix $\mathscr{J}$ and therefore $\mathbf{f}_1, \ldots, \mathbf{f}_5$ corresponds to our choice of anti-diagonal basis for $V^{3,2}$.

Now, we wish to observe how $\tilde{a}_1(t)$ and $\tilde{a}_2(t)$ act on W_0 through the defined isomorphism Φ. We see that

$$\begin{aligned} \tilde{a}_1(t)(\mathbf{f}_1) &= \tilde{a}_1(t)(\mathbf{e}_1) \wedge \tilde{a}_1(t)(\mathbf{e}_2) = \mathbf{e}_1 \wedge \mathbf{e}_2 = \mathbf{f}_1 \\ \tilde{a}_1(t)(\mathbf{f}_2) &= \tilde{a}_1(t)(\mathbf{e}_1) \wedge \tilde{a}_1(t)(\mathbf{e}_3) = \mathbf{e}_1 \wedge (t\mathbf{e}_2 + \mathbf{e}_3) = t\mathbf{f}_1 + \mathbf{f}_2 \\ \tilde{a}_1(t)(\mathbf{f}_2) &= \frac{1}{\sqrt{2}}(\tilde{a}_1(t)(\mathbf{e}_1) \wedge \tilde{a}_1(t)(\mathbf{e}_4) + \tilde{a}_1(t)(\mathbf{e}_2) \wedge \tilde{a}_1(t)(\mathbf{e}_3)) \\ &= \frac{1}{\sqrt{2}}(\mathbf{e}_1 \wedge \mathbf{e}_4 + \mathbf{e}_2 \wedge (t\mathbf{e}_2 + \mathbf{e}_3)) = \mathbf{f}_3 \\ \tilde{a}_1(t)(\mathbf{f}_4) &= \tilde{a}_1(t)(\mathbf{e}_2) \wedge \tilde{a}_1(t)(\mathbf{e}_4) = \mathbf{e}_2 \wedge \mathbf{e}_4 = \mathbf{f}_4 \\ \tilde{a}_1(t)(\mathbf{f}_5) &= \tilde{a}_1(t)(\mathbf{e}_3) \wedge \tilde{a}_1(t)(\mathbf{e}_4) = (t\mathbf{e}_2 + \mathbf{e}_3) \wedge \mathbf{e}_4 = t\mathbf{f}_4 + \mathbf{f}_5. \end{aligned} \tag{3.113}$$

Thus, as a matrix in $\mathsf{SO}(V^{3,2})$ with respect to the anti-diagonal basis, we see that

$$\Phi(\tilde{a}_1(t)) = \begin{pmatrix} 1 & t & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = a_1(t). \tag{3.114}$$

Similarly, we have that

$$\begin{aligned}
\tilde{a}_2(t)(\mathbf{f}_1) &= \tilde{a}_2(t)(\mathbf{e}_1) \wedge \tilde{a}_2(t)(\mathbf{e}_2) = \mathbf{e}_1 \wedge (t\mathbf{e}_1 + \mathbf{e}_2) = \mathbf{f}_1 \\
\tilde{a}_2(t)(\mathbf{f}_2) &= \tilde{a}_2(t)(\mathbf{e}_1) \wedge \tilde{a}_2(t)(\mathbf{e}_3) = \mathbf{e}_1 \wedge \mathbf{e}_3 = \mathbf{f}_2 \\
\tilde{a}_2(t)(\mathbf{f}_2) &= \frac{1}{\sqrt{2}}(\tilde{a}_2(t)(\mathbf{e}_1) \wedge \tilde{a}_2(t)(\mathbf{e}_4) + \tilde{a}_2(t)(\mathbf{e}_2) \wedge \tilde{a}_2(t)(\mathbf{e}_3)) \\
&= \frac{1}{\sqrt{2}}(\mathbf{e}_1 \wedge (t\mathbf{e}_3 + \mathbf{e}_4) + (t\mathbf{e}_1 + \mathbf{e}_2) \wedge \mathbf{e}_3) \\
&= \frac{1}{\sqrt{2}}(2t(\mathbf{e}_1 \wedge \mathbf{e}_3) + (\mathbf{e}_1 \wedge \mathbf{e}_4 + \mathbf{e}_2 \wedge \mathbf{e}_3)) = \sqrt{2}t\mathbf{f}_2 + \mathbf{f}_3 \\
\tilde{a}_2(t)(\mathbf{f}_4) &= \tilde{a}_2(t)(\mathbf{e}_2) \wedge \tilde{a}_2(t)(\mathbf{e}_4) = (t\mathbf{e}_1 + \mathbf{e}_2) \wedge (t\mathbf{e}_3 + \mathbf{e}_4) \\
&= t^2\mathbf{f}_2 + \sqrt{2}t\mathbf{f}_3 + \mathbf{f}_4 \\
\tilde{a}_2(t)(\mathbf{f}_5) &= \tilde{a}_2(t)(\mathbf{e}_3) \wedge \tilde{a}_2(t)(\mathbf{e}_4) = \mathbf{e}_3 \wedge (t\mathbf{e}_3 + \mathbf{e}_4) = \mathbf{f}_5.
\end{aligned} \tag{3.115}$$

So as a matrix in $\mathsf{SO}(V^{3,2})$ with respect to the anti-diagonal basis, we see that

$$\Phi(\tilde{a}_2(t)) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & \sqrt{2}t & t^2 & 0 \\ 0 & 0 & 1 & \sqrt{2}t & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = a_1(t). \tag{3.116}$$

By the definitions of $U_+^{>0}$ and $\tilde{U}_+^{>0}$, showing that $\Phi(\tilde{a}_1(t)) = a_1(t)$ and $\Phi(\tilde{a}_2(t)) = a_2(t)$ is sufficient to prove that $\Phi(\tilde{U}_+^{>0}) = U_+^{>0}$. We conclude that $\tilde{U}_+^{>0} \cong U_+^{>0}$ as desired.

Now we wish to characterize $\tilde{U}_+^{>0}$ as the subset of matrices in $\tilde{U}_+$ whose non-trivial minors are strictly positive.

Lemma 3.5.3. *The positive sub-semigroup $\tilde{U}_+^{>0}$ is the subset of all matrices in $\tilde{U}_+$ such that all their minors which are not zero by upper triangularity are strictly positive.*

Proof. We first wish to show that all non-trivial minors of $\tilde{a}_2(t_1)\tilde{a}_1(t_2)\tilde{a}_2(t_3)\tilde{a}_1(t_4)$, expressed in matrix form in (3.107), are strictly positive. As was the case for $\mathsf{SO}(V^{3,2})$, Theorem 3.2.3 says that we need only check that all connected minors that contain the first row are strictly positive. Similarly, any such minor that also contains the first column is necessarily strictly positive as they are the determinants of upper triangular matices with 1's along the diagonal. Furthermore, all non-trivial entries in the matrix are positive by definition and so we only need to check for sub-matrices size 2×2 or larger. We are then left with having to confirm that the following minors are strictly positive.

$$\mathsf{det}\begin{pmatrix} t_1 + t_3 & t_3t_4 + t_1(t_2 + t_4) \\ 1 & t_2 + t_4 \end{pmatrix} = t_2t_3 > 0 \tag{3.117}$$

$$\det\begin{pmatrix} t_1+t_3 & t_3t_4+t_1(t_2+t_4) & t_1t_2t_3 \\ 1 & t_2+t_4 & t_2t_3 \\ 0 & 1 & t_1+t_3 \end{pmatrix} = t_1t_2t_3 > 0 \tag{3.118}$$

$$\det\begin{pmatrix} t_3t_4+t_1(t_2+t_4) & t_1t_2t_3 \\ t_2+t_4 & t_2t_3 \end{pmatrix} = t_2t_3^2t_4 > 0 \tag{3.119}$$

Thus all non-trivial minors of any element of the positive sub-semigroup $\tilde{U}_+^{>0}$ are strictly positive.

Conversely, any element of $\tilde{U}_+$ whose non-trivial minors are strictly positive can be written as

$$M = \begin{pmatrix} 1 & a & ad-e & c \\ 0 & 1 & d & e \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.120}$$

where all entries above the diagonal are strictly positive as well as $ade - e^2 - cd > 0$, $a^2d - ae - c > 0$, and $ae - c > 0$. The fact that $M_{1,2} = M_{3,4}$ and $M_{1,3} = ad - e$ comes from the fact that M must preserve the symplectic form ω. That M has 1's along the diagonal is because M is unipotent and all non-trivial entries of our matrix must be positive.

By setting

$$t_1 = \frac{c}{e}, \qquad t_2 = \frac{e^2}{ae-c}, \qquad t_3 = \frac{ae-c}{e}, \qquad t_4 = \frac{ade-e^2-cd}{ae-c}, \tag{3.121}$$

we see that $\tilde{a}_2(t_1)\tilde{a}_1(t_2)\tilde{a}_2(t_3)\tilde{a}_1(t_4) = M$. By the conditions on M, we clearly have that $t_1, t_2, t_3, t_4 > 0$. Thus, we conclude that M is in te positive sub-semigroup $\tilde{U}_+^{>0}$. $\square$

3.5.3 Coordinates on positive triples of flags in $\mathsf{PSp}(4, \mathbb{R})$

In this section, we use the isomorphism $\Phi : \mathsf{PSp}(4, \mathbb{R}) \xrightarrow{\sim} \mathsf{SO}^0(V^{3,2})$ to understand how the coordinates (x, y) parametrize the totally positive part on the configuration space of triples of isotropic flags in $\mathsf{PSp}(4, \mathbb{R})$.

As in the previous section, we take (V, ω) to be a 4-dimensional real symplectic vector space with basis $\mathbf{e}_1, \ldots, \mathbf{e}_4$ such that $\omega(\mathbf{e}_1, \mathbf{e}_j) = \Omega_{ij}$. We fix a dual exterior bivector $\omega^* = \mathbf{e}_1 \wedge \mathbf{e}_4 - \mathbf{e}_2 \wedge \mathbf{e}_3$. The restriction of the bilinear form $B(,)$ for $\Lambda^2(V)$ to $W_0 = (\omega^*)^\perp$ has signature $(3, 2)$. The basis $\mathbf{f}_1, \ldots, \mathbf{f}_5$ defined in the previous section corresponds to our choice of anti-diagonal basis for $V^{3,2}$.

Definition 3.5.4. A pair of flags (F_1, F_2) in $\tilde{\mathscr{F}}$ is said to be *transverse* if $\mathsf{dim}(\mathrm{span}(F_1^i, F_2^j)) = \mathsf{min}(i + j, 4)$.

Definition 3.5.5. The *space of configurations of k-tuples of isotropic flags*, denoted $\mathsf{Conf}^{(k)}(\tilde{\mathscr{F}})$, is the space of generic k-tuples of isotropic flags in Ein. More precisely, it is the quotient of the set of k-tuples of isotropic flags $\tilde{\mathscr{F}}^{(k)}$ which are pairwise transverse by the diagonal action of $\mathsf{PSp}(4, \mathbb{R})$.

Recall the standard pair of transverse flags $F_0, F_\infty \in \mathscr{F}$. Consider the the following flags in $\tilde{\mathscr{F}}$

$$\tilde{F}_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tilde{B}_+, \qquad \tilde{F}_\infty = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tilde{B}_+. \tag{3.122}$$

With the the induced isomorphism on the flag manifolds $\widehat{\Phi} : \tilde{\mathscr{F}} \xrightarrow{\sim} \mathscr{F}$, we can easily confirm that $\widehat{\Phi}(\tilde{F}_0) = F_0$ and $\widehat{\Phi}(\tilde{F}_\infty) = F_\infty$ as flags. For this reason, we will refer to both F_0 and $\tilde{F}_0$ as F_0 when the context is clear, and similarly for F_∞.

Since we have shown that $\tilde{F}_0$, $\tilde{F}_\infty$, and $\tilde{U}_+^{>0}$ each correspond to F_0, F_∞, and $U_+^{>0}$, respectively, we see that the following definition of positivity for tuples of flags in $\mathsf{PSp}(4,\mathbb{R})$ agrees with the definition for tuples of flags in $\mathsf{SO}(V^{3,2})$.

Definition 3.5.6 ([FG06]). A tuple of isotropic flags $(F_1, F_2, F_3, \dots, F_k)$ with $F_i \in \tilde{\mathscr{F}}$ is *positive* if and only if it is in the $\mathsf{PSp}(4,\mathbb{R})$ orbit of a tuple of the form

$$(\tilde{F}_0, \tilde{F}_\infty, u_1\tilde{F}_\infty, u_1 \cdot u_2\tilde{F}_\infty, \dots, (u_1 \cdot \dots \cdot u_{k-2})\tilde{F}_\infty) \tag{3.123}$$

with $u_i \in \tilde{U}_+^{>0}$.

Now a similar argument to that for Lemma 3.3.3 gives us the following result.

Lemma 3.5.7. *The group* $\mathsf{PSp}(4,\mathbb{R})$ *acts transitively on pairs of transverse isotropic flags. The stabilizer of such a pair is a Cartan subgroup H of* $\mathsf{PSp}(4,\mathbb{R})$ *isomorphic to* $(\mathbb{R}^*)^2$.

By using the diagonal action of $\mathsf{PSp}(4,\mathbb{R})$ on a positive triple of flags

(F_1, F_2, F_3) in $\tilde{\mathscr{F}}^{(3)}$, Lemma 3.5.7 says that we can always get the *normalized form* $(\tilde{F}_0, \tilde{F}_\infty, \tilde{F})$.

The flag $\tilde{F}$ can be expressed as a 4×2 matrix

$$\tilde{F} = \begin{pmatrix} 1 & 0 \\ a & 1 \\ ad - e & d \\ c & e \end{pmatrix} \tag{3.124}$$

by transversailty of our triple and that fact that $\tilde{F}^2$ must be isotropic. Note that the stabilizer of $(\tilde{F}_0, \tilde{F}_\infty)$ is

$$\mathsf{Stab}((\tilde{F}_0, \tilde{F}_\infty)) = \left\{ \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & \mu^{-1} & 0 \\ 0 & 0 & 0 & \lambda^{-1} \end{pmatrix} \middle| \lambda, \mu \in \mathbb{R}^* \right\}. \tag{3.125}$$

By the diagonal action of the stabilizer element with $\lambda = \sqrt[4]{\frac{e^2}{d}(ae - c)}$ and $\mu = \frac{e}{\lambda}$, we get a normalized form for the triple $(\tilde{F}_0, \tilde{F}_\infty, \tilde{F})$ with

$$\tilde{F} = \begin{pmatrix} 1 & 0 \\ \frac{ae}{\sqrt{\frac{e^2}{d}(ae-c)}} & 1 \\ \frac{ad}{e} - 1 & \frac{d\sqrt{\frac{e^2}{d}(ae-c)}}{e^2} \\ \frac{c}{\sqrt{\frac{e^2}{d}(ae-c)}} & 1 \end{pmatrix}. \tag{3.126}$$

Note that as was the case for $\mathsf{SO}(V^{3,2})$, we assume that $d > 0$, $e > 0$, and $ae - c > 0$

otherwise we would not have a chance for our triple to be positive. By setting

$$\frac{y}{\sqrt{2}} = \frac{ae}{\sqrt{\frac{e^2}{d}(ae-c)}} \qquad \text{and} \qquad x = \frac{d\sqrt{\frac{e^2}{d}(ae-c)}}{e^2}, \tag{3.127}$$

we can simplify to get the final normalized form of $\tilde{F}$ to be

$$\tilde{F} = \begin{pmatrix} 1 & 0 \\ \frac{y}{\sqrt{2}} & 1 \\ \frac{xy}{\sqrt{2}} - 1 & x \\ \frac{y}{\sqrt{2}} - x & 1 \end{pmatrix}. \tag{3.128}$$

Now that we know that $\tilde{F}$ can be expressed in the form

$$\tilde{F} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{y}{\sqrt{2}} & 1 & 0 & 0 \\ \frac{xy}{\sqrt{2}} - 1 & x & 1 & 0 \\ \frac{y}{\sqrt{2}} - x & 1 & \frac{y}{\sqrt{2}} & 1 \end{pmatrix} \tilde{B}_+, \tag{3.129}$$

we can can describe its image under the isomorphism $\widehat{\Phi} : \tilde{\mathscr{F}} \xrightarrow{\sim} \mathscr{F}$. A tedious calculation similar to those done for $\tilde{a}_i(t)$ in the previous section gives that

$$\widehat{\Phi}(\tilde{F}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ x & 1 & 0 & 0 & 0 \\ \sqrt{2} & y & 1 & 0 & 0 \\ x & \frac{y^2}{2} & y & 1 & 0 \\ x^2 - 1 & \frac{xy^2}{2} + x - \sqrt{2}y & xy - \sqrt{2} & x & 1 \end{pmatrix} B_+ = F. \tag{3.130}$$

Thus we have that $\widehat{\Phi}((\tilde{F}_0, \tilde{F}_\infty, \tilde{F})) = (F_0, F_\infty, F)$, the normalized form of a positive triple of flags in $\mathsf{SO}(V^{3,2})$.

We can then associate to this positive element of $\mathsf{Conf}^{(3)}(\tilde{\mathscr{F}})$ the coordinates (x, y).

We now recognize that Theorem 3.3.8 gives the following corollary.

Corollary 3.5.8. *The totally positive part of $\mathsf{Conf}^{(3)}(\tilde{\mathscr{F}})$ will then be parametrized by the quadrant $\{(x, y) \in \mathbb{R}^2 \mid x > 1$ and $y > \sqrt{2}x\}$.*

Chapter 4: Piecewise circular wavefronts

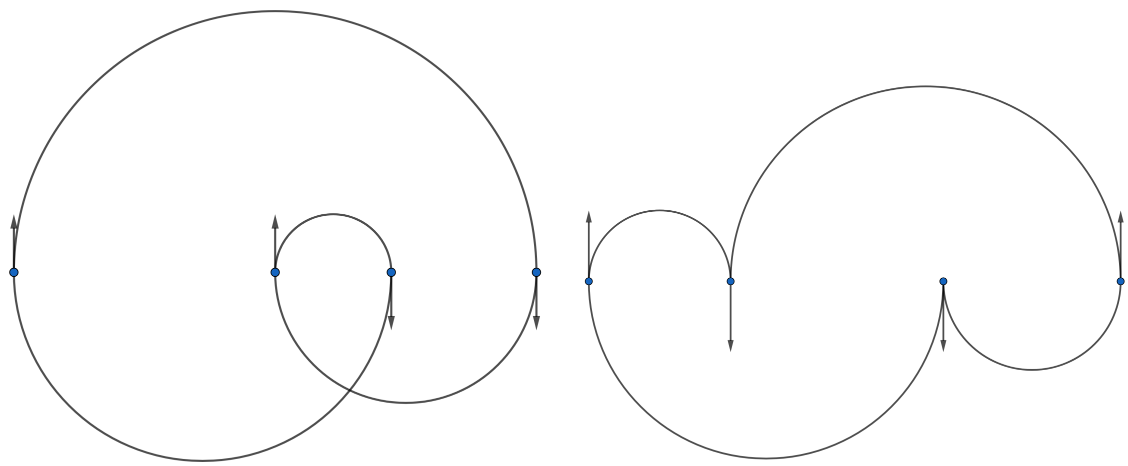
In this chapter, we relate the configuration spaces of isotropic flags in $\mathsf{PSp}(4,\mathbb{R})$, with a class of curves in the 2-sphere. The class of curves we consider are the labeled, oriented piecewise circular curves in the 2-sphere up to conformal transformations, equidistant transformations, and the orientation-reversing transformation. As a reminder that we are considering these curves up to the action by conformal transformations, equidistant transformations, and orientation-reversing transformations, we will refer to the piecewise circular curves as piecewise circular wavefronts. The term wavefronts comes from the notion of wavefront propagation as the equidistant transformations will be shown to act on the space of oriented circles by uniformly adding a value r to the signed radii of all circles. As in the case for flags in $\mathsf{SL}(3,\mathbb{R})$ discussed in [FG07], the positivity of the configuration of a triple of isotropic flags relates to a certain notion of positivity of such a curve with six arcs. By the equivalence of the complete isotropic flag manifolds for $\mathsf{PSp}(4,\mathbb{R})$ and $\mathsf{SO}(V^{3,2})$ and their respective notions of positivity, we see that the coordinates on the totally positive part of the configuration space of triples of isotropic flags parametrizes the space of labeled, positive, oriented piecewise circular hexagons.

Definition 4.0.1. A *piecewise circular curve* γ in S^2 is a closed curve consisting of finitely many circular arcs with matching tangent lines at the intersections of adjacent arcs. We will refer to the circular pieces as *edges* or *arcs* of the curve and the junctions between adjacent arcs as the *vertices.*

An *orientation* on a piecewise circular curve is a continuous choice of unit vector tangent to γ a each point of γ. Note that if a piecewise circular curve is orientable, then it will necessarily have exactly two orientations.

Remark. All piecewise circular triangles are not orientable. Any piecewise circular triangle is given by three distinct circles which are pairwise tangent. If we were able to give a triangle an orientation, then we would have three distinct oriented circles which are pairwise tangent. By considering the correspondence between the projective and Lie circles model of Ein given in Table 2.1, we see that this would imply that we have three distinct points $[\mathbf{u}], [\mathbf{v}], [\mathbf{w}] \in \mathsf{Ein}$ which are pairwise incident. However, this means that $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are lightlike and $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle = 0$. Therefore, $\text{span}(\mathbf{u}, \mathbf{v}, \mathbf{w})$ would be a 3-dimensional isotropic subspace of $V^{3,2}$, a contradiction by Lemma 2.1.3.

Remark. We allow for piecewise circular curves to have singularities and points of self-intersection as shown in Figures 4.1(a) and 4.1(b).



(a) An oriented piecewise circular quadrilateral with a self-intersection.

(b) An oriented piecewise circular quadrilateral with singularities.

Figure 4.1: Oriented piecewise circular quadrilaterals.

4.1 Transformations on oriented piecewise circular curves

In this section, we describe a group of transformations acting on the space of oriented piecewise circular curves. We show that this group of transformations can be understood analytically as the group $\mathsf{PSp}^{\pm}(4,\mathbb{R}) \cong \mathsf{SO}(V^{3,2})$ acting on the unit tangent bundle $T^1(S^2)$.

The set of oriented piecewise circular curves is invariant under the conformal group of the sphere $\mathsf{PSL}(2,\mathbb{C})$ since conformal maps send circles to circles and preserve orientation. There is a larger group of transformations, the group of *projective contactomorphisms* of $\mathbb{RP}^3 \cong T^1(S^2)$, which acts on these curves.

Remark. The relationship between $\mathbb{RP}^3$ and the unit tangent bundle of the sphere $T^1(S^2)$ was discussed in Table 2.1. To further understand that this relationship is a diffeomorphism, we refer to [Ste51] for a proof that $\mathbb{RP}^3 \cong \mathsf{SO}(3)$ and [KS75] for a proof that $\mathsf{SO}(3) \cong T^1(S^2)$.

These transformations are referred to as contactomorphisms as they preserve a contact structure on $\mathbb{RP}^3$. The contact structure we consider is defined as follows: given a point $p \in \mathbb{RP}^3$, look at its orthogonal subspace with respect to the symplectic form $\omega(,)$. This subspace is a 2-dimensional projective plane through the point p. The vectors tangent to this projective plane based at p form a hyperplane in $T_p(\mathbb{RP}^3)$. These tangent vectors form the contact plane at p. Transformations preserving this contact structure will necessarily preserve tangency of oriented circles by considering the correspondence between the Lie circles and Lagrangian Grassmannian models of Ein.

The group of projective contactomorphisms of $\mathbb{RP}^3 \cong T^1(S^2)$ is generated by three types of transformations. The first type is the Möbius transformations $\mathsf{PSL}(2,\mathbb{C})$. Next we have the one-parameter group of *equidistant transformations*. These transformations are described as follows: in each of the affine charts $S^2 \backslash p_0$, add the same quantity r to the signed radius of all oriented circles. If $r > 0$, this will increase the radius of every positively-oriented circle by the quantity r and decrease the radius of every negatively-oriented circle by the quantity r. The final type of transformation generating the group of projective contactomorphisms is the *orientation-reversing transformation*. This transformation switches the orientation of all oriented circles in S^2. Circles of radius zero will remain fixed under this transformation. Each of these types of transformations clearly preserve oriented tangencies, and therefore produces a new oriented piecewise circular curve. We therefore have that the group of projective contactomorphisms generated by these transformations will act on the set of oriented piecewise circular curves.

We will show that the group of projective contactomorphisms can be understood analytically as the group $\mathsf{PSp}^{\pm}(4,\mathbb{R}) \cong \mathsf{SO}(V^{3,2})$ acting on the unit tangent bundle $T^1(S^2)$ and therefore on the lift of an oriented piecewise circular curve into $T^1(S^2)$.

Definition 4.1.1. The group $\mathsf{PSp}^{\pm}(4,\mathbb{R})$ is defined to be the quotient of the union of the set of symplectic matrices in $\mathsf{GL}(4,\mathbb{R})$ with the set of anti-symplectic matrices in $\mathsf{GL}(4,\mathbb{R})$ by $\{\pm \mathrm{I}\}$. In other words,

$$\mathsf{PSp}^{\pm}(4,\mathbb{R}) = {\{A \in \mathsf{GL}(4,\mathbb{R}) \mid A^t\Omega A = \Omega\} \cup \{A \in \mathsf{GL}(4,\mathbb{R}) \mid A^t\Omega A = -\Omega\}}\big/_{\{\pm \mathrm{I}\}} \tag{4.1}$$

where $\Omega_{ij} = \omega(\mathbf{e}_i, \mathbf{e}_j)$ for our choice of symplectic form $\omega(,)$ and basis $\mathbf{e}_1, \ldots, \mathbf{e}_4$.

Lemma 4.1.2. *The group of projective contactomorphisms of* $\mathbb{RP}^3 \cong T^1(S^2)$ *described above is exactly the group* $\mathsf{PSp}^{\pm}(4,\mathbb{R}) \cong \mathsf{SO}(V^{3,2})$.

Proof. For the duration of this proof we will fix the basis of $V^{3,2}$ to be the diagonal basis such that the bilinear form $\langle , \rangle$ is given by the Gram matrix $J = \mathsf{diag}(1,1,1,-1,-1)$. Recall that in this case we write $\mathsf{SO}(3,2)$ rather than $\mathsf{SO}(V^{3,2})$.

We first wish to show that $\mathsf{SO}^0(3,2)$ is generated by the Möbius transformations together with the equidistant transformations. To do this we compute the Lie algebra corresponding to $\mathsf{SO}(3,2)$ as

$$\mathfrak{so}(3,2) = \left\{ \begin{pmatrix} 0 & b & c & d & e \\ -b & 0 & h & i & j \\ -c & -h & 0 & n & p \\ d & i & n & 0 & u \\ e & j & p & -u & 0 \end{pmatrix} \middle| \, b,c,d,e,h,i,j,n,p,u \in \mathbb{R} \right\}. \tag{4.2}$$

We refer to Ratcliffe's exposition in [Rat06] showing there exists an isomorphism $\mathsf{PSL}(2,\mathbb{C}) \cong \mathsf{SO}^0(3,1)$. Therefore, we can think of the group of Möbius transformations as the group $\mathsf{SO}^0(3,1)$. There is a Lie subalgebra of $\mathfrak{so}(3,2)$ which is isomorphic to $\mathfrak{so}(3,1)$ given by

$$\mathfrak{so}(3,1) = \left\{ \begin{pmatrix} 0 & b & c & 0 & e \\ -b & 0 & h & 0 & j \\ -c & -h & 0 & 0 & p \\ 0 & 0 & 0 & 0 & 0 \\ e & j & p & 0 & 0 \end{pmatrix} \middle| \, b,c,e,h,j,p \in \mathbb{R} \right\}. \tag{4.3}$$

Now we wish to characterize the one-parameter group of equidistant transformations. Without loss of generality, let us choose the affine patch given by taking $[\mathbf{q}]$ to be the north pole as we did in Lemma 2.2.13. Consider the Lie subalgebra of

$\mathfrak{so}(3,2)$ given by

$$\mathfrak{e} = \left\{ \left. \begin{pmatrix} 0 & 0 & 0 & t & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ t & 0 & 0 & 0 & t \\ 0 & 0 & 0 & -t & 0 \end{pmatrix} \right| t \in \mathbb{R} \right\}. \tag{4.4}$$

By exponentiation, we get the the corresponding Lie group is

$$E = \left\{ \left. \begin{pmatrix} \frac{t^2}{2}+1 & 0 & 0 & t & \frac{t^2}{2} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ t & 0 & 0 & 1 & t \\ -\frac{t^2}{2} & 0 & 0 & -t & 1-\frac{t^2}{2} \end{pmatrix} \right| t \in \mathbb{R} \right\}. \tag{4.5}$$

Now consider the action of E on the oriented circles in the Lie circles model of Ein.

$$\begin{pmatrix} \frac{t^2}{2}+1 & 0 & 0 & t & \frac{t^2}{2} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ t & 0 & 0 & 1 & t \\ -\frac{t^2}{2} & 0 & 0 & -t & 1-\frac{t^2}{2} \end{pmatrix} \begin{pmatrix} \frac{1+r^2-x_0^2-y_0^2}{2} \\ x_0 \\ y_0 \\ r \\ \frac{1-r^2+x_0^2+y_0^2}{2} \end{pmatrix} = \begin{pmatrix} \frac{1+(r+t)^2-x_0^2-y_0^2}{2} \\ x_0 \\ y_0 \\ r+t \\ \frac{1-(r+t)^2+x_0^2+y_0^2}{2} \end{pmatrix}. \tag{4.6}$$

Therefore we have that E is in fact the group of equidistant transformations for our choice of affine patch.

Now we wish to show that $\mathfrak{so}(3,1)$ and $\mathfrak{e}$ described above generate $\mathfrak{so}(3,2)$ through the Lie bracket. A natural choice of basis for $\mathfrak{so}(3,2)$ are the elements where one variable is equal to 1 and the rest are 0. The variables b, c, e, h, j, p are each accounted for in $\mathfrak{so}(3,1)$. Now we can get the remaining elements of the basis of $\mathfrak{so}(3,2)$ as follows. First we consider taking the Lie bracket of the single basis vector of $\mathfrak{e}$ with particular basis vectors from $\mathfrak{so}(3,1)$. We can get the elements of the basis of $\mathfrak{so}(3,2)$ corresponding to i and n with

$$\left[\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.7}$$

and

$$\left[\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.8}$$

Now by bracketing the basis element corresponding to b with the basis element corresponding to i which we got in the first bracket above, we can get the basis element corresponding to d.

$$\left[\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.9}$$

Finally, we see that subtracting the basis element corresponding to d from the basis element of $\mathfrak{e}$, we get the finally remaining basis element which corresponds to u.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix} \tag{4.10}$$

Therefore, we have shown that $\mathfrak{so}(3,1)$ together with $\mathfrak{e}$ generate $\mathfrak{so}(3,2)$. We can then conclude that the group of Möbius transformations $\mathsf{SO}^0(3,1)$ together with the equidistant transformations E generate $\mathsf{SO}^0(3,2)$.

The remaining type of transformation to consider is the orientation-reversing transformation. The group of orientation-reversing transformations is clearly isomorphic to the 2 element group $\mathbb{Z}_2$. It is easy to verify that the generator is given by the matrix

$$R = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{4.11}$$

which clearly changes the orientation of all the oriented circles in the 2-sphere. Furthermore, R is contained in $\mathsf{SO}(3,2)$, but is not in the identity component $\mathsf{SO}^0(3,2)$. Since $\mathsf{SO}(3,2)$ has exactly two components, we know that $\mathsf{SO}^0(3,2)$ together with the non-identity component element will generate all of $\mathsf{SO}(3,2)$. We conclude that, in fact, the group of projective contactomorphisms as described is isomorphic to the group $\mathsf{SO}(3,2)$.

We now wish to show that $\mathsf{PSp}^{\pm}(4,\mathbb{R})$ is isomorphic to $\mathsf{SO}(3,2)$ and therefore isomorphic to the group of projective contactomorphisms. Note that $\mathsf{SO}(3,2) \cong \mathsf{PO}(3,2)$ as the total dimension of our vector space is odd. So it suffices to show that $\mathsf{PSp}^{\pm}(4,\mathbb{R}) \cong \mathsf{PO}(3,2)$. We first recall the isomorphism $\mathsf{PSp}(4,\mathbb{R}) \cong \mathsf{SO}^0(3,2)$ which means that $\mathsf{PSp}(4,\mathbb{R}) \cong \mathsf{PO}^0(3,2)$. Therefore, we need only show that there is an element in the anti-symplectic part of $\mathsf{PSp}(4,\mathbb{R})$ which, by the extension of the isomorphism $\mathsf{PSp}(4,\mathbb{R}) \cong \mathsf{SO}^0(3,2)$, maps to the element in $\mathsf{PO}(3,2)$ corresponding to the orientation-reversing transformation.

Our explicit computations using the isomorphism $\mathsf{PSp}(4,\mathbb{R}) \cong \mathsf{SO}^0(3,2)$ in Section 3.5.2 were done in terms of the anti-diagonal basis for $V^{3,2}$. Recall the change of basis matrix P from the diagonal basis to the anti-diagonal basis from

Section 2.2.2. We have that the orientation-reversing transformation in terms of the anti-diagonal basis of $V^{3,2}$ is

$$PRP^{-1} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}. \tag{4.12}$$

Now consider the matrix

$$\tilde{R} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in \mathsf{PSp}^{\pm}(4, \mathbb{R}). \tag{4.13}$$

As in Section 3.5.2, we observe the induced action of the matrix $\tilde{R}$ on the basis $\mathbf{f}_1, \ldots, \mathbf{f}_5$ of $W_0 \cong V^{3,2}$ and see that this matrix corresponds to

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = -PRP^{-1}. \tag{4.14}$$

Now $-PRP^{-1}$ is a representative for the element of $\mathsf{PO}(3,2)$ corresponding to PRP^{-1} and, together with $\mathsf{PO}^0(3,2)$ generates all of $\mathsf{PO}(3,2)$. Similarly, $\tilde{R}$ together with $\mathsf{PSp}(4,\mathbb{R})$ generates all of $\mathsf{PSp}^{\pm}(4,\mathbb{R})$. It follows that

$\mathsf{PSp}^{\pm}(4,\mathbb{R}) \cong \mathsf{PO}(3,2) \cong \mathsf{SO}(3,2)$. Finally, we conclude that $\mathsf{PSp}^{\pm}(4,\mathbb{R})$ is isomorphic to the group of projective contactomorphisms. $\square$

Definition 4.1.3. The set of oriented piecewise circular curves up to the action of $\mathsf{PSp}^{\pm}(4,\mathbb{R})$ is referred to as the set of *oriented piecewise circular wavefronts*.

Our use of the term wavefronts to describe the set of oriented piecewise circular curves up to the action of $\mathsf{PSp}^{\pm}(4,\mathbb{R})$ is meant to remind the reader that the group of transformations we consider is larger than the group of Möbius transformations. The equidistant transformations we also consider can be thought of as uniform wave propagation of circular wavefronts.

4.2 Positive triples of flags and convex triangles in $\mathbb{RP}^2$

In this section, we provide a brief exposition of the correspondence between positive triples of flags and convex triangles in $\mathbb{RP}^2$ which was first proved by Fock and Goncharov in [FG07]. We refer the reader to [CTT18] for a nice, self-contained treatment of the results of Fock and Goncharov for $\mathbb{RP}^2$. This section is entirely expository and serves the purpose of motivating the definition of positivity for an oriented piecewise circular curve in the 2-sphere given in the following section.

In this section, we denote points and lines of $\mathbb{RP}^2$ by column and row vectors, respectively. The line $[a : b : c]$ will correspond to the set of points $[x : y : z]^t \in \mathbb{RP}^2$ such that $ax + by + cz = 0$.

A *flag* in $\mathbb{RP}$ can be thought of as a pair $F_i = (V_i, \eta_i)$ consisting of a point $V_i \in \mathbb{RP}^2$ and a line $\eta_i \subset \mathbb{RP}^2$ passing through V_i. Alternatively, we can express a flag F_i in $\mathbb{RP}^2$ as a 3×3 matrix in $\mathsf{SL}(3,\mathbb{R})$ where the first column is a representative vector for V_i and the projectivization of the first two columns is η_i. A k-tuple of flags $(F_1,\ldots,F_k)$ is in *general position* if no three points are collinear, no three lines are coincident, and $V_i \notin \eta_j$ for all $i \neq j$.

Notation. For distinct projective lines η_i, η_j, we denote by $\eta_i\eta_j$ the point at which η_i and η_j intersect. Similarly, for distinct points $V_i, V_j \in \mathbb{RP}^2$, we denote by V_iV_j the line containing both V_i and V_j.

We say that a non-degenerate polygonal curve $\gamma \subset \mathbb{RP}^2$ is *convex* if any projective line intersects the curve γ in either a connected line segment or in no more than two points. For a pair of non-degenerate convex triangles σ_1, σ_2, we say that σ_1 is *strictly inscribed* in σ_2 if each edge of σ_2 contains one vertex of σ_1 in its interior. A pair of oriented convex triangles, one strictly inscribed in the other, is said to be *oriented* if they have the same orientation.

We denote by $\mathcal{P}_3$ the space of oriented pairs of strictly inscribed convex triangles in $\mathbb{RP}^2$, modulo the action of $\mathsf{SL}(3,\mathbb{R})$. Similarly, we denote by Conf_3 the space of cyclically ordered triples of flags in general position, modulo the action of $\mathsf{SL}(3,\mathbb{R})$. There is a natural map between these two spaces

$$\xi_3 : \mathcal{P}_3 \longrightarrow \mathsf{Conf}_3 \tag{4.15}$$

defined by deleting the interior of the edges of the inscribed triangle and extending the edges of the circumscribed triangle to projective lines. Note that the condition that the triangles are strictly inscribed one in the other ensures that the corresponding triple of flags is in general position. Additionally, two pairs of polygons

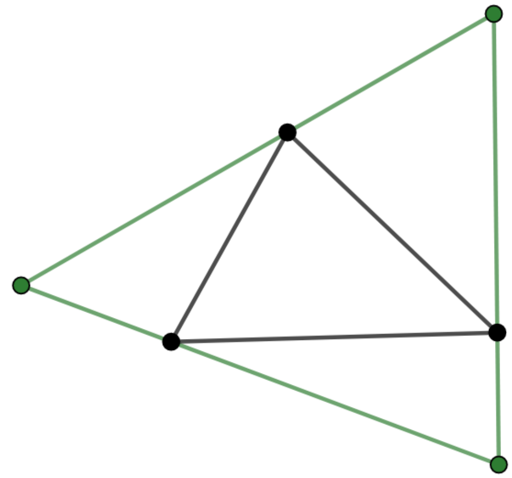


Figure 4.2: A pair of strictly inscribed convex triangles in $\mathbb{RP}^2$.

are equivalent under the action of $\mathsf{SL}(3,\mathbb{R})$ if and only if the associated flags are equivalent under the action of $\mathsf{SL}(3,\mathbb{R})$. Therefore, ξ_3 is injective and $\mathcal{P}_3$ can be identified with its image in Conf_3.

Similarly to the proof for $\mathsf{SO}(V^{3,2})$, one can show that $\mathsf{SL}(3,\mathbb{R})$ acts 2-transitively on flags in $\mathbb{RP}^2$. Therefore, every element of Conf_3 has a representative triple with the first two flags being

$$F_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad F_\infty = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{4.16}$$

Just as we did to describe the normalized form of triples in $\mathsf{SO}(V^{3,2})$, we can use the action of the stabilizer

$$\mathsf{Stab}(F_0, F_\infty) = \left\{ \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_1^{-1}\lambda_2^{-1} \end{pmatrix} \middle| \lambda_1, \lambda_2 \in \mathbb{R}^* \right\} \tag{4.17}$$

to show that every element of Conf_3 has a unique representative of the form

$$\mathcal{F} = (F_1, F_2, F_3) = \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1+t & 1 \end{pmatrix} \right). \tag{4.18}$$

Note that this triple is in general position if and only if $t \neq 0, -1$ as these are the cases where η_3 would pass through the points V_1 and $\eta_1\eta_2$, respectively. The case in which $t = \infty$ also violates general position as here we would have η_3 passing through V_2.

Definition 4.2.1. We refer to the unique representative (4.18) as the *normalized form* for a element in Conf_3. We associate to this element the coordinate t.

Similar to the case for $\mathsf{SO}(V^{3,2})$, the triple $\mathcal{F} = (F_1, F_2, F_3)$ will be positive if and only if the matrix representing F_3 is lower strictly totally positive. By Theorem 3.2.3, we must only check that the connected minors containing an element of the first column are strictly positive. As the entries in the first row are all clearly positive, and the matrix is an element of $\mathsf{SL}(3, \mathbb{R})$, we only need to check the following two minors:

$$\mathsf{det} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = 1, \qquad \mathsf{det} \begin{pmatrix} 1 & 1 \\ 1 & 1+t \end{pmatrix} = t. \tag{4.19}$$

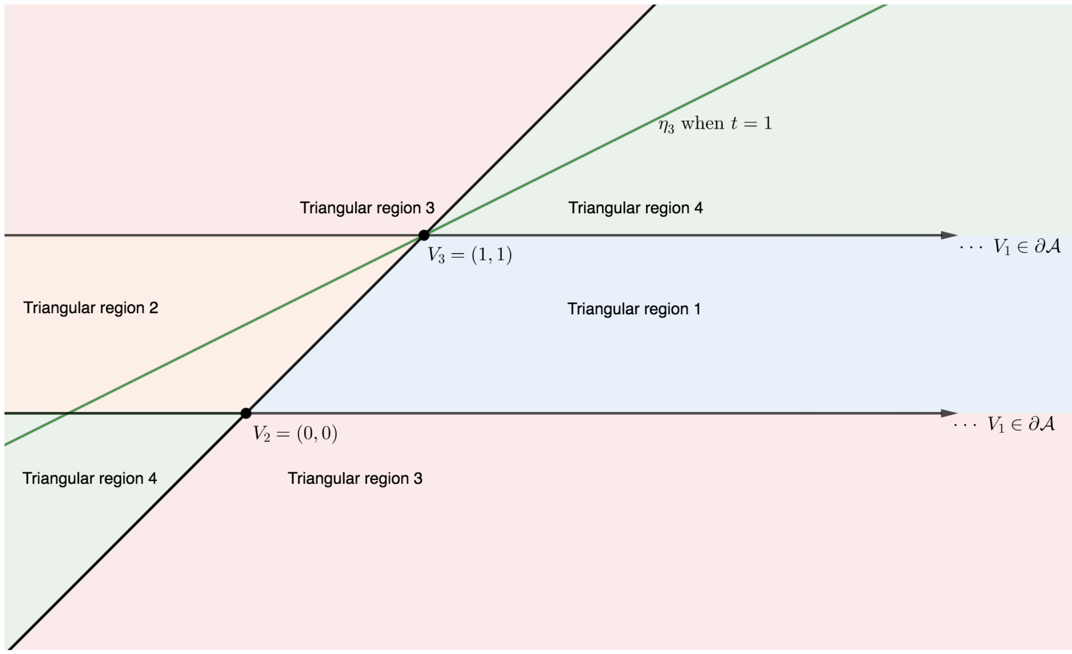


Figure 4.3: Triangular regions in the affine patch $\mathcal{A} = \{[x : y : 1]^t \mid x, y \in \mathbb{R}\} \subset \mathbb{RP}^2$ with vertices V_1, V_2, and V_3.

This proves the following theorem.

Theorem 4.2.2. *In terms of the coordinate t, the totally positive part of Conf_3 is given by $t > 0$.*

Now there are four triangular regions in $\mathbb{RP}^2$ with vertices V_1, V_2, and V_3 (See Figure 4.3). The triple $\mathcal{F}$ is in the image of ξ_3 if and only if one of these four triangular regions is disjoint from each of the lines η_1, η_2, and η_3. One can easily check that η_1 intersects triangular regions 3 and 4 in Figure 4.3. Also, η_2 intersects triangular regions 2 and 3. The only remaining option is triangular region 1. As the projective line η_3 cannot contain V_1 or V_2, it must intersect the line V_1V_2 at a point of the form $[x : 0 : 1]^t$ with $x \neq 0$. By considering the affine patch $\mathcal{A} = \{[x : y : 1]^t \mid x, y \in \mathbb{R}\} \subset \mathbb{RP}^2$ pictured in Figure 4.3, we see that η_3 will be

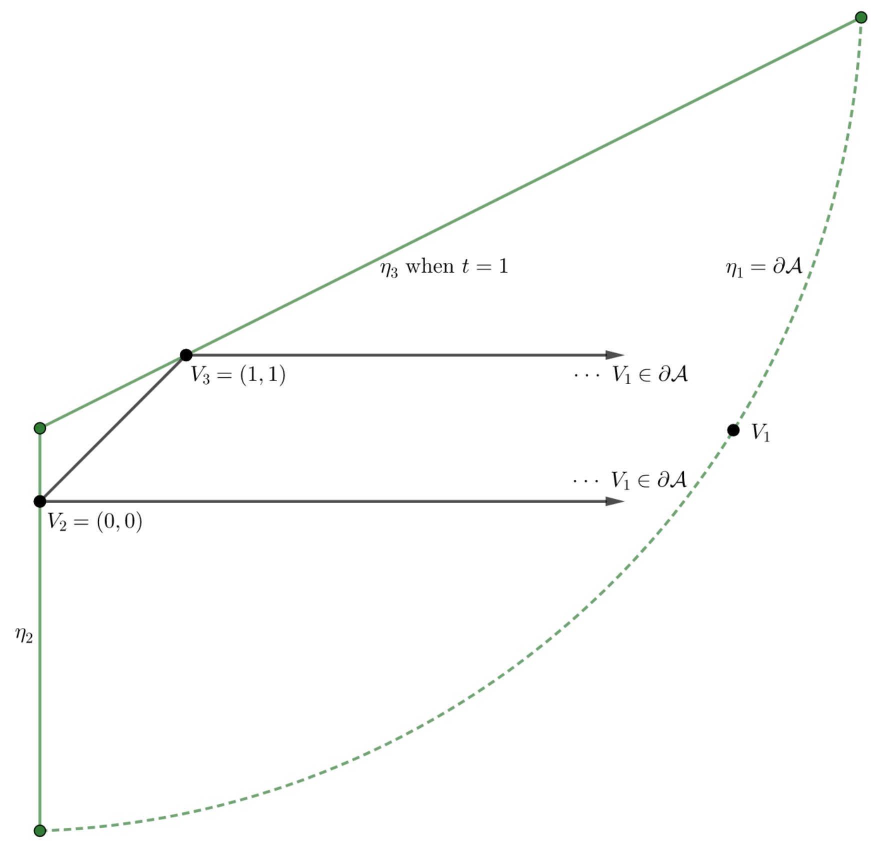


Figure 4.4: The pair of strictly inscribed convex triangles in the affine patch $\mathcal{A} = \{[x : y : 1]^t \mid x, y \in \mathbb{R}\} \subset \mathbb{RP}^2$ corresponding to the normalized triple of flags when $t = 1$.

disjoint from triangular region 1 if and only if $x < 0$. Therefore, $\mathcal{F} \in \xi_3(\mathcal{P}_3)$ if and only if η_3 intersects the line V_1V_2 at a point of the form $[x : 0 : 1]^t$, where $x < 0$. This is the case if and only if $t > 0$. This along with Theorem 4.2.2 gives the following result.

Theorem 4.2.3. *The map* $\xi_3 : \mathcal{P}_3 \longrightarrow \mathsf{Conf}_3$ *identifies the space of oriented pairs of strictly inscribed convex triangles in* $\mathbb{RP}^2$*, modulo the action of* $\mathsf{SL}(3, \mathbb{R})$*, with the totally positive part of* Conf_3.

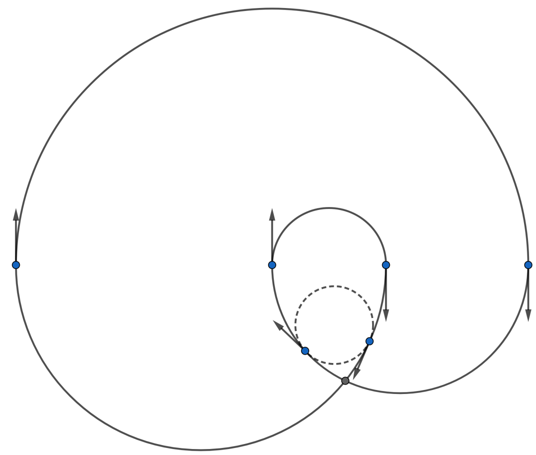
4.3 Positive piecewise circular wavefronts

In this section, we define a notion of positivity for oriented piecewise cicular wavefronts. This definition of positivity is motivated by the notion of a convex curve in the projective plane. In [FG07], Fock and Goncharov showed a relationship between positive tuples of flags in $\mathsf{PSL}(3,\mathbb{R})$ and convex polygons in $\mathbb{RP}^2$. Similarly, we show a relationship between positive triples of flags in $\mathsf{PSp}^{\pm}(4,\mathbb{R}) \cong \mathsf{SO}(3,2)$ and labeled, positive, oriented piecewise circular wavefronts. We conclude that the space of labeled, positive, oriented piecewise circular wavefronts up to projective contactomorphisms is parametrized by $\mathbb{R}^2$.

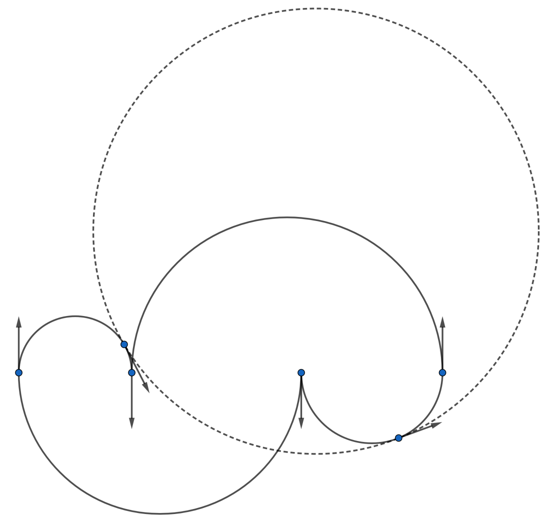
For this section, recall that we take V to be a 4-dimensional real symplectic vector space with symplectic form $\omega(,)$. We fix the basis $\mathbf{e}_1,\ldots,\mathbf{e}_4$ such that $\omega(\mathbf{e}_i,\mathbf{e}_j) = \Omega_{ij}$ where

$$\Omega = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{4.20}$$

In the previous section, we discussed the correspondence between positive triples of flags and pairs of convex triangles one inscribed in the other in the projective plane. In the example of $\mathbb{RP}^2$, we assigned to each positive element in the configuration space of triples of flags a unique, up to the action of $\mathsf{SL}(3,\mathbb{R})$, pair of convex, oriented triangles, one inscribed in the other. Similarly, we wish to assign to each positive element in $\mathsf{Conf}^{(3)}(\mathscr{F})$ a unique, up to the action of projective



(a) The oriented piecewise circular quadrilateral in Figure 4.1(a) is not positive.



(b) The oriented piecewise circular quadrilateral in Figure 4.1(b) is not positive.

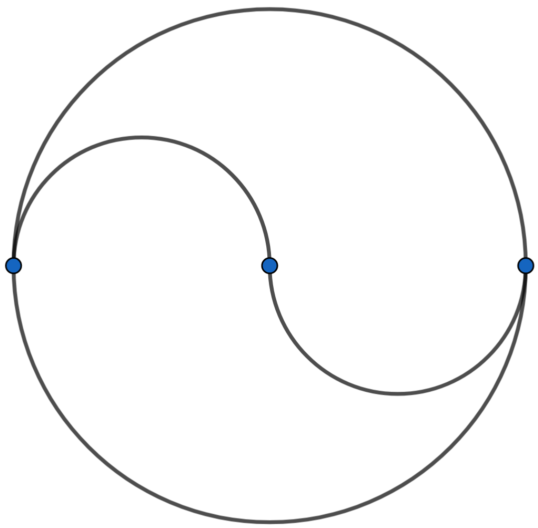
Figure 4.5: Examples of oriented piecewise circular quadrilaterals which are not positive.

contactomorphisms, geometric object using incidence relations. We then look to characterize positivity in terms of a geometric property of such objects. This fact motivates the following definition of positivity for oriented piecewise circular curves in the 2-sphere.

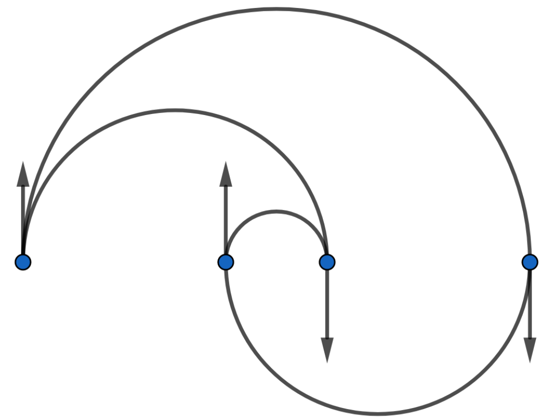
Definition 4.3.1. An oriented piecewise circular curve γ in the 2-sphere is *positive* if the set of tangency points between any oriented circle and γ is either a single point or an arc of γ. Here, tangency is understood to imply matching orientations, and we allow zero radius circles as arcs of the curve γ.

Example 4.3.2. The oriented piecewise circular quadrilaterals in Figures 4.1(a) and 4.1(b) are not positive as shown by Figures 4.5(a) and 4.5(b).

Example 4.3.3. The Yin-Yang symbol (Figure 4.6(a)) is made out of two inter-



(a) The Yin-Yang symbol is made out of two interlocking piecewise circular triangles.



(b) One piece of the Yin-Yang symbol after an equidistant transformation is orientable.

Figure 4.6: The oriented Yin-Yang curve is positive.

locking piecewise circular triangles. Since they are piecewise circular triangles, they are not orientable. However, if we interpret the central point as a zero radius circle, then we can endow the curve with an orientation. By perturbing the circular wavefronts by an equidistant transformation, then we obtain a positive, oriented piecewise circular wavefront as seen in Figure 4.6(b).

Definition 4.3.4. A *labeling* of a piecewise circular curve is a choice of vertex and one of the arcs to which it is adjacent. A labeling gives an ordering of the vertices starting with the labeled vertex and following the piecewise circular curve in the direction of the chosen arc.

Evidently, positive, oriented piecewise circular curves are intimately related to positive configurations of flags in $\mathsf{PSp}^{\pm}(4,\mathbb{R}) \cong \mathsf{SO}(3,2)$. We wish to explicitly define this relationship. We begin with two useful lemmas.

Lemma 4.3.5. *Let $\mathbf{u}, \mathbf{v}, \mathbf{u}', \mathbf{v}' \in V$. There is no orthogonal pair $\mathbf{p}, \mathbf{q}$ with $\mathbf{p} \in \{k\mathbf{u} + l\mathbf{v} \mid k, l > 0\}$ and $\mathbf{q} \in \{k'\mathbf{u}' + l'\mathbf{v}' \mid k', l' > 0\}$ if and only if $\omega(\mathbf{u}, \mathbf{u}')$, $\omega(\mathbf{u}, \mathbf{v}')$, $\omega(\mathbf{v}, \mathbf{u}')$, and $\omega(\mathbf{v}, \mathbf{v}')$ all have the same sign, possibly zero, but not all zero.*

Proof. There exists such an orthogonal pair $\mathbf{p}, \mathbf{q}$ if and only if the equation

$$\omega(\mathbf{p}, \mathbf{q}) = kk'\omega(\mathbf{u}, \mathbf{u}') + kl'\omega(\mathbf{u}, \mathbf{v}') + lk'\omega(\mathbf{v}, \mathbf{u}') + ll'\omega(\mathbf{v}, \mathbf{v}') = 0 \tag{4.21}$$

admits a solutions. Since $k, l, k', l' > 0$, it necessarily follows that this equation will admit a solution if and only if two the coefficients $\omega(\mathbf{u}, \mathbf{u}')$, $\omega(\mathbf{u}, \mathbf{v}')$, $\omega(\mathbf{v}, \mathbf{u}')$, and $\omega(\mathbf{v}, \mathbf{v}')$ have opposite signs. □

Definition 4.3.6. For $\mathbf{u}, \mathbf{v}, \mathbf{u}', \mathbf{v}' \in V$, we say that $\{k\mathbf{u} + l\mathbf{v} \mid k, l > 0\}$ and $\{k'\mathbf{u}' + l'\mathbf{v}' \mid k', l' > 0\}$ are *transverse* if there does not exist any pair $\mathbf{p} \in \{k\mathbf{u} + l\mathbf{v} \mid k, l > 0\}$ and $\mathbf{q} \in \{k'\mathbf{u}' + l'\mathbf{v}' \mid k', l' > 0\}$ with $\omega(\mathbf{p}, \mathbf{q}) = 0$. In this case, the corresponding line segments in $\mathbb{RP}^3$ are also said to be *transverse*.

Remark. Recall the relationships between the Lie circles and Lagrangian Grassmannian models of Ein displayed in Table 2.1. It is clear through the identification of these two models that the set of piecewise circular curves in the 2-sphere corresponds exactly to the the set of piecewise linear curves in $\mathbb{RP}^3$ such that each segment is Legendrian, i.e. each linear segment corresponds to a Lagrangian plane in (V, ω). For a piecewise circular curve γ, we will denote by $\hat{\gamma}$ the corresponding Legendrian piecewise linear lift of γ.

Lemma 4.3.7. *Let $\mathbf{v}_1, \ldots, \mathbf{v}_{2k} \in V$ such that the oriented piecewise linear curve $\gamma \subset \mathbb{RP}^3$ given by positive linear combinations of $\mathbf{v}_i, \mathbf{v}_{i+1}$ for $i = 1, \ldots, 2k-1$ and* *positive linear combinations of* $\mathbf{v}_{2k}, \varepsilon\mathbf{v}_1$ *(*$\varepsilon = \pm 1$*) is Legendrian, i.e.* $\omega(\mathbf{v}_i, \mathbf{v}_{i+1}) = 0$ *for* $i = 1, \dots, 2k-1$ *and* $\omega(\mathbf{v}_{2k}, \varepsilon\mathbf{v}_1) = 0$*. Then,* γ *is positive if and only if* $\varepsilon = -1$ *and* $\omega(\mathbf{v}_i, \mathbf{v}_j)$ *have the same sign for all* $j > i+1$*.*

Proof. Suppose that γ is a positive curve. Through the correspondence between oriented piecewise circular curves in S^2 and oriented piecewise linear curves with Legendrian segments in $\mathbb{RP}^3$, the notion of positivity of γ means that non-adjacent edges of γ are transverse. By Lemma 4.3.5, the products $\omega(\mathbf{v}_i, \mathbf{v}_j)$ must all have the same sign for $j > i+1$. By skew symmetry of $\omega(,)$, the product $\omega(\mathbf{v}_{2k}, \mathbf{v}_1)$ must have the opposite sign. Now in order for the segment $\{a\mathbf{v}_{2k} + b(\varepsilon\mathbf{v}_1) \mid a, b > 0\}$ to be transverse to the other segments, it must follow that $\varepsilon = -1$.

Conversely, applying Lemma 4.3.5 again, we have that each pair of segments $\overline{\mathbf{v}_i\mathbf{v}_{i+1}}$, $\overline{\mathbf{v}_j\mathbf{v}_{j+1}}$ for $j > i+1$ is necessarily transverse. Now it only remains to show that each segment $\overline{\mathbf{v}_i\mathbf{v}_{i+1}}$ with $1 < i < 2k-1$ is transverse to $\overline{\mathbf{v}_{2k}(-\mathbf{v}_1)}$. By hypothesis, we know that $\omega(\mathbf{v}_i, \mathbf{v}_{2k})$ all have the same sign for $i < 2k-1$ (for $i = 2k-2$, we have that $\omega(\mathbf{v}_{i+1}, \mathbf{v}_{2k}) = 0$ which allowed). Furthermore, for $i > 2$ we have that $\omega(\mathbf{v}_i, -\mathbf{v}_1) = \omega(\mathbf{v}_1, \mathbf{v}_i)$ will share the same sign (similarly, $\omega(\mathbf{v}_2, -\mathbf{v}_1) = 0$ which is allowed). We conclude that each segment $\overline{\mathbf{v}_i\mathbf{v}_{i+1}}$ with $1 < i < 2k-1$ is transverse to $\overline{\mathbf{v}_{2k}(-\mathbf{v}_1)}$. We have thus shown that γ is a positive piecewise linear curve in $\mathbb{RP}^3$. $\square$

For a given labeled, oriented piecewise circular curve γ with $2k$ vertices $p_1, \dots, p_{2k}$, we will associate to γ a k-tuple of flags in $\mathsf{PSp}^{\pm}(4, \mathbb{R}) \cong \mathsf{SO}(V^{3,2})$ as follows. Since γ is oriented, we have a unit tangent vector for the curve γ based at each vertex p_i. Denote by p_i the points in $\mathbb{RP}^3$ corresponding to the unit tangent vectors at p_i by $\mathbb{RP}^3 \cong T^1(S^2)$. Let P_i be the 1-dimensional subspaces of $V \cong \mathbb{R}^4$ corresponding to $p_i \in \mathbb{RP}^3$.

Definition 4.3.8. For a labeled, oriented piecewise circular curve γ with $2k$ vertices $p_1, \ldots, p_{2k}$, we define the *associated k-tuple of flags* in $\mathsf{PSp}^{\pm}(4,\mathbb{R}) \cong \mathsf{SO}(V^{3,2})$ to be $F_1 = (P_1, \mathrm{span}(P_1, P_2)), F_2 = (P_3, \mathrm{span}(P_3, P_4)), \ldots, F_k = (P_{2k-1}, \mathrm{span}(P_{2k-1}, P_{2k}))$.

Proposition 4.3.9. *If γ is a positive, oriented piecewise circular hexagon, then for any labeling of γ the associated triple of flags (F_1, F_2, F_3) is positive.*

Proof. To rephrase the statement of the proposition in terms of the Legendrian piecewise linear lift $\hat{\gamma}$ of γ, we want to show that if no two non-adjacent segments of $\hat{\gamma}$ contain an orthogonal pair, the (F_1, F_2, F_3) is positive.

For a given labeling of γ, we will denote the vertices of γ by $p_1, \ldots, p_6$. Recall that the action of $\mathsf{PSp}^{\pm}(4,\mathbb{R}) \cong \mathsf{SO}(V^{3,2})$ on the complete flag manifold is 2-transitive. Therefore, by applying as element of $\mathsf{PSp}^{\pm}(4,\mathbb{R}) \cong \mathsf{SO}(V^{3,2})$, we can assume without loss of generality that $F_1 = F_0$ and $F_3 = F_\infty$. We fix the representative of P_1 to be $\mathbf{v}_1 = \mathbf{e}_1$, and then choose successive representative of P_i such that the Legendrian line segements are given by positive linear combinations of the representatives, except for the segment $\overline{\mathbf{v}_6\mathbf{v}_1}$.

Since we fixed the flags F_1 and F_3 to be F_0 and F_∞, respectively, we know that the representative $\mathbf{v}_2$ of P_2 has the form $\mathbf{v}_2 = a\mathbf{e}_1 + b\mathbf{e}_2$. Similarly, $\mathbf{v}_5 = \delta\mathbf{e}_4$ and $\mathbf{v}_6 = \varepsilon\mathbf{e}_3$ with $\delta, \varepsilon = \pm 1$. Note that the form for $\mathbf{v}_6$ does not include $\mathbf{e}_4$ as $\mathbf{v}_6$ and $\mathbf{v}_1$ must be orthogonal by assumption.

By applying $\mathsf{diag}(1,-1,-1,1) \in \mathsf{Stab}(F_0, F_\infty) \subset \mathsf{PSp}^{\pm}(4,\mathbb{R})$, we can assume without loss of generality that $\varepsilon = -1$. Similarly, by applying the element $\mathsf{diag}(1,-1,1,-1) \in \mathsf{Stab}(F_0, F_\infty) \subset \mathsf{PSp}^{\pm}(4,\mathbb{R})$, we can assume without loss of generality that $\delta = 1$.

Now we consider $\mathbf{v}_3 = \sum x_i \mathbf{e}_i$ and $\mathbf{v}_4 = \sum y_i \mathbf{e}_i$ for the representatives of p_3 and p_4. Since $\overline{\mathbf{v}_4\mathbf{v}_5}$ is a Legendrian segment, we know that $y_1 = 0$. Therefore, the flag F_2 is given by the matrix

$$M = \begin{pmatrix} x_1 & 0 & 0 & 0 \\ x_2 & y_2 & 0 & 0 \\ x_3 & y_3 & 1 & 0 \\ x_4 & \frac{x_2y_3 - x_3y_2}{x_1} & \frac{x_2}{x_1} & 1 \end{pmatrix} \tag{4.22}$$

where we have used the fact that F_2^2 is isotropic to get $y_4 = \frac{x_2y_3 - x_3y_2}{x_1}$ and $F_2^3 = (F_2^1)^\perp$ to simplify the third column. As we saw in the proof of Theorem 3.3.8, the triple (F_0, F_2, F_∞) will be positive if this matrix is lower strictly totally positive. By Theorem 3.2.3, we need only check that all the connected minors containing an element of the first column are strictly positive.

Here it it suffices to check that $x_i > 0$, $x_1y_2 > 0$, $x_2y_3 - x_3y_2 > 0$, and $x_3y_4 - x_4y_3 > 0$. This is because the 3×3 minors simplify to give

$$\mathsf{det}\begin{pmatrix} x_1 & 0 & 0 \\ x_2 & y_2 & 0 \\ x_3 & y_3 & 1 \end{pmatrix} = x_1y_2 \qquad \text{and} \qquad \mathsf{det}\begin{pmatrix} x_2 & y_2 & 0 \\ x_3 & y_3 & 1 \\ x_4 & \frac{x_2y_3 - x_3y_2}{x_1} & \frac{x_2}{x_1} \end{pmatrix} = x_4y_2. \tag{4.23}$$

If we have shown that $x_1 > 0$, $x_4 > 0$, and $x_1y_2 > 0$, then clearly $x_4y_2 > 0$.

As the curve γ was assumed to be positive, we can apply Lemma 4.3.7 to get that each of the following products must have the same sign.

$$\begin{gathered}\omega(\mathbf{v}_1,\mathbf{v}_3)=-x_4,\qquad \omega(\mathbf{v}_1,\mathbf{v}_4)=-y_4,\qquad \omega(\mathbf{v}_1,\mathbf{v}_5)=-\delta=-1,\\ \omega(\mathbf{v}_1,\mathbf{v}_6)=0,\qquad \omega(\mathbf{v}_2,\mathbf{v}_4)=-ay_4+by_3,\qquad \omega(\mathbf{v}_2,\mathbf{v}_5)=-a\delta=-a,\\ \omega(\mathbf{v}_2,\mathbf{v}_6)=b\varepsilon=-b,\qquad \omega(\mathbf{v}_3,\mathbf{v}_5)=-\delta x_1=-x_1,\\ \omega(\mathbf{v}_3,\mathbf{v}_6)=\varepsilon x_2=-x_2,\qquad \omega(\mathbf{v}_4,\mathbf{v}_6)=\varepsilon y_2=-y_2\end{gathered}\tag{4.24}$$

Therefore, we must have that $x_1,x_2,x_4>0$, $y_2,y_4>0$, $ay_4-by_3>0$, and $a,b>0$.

This immediately gives that $x_1y_2>0$. Now the segment $\overline{\mathbf{v}_2\mathbf{v}_3}$ is Legendrian and therefore $x_3=\frac{a}{b}x_4>0$. Similarly, $\overline{\mathbf{v}_3,\mathbf{v}_4}$ is Legendrian and so $x_2y_3-x_3y_2=x_1y_4>0$. Lastly, since $x_4>0$, $b>0$, $ay_4-by_3>0$, and $x_3=\frac{a}{b}x_4$ we see that

$$x_3y_4-x_4y_3=\frac{a}{b}x_4y_4-x_4y_3=\frac{x_4}{b}(ay_4-by_3)>0.\tag{4.25}$$

We conclude that the matrix M is lower strictly totally positive. Therefore, the triple $(F_1,F_2,F_3)=(F_0,F_2,F_\infty)$ is positive. $\square$

Proposition 4.3.10. *Let (F_1,F_2,F_3) be a positive triple of flags in $\mathsf{PSp}^\pm(4,\mathbb{R})\cong\mathsf{SO}(V^{3,2})$. Then, there is a unique labeled, positive, oriented piecewise circular hexagon with vertices p_i such that $F_i=(P_{2i-1},\mathrm{span}(P_{2i-1},P_{2i}))$.*

Proof. As the action of $\mathsf{PSp}^\pm(4,\mathbb{R})\cong\mathsf{SO}(V^{3,2})$ preserves both positivity of triples and positivity of a oriented piecewise circular curve, we can use the 2-transitivity of the action of $\mathsf{PSp}^\pm(4,\mathbb{R})\cong\mathsf{SO}(V^{3,2})$ on the complete flag manifold. Therefore, we can assume, without loss of generality that $F_1=F_0$, $F_3=F_\infty$, and

$F_2 = \begin{pmatrix} 1 & 0 \\ a & 1 \\ b & d \\ c & ad-b \end{pmatrix}$ where we have the conditions for positivity requiring that $a, b, c, d > 0$, $ad - b > 0$, and $abd - b^2 - cd > 0$.

We consider the Legendrian piecewise linear curve with vertices $p_i \in \mathbb{RP}^3$ given by the columns $\mathbf{v}_i$ of the matrix

$$\begin{pmatrix} 1 & b & 1 & 0 & 0 & 0 \\ 0 & c & a & 1 & 0 & 0 \\ 0 & 0 & b & d & 0 & -1 \\ 0 & 0 & c & ad-b & 1 & 0 \end{pmatrix} \tag{4.26}$$

such that each segment is given by positive linear combinations of adjacent columns, except the segment $\overline{\mathbf{v}_6\mathbf{v}_1}$ which is given by linear combinations with opposite signs of the first and sixth columns.

Now consider the following products of the $\mathbf{v}_i$:

$$\begin{gathered} \omega(\mathbf{v}_1, \mathbf{v}_3) = -c, \qquad \omega(\mathbf{v}_1, \mathbf{v}_4) = -(ad-b), \qquad \omega(\mathbf{v}_1, \mathbf{v}_5) = -1, \\ \omega(\mathbf{v}_1, \mathbf{v}_6) = 0, \qquad \omega(\mathbf{v}_2, \mathbf{v}_4) = -abd + b^2 + cd, \qquad \omega(\mathbf{v}_2, \mathbf{v}_5) = -b, \\ \omega(\mathbf{v}_2, \mathbf{v}_6) = -c, \qquad \omega(\mathbf{v}_3, \mathbf{v}_5) = -1, \\ \omega(\mathbf{v}_3, \mathbf{v}_6) = -a, \qquad \omega(\mathbf{v}_4, \mathbf{v}_6) = -1. \end{gathered} \tag{4.27}$$

By the conditions required for positivity, we have that each of these products are strictly negative, except for $\omega(\mathbf{v}_1, \mathbf{v}_6) = 0$. Therefore, $\omega(\mathbf{v}_i, \mathbf{v}_j)$ have the same sign for all $j > i + 1$. Furthermore, our choice of segment which connects the vertices p_6

and p_1 was designed such that it is given by positive linear combinations of $\mathbf{v}_6$, $\varepsilon\mathbf{v}_1$ such that $\varepsilon = -1$. By applying Lemma 4.3.7, we conclude that our chosen curve is positive.

We now wish to show uniqueness of this curve. Since our curve is a hexagon, we have that for non-adjacent edges $\overline{\mathbf{v}_i\mathbf{v}_{i+1}}$, $\overline{\mathbf{v}_j\mathbf{v}_{j+1}}$ at most one of the products $\omega(\mathbf{v}_i, \mathbf{v}_j)$, $\omega(\mathbf{v}_i, \mathbf{v}_{j+1})$, $\omega(\mathbf{v}_{i+1}, \mathbf{v}_j)$, $\omega(\mathbf{v}_{i+1}, \mathbf{v}_{j+1})$ can be zero. This is because otherwise we would have more than one pair of vertices of these distinct non-adjacent edges which are connected by a single edge. This is only possible in a quadrilateral.

As the curve is positive, we have that non-adjacent edges $\overline{\mathbf{v}_i\mathbf{v}_{i+1}}$, $\overline{\mathbf{v}_j\mathbf{v}_{j+1}}$ are necessarily transverse. By replacing one or both segments $\overline{\mathbf{v}_i\mathbf{v}_{i+1}}$, $\overline{\mathbf{v}_j\mathbf{v}_{j+1}}$ by their opposite segments, the pair of segments would no longer be transverse as a consequence of Lemma 4.3.5. Therefore, switching which segment connects adjacent vertices would result in a curve which is not positive. When choosing a labeled, oriented piecewise circular curve compatible with (F_1, F_2, F_3), the only freedom we had was in choosing which segments connect the vertices. Therefore the corresponding positive curve must be unique. $\square$

As a consequence of the previous two propositions, we have explicitly defined a bijection

$$\left\{\begin{array}{c}\text{Labeled, positive, oriented}\\ \text{piecewise circular hexagons}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{Positive triples of flags}\\ \text{in } \mathsf{PSp}^{\pm}(4,\mathbb{R}) \cong \mathsf{SO}(V^{3,2})\end{array}\right\} \tag{4.28}$$

sending a curve γ with vertices $p_1, \ldots, p_6$ to its associated triple of flags $(F_1, F_2, F_3) = ((P_1, \mathrm{span}(P_1, P_2)), (P_3, \mathrm{span}(P_3, P_4)), (P_5, \mathrm{span}(P_5, P_6)))$. The group action of $\mathsf{PSp}^{\pm}(4,\mathbb{R}) \cong \mathsf{SO}(V^{3,2})$ on each of these spaces respects this bijection.

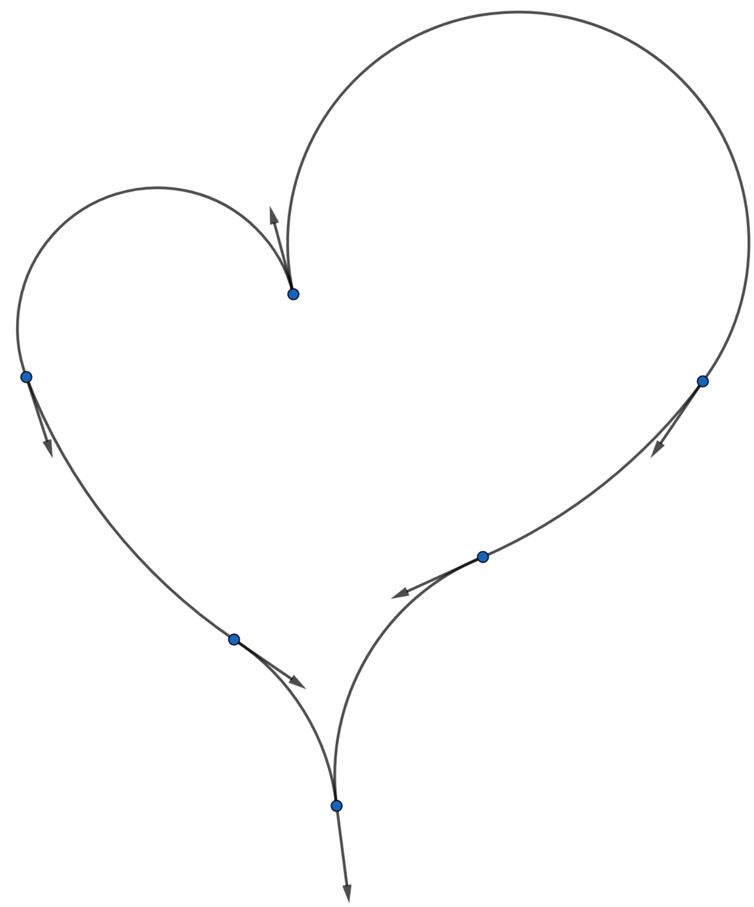


Figure 4.7: A positive, oriented piecewise circular hexagon.

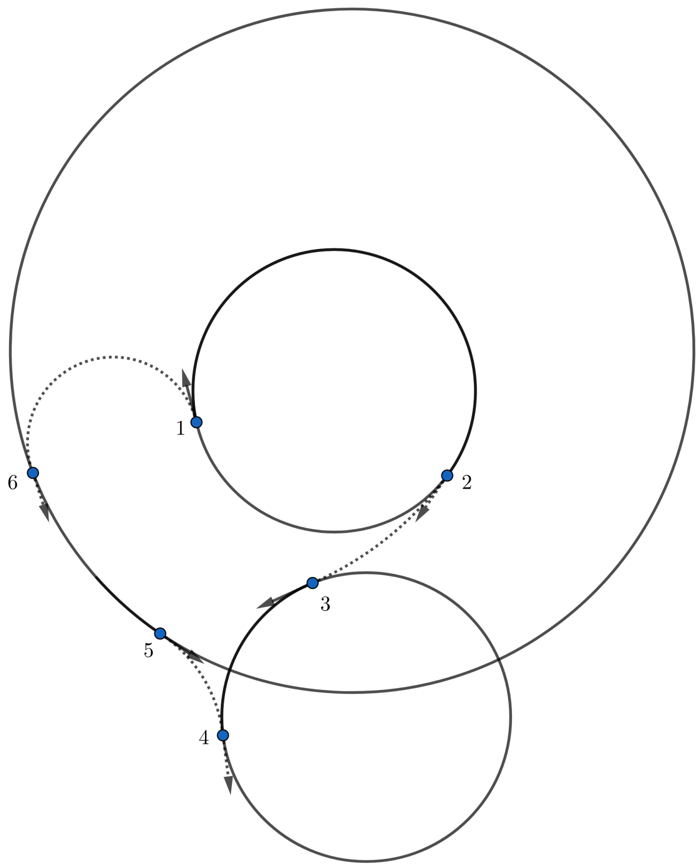


Figure 4.8: The triple of flags associated to a labeled, positive, oriented piecewise circular hexagon.

We therefore have the following result relating labeled, positive, oriented piecewise circular hexagons up to projective contactomorphisms with the totally positive part of $\mathsf{Conf}^{(3)}(\mathscr{F})$.

Theorem 4.3.11. *The induced map*

$$\begin{aligned} \left\{ \begin{matrix} \textit{labeled, positive, oriented} \\ \textit{piecewise circular hexagons} \end{matrix} \right\} \Big/ \mathsf{PSp}^{\pm}(4,\mathbb{R}) &\longrightarrow \mathsf{Conf}_{+}^{(3)}(\mathscr{F}) \\ [\gamma] &\longmapsto [(F_1, F_2, F_3)] \end{aligned} \tag{4.29}$$

where $\mathsf{Conf}_{+}^{(3)}(\mathscr{F})$ *is the totally positive part of* $\mathsf{Conf}^{(3)}(\mathscr{F})$ *is a bijection.*

Recall from Theorems 3.3.8 and 3.4.4 that we have two types of coordinates on the totally positive part of $\mathsf{Conf}^{(3)}(\mathscr{F})$ which parametrize by it quadrants homeomorphic to $\mathbb{R}^2$.

Corollary 4.3.12. *The space of labeled, positive, oriented piecewise circular hexagons in the 2-sphere up to Möbius transformations, equidistant transformations, and the orientation reversing transformation is parametrized by* $\mathbb{R}^2$.

4.4 Future work

The results of Fock and Goncharov [FG07] for $\mathbb{RP}^2$ parametrize the totally positive part of the configuration space k-tuples of flags. They associated to each k-tuple of flags a particular k-sided convex polygon in the projective plane. After triangulating this polygon, they assign a single coordinate to the face of each triangle and two coordinates to each shared edge. The face coordinates correspond to coordinates on the totally positive part of the configuration space of triples of flags while the edge coordinates give instructions on how to combine two positive triples of flags into a positive 4-tuple of flags.

In the future, we wish to describe similar edge coordinates for the totally positive part of the configuration space of k-tuples of flags in $\mathsf{SO}(V^{3,2}) \cong \mathsf{PSp}^{\pm}(4,\mathbb{R})$. We expect the following result.

Conjecture 4.4.1. *There are geometrically defined invariants giving coordinates on the totally positive part of* $\mathsf{Conf}^{(k)}(\mathscr{F})$ *which parametrizes the totally positive part of* $\mathsf{Conf}^{(k)}(\mathscr{F})$ *by* $\mathbb{R}^{4k-10}$.

We then wish to understand how this result will apply to labeled, positive, oriented, piecewise circular wavefronts. We believe we should be able to describe a process akin to gluing two convex triangles in the projective plane along a common edge to get a quadrilateral. Here we would like to understand how to glue two labeled, positive, oriented piecewise circular hexagons to get a labeled, positive, oriented piecewise circular 8-gon. We expect this process to give the following result.

Conjecture 4.4.2. *The space of labeled, positive, oriented piecewise circular* $2k$*-gons in the 2-sphere up to Möbius transformations, equidistant transformations, and the orientation reversing transformation is parametrized by* $\mathbb{R}^{4k-10}$.

Additionally, we wish to study the case of infinite positive subsets of complete isotropic flag manifold. In order to do this we will generalize the notion of positivity to piecewise C^2 curves in the 2-sphere. These curves should be the limit curves of positive representations into $\mathsf{SO}(3,2)$. Here the limit curve of a positive representation $\rho : \pi_1(S) \to \mathsf{SO}(3,2)$ can be understood as a map $f : S^1 \cong \partial\mathbb{H}^2 \to \mathscr{F}$ which is ρ-equivariant, i.e. for every $g \in \pi_1(S)$, we have $f(gp) = \rho(g)f(p)$. In this way we wish to provide a characterization of the higher Teichmüller space for $\mathsf{SO}(3,2)$.

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