AN APPROXIMATION TO TRANSONIC FLOW OF A POLYTROPIC GAS

by
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Introduction. The equation of state of a gas whose flow is governed by Frankl's equation[1] $\psi_{\sigma\sigma} + K(\sigma)\ \psi_{\theta\theta} = 0$ is determined by the choice of the function $K(\sigma)$. With the proper choice of units, for a polytropic gas

$$K(\sigma) = \sigma - b\,\sigma^2 + \ldots \qquad , \quad b = \frac{2\gamma + 5}{2\sqrt[3]{\gamma + 1}}$$

where γ is the adiabatic exponent.

By retaining only the first term in this expansion we obtain $K(\sigma) = \sigma$, and Frankl's equation becomes Tricomi's equation, used by various authors[2] in the study of transonic flows. The gas determined by taking $K(\sigma) = \sigma$ is called the Tricomi gas,[3] or a T_1-gas.

In section 2 we show that if two functions $K(\sigma)$ have the same expansions in positive integral powers of σ up to and including the term in σ^n, the graphs in the (ρ, p)-plane of the corresponding equations of state have contact of order at least $n+1$ at the sonic point (ρ_*, p_*). Thus for example if we approximate $K(\sigma)$ above for a polytropic gas by retaining only the first two terms in its expansion, the equation of state corresponding to $K(\sigma) = \sigma - b\,\sigma^2$ can be made to have contact of the third order at the sonic point (ρ_*, p_*). A gas with an equation of state corresponding to $K(\sigma) = \sigma - b\,\sigma^2$ is called a T_2-gas and in section 4 we study its equation of state. In the concluding section we study a particular example of transonic flow for a T_2-gas in the physical plane.

1. Preliminary Considerations. Employing the pressure p and the stream function ψ as independent variables Martin[4] has shown that, given a Bernoulli function $q = q(p, \psi)$ and a direction function $\theta = \theta(p, \psi)$ which jointly satisfy

$$(1) \qquad q\left(\frac{q_{pp} - q\theta_p^2}{\theta_\psi}\right)_\psi + \left(q^2 \theta_p\right)_p = 0 \quad ,$$

a flow is presented in the physical plane $z = x + iy$ by

$$(2) \qquad z = \int e^{i\theta}\left\{-\frac{q_{pp} - q\theta_p^2}{\theta_\psi}\,dp + \left(q\theta_p - i\, q_p\right) d\psi\right\}$$

The density ρ and Mach number M are given[5] by

$$(3) \qquad \rho = -(q q_p)^{-1} \quad , \quad M^2 = 1 + q\, q_p^{-2}\, q_{pp} \; .$$

Irrotational flows are characterized by a Bernoulli function of the form $q = q(p)$ and we see from (3) that q is a decreasing function of p for $\rho > 0$, $q > 0$. Furthermore, the flow is subsonic, sonic, or supersonic according as $q_{pp} \lesseqqgtr 0$. The sonic speed is given by $q_* = q(p_*)$, where p_*, the sonic pressure, is defined by $q_{pp}(p_*) = 0$.

If we introduce a new variable

$$(4) \qquad \sigma = \int_{p_*}^{p} \frac{dp}{q^2} = \sigma(p)$$

in place[6] of p, and set

$$(5) \qquad K(\sigma) = -q^3\, q_{pp}$$

the variable p being eliminated from the second member with the aid of (4), equation (1) is replaced by

$$(6) \qquad \theta_{\sigma\sigma} = \left(\frac{K + \theta_\sigma^2}{\theta_\psi}\right)_\psi$$

and (2) by

$$(7) \qquad z = \int \mathcal{H} e^{i\theta} \left\{ \frac{K+\theta_\sigma^2}{\theta_\psi} d\sigma + \left(\theta_\sigma + i \, \frac{\mathcal{H}_\sigma}{\mathcal{H}} \right) d\psi \right\} ,$$

where the function

$$(8) \qquad \mathcal{H} = \mathcal{H}(\sigma) = q^{-1}$$

is obtained by eliminating p from $q = q(p)$ again with the help of (4).

$K(\sigma)$ as defined by (5) and (4) is identical with $K(\sigma)$ in Frankl's equation $\psi_{\sigma\sigma} + K(\sigma)\,\psi_{\theta\theta} = 0$, equivalent to (6). It follows from (4), (5), and (8) that

$$(9) \qquad \mathcal{H}_{\sigma\sigma} - K(\sigma)\,\mathcal{H} = 0$$

A solution $\theta = \theta(\sigma, \psi)$ of (6) when inserted in (7) yields a mapping $z = z(\sigma, \psi)$ of the (σ, ψ)-plane upon the physical plane, which carries the straight lines $\sigma = \text{const.}$ into the isovels (isobars) and the straight lines $\psi = \text{const.}$ into the streamlines.

2. <u>The equation of state. Order of contact</u>. Alternatively, given $K(\sigma)$, corresponding to a solution $\mathcal{H}(\sigma)$ of (9) we find from (8), (3), and (4) that

$$(10) \qquad q = \mathcal{H}^{-1} = q(\sigma) ,$$

$$(11) \qquad \rho = \mathcal{H}\,\mathcal{H}_\sigma^{-1} = \rho(\sigma) ,$$

and

$$(12) \qquad p = C + \int_0^\sigma \mathcal{H}^{-2}\, d\sigma = p(\sigma)$$

From (4) it is clear that $\sigma = 0$ yields the sonic values q_*, ρ_*, p_*.

When σ is eliminated from (10), (12) we obtain the Bernoulli function $q = q(p)$. Equations (11) and (12) constitute parametric equations for the equation of state, the elimination of the parameter σ leading to the equation of state in the usual form $p = p(\rho)$. For a given $K(\sigma)$ the equation of state (11) (12) is uniquely determined by the choice of the initial values $\mathcal{H}(0)$, $\mathcal{H}'(0)$ of the solution $\mathcal{H} = \mathcal{H}(\sigma)$ of (9), and the constant C in (12).

We now prove the

THEOREM If $K(\sigma) = \sum_{r=1}^{\infty} k_r \sigma^r$, $\bar{K}(\sigma) = \sum_{r=1}^{\infty} \bar{k}_r \sigma^r$, $k_{n+1} \neq \bar{k}_{n+1}$, $k_i = \bar{k}_i$ $(i = 1, 2, \ldots, n)$ and if $\mathcal{H}(\sigma)$, $\bar{\mathcal{H}}(\sigma)$ are solutions of

$$\mathcal{H}'' - K\mathcal{H} = 0 \quad , \quad \bar{\mathcal{H}}'' - \bar{K}\bar{\mathcal{H}} = 0 \quad ,$$

respectively, meeting the same initial conditions

$$\mathcal{H}(0) = \bar{\mathcal{H}}(0) = \text{const.}, \qquad \mathcal{H}'(0) = \bar{\mathcal{H}}'(0) = \text{const.},$$

the curves

$$C_1 : \rho = \frac{\mathcal{H}(\sigma)}{\mathcal{H}'(\sigma)} = \rho(\sigma) \quad , \quad p = p_* + \int_0^{\sigma} \frac{d\sigma}{\mathcal{H}^2(\sigma)} = p(\sigma) \; ,$$

$$C_2 : \bar{\rho} = \frac{\bar{\mathcal{H}}(\sigma)}{\bar{\mathcal{H}}'(\sigma)} = \bar{\rho}(\sigma) \quad , \quad \bar{p} = p_* + \int_0^{\sigma} \frac{d\sigma}{\bar{\mathcal{H}}^2(\sigma)} = \bar{p}(\sigma) \; ,$$

have contact of order(7) at least $n+1$ at the point (ρ_*, p_*) in the (ρ, p)-plane, where

$$\rho_* = \rho(0) = \bar{\rho}(0) \quad , \quad p_* = p(0) = \bar{p}(0) \; ,$$

are the sonic densities and sonic pressures.

Proof: By Maclaurin's theorem the first n derivatives of K and $\overline{K}$ agree at $\sigma = 0$ but $K^{(n+1)}(0) \neq \overline{K}^{(n+1)}(0)$. By taking the r^{th} derivatives of the differential equations satisfied by $\mathcal{H}$, $\overline{\mathcal{H}}$ it is easy to see from Leibnitz's theorem that the $(r+2)^{nd}$ derivative of each of $\mathcal{H}$ $(\overline{\mathcal{H}})$ can be expressed in terms of at most the r^{th} derivative of $K(\overline{K})$ and $\mathcal{H}$ $(\overline{\mathcal{H}})$. In particular, since

$$\mathcal{H}(0) = \overline{\mathcal{H}}(0) \quad , \quad \mathcal{H}'(0) = \overline{\mathcal{H}}'(0) \quad ,$$

the differential equations assume $\mathcal{H}''(0) = \overline{\mathcal{H}}''(0)$ and by taking $r = 1$ we see that $\mathcal{H}'''(0) = \overline{\mathcal{H}}'''(0)$. Continuing in this way we see that

$$\mathcal{H}^{(i)}(0) = \overline{\mathcal{H}}^{(i)}(0) \qquad (i = 0, 1, \ldots, n+2).$$

By differentiating the equations in C_1, C_2, $n+1$ times and evaluating for $\sigma = 0$ it follows that the $(n+1)^{st}$ derivatives of $\rho, \overline{\rho}$ and of $p, \overline{p}$ agree for $\sigma = 0$, to establish the theorem.

3. <u>An approximation to a polytropic gas.</u> For a polytropic gas

(13) $$\rho = k p^n \quad , \quad 0 < n = \gamma^{-1} < 1 \quad ,$$

and

(14) $$q^2 = \hat{q}^2 - \frac{2 p^{1-n}}{k(1-n)}$$

where k and $\hat{q}$ denote constants for irrotational flow, $\hat{q}$ being the maximum speed. For the acoustic pressure p_* we find

(15) $$p_* = \left(kn \frac{1-n}{1+n} \hat{q}^2\right)^{\frac{1}{1-n}}$$

from which we find

(16) $$q_* = \sqrt{\frac{1-n}{1+n}}\, \hat{q} \quad , \quad \rho_* = k\left(kn \frac{1-n}{1+n} \hat{q}^2\right)^{\frac{n}{1-n}} .$$

If $K(\sigma)$ is expanded in a power series about $\sigma = 0$,

(17) $$K(\sigma) = a\sigma - b\sigma^2 + \dots ,$$

there being no constant term due to (4) and (5). Clearly

$$a = K'(0) = \frac{K_p}{\sigma_p}\bigg|_{p=p_*} = -g^5 g_{ppp}\bigg|_{p=p_*}$$

We now employ (14) to find

(18) $$a = \frac{\gamma+1}{\rho_*^3}$$

By suitable choice of units we can realize

(19) $$\rho_*^3 = \gamma + 1 \quad , \quad i.e. \quad a = 1$$

For the first approximation $K(\sigma) = \sigma$ to (17) the program outlined in sections 1 and 2 for obtaining a flow in the physical plane was carried out by Martin and Thickstun[8] for the particular solution $\theta = \sigma\psi + \frac{\psi^3}{3}$ of (6). It is the purpose of this paper to carry out a similar investigation beginning with the second approximation

$$K = \sigma - b\sigma^2$$

to (17)

Accordingly, we again employ (4) and (5) to obtain

$$b = -\tfrac{1}{2}K''(0) = \left[\frac{g^7}{2} g_{pppp} - 4 g g_p \left(-g^5 g_{ppp}\right)\right]_{p=p_*}$$

which with (14), (3), (15), (16), and (19) yields

(20) $$b = \frac{2\gamma+5}{2\sqrt[3]{\gamma+1}}$$

For air $\gamma = \frac{7}{5}$ (approx.) and $b = 2.9129$ (approx.). It is convenient to set $\delta^2 = 2b$ and from now on we shall take

(21) $$K(\sigma) = \sigma - \frac{\delta^2}{2}\sigma^2$$

By a T_2-gas we understand a fluid whose equation of state $p = p(\rho)$ is defined by (11) and (12) for K as given in (21). Equation (9) now becomes

$$\mathcal{H}_{\sigma\sigma} - \left(\sigma - \frac{\delta^2}{2}\sigma^2\right)\mathcal{H} = 0 \tag{22}$$

and by suitably adjusting the two arbitrary constants in the general solution of (22) and taking $C = p_*$ in (12) we can realize

$$q(0) = q_*\ ,\quad \rho(0) = \rho_*\ ,\quad p(0) = p_* \tag{23}$$

Thus the speed, density, and pressure of a T_2-gas along the sonic line in the physical plane can be brought into agreement with the acoustic values of these quantities for a polytropic gas.

Using (17) and (21) as K and $\bar{K}$ in the theorem of section 2 it follows that the graphs of the equations of state for a T_2-gas and for a polytropic gas have contact of order at least three at (ρ_*, p_*). Further computation shows that the contact is exactly or order 3.

4. The Equation of State for a T_2-Gas. To investigate the equation of state for a T_2-gas we use (22) to study the manner in which $\mathcal{H}$ varies with σ. The existence theorem for linear equations(9) assures us of a unique solution $\mathcal{H} = \mathcal{H}(\sigma)$, once we prescribe

$$\mathcal{H}(0) = \frac{1}{q_*} \quad \text{and} \quad \mathcal{H}_\sigma(0) = \frac{1}{\rho_* q_*}$$

As Figure 1 indicates and (22) implies the graph of $\mathcal{H} = \mathcal{H}(\sigma)$ has an inflection point at $(0, \frac{1}{q_*})$, is concave downward for $\sigma < 0$, and cuts the σ-axis at an acute angle at a point $(\bar{\sigma}, 0)$, where $-\rho_* < \bar{\sigma} < 0$. For $0 < \sigma < \frac{2}{\delta^2}$, the graph is concave upward with a second point of inflection at $\sigma = \frac{2}{\delta^2}$ after which point the graph is concave downward.

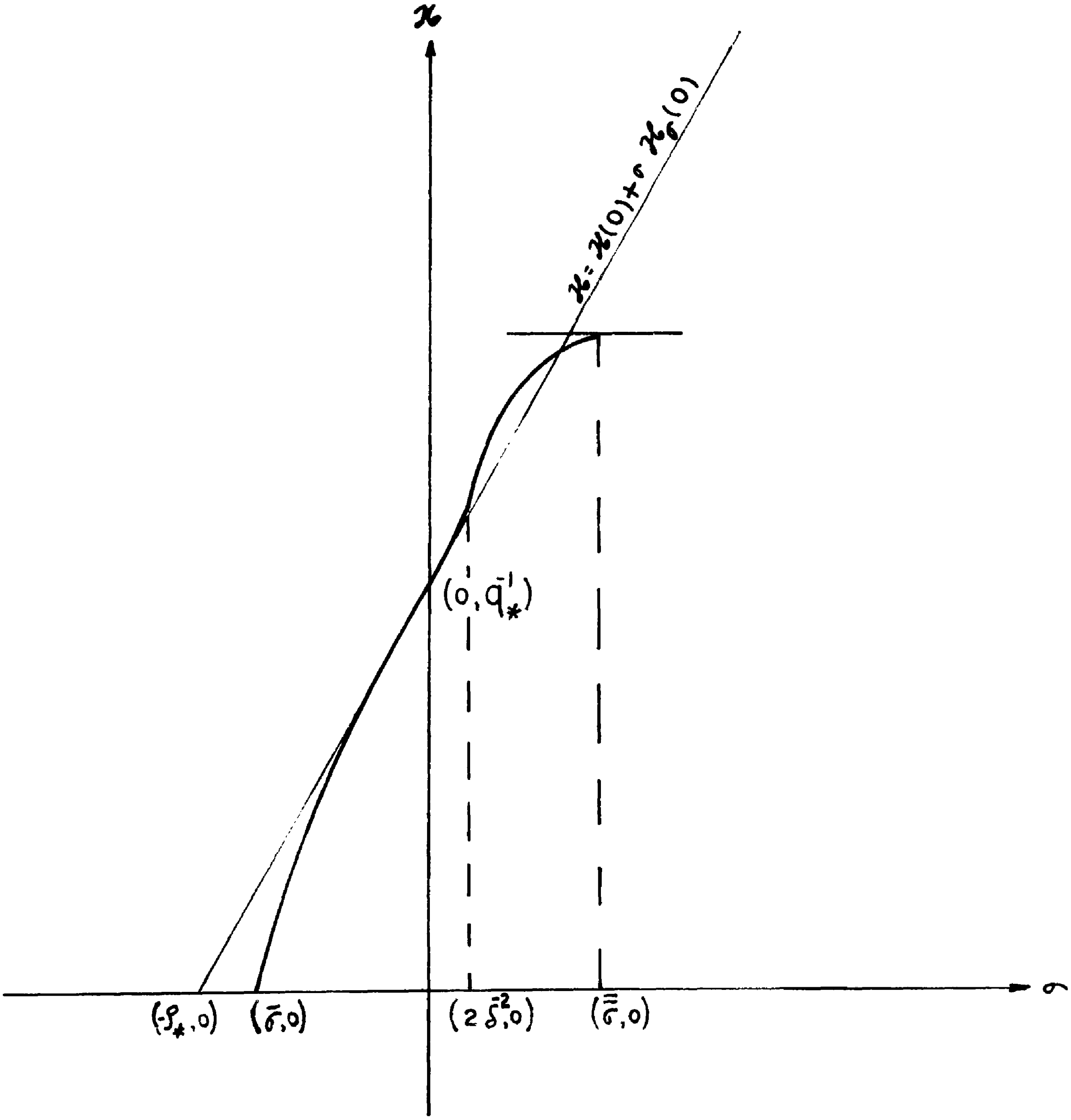


FIGURE 1

The graph has a horizontal tangent for $\sigma = \bar{\bar{\sigma}}$ where $\bar{\bar{\sigma}} > \frac{2}{\delta^2}$. To see this we write (22) in the form

$$\mathcal{H}_\sigma = \mathcal{H}_\sigma\left(\frac{2}{\delta^2}\right) + \frac{\delta^2}{2}\int_{2/\delta^2}^{\sigma} \sigma\left(\frac{2}{\delta^2} - \sigma\right) \mathcal{H}\, d\sigma .$$

We may assume that as σ increases $\mathcal{H} > \mathcal{H}\left(\frac{2}{\delta^2}\right)$, otherwise $\mathcal{H}_\sigma$ would vanish by Rolle's theorem and the assertion is immediately true. Under this assumption it is readily seen that the second member of the above equation eventually becomes negative so that $\mathcal{H}_\sigma = 0$ must hold for some $\sigma = \bar{\bar{\sigma}}$ as stated.

From (8) the speed of flow is infinite for $\sigma = \bar{\sigma}$ and from (11) the density is infinite for $\sigma = \bar{\bar{\sigma}}$. We shall accordingly restrict ourselves to values of σ between $\bar{\sigma}$ and $\bar{\bar{\sigma}}$.

To study ρ as a function of σ in the interval $(\bar{\sigma}, \bar{\bar{\sigma}})$, we note from (9) and (11) that ρ satisfies the Riccati differential equation

$$\rho' = 1 - K\rho^2 = 1 + \sigma\left(\frac{\delta^2}{2}\sigma - 1\right)\rho^2 . \tag{24}$$

As Figure 2 indicates ρ increases monotonically from 0 to $+\infty$ as σ ranges from $\bar{\sigma}$ to $\bar{\bar{\sigma}}$. To verify that $\rho' > 0$ in the interval $(\bar{\sigma}, \bar{\bar{\sigma}})$ we first observe from (24) that $\rho' = 1$ for $\sigma = 0$. Moreover ρ' cannot vanish in this interval. Indeed, if $\rho'(\sigma_0) = 0$ we shall have from (24)

$$\rho^2(\sigma_0) = \frac{1}{\sigma_0\left(1 - \frac{\delta^2}{2}\sigma_0\right)} . \tag{25}$$

This implies that $0 < \sigma_0 < \frac{2}{\delta^2}$ and since $\rho' \leq 1$ in the closed interval $(0, \frac{2}{\delta^2})$ by (24), the curve $\rho = \rho(\sigma)$ lies below the straight line $\rho = \sigma + \rho_*$ in the interval under consideration. It is easy to show that the curve $\rho^2 = \frac{1}{\sigma\left(1 - \frac{\delta^2}{2}\sigma\right)}$ lies entirely above this same line which would contradict (25). Thus ρ is an increasing function of σ in $(\bar{\sigma}, \bar{\bar{\sigma}})$.

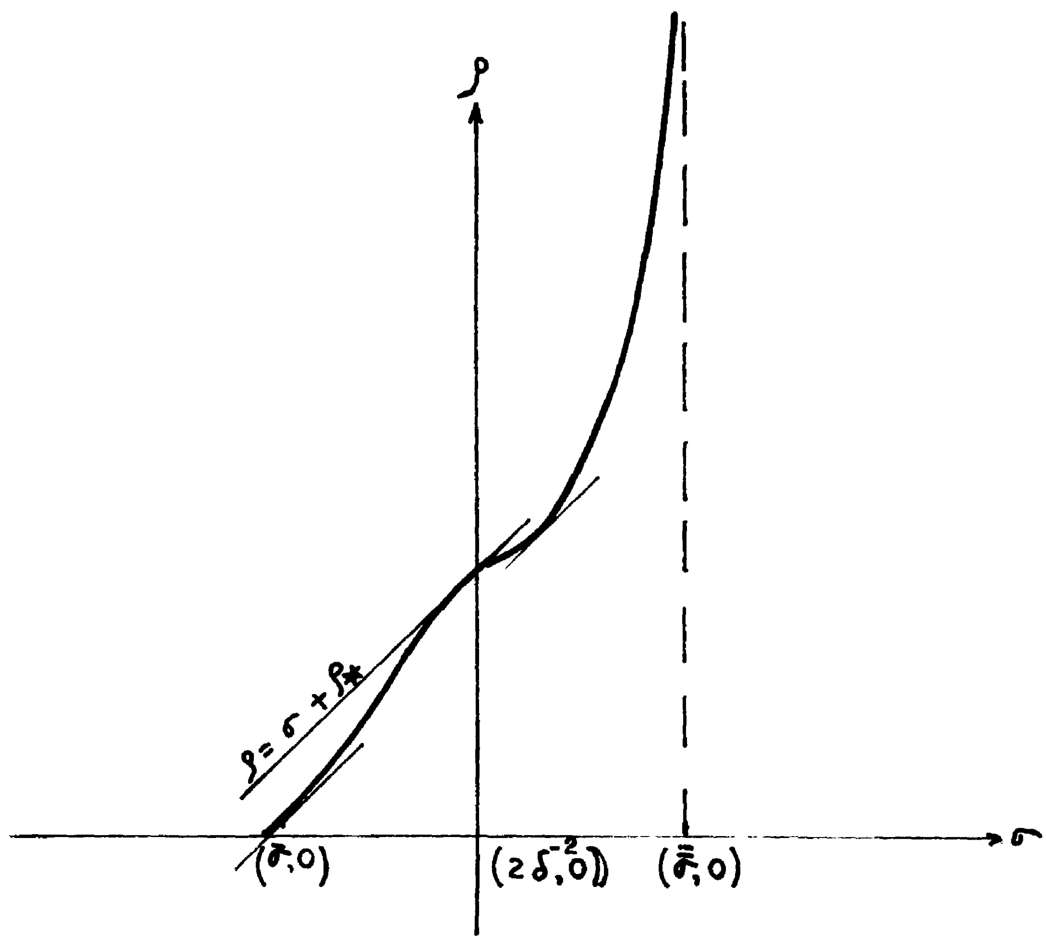


FIGURE 2

From (12) and (23) the pressure p is given by

$$p = p_* + \int_0^{\sigma} \frac{d\sigma}{\mathcal{H}^2} .$$

It is clear that p is an increasing function of σ and from Figure 1 we see that as σ tends to $\bar{\bar{\sigma}}$ p tends to a finite value p_1. As σ tends to $\bar{\sigma}$, p tends to $-\infty$ since the expansion

$$\mathcal{H}(\sigma) = a(\sigma - \bar{\sigma}) + \dots , \quad a \neq 0 ,$$

is valid about $\sigma = \bar{\sigma}$, since the graph of $\mathcal{H} = \mathcal{H}(\sigma)$ cuts the σ-axis at an acute angle at $(\bar{\sigma}, 0)$.

The graph $p = p(\rho)$ of the equation of state is shown in Figure 3 and is obtained by a comparison of Figures 1 and 2.

5. The Direction Function. We seek solutions to (6) with K defined by (21) of the form

$$\theta = \Psi_0 + \Psi_1 \sigma + \overline{\Psi}_2{}' \sigma^2 \tag{26}$$

where Ψ_0, Ψ_1, and $\overline{\Psi}_2$ denote unknown functions. Substituting from (26) into (6) and integrating with respect to ψ, we find

$$\sigma - \frac{\delta^2}{2} \sigma^2 = \left(\Sigma_0 + 2\overline{\Psi}_2\right)\left(\Psi_0' + \Psi_1' \sigma + \overline{\Psi}_2'' \sigma^2\right) - \left(\Psi_1 + 2\overline{\Psi}_2' \sigma\right)^2$$

where the arbitrary function $\Sigma_0 = \Sigma_0(\sigma)$ is introduced by the integration.

We restrict ourselves to the special case Σ_0 = constant and set

$\Sigma_0 + 2\overline{\Psi}_2 = \Psi_2$ to obtain

$$\sigma - \frac{\delta^2}{2}\sigma^2 = \Psi_2\left(\Psi_0' + \Psi_1'\sigma + \frac{1}{2}\Psi_2''\sigma^2\right) - \left(\Psi_1 + \Psi_2'\sigma\right)^2$$

which, on equating coefficients of like powers of σ, yields the following system of differential equations for Ψ_0, Ψ_1, Ψ_2.

(27) (a) $$\Psi_0' \Psi_2 - \Psi_1^2 = 0$$

(b) $$\Psi_1' \Psi_2 - 2\Psi_1 \Psi_2' = 1$$

(c) $$\Psi_2 \Psi_2'' - 2\Psi_2'^2 = -\delta^2$$

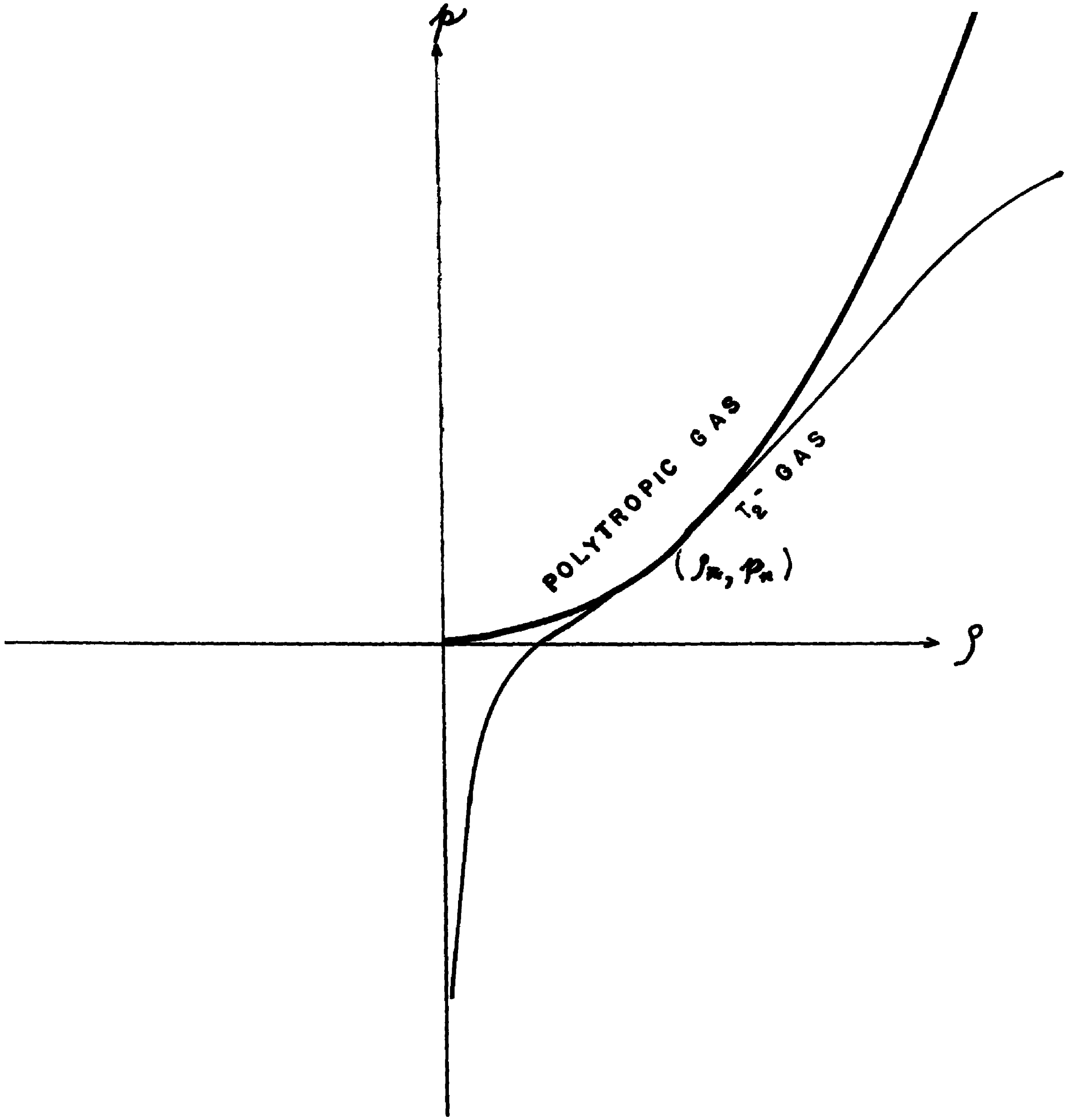


FIGURE 3

To integrate (27c) we set $\Psi_2 = \frac{1}{\Phi}$ and this equation is replaced by

$$\Phi'' - \delta^2 \Phi^3 = 0$$

a first integral of which is

$$\Phi'^2 = \frac{\delta^2}{2} \Phi^4 - C \quad , \qquad C \text{ const.}$$

If we set $C = \frac{\delta^2}{2}$ we find

$$\frac{\delta}{\sqrt{2}} \psi = \int_1^{\Phi} \frac{dt}{\sqrt{(t^2-1)(t^2+1)}} \quad , \delta > 0 ,$$

from which we see(10) that Ψ_2 is the elliptic function

$$\Psi_2 = \operatorname{cn} \delta \psi \quad ,$$

with modulus $k = \frac{1}{\sqrt{2}}$.

To find Ψ_1 we multiply (27b) through by $\frac{1}{\Psi_2^2}$ and integrate, to obtain

$$\Psi_1 = \Psi_2^2 \int \frac{d\psi}{\Psi_2^3}$$

which, on substituting for Ψ_2 and integrating, yields

$$\Psi_1 = \frac{1}{\delta} \operatorname{sn} \delta\psi \ \operatorname{dn} \delta\psi \quad ,$$

provided we assume $\Psi_1 (0) = 0$.

From (27a) we find, on substituting for Ψ_1, Ψ_2 as given above, that

$$\Psi_0' = \frac{1}{2\delta^2} \left(\operatorname{nc} \delta\psi - \operatorname{cn}^3 \delta\psi \right) \quad ,$$

and this, with the aid of the formulas(11)

$$\int \operatorname{cn}^3 u \, du = \operatorname{sn} u \operatorname{dn} u \ ; \quad \int \operatorname{nc} u \, du = \sqrt{2} \ln \frac{\operatorname{sn} u + \sqrt{2} \operatorname{dn} u}{\sqrt{2} \operatorname{cn} u}$$

yields

$$\Psi_0 = \frac{1}{\sqrt{2}\,\delta^3} \ln \frac{\operatorname{sn} \delta\psi + \sqrt{2} \operatorname{dn} \delta\psi}{\sqrt{2} \operatorname{cn} \delta\psi} - \frac{\operatorname{sn} \delta\psi \ \operatorname{dn} \delta\psi}{2\delta^3} \quad ,$$

provided, we again assume $\Psi_0 (0) = 0$.

On substituting for Ψ_0, Ψ_1, Ψ_2 in (26) we find the solution

$$(28) \qquad \theta = \frac{1}{2\delta^3} \left[\sqrt{2} \ln \frac{\operatorname{sn} \delta\psi + \sqrt{2} \operatorname{dn} \delta\psi}{\sqrt{2} \operatorname{cn} \delta\psi} - \operatorname{sn} \delta\psi \operatorname{dn} \delta\psi \left(1 - \delta^2 \sigma\right)^2 \right]$$

to (6) for K as given in (21).

It seems unlikely that this solution could be obtained from Frankl's equation by any method which depends upon seeking for solutions explicitly of the form $\psi = \psi(\theta, \sigma)$.

It is interesting to note that if we let δ approach 0 in (21), (28), we obtain

$$K = \sigma \quad , \qquad \theta = \sigma\psi + \frac{\psi^3}{3} \quad ,$$

i.e. the solution to Tricomi's equation treated by Martin and Thickstun[(3)].

6. <u>The Mapping from the (σ, ψ)-plane to the Physical Plane.</u> A computation based on (7) reveals that

$$J = \frac{\partial(x, y)}{\partial(\sigma, \psi)} = \frac{K + \theta_\sigma^2}{\theta_\psi} \mathcal{H}_0 \mathcal{H}_{0\sigma}$$

which for K as in (21), and for the special solution (28), reduces to

$$J = \frac{cn\, \delta\psi}{\rho\, g^2} \tag{29}$$

We study the mapping upon the physical plane of the region

$$\bar{\sigma} < \sigma < \bar{\bar{\sigma}} \; ; \quad -\frac{\mathcal{K}}{\delta} < \psi < \frac{\mathcal{K}}{\delta} \, , \quad \mathcal{K} = 1.85407 \ (approx.)$$

where $\mathcal{K}$ is the quarter-period of the elliptic function $cn\, u$ with modulus $k = \frac{1}{\sqrt{2}}$. It is clear from (29) that $0 < J < +\infty$ at every point of this region inasmuch as from section 4, ρ and g remain finite and positive for $\bar{\sigma} < \sigma < \bar{\bar{\sigma}}$. Thus the mapping upon the physical plane is on-to-one locally although a region of the physical plane may be covered more than once. The streamlines and isovels in the physical plane are the transforms of the straight lines ψ = const., σ = const. in the (σ, ψ)-plane.

The required mapping

$$z = \int \frac{e^{i\theta}}{g} \left\{ cn\, \delta\psi \; d\sigma + \left[\frac{1}{\delta} sn\, \delta\psi \; dn\, \delta\psi \left(1 - \delta^2\sigma\right) + i\, \frac{1}{\rho} \right] d\psi \right\} \, , \tag{30}$$

where θ is given by (28), is obtained by substituting for θ_σ, θ_ψ from (28), for K from (21), for $\mathcal{H}$ from (10), and for $\frac{\mathcal{H}_\sigma}{\mathcal{H}}$ from (11) into (7).

To obtain a streamline in the physical plane the line integral in (30) is evaluated along the path OAP in Figure 4 for fixed A and variable B; to obtain an isovel the integration is carried out along the path OBP, with B fixed and A variable.

Since $z(\sigma,-\psi) = \bar{z}(\sigma,\psi)$, from (30), it is clear that the flow is symmetric with respect to the x-axis and we accordingly restrict our attention to the upper half of the physical plane.

To obtain the sonic line in the physical plane we set $\sigma = 0$ in (30), and find

(31) $$z = \frac{1}{g_*}\int_0^\psi e^{i\theta}\left[\frac{1}{\delta}\,\mathrm{sn}\,\delta\psi\,\mathrm{dn}\,\delta\psi + i\,\frac{1}{\rho_*}\right]d\psi$$

where, from (28),

$$\theta = \frac{1}{2\delta^3}\left\{\sqrt{2}\,\ln\frac{\mathrm{sn}\,\delta\psi + \sqrt{2}\,\mathrm{dn}\,\delta\psi}{\sqrt{2}\,\mathrm{cn}\,\delta\psi} - \mathrm{sn}\,\delta\psi\,\mathrm{dn}\,\delta\psi\right\}.$$

Thus we obtain parametric equations for the sonic line with ψ serving as parameter.

If we denote the inclination of the tangent to the sonic line by ϕ

$$\phi = \arg z_\psi = \theta + \beta$$

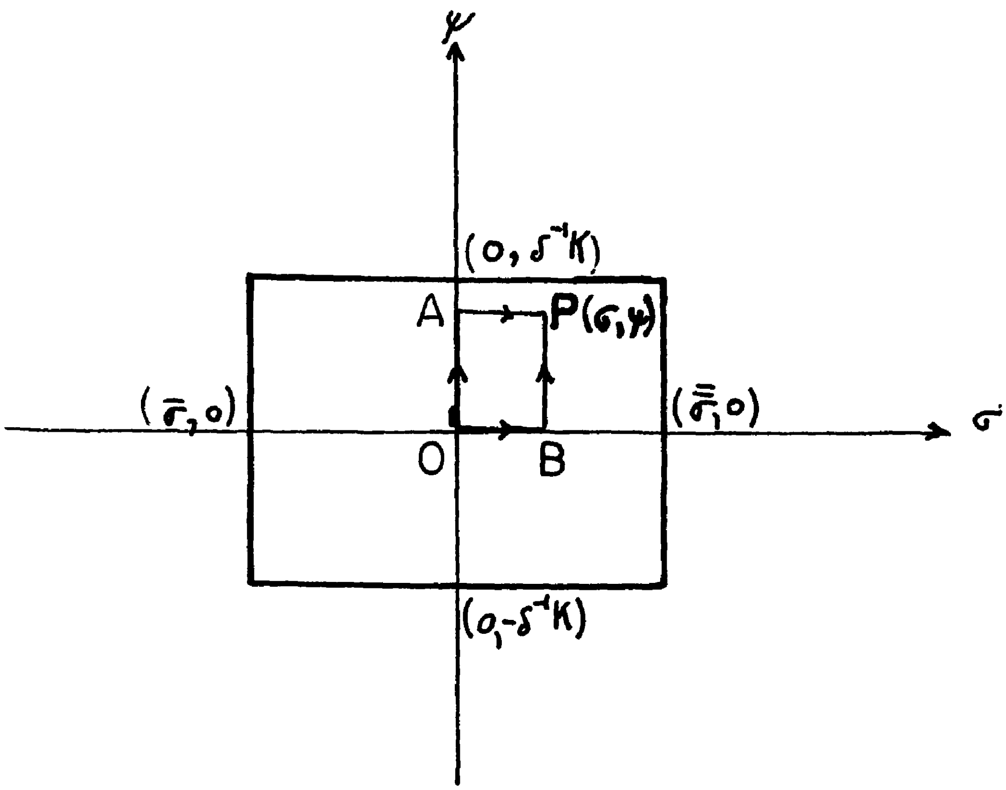
where
$$\beta = \mathrm{arc\,cot}\left(\frac{\rho_*}{\delta}\,\mathrm{sn}\,\delta\psi\,\mathrm{dn}\,\delta\psi\right).$$

The derivatives of θ and β with respect to ψ are given by

$$\theta_\psi = \frac{1-\mathrm{cn}^4\delta\psi}{2\delta^2\,\mathrm{cn}\,\delta\psi}, \quad \beta_\psi = \frac{-\delta^2\rho_*\,\mathrm{cn}^3\delta\psi}{\delta^2+\rho_*^2\,\mathrm{sn}^2\delta\psi\,\mathrm{dn}^2\delta\psi}$$

When $\psi = 0$, $\beta = \frac{\pi}{2}$, $\theta = 0$ and $\phi = \frac{\pi}{2}$. As ψ varies from 0 to $\frac{K}{\delta}$, θ increases monotonely from 0 to $+\infty$ while β decreases monotonely from $\frac{\pi}{2}$ to the first quadrant angle arc cot $\frac{\rho_*}{\delta\sqrt{2}}$.

It follows that ϕ must decrease



ψ
$(0, \delta^{-1}K)$
A
$P(\sigma, \psi)$
$(\bar{\sigma}, 0)$
$(\bar{\bar{\sigma}}, 0)$
σ
O
B
$(0, -\delta^{-1}K)$

FIGURE 4

until a point of inflection is reached and thereafter increases without limit. The length of the sonic line measured from the origin is given by

$$s = \frac{1}{\rho_* \gamma_* \delta} \int_0^{\psi} \sqrt{\rho_*^2 sn^2 \delta t \, dn^2 \delta t + \delta^2} \quad dt \,,$$

from which we conclude that the length of the curve cannot exceed

$$\frac{\mathcal{K}}{\rho_* \gamma_* \delta^2} \sqrt{\frac{\rho_*^2 + 2\delta^2}{2}}$$

Hence the sonic line spirals into a finite point as is shown in Figure 5.

To obtain the streamline $\psi = 0$ we set $\psi = 0$ in (30) to obtain

$$x = \int_0^{\sigma} \mathcal{H}(t) \; dt = x(\sigma)$$

i.e., the segment $\bar{x} < x < \bar{\bar{x}}$ of the x-axis where $\bar{x} = x(\bar{\sigma})$ and $\bar{\bar{x}} = x(\bar{\bar{\sigma}})$ are finite points, since the corresponding areas under the curve $\mathcal{H} = \mathcal{H}(\sigma)$ in Figure 1 are finite.

To obtain an isovel $\sigma = \sigma_1$ = constant we set $\sigma = \sigma_1$ in (30) and integrate along OBP in Figure 4 with OB = σ_1 and A variable. This yields

$$z = x(\sigma_1) + \mathcal{H}(\sigma_1) \int_0^{\psi} e^{i\theta} \left[\frac{1}{\delta} sn \, \delta\psi \, dn \, \delta\psi \left(1 - \delta^2 \sigma_1\right) + i \frac{1}{\rho(\sigma_1)} \right] d\psi .$$

Then

$$\arg z_{\psi} = \theta + \beta = \phi$$

where

$$\theta = \theta(\sigma_1, \psi) \quad , \; \beta = \operatorname{arc\,cot} \left[\frac{1}{\delta} \rho(\sigma_1) \, sn \, \delta\psi \, dn \, \delta\psi \left(1 - \delta^2 \sigma_1\right) \right] .$$

By an argument similar to the one employed in studying the sonic line we find that all the isovels are spiral in character and intersect the x-axis at right angles. The isovel $\sigma = \frac{1}{\delta^2}$, shown in Figure 5, is orthogonal to all the streamlines since $\beta = \frac{\pi}{2}$ at every point of this curve.

A streamline $\psi = \psi_1$ = constant is found by setting $\psi = \psi_1$ in (30) and integrating along OAP in Figure 4 with OA = ψ_1 and B variable.

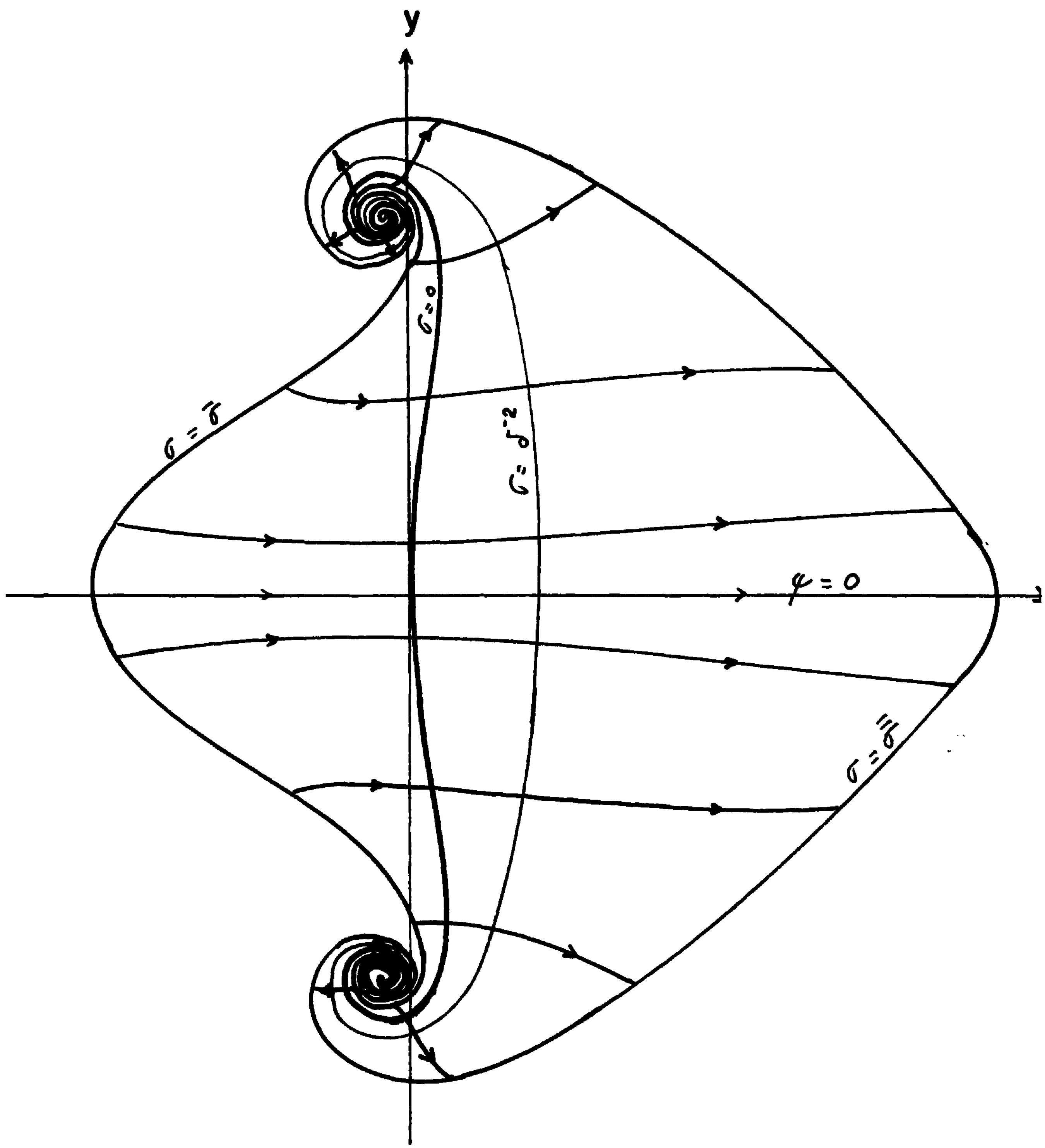


FIGURE 5

We find

$$z = z_1 + cn\,\delta\psi_1 \int_0^{\sigma} \mathcal{H}\, e^{i\theta}\, d\sigma$$

where θ is obtained from (28) by setting $\psi = \psi_1$ and where z_1 is the point on the sonic line corresponding to $\psi = \psi_1$.

From (28) we find that

$$\theta_\sigma = \frac{1}{2\delta}\, sn\,\delta\psi\, dn\,\delta\psi\, (1 - \delta^2\sigma).$$

Thus along the streamline $\psi = \psi_1$, θ increases as σ varies from $\bar{\sigma}$ to $\frac{1}{\delta^2}$ and then decreases until $\sigma = \bar{\bar{\sigma}}$.

The arc length s along $\psi = \psi_1$ measured from $z = z_1$ to an arbitrary isovel $\sigma = \sigma_0$ is given by

$$s = x_0\, cn\,\delta\psi_1 \quad , \quad x_0 = \int_0^{\sigma_0} \mathcal{H}(t)\, dt$$

where x_0 is the distance along the x-axis between the sonic line and the isovel $\sigma = \sigma_0$. It is apparent that, as ψ_1 approaches $\frac{\mathcal{K}}{\delta}$, the two isovels approach each other and consequently all isovels spiral into the same point.

The flow begins at the "starting line" $\sigma = \bar{\sigma}$ along which the speed is infinite. The flow particles move toward the sonic line at supersonic speed after which they move toward the isovel $\sigma = \bar{\bar{\sigma}}$ at subsonic speed upon which the density ρ becomes infinite in view of (11). The flow is illustrated in Figure 5.

APPENDIX

It is interesting to compare the graphs of the functions $K = K(\sigma)$ for a T_1-gas and a T_2-gas with the graph for a polytropic gas. For a T_1-gas $K(\sigma) = \sigma$ and for a T_2-gas $K(\sigma)$ is given by (21). To graph $K = K(\sigma)$ for a polytropic gas we have from (24)

$$K = \frac{1}{\rho^2}\left(1 - \frac{\frac{d\rho}{dp}}{\frac{d\sigma}{dp}}\right)$$

and then employ (4), (13), and (14) to obtain

$$K = \frac{(1+n)\, p^{1-n} - (1-n)\, k n\, \hat{\rho}^2}{(1-n)\, k^2\, p^{1+n}} = K(p)$$

$$\sigma = \int_{p_*}^{p} \frac{dp}{\rho^2} = \sigma(p)$$

These are parametric equations of the required curve with the pressure p serving as parameter. When $p = 0$, $K = -\infty$ and as p increases to p_0 (stagnation pressure) K increases monotonely from

$-\infty$ to $\frac{1}{k^2 p_0^{2n}}$; when $p = p_*$, $K = 0$.

Since $\frac{d\sigma}{dp} = \frac{1}{\rho^2}$ we see that as p increases from $p = 0$ to $p = p_0$, σ increases monotonely from a negative constant to $+\infty$; when $p = p_*$, $\sigma = 0$. Therefore, as σ varies from a negative constant to $+\infty$, $K(\sigma)$ for a polytropic gas increases monotonely from $-\infty$ to $\frac{1}{k^2 p_0^{2n}}$. The graphs of the three functions $K(\sigma)$ are exhibited in Figure 6. The shape of the curve $K = K(\sigma)$ for a polytropic gas may suggest other possible functions $K(\sigma)$ as approximations.

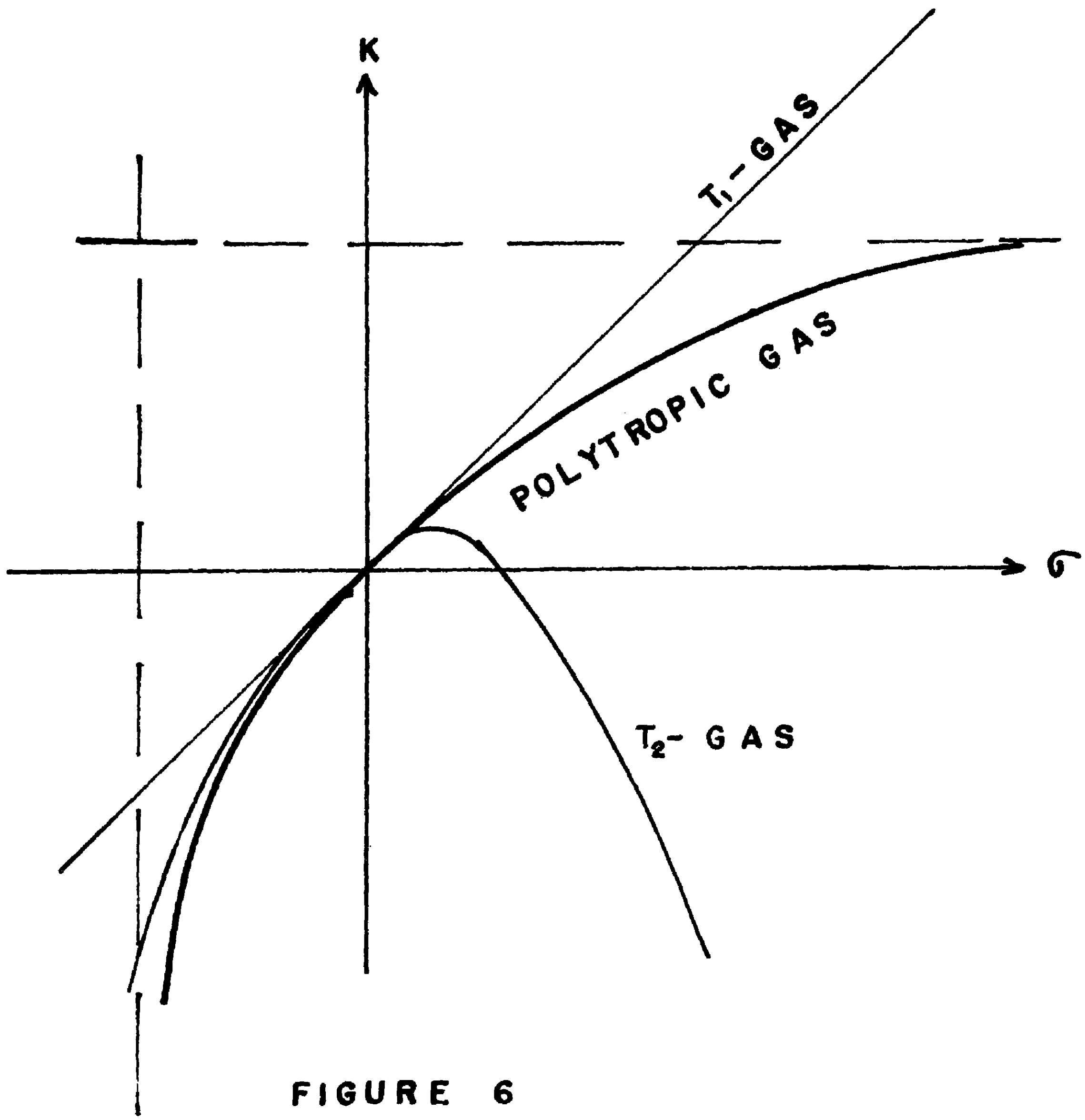


FIGURE 6

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