ABSTRACT

Title of dissertation: EXPONENTIATION OF MOTIVIC
                                    ZETA FUNCTIONS

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We provide a formula for the generating series of the Weil zeta function $Z(X,t)$ of symmetric powers $\mathrm{Sym}^n X$ of varieties X over finite fields. This realizes the zeta function $Z(X,t)$ as an exponentiable measure whose associated Kapranov motivic zeta function takes values in $W(R)$ the big Witt ring of $R = W(\mathbb{Z})$. We apply our formula to compute $Z(\mathrm{Sym}^n X, t)$ in a number of explicit cases. Any motivic zeta function ζ_μ of a measure μ factoring through the Grothendieck ring of Chow motives is itself exponentiable; in fact, this applies to ζ_μ as a motivic measure itself. We prove a condition for which any motivic measure taking values in a λ-ring has an associated motivic zeta function $Z = \zeta_\mu$ that is itself an exponentiable measure $\mu_Z = Z$, and this process is shown to iterate indefinitely. This involves a study of the case of λ-ring-valued motivic measures. Finally, we provide an understanding of MacDonald's formula in this context.

EXPONENTIATION OF MOTIVIC ZETA FUNCTIONS

by

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Dedication

To my parents.

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Introduction

Zeta functions play a central role in number theory; the theme of many conjectures is that the special values of certain zeta functions, which are analytically defined objects, can be expressed in terms of algebraic invariants. These algebraic invariants (ℓ-adic cohomology, Weil-étale cohomology,...) satisfy usual properties, for example they are typically additive and multiplicative (via a Künneth formula, for instance). These are so-called Euler-Poincaré characteristics on the category of certain spaces (varieties over $\operatorname{Spec} k$, schemes of finite type over $\operatorname{Spec} \mathbb{Z}$,...). If we view the zeta function as a map on the category of such spaces, then it is natural to ask if zeta functions themselves can be considered Euler-Poincaré characteristics. This would shed light on why the special values of zeta functions take the form of Euler characteristic formulas: the zeta function is itself an Euler characteristic.

To better understand the relation of zeta functions to these algebraic invariants, an increasingly important object of study in recent years has been the Grothendieck ring of varieties $K_0(\operatorname{Var}_k)$. This is the value group of the universal Euler-Poincaré characteristics on the category Var_k of varieties over $\operatorname{Spec} k$, as Heinloth describes in [12]. Ring homomorphisms on $K_0(\operatorname{Var}_k)$ are called *motivic measures*. Given a motivic measure μ, Kapranov in [15] constructs a *motivic zeta function*

$\zeta_\mu(X, t)$ for $X \in \mathrm{Var}_k$ using these algebraic invariants $\mu(X)$. It is the generating series for (the measure of) symmetric powers $\mathrm{Sym}^n X$: for $\mu : K_0(\mathrm{Var}_k) \to R$,

$$\zeta_\mu(X, t) = \sum_{n=0}^{\infty} \mu(\mathrm{Sym}^n X) t^n \in R[[t]].$$

This is a group homomorphism on $K_0(\mathrm{Var}_k)$ taking values in the group of invertible power series $(1 + tR[[t]], \times)$. To understand ζ_μ as an Euler-Poincaré characteristic, we must study its (possible) structure as a ring homomorphism. Indeed, the Grothendieck ring of varieties is a ring where multiplication is induced by $X \times_k Y$. We can endeavor to describe the product $\zeta_\mu(X \times_k Y)$ using a product structure on $(1 + tR[[t]], \times)$. As it turns out, the correct product structure is often given by the big Witt ring $W(R)$.

Take for instance the case of varieties over finite fields, $\mathrm{Var}_{\mathbb{F}_q}$. The counting measure $\mu_\#$ is the motivic measure that counts the number of points over $\mathbb{F}_q$: given a variety $X \in \mathrm{Var}_{\mathbb{F}_q}$, set $\mu_\#(X) = \#X(\mathbb{F}_q)$, taking values in the ring $\mathbb{Z}$. The zeta function of a variety over a finite field is the Kapranov motivic zeta function associated to the counting measure, $Z(X, t) = \zeta_{\mu_\#}$. In [22], Ramachandran points out that for varieties over finite fields, the zeta function $Z(X \times Y, t)$ is the product $Z(X, t) *_W Z(Y, t)$ in the Witt ring $W(\mathbb{Z})$. This leads to the following definition: a measure $\mu : K_0(\mathrm{Var}_k) \to R$ is *exponentiable* (see [22] and [23]) if the product structure of $K_0(\mathrm{Var}_k)$ is reflected by the product in $W(R)$. That is, motivic zeta functions (of exponentiable measures) are Euler-Poincaré characteristics $\zeta_\mu : K_0(\mathrm{Var}_k) \to W(R)$ taking values in the big Witt ring.

In this thesis, we answer several questions posed in [22]. Specifically, our main

results are as follows:

- We prove a formula for the generating series of the Weil zeta function $Z(X, t)$ of symmetric powers $\mathrm{Sym}^n X$ of varieties X over finite fields (Theorem 2.3). This shows that the induced motivic zeta function measure μ_Z is exponentiable for $\mathrm{Var}_{\mathbb{F}_q}$.

- We prove various conditions for lambda-ring-valued exponentiable motivic measures μ to have an induced motivic zeta function measure μ_Z that is itself exponentiable, and show that this process iterates (Theorem 3.5, Corollary 3.6, and Theorem 3.8).

- We relate these results to the classical MacDonald formula in the setting of λ-ring valued measures and argue that the formula in Theorem 2.3 provides a "MacDonald" formula for the Weil zeta function $Z(X, t)$ (Section 3.3).

As the motivic zeta function ζ_μ can be thought of as a motivic measure, we can ask whether ζ_μ is itself an exponentiable measure. We show that this is the case for varieties over finite fields $\mathrm{Var}_{\mathbb{F}_q}$ and the Weil zeta function $Z(X, t)$. We prove a formula reminiscent of the Euler-Poincaré characteristic formula for the Weil zeta function as an alternating sum of *Teichmüller elements* $[\alpha]$:

$$Z(X, t) = \sum_{i,j} (-1)^{i+1} [\alpha_{ij}]$$

where this sum is taking place in the Witt ring $W(\mathbb{Z})$. As $Z(X, t) = \zeta_{\mu_\#}(X, t)$ takes values in the ring $A = W(\mathbb{Z})$, we show that the zeta function measure $\mu_Z(X) = Z(X, t)$ is exponentiable and its associated motivic zeta function ζ_{μ_Z} takes values in

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$W(A) = W(W(\mathbb{Z}))$. To do so, we provide a closed formula for the generating series of the zeta function of symmetric powers,

$$\zeta_{\mu_Z}(X, u) = \sum_{n=0}^{\infty} Z(\mathrm{Sym}^n X, t) u^n.$$

Specifically, the formula put forth in Theorem 2.3 presents ζ_{μ_Z} in $W(W(\mathbb{Z}))$ as a sum of *double Teichmüller elements* $[[\,[\alpha]\,]]$:

$$\zeta_{\mu_Z}(X, u) = \sum_{i,j} (-1)^{i+1} [[\,[\alpha_{ij}]\,]].$$

We then generalize these results for $\mu = \mu_\#$ and varieties over finite fields to the general case of abstract motivic measures μ. We prove that if μ is any motivic measure taking values in a λ-ring R and its motivic zeta function factors through μ via the opposite λ-ring structure $\sigma_t : R \to W(R)$, the motivic zeta function is itself an exponentiable measure $\mu_Z(X) := \zeta_\mu(X, t)$ with its associated motivic zeta function ζ_{μ_Z} taking values in $W(W(R))$. This involves a study of λ-ring-valued motivic measures.

The idea of studying the Grothendieck ring of varieties using motivic measures is well-established in the literature. Moreover, the importance of λ-ring-valued motivic measures is observed in Larsen and Lunts [17], Heinloth [12] [13], Gusein-Zade, Lluengo, and Melle-Hernandez [10] [9], del Baño Rollin [2] [3], and Maxim and Schurmann [20]. Although Witt-type phenomena of zeta functions appears in prior studies, identifying the Kapranov motivic zeta function as an object in the big Witt ring is only earnestly present in Ramachandran [22] and Ramachandran-Tabuada [23]. In this work, we make heavy use of the fact that the big Witt ring $W(R)$ is a λ-ring and study zeta functions as measures taking values in this ring.

We begin in Chapter 1 with a brief survey of preliminary results on λ-rings, Witt rings, motivic measures and motivic zeta functions. We then analyze the finite field case, proving a generating series formula for the Weil zeta function of symmetric powers of a variety, and we then apply this formula in a number of explicit cases (Chapter 2). This shows that the zeta function for varieties over finite fields, when considered as a motivic measure, is exponentiable. In Chapter 3 we study exponentiation of λ-ring-valued motivic measures along the lines of Ramachandran and Tabuada in [23]. We show for certain motivic measures, the induced motivic zeta function measure $\mu_Z = \zeta_\mu$ is exponentiable, and this process iterates indefinitely. Finally, we contrast this with the existence of a MacDonald formula by reframing these in the setting of λ-rings.

Chapter 1 Preliminaries

This chapter is a survey of some well-known results on λ-rings and Witt rings, the Grothendieck ring of varieties $K_0(\mathrm{Var}_k)$, and motivic measures and the Kapranov motivic zeta functions. We end with an explanation of what it means for a motivic measure to be exponentiable.

1.1 Lambda rings and Witt rings

We begin with a review of the theory of λ-rings and Witt rings. For a more comprehensive introduction, the reader may consult Knutson [16], Yao [25], and Hazewinkel [11]. Grothendieck originally introduced the general concept of a λ-ring in order to study the Riemann-Roch theorem.

Let A be a commutative ring with identity. We denote by $\Lambda(A)$ the following subgroup of invertible power series under the usual multiplication of power series:

$$\Lambda(A) := (1 + tA[[t]], \times).$$

We will seek to endow $\Lambda(A)$ with a product structure in the sequel; however, unless otherwise stated, $\Lambda(A)$ will mean the abelian group under multiplication of power series.

Given a ring map $f : R \to S$, the induced map $\Lambda_f : \Lambda(R) \to \Lambda(S)$ acts on each coefficient:

$$\Lambda_f(a_0 + a_1 t + a_2 t^2 + \cdots) = f(a_0) + f(a_1)t + f(a_2)t^2 + \cdots$$

Note that f needs to be a ring homomorphism in order for Λ_f to be a group homomorphism.

We shall make heavy use of the ghost map $\mathrm{gh} : \Lambda(A) \longrightarrow A^{\mathbb{N}}$. Given $P(t) \in \Lambda(A)$, we define

$$\mathrm{gh}(P(t)) = (b_1, b_2, b_3, \ldots), \quad \text{where} \quad \frac{t}{P}\frac{dP}{dt} = \sum_{n=1}^{\infty} b_n t^n.$$

This is a functorial group homomorphism. Essentially, instead of considering the power series $P(t) = a_0 + a_1 t + a_2 t^2 \cdots$, we use another sequence of coordinates $(b_1, b_2, \ldots)$ to represent $P(t)$ in $A^{\mathbb{N}}$. These coordinates b_n are called the ghost coordinates gh_n of $P(t)$ where $\mathrm{gh}_n(P(t)) = b_n$ where $\mathrm{gh}_n : A \to A$. As we will see, it is often more convenient to use the ghost coordinates of a power series.

The relation between the power series coefficients and the ghost coordinates can be made explicit:

Lemma 1.1. *For $P(t) = \sum_n a_n t^n \in \Lambda(A)$ and $b_n = \mathrm{gh}_n(P(t))$, we have*

$$na_n = b_n + a_1 b_{n-1} + \cdots a_{n-1} b_1.$$

This relation uniquely determines $P(t)$ in terms of its ghost coordinates b_n in the case where A has no $\mathbb{Z}$-torsion.

Proof. By definition, $\dfrac{t}{P}\dfrac{dP}{dt} = \sum_{n=1}^{\infty} b_n t^n$ so that

$$P(t)\left(\sum_{n=1}^{\infty} b_n t^n\right) = t\frac{dP}{dt}$$

$$(a_0 + a_1 t + a_2 t^2 + \cdots)(b_1 t + b_2 t^2 + \cdots) = t(a_1 + 2a_2 t + 3a_3 t^2 \cdots)$$

$$= a_1 t + 2a_2 t^2 + 3a_3 t^3 + \cdots$$

and we simply identify coefficients. $\square$

Lemma 1.2. *For $A = \mathbb{Z}$, the ghost coordinates of*

$$P(t) = \exp\left[\sum_{r=1}^{\infty} b_r \frac{t^r}{r}\right]$$

are given by $\mathrm{gh}(P) = (b_1, b_2, b_3, \ldots)$. *This is true for any ring A where* exp *makes sense.*

For example,

$$\left(\frac{1}{1-at}\right) = \exp\left[\sum_{r=1}^{\infty} a^r \frac{t^r}{r}\right] \quad \text{and} \quad \left(\frac{1}{1-t}\right)^a = \exp\left[\sum_{r=1}^{\infty} a \frac{t^r}{r}\right].$$

and thus $\mathrm{gh}_r\left((1-at)^{-1}\right) = a^r$ and $\mathrm{gh}_r\left((1-t)^a\right) = a$.

1.1.1   Multiplicative structure on $\Lambda(A)$

The first multiplicative structure we will study on $\Lambda(A)$ comes from certain universal polynomials. Given two power series $f(t) = \sum_i a_i t^i$ and $g(t) = \sum_i b_i t^i$, we define their product as the power series

$$f(t) *_\Lambda g(t) = \sum_n P_n(a_1, \ldots, a_n; b_1, b_2, \ldots, b_n) t^n$$

where the P_n are the universal polynomials described in Appendix A. For example,

$$P_0 \;=\; 1$$

$$P_1(a_1; b_1) \;=\; a_1 b_1$$

$$P_2(a_1, a_2; b_1 b_2) \;=\; a_1^2 b_2 - 2a_2 b_2 + a_2 b_1^2$$

This commutative product is inspired by the behavior of the exterior product $\Lambda^n V$ on vector spaces V (see Appendix A). We will refer to this as the universal multiplicative structure on $\Lambda(A)$.

1.1.2 Lambda-rings

We wish to define operations λ^n on A that behave like exterior powers on vector spaces, and we will use the power series coefficients in $\Lambda(A)$ to enumerate them.

Definition 1.3. A *pre-λ-ring structure* on A is a group homomorphism $\lambda_t : A \to \Lambda(A)$ denoted by

$$\lambda_t(a) = 1 + \sum_{n=1}^{\infty} \lambda^n(a) t^n, \quad \text{such that } \lambda^1(a) = a.$$

The group homomorphism requirement is equivalent to $\lambda_t(a + b) = \lambda_t(a)\lambda_t(b)$ the product of power series. Thus we obtain a sequence of maps $\lambda^n : A \to A$ such that

$$\lambda^n(a + b) = \sum_{i+j=n} \lambda^i(a)\lambda^j(b) \quad \text{for all } n.$$

The pair (A, λ_t) is called a *pre-λ-ring*.

Given two pre-λ-rings (A, λ_t) and (B, λ'_t), a ring homomorphism $f : A \to B$ is a map of pre-λ-rings if it preserves the pre-λ-ring structure map. That is, $\Lambda_f \circ \lambda'_t = \lambda_t \circ f$, i.e.

$$
\begin{array}{ccc}
A & \xrightarrow{\lambda_t} & \Lambda(A) \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \Lambda_f} \\
B & \xrightarrow{\lambda'_t} & \Lambda(B)
\end{array}
$$

commutes.

Examples. Here are a few examples of rings with pre-λ-ring structure:

- $\mathbb{Z}$ with $\lambda_t(a) = (1 + t)^a$ for $a \in \mathbb{Z}$. In this case, the λ maps are given by binomial coefficients $\lambda^n(a) = \begin{pmatrix} a \\ n \end{pmatrix}$.

- $\mathbb{R}$ with $\lambda_t(x) = (1 + t)^x$.

- $\mathbb{R}$ with $\lambda_t(x) = e^{xt}$.

- $K_0(F)$ the Grothendieck ring of (isomorphism classes of) finite dimensional (virtual) vector spaces over a field F. In this case, $\lambda^n[V] = [\Lambda^n V]$.

Definition 1.4. Given a pre-λ-ring A , the *nth Adams operation* $\Psi_n : A \to A$ is defined as

$$
\frac{t}{\lambda_t(a)} \frac{d\lambda_t(a)}{dt} = \sum_{n=1}^{\infty} (-1)^{n+1} \Psi_n(a) t^n
$$

The nth Adams operation is simply (up to $\pm$) the nth ghost coordinate of $\lambda_t(a) \in \Lambda(A)$:

$$
\Psi_n(a) = (-1)^{n+1} \operatorname{gh}_n(\lambda_t(a)).
$$

Thus, we have $\Psi_n(a+b) = \Psi_n(a) + \Psi_n(b)$, i.e. the Adams operations are group homomorphisms, since λ_t is a group homomorphism.

Examples. Here are a few examples of Adams operations:

- In the case of $\mathbb{Z}$ with $\lambda_t(a) = (1+t)^a$, $\Psi_n(a) = a$.

- In the case of $\mathbb{R}$ with $\lambda_t(x) = (1+t)^x$, $\Psi_n(x) = x$.

- In the case of $\mathbb{R}$ with $\lambda_t(x) = e^{xt}$,

$$\Psi_n(x) = \begin{cases} x & \text{for } n = 1 \\ 0 & \text{for } n > 1 \end{cases}$$

- If $\lambda_t(a) = 1 + at$, then $\Psi_n(a) = a^n$.

- In the case of $K_0(F)$, the Adams operations are the classical Adams operations on vector spaces.

Example 1.5. Recall from Section 1.1.1 that the abelian group $\Lambda(A)$ is a commutative ring, given a universal multiplicative structure. It has a natural pre-λ-ring structure denoted $(\Lambda(A), \boldsymbol{\lambda}_u)$. For $m \geqslant 1$ the λ-ring maps $\boldsymbol{\lambda}^{\boldsymbol{m}} : \Lambda(A) \to \Lambda(A)$ are given by

$$\boldsymbol{\lambda}^{\boldsymbol{m}}(1 + a_1 t + a_2 t^2 + \cdots) = 1 + \sum_{n=1}^{\infty} P_{n,m}(a_1, a_2, \ldots, a_{nm}) t^n \in \Lambda(A)$$

with $\boldsymbol{\lambda}^{\boldsymbol{0}} = 1$ and $\boldsymbol{\lambda}_{\boldsymbol{u}} = \sum_{m \geqslant 0} \boldsymbol{\lambda}^{\boldsymbol{m}} u^m \in \Lambda(\Lambda(A))$, where the $P_{n,m}$ are the universal polynomials described in Appendix A. For example,

$$P_{0,m} = 1$$

$$P_{1,m}(a_1, a_2, \ldots, a_m) = a_m$$

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These polynomials can quickly get complicated. For example,

$$P_{4,2}(a_1, a_2, \ldots, a_8) = a_1 a_3 a_4 - 3a_1 a_2 a_5 + a_1^3 a_5 - a_1^2 a_6 + a_1 a_7 + 2a_2 a_6 - a_8.$$

For a table of polynomials $P_{n,m}$, see Hopkinson ([14]).

Proposition 1.6. This pre-λ-ring structure is functorial in the following sense: given any ring homomorphism $f : A \to A'$, the induced map $\Lambda_f : \Lambda(A) \to \Lambda(A')$ is a map of pre-λ-rings.

Proof. For $Q = 1 + a_1 t + a_2 t^2 + \cdots \in \Lambda(A)$, we have

$$\boldsymbol{\lambda^m}(\Lambda_f Q) = \boldsymbol{\lambda^m}(f(1) + f(a_1 t + f(a_2)t^2 + \cdots))$$

$$= 1 + \sum_{n=1}^{\infty} P_{n,m}(f(a_1), f(a_2), \ldots, f(a_{nm}))t^n$$

$$= f(1) + \sum_{n=1}^{\infty} f(P_{n,m}(a_1, a_2, \ldots, a_{nm}))t^n = \Lambda_f(\boldsymbol{\lambda^m}(Q))$$

$\square$

Definition 1.7. A pre-λ-ring (A, λ_t) is a λ-*ring* if λ_t is a map of pre-λ-rings. Specifically, this means for $(\Lambda(A), \boldsymbol{\lambda}_u)$

$$\Lambda_{\lambda_t}(\lambda_t(a)) = \boldsymbol{\lambda}_u(\lambda_t(a)).$$

and the diagram

$$
\begin{array}{ccc}
A & \xrightarrow{\lambda_t} & \Lambda(A) \\
{\scriptstyle \lambda_t}\downarrow & & \downarrow{\scriptstyle \Lambda_{\lambda_t}} \\
\Lambda(A) & \xrightarrow[\boldsymbol{\lambda_u}]{} & \Lambda(\Lambda(A))
\end{array}
$$

commutes. A map of λ-rings is simply a map of the underlying pre-λ-rings.

Note. In the literature, pre-λ-rings are sometimes known as λ-rings and λ-rings sometimes known as special λ-rings.

Proposition 1.8. A pre-λ-ring (A, λ_t) is a λ-ring if and only if both conditions on Adams operations $\Psi_n : A \to A$ hold

- $\Psi_n(ab) = \Psi_n(a)\Psi_n(b)$ for all n.

- $\Psi_n \circ \Psi_m = \Psi_{nm}$.

Proof. The first condition says that λ_t is a ring homomorphism. The second says that it is a map of pre-λ-rings. See Knutson [16, p. 49]. $\square$

Examples. The pre-λ-ring structures described above are λ-rings:

- $\mathbb{Z}$ with $\lambda_t(a) = (1+t)^a$ for $a \in \mathbb{Z}$ is a λ-ring. This is in fact the unique λ-ring structure on $\mathbb{Z}$.

- $\mathbb{R}$ with $\lambda_t(x) = (1+t)^x$ is a λ-ring. $\mathbb{R}$ with $\lambda_t(x) = e^{xt}$ is not a λ-ring (see the Adams operations $\Psi_n(x)$ in this case).

- $K_0(F)$ with $\lambda^n[V] = [\Lambda^n V]$ is a λ-ring. In fact, $K_0(R)$ is a λ-ring. See Grothendieck [8].

The following propositions will be useful later:

Proposition 1.9. The pre-λ-ring structure defined on $\Lambda(A)$ is a λ-ring structure.

Proof. See Knutson [16]. $\square$

Proposition 1.10. For A a $\mathbb{Z}$-torsion free ring, any λ-ring structure λ_t on A is uniquely determined by the Adams operations Ψ_n. Moreover, given $f : A \to A'$ a ring homomorphism between λ-rings (A, λ_t) and (A', λ'_t), f is a λ-ring map if and only if f respects the Adams operations. That is, for Ψ_n and Ψ'_n on A and A' respectively, $f \circ \Psi_n = \Psi'_n \circ f$ for all n.

Proof. See Yau [25, Thm 3.15 and Cor 3.16]. $\qquad\square$

Proposition 1.11. If A is a λ-ring then so is the polynomial ring $A[z]$. For $A = \mathbb{Z}$, we have

$$\lambda_t(a_0 + a_1 z + a_2 z^2 + \cdots a_n z^n) = \prod_i (1 + z^i t)^{a_i}$$

For $P[z] \in A[z]$, the Adams operations are $\Psi_n(P(z)) = P(z^n)$.

Proof. Define the λ-ring structure on $A[z]$ by $\lambda_t(z) = (1+zt)$. Now use the universal polynomials to define the λ-maps on products az^i, using the λ-ring structure on A. In particular, for $A = \mathbb{Z}$,

$$\lambda_t(nz) = \lambda_t(z + z + \cdots + z) = (1 + zt)^n,$$

and a straightforward computation shows we have $\Psi_n(z) = z^n$. $\qquad\square$

Definition 1.12. Given a λ-ring (A, λ_t), the *opposite λ-ring structure map* σ_t is defined as $\sigma_t = \lambda_{-t}^{-1}$, with $\sigma^n : A \to A$ given by

$$\sigma_t(a) = \sum_{n=0}^{\infty} \sigma^n(a) t^n.$$

Note that this is not a ring homomorphism into $\Lambda(A)$. Whereas the the λ-ring structure map λ_t is inspired by exterior powers of vector spaces, the opposite λ-ring structure map corresponds to symmetric powers. Note that $\Psi_n(a) = \mathrm{gh}_n(\sigma_t(a))$.

1.1.3 Witt rings

We now describe another multiplicative structure on $\Lambda(A)$ given by the universal ring of Witt vectors. While the universal multiplicative structure on $\Lambda(A)$ described above is inspired by the behavior of exterior products on vector spaces, the Witt ring is inspired by the behavior of Chern character $Ch(V)$ of vector bundles V. Our presentation here follows Bloch [1] and Ramachandran [22].

Definition 1.13. For a commutative ring A with identity, the *big Witt ring* $W(A)$ has underlying additive group $\Lambda(A)$ and has multiplication $*_W$ defined such that for $a, b \in A$,

$$(1 - at)^{-1} *_W (1 - bt)^{-1} = (1 - abt)^{-1}.$$

We require that the association $A \mapsto W(A)$ is functorial–the induced map $W(f) : W(A) \to W(A')$ given by Λ_f is a ring homomorphism and the resulting diagram commutes. This uniquely determines the structure of a commutative ring on $W(A)$.

For $a \in A$, the Teichmüller element $[a] \in W(A)$ is

$$[a] = (1 - at)^{-1}.$$

Multiplication on $W(A)$ is defined such that $[a] *_W [b] = [ab]$. The Teichmüller element $[1]$ is the multiplicative identity in $W(A)$; the additive identity element in $W(A)$ is $1 = [0] \in A$.

It is often convenient to recast the big Witt ring in terms of ghost coordinates. In fact, one reason to consider the Witt product is that the ghost map

(previously defined) induces a ring homomorphism $\mathrm{gh} : W(A) \to A^{\mathbb{N}}$, where $A^{\mathbb{N}}$ has a component-wise ring structure. Thus for P and Q in $W(A)$,

$$\mathrm{gh}_n(PQ) = \mathrm{gh}_n(P) + \mathrm{gh}_n(Q), \quad \text{and} \quad \mathrm{gh}_n(P *_W Q) = \mathrm{gh}_n(P)\,\mathrm{gh}_n(Q).$$

Note that on Teichmüller elements, $\mathrm{gh}_n([a]) = a^n$.

Note. The big Witt ring is traditionally described in terms of Witt coordinates $a_i \in A$, and $P = (a_1, a_2, \ldots) \in A^{\mathbb{N}}$. These are related to the ghost coordinates by the Witt polynomials

$$w_n = \sum_{d \mid n} d(a_d)^{n/d} = \mathrm{gh}_n(P).$$

Thought of as a power series $P \in \Lambda(A)$ is recovered from the Witt coordinates as

$$P(t) = \prod_{n=1}^{\infty} (1 - a_n t^n)^{-1}.$$

The Witt polynomials w_n are used to describe addition and multiplication in the Witt ring.

Note. The Witt ring $W(A)$ has descending filtration $\mathrm{Filt}^n W(A) = (1 + t^{n+1} A[[t]])$ and $W_n(A) = W(A)/\mathrm{Filt}^n W(A)$. We have $W(A) = \varprojlim W_n(A)$, giving the Witt ring an induced topology where each $W_n(A)$ is given the discrete topology.

Definition 1.14. For each positive integer n, the *Frobenius* $\mathrm{Fr}_n : W(A) \to W(A)$ is a ring endomorphism defined as

$$\mathrm{Fr}_m(P(t)) = \prod_{\zeta^m = 1} P(\zeta t^{1/m})$$

On Teichmüller elements, $\mathrm{Fr}_n([a]) = [a^n]$. On ghost coordinates,

$$\mathrm{gh}_m(\mathrm{Fr}_n(P)) = \mathrm{gh}_{nm}(P)$$

Clearly, we have $\mathrm{Fr}_n \circ \mathrm{Fr}_m = \mathrm{Fr}_{nm}$.

1.1.4 Witt rings of lambda-rings

The Witt ring $W(A)$ has a natural λ-ring structure $\boldsymbol{\lambda_u}$ (see Knutson, [16, p. 18]). On Teichmüller elements, it is given by

$$\boldsymbol{\lambda^0}([a]) = [1], \quad \boldsymbol{\lambda^1}([a]) = [a], \quad \boldsymbol{\lambda^n}([a]) = 1, \ n > 1$$

and so $\boldsymbol{\lambda_u}([a]) = [1] + [a]u \in \Lambda(W(A))$. The Frobenius morphism $\mathrm{Fr}_n : W(A) \to W(A)$ is equal to the nth Adams operation Ψ_n (see Hazewinkel [11]). Note that the Adams operation requirement $\Psi_n \circ \Psi_m = \Psi_{nm}$ in Proposition 1.8 is readily satisfied by the Frobenius on $W(A)$, as noted above. To compare the λ-ring structures on the universal λ-ring $\Lambda(A)$ and the Witt ring $W(A)$, there is a ring isomorphism $\iota : \Lambda(A) \to W(A)$, $P(t) \mapsto P(-t)^{-1}$ induced by the so-called Artin-Hasse exponential.

Proposition 1.15. Given its natural λ-ring structure $\boldsymbol{\lambda_u}$, $W(A)$ has opposite λ-ring structure map $\boldsymbol{\sigma_u} : W(A) \to W(W(A))$ satisfying

$$\boldsymbol{\sigma_u}([a]) = ([1] - [a]u)^{-1}$$

This may be denoted by the *double Teichmüller* $[[\,[a]\,]]$. Note that this only holds for Teichmüller elements $[a]$ and not arbitrary $P(t) \in W(A)$.

Proposition 1.16. Given any ring homomorphism $f : A \to A'$, the induced map $W(f) : W(A) \to W(A')$ is a λ-ring map.

Proof. See Example 1.5. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

We are particularly interested in the case of Witt rings $W(A)$ where the commutative ring A is a λ-ring, (A, λ_t). Just as in the case of the λ-ring structure on

$\Lambda(A)$, there are two λ-ring structures: (A, λ_t) and $(W(A), \boldsymbol{\lambda_t})$. As A is a λ-ring, $\lambda_t : A \rightarrow \Lambda(A)$ is a ring homomorphism. Using the isomorphism $\iota : \Lambda(A) \rightarrow W(A)$, we see that $\sigma_t : A \rightarrow W(A)$ is also a ring homomorphism. In fact, we have

Proposition 1.17. If A is a λ-ring, its opposite λ-structure map $\sigma_t : A \rightarrow W(A)$ is a λ-ring map. In particular,

$$\Lambda_{\sigma_t}(\sigma_t(a)) = \boldsymbol{\sigma_u}(\sigma_t(a))$$

where $\boldsymbol{\sigma_u}$ is the opposite λ-structure map on $W(A)$. Notice that Λ_{σ_t} was also identified previously as $W(\sigma_t)$.

1.2   The Grothendieck Ring of Varieties

Consider Var_k, the category of algebraic varieties X over a field k. For us, an algebraic variety over k will mean a reduced scheme of finite type over $\text{Spec}\,k$. In particular, these are not necessarily separated nor irreducible. For simplicity let k be perfect; thus, for X and Y reduced, the product $X \times_k Y$ is also reduced.

Definition 1.18. The *Grothendieck ring of varieties* $K_0(\text{Var}_k)$ is the abelian group generated by symbols $[X]$ of isomorphism classes of $X \in \text{Var}_k$ subject to the scissor relation $[X] = [Y] + [X \setminus Y]$ for Y any closed subvariety of X. $K_0(\text{Var}_k)$ is a commutative ring under the product $[X] \cdot [Y] = [X \times_k Y]$.

The idea of studying such a ring originates in a letter of Grothendieck to Serre in 1964; it behaves as a shadow (decategorification) of a (still conjectural) category of motives and is sometimes known as the ring of "baby motives." Any

additive invariant on varieties (known generally as Euler-Poincaré characteristics, see below), must factor through this universal value group $K_0(\text{Var}_k)$. The scissor relation is a violent operation on varieties making it possible to formally add and subtract varieties. For a comprehensive survey of the Grothendieck ring of varieties, see Mustata [21, Chap 7].

Examples. Here are a few examples of elements in $K_0(\text{Var}_k)$:

- $X = \text{Spec } k = pt$ and $[pt] \in K_0(\text{Var}_k)$ is the multiplicative identity 1. The empty set $[\emptyset] \in K_0(\text{Var}_k)$ is the additive identity 0.

- For $X = \mathbb{A}^1$ the affine line, the class $[\mathbb{A}^1] \in K_0(\text{Var}_k)$ is often denoted $\mathbb{L}$, the Lefschetz motive.

- For $\mathbb{P}^n$ projective space and $\mathbb{A}^n$ affine space, $[\mathbb{P}^1] = [\mathbb{A}^1] + [pt]$ and $[\mathbb{P}^n] = [\mathbb{A}^n] + \cdots [\mathbb{A}^1] + [pt]$.

We now identify certain important properties of $K_0(\text{Var}_k)$:

Proposition 1.19. For the Grothendieck ring of varieties:

- Every map of fields $k \rightarrow k'$ induces a base change ring homomorphism on $b : K_0(\text{Var}_k) \rightarrow K_0(\text{Var}_{k'})$.

- $K_0(\text{Var}_k)$ is additively generated by the classes $[X]$ for X quasi-projective.

Proof. See Mustata [21, Lem 7.6, Prop 7.27]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

In the case of char $k = 0$, Bittner provides a presentation of the Grothendieck ring as generated by smooth projective varieties and smooth closed subvarieties.

Proposition 1.20 (Bittner [12]). The Grothendieck group $K_0(\text{Var}_k)$ is the abelian group generated by isomorphism classes $[X]$ of smooth projective varieties $X \in \text{Var}_k$ subject to the scissor relation $[X] = [Y] + [X \setminus Y]$ for Y any closed smooth subvariety of X.

1.3   Motivic Measures and Motivic Zeta Functions

A motivic measure μ is a map that associates to every $X \in \text{Var}_k$ up to isomorphism an element $\mu(X)$ in a ring R such that

- $\mu(X) = \mu(X_{red})$.

- $\mu(X \setminus Y) = \mu(X) - \mu(Y)$ for Y a closed subvariety of X.

- $\mu(X \times_k Y) = \mu(X)\mu(Y)$.

Thus every motivic measure determines a ring homomorphism on the Grothendieck ring of varieties, also denoted $\mu : K_0(\text{Var}_k) \to R$. These are sometimes described as Euler-Poincaré characteristics, reflecting the theme that all Euler-Poincaré characteristics should factor through $K_0(\text{Var}_k)$, the universal value group.

Examples. Here are some examples of motivic measures

- Consider the case of $k = \mathbb{C}$. Then the (classical) Euler characteristic $\chi_c(X)$ defined as the the alternating sum of ranks of cohomology with compact support $b_i(X) = \text{rank}\, H_c^i(X(\mathbb{C}))$,

$$\chi_c(X) = \sum_{i=0}^{2m}(-1)^i b_i(X)$$

  where m is the dimension of X, is a motivic measure taking values in $R = \mathbb{Z}$.

- For $X \in \mathrm{Var}_{\mathbb{C}}$ and m the dimension of X, the Poincaré polynomial $P(X, z)$,

$$P(X, z) = \sum_{i=0}^{2m} (-1)^i b_i(X) z^i,$$

defines a motivic measure $\mu_P(X) = P(X, z)$ taking values in the polynomial ring $R = \mathbb{Z}[z]$.

- Consider the case of $k = \mathbb{F}_q$. The counting measure $\mu_\# : K_0(\mathrm{Var}_{\mathbb{F}_q}) \to \mathbb{Z}$ returns the number of points over $\mathbb{F}_q$: $\mu_\#(X) = \#(X)(\mathbb{F}_q)$.

- For any k, there is the universal motivic measure $u : K_0(\mathrm{Var}_k) \to K_0(\mathrm{Var}_k)$, given by the identity map.

Given a motivic measure μ, Kapranov in [15] associates to μ a motivic zeta function ζ_μ defined on quasi-projective varieties X as

$$\zeta_\mu(X, t) = \sum_{n=0}^{\infty} \mu(\mathrm{Sym}^n X) t^n$$

where $\mathrm{Sym}^n X$ is the nth symmetric power of X. This is a power series in $1 + tR[[t]]$, and as the Grothendieck ring of varieties is additively generated by quasi-projective varieties, it defines a map

$$\zeta_\mu : K_0(\mathrm{Var}_k) \longrightarrow \Lambda(R)$$

Recall this subgroup of invertible power series $\Lambda(R) = (1 + tR[[t]], \times)$ is the underlying additive group for the Witt ring $W(R)$. For all μ, the Kapranov motivic zeta function is a group homomorphism.

Note. In the case of char $k > 0$, in order to properly define the motivic zeta function, we need to consider a quotient $\widetilde{K_0}(\mathrm{Var}_k)$ of $K_0(\mathrm{Var}_k)$ where $[X] = [Y]$ if there exists

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a so-called radicial morphism $f : X \to Y$. This is required in order to define a group homomorphism on $K_0(\mathrm{Var}_k)$ using symmetric products $\mathrm{Sym}^n X$ (as these fail to behave nicely under the scissor relation). See Appendix B or Mustata [21] for details. For char $k = 0$, we have $\widetilde{K_0}(\mathrm{Var}_k) \cong K_0(\mathrm{Var}_k)$. Moreover, for char $k > 0$, our focus will be on ζ_μ for the counting measure $\mu = \mu_\#$, which factors through $\widetilde{K_0}(\mathrm{Var}_k)$.

Examples. Here are a few examples of motivic zeta functions:

1. For any motivic measure μ, $\zeta_\mu(\mathrm{Spec}\, k, t) = (1 - t)^{-1}$ and $\zeta_\mu(\emptyset, t) = 1$.

2. For $\mu = u$, ζ_u is sometimes denoted Z_{mot} the universal motivic zeta function taking values in $\Lambda(K_0(\mathrm{Var}_k))$. Here we have

$$Z_{mot}(\mathbb{L}, t) = \frac{1}{(1 - \mathbb{L}t)} \quad \text{and} \quad Z_{mot}(\mathbb{P}^1, t) = \frac{1}{(1 - t)(1 - \mathbb{L}t)}$$

1.4 Exponentiable Motivic Measures

Since the Grothendieck ring is in fact a ring, it is natural to ask how products behave under the motivic zeta function map. It is pointed out in Ramachandran [22] that in the case of varieties over finite fields, the Weil zeta function takes products in the Witt ring $W(\mathbb{Z})$. Thus one may ask that for the Kapranov motivic zeta function, the product structure of $K_0(\mathrm{Var}_k)$ be reflected by the Witt product structure on $\Lambda(R)$.

Definition 1.21. A motivic measure $\mu : K_0(\mathrm{Var}_k) \to R$ is *exponentiable* or *expo-*

nentiates[1] if its associated Kapranov motivic zeta function ζ_μ defines a ring homomorphism

$$\zeta_\mu : K_0(\mathrm{Var}_k) \longrightarrow W(R).$$

Lemma 1.22. *A motivic measure μ is exponentiable if and only if the ghost coordinates $\mathrm{gh}_n(\zeta_\mu)$ are multiplicative for all n.*

Example 1.23. Let $k = \mathbb{C}$. Consider $\mu_P(X) = P(X, z)$ the Poincaré polynomial measure. Then a classical formula of MacDonald [18] provides a closed form for the generating series of symmetric powers $P(\mathrm{Sym}^n X, z)$:

$$\sum_{n=0}^{\infty} P(\mathrm{Sym}^n X, z)t^n = \frac{(1 - z^1 t)^{b_1}(1 - z^3 t)^{b_3} \cdots (1 - z^{2m-1}t)^{b_{2m-1}}}{(1 - z^1 t)^{b_1}(1 - z^2 t)^{b_2} \cdots (1 - z^{2m}t)^{b_{2m}}}$$

$$= \exp\left[\sum_{r=1}^{\infty} P(X, z^r)\frac{t^r}{r}\right]$$

where m is the dimension of X. This shows that $gh_n(\zeta_{\mu_P}(X, t)) = P(X, z^n)$, the Poincaré polynomial in z^n, which is multiplicative in X, and thus $P(X, t)$ is exponentiable. Moreover, the associated motivic zeta function may be written in the Witt ring $W(\mathbb{Z}[z])$ as

$$\zeta_{\mu_P}(X, t) = \sum_{i=1}^{2m}(-1)^i b_i(X)[z^i].$$

Example 1.24. The previous example specializes to the case of $\mu = \chi_c$ where the formula due to MacDonald [18] takes the following form

$$\sum_{n=1}^{\infty} \chi_c(\mathrm{Sym}^n X)t^n = \left(\frac{1}{1 - t}\right)^{\chi_c(X)} = \exp\left[\sum_{r=1}^{\infty} \chi_c(X)\frac{t^r}{r}\right].$$

---

[1]This terminology comes from the observation that the symmetric power $\mathrm{Sym}^n X$ corresponds to $\frac{x^n}{n!}$ since $n!$ is the size of the symmetric group S_n and the motivic zeta function thus corresponds to $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Thus the motivic zeta function ζ_{χ_c} may be written in the Witt ring $W(\mathbb{Z})$ as $\chi_c(X)[1]$ (and its ghost coordinates are constant $\chi_c(X)$), and exponentiability follows from the fact that the measure χ_c is a ring homomorphism.

There are many benefits of understanding the multiplicative structure expressed by the motivic zeta function. The class of exponentiable motivic measures is also well-behaved. Here are a few of the results detailed in [22] and [23]:

Corollary 1.25. *If μ is an exponentiable motivic measure $\mu : K_0(\mathrm{Var}_k) \to R$, then ζ_μ is a motivic measure $\zeta_\mu : K_0(\mathrm{Var}_k) \to W(R)$.*

Proof. The zeta function ζ_μ is always a group homomorphism; μ being exponentiable makes it a ring homomorphism. $\square$

Corollary 1.26. *If μ is an exponentiable motivic measure and μ' factors through μ, then μ' is an exponentiable measure.*

Proof. Suppose that $\mu : K_0(\mathrm{Var}_k) \to R$ and $\mu' : K_0(\mathrm{Var}_k) \to S$ with

$$
\begin{array}{ccc}
K_0(\mathrm{Var}_k) & \xrightarrow{\;\mu\;} & R \\
& \searrow{\scriptstyle \mu'} & \downarrow{\scriptstyle f} \\
& & S
\end{array}
$$

for $f : R \to S$ some ring homomorphism. Then

$$
\zeta_{\mu'}(X, t) = \sum_{n=0}^{\infty} \mu'(\mathrm{Sym}^n X) t^n = \sum_{n=0}^{\infty} f \circ \mu(\mathrm{Sym}^n X) t^n = W(f)(\zeta_\mu)
$$

Thus $\zeta_{\mu'}$ is a composition of ring homomorphisms $W(f)$ and ζ_μ. $\square$

Proposition 1.27. *For μ an exponentiable measure, if $\zeta_\mu(X)$ and $\zeta_\mu(Y)$ are rational functions, then so is $\zeta_\mu(X \times Y)$.*

Proof. The subset of rational functions in $W(R)$ is an ideal, in particular, closed under multiplication. □

In the following chapters, we will be exploring certain examples of motivic measures which are exponentiable. We end this chapter with an example of a non-exponentiable measure.

Example 1.28. For $k = \mathbb{C}$ and X a smooth projective surface of Kodaira dimension $\geqslant 0$, Larsen-Lunts construct a measure μ_{LL} such that the associated motivic zeta function $\zeta_{\mu_{LL}}(X, t)$ is not rational (Theorem 7.6 in [17]). However, the universal motivic zeta function for a smooth projective curve C is always rational (see [15]), and so $\zeta_\mu(C, t)$ is rational for any motivic measure μ. Applying this to the smooth projective surface $C_1 \times C_2$ implies that $\zeta_{\mu_{LL}}$ cannot take values in the Witt ring, since in the Witt ring, the product of rational functions is rational.

Note. The Grothendieck ring of varieties has some interesting properties which can be probed using motivic measures and motivic zeta functions. For instance, $K_0(\text{Var}_k)$ is a pre-λ-ring via the universal motivic zeta function ζ_u

$$\zeta_u(X) = \sum_{n=0}^{\infty} [\text{Sym}^n X] t^n \in \Lambda(K_0(\text{Var}_k)).$$

This is simply because $\zeta_u(X)$ is a group homomorphism $\zeta_u : K_0(\text{Var}_k) \to \Lambda(K_0(\text{Var}_k))$. However, this motivic zeta function is not a λ-ring structure map since it is not necessarily a ring homomorphism into $W(K_0(\text{Var}_k))$. For instance, in the case $k = \mathbb{C}$ the construction of the Larsen-Lunts measure suffices to show ζ_u is not a suitable (opposite) λ-structure map since $\zeta_\mu = \Lambda_\mu \circ \zeta_u$ and Λ_μ preserves rationality. See [17] Section 8.

Chapter 2   The Case of Varieties over Finite Fields

In this chapter, we provide a formula for the generating series of the zeta function $Z(X,t)$ of symmetric powers of varieties over finite fields. The zeta function $Z(X,t)$ is the Kapranov motivic zeta function $\zeta_{\mu_\#}$ associated to the counting measure $\mu_\#$ and counting measure $\mu_\#$ is exponentiable; that is, $Z(X,t)$ takes values in the big Witt ring $W(\mathbb{Z})$. We use our formula to prove that the zeta function, when thought of as a motivic measure itself $\mu_Z(X) = Z(X,t)$, is also exponentiable, so that its motivic zeta function ζ_{μ_Z} takes values in $W(W(\mathbb{Z}))$. A number of explicit examples of zeta functions of symmetric powers are computed.

Given a variety X over a finite field $\mathbb{F}_q$, the Weil zeta function (also known as the Hasse-Weil zeta function or simply the zeta function) is defined as

$$\zeta_X(s) = \prod_{x \in |X|} (1 - N(x)^{-s})^{-1}$$

where the product ranges over all closed points $x \in |X|$ and $N(x)$ is the degree of the residue field $\kappa(x)$. Setting $t = q^{-s}$, we may rewrite the zeta function as $\zeta_X(s) = Z(X, q^{-s})$,

$$Z(X,t) = \prod_{x \in |X|} (1 - t^{\deg x})^{-1} = \exp\left[\sum_{r=1}^{\infty} N_r(X)\frac{t^r}{r}\right] \in 1 + t\mathbb{Z}[[t]] \qquad (2.1)$$

where $N_r(X) = \#X(\mathbb{F}_{q^r})$ the number of points over $\mathbb{F}_{q^r}$.

Due to the work of Grothendieck and others on the Weil conjectures, we can understand the zeta function in terms the action of the Frobenius on (compactly supported) étale cohomology. Namely, there is an algebraic Lefschetz fixed point theorem, for $\ell \neq p$, whereby the number of points is given by an alternating trace

$$N_r(X) = \sum_i (-1)^i \operatorname{Tr}(\Phi^r | H^i_{et,c}(\overline{X}; \mathbb{Q}_\ell))$$

for Φ the Frobenius on X. This can be used to show that

$$Z(X,t) = \frac{\prod_i P_{2i+1}(t)}{\prod_i P_{2i}(t)}, \quad P_i(t) = \prod_j (1 - \alpha_{ij} t)^{-1} \qquad (2.2)$$

where α_{ij} are the inverse eigenvalues of the Frobenius action Φ on $H^i_{et,c}(\overline{X}; \mathbb{Q}_\ell)$. These are so-called Weil q-numbers $|\alpha_{i,j}| = q^{r/2}$ for some $r \leqslant i$ (Deligne) and $Z(X,t)$ is a rational function (Dwork).

The Weil zeta function $Z(X,t)$ is the Kapranov motivic zeta function associated to the counting measure $\mu_\#$. This was Kapranov's initial motivation as well as justification for calling the generating series for symmetric powers a zeta function.

Proposition 2.1. For X a variety over $\mathbb{F}_q$,

$$\zeta_{\mu_\#}(X,t) = \sum_{n=0}^\infty \#(\operatorname{Sym}^n X)(\mathbb{F}_q) t^n = Z(X,t)$$

Proof. We may rewrite the zeta function as

$$Z(X,t) = \prod_{x \in |X|} \frac{1}{(1 - t^{\deg x})} = \sum_{Y \subset X} t^{\deg Y}$$

where Y ranges over all effective zero cycles $Y \subset X$. These are classified by $\#(\operatorname{Sym}^n X)(\mathbb{F}_q)$. See Mustata [21, Prop 7.31]. $\qquad \square$

From the form of $Z(X, t)$ in (2.1), it is clear that the zeta function is additive

over closed subvarieties $Y \subset X$ in $\Lambda(\mathbb{Z})$. The zeta function $Z(X, t)$ takes values

in $W(\mathbb{Z})$; that is, the zeta function of a product is the Witt product of the zeta

functions. Again, from (2.1) we have that $\mathrm{gh}_r(Z(X, t)) = N_r(X)$ (see Lemma 1.2)

and $N_r(X)$ is multiplicative which suffices to prove that $\mu_\#$ is exponentiable. In

fact, the following is shown in [22, Thm 2.1]:

Proposition 2.2. For X and $Y \in \mathrm{Var}_{\mathbb{F}_q}$, the zeta function of the product $X \times_{\mathbb{F}_q} Y$

is the Witt product of the zeta functions

$$Z(X \times_{\mathbb{F}_q} Y, t) = Z(X, t) *_W Z(Y, t).$$

Moreover, the Frobenius operator Fr_n on $W(\mathbb{Z})$ corresponds to base change $X_{\mathbb{F}_{q^n}}$,

so that $\mathrm{Fr}_n(Z(X/\mathbb{F}_q, t)) = Z(X/\mathbb{F}_{q^r}, t)$. The base change b_m to $\mathbb{F}_{q^m}$ induces a map

on the Grothendieck ring, and the following

$$
\begin{array}{ccc}
K_0(\mathrm{Var}_{\mathbb{F}_q}) & \xrightarrow{\;b_m\;} & K_0(\mathrm{Var}_{\mathbb{F}_{q^m}}) \\
{\scriptstyle Z}\big\downarrow & & \big\downarrow{\scriptstyle Z} \\
W(\mathbb{Z}) & \xrightarrow{\;\mathrm{Fr}_m\;} & W(\mathbb{Z})
\end{array}
$$

is a commutative diagram of ring homomorphisms.

2.1   Exponentiation of the Weil zeta function

The Weil zeta function of a variety X over $\mathbb{F}_q$ can be written in the Witt ring

$W(\mathbb{Z})$ as

$$Z(X, t) = \sum_{i,j} (-1)^{i+1}[\alpha_{ij}]$$

where the sum is taking place in the Witt ring and $[\alpha]$ is the Teichmüller element $(1-\alpha t)^{-1}$. This directly follows from the presentation in (2.2) of the zeta function as an alternating product of polynomials $P_i = (1 - \alpha_{ij}t)^{-1}$ and Proposition 2.2 above. Although each Teichmüller element $[\alpha_{ij}]$ is in $W(\overline{\mathbb{Q}_\ell})$, the sum nevertheless lies in the subring $W(\mathbb{Z})$.

We now state our main result.

Theorem 2.3 (Main Theorem). *The Weil zeta function of a variety X over $\mathbb{F}_q$, when considered as a motivic measure $\mu_Z(X) = Z(X,t)$, has an associated motivic zeta function ζ_{μ_Z} taking the form*

$$\zeta_{\mu_Z}(X, u) = \sum_n Z(\mathrm{Sym}^n X, t)u^n = \sum_{i,j}(-1)^{i+1}[\![\,[\alpha_{ij}]\,]\!]$$

where the second sum is taking place in the Witt ring $W(W(\overline{\mathbb{Q}_\ell}))$.

This can be viewed as a closed product formula for the generating function for zeta series of symmetric powers. We call the induced measure μ_Z the Weil zeta function measure.

2.2   Examples

Before proving the main theorem, we exhibit some explicit examples. We apply the generating series formula to compute $Z(\mathrm{Sym}^n X, t)$ for various cases of varieties X over finite fields.

2.2.1 Affine and projective space

Consider n-dimensional affine space $\mathbb{A}^n$ and n-dimensional projective space $\mathbb{P}^n$ over $\mathbb{F}_q$. Then $N_r(\mathbb{A}^n) = q^{nr}$ and we have

$$Z(\mathbb{A}^n, t) = \frac{1}{(1 - q^n t)}$$

It is also easy to show that $N_r(\mathbb{P}^n) = q^{nr} + q^{(n-1)r} + \cdots + q^r + 1$ so that

$$Z(\mathbb{P}^n, t) = \frac{1}{(1 - t)(1 - qt) \cdots (1 - q^m t)}$$

In the Witt ring $W(\mathbb{Z})$, these zeta functions are $Z(\mathbb{A}^n, t) = [q^n]$ and $Z(\mathbb{P}^n, t) = [q^m] +_W [q^{m-1}] +_W \cdots +_W [q] +_W [1]$.

Recall that $\mathrm{Sym}^n \mathbb{A}^1 = \mathbb{A}^n$ and $\mathrm{Sym}^n \mathbb{P}^1 = \mathbb{P}^n$, and that

$$Z(\mathbb{A}^1, t) = \frac{1}{1 - qt} = [q] \quad \text{and} \quad Z(\mathbb{P}^1, t) = \frac{1}{(1 - t)(1 - qt)} = [1] +_W [q]$$

The formula provided in Theorem 2.3 predicts that $Z(\mathrm{Sym}^n \mathbb{A}^1, t)$ is the coefficient of u^n in $[[\,[q]\,]] \in W(W(\mathbb{Z}))$. Similarly, that $Z(\mathrm{Sym}^n \mathbb{P}^1, t)$ is the coefficient of u^n in $[[\,[1]\,]] +_W [[\,[q]\,]] \in W(W(\mathbb{Z}))$. We have

$$[[\,[q]\,]] = \frac{1}{[1] - [q]u} = [1] + [q]u + [q^2]u^2 + \cdots$$

and

$$[[\,[1]\,]] +_W [[\,[q]\,]] = \left(\frac{1}{[1] - [1]u}\right)\left(\frac{1}{[1] - [q]u}\right) = [1] + ([1] + [q])u + \cdots$$

which agrees with the zeta functions $Z(\mathbb{A}^n, t)$ and $Z(\mathbb{P}^n, t)$ described above:

2.2.2 Elliptic curves

Let E be an elliptic curve over $\mathbb{F}_q$. In this case, we have

$$Z(E, t) = \frac{(1 - \alpha t)(1 - \beta t)}{(1 - t)(1 - qt)}$$

where $\alpha + \beta = a \in \mathbb{Z}$ and $\alpha\beta = q$. Written in the Witt ring $W(\mathbb{Z})$, this appears as

$$Z(E, t) = [1] - [\alpha] - [\beta] + [q],$$

where $[\alpha] = (1 - \alpha t)^{-1}$ the Teichmüller element. The formula provided by Theorem 2.3 is

$$Z(\mathrm{Sym}^n E, t) = \text{coefficient of } u^n \text{ in } \left[\frac{(1 - [\alpha]u)(1 - [\beta]u))}{(1 - [1]u)(1 - [q]u))} \right].$$

The results in the section in fact work for any curve C over $\mathbb{F}_q$. In this general case, the eigenvalues α and β for the action on $H^1(C, \overline{\mathbb{Q}_\ell})$ come in pairs α_i and β_i for $i = 1, 2, \ldots, g$ where g is the genus. In fact, the case of symmetric powers of smooth projective curves was worked out by MacDonald in [19] in 1962. Here we work with the elliptic curve case for simplicity.

2.2.2.1 Small cases $n = 2, 3$

In the case of $\mathrm{Sym}^2 E$, the formula states that

$$Z(\mathrm{Sym}^2 E, t) = \frac{(1 - \alpha t)(1 - \beta t)(1 - q\alpha t)(1 - q\beta t)}{(1 - t)(1 - qt)(1 - qt)(1 - q^2 t)}$$

The ghost coordinates of $Z(X, t)$ are given by the number of points over $\mathbb{F}_{q^r}$, which we denote by $N_r(X)$. We verify the formula by counting the number of points over $\mathbb{F}_{q^r}$. For a Teichmüller element $[\alpha]$ the rth ghost coordinate is α^r. The formula

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exhibits $Z(\mathrm{Sym}^2 E, t)$ as a combination of Teichmüller elements, thus whose ghost coordinates are a combination of powers. We will verify this combination with a counting argument for $\# \mathrm{Sym}^2 E(\mathbb{F}_{q^r})$.

By the formula for $\mathrm{Sym}^2 E$ above, the number of points over $\mathbb{F}_{q^r}$ should be

$$N_r(\mathrm{Sym}^2 E) = 1^r + q^r + q^r + (q^2)^r - \alpha^r - \beta^r - (q\alpha)^r - (q\beta)^r \qquad (2.3)$$

This is what we will verify. Recall that $N_r(E) = 1^r - \alpha^r - \beta^r + q^r$. For notational convenience, we will use $N_r = N_r(E)$.

There is another combinatorial way of counting points on $\mathrm{Sym}^2 E$.

Proposition 2.4.

$$N_r(\mathrm{Sym}^2 E) = \frac{1}{2} N_r(N_r + 1) + \frac{1}{2}(N_{2r} - N_r)$$

Proof. We will need the following well-known combinatorial lemma:

Lemma 2.5. *The number of unordered k-tuples of a set of n elements is given by the binomial coefficient* $\begin{pmatrix} n + k - 1 \\ k \end{pmatrix}$.

Thus the first term represents the number of unordered pairs of points in $E(\mathbb{F}_{q^r})$. The second term counts the extra points on $E(\mathbb{F}_{q^{2r}})$ of the form $(P, \overline{P})$ where $\overline{P}$ is the conjugate in the quadratic extension $\mathbb{F}_{q^{2r}}$. These points are fixed by Frobenius since $(P, \overline{P}) = (\overline{P}, P)$ in $\mathrm{Sym}^2 E$ and thus they must be counted in $\mathrm{Sym}^2 E(\mathbb{F}_{q^r})$. $\qquad \square$

The first term in Proposition 2.4 works out as follows

$$\binom{N_r + 1}{2} = \frac{1}{2} N_r (N_r + 1)$$

$$= \frac{1}{2}(1 - \alpha^r - \beta^r + q^r)(2 - \alpha^r - \beta^r + q^r)$$

The second term works out as follows:

$$\frac{1}{2}(N_{2r} - N_r) = \frac{1}{2}\left[(1 - \alpha^{2r} - \beta^{2r} + q^{2r}) - (1 - \alpha^r - \beta^r + q^r)\right]$$

$$= \frac{1}{2}(\alpha^r - \alpha^{2r} + \beta^r - \beta^{2r} + q^{2r} - q^r)$$

Adding the terms yields

$$N_r(\mathrm{Sym}^2 E) = \frac{1}{2} N_r(N_r + 1) + \frac{1}{2}(N_{2r} - N_r)$$

$$= 1 - \alpha^r - \beta^r + q^r + \alpha^r \beta^r - \alpha^r q^r + q^{2r} - \beta^r q^r$$

which is the number predicted by the formula (2.3).

We can repeat this procedure for $\mathrm{Sym}^3 E$. In this case, the formula states that

$$Z(\mathrm{Sym}^3 E, t) = \frac{(1 - \alpha t)(1 - \beta t)(1 - \alpha q t)(1 - \beta q t)(1 - \alpha q^2 t)(1 - \beta q^2 t}{(1 - t)(1 - qt)(1 - q^2 t)(1 - \alpha \beta t)(1 - \alpha \beta q t)(1 - q^3 t)}$$

which implies that

$$N_r(\mathrm{Sym}^3 E) = 1^r + q^r - \alpha^r - \beta^r - \alpha^r q^r - \beta^r q^r + q^{2r} + \alpha^r \beta^r + \alpha^r \beta^r q^r - \alpha^r q^{2r} - \beta^r q^{2r} + q^{3r}$$

We will verify this against the combinatorial count:

Proposition 2.6.

$$N_r(\mathrm{Sym}^3 E) = \binom{N_r + 2}{3} + \frac{1}{2}(N_{2r} - N_r)N_r + \frac{1}{3}(N_{3r} - N_r)$$

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Proof. The first two terms are similar to the terms in the count for $\mathrm{Sym}^2 E$, where instead we count the number of unordered triples. The third term counts the points of the form $(P, \sigma P, \sigma^2 P)$ in a cubic extension. $\qquad\square$

The first term works out as follows

$$
\binom{N_r + 2}{3} = \frac{1}{6}(N_r + 2)(N_r + 1)(N_r)
$$

$$
= \frac{1}{6}(3 - \alpha^r - \beta^r + q^r)(2 - \alpha^r - \beta^r + q^r)(1 - \alpha^r - \beta^r + q^r)
$$

$$
= \frac{1}{6}(6 - 11\alpha^r - 11\beta^r + 11q^r + 6\alpha^{2r} + 12\alpha^r\beta^r - 12\alpha^r q^r - 12\beta^r q^r
$$

$$
+ 6\beta^{2r} + 6q^{2r} - \alpha^{3r} - 3\alpha^{2r}\beta^r - 3\beta^{2r}\alpha^r + 6q^r\alpha^r\beta^r + 3\alpha^{2r}q^r
$$

$$
+ 3\beta^{2r}q^r - \beta^{3r} + q^{3r})
$$

The second term works out as follows:

$$
\frac{1}{2}(N_{2r} - N_r)N_r = \frac{1}{2}(\alpha^r - \alpha^{2r} + \beta^r - \beta^{2r} + q^{2r} - q^r)(1 - \alpha^r - \beta^r + q^r)
$$

$$
= \frac{1}{2}(\alpha^r - 2\alpha^{2r} + \beta^r - 2\beta^{2r} - q^r + \alpha^{3r} - 2\alpha^r\beta^r + \alpha^r\beta^{2r} + \alpha^{2r}\beta^r
$$

$$
+ \beta^{3r} - \alpha^r q^{2r} - \beta^r q^{2r} + 2\alpha^r q^r + 2\beta^r q^r - q^r\alpha^{2r} - q^r\beta^{2r} + q^{3r})
$$

The third term is

$$
\frac{1}{3}(N_{3r} - N_r) = \frac{1}{3}(\alpha^r - \alpha^{3r} + \beta^r - \beta^{3r} + q^{3r} - q^r)
$$

Adding these terms yields the count predicted by the formula.

2.2.2.2 General case

The combinatorial propositions counting the number of points on $\mathrm{Sym}^2 E$ and $\mathrm{Sym}^3 E$ generalizes to arbitrary varieties X over $\mathbb{F}_q$. They reduce to the relation

provided by the Newton identities:

$$na_n = b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1$$

These relations allow us to recover a_n from b_i; for example

$$a_1 = b_1, \quad a_2 = \frac{1}{2} b_2 + b_1^2, \quad a_3 = \frac{1}{6} b_1^3 + \frac{1}{2} b_1 b_2 + \frac{1}{3} b_3$$

Setting $a_n = N_r(\text{Sym}^n X)$ and $b_n = N_{nr}(X)$, here are some small cases $n = 2$ and

$n = 3$:

$$
\begin{aligned}
N_r(\text{Sym}^2 X) &= \frac{1}{2} N_{2r}(X) + \frac{1}{2} N_r(X)^2 \\
&= \frac{1}{2}(N_r)(N_r + 1) + \frac{1}{2}(N_{2r} - N_r)
\end{aligned}
$$

and

$$
\begin{aligned}
N_r(\text{Sym}^3 X) &= \frac{1}{6} N_r(X)^3 + \frac{1}{2} N_r(X) N_{2r}(X) + \frac{1}{3} N_{3r}(X) \\
&= \frac{1}{6}(N_r + 2)(N_r + 1) N_r + \frac{1}{2}(N_{2r} - N_r) N_r + \frac{1}{3}(N_{3r} - N_r).
\end{aligned}
$$

Notice these are exactly the counts provided by the combinatorial propositions above. Newton's identities provide a relationship between $N_r(\text{Sym}^n X)$ to $N_{ir}(X)$ for $i = 1, 2, \ldots . n$; this relationship will be formalized using λ-rings in the sequel.

While the previous examples served to verify the formula provided in Theorem 2.3, we now directly apply the formula to compute zeta functions of symmetric products. For notational convenience, we will suppress $+_W$, and sums in $W(R)$ will be understood as such.

2.2.3 Symmetric powers of affine and projective space

We use the formula in Theorem 2.3 to compute the zeta functions $Z(\operatorname{Sym}^n \mathbb{A}^m, t)$ and $Z(\operatorname{Sym}^n \mathbb{P}^m, t)$. Note that these varieties are no longer smooth once $m > 1$. Recall that

$$Z(\mathbb{A}^m, t) = \frac{1}{(1 - q^m t)} = [q^m] \quad \text{and} \quad Z(\mathbb{P}^m, t) = \frac{1}{(1 - t)(1 - [q^m])} = [1] + [q^m]$$

Then by the formula we have for affine space

$$Z(\operatorname{Sym}^n \mathbb{A}^m, t) = \text{ coefficient of } u^n \text{ in } [[\, [q^m]\,]] = [q^{nm}],$$

which agrees with the fact that $[\operatorname{Sym}^n \mathbb{A}^m] = [\mathbb{A}^{nm}]$ (see Appendix B). For projective space,

$$Z(\operatorname{Sym}^n \mathbb{P}^m, t) = \text{ coefficient of } u^n \text{ in } [[\, [1]\,]] + [[\, [q]\,]] + \cdots + [[\, [q^m]\,]].$$

Note that this coefficient is not $[1] + [q] + \cdots + [q^{nm}]$ as $\operatorname{Sym}^n \mathbb{P}^m$ is not $\mathbb{P}^{nm}$. For instance, consider $n = 2$ and $m = 2$,

$$Z(\operatorname{Sym}^2 \mathbb{P}^2, t) = [1] + [q] + 2[q^2] + [q^3] + [q^4]$$

whereas

$$Z(\mathbb{P}^4, t) = [1] + [q] + [q^2] + [q^3] + [q^4].$$

2.2.4 Product of elliptic curves

Consider $X = E_1 \times E_2$ a product of elliptic curves over $\mathbb{F}_q$. We use the formula in Theorem 2.3 to compute $Z(\operatorname{Sym}^n X, t)$ in terms of $Z(E_1, t)$ and $Z(E_2, t)$. Let

$$Z(E_1, t) = \frac{(1 - \alpha_1 t)(1 - \beta_1 t)}{(1 - t)(1 - qt)} \quad \text{and} \quad Z(E_2, t) = \frac{(1 - \alpha_2 t)(1 - \beta_2 t)}{(1 - t)(1 - qt)}$$

and recall that in the Witt ring, these can be written as

$$Z(E_1, t) = [1] - [\alpha_1] - [\beta_1] + [q] \quad \text{and} \quad Z(E_2, t) = [1] - [\alpha_2] - [\beta_2] + [q].$$

Then since the zeta function takes products in the Witt ring, we may compute $Z(E_1 \times E_2, t)$ as

$$
\begin{aligned}
Z(E_1 \times E_2, t) &= ([1] - [\alpha_1] - [\beta_1] + [q]) * ([1] - [\alpha_2] - [\beta_2] + [q]) \\
&= [1] - [\alpha_1] - [\alpha_2] - [\beta_1] - [\beta_2] + [\alpha_1\beta_2] + [\alpha_2\beta_1] + [\alpha_1\alpha_2] \\
&\quad + [\beta_1\beta_2] - [\alpha_1 q] - [\alpha_2 q] - [\beta_1 q] - [\beta_2 q] + [q^2].
\end{aligned}
$$

Thus the motivic zeta function generating series is

$$
\begin{aligned}
\zeta_Z(E_1 \times E_2, u) &= \sum_{n=0}^{\infty} Z(\mathrm{Sym}^n(E_1 \times E_2), t) u^n \\
&= [[\, [1]\,]] - [[\, [\alpha_1]\,]] - [[\, [\alpha_2]\,]] - [[\, [\beta_1]\,]] - [[\, [\beta_2]\,]] \\
&\quad + [[\, [\alpha_1\beta_2]\,]] + [[\, [\alpha_2\beta_1]\,]] + [[\, [\alpha_1\alpha_2]\,]] + [[\, [\beta_1\beta_2]\,]] \\
&\quad - [[\, [\alpha_1 q]\,]] - [[\, [\alpha_2 q]\,]] - [[\, [\beta_1 q]\,]] - [[\, [\beta_2 q]\,]] + [[\, [q^2]\,]].
\end{aligned}
$$

Note that these equalities are being written in $W(W(A))$. These methods similarly also work for n-fold products of smooth projective curves; however, the formulas quickly become unwieldy.

2.2.5 Grassmannians

Let $\mathrm{Gr}(n, d)$ be the Grassmannian of d-dimensional subspaces of a n-dimensional vector space over $\mathbb{F}_q$. In order count the number of points $\# \mathrm{Gr}(n, d)(\mathbb{F}_{q^r})$, we will need the Gaussian binomial coefficient. For a prime power q, this is the q-analog of

the usual binomial coefficient, defined as

$$\binom{n}{d}_q = \frac{(q^n - 1) \cdots (q^{n-d+1} - 1)}{(q^d - 1) \cdots (q - 1)},$$

and it is a polynomial in q of degree $d(n-d)$. Let $b_i(q)$ be the coefficient of q^i in

$\binom{n}{d}_q$ so that

$$\binom{n}{d}_q = \sum_{i=0}^{d(n-d)} b_i(q)\, q^i$$

It turns out that these coefficients $b_i(q) = b_i$ are independent of the power q. For a

detailed description of these computations, see Mustata [21] Section 2.6. For a nice

collection of computations of zeta functions of Grassmannians, see Kolhatkar [?].

The number of elements over $\mathbb{F}_{q^r}$ is in fact the Gaussian binomial coefficient:

$$N_r(\mathrm{Gr}(n,d)) = \binom{n}{d}_{q^r} = \sum_{i=0}^{\infty} d(n-d) b_i(q^r)^i.$$

Thus we have

$$Z(\mathrm{Gr}(n,d),t) = \exp\left[\sum_{r=1}^{\infty} N_r(\mathrm{Gr}(n,d)) \frac{t^r}{r} \right]$$

However, since the coefficients b_i remain the same for all powers q^r, we can realize

the zeta function as

$$Z(\mathrm{Gr}(n,d)) = \frac{1}{(1-t)^{b_0}(1-qt)^{b_1} \cdots (1-q^{d(n-d)}t)^{b_{d(n-d)}}} = \sum_{i=0}^{d(n-d)} b_i[q^i].$$

where the sum in the second equality is taking place in the Witt ring.

For example, $\mathrm{Gr}(n,1) = \mathbb{P}^{n-1}$ and the Gaussian binomial coefficient is

$$\begin{pmatrix} n \\ 1 \end{pmatrix}_q = \frac{(q^n - 1)}{(q - 1)} = 1 + q + q^2 + \cdots + q^{n-1},$$

so we recover

$$Z(\mathrm{Gr}(n,1), t) = \frac{1}{(1-t)(1-qt)\cdots(1-q^{(n-1)}t)} = \sum_{i=0}^{(n-1)} [q^i].$$

Another small example is $\mathrm{Gr}(4,2)$ where the Gaussian binomial coefficient is

$$\begin{pmatrix} 4 \\ 2 \end{pmatrix}_q = \frac{(q^4 - 1)(q^3 - 1)}{(q^2 - 1)(q - 1)} = 1 + q + 2q^2 + q^3 + q^4,$$

so we have

$$Z(\mathrm{Gr}(4,2), t) = \frac{1}{(1-t)(1-qt)(1-q^2t)^2(1-q^3t)(1-q^4t)}$$

$$= [1] + [q] + 2[q^2] + [q^3] + [q^4].$$

Using the formula from Theorem 2.3, we can compute the zeta function for the symmetric powers $\mathrm{Sym}^m \mathrm{Gr}(n,d)$. Here we only work out the zeta function $Z(\mathrm{Sym}^2 \mathrm{Gr}(4,2), t)$ as the coefficient of u^2 in

$$\frac{1}{([1] - [1]u)([1] - [q]u)([1] - [q^2]u)^2([1] - [q^3]u)([1] - [q^4]u)}.$$

2.3   Proof of the Main Theorem

We now proceed to the proof of Theorem 2.3. First, we will need to recall the result on Newton's identities from Lemma 1.1: in $\Lambda(A)$, for $P(t) = \sum_n a_n t^n$ and $b_n = \mathrm{gh}_n(P(t))$, we have

$$na_n = b_n + a_1 b_{n-1} + \cdots a_{n-1} b_1$$

Recursively, this gives a way to recover a_n from the ghost coordinates $b_1, b_2, \ldots, b_n$. That is, a_n can be written purely in terms of b_i for $i = 1, 2, \ldots n$ by replacing each a_i in the above relation. Let us call this relation ϕ so that

$$a_n = \phi(b_1, b_2, \ldots, b_n). \tag{2.4}$$

Lemma 2.7. *For a variety X over a finite field $\mathbb{F}_q$, let $N_r(X)$ be the number of points over $\mathbb{F}_{q^r}$, $\#X(\mathbb{F}_{q^r})$. Then*

$$N_r(\operatorname{Sym}^n X) = \phi(N_r(X), N_{2r}(X), \ldots, N_{nr}(X)) \tag{$*_r$}$$

Proof. Recall that for the zeta function,

$$Z(X, t) = \sum_n N_1(\operatorname{Sym}^n X)t^n = \exp\left[\sum_r N_r(X)\frac{t^r}{r}\right] \tag{2.5}$$

Recall that this implies $\operatorname{gh}_r(Z(X, t)) = N_r(X)$. Using the relation (2.4), this shows that

$$N_1(\operatorname{Sym}^n X) = \phi(N_1(X), N_2(X), \ldots, N_n(X)),$$

which is $(*_1)$. To obtain $(*_r)$ from $(*_1)$, we now use the Frobenius operator Fr_r for $W(\mathbb{Z})$ on the equation (2.5). The proof follows now follows from the observation that on ghost coordinates, $\operatorname{gh}_n(\operatorname{Fr}_r(P)) = \operatorname{gh}_{nr}(P)$ and for the Weil zeta function, $\operatorname{Fr}_r Z(X/\mathbb{F}_q, t) = Z(X/\mathbb{F}_{q^r})$. □

Proof of Theorem 2.3. Recall the statement in Theorem 2.3:

$$\zeta_Z(X, t) = \sum_n Z(\operatorname{Sym}^n X, t)u^n = \sum_{i,j}(-1)^{i+1}[[\, [\alpha_{ij}]\,]] \tag{2.6}$$

Using the ghost map gh^u on $W(W(\mathbb{Z}))$, let β_n be the ghost coordinate of the right hand side of Equation 2.6:

$$\beta_n = \mathrm{gh}_n^u\left(\sum_{i,j}(-1)^{i+1}[[\,[\alpha_{ij}]\,]]\right) = \sum_{i,j}(-1)^{i+1}[\alpha_{ij}]^n \in W(\mathbb{Z}).$$

We wish to show

$$Z(\mathrm{Sym}^n X, t) = \phi(\beta_1, \beta_2, \ldots, \beta_n).$$

This relation is taking place in $W(\mathbb{Z})$. We now use the ghost map gh^t on $W(\mathbb{Z})$. It suffices to show for all r,

$$\mathrm{gh}_r^t Z(\mathrm{Sym}^n X, t) = \phi(\mathrm{gh}_r^t \beta_1, \mathrm{gh}_r^t \beta_2, \ldots, \mathrm{gh}_r^t \beta_n).$$

We have

$$\mathrm{gh}_r^t \beta_n = \mathrm{gh}_r^t\left(\sum_{i,j}(-1)^{i+1}[\alpha_{ij}]^n\right) = \sum_{i,j}(-1)^{i+1}\alpha_{ij}^{nr} = N_{nr}(X) \in \mathbb{Z}$$

Moreover, $\mathrm{gh}_r^t Z(\mathrm{Sym}^n X, t) = N_r(\mathrm{Sym}^n X)$. Now recall from Lemma 2.7

$$N_r(\mathrm{Sym}^n X) = \phi(N_r(X), N_{2r}(X), \ldots, N_{nr}(X)).$$

As this holds for all r, the proof follows. $\qquad\square$

One of the consequences of Proposition 2.1, as pointed out in [22], is that the Weil zeta function $Z(X, t)$ is a motivic measure $\mu_Z : K_0(\mathrm{Var}_{\mathbb{F}_q}) \to W(\mathbb{Z})$. Using the formula for its associated motivic zeta function, it is now easily shown that μ_Z is in fact an exponentiable measure.

Corollary 2.8. *The Weil zeta function $Z(X, t)$ is an exponentiable motivic measure taking values in $A = W(\mathbb{Z})$. Its associated motivic zeta function ζ_Z takes values in $W(A) = W(W(\mathbb{Z}))$.*

Proof. Multiplication on $W(W(\overline{\mathbb{Q}_\ell}))$ is defined so that $[[\,[\alpha]\,]] *_W [[\,[\beta]\,]] = [[\,[\alpha\beta]\,]]$.

$\square$

Chapter 3 Motivic Measures with Values in Lambda-Rings

In this chapter, we study λ-ring-valued motivic measures and their associated motivic zeta functions. We show that for certain motivic measures, the motivic zeta function is always an exponentiable measure. In particular, the case of the Weil zeta function can be reformulated in this setting. We also recast in terms of λ-rings what it means for a motivic measure μ to have a MacDonald formula.

Let $\mu : K_0(\text{Var}_k) \to R$ be a motivic measure. Recall that we say μ exponentiates if the associated motivic zeta function

$$\zeta_\mu : K_0(\text{Var}_k) \longrightarrow (1 + tR[[t]], \times)$$

defines a ring homomorphism taking values in the big Witt ring $W(R)$. As all motivic zeta functions are group homomorphisms into $\Lambda(R)$, this is the statement that products of varieties are realized as products in the Witt ring:

$$\zeta_\mu(X \times_k Y) = \zeta_\mu(X) *_W \zeta_\mu(Y).$$

Thus motivic zeta functions of exponentiable measures $\mu : K_0(\text{Var}_k) \to R$ are themselves motivic measures $\zeta_\mu : K_0(\text{Var}_k) \to W(R)$. Our focus here is on motivic measures taking values in R a λ-ring; our main observation is that for any ring A, the big Witt ring $W(A)$ is a λ-ring.

For motivic measures μ taking values in a λ-ring, Ramachandran and Tabuada in [23] formulate a condition for exponentiation. For char $k = 0$, this condition is satisfied by the Gillet-Soulé motivic measure μ_{GS} described below. Thus exponentiability is demonstrated for a broad class of motivic measures: all motivic measures factoring through μ_{GS}. In particular, the Euler characteristic measure χ_c and the Poincaré polynomial $\mu_P(X) = P(X, z)$ factor through μ_{GS} and are thus exponentiable. We begin by studying λ-ring valued motivic measures.

3.1 Lambda-ring valued motivic measures

Let R be a λ-ring. It is shown by Ramachandran and Tabuada [23], that a motivic measure $\mu : K_0(\text{Var}_k) \to R$ is exponentiable if the associated motivic zeta function ζ_μ factors through its defining measure as

$$
\begin{array}{ccc}
K_0(\text{Var}_k) & \xrightarrow{\ \mu\ } & R \\
& {\scriptstyle \zeta_\mu} \searrow & \downarrow {\scriptstyle \sigma_t} \\
& & W(R)
\end{array}
$$

where σ_t is the opposite λ-ring structure map (see Definition 1.12). Namely, for X quasi-projective,

$$
\zeta_\mu(X, t) = \sum_{n=0}^{\infty} \mu(\text{Sym}^n X) t^n = \sum_{n=0}^{\infty} \sigma^n \mu(X) t^n = \sigma_t(\mu(X)).
$$

We gather these properties into:

Condition L. The measure $\mu : K_0(\text{Var}_k) \to R$ satisfies one of the equivalent conditions:

- μ is a map of pre-λ-rings.

- $\mu(\mathrm{Sym}^n X) = \sigma^n \mu(X)$ for all n and X quasi-projective.

- $\zeta_\mu = \sigma_t \circ \mu$.

This condition is that the associated motivic zeta function factors through its defining measure.

Proposition 3.1 (Ramachandran, Tabuada [23]). *Let R be a λ-ring and σ_t its opposite λ-ring structure map. If a motivic measure $\mu : K_0(\mathrm{Var}_k) \to R$ satisfies Condition L, then μ is exponentiable.*

Proof. Note that ζ_μ factors as $\zeta_\mu = \sigma_t \circ \mu$ and σ_t is a ring homomorphism when R is a λ-ring. Thus, the composition is a ring homomorphism and μ is exponentiable. $\square$

This allows us to prove $\zeta_\mu \in W(R)$ for a large class of motivic measures— namely, those measures taking values in λ-rings satisfying Condition L. Moreover, the condition is closed under composition of λ-ring maps $f : R \to S$.

Note. Condition L is not equivalent to a motivic measure being exponentiable. That is, there exist measures μ which are exponentiable but fail to define pre-λ-ring maps $\mu : K_0(\mathrm{Var}_k) \to R$. For example,

Proposition 3.2. Consider $k = \mathbb{F}_q$ and the counting measure $\mu_\#(X) = \#X(\mathbb{F}_q)$. Counting measure is exponentiable but fails to satisfy Condition L.

Proof. Recall from Section 1.1.2 the unique λ-ring structure on $\mathbb{Z}$: for $a \in \mathbb{Z}$,

$$\lambda_t(a) = (1+t)^a, \quad \lambda^n(a) = \binom{a}{n}, \quad \sigma^n(a) = \binom{a+n-1}{n}.$$

However, it is easy to verify that $\mu_{\#}(\mathrm{Sym}^n X) \neq \sigma^n(\mu_{\#}(X))$. For instance, $X = \mathbb{A}^1$,

$\mathrm{Sym}^n \mathbb{A}^1 = \mathbb{A}^n$ and $\mu_{\#}(\mathbb{A}^1) = q$ whereas $\mu_{\#}(\mathbb{A}^n) = q^n \neq \sigma^n(q)$. As this is the only

possible λ-ring structure on $\mathbb{Z}$, counting measure can not be a pre-λ-ring map. That

is, $Z(X, t) \neq \sigma_t \circ \mu_{\#}$ and $\mu_{\#}$ fails to satisfy Condition L. $\qquad \square$

In general, recall that by Corollary 1.26, any μ' factoring through an exponentiable measure μ is also exponentiable. This does not require $f : R \to S$ to be a λ-ring map, and thus $\mu' = f \circ \mu$ need not be a pre-λ-ring map.

3.1.1   The Gillet-Soulé Measure

In a seminal paper [5], Gillet and Soulé extend the theory of Chow motives to a motivic measure on the Grothendieck ring of varieties in the case of $\mathrm{char}(k) = 0$. Associated to the functor

$$\mathfrak{h} : SmProj(k) \longrightarrow Chow(k)_{\mathbb{Q}}$$

from the category of smooth projective varieties over k to the category of rational pure effective Chow motives, they construct a motivic measure

$$\mu_{GS} : K_0(\mathrm{Var}_k) \longrightarrow K_0(Chow(k)_{\mathbb{Q}})$$

such that in the case X is smooth and projective, $\mu_{GS}([X]) = [\mathfrak{h}(X)]$ the class of the motive.

Proposition 3.3 (Ramachandran, Tabuada [23])**.** The motivic zeta function $\zeta_{\mu_{GS}}$ factors through μ_{GS} and thus μ_{GS} is an exponentiable measure.

Proof. The ring $K_0(\mathcal{C})$ is a λ-ring for any $\mathbb{Q}$-linear additive pseudo-abelian symmetric monoidal category (Heinloth [13]). Results of del Baño Rollin and Aznar [4] show that $\mathfrak{h}(\operatorname{Sym}^n X) = \operatorname{Sym}^n \mathfrak{h}(X)$ in $Chow(k)_{\mathbb{Q}}$ for smooth projective X. Due to the presentation of $K_0(\operatorname{Var}_k)$ by Heinloth (Proposition 1.20) in terms of smooth projective varieties, this suffices to prove the same holds for μ_{GS}. Therefore, $\zeta_{\mu_{GS}} = \sigma_t \circ \mu_{GS}$ and exponentiability immediately follows. $\qquad\square$

Thus, for a broad variety of motivic measures, the Witt product of zeta functions reflects the multiplicative structure on $K_0(\operatorname{Var}_k)$.

Corollary 3.4. *Every motivic measure μ factoring through μ_{GS} is exponentiable. For example, the motivic measures χ_c and μ_P are thus shown to be exponentiable.*

3.2   Exponentiation of motivic zeta functions

If $\mu : K_0(\operatorname{Var}_k) \to A$ is exponentiable then its associated motivic zeta function defines a measure $\zeta_\mu : K_0(\operatorname{Var}_k) \to W(A)$; we call this measure the induced motivic zeta function measure $\mu_Z(X) = \zeta_\mu(X, t)$. We now show that any motivic measure μ satisfying Condition L has an induced motivic zeta function measure μ_Z which also satisfies Condition L and is thus itself exponentiable.

Theorem 3.5. *Let $\mu : K_0(\operatorname{Var}_k) \to A$ be an exponentiable motivic measure and $\mu_Z : K_0(\operatorname{Var}_k) \to W(A)$ its induced motivic zeta function measure. If μ satisfies Condition L, then μ_Z satisfies Condition L. Namely,*

$$\zeta_{\mu_Z}(X, u) = \sum_n \zeta_\mu(\operatorname{Sym}^n X, t)\, u^n = \boldsymbol{\sigma}_u(\zeta_\mu(X, t))$$

and thus μ_Z is exponentiable with ζ_{μ_Z} taking values in $W(R) = W(W(A))$.

Proof. This follows from the behavior of σ_t and $\boldsymbol{\sigma_u}$ the opposite λ-ring structure maps on A and $W(A)$, respectively. Namely, $\sigma_t : A \to W(A)$ is a map of pre-λ-rings by Proposition 1.17 and thus the composition $\zeta_\mu = \sigma_t \circ \mu$ of pre-λ-ring maps satisfies Condition L.

Specifically, recall that in this case, $Z = \sigma_t \circ \mu$. We have,

$$\sum_n \zeta_\mu(\mathrm{Sym}^n X, t) u^n = \sum_n \sigma_t(\mu(\mathrm{Sym}^n X)) u^n = \Lambda_{\sigma_t}(Z(X, t))$$

and now apply Proposition 1.17. Exponentiability of μ_Z follows from Proposition 3.1. □

Corollary 3.6. *The zeta function ζ_{μ_Z} is itself a motivic measure satisfying Condition L and thus ζ_{μ_Z} is exponentiable; this process may be iterated. To wit, ζ_{μ_Z} is itself an exponentiable motivic measure, ad infinitum.*

We may summarize the situation as follows: let $\mu : K_0(\mathrm{Var}_k) \to R$ be a motivic measure satisfying Condition L. Then its motivic zeta function can be considered a motivic measure $\mu_Z = \zeta_\mu$ and we have

$$
\begin{array}{ccc}
K_0(\mathrm{Var}_k) & \xrightarrow{\ \mu\ } & R \\
 & \searrow^{\zeta_\mu} & \downarrow^{\sigma_t} \\
 & \searrow_{\zeta_{\mu_Z}} & W(R) \\
 & & \downarrow^{\boldsymbol{\sigma_u}} \\
 & & W(W(R))
\end{array}
$$

where this diagram commutes and extends infinitely downward.

3.2.1 The Weil zeta function, revisited

There are instances where μ does not satisfy Condition L while ζ_μ does. Recall from Propostion 3.2, the counting measure $\mu_\#$, while exponentiable, is not a pre-λ-ring map and $Z(X,t) \neq \sigma_t \circ \mu_\#$. However, in Theorem 2.3, the motivic zeta function associated to the Weil zeta function measure $\mu_Z(X) = Z(X,t)$ was shown to have the form

$$\zeta_{\mu_Z}(X,u) = \sum_{i,j}(-1)^{i+1}[\![\, [\alpha_{ij}] \,]\!],$$

an alternating sum of double Teichmüller elements.

Proposition 3.7. The motivic zeta function ζ_{μ_Z} factors through $\mu_Z = Z(X,t)$ as $\zeta_{\mu_Z} = \boldsymbol{\sigma_u} \circ \mu_Z$. Thus the Weil zeta function measure μ_Z satisfies Condition L and exponentiates as in Theorem 3.5.

Proof. Recall that $\boldsymbol{\sigma_u}([\alpha]) = [\![\, [\alpha] \,]\!]$. Since $\boldsymbol{\sigma_u}$ is a ring homomorphism, the formula in Theorem 2.3 suffices to prove the factorization of ζ_{μ_Z}. $\qquad\qquad\square$

Now suppose that $\mu : K_0(\mathrm{Var}_k) \to R$ satisfies Condition L (i.e. it is a pre-λ-ring map) and $f : R \to S$ is *any* ring homomorphism. Then the measure $\mu' : K_0(\mathrm{Var}_k) \to S$, $\mu' = f \circ \mu$ need not be a pre-λ-ring map. However, the associated zeta function ζ'_μ will still satisfy Condition L. That is,

Theorem 3.8. *Let $\mu : K_0(\mathrm{Var}_k) \to R$ be a motivic measure satisfying Condition L and $\mu' : K_0(\mathrm{Var}_k) \to S$ factor through μ via any ring homomorphism $f : R \to S$. Then the induced motivic zeta function measure $\mu'_Z = \zeta_{\mu'}$ satisfies Condition L even though μ' itself may not.*

Proof. Recall from Proposition 1.16 that for any ring homomorphism $f : R \to S$, the induced map $W(f) : W(R) \to W(S)$ is a λ-ring map. We have

$$\mu'_Z(X) = \zeta_{\mu'}(X, t) == \sum_{n=0}^{\infty} \mu'(\text{Sym}^X)t^n = \sum_{n=0}^{\infty} f \circ \mu(\text{Sym}^n X)t^n$$

Thus, $\mu'_Z(X) = W(f) \circ \zeta_\mu$ is a composition of pre-λ-ring maps, hence satisfies the condition. $\square$

This proposition may be summarized as

where the top square does not commute while the bottom square does. Again this diagram extends infinitely downward. We thus see that if μ' is any measure factoring through a measure μ which satisfies Condition L, the associated motivic zeta function measure $\mu_Z(X) = \zeta_{\mu'}(X, t)$ satisfies Condition L and is thus exponentiable. This in fact is the case for the Weil zeta function:

Corollary 3.9. *The Weil zeta function measure μ_Z satisfies Condition L*

Proof. The Weil zeta function is the motivic zeta function associated to counting measure $\mu_\#$. Recall the algebraic Lefschetz fixed point theorem

$$N_r(X) = \sum_i (-1)^i \text{Tr}(\Phi^r | H^i_{et,c}(\overline{X}; \mathbb{Q}_\ell))$$

This shows that counting measure $\mu_\#(X) = N_1(X)$ factors as $\mathrm{Tr} \circ \mu_\ell$ where μ_ℓ is the ℓ-adic cohomology measure

$$\mu_\ell(X) = (H^i_{et,c}(\overline{X}; \mathbb{Q}_\ell), \Phi) \in K_0(End_{\mathbb{Q}_\ell})$$

where Φ is the Frobenius on X. This measure is a pre-λ-ring map whereas the the alternating trace Tr is not. $\qquad\square$

3.3 MacDonald formulae

We now wish to reframe the classical MacDonald formulae in the setting of λ-ring valued measures. We recast the classical MacDonald formulae using the λ-ring structures on $\mathbb{Z}[z]$ and $\mathbb{Z}$. We show that these classical formulae prove that the measures χ_c and μ_P satisfy Condition L.

Let $\mathcal{C}$ be a k-linear tensor category, in the sense of a k-linear additive pseudo-abelian symmetric monoidal category where $\otimes$ is k-linear. Due to Heinloth [13], the Grothendieck ring $K_0(\mathcal{C})$ is a λ-ring. From work of del Baño Rollin [2] and Maxim-Schürmann [20], it is clear that many closed formulae for various generating series of measures taking values in $K(\mathcal{C})$ follow directly from the opposite λ-ring structure map σ_t. In fact, our discussion here for the classical MacDonald formulae analogously holds for these motivic MacDonald formulae. Thus, the motivic measures described in [2] and [20] also satisfy Condition L.

3.3.1 The classical MacDonald formulae

Let $X \in \mathrm{Var}_{\mathbb{C}}$ and m the dimension of X. Recall from Example 1.23 and Example 1.24, for the measures χ_c and μ_P

$$\chi_c(X) = \sum_{i=0}^{2m}(-1)^i b_i(X), \quad \mu_P(X) = P(X, z) = \sum_{i=0}^{2m}(-1)^i b_i(X) z^i,$$

we have the classical MacDonald formulae

$$\sum_{n=0}^{\infty} \chi(\mathrm{Sym}^n X) t^n = \frac{(1-t)^{b_1} \cdots (1-t)^{b_{2m-1}}}{(1-t)^{b_0} \cdots (1-t)^{b_{2m}}} \tag{3.1}$$

and

$$\sum_{n=0}^{\infty} P(\mathrm{Sym}^n X, z) t^n = \frac{(1-z^1 t)^{b_1}(1-z^3 t)^{b_3} \cdots (1-z^{2m-1} t)^{b_{2m-1}}}{(1-z^1 t)^{b_1}(1-z^2 t)^{b_2} \cdots (1-z^{2m} t)^{b_{2m}}}. \tag{3.2}$$

As pointed out previously, these formulae provide a proof that the zeta function should take values in the Witt ring . There, we noted that when written in the Witt ring, the zeta functions take the form of Euler characteristics themselves, as alternating sums. That is, Equation (3.1) becomes

$$\zeta_{\chi_c}(X, t) = \chi_c(X)[1] \in W(\mathbb{Z}) \tag{3.3}$$

and Equation (3.2) becomes

$$\zeta_{\mu_P}(X, t) = \sum_i (-1)^{i+1} b_i(X)[z^i] \in W(\mathbb{Z}[z]) \tag{3.4}$$

and these have the multiplicative structure reflected by the Witt product. Indeed the ghost coordinates are

$$\mathrm{gh}_n(\zeta_{\chi_c}(X, t)) = \chi_c(X) \quad \text{and} \quad \mathrm{gh}_n(\zeta_{\mu_P}(X, t)) = P(X, z^n);$$

both of these are multiplicative which suffices to prove exponentiability.

The measures χ_c and μ_P take values in the λ-rings $\mathbb{Z}$ and $\mathbb{Z}[z]$, respectively, and these λ-ring structures explain the form that the MacDonald formulae take. We observe that

Lemma 3.10. *Given the λ-ring structure on $\mathbb{Z}$, we have $\sigma_t(\chi(X)) = \chi(X)[1]$. Given the λ-ring structure on $\mathbb{Z}[z]$, we have $\sigma_t(P(X,z)) = \sum_i(-1)^i b_i(X)[z^i]$.*

Proof. The λ-ring structure on $\mathbb{Z}$ is $\lambda_t(a) = (1+t)^a$ so that

$$\sigma_t(a) = (1-t)^{-a} = a[1] \in W(\mathbb{Z}),$$

and as σ_t is a ring homomorphism, the first statement follows. Recall from Proposition 1.11, the λ-ring structure on $\mathbb{Z}[z]$ is defined so that $\lambda_t(z) = (1+zt)$. This implies that for $a \in \mathbb{Z}$,

$$\sigma_t(az) = (1-zt)^{-a} = a[z] \in W(\mathbb{Z}[z]),$$

whence the second statement. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We can now reframe the MacDonald formulae in terms of λ-ring-valued measures satisfying Condition L. For $\mu = \chi_c$ the Euler characteristic measure, combining Equation 3.1 and Equation 3.3 we see by the above lemma that the MacDonald formula is equivalent to

$$\zeta_{\chi_c}(X,t) = \sigma_t(\chi_c(X))$$

where σ_t is the opposite λ-ring structure map on $\mathbb{Z}$. Similarly for μ_P, combining Equation 3.2 and Equation (3.4) the MacDonald formula is

$$\zeta_{\mu_P}(X,t) = \sigma_t(P(X,z))$$

where σ_t is the opposite λ-ring structure map on $\mathbb{Z}[z]$.

Thus, we have shown the classical MacDonald formulae (3.1) and (3.2) are equivalent to the following:

Proposition 3.11. The measures χ and μ_P are pre-λ-ring maps,

$$\chi : K_0(\mathrm{Var}_k) \to \mathbb{Z} \quad \text{and} \quad \mu_P : K_0(\mathrm{Var}_k) \to \mathbb{Z}[z],$$

and thus satisfy Condition L, i.e. $\zeta_{\mu_P} = \sigma_t \circ \mu_p$ and $\zeta_{\chi_c} = \sigma_t \circ \chi_c$.

To wit, the MacDonald formulae are simply reflecting the fact that the measures χ_c and μ_P satisfy Condition L.

3.3.2 The Weil zeta function, revisited again

Finally, we argue here that the formula produced in Theorem 2.3 should be considered a "MacDonald formula" for the Weil zeta function measure $\mu_Z(X) = Z(X, t)$ for varieties over finite fields. Indeed, note that:

- Theorem 2.3 presents ζ_{μ_Z} as a finite sum of double Teichmüller elements,

$$\zeta_{\mu_Z}(X, u) = \sum_{i,j} (-1)^{i+1} [[\, [\alpha_{ij}]\,]]$$

  in $W(W(\overline{\mathbb{Q}_\ell}))$, which appears as a closed (i.e. finite) product in $\Lambda(W(\overline{\mathbb{Q}_\ell}))$.

- Proposition 3.7 shows that $\zeta_{\mu_Z} = \boldsymbol{\sigma_u} \circ \mu_Z$, that is μ_Z satisfies Condition L.

The classical MacDonald formulae similarly arise from the fact that they are each finite sums and satisfy Condition L.

Appendices

Appendix A: Symmetric Polynomials

In this appendix, we detail some facts about symmetric polynomials, in particular the universal symmetric polynomials appearing in the theory of λ-rings. For a more comprehensive approach, see Knutson [16] or Yau [25].

The polynomial ring $R[x_1, x_2, \ldots, x_n]$ in n variables carries an evident action of the symmetric group S_n by permuting the variables. A polynomial f is called a *symmetric polynomial* if it is fixed under this action; that is, for all permutations $\pi \in S_n$, we have $\pi f(x_1, x_2, \ldots, x_n) = f(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)}) = f(x_1, x_2, \ldots, x_n)$.

The elementary symmetric polynomials s_i in the variables $x_1, x_2, \ldots, x_n$ are defined as the coefficients of t^i in the function $h(t)$,

$$h(t) = \prod_j (1 - x_j t) = s_0 + s_1 t + s_2 t^2 \cdots + s_n t^n$$

Specifically, we have

$$s_1 = x_1 + x_2 + \cdots + x_n$$

$$s_2 = x_1 x_2 + x_1 x_3 + \cdots + x_{n-1} x_n$$

$$s_n = x_1 x_2 \cdots x_n$$

Thus s_j is the sum of unordered j-tuples of distinct variables x_i.

The elementary symmetric polynomials generate all symmetric polynomials in

$R[x_1, x_2, \ldots, x_n]$.

Theorem A.1 (Fundamental Theorem of Symmetric Polynomials). *Every symmetric polynomial $f \in R[x_1, x_2, \ldots, x_n]$ may be written as a polynomial $f = P(s_1, s_2, \ldots, s_n)$ in the elementary symmetric polynomials.*

A.1   The universal polynomials P_n and $P_{n,m}$

There are two generalizations of the elementary symmetric polynomials. We may take a product over all unordered m-tuples of distinct variables x_i. We may also take a product over all pairs in two variable sets. Both result in symmetric functions which may be expressed in terms of elementary symmetric polynomials.

Consider the variable set $x_1, x_2, \ldots, x_{nm}$ and let $m \leqslant n$. The coefficients of t^n of the function

$$h(t) = \prod_{1 \leqslant i_1 < i_2 < \cdots < i_m \leqslant nm} (1 + x_{i_1} x_{i_2} \cdots x_{i_m} t)$$

are symmetric in the x_i. Thus they can be written in terms of the elementary symmetric polynomials. The universal polynomials $P_{n,m}$ are defined such that

$$h(t) = 1 + P_{1,m} t + P_{2,m} t^2 + \cdots P_{n,m} t^{nm}.$$

These are polynomials $P_{n,m}$ in nm variables. Here we set $P_{0,m} = 1$. We have, for example, $P_{1,m}$ is the coefficient of t in $h(t)$, so

$$P_{1,m}(s_1, s_2, \ldots, s_m) = s_m$$

the sum of unordered m-tuple products of distinct variables x_i.

Consider two sets of variables $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$. The coefficients of t^n of the function

$$g(t) = \prod_{i,j}(1 + x_i y_j t)$$

are symmetric in both x_i and y_i. Thus, they can be written in terms of the elementary symmetric polynomials s_i in the x_i and the elementary symmetric polynomials σ_j in the y_j. The universal polynomial P_n is the polynomial such that

$$h(t) = 1 + P_1 t + P_2 t^2 + \cdots + P_{n^2} t^{n^2}$$

These are polynomials in $2n$ variables. Here we set $P_0 = 1$. We have, for example, P_1 is the coefficient of t in $g(t)$, so

$$P_1(s_1, \sigma_1) = s_1 \sigma_1$$

the product $(x_1 + x_2 + \cdots + x_n)(y_1 + y_2 + \cdots + y_n)$, the sum of all pairs $x_i y_j$ for all $i, j = 1, \ldots, n$.

A.2 The complete homogeneous and power symmetric polynomials

There is another pair of symmetric polynomials which are relevant. These are h_n the complete homogeneous symmetric functions and p_n be the power symmetric functions. Consider the variable set $x_1, x_2, \ldots, x_m$. Then

$$h_n = \sum_{1 \leqslant i_1 \leqslant i_2 \leqslant \cdots \leqslant i_n} x_{i_1} x_{i_2} \cdots x_{i_n} \quad \text{and} \quad p_n = x_1^n + x_2^n + \cdots x_m^n.$$

For example,

$$h_1 = x_1 + x_2 + \cdots x_m$$

$$h_2 = x_1^2 + x_1 x_2 + \cdots x_1 x_m + x_2^2 + \cdots$$

whereas

$$p_1 = x_1 + x_2 + \cdots x_m$$

$$p_2 = x_1^2 + x_2^2 + \cdots x_m^2.$$

Notice we are allowed all n-tuples of variables with repetition for h_n and simple nth powers for p_n.

These polynomials satisfy Newton's identities:

$$n h_n = p_n + h_1 p_{n-1} + \cdots + h_{n-1} p_1.$$

A.3 Relation to vector spaces

Let V and W be two finite dimensional vector spaces over a field F. For simplicity, suppose $\dim_F V = \dim_F W = n$. The universal polynomials P_n and $P_{n,m}$ are inspired by the exterior power of the tensor product $\Lambda^k(V \otimes W)$ and exterior power of an exterior power $\Lambda^j(\Lambda^k V)$.

Let $V = V_1 \oplus V_2 \oplus \cdots \oplus V_n$ and $W = W_1 \oplus W_2 \oplus \cdots \oplus W_n$ where the V_i and W_j are one-dimensional subspaces. Then

$$V \otimes W = \bigoplus_{i,j} V_i \otimes W_j$$

Note that $V_i \otimes W_j$ is also one-dimensional, so that $\Lambda^k(V_i \otimes W_j)$ is 0 once $k > 1$. If we want to enumerate the exterior powers of $V \otimes W$ using a power series, we should have

$$1 + \Lambda^1(V \otimes W)t + \Lambda^2(V \otimes W)t^2 + \cdots = \prod_{i,j}(1 + V_i \otimes W_j t)$$

The right hand side is symmetric in the V_i and W_j; hence so are the polynomials $P_{n,m}$.

Similarly, if $V = V_1 \oplus V_2 \oplus \cdots \oplus V_{nm}$, where each V_i is one dimensional, then

$$\Lambda^m(V) = \bigoplus_{1 \leqslant i_1 < i_2 < \cdots < i_m \leqslant nm} V_{i_1} \otimes V_{i_2} \otimes \cdots \otimes V_{i_n}$$

and $\Lambda^m(V)$ is thus a sum of one-dimensional vector spaces. Thus if we want to enumerate the exterior powers of $\Lambda^m(V)$ we should have

$$1 + \Lambda^1(\Lambda^m(V))t + \Lambda^2(\Lambda^m(V))t^2 + \cdots = \prod_{1 \leqslant i_1 < i_2 < \cdots < i_m \leqslant nm} (1 + V_{i_1} \otimes V_{i_2} \otimes \cdots \otimes V_{i_n} t)$$

The right hand side is symmetric in the V_i; hence so are the polynomials P_n.

Finally, if V is one-dimensional, then $\Psi_n(V) = V^{\otimes n}$. As these are ring homomorphisms, if $V = V_1 \oplus V_2 \oplus \cdots \oplus V_m$, where each V_i is one-dimensional, then

$$\Psi_n(V) = V_1^{\otimes n} \oplus V_2^{\otimes n} \oplus \cdots \oplus \mathbb{V}_m^{\otimes n}.$$

Thus the Adams isomorphisms can be described by the power symmetric polynomials p_n.

Appendix B: Symmetric Powers

In this appendix, we review details about symmetric powers of finite dimensional vector spaces V, finite graded vector spaces M_*, and varieties X. Let S_n denote the symmetric group on n elements.

B.1   Symmetric powers of vector spaces

For V a finite dimensional vector space over a field F, consider the n-fold tensor product $V^{\otimes n} = V \otimes V \otimes \cdots \otimes V$. This admits an obvious action of the symmetric group S_n by permuting the factors. The n-fold symmetric power of V is defined to be the quotient space of $V^{\otimes n}$ under this action

$$\mathrm{Sym}^n V = (V^{\otimes n})/\langle v - \sigma(v) | v \in V^{\otimes n}, \ \sigma \in S_n \rangle$$

In the case char $F = 0$, we can identify $\mathrm{Sym}^n V$ as the subspace of symmetric tensors, i.e. the subspace of all tensors $v \in V^{\otimes n}$ fixed under the action of S_n, as the image of the symmetrization map

$$v_1 \otimes v_2 \otimes \cdots \otimes v_n \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)}.$$

If $\dim_F V = d$, then dimension of $\mathrm{Sym}^n V$ is

$$\dim_F \mathrm{Sym}^n V = \begin{pmatrix} d+n-1 \\ n \end{pmatrix} = \begin{pmatrix} d+n-1 \\ d-1 \end{pmatrix}.$$

This is the number of unordered n-tuples of a set of d elements.

B.2   Symmetric powers of graded vector spaces

For $V_* = \bigoplus_{i \geqslant 0} V_i$ a (finite, non-negatively) graded (finite dimensional) vector space over a field F, we can similarly consider the n-fold tensor product $(V_*)^{\otimes n}$. Recall that given two graded vector spaces V_* and W_*, the graded tensor product $V_* \otimes W_*$ is defined as

$$(V_* \otimes W_*)_k = \bigoplus_{i+j=k} V_i \otimes W_j.$$

Thus, $(V_*)^{\otimes n}$ is a graded vector space

$$\left[(V_*)^{\otimes n} \right]_k = \bigoplus_{i_1+i_2+\cdots+i_n=k} (V_{i_1} \otimes V_{i_2} \otimes \cdots \otimes V_{i_n}).$$

The group S_n acts on each component of $\left[(V_*)^{\otimes n} \right]_k$ by permuting the factors in each summand. Thus for each $\sigma \in S_n$ we obtain a map

$$V_{i_1} \otimes V_{i_2} \otimes \cdots \otimes V_{i_n} \longrightarrow V_{\sigma^{-1}(i_1)} \otimes V_{\sigma^{-1}(i_2)} \otimes \cdots \otimes V_{\sigma^{-1}(i_n)}.$$

This may send one summand of $\left[(V_*)^{\otimes n} \right]_k$ to another. Moreover, the action is defined to keep track of the degrees involved i_k via a sign:

$$\sigma(v_{i_1} \otimes v_{i_2} \otimes \cdots v_{i_n}) = (-1)^\epsilon v_{\sigma^{-1}(i_1)} \otimes v_{\sigma^{-1}(i_2)} \otimes \cdots \otimes v_{\sigma^{-1}(i_n)}$$

where $\epsilon = \epsilon(\sigma)$ the signature (see MacDonald [18]).

The n-fold symmetric power of V_* is defined to be the subspace of $(V_*)^{\otimes n}$ fixed by this action, so that

$$[\mathrm{Sym}^n V_*]_k = \bigoplus_{i_1+i_2+\cdots+i_n=k} (V_{i_1} \otimes V_{i_2} \otimes \cdots \otimes V_{i_n}).$$

B.3 Symmetric powers of varieties

Let Var_k the category of varieties over $\mathrm{Spec}\, k$, i.e. reduced schemes of finite type over $\mathrm{Spec}\, k$. Recall that a quasi-projective variety X is a locally-closed subvariety of projective space.

Definition B.1. For $X \in \mathrm{Var}_k$ quasi-projective, the *n-th symmetric power* of X is $\mathrm{Sym}^n X$,

$$\mathrm{Sym}^n X := (X \times_k X \times_k \cdots \times_k X)/S_n,$$

the quotient variety of the n-fold product under the action of S_n. $\mathrm{Sym}^n X$ is also quasi-projective.

The construction of a quotient variety Y/G under the action of a finite group G is demonstrated in SGA I, V.1 [7]. We require that every $y \in Y$ be contained in an open affine $A \subset Y$ preserved under the action by G. This is true for Y quasi-projective. Here, we have $Y = X \times_k X \times_k \cdots \times_k X$, which is quasi-projective when X is, and $G = S_n$. See Mustata Appendix A.I [21].

Example B.2. $\mathrm{Sym}^n \mathbb{A}^1 \cong \mathbb{A}^n$. This statement is a reformulation of the Fundamental Theorem of Symmetric Polynomials (see Appendix A).

Example B.3. $\mathrm{Sym}^n \mathbb{P}^1 \cong \mathbb{P}^n$.

B.3.1 Symmetric powers in $K_0(\mathrm{Var}_k)$

For $X \in \mathrm{Var}_k$ quasiprojective, we define $\mathrm{Sym}^n[X] \in K_0(\mathrm{Var}_k)$ as $[\mathrm{Sym}^n X]$. This extends to all elements in $K_0(\mathrm{Var}_k)$ as $K_0(\mathrm{Var}_k)$ is additively generated by quasi-projective varieties (see Mustata [21]). Moreover, if $Y \subset X$ is a closed subvariety and $U = X \setminus Y$, we have

$$[\mathrm{Sym}^n X] = \sum_{i+j=n} [\mathrm{Sym}^i Y][\mathrm{Sym}^j U]$$

in $K_0(\mathrm{Var}_k)/\sim$, where $[X] \sim [Y]$ if there exists a surjective radicial morphism $X \to Y$. In the case $\mathrm{char}\, k = 0$, $K_0(\mathrm{Var}_k) \cong K_0(\mathrm{Var}_k)/\sim$. In the finite field case, $\mu_\#$ factors through $K_0(\mathrm{Var}_k)/\sim$. This is required to prove that motivic zeta functions are group homomorphisms into $\Lambda(R)$.

Example B.4. In $K_0(\mathrm{Var}_k)$, $[\mathrm{Sym}^n \mathbb{A}^m] = [\mathbb{A}^{nm}]$. This follows from an observation due to Totaro (see Göttsche [6] that in $K_0(\mathrm{Var}_k)$, $[\mathrm{Sym}^n(X \times \mathbb{A}^1)] = [\mathrm{Sym}^n(X)] \times [\mathbb{A}^n]$. In fact, $\mathrm{Sym}^n \mathbb{A}^m$ is piecewise isomorphic to $\mathbb{A}^{mn}$ (see Vakil's notes at [24]).

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