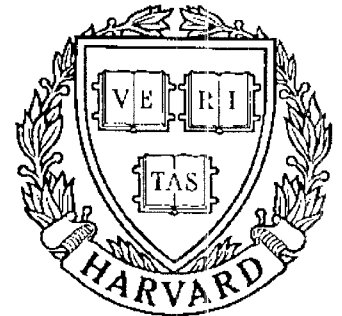


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Internal Model Control: Robust Digital Controller Synthesis for Multivariable Open-Loop Stable or Unstable Processes

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Internal Model Control: Robust Digital Controller Synthesis for Multivariable Open-Loop Stable or Unstable Processes

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Abstract

The two-step Internal Model Control (IMC) procedure is presented for the synthesis of multivariable discrete controllers. This paper adds the following features to the IMC design methodology: (i) Extension to open-loop unstable plants. (ii) Design of the first-step (no model error) IMC controller so that the L_2 -error (sum of squared errors) is minimized for every setpoint or disturbance vector in a designer-specified set and their linear combinations. (iii) The second-step (model-plant mismatch) multivariable low-pass filter is designed for robust stability and performance by minimizing a non-conservative robustness measure, the Structured Singular Value. (iv) The potential problem of intersample rippling is avoided by introducing a modification in the first-step controller and formulating the robust performance objective for the continuous plant output.

1 Preliminaries

1.1 Sampled-Data Feedback

The block diagram of a typical sampled-data feedback loop is shown in Fig. 1A. Thick lines are used to represent the paths along which the signals are continuous. $C(z)$ denotes

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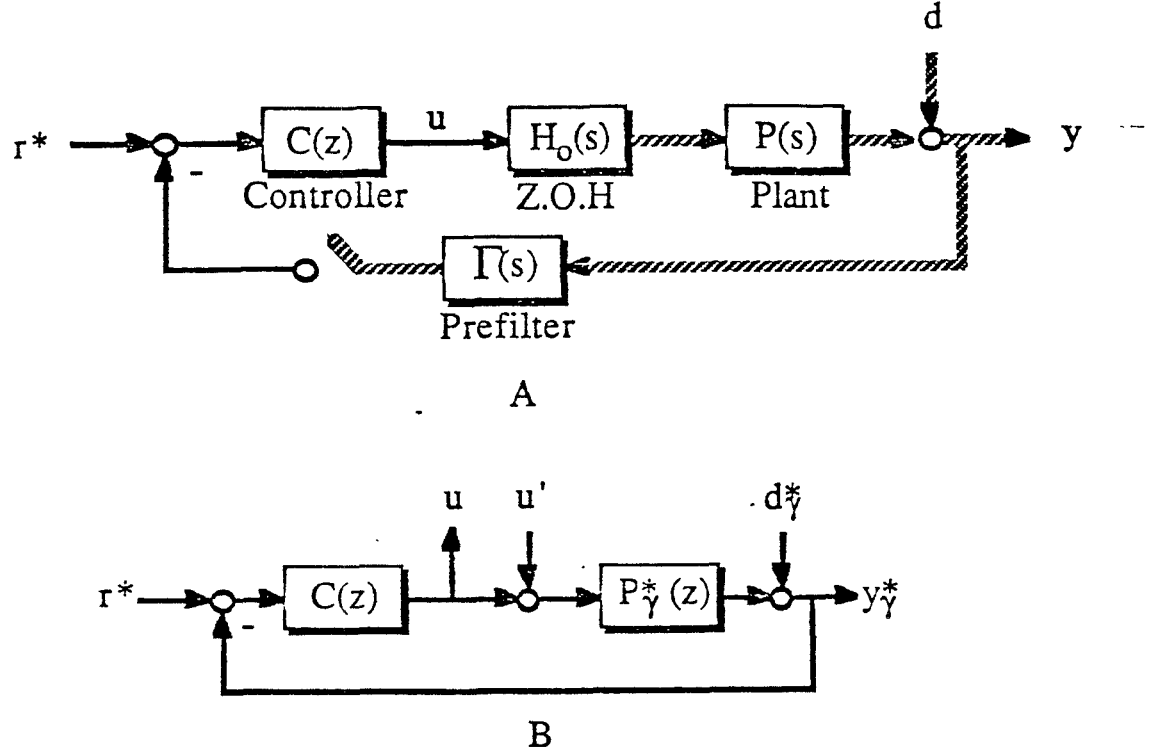


Figure 1: Block diagram of computer controlled system A: Sampled-data structure with thick lines indicating analog signals. B: Discrete structure with all signals digital.

the discrete controller implemented through a digital computer. $H_0(s)$ models the D/A converter. We have

$$H_0(s) = h_0(s)I = \frac{1 - e^{-sT}}{s}I$$

where $h_0(s)$ is the zero-order hold and I is the identity matrix with dimension equal to the number of controller outputs. The block $\Gamma(s)$ represents an anti-aliasing prefilter. The problem of aliasing is discussed in the literature (e. g., Astrom and Wittenmark, 1984). Assuming that the same sampling time is used for all the process outputs, it is reasonable to choose $\Gamma(s) = \gamma(s)I$, where I has dimension equal to the number of the process outputs. $P(s)$ is the continuous system transfer matrix.

When the continuous output y is not observed directly but after the prefilter and only at the sampling points, then Fig. 1A can be simplified to Fig. 1B, where

$$d_\gamma^*(z) = \mathcal{Z}\mathcal{L}^{-1}\{\Gamma(s)d(s)\} = \mathcal{Z}\mathcal{L}^{-1}\{\gamma(s)d(s)\}$$

$$y_\gamma^*(z) = \mathcal{Z}\mathcal{L}^{-1}\{\Gamma(s)y(s)\} = \mathcal{Z}\mathcal{L}^{-1}\{\gamma(s)y(s)\}$$

and all signals are discrete. We define

$$P_\gamma^*(z) = \mathcal{Z}\mathcal{L}^{-1}\{\Gamma(s)P(s)H_0(s)\} = \mathcal{Z}\mathcal{L}^{-1}\{\gamma(s)P(s)h_0(s)\} \quad (1)$$

$$P^*(z) = \mathcal{Z}\mathcal{L}^{-1}\{P(s)H_0(s)\} = \mathcal{Z}\mathcal{L}^{-1}\{h_0(s)P(s)\}$$

Assuming that no unstable poles of the continuous process have become unobservable after sampling, the internal stability of the system in Fig. 1A can be assessed from the internal stability of the system in Fig. 1B.

Theorem 1 (based on Callier and Desoer, 1982) *The discrete system in Fig. 1B is internally stable if and only if the transfer matrix in (2)*

$$\begin{pmatrix} y_\gamma^* \\ u \end{pmatrix} = \begin{pmatrix} P_\gamma^* C (I + P_\gamma^* C)^{-1} & (I + P_\gamma^* C)^{-1} P_\gamma^* \\ C (I + P_\gamma^* C)^{-1} & -C (I + P_\gamma^* C)^{-1} P_\gamma^* \end{pmatrix} \begin{pmatrix} r^* \\ u' \end{pmatrix} \quad (2)$$

is stable – i.e. if and only if all its poles are strictly inside the unit circle.

1.2 Internal Model Control

The block diagram of the sampled-data Internal Model Control (IMC) structure is shown in Fig. 2A, where

$$\tilde{P}_\gamma^*(z) = \mathcal{ZL}^{-1}\{\Gamma(s)\tilde{P}(s)H_0(s)\} = \mathcal{ZL}^{-1}\{\gamma(s)\tilde{P}(s)h_0(s)\} \quad (3)$$

$$\tilde{P}^*(z) = \mathcal{ZL}^{-1}\{\tilde{P}(s)H_0(s)\} = \mathcal{ZL}^{-1}\{\tilde{P}(s)h_0(s)\}$$

and $\tilde{P}(s)$ is the Laplace transfer matrix for the process model. When the IMC controller Q and the feedback controller C are related through

$$C = Q(I - \tilde{P}_\gamma^* Q)^{-1} \quad (4)$$

$$Q = C(I + \tilde{P}_\gamma^* C)^{-1}$$

then $u(z)$ and $y(s)$ react to inputs $r^*(z)$ and $d(s)$ in exactly the same way for both the classic feedback and the IMC structure.

Figure 2B is a different representation of the sampled-data IMC structure, which is equivalent to that in Fig. 2A, but not suitable for computer implementation because of the presence of the continuous model $\tilde{P}(s)$. If only the sampled signals are of interest, then Fig. 2A is equivalent to Fig. 2C, where all signals are digital.

Some advantages of using the IMC structure can be seen by examining the structure for $P = \tilde{P}$ and $P \neq \tilde{P}$:

$P = \tilde{P}$.

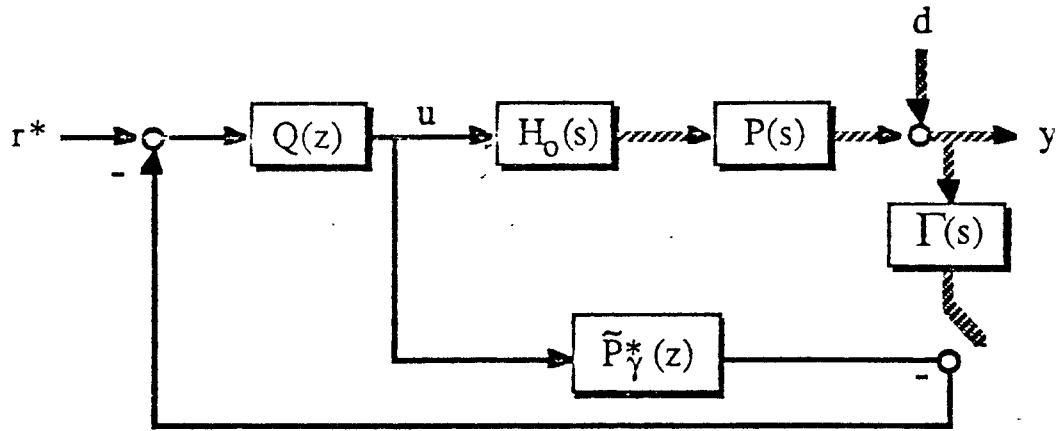
In this case the IMC structure becomes effectively open-loop and the design of Q is simplified. Note that the IMC controller is identical to the parameter of the Q -parametrization (Zames, 1981). Also the addition of a diagonal filter F by writing

$$Q = \tilde{Q}F$$

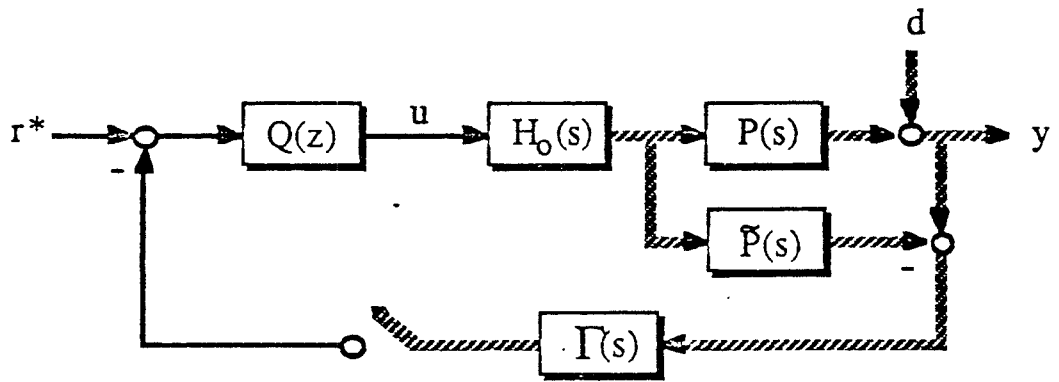
introduces parameters (the filter time constants) which can be used for adjusting on-line the speed of response for each process output.

$P \neq \tilde{P}$.

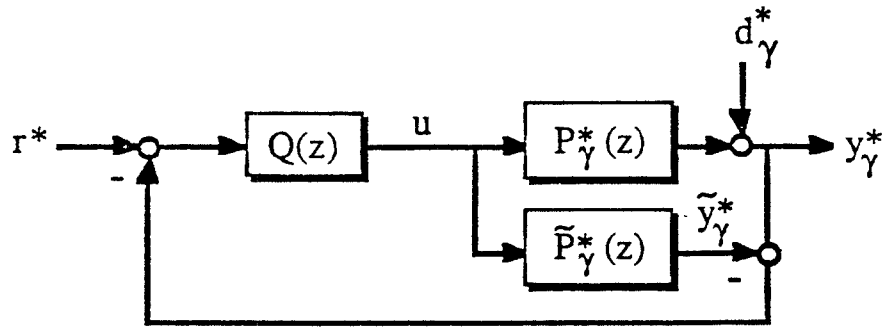
The model-plant mismatch generates a feedback signal in the IMC structure which can cause performance deterioration or even instability. Since the relative modeling error is larger at higher frequencies, the addition of the low-pass filter F adds robustness characteristics into the control system.



A



B



C

Figure 2: IMC structure: A: Sampled-data structure; B: Structure equivalent to (A) but not implementable; C: Discrete structure (all signals discrete).

Hence the IMC structure gives rise rather naturally to a two-step design procedure:

Step 1: Design \tilde{Q} , assuming $P = \tilde{P}$.

Step 2: Design F so that the closed-loop characteristics that \tilde{Q} was designed to produce in Step 1, are preserved in the presence of model-plant mismatch ($P \neq \tilde{P}$).

In previous IMC work, Garcia and Morari (1985a) proposed for open-loop stable plants the direct design of the closed-loop transfer function. In Step 1 this approach was rigorously quantified in its general form by Zafriou and Morari (1987b) by using the concept of zero directions, who also took into account intersample rippling and modified the approach so that time delays and outside the unit circle (UC) zeros were considered in one single step. However, the extension of the above approach or of the impulse response formulation of the problem (Garcia and Morari, 1985b) to open-loop unstable systems is very awkward. The method proposed in this paper takes care of both open-loop stable and unstable plants in a general way by minimizing the Sum of Squared Errors (SSE) for a set of setpoints or disturbance vectors and their linear combinations.

With respect to the second step, this paper proposes the design of the IMC filter by minimizing an appropriate robustness condition. Potential modeling errors, described as uncertainty associated with the process model, can appear in different forms and places in a multivariable model. This fact makes the derivation of non-conservative conditions that guarantee robustness with respect to model-plant mismatch quite difficult. The Structured Singular Value (SSV), introduced by Doyle (1982), has gained a lot of popularity because it takes into account the structure of the model uncertainty and it allows the non-conservative quantification of the concept of robust performance. The objective function for the minimization problem is formulated in such a way so that the continuous plant output is considered and the problem of intersample rippling does not occur.

1.3 Structured Singular Value

For a constant complex matrix M , the definition of the SSV $\mu_{\Delta}(M)$ depends also on a certain set Δ . Each element Δ of Δ is a block diagonal complex matrix with a specified dimension for each block, i.e.

$$\Delta = \{diag(\Delta_1, \Delta_2, \dots, \Delta_n) | \Delta_j \in \mathbb{C}^{m_j \times m_j}\}$$

Then

$$\frac{1}{\mu_{\Delta}(M)} = \min_{\Delta \in \Delta} \{\bar{\sigma}(\Delta) | det(I - M\Delta) = 0\}$$

and $\mu_{\Delta}(M) = 0$ if $det(I - M\Delta) \neq 0 \quad \forall \Delta \in \Delta$. Note that $\bar{\sigma}$ is the maximum singular value of the corresponding matrix.

Details on how the SSV can be used for studying the robustness of a control system can be found in Doyle (1985), where a discussion of the computational problems is also given. For three or fewer blocks in each element of Δ , the SSV can be computed from

$$\mu_{\Delta}(M) = \inf_{D \in \mathbf{D}} \bar{\sigma}(DM D^{-1}) \quad (5)$$

where

$$\mathbf{D} = \{diag(d_1 I_{m_1}, d_2 I_{m_2}, \dots, d_n I_{m_n}) | d_j \in \mathbb{R}_+\}$$

and I_{m_j} is the identity matrix of dimension $m_j \times m_j$. For more than three blocks, (5) still gives an upper bound for the SSV.

2 Step 1: Design of the Nominal IMC Controller $\tilde{Q}(z)$

Throughout this section the assumption is made that $P = \tilde{P}$. Details on the definition of multivariable zeros and poles and their degrees and orders can be found in the literature (Desoer and Schulman, 1974). In general a pole of an element of $P(z)$ is also a pole of $P(z)$ and the roots of $\det[P(z)] = 0$ are zeros of the matrix $P(z)$.

2.1 Parametrization of All Stabilizing Q s

When there is no modeling error, substitution of (4) into (2) yields for the internal stability matrix

$$S = \begin{pmatrix} P_\gamma^* Q & (I - P_\gamma^* Q) P_\gamma^* \\ Q & -Q P_\gamma^* \end{pmatrix} \quad (6)$$

All four transfer matrices in (6) have to be stable for nominal stability of the classic feedback structure in Fig. 1B.

Theorem 2 provides a parametrization of all proper stabilizing controllers in terms of a stable transfer matrix Q_1 . This parametrization is equivalent to other parametrizations that can be found in the literature, but it is stated in a form that is convenient for use in the proofs of the theorems of Sec. 2.3. We will make the following two assumptions.

Assumption A1. *If π is a pole of \tilde{P}^* outside the UC, then (a) The order of π is equal to 1 and (b) \tilde{P} has no zeros at $z = \pi$.*

Assumption A1a is made solely to simplify the notation. Assumption A1b is not very restrictive because the presence of a zero at $z = \pi$ implies an exact cancellation in $\det(\tilde{P}^*(z))$, which is a nongeneric property — i.e., it does not happen after a slight perturbation in the coefficients of \tilde{P}^* is introduced.

Assumption A1a is not made for poles at $z = 1$ because more than one such pole may appear in an element of \tilde{P}^* , introduced by capacitances present in the process. However, the following assumption is made which is true for all practical process control problems.

Assumption A2. *Any poles of \tilde{P}^* or P^* on the UC are at $z = 1$. Also \tilde{P}^* has no zeros on the UC.*

These two assumptions allow us to derive the following theorem.

Theorem 2 *Assume that Assns. A1 and A2 hold and that $Q_0(z)$ is a proper transfer matrix that stabilizes \tilde{P}^* — i.e., it yields a stable S . Then all proper Q s that make S stable are given by*

$$Q(z) = Q_0(z) + Q_1(z) \quad (7)$$

where $Q_1(z)$ is any proper and stable transfer matrix such that $P^*(z)Q_1(z)P^*(z)$ is stable.

Proof. See Appendix A.

2.2 H_2 -Performance Objectives

We define as L_2^n the Hilbert space of complex valued vector functions $y(z)$ with n elements, defined on the unit circle and square integrable with respect to θ — i.e., for which the

following quantity is finite:

$$\|y\|_2 = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} y(e^{i\theta})^H y(e^{i\theta}) d\theta \right)^{1/2} \quad (8)$$

where the superscript H indicates complex conjugate transpose. Note that (8) defines a norm on L_2^n . In the case where $y(z)$ has no poles outside the UC, Parseval's theorem yields a time domain expression for $\|y\|_2$:

$$\|y\|_2 = \left(\sum_{k=0}^{\infty} y_k^T y_k \right)^{1/2}$$

where T indicates transpose. For matrix valued functions $G(z)$ of dimensions $n \times m$, the space $L_2^{n \times m}$ is defined similarly with norm

$$\|G\|_2 = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \text{trace}[G(e^{i\theta})^H G(e^{i\theta})] d\theta \right)^{1/2}$$

The spaces H_2^n and $H_2^{n \times m}$ are defined as subspaces of the corresponding L_2 spaces (see Appendix B).

The H_2 performance objective is to minimize over all stabilizing \tilde{Q} the weighted sum of squared errors for the response to an external input or a set of inputs of interest. Several H_2 -type objective functions will be considered.

From Fig. 1A we get for $P = \tilde{P}$

$$y(s) = h_0(s)P(s)Q(e^{sT})(r^*(e^{sT}) - d_\gamma^*(e^{sT})) + d(s) \quad (9)$$

where $z = e^{sT}$ is used. Sampling of (9) yields

$$y^*(z) = P^*(z)Q(z)(r^*(z) - d_\gamma^*(z)) + d^*(z) \quad (10)$$

from which we can obtain the *pulse* sensitivity function, relating $e^*(z)$ to $r^*(z)$ and $d^*(z)$ (for $\gamma(s) = 1$) or $d_\gamma^*(z)$ (when d_γ^* is substituted for d^* in (10)):

$$e^*(z) = \tilde{E}^*(z)v^*(z) \triangleq (I - P^*(z)Q(z))v^*(z)$$

where v^* is an external input (r^* or d^* or d_γ^*). Define

$$\Phi(v^*) \triangleq \|W e^*\|_2^2 = \|W \tilde{E}^* v^*\|_2^2 = \|W(I - P^* \tilde{Q})v^*\|_2^2 \quad (11)$$

where W is a frequency dependent matrix or scalar weight. One objective could be

Objective O1:

$$\min_{\tilde{Q}} \Phi(v^*)$$

for a particular input $v^* = (v_1 \ v_2 \ \dots \ v_n)^T$.

A more meaningful objective would be to minimize $\Phi(v^*)$ not just for one input vector v^* , but for every input in a set \mathcal{V} :

$$\mathcal{V} = \{v^i(z) | i = 1, \dots, n\}$$

where $v^1(z), \dots, v^n(z)$ are vectors that describe the directions and the frequency content of the expected external system inputs and n is the dimension of P , which is assumed to be square. Thus, the objective is

Objective O2:

$$\min_{\tilde{Q}} \Phi(v^*) \quad \forall v^* \in \mathcal{V}$$

However a linear time invariant $\tilde{Q}(z)$ that solves O2 does not always exist. The conditions necessary for its existence are expressed in Thm. 5. An alternative is

Objective O3:

$$\min_{\tilde{Q}} [\Phi(v^1) + \dots + \Phi(v^n)]$$

In this case the objective is the sum of the squared errors caused by each of the v^i 's, when applied separately.

For every external input v^* that will be considered in this chapter the following assumptions will be made. These assumptions are similar to the ones in Section 2.1.

Assumption A3. *Every nonzero element of v^* includes all the poles of \tilde{P}^* outside the UC, each with degree 1, and those are the only poles of v^* outside the UC.*

Assumption A4. *Let ℓ_i be the maximum number of poles at $z = 1$ of any element in the i^{th} row of P^* . Then the i^{th} element of v^* , v_i , has at least ℓ_i poles at $z = 1$. Also v^* has no other poles on the UC and its elements have no zeros on the UC.*

Assumptions A3 and A4 are not restrictive when v^* is an output disturbance d^* , generated by an input disturbance that has passed through the process. In that case d^* usually includes all the unstable process poles as postulated by Assns. A3 and A4. Note that the control system will also reject other disturbances with fewer unstable poles, without producing steady-state offset. Assumption A4 is different for poles at $z = 1$ because their number in each row of \tilde{P}^* can be different (capacitancies may be associated only with certain process outputs). Also, the output disturbance may have more poles at $z = 1$ than the process.

Assumptions A3 and A4 may be restrictive for setpoints. Then, however, we can employ the two-degree-of-freedom structure which allows us to disregard the existence of any unstable poles of \tilde{P}^* and Assns. A3 and A4 are not needed. A discussion of the properties of this structure can be found in the literature (Vidyasagar, 1985; Zafiriou and Morari, 1987a).

For the case, where a set \mathcal{V} of inputs is considered, define

$$V \triangleq (v^1 \quad v^2 \quad \dots \quad v^n)$$

where v^1, \dots, v^n satisfy Assn. A3. An additional assumption on V is needed:

Assumption A5. *V has no zeros at the location of its unstable poles or on the UC and V^{-1} cancels the unstable poles of \tilde{P}^* in $V^{-1}\tilde{P}^*$.*

Assumptions A3 and A4 for each column of V do not necessarily imply that Assn. A5 is satisfied. A matrix V which satisfies Assn. A5 can be easily constructed. One way is to obtain V as \tilde{P}^* times a matrix with no open RHP poles and no finite zeros on the imaginary axis. This case corresponds to the physically meaningful situation, where the output disturbances v^i are generated by disturbances at the plant input. Another way is to use a diagonal V , in which case satisfaction of Assns. A3 and A4 by every column of V implies satisfaction of Assn. A5.

Note that Assns. A1-A5 are only relevant for unstable plants and impose no restrictions on stable plants.

2.3 H_2 -Optimal Controller

The plant P^* can be factored into an allpass portion P_A^* and a minimum phase portion P_M^* :

$$P^* = P_A^* P_M^* \quad (12)$$

Here P_A^* is stable and such that $P_A^*(e^{i\theta})^H P_A^*(e^{i\theta}) = I$. Also $(P_M^*)^{-1}$ is stable. P_M^* has the additional property that both P_M^* and $(P_M^*)^{-1}$ are proper. In the case where P^* is scalar, this factorization can be easily accomplished as follows.

$$P_A^*(z) = z^{-N} \prod_{j=1}^h \frac{(1 - \zeta_j^{-1})(z - \zeta_j)}{(1 - \zeta_j)(z - \zeta_j^{-1})} \quad (13)$$

where $\zeta_j, j = 1, \dots, h$ are the zeros of $P^*(z)$ which are outside the UC. The positive integer N is chosen such that $z^N P^*(z)$ is semi-proper, i. e., its numerator and denominator have the same degree.

In the general multivariable case, this “inner-outer factorization” can be accomplished as described in Section 2.6.

Objective O1: Specific Input

Let $v_0(z)$ be the scalar allpass with the property $v_0(1) = 1$, which includes the *common* zeros outside the UC and the *common* delays of the elements of $v^*(z)$. Write

$$v^*(z) = v_0(z) \hat{v}(z) \quad (14)$$

where $\hat{v}(z)$ is a vector. Hence \hat{v} is proper with at least one element semi-proper and there is no point z outside the UC where \hat{v} becomes identically zero.

Theorem 3 *Assume that Assns. A1-A4 hold. Any stabilizing \tilde{Q} that solves Obj. O1 satisfies*

$$\tilde{Q} \hat{v} = z(W P_M^*)^{-1} \{z^{-1} W (P_A^*)^{-1} \hat{v}\}_* \quad (15)$$

where the operator $\{\cdot\}_*$ denotes that after a partial fraction expansion of the operand, only the strictly proper terms are retained except those corresponding to poles of $(P_A^*)^{-1}$. Furthermore, for $n \geq 2$ the number of stabilizing controllers that satisfy (15) is infinite. Guidelines for the construction of such a controller are given in the proof.

Proof. See Appendix B.

Note that not every \tilde{Q} satisfying (15) is necessarily a stabilizing controller.

Objectives O2 and O3: Set of v^ s.*

Factor V similarly to P^* (see Sec. 2.6):

$$V = V_M V_A \quad (16)$$

Theorem 4 Assume that Assns. A1-A5 hold. The controller

$$\tilde{Q} = z(WP_M^*)^{-1}\{z^{-1}W(P_A^*)^{-1}V_M\}_*V_M^{-1} \quad (17)$$

is the unique solution to O3. Here the operator $\{\cdot\}_*$ denotes that after a partial fraction expansion of the operand, only the strictly proper terms are retained except those corresponding to poles of $(P_A^*)^{-1}$.

Proof. See Appendix C

A more meaningful objective would be to solve Obj. O2. However a \tilde{Q} that solves Obj. O2 will also solve Obj. O3. Then from Thm. 4 it follows that if a solution to O2 exists, it is given by (17). Factor each of the v^i in the same way as in (14):

$$v^i(z) = v_0^i(z)\hat{v}^i(z)$$

Define

$$\hat{V} \triangleq (\hat{v}^1 \quad \hat{v}^2 \quad \dots \quad \hat{v}^n)$$

Theorem 5 Assume that Assns. A1-A5 hold.

- (i) If $\hat{V}(z)$ is non-minimum phase (i.e., \hat{V}^{-1} is unstable or improper), then there exists no solution to Obj. O2.
- (ii) If $\hat{V}(z)$ is minimum phase, then use of \hat{V} instead of V_M in (17) yields exactly the same \tilde{Q} , which also solves Obj. O2. In addition \tilde{Q} minimizes $\Phi(v^*)$ for any $v^*(z)$ that is a linear combination of v^i s that have the same v_0^i s.

Proof. See Appendix D.

The following corollary to Thm. 5 holds for a specific choice of V .

Corollary 1 Let

$$V = \text{diag}\{v_1, v_2, \dots, v_n\}$$

where $v_1(z), \dots, v_n(z)$ are scalars. Then use of \hat{V} instead of V_M in (17) yields exactly the same \tilde{Q} , which minimizes $\Phi(v^*)$ for the following n vectors:

$$v^* = \begin{pmatrix} v_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ v_2 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ v_n \end{pmatrix}$$

and their multiples, as well as for the linear combinations of those directions that correspond to v_i s with the same zeros outside the UC with the same degree and the same time delays.

Example 1 Minimum phase P^* .

$P^*(z)$ cannot be truly MP for a physical system. Even if the Laplace transfer matrix representing the continuous plant is MP but strictly proper, the discretized plant $P^*(z)$ will still have a delay of one unit because of sampling. Hence $P_A^* = z^{-1}I$, $P_M^* = zP^*$ and (17) yields for $W = \text{constant}$

$$\tilde{Q} = (P^*)^{-1}(I - KV_M^{-1}) \quad (18)$$

where K is the constant term in a partial fraction expansion of V_M . This is equal to the first non-zero matrix in the impulse response description of $V(z)$, which can be obtained by long division.

Consider the continuous MP system

$$P(s) = \frac{b}{-s + b}, \quad b > 0$$

and assume that a step disturbance acts at the process input, i.e., the continuous output disturbance is

$$d(s) = \frac{b}{s(-s + b)}$$

Then for a sampling time T we have

$$P^*(z) = \frac{1 - e^{bT}}{z - e^{bT}}$$

$$V(z) = \frac{(1 - e^{bT})z}{(z - 1)(z - e^{bT})}$$

$$V_M(z) = zV(z)$$

Note that $e^{bT} > 1$ since $b > 0$. We have $K = 1 - e^{bT}$. From (18) we obtain

$$\tilde{Q}(z) = \frac{(z - e^{bT})((1 + e^{bT})z - e^{bT})}{(1 - e^{bT})z^2}$$

□

2.4 Setpoint Prediction

In the case of setpoint tracking, future values of r^* are often known and supplied to the computer ahead of time. If at time t the setpoint value that is provided to the control algorithm as $\mathcal{Z}^{-1}\{r^*(z)\}$ is the one we wish the plant output to reach at time $t + NT$, then the objective function has to be modified to:

$$\Phi_N(r^*) = \|W(z^{-N}I - P^*\tilde{Q})r^*\|_2^2 \quad (19)$$

If the above objective function is used for Objs. O1, O2, O3, then the resulting expressions for the H_2 -optimal controller are the same as in Thms. 3, 4, and 5, but with the term z^{-N-1} instead of z^{-1} inside $\{\cdot\}_*$. All the steps in the proofs remain the same when (19) is used rather than (11).

2.5 Intersample Rippling

The H_2 -optimal controller minimizes the sum of squared errors and completely disregards the plant's output behavior between the sample points. Therefore the performance of the H_2 -optimal controller may be excellent at the sample points but may suffer from severe intersample rippling. This problem was demonstrated for scalar systems in Zafiriou and Morari (1985), where a modification was introduced to substitute poles in \tilde{Q} close to $(-1, 0)$ with poles at $z = 0$. The new \tilde{Q} was shown to be free of the problem of intersample

rippling and to combine desirable deadbeat type characteristics with those of the H_2 -optimal controller. This section extends the modification to MIMO systems and general open-loop stable and unstable plants. It should be pointed out that this modification is sufficient only if there are no open-loop oscillatory poles in the continuous plant transfer function, which have become unobservable after sampling.

Let $\tilde{Q}_H(z)$ be the H_2 -optimal \tilde{Q} obtained according to the previous sections. Also let $\delta(z)$ be the least common denominator of the elements of $\tilde{Q}_H(z)$, and $\kappa_i, i = 1, \dots, \rho$ be the roots of $\delta(z)$ close to $(-1, 0)$ (or in general with negative real part). Define

$$\tilde{q}_-(z) = z^{-\rho} \prod_{j=1}^{\rho} \frac{z - \kappa_j}{1 - \kappa_j}$$

Then \tilde{Q}_H is modified as follows:

$$\tilde{Q}(z) = \tilde{Q}_H(z) \tilde{q}_-(z) B(z) \quad (20)$$

where the scalar $B(z)$ is selected so that the matrix S (6) and $(I - P^* \tilde{Q})V$ remain stable. Let $\pi_i, i = 1, \dots, \xi$ be the unstable roots (including $\pi_1 = 1$) of the least common denominator of $P^*(z), V(z)$. Let the multiplicity of each of them be m_i . Note that the poles outside the UC have multiplicity one, according to Assns. A1 and A3. Remember also that according to Assns. A3 and A4, V has at least as many poles at $z = 1$ as P^* and that each pole of V outside the UC is also a pole of P^* . Then, since \tilde{Q}_H makes S and $(I - P^* \tilde{Q}_H)V$ stable, it follows that the requirements on $B(z)$ are:

$$\left. \frac{d^k}{dz^k} (1 - \tilde{q}_-(z) B(z)) \right|_{z=\pi_i} = 0, \quad k = 0, \dots, m_i - 1; \quad i = 1, \dots, \xi \quad (21)$$

We can write

$$B(z) = \sum_{j=0}^{M-1} b_j z^{-j}$$

where

$$M = \sum_{i=1}^{\xi} m_i$$

and then compute the coefficients $b_j, j = 0, \dots, M - 1$ from (21). Note that since none of the π_i s is 0 or ∞ , (21) is equivalent to

$$\left. \frac{d^k}{d\lambda^k} (1 - \tilde{q}_-(\lambda^{-1}) B(\lambda^{-1})) \right|_{\lambda=\pi_i^{-1}} = 0, \quad k = 0, \dots, m_i - 1; \quad i = 1, \dots, \xi \quad (22)$$

Both $\tilde{q}_-(\lambda^{-1})$ and $B(\lambda^{-1})$ are polynomials in λ and therefore their derivatives with respect to λ can be computed easily. Then (22) yields a system of M linear equations with M unknowns (b_0, b_1, \dots, b_{M-1}). The resulting controller \tilde{Q} combines the desirable properties of the H_2 -optimal controller and deadbeat type controllers.

Example 2 Intersample Rippling.

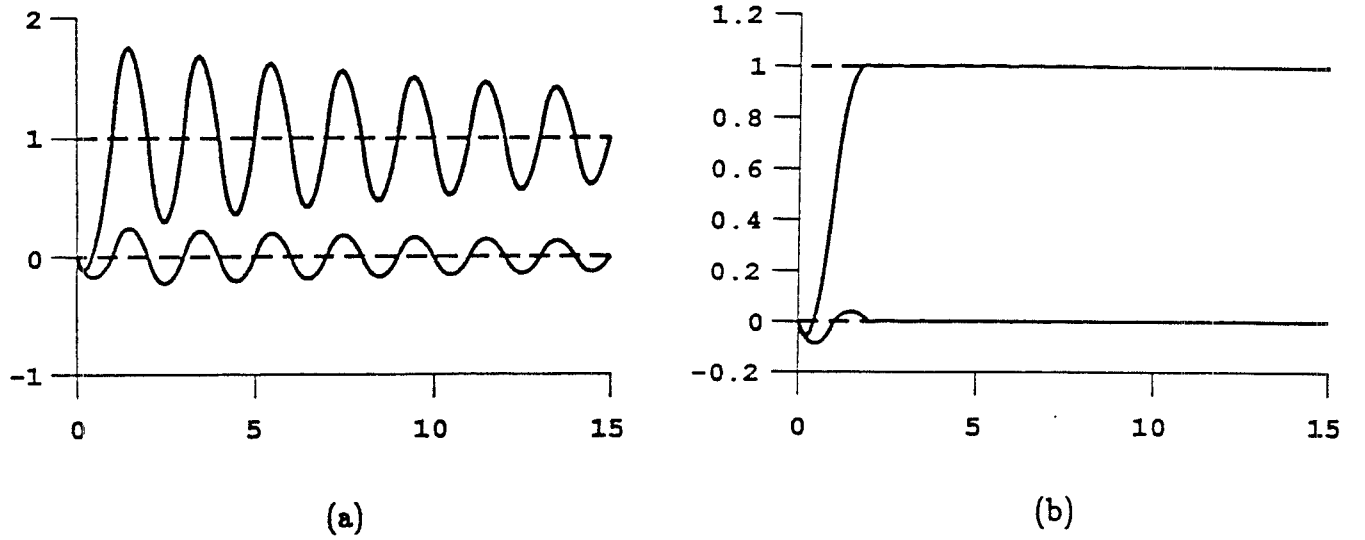


Figure 3: Response to unit setpoint change $r = (1/s, 0)^T$. A: H_2^* -optimal controller, B: IMC.

This example is presented to demonstrate the problem of intersample rippling in the H_2 -optimal controller and the modification discussed above. Consider the continuous system

$$P(s) = \begin{pmatrix} \frac{0.50}{s+1} & \frac{1.42}{6s+1} \\ \frac{1.00}{2s+1} & \frac{1.00}{4s+1} \end{pmatrix}$$

The discretized system (zero order hold included) for a sampling time of $T = 1$, is

$$P^*(z) = \begin{pmatrix} \frac{0.316}{z-0.368} & \frac{0.218}{z-0.846} \\ \frac{0.393}{z-0.607} & \frac{0.221}{z-0.779} \end{pmatrix} \quad (23)$$

Computation of the roots of $\det P(z) = 0$ shows that the system in (23) has two finite zeros, at $a_1 = -0.95$ and $a_2 = 0.75$. The first zero is close to $(-1, 0)$ and is expected to cause intersample rippling when the H_2 -optimal controller is used.

We find from (23) that $P_A^* = z^{-1}I$, $P_M^* = zP^*$. We shall consider step setpoint changes as external inputs – i.e.,

$$V(z) = \frac{z}{z-1}I$$

Then (17) yields

$$\tilde{Q}_H(z) = z^{-1}(P^*)^{-1}$$

Figure 3A shows the time response of this control system for a unit step change in the setpoint of output 1:

$$v^*(z) = r^*(z) = \begin{pmatrix} z/(z-1) \\ 0 \end{pmatrix}$$

The prediction of intersample rippling is verified. Note that at the sample points the outputs are indeed exactly at the setpoints yielding the minimum sum of squared errors.

The IMC controller is now obtained from (20) with $B(z) = 1$ and

$$\tilde{q}_-(z) = \frac{z + 0.95}{1.95z}$$

The response for this control system is shown in Fig. 3B. Clearly the rippling problem has disappeared. Note the inverse responses caused by the RHP zero of the continuous system $P(s)$. \square

2.6 Inner-Outer Factorization

Let the following be a state space description (realization) of a z-transfer matrix $G^*(z)$:

$$x(kT + T) = \Phi x(kT) + \Gamma u(kT)$$

$$y(kT) = Cx(kT) + Du(kT)$$

Then

$$G^*(z) = C(zI - \Phi)^{-1}\Gamma + D$$

Theorem 6 will be used to obtain an inner-outer factorization of $G^*(z)$. To use that theorem, one needs to employ the bilinear transformation

$$z = \frac{1 + s}{1 - s} \quad (24)$$

to reduce the problem into one on which Thm. 6 can be applied. Theorems 7, 8 provide the formulas for the transformation of state space descriptions implied by (24) or its inverse

$$s = \frac{-1 + z}{1 + z} \quad (25)$$

Theorem 6 (Chu, 1985) *Let $G(s) = C(sI - A)^{-1}B + D$ be a minimal realization of the square transfer matrix $G(s)$, and let $G(s)$ have no zeros on the $i\omega$ -axis including infinity. Then we have*

$$G(s) = N(s)M(s)^{-1} \quad (26)$$

where N, M are stable and $N(i\omega)^H N(i\omega) = I$. $N(s)$ and $M(s)^{-1}$ are given by

$$N(s) = (C - QF)(sI - (A - BR^{-1}F))^{-1}BR^{-1} + Q \quad (27)$$

$$M(s)^{-1} = F(sI - A)^{-1}B + R$$

where

$$D = QR$$

is the QR factorization of D into an orthogonal matrix Q ($Q^T Q = I$) and an upper triangular matrix R , and

$$F = Q^T C + (BR^{-1})^T X$$

with X the stabilizing [i.e., it makes $(A - BR^{-1}F)$ stable] real symmetric solution of the following algebraic Riccati equation (ARE):

$$(A - BR^{-1}Q^T C)^T X + X(A - BR^{-1}Q^T C) - X(BR^{-1})(BR^{-1})^T X = 0 \quad (28)$$

Such a solution X exists for (28).

Laub (1979) and Molinari (1973) provide methods for solving AREs. Several control software packages include programs for that purpose. Details on the QR factorization can be found in the literature (e. g., Dahlquist and Bjorck, 1974).

Note that $N(s)$ in (27) is unique only up to premultiplication with a constant unitary matrix U ($U^H U = 1$).

Theorem 7 *Let $G^*(z) = C(zI - \Phi)^{-1}\Gamma + D$ have no poles at $z = -1$. Then*

$$\hat{G}(s) \triangleq G^*\left(\frac{1+s}{1-s}\right) = \hat{C}(sI - \hat{A})^{-1}\hat{B} + \hat{D}$$

where

$$\hat{A} = (\Phi + I)^{-1}(\Phi - I)$$

$$\hat{B} = 2(\Phi + I)^{-2}\Gamma$$

$$\hat{C} = C$$

$$\hat{D} = D - C(\Phi + I)^{-1}\Gamma$$

Proof. See Appendix E.

Theorem 8 *Let $\hat{G}(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B} + \hat{D}$ have no poles at $s = 1$. Then*

$$G^*(z) \triangleq \hat{G}\left(\frac{-1+z}{1+z}\right) = C(zI - \Phi)^{-1}\Gamma + D$$

where

$$\Phi = (I - \hat{A})^{-1}(I + \hat{A})$$

$$\Gamma = 2(I - \hat{A})^{-2}\hat{B}$$

$$C = \hat{C}$$

$$D = \hat{D} + \hat{C}(I - \hat{A})^{-1}\hat{B}$$

Proof. See Appendix F.

The factorization

$$P^*(z) = P_A^*(z)P_M^*(z)$$

involves the following steps:

Step 1: Use the variable transformation (24) on $P^*(z)$ to obtain $\hat{P}(s)$. Note that the assumption of Thm. 7 that $P^*(z)$ has no poles at $z = -1$ holds for the P^* s under consideration in this section because of Assn. A2.

Step 2: Apply Thm. 6 on $\hat{P}(s)$ to obtain the factorization

$$\hat{P}(s) = \hat{P}_A(s)\hat{P}_M(s)$$

Note that for a strictly proper system $D = 0$ and therefore from Thm. 7 we have $\hat{D} = -C(\Phi + I)^{-1}\Gamma = P^*(-1)$. According to Assn. A2, $P^*(z)$ has no zeros on the UC and

therefore $P^*(-1)$ has full rank. Hence, the assumption of no zeros on the imaginary axis including infinity in Thm. 6, holds for $\hat{P}(s)$.

Step 3: Use the variable transformation (25) on $\hat{P}_A(s)$ and $\hat{P}_M(s)$ to obtain $P_A^*(z)$ and $P_M^*(z)$ correspondingly. Note that $\hat{P}_A(s)$ satisfies the assumption of no poles at $s = 1$, since by construction all its poles are in the LHP. Also, $\hat{P}_M(s)$ has the poles of $\hat{P}(s)$, which do not include a pole at $s = 1$, since $P^*(z)$ has no poles at $z = \infty$.

The result of the above steps is a stable, all-pass P_A^* and a minimum phase P_M^* . Both P_A^* and P_M^* are proper because \hat{P}_A and \hat{P}_M have no poles at $s = 1$. Also note that since $\hat{P}_M(s)$ is minimum phase, it does not have a zero at $s = 1$ and therefore $P_M^*(z)$ has no zero at $z = \infty$, which means that $(P_M^*(z))^{-1}$ is proper.

A comparison of (26) and (16) indicates that Thm. 6 cannot be directly applied to $G^*(z) = V(z)$. We can, however, apply the above procedure to $\tilde{V}(z) \triangleq V^T(z)$ to obtain

$$\tilde{V}(z) = V^T(z) = \tilde{V}_A(z)\tilde{V}_M(z) \quad (29)$$

with $\tilde{V}_A, \tilde{V}_M^{-1}$ stable and $\tilde{V}_A(i\omega)^H \tilde{V}_A(i\omega) = I$. From (29) we find

$$V(z) = \tilde{V}_M^T(z)\tilde{V}_A^T(z)$$

Since $\tilde{V}_A^T(z), (\tilde{V}_M^T(z))^{-1}$ are stable and

$$\tilde{V}_A^T(i\omega)(\tilde{V}_A^T(i\omega))^H = (\tilde{V}_A^H(i\omega)\tilde{V}_A(i\omega))^T = I$$

we can select V_A, V_M as

$$V_A(z) = \tilde{V}_A^T(z)$$

$$V_M(z) = \tilde{V}_M^T(z)$$

2.7 Application: Distillation Column Base Level Control

When the steam input to a distillation column is increased, the liquid level in the column base can exhibit inverse response behavior. The following type of model

$$\tilde{P}(s) = \frac{1}{s}(1 - 2e^{-s\theta})$$

was found to describe the behavior of many industrial columns adequately (Buckley *et al.*, 1985). The deadtime θ is equal to the sum of the hydraulic time constants of the individual trays and it approximates a large number of first order lags modeling the tray behavior. Let us select a sampling time $T = \theta/N$, where N is an integer. Then

$$\tilde{P}^*(z) = (1 - 2z^{-N})\frac{T}{z - 1} \quad (30)$$

In this very special case we find that if ζ is a finite zero of $\tilde{P}(s)$, then $e^{\zeta T}$ is a zero of $\tilde{P}^*(z)$ and therefore ζ is a zero of $\tilde{P}^*(e^{sT})$. This mapping does not hold for zeros in general, although it is always true for the poles of $\tilde{P}(s)$ and $\tilde{P}^*(z)$.

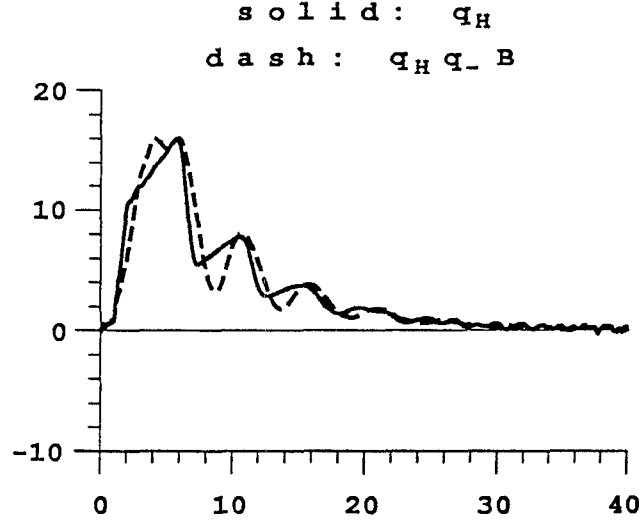


Figure 4: Distillation column base level control; response to $d(s) = s^{-2}$. Solid line: \tilde{Q}_H . Dashed line: $\tilde{Q}_H \tilde{q}_- B$.

Because of this mapping the zeros of $\tilde{P}^*(z)$ that appear as poles of $\tilde{Q}_H(z)$ are cancelled by zeros of $\tilde{P}(s)$ in the IMC structure (Figs. 2A and B) for $P(s) = \tilde{P}(s)$ and do not affect the continuous plant output $y(s)$. Therefore, any such zeros close to $(-1, 0)$ do not produce intersample rippling even if the modification described in Sec. 2.5 is not made. However, this does not mean that the behavior of the control system will deteriorate if the suggested modification is introduced. Indeed, the steps proposed in Sec. 2 will result in a well-performing controller, regardless of whether $\tilde{Q}_H(z)$ suffers from rippling problems or not.

Let us proceed to illustrate this point by simulating the response for the two controllers for a ramp disturbance $d(s) = s^{-2}$. For the simulations we choose $\theta = 5$ and $T = 1$, which implies that $N = 5$. It follows from (30) that the zeros of $\tilde{P}^*(z)$ are located at $2^{1/5} e^{i2k\pi/5}$, $k = 0, 1, 2, 3, 4$, where $2^{1/5}$ indicates the real fifth root of 2. Two of these zeros (the ones that correspond to $k = 2, 3$) have negative real parts and will give rise to poles of $\tilde{Q}_H(z)$ with negative real parts. The procedure of Sec. 2 yields

$$\tilde{Q}_H(z) = \frac{z^3(17z - 16)(z - 1)}{(-2z^5 + 1)}$$

$$\tilde{q}_-(z) = \frac{z^2 + 1.8586z + 1.3195}{4.1781z^2}$$

$$B(z) = 2.0765 - 1.0765z^{-1}$$

Figure 4 shows the responses to $d(s) = s^{-2}$ for both $Q = \tilde{Q}_H$ and $Q = \tilde{Q}_H \tilde{q}_- B$. Clearly, \tilde{Q}_H produces no intersample rippling. One can also see that when the modification of Sec. 2.5 is made anyway, the response is essentially unaffected.

Finally, note that because the open-loop system is unstable, the controller has to be implemented in the classic feedback structure. The expression can be obtained from (4) and its implementation presents no problem.

3 Step 2: Design of the IMC Filter $F(z)$

This section deals with the design of the IMC filter $F(z)$ so that the performance characteristics obtained in Step 1 are preserved in the presence of model-plant mismatch.

3.1 Model Uncertainty

The IMC structure of Fig. 2C can be written as that of Fig. 5A, where $v^* = d_\gamma^* - r^*$, $e^* = y_\gamma^* - r^*$ and

$$G^*(F) = \begin{pmatrix} -\tilde{Q}(z)F(z) & -\tilde{Q}(z)F(z) \\ I - \tilde{P}_\gamma^* \tilde{Q}(z)F(z) & I - \tilde{P}_\gamma^* \tilde{Q}(z)F(z) \end{pmatrix}$$

F is used as a parameter of G^* to denote that at this second step of the IMC design procedure, F is the design variable. The block $(P_\gamma^* - \tilde{P}_\gamma^*)$ represents the model-plant mismatch. In order to design a control system that takes into account this modeling error, we need to have some information on how large this mismatch can be. For example we might know a bound $l_a^*(\omega)$, where ω is the frequency ($z = e^{i\omega T}$), on the additive error:

$$\bar{\sigma}(P_\gamma^* - \tilde{P}_\gamma^*) \leq l_a^* \quad (31)$$

where $\bar{\sigma}(\cdot)$ is the maximum singular value of (\cdot) . However (31) represents only a very simple way to describe model uncertainty. For multivariable systems, such uncertainty may appear in many different places in the matrix, like specific parameters, elements of P_γ^* , the inputs or outputs of P_γ^* , etc. It may then be very conservative to lump this information into (31). However, provided that we can write P_γ^* as a linear fractional transformation of its uncertainties, the structure in Fig. 5A can always be transformed into that in Fig. 5B, where Δ is a block diagonal matrix with the additional property that

$$\bar{\sigma}(\Delta) \leq 1 \quad \forall \omega \quad (32)$$

The superscript u in $G^{*,u}$ denotes the dependence on $G^{*,u}$ not only on G^* but also on the specific uncertainty description available for the model \tilde{P}_γ^* . We shall not demonstrate in detail here, how $G^{*,u}$ can be obtained from G^* . For some common cases of model uncertainty, the expressions can be found in the literature (Zafriou and Morari, 1988). For the simple case described by (31), this can be accomplished by simply multiplying the first row of G^* with l_a^* .

Note that for $P_\gamma^*, \tilde{P}_\gamma^*$, z-transforms and therefore periodic in ω , the block Δ will also be periodic. Hence in this case only the frequencies from 0 to π/T need be considered in (32). In Sec. 3.3 it will become apparent that in order to avoid bad intersample behavior, we also have to consider the continuous plant, described by some Laplace transfer function $P(s)$ (and $\tilde{P}(s)$ for the model). Clearly the modeling error in the description of the discretized plant is related to that in the continuous plant description. For example, let us assume that we have a bound on the additive uncertainty for the continuous plant:

$$\bar{\sigma}(P(i\omega) - \tilde{P}(i\omega)) \leq l_a(\omega) \quad (33)$$

Then for the discretized plant we have from (1), (3):

$$P_\gamma^*(z) - \tilde{P}_\gamma^*(z) = \mathcal{ZL}^{-1}\{h_0(s)\gamma(s)(P(s) - \tilde{P}(s))\} \quad (34)$$

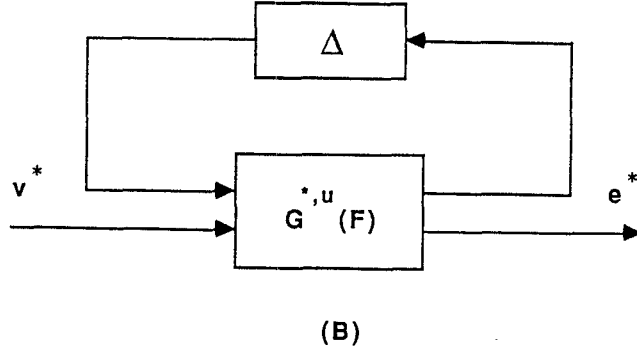
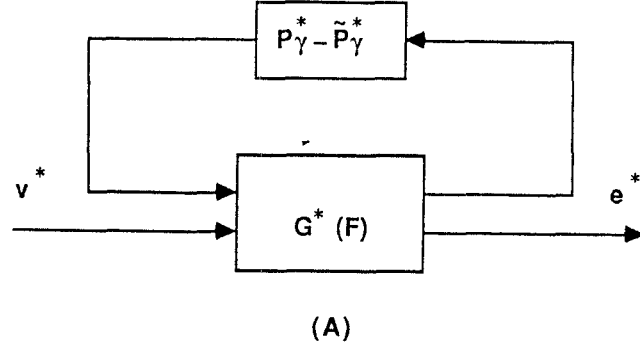


Figure 5: Structured Singular Value block diagram.

Every z-transform $a(z) = \mathcal{Z}\mathcal{L}^{-1}\{a(s)\}$ has the property:

$$a(e^{i\omega T}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} a(i\omega + ik2\pi/T) \quad (35)$$

From (33), (34), (35) and the singular value property $\bar{\sigma}(A + B) \leq \bar{\sigma}(A) + \bar{\sigma}(B)$, it follows that

$$\bar{\sigma}(P_{\gamma}^*(e^{i\omega T}) - \tilde{P}_{\gamma}^*(e^{i\omega T})) \leq \frac{1}{T} \sum_{k=-\infty}^{\infty} |h_0(i\omega + ik2\pi/T)| l_a(i\omega + ik2\pi/T) \quad (36)$$

$h_0(s)$ is small at frequencies higher than π/T and goes to 0 as fast as $1/\omega$ as $\omega \rightarrow \infty$. Therefore only a few terms around $k = 0$ are important in the infinite sum. Also note that for a physical system, $l_a(\omega) \rightarrow 0$ at least as fast as $1/\omega$, as $\omega \rightarrow \infty$, and hence the sum converges. Then the bound l_a^* in (31) can be set equal to the right hand side of (36).

However, it is not always possible to obtain in a non-conservative way a mathematical description for the uncertainty in the z-domain, starting from the uncertainty in the s-domain. If first principles models are not available, these descriptions may be the result of experiments conducted with different sampling times, of which one is small enough to approximate the continuous system. A discussion of identification techniques is beyond the scope of this paper. Details on such methods and the resulting modelling error can be found in the literature (e.g., Jenkins and Watts (1969), Astrom and Wittenmark (1984)).

3.2 Robust Stability Condition

We now require that the internal stability matrix in (2) is stable for all possible plants P . The design of \tilde{Q} according to Sec. 2 resulted in a stable matrix for $P = \tilde{P}$. In order for the matrix to remain stable we need to satisfy the requirements that as we move in a “continuous” way from the model \tilde{P} to the plant P , no closed-loop poles cross the UC and no such poles suddenly appear outside the UC. The latter requirement is satisfied if we assume that the model and the plant have the same number of poles outside the UC. If this is not the case, another sufficient condition is that $G^{*,u}(F)$ is a stable matrix and only stable Δ s are possible.

Let $G^{*,u}(F)$ be partitioned as

$$G^{*,u}(F) \triangleq \begin{pmatrix} G_{11}^{*,u}(F) & G_{12}^{*,u}(F) \\ G_{21}^{*,u}(F) & G_{22}^{*,u}(F) \end{pmatrix}$$

The SSV can be used to determine if any crossings of the UC occur. The system is stable for any of the plants in the set defined from the bounds on the model uncertainty and which have the same number of outside the UC poles as the model, if and only if (Doyle, 1985)

$$\mu_{\Delta}(G_{11}^{*,u}(F)) < 1, \quad 0 \leq \omega \leq \pi/T \quad (37)$$

3.3 Robust Performance Condition

If one only cared about the performance at the sample points, then one could use Fig. 5B to state the robust performance conditions. However, because of the intersample rippling problem, one has to consider the continuous output of the plant. The problem of intersample rippling was addressed for the first step of the design procedure in Sec. 2.5. There, a simple modification was sufficient because the model $\tilde{P}^*(z)$ was known exactly. In this section however we have to consider the situation where $P \neq \tilde{P}$ and as a result we have to examine the continuous plant output $y(s)$ in order to avoid bad intersample behavior. The obstacle in doing so is the fact that the relation between $y(s)$ and $r(s)$ or $d(s)$ (continuous setpoint and disturbance descriptions) is linear but time varying because of the sampling, and so there is no transfer function that describes this relation. Dailey (1987) and Thompson (1982) propose design approaches that bound appropriate time varying operators, like the sensitivity operator or the sampling switch, with “conic sectors”. The approach that will be followed in this paper is to obtain a transfer function approximation for the frequencies of interest.

From Fig. 2A it follows that

$$\begin{aligned} e(s) &\triangleq y(s) - r(s) \\ &= (d(s) - r(s)) - P(s)H_0(s)Q(e^{sT}). \\ &= (I + (P_{\gamma}^*(e^{sT}) - \tilde{P}_{\gamma}^*(e^{sT}))Q(e^{sT}))^{-1}(d_{\gamma}^*(z) - r^*(z)) \end{aligned} \quad (38)$$

We shall now obtain an approximation to (38) by considering the frequencies $0 \leq \omega \leq \pi/T$. Note that because of the periodicity of $Q(z)$, these are the only frequencies which one can influence independently by using a digital controller. It follows from (35) that if $a(s)$ is small for $\omega > \pi/T$, then

$$a^*(e^{i\omega T}) \cong \frac{1}{T}a(i\omega), \quad 0 \leq \omega \leq \pi/T \quad (39)$$

Use of (39) for all the z -transforms in (38) except for r^* for which we assume $r(s) = h_0(s)r^*(e^{sT})$ (staircase function), yields the approximation

$$e(i\omega) \cong E_d(i\omega)d(i\omega) - E_r(i\omega)r(i\omega), \quad 0 \leq \omega \leq \pi/T$$

where

$$E_r(i\omega) \triangleq I - P(i\omega)Q(e^{i\omega T}).$$

$$\left[I + (P(i\omega) - \tilde{P}(i\omega))Q(e^{i\omega T})\gamma(i\omega)h_0(i\omega)/T \right]^{-1} \quad (40)$$

$$E_d(i\omega) \triangleq I - P(i\omega)Q(e^{i\omega T})\gamma(i\omega)h_0(i\omega)/T.$$

$$\left[I + (P(i\omega) - \tilde{P}(i\omega))Q(e^{i\omega T})\gamma(i\omega)h_0(i\omega)/T \right]^{-1} \quad (41)$$

Note that the above approximation is valid when the input signals r and d are small for $\omega > \pi/T$. If we assume that $r(t)$ is a staircase function then it has the desired property. If one expects disturbances with high frequency content at $\omega > \pi/T$ then one should reduce T or use the anti-aliasing prefilter whose function is to cut off signals with frequencies higher than π/T .

Let us use the notation $\tilde{E}_v(i\omega) = E_v(i\omega)$ when $P = \tilde{P}$, where $v = r$ or d . In the first step of the IMC design procedure, \tilde{Q} is obtained so that it produces satisfactory disturbance rejection and/or setpoint tracking. Since \tilde{E}_v connects the external inputs to the error e , a well-designed control system produces a relatively "small" \tilde{E}_v . A measure of the magnitude of the known \tilde{E}_v is its maximum singular value. Let $b(\omega)$ be a frequency function such that

$$\bar{\sigma}(\tilde{E}_v(i\omega)) < b(\omega), \quad 0 \leq \omega \leq \pi/T$$

In order for the performance of the control system to remain robust with respect to model-plant mismatch we have to keep e small in spite of the modeling error. Then we require:

$$\max_{0 \leq \omega \leq \pi/T} \bar{\sigma}(b(\omega)^{-1}E_v(i\omega)) < 1 \quad \forall \Delta \in \Delta \quad (42)$$

A more general form of (42) is

$$\max_{0 \leq \omega \leq \pi/T} \bar{\sigma}(W(\omega)E_v(i\omega)) < 1 \quad \forall \Delta \in \Delta \quad (43)$$

where $W(\omega)$ is a frequency weighting matrix.

The use of (39) in the derivation of (40), (41), is equivalent to approximating the function of the sampling operator by $1/T$ for $0 \leq \omega \leq \pi/T$. This approximation is reasonable for signals with small power for $\omega > \pi/T$. Use of $1/T$ in the place of the sampling switch in Fig. 2B, allows us to derive the matrix $G_v(F)$ in the block diagram in Fig. 6A.

For $v = d$, $e = y$, we have

$$G_d(F) = \begin{pmatrix} -\frac{h_0\gamma}{T}\tilde{Q}(z)F(z) & -\frac{h_0\gamma}{T}\tilde{Q}(z)F(z) \\ I - \frac{h_0\gamma}{T}\tilde{P}^*\tilde{Q}(z)F(z) & I - \frac{h_0\gamma}{T}\tilde{P}^*\tilde{Q}(z)F(z) \end{pmatrix}$$

For $v = -r$, $e = y - r$, with $r(s) = h_0(s)r^*(e^{sT})$:

$$G_r(F) = \begin{pmatrix} -\frac{h_0\gamma}{T}\tilde{Q}(z)F(z) & -\tilde{Q}(z)F(z) \\ I - \frac{h_0\gamma}{T}\tilde{P}^*\tilde{Q}(z)F(z) & I - \tilde{P}^*\tilde{Q}(z)F(z) \end{pmatrix}$$

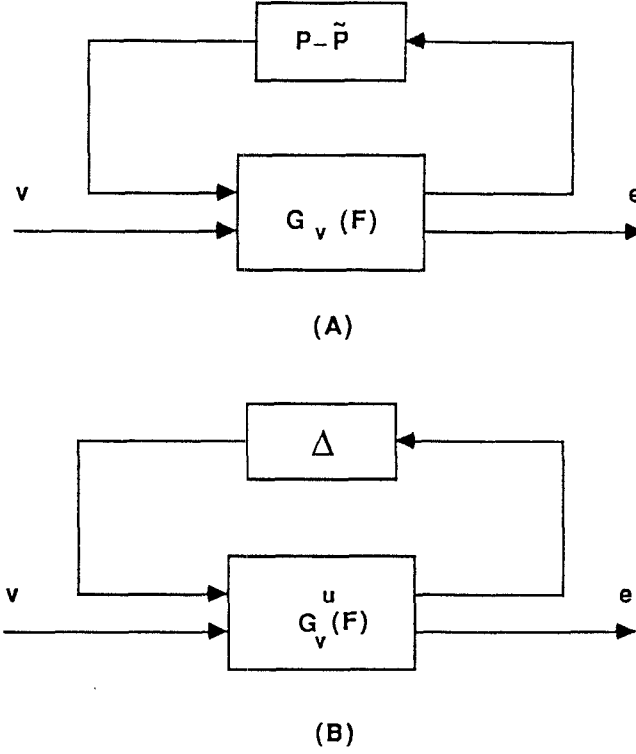


Figure 6: SSV block diagram for Robust Performance.

For some specific model uncertainty description, we can transform Fig. 6A into 6B (e. g., Zafiriou and Morari, 1988).

Note that in this case the uncertainty block Δ represents continuous transfer functions.

We can now use the properties of the SSV (Doyle, 1985) to obtain

$$\max_{0 \leq \omega \leq \pi/T} \bar{\sigma}(W(\omega)E_v(i\omega)) < 1 \quad \forall \Delta \in \Delta \iff \max_{0 \leq \omega \leq \pi/T} \mu_{\Delta^0}(G_v^W) < 1 \quad (44)$$

where

$$G_v^W = \begin{pmatrix} I & 0 \\ 0 & W \end{pmatrix} G_v$$

$$\Delta^0 = \{diag(\Delta, \Delta^0) | \Delta \in \Delta, \Delta^0 \in C^{n \times n}\}$$

3.4 Filter Structure

First we will postulate reasonable filter structures. Then in Sec. 3.5 we will define appropriate minimization problems to be solved over the filter parameters.

The structure of F can be as general as the designer wishes. However, in order to keep the number of variables in the optimization problems small, a rather simple structure like a diagonal F with first- or second-order terms would be recommended. In most cases this is not restrictive because the controller \tilde{Q} that was designed in the first step of the IMC procedure is in general a full high-order transfer matrix. More complex filter structures may be necessary in cases of ill-conditioned systems ($\bar{\sigma}(\tilde{P}^*)/\underline{\sigma}(\tilde{P}^*)$ very large). For such systems a two-filter structure discussed in detail in Zafiriou and Morari (1987c). The elements of each of the two filters in that structure can be designed as described below.

The filter $F(z)$ is chosen to be a diagonal rational function that satisfies the following requirements in addition to the robustness conditions..

- a. Internal Stability. The matrix in (2) must be stable for $P = \tilde{P}$.
- b. Asymptotic setpoint tracking and/or disturbance rejection. $(I - \tilde{P}^* \tilde{Q} F)v^*$ must be stable.

Write

$$F(z) = \text{diag}\{f_1(z), \dots, f_n(z)\}$$

Then, Assns. A1-A5 and the fact that by construction $\tilde{Q}(z)$ satisfy the above requirements, imply that the requirements on an element f_ℓ of F are:

$$\left. \frac{d^j}{dz^j} (1 - f_\ell(z)) \right|_{z=\pi_1} = 1, \quad j = 0, \dots, m_{1\ell} - 1 \quad (45)$$

$$f_\ell(\pi_i) = 1, \quad i = 2, \dots, \xi \quad (46)$$

where $\pi_1 = 1$ and $m_{1\ell}$ is the highest multiplicity of such a pole in any element of the ℓ^{th} row of V and $\pi_i, i = 2, \dots, \xi$ are the poles of \tilde{P}^* outside the UC, each with multiplicity 1, according to Assn. A1.

One can now select the filter elements to be of the form:

$$f(z) = \phi(z)f_1(z) \quad (47)$$

where

$$f_1(z) = \frac{(1 - \alpha)z}{z - \alpha}$$

$$\phi(z) = \sum_{j=0}^w \beta_j z^{-j}$$

and the coefficients β_0, \dots, β_w are computed so that (45), (46) are satisfied for some specified α . The parameter α can be used as a tuning parameter to satisfy the robustness conditions.

Note that for $\xi = 1, \pi_1 = 1, m_{1\ell} = 1$, we only need $\phi(z) = 1$. For the general case, a system of M_ℓ linear equations with β_0, \dots, β_w as unknowns where M_ℓ is given by

$$M_\ell = m_{1\ell} + \xi$$

These equations are described by the following theorem:

Theorem 9 For $\pi_1 = 1, \xi \geq 2, m_i = 1$ for $i = 2, \dots, \xi$, the coefficients β_0, \dots, β_w must satisfy

$$\beta_0 = 1 - \beta_1 - \dots - \beta_w$$

$$\left(\begin{array}{c} \Pi \\ N_w \end{array} \right) \left(\begin{array}{c} \beta_1 \\ \vdots \\ \beta_w \end{array} \right) = \left\{ \begin{array}{c} f_1(\pi_\xi)^{-1} - 1 \\ \vdots \\ f_1(\pi_2)^{-1} - 1 \\ -\alpha/(1 - \alpha) \\ 0 \\ \vdots \\ 0 \end{array} \right\} \left\{ \begin{array}{c} \xi - 1 \\ \\ \\ m_{1\ell} - 1 \end{array} \right\} \triangleq \chi$$

where

$$\Pi = \begin{pmatrix} \pi_\xi^{-1} - 1 & \dots & \pi_\xi^{-w} - 1 \\ \vdots & & \vdots \\ \pi_2^{-1} - 1 & \dots & \pi_2^{-w} - 1 \end{pmatrix}$$

and the elements ν_{ij} of the $(m_{1\ell} - 1) \times w$ matrix N_w are defined by:

$$\nu_{ij} = \begin{cases} 0 & \text{for } i > j \\ \frac{j!}{(j-i)!} & \text{for } i \leq j \end{cases}$$

Proof. Follows directly from (9.3-5), the fact that $f_1(1) = 1$ and Thm. 3 of Zafiriou and Morari (1986). \square

In general one should select $w \geq M_\ell - 1 = \xi + m_{1\ell} - 2$ and obtain β_1, \dots, β_w as the minimum norm solution:

$$\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_w \end{pmatrix} = A^T(AA^T)^{-1}\chi \quad (48)$$

where

$$A \triangleq \begin{pmatrix} \Pi \\ N_w \end{pmatrix}$$

Note that from a numerical point of view it is preferable to compute the pseudo-inverse in (48) from a singular value decomposition of $\begin{pmatrix} \Pi \\ N_w \end{pmatrix}$.

Example 3 (Zafiriou and Morari (1986)).

Assume that $\xi = 0$, $\pi_1 = 1$ and that $m_{1\ell} = 2$. Then the minimum norm solution is

$$\beta_j = -\frac{\kappa_j \alpha}{(1 - \alpha_j w(w+1)(2w+1))}, \quad j = 1, \dots, w$$

$$\beta_0 = 1 - \sum_{j=1}^w \beta_j$$

The norm of this solution goes to 0 and β_0 goes to 1 as $w \rightarrow \infty$ and so the properties of $f(z)$ are similar to those of $f_1(z)$ when w is large enough. \square

Example 4 *Qualitative Interpretation of the Filter Function.*

For open-loop unstable systems, the filter has to be unity at the unstable system poles, which limits the range of filter parameters α that can be chosen for reasonable performance, as we shall show next. For the effect of the unstable poles to become negligible $f(z)$ has to approach $f_1(z)$, or in other words $\phi(z)$ has to approach unity. We will study the behavior of $\phi(z)$ for $w \rightarrow \infty$.

Consider the system studied in Ex. 1. For internal stability and asymptotically error-free disturbance compensation we require

$$f(1) = f(e^{bT}) = 1$$

We have in this case $\xi = 2$, $\pi_1 = 1$, $\pi_2 = e^{bT}$, $m_1 = m_2 = 1$. Hence

$$A = (e^{-bT} - 1 \quad \dots \quad e^{-wbT} - 1)$$

$$\chi = f_1(e^{bT})^{-1} - 1 = \frac{\alpha(1 - e^{-bT})}{1 - \alpha}$$

From (48) it follows that

$$\sum_{j=1}^w \beta_j^2 = \chi^T (AA^T)^{-1} \chi = \frac{\alpha^2(1 - e^{-bT})^2}{(1 - \alpha)^2 \Sigma_1}$$

where

$$\begin{aligned} \Sigma_1 &\triangleq \sum_{j=1}^w (e^{-jbT} - 1)^2 \\ &= \sum_{j=1}^w e^{-j2bT} - 2 \sum_{j=1}^w e^{-jbT} + w \\ &= \frac{1 - e^{-2bTw}}{e^{2bT} - 1} - 2 \frac{1 - e^{-bTw}}{e^{bT} - 1} + w \end{aligned} \quad (49)$$

Since $|e^{-bT}| < 1$, it follows from (49) that $\lim_{w \rightarrow \infty} \Sigma_1 = \infty$ and $\lim_{w \rightarrow \infty} \sum_{j=1}^w \beta_j^2 = 0$. This fact, however, is not sufficient to produce an $f(z)$ that approximates the behavior of $f_1(z)$. For this to happen we need $\lim_{w \rightarrow \infty} \beta_0 = 1$. Let us compute this limit. From (48) we get

$$\beta_k = \frac{\alpha(1 - e^{-bT})(e^{-kbT} - 1)}{(1 - \alpha)\Sigma_1}, \quad k = 1, \dots, w$$

Then

$$\beta_0 = 1 - \frac{\alpha(1 - e^{-bT})\Sigma_2}{(1 - \alpha)\Sigma_1} \quad (50)$$

where

$$\Sigma_2 \triangleq \sum_{j=1}^w (e^{-jbT} - 1) = \frac{1 - e^{-bTw}}{e^{bT} - 1} - w \quad (51)$$

From (49), (51) it follows that $\lim_{w \rightarrow \infty} \Sigma_2/\Sigma_1 = -1$. Then (50) yields

$$\lim_{w \rightarrow \infty} \beta_0 = \frac{1 - \alpha e^{-bT}}{1 - \alpha}$$

By writing $\alpha = e^{-T/\lambda}$ we get

$$\lim_{w \rightarrow \infty} \beta_0 = \frac{1 - e^{-T(1/\lambda + b)}}{1 - e^{-T/\lambda}}$$

Hence in order for $\lim_{w \rightarrow \infty} \beta_0 \cong 1$ we need $1/\lambda \gg b$ or $\lambda b \ll 1$. In this case the behavior of $f(z)$ approaches that of $f_1(z)$ and if a λ in that range is sufficient for robustness, the unstable pole b produces no significant effect on the system behavior. If, however, one chooses a λ for which $\lambda b \gg 1$, then the $\lim_{w \rightarrow \infty} \beta_0$ is very far from 1.

This is illustrated in Fig. 7, where amplitude plots of f_1 and f are shown for different values of λ and w . We see that as w increases, f tends towards f_1 . For $\lambda b \ll 1$, the approximation is very good, while for $\lambda b \gg 1$, the closer we get to f_1 , the higher the peak in $|f|$ becomes. \square

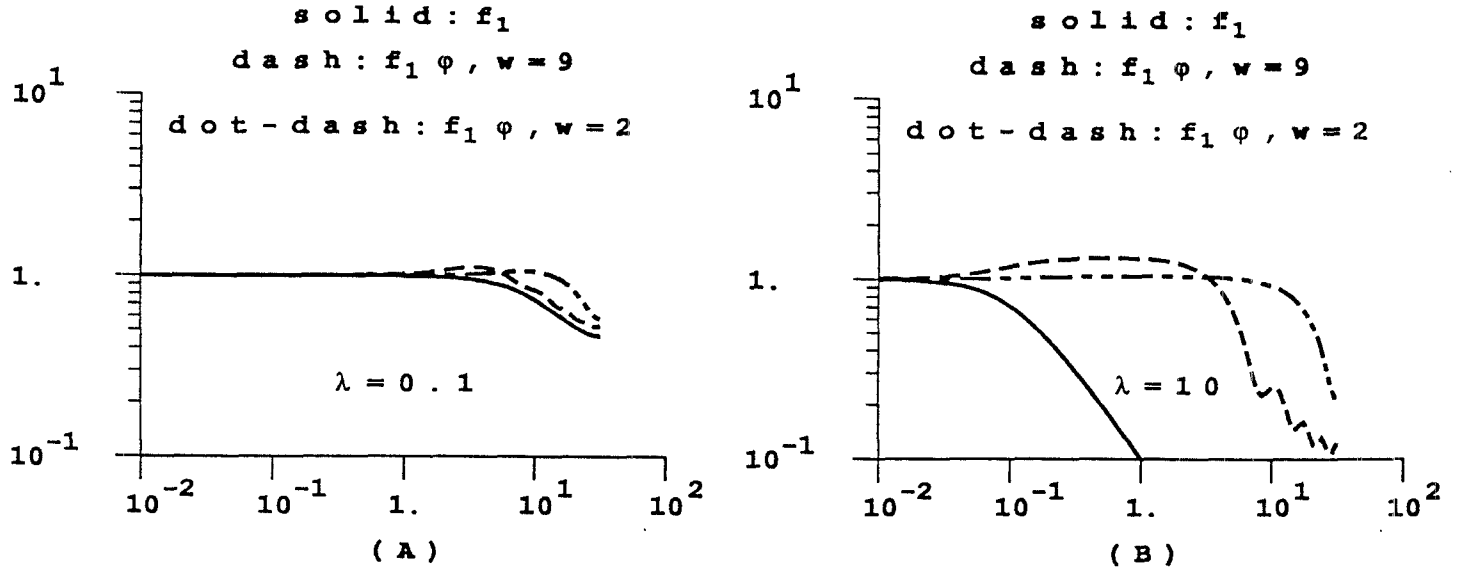


Figure 7: Effect of a RHP pole on the discrete IMC filter. $T = 0.1, b = 1$. Solid: f_1 , Dash: $f_1\phi, w = 9$, Dot-Dash: $f_1\phi, w = 2$.

3.5 Filter Parameter Optimization

The next step is to find filter parameters that satisfy (44) (the robust performance condition) and (37) (the robust stability condition). We can write

$$F \triangleq F(z; \Lambda)$$

where Λ is an array with the adjustable filter parameters. The filter design problem can be formulated as a minimization problem over the elements of Λ . In the filter structure proposed in Sec. 3.4, there is one adjustable parameter α for each element of the diagonal filter. Each one of these real parameters, say α_j , has to be inside the UC for F to be stable. The stability constraints can be removed from the minimization problem by setting

$$\alpha_j = e^{-T/\lambda_j^2} \quad (52)$$

where λ_j is an element of Λ . Then any λ_j in $(-\infty, \infty)$ produces an α_j in $[0, 1]$. Note that if the parametrization (52) is used, then it is λ_j^2 and not λ_j that corresponds to a time constant. If one wishes to use a higher than first order $f_1(z)$ one can write the denominator of each element of F as the product of polynomials of degree 2 and one of degree 1 if the order is odd. A polynomial of degree 2 with roots inside the UC can be written as $z^2 - (e^{Tp_1} + e^{Tp_2})z + e^{Tp_1+Tp_2}$, where p_1, p_2 are the roots of $\lambda_2^2 x^2 + \lambda_1^2 x + 1 = 0$ for some value of λ_1, λ_2 . In this way, the optimization problem is unconstrained in the optimization variables λ_1, λ_2 , which can take any value in $(-\infty, \infty)$.

Our goal is to satisfy (44). The filter parameters can be obtained by solving

Objective O4:

$$\min_{\Lambda} \max_{0 \leq \omega \leq \pi/T} \mu_{\Delta^0}(G_v^W)$$

It should be noted however that the optimal solution for Obj. O4, may still not satisfy (44). The reason is usually that the performance requirements set by the selection of W

in (43) are too tight to be satisfied in the presence of the model-plant mismatch. In this case one should choose a less tight W and solve Obj. O4 again.

Another important point is that satisfaction of the robust performance condition (44) does not necessarily imply satisfaction of the robust stability condition (37), which was the case in the continuous controller design. This is so even if the uncertainty descriptions for the continuous plant [used in (44)] and the discretized plant [used in (37)] correspond to exactly the same sets of possible plants. The reason is that (44) was obtained by using the approximations discussed in detail in Sec. 3.1, while there are no approximations involved in the derivation of (37). Note however, that if the uncertainty descriptions for the continuous and the discrete plant are equivalent in the sense discussed in Sec. 3.1, then satisfaction of (44) is usually sufficient for satisfaction of (37), although this is not guaranteed. As a result of the above discussed possibility, when a solution to Obj. O4 is found, one should check if (37) holds. If this does not happen, then one can always substitute the robust stability μ (37) in Obj. O4 and proceed with the minimization until (37) becomes less than one.

By considering only a finite number of frequencies in the computation of the maximum in Obj. O4, we can write the optimization problem as

Objective O4':

$$\min_{\Lambda} \max_{\omega \in \Omega} \mu_{\Delta^o}(G_v^W)$$

where Ω is a set containing a finite number of frequencies in $[0, \pi/T]$. The optimization problem defined in Obj. O4', is identical to that defined by eq. (46) in Zafiriou and Morari (1988). For a discussion of the computational issues the reader is referred to that paper.

4 Illustration of the Design Procedure

The purpose of this section is to demonstrate the IMC design procedure by applying it to a 2×2 open-loop unstable system.

4.1 System Description

Let the continuous system be modeled by

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

where

$$A = \begin{pmatrix} 2.375 & 0.857 & 1.000 \\ -17.719 & -5.500 & -5.250 \\ -14.766 & -6.750 & -7.375 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ -1 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 0.3 & 1.8 \\ 0 & 0 & -4.0 \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

The eigenvalues of A , which are the poles of the continuous system are located at -1 , -10 , $+0.5$. Hence the open-loop system has an unstable pole of multiplicity 1 at 0.5 .

The transfer matrix of the system is

$$\tilde{P}(s) = \begin{pmatrix} \frac{-1.5(s-0.2)}{(s-0.5)(s+1)} & \frac{0.3(6s+7.5)}{(s-0.5)(s+10)} \\ \frac{4s-0.5}{(s-0.5)(s+1)} & \frac{-(4s+8.5)}{(s-0.5)(s+10)} \end{pmatrix}$$

Note that the unstable pole ($s = 0.5$) appears in *all* elements of $\tilde{P}(s)$, though it has only multiplicity 1. This is not an artifact of the example but rather the generic case.

Let us now compute the zero-order hold discrete equivalent of $\tilde{P}(s)$ for a sampling time of $T = 0.1$. This is a reasonable choice, equal to $1/10$ of the dominant stable time constant and $1/20$ of the unstable time constant of the system.

$$\tilde{P}^*(z) = C(zI - \Phi)^{-1}\Gamma + D$$

where

$$\Phi = \begin{pmatrix} 1.2757 & 1.1138 & 1.0 \\ -0.15462 & 0.44053 & -0.41687 \\ -0.079536 & -0.44598 & 0.60772 \end{pmatrix}$$

$$\Gamma = \begin{pmatrix} 0 & 0 \\ 0.071429 & 0 \\ -0.094864 & 0.071429 \end{pmatrix}$$

For the design we need some information on the potential model error. We will assume a diagonal input multiplicative uncertainty

$$P^*(z) = \tilde{P}^*(z)(I + L_I^*(z)) \quad (53)$$

where

$$L_I^*(z) = \text{diag}\{\ell_1^*(z), \ell_2^*(z)\} \quad (54)$$

and ℓ_1^*, ℓ_2^* are bounded by

$$|\ell_i^*(z)| \leq \bar{\ell}_i^*(z) = \left| 0.2 \frac{z - p_i}{z - p_i^{10}} \cdot \frac{1 - p_i^{10}}{1 - p_i} \right| \quad (55)$$

with $p_i = e^{-T/\tau_i}$. We will also assume that *all* plants $P^*(z)$ have exactly one unstable pole. The bound (55) implies that the uncertainty starts to increase around $\omega = 1/\tau_i$ with slope 1 and flattens out after one decade. Also, the low frequency uncertainty can be as much as 20%. The τ_i s are selected here to correspond to the dominant stable time constants of $\tilde{P}(s)$ associated with the respective inputs, i.e., $\tau_1 = 1$ and $\tau_2 = 0.1$. Bode plots of $\bar{\ell}_1^*, \bar{\ell}_2^*$ are shown in Fig. 8, for $0 \leq \omega \leq \pi/T$.

4.2 Design of \tilde{Q}

First one has to decide on the type of external input v for which \tilde{Q} will be designed. Here we will consider *step-like* disturbances entering at the plant inputs. The diagonal $V(z)$ is of the form described in Cor. 1 with

$$v_1(z) = v_2(z) = v^*(z) = \mathcal{ZL}^{-1}\{v(s)\}$$

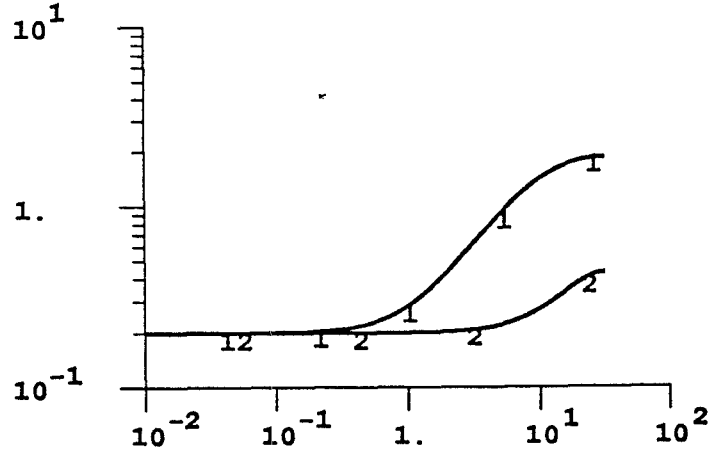


Figure 8: Multiplicative uncertainties $\bar{\ell}_1^*$ and $\bar{\ell}_2^*$.

where $v(s)$ is an appropriate transfer function. Since the v_i s represent the effect of step-like inputs on the plant outputs, $v(s)$ should include both an integrator and a pole at $s = 0.5$. We select

$$v(s) = \frac{s + 0.5}{s(-s + 0.5)}$$

The next task is the factorization of \tilde{P}^* into P_A^* and P_M^* (12). We follow the steps described in Sec. 2.6. This procedure yields the matrices $\Phi_A, \Gamma_A, C_A, D_A$ and $\Phi_M, \Gamma_M, C_M, D_M$ that define $P_A^*(z)$ and $P_M^*(z)$ respectively:

$$\Phi_A = \begin{pmatrix} 1.54714 & 1.50513 & 1.41162 \\ -0.69253 & -0.67372 & -0.63186 \\ -0.098133 & -0.095468 & -0.089537 \end{pmatrix}$$

$$\Gamma_A = \begin{pmatrix} -8.27667 & -3.06852 \\ -15.61316 & -4.02293 \\ -3.78625 & -3.05041 \end{pmatrix}$$

$$C_A = \begin{pmatrix} -7.0645 \times 10^{-4} & -0.012260 & 0.20435 \\ 0.027830 & 0.13477 & -0.46555 \end{pmatrix}$$

$$D_A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Phi_M = \begin{pmatrix} 1.27570 & 1.11380 & 1.0 \\ -0.15462 & 0.44053 & -0.41687 \\ -0.079536 & -0.44598 & 0.60772 \end{pmatrix}$$

$$\Gamma_M = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ -1.32810 & 1 \end{pmatrix}$$

$$C_M = \begin{pmatrix} -0.017300 & -0.060253 & 0.050708 \\ 0.021810 & 0.12528 & -0.18355 \end{pmatrix}$$

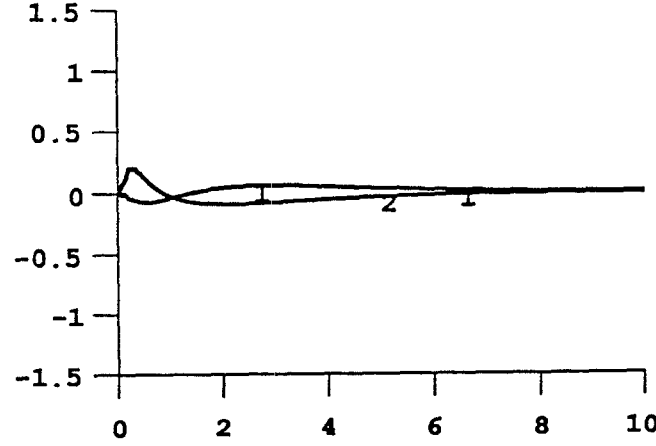


Figure 9: Nominal response to a step change at the plant input. (No filter).

$$D_M = \begin{pmatrix} -0.13652 & 0.086180 \\ 0.39111 & -0.30647 \end{pmatrix}$$

We also need to factor $V(z)$ according to (16), but this is trivial since $V(z)$ is diagonal and $v^*(z)$ can be factored as described by (13).

The final task is to determine $\tilde{Q}(z)$ from (17). For $W = I$ a state space description of $\tilde{Q}(z)$ is given by

$$\Phi_Q = \begin{pmatrix} 1.27570 & 1.11380 & 1 & 0 & 0 \\ -0.57541 & -0.50239 & -0.45106 & -2.25465 & -0.80172 \\ 0.013486 & 0.011775 & 0.010572 & 0.058490 & 3.2232 \times 10^{-3} \\ 0 & 0 & 0 & 0.94873 & 0 \\ 0 & 0 & 0 & 0 & 0.94873 \end{pmatrix}$$

$$\Gamma_Q = \begin{pmatrix} 0 & 0 \\ 81.6566 & 26.2327 \\ -3.09800 & 2.64970 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$C_Q = \begin{pmatrix} 0.42079 & 0.94292 & 0.034188 & 2.25465 & 0.80172 \\ 0.46584 & 0.79454 & 0.64255 & 2.93590 & 1.06154 \end{pmatrix}$$

$$D_Q = \begin{pmatrix} -81.6556 & -26.2327 \\ -105.3488 & -37.4893 \end{pmatrix}$$

Figure 9 shows the response with this controller for the disturbance

$$u'(s) = \begin{pmatrix} s^{-1} \\ s^{-1} \end{pmatrix} \quad (56)$$

entering at the plant inputs, when $P = \tilde{P}$. (The same disturbance will be used in all subsequent simulations).

4.3 Design of F

In this section we will design a filter $F(z)$ that guarantees robust stability in the presence of the model-plant mismatch described by (53). The condition for robust stability is given by (37). Here Δ consists of two scalar blocks and

$$G_{11}^{*,u}(F) = -\tilde{Q}F\tilde{P}^*\tilde{L}_I^*$$

where

$$\tilde{L}_I^* = \text{diag}\{\bar{\ell}_1^*, \bar{\ell}_2^*\}$$

The selection of the filter structure follows Sec. 3.4. A simple scalar filter will be used:

$$F(z) = f(z)I$$

where $f(z)$ is given by (47) with $w = 29$. The tuning parameter α in $f_1(z)$ must be in $[0,1]$ and can be parametrized as

$$\alpha = e^{-T/\lambda} \quad (57)$$

where λ is a positive time constant which becomes the new tuning parameter. We prefer λ over α because λ has a clear physical meaning and effect as was illustrated in Ex. 4. Note that the coefficients of $\phi(z)$ are functions of λ and are obtained from Thm. 9. If one wishes to remove the positivity constraint from the design parameter λ , then one should use (52) instead of (57). In this example however, we only have a single design variable to search over, which is a simple optimization problem. Hence (57) is used here to maintain a clear physical meaning for the optimization variable λ .

For $F = I$ ($\lambda = \alpha = 0$) we find $\mu(G_{11}^{*,u}) = 3.75$, which implies that there exist plants among those described by (53) for which the closed loop system is unstable. A plot of μ is shown in Fig. 10. A search over the parameter λ shows that one has to increase λ to at least 0.5 to get $\mu = 1.0$ so that robust stability is guaranteed. Further increase of λ can reduce $\mu(G_{11}^{*,u})$ even further. Plots of μ for $\lambda = 0.5$ and $\lambda = 1$ can be seen in Fig. 10.

Note, however, that the lower μ for $\lambda = 1$ does not necessarily mean that the performance of the system is superior because $\mu(G_{11}^{*,u})$ is not the robust performance index. For determining robust performance, one has to select an appropriate performance weight W and compute $\mu(G^W)$ (Sec. 3.3). For our particular example, $\tilde{P}(s)$ has an unstable pole at $s = 0.5$ and the uncertainty becomes significant for $\omega > 1$. Therefore there is not much room for performance improvement. The question of robust performance will not be examined any further in this illustration.

Let us now look at some simulations to examine the behavior of the control system when there is model-plant mismatch. The following transfer function was chosen for the “real” continuous plant $P(s)$:

$$P(s) = \tilde{P}(s)(I + L_I(s)) \quad (58)$$

where

$$L_I(s) = \begin{pmatrix} -0.2\frac{s+1}{0.1s+1} & 0 \\ 0 & -0.2\frac{0.1s+1}{0.01s+1} \end{pmatrix} \quad (59)$$

Note that this $L_I(s)$ does not generate a plant that falls exactly in the class described by (53, 54, 55), although the steady-state gains and time constants of $L_I(s)$ match those used in (55) exactly. The reason is that no simple and non-conservative method is available

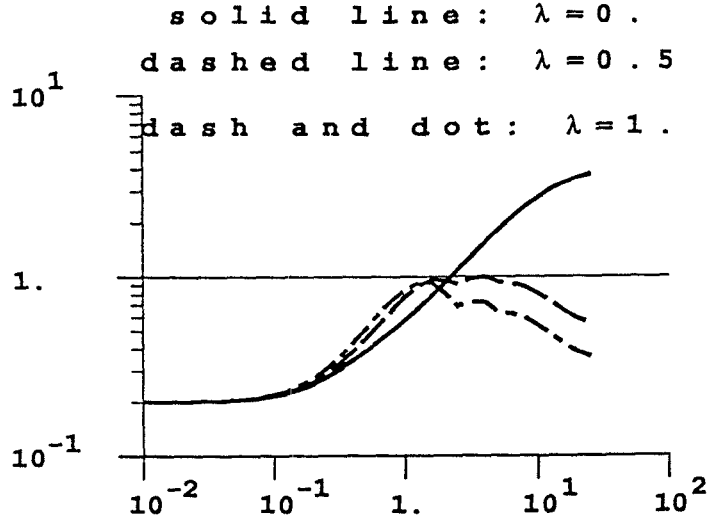


Figure 10: μ (Robust Stability) for different filter parameters λ .

for translating a type of uncertainty description (input multiplicative in this case) from the s -domain to exactly the same type in the z -domain. As explained in Sec. 3.1, such descriptions may be obtained either from first – principles models or via experiments conducted with different sampling rates. For the purposes of this example, (59) yields a plant sufficiently close to the class described by (53) to serve our illustration goals.

The responses to the input disturbance (56) are shown in Fig. 11 for $\lambda = 0.5$ and in Fig. 12 for $\lambda = 1$, for both the nominal case ($P = \tilde{P}$) and the case of model-plant mismatch with P given by (58). Without the IMC filter, the system is unstable for the “real” plant P in (58) as expected from the large value of $\mu(G_{11}^{*,u})$. The nominal response is shown in Fig. 9. The responses for $\lambda = 1$ are not significantly better than that for $\lambda = 0.5$, although the robust stability μ is smaller for $\lambda = 1$. This is not surprising because $\mu(G_{11}^{*,u})$ is an indicator of stability only.

5 Concluding Remarks

This paper provides a synthesis procedure for digital multivariable controllers. The practical appeal of the methodology derives to a great extent from the use of the two-step IMC design concept. The previous status of the IMC procedure left unanswered several important questions, which are addressed in this paper. The methodology is extended to open-loop unstable processes and the limitations imposed by open-loop unstable poles on achievable performance and robustness are quantified. This extension would not have been possible without a change in the first-step design. Instead of directly specifying the structure (zero / nonzero elements) of the closed-loop transfer matrix, in the first step the controller is designed to minimize the error for every setpoint or disturbance vector in a designer-specified set and their linear combinations. This controller combines desirable properties of the H_2 -optimal and deadbeat type controllers so that it is free of intersample rippling problems.

In the second step of the IMC design, the model-plant mismatch is taken into account by designing a low-pass discrete IMC filter. No rigorous procedure for the determination

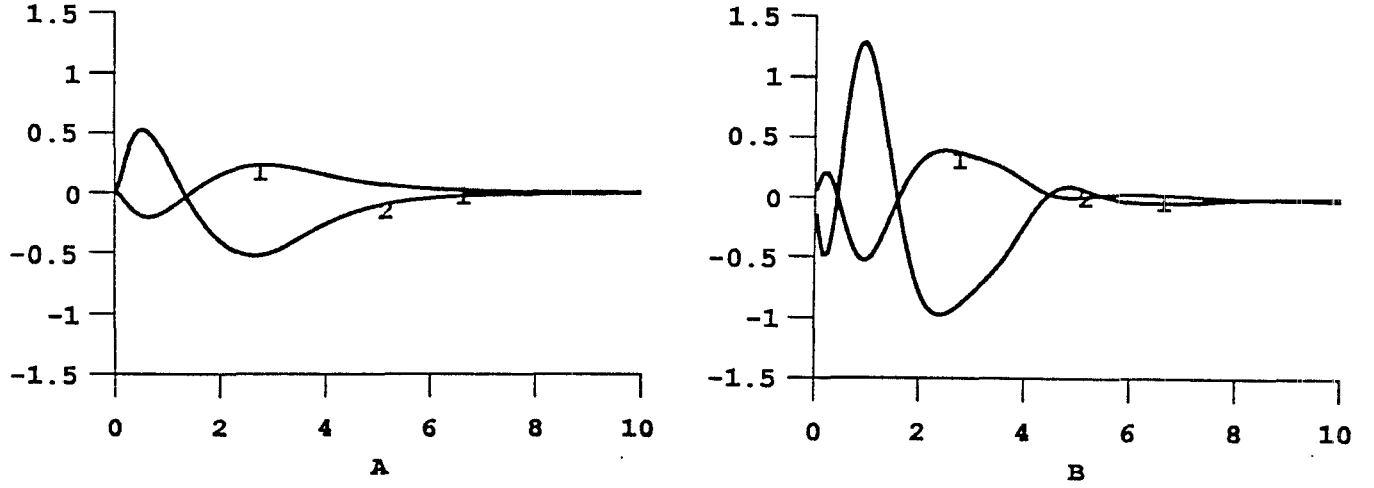


Figure 11: Responses (A) for nominal system and (B) the plant given by (58) for IMC filter time constant $\lambda = 0.5$.

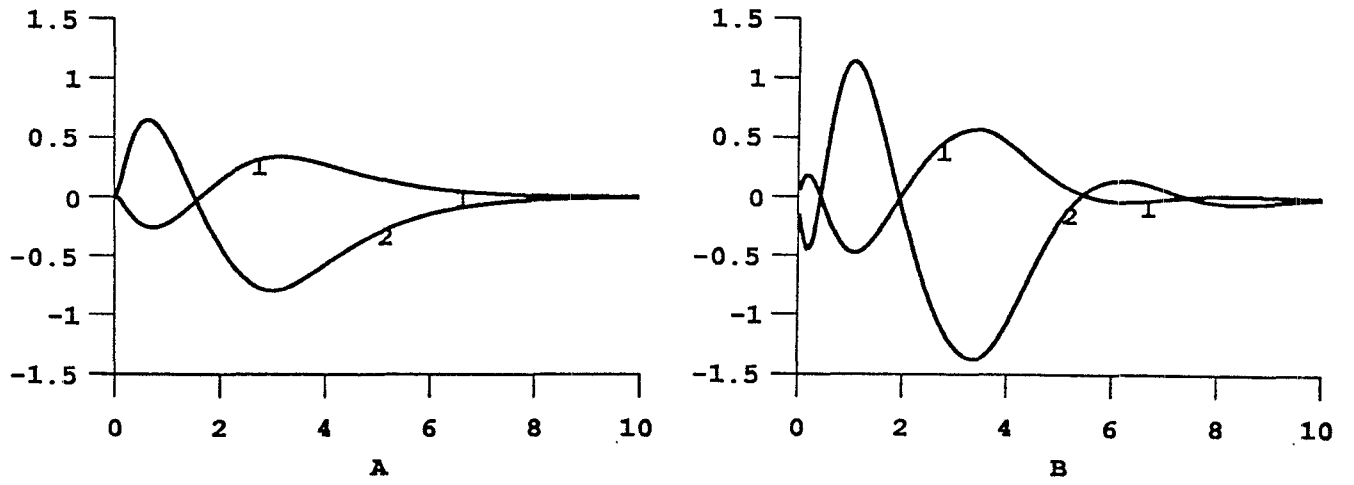


Figure 12: Responses (A) for nominal system and (B) the plant given by (58) for IMC filter time constant $\lambda = 1.0$.

of the values of the parameters of a multivariable filter existed. This paper proposes to obtain these values as the result of the optimization of an SSV based objective function, which reflects the performance of the continuous plant outputs, so that bad intersample behavior is avoided. The use of the SSV allows the treatment of general types of model-plant mismatch. It should also be pointed out that the filter parameters still have a clear physical meaning and effect through their relationship to the desired closed-loop speed of response and therefore they can be used as on-line tuning parameters, if so desired.

A Proof of Thm. 2

The fact that Q_1 has to be proper in order for Q to be proper and vice versa, follows from the properness of Q_0 . For the following part of the proof we will use the fact that P^* and P_γ^* have the same unstable poles.

\Rightarrow We shall show that any Q given by (7) makes S stable. From substitution of (7) into (6) it follows that all that is required is that $\begin{pmatrix} P^*Q_1 & Q_1P^* & P^*Q_1P^* \end{pmatrix}$ be stable. From the properties of Q_1 , it follows that the third element in the above matrix is stable. Stability of the other two elements follows by pre- and post-multiplication of $P^*Q_1P^*$ by $(P^*)^{-1}$, since according to assumptions A1 and A2, P^* has no zeros at the location of its unstable poles and these are the only possible unstable poles of S .

\Leftarrow Assume that Q makes S stable. Then the difference matrix

$$\Delta S = S(Q) - S(Q_0) = \begin{pmatrix} P^*(Q - Q_0) & P^*(Q - Q_0)P^* \\ (Q - Q_0) & (Q - Q_0)P^* \end{pmatrix}$$

is stable. This implies that $(Q - Q_0) = Q_1$ and $P^*Q_1P^*$ are stable. \square

B Proof of Thm. 3

We shall assume $W = I$ for simplicity. The proof of the weighted case follows exactly the same steps. Let V_0 be a diagonal matrix where each column satisfies Assn. A3 and every element has ℓ_v poles at $z = 1$, where ℓ_v is the maximum number of such poles in any element of v . Assume that there exists Q_0 , which stabilizes P^* in the sense of Thm. 2 and also makes $(I - P^*Q_0)V_0$ stable. Its existence will be proven by construction. Substitution of (7) into (11) and use of the fact that pre- or post-multiplication of a function with an allpass does not change its L_2 -norm, yields:

$$\Phi(v^*) = \|z^{-1}(P_A^*)^{-1}(I - P^*Q_0)\hat{v} - z^{-1}P_M^*Q_1\hat{v}\|_2^2 \triangleq \|f_1 - f_2Q_1\hat{v}\|_2^2$$

The term f_1 has no poles at $z = 1$ because $(I - P^*Q_0)V_0$ has no such poles. Any rational function $f_1(z)$ with no poles on the UC, can be uniquely decomposed into a strictly proper, stable part $\{f_1\}_+$ in H_2^n and a strictly unstable part $\{f_1\}_-$ in $(H_2^n)^\perp$:

$$f_1 = \{f_1\}_+ + \{f_1\}_-$$

where the closed subspace of L_2^n of functions having analytic continuations inside the UC is

defined as $(H_2^n)^\perp$; its orthogonal complement is denoted by H_2^n .¹ Note that with the above definitions a constant function is in $(H_2^n)^\perp$. $(H_2^n)^\perp$ also includes all rational z -transfer functions that are strictly unstable — i.e., which have all their poles strictly outside the UC [including poles at $z = \infty$ (improper transfer functions)]. All strictly proper, stable rational z -transfer functions are in H_2^n . Note that according to the definition of H_2^n , $(H_2^n)^\perp$, any improper terms as well as the constant term in a partial fraction expansion of f_1 , belong in $\{f_1\}_-$. Next we want to show that $f_2 Q_1 \hat{v}$ has to be stable. The fact that $(I - P^* Q_0) V_0$ is stable implies that $(I - P^* Q_0) \hat{v}$ is stable. We require that $(I - P^* Q) v$ has no poles outside the UC and therefore that $(I - P^* Q) \hat{v} = (I - P^* Q_0) \hat{v} - P^* Q_1 \hat{v}$ have no poles outside the UC. But since $(I - P^* Q_0) \hat{v}$ is stable, this requirement reduces to $P^* Q_1 \hat{v}$ having no poles outside the UC. Also in order for $\Phi(v^*)$ to be finite, Q_1 must be such that $(I - P^* Q) \hat{v}$ has no poles on the UC. But since $(I - P^* Q_0) \hat{v}$ is stable, this is equivalent to $P^* Q_1 \hat{v}$ having no poles on the UC. Hence the optimal Q_1 must be such that $P^* Q_1 \hat{v}$ is stable. Then the only possible unstable poles of $f_2 Q_1 \hat{v} = z^{-1} (P_A^*)^{-1} P^* Q_1 \hat{v}$ are the poles of $(P_A^*)^{-1}$. But Assns. A1, A2 imply that the poles of $(P_A^*)^{-1}$ are not among those of $f_2 Q_1 \hat{v}$ and therefore $f_2 Q_1 \hat{v}$ has to be stable. To proceed we will assume that Q_1 has this property. We will verify later that the solution indeed has this property.

Hence we can write

$$\Phi(v^*) = \|\{f_1\}_-\|_2^2 + \|\{f_1\}_+ - f_2 Q_1 \hat{v}\|_2^2 \quad (60)$$

The first term on the RHS of (60) does not depend on Q_1 . Hence for solving O1 we only have to look at the second term. The obvious solution is

$$Q_1 \hat{v} = f_2^{-1} \{f_1\}_+ \quad (61)$$

Clearly such a Q_1 produces a stable $f_2 Q_1 \hat{v}$ as was assumed. It should now be proved that Q_1 s that satisfy the internal stability requirements exist among those described by (61), so that the obvious solution is a true solution. For $n = 1$, (61) yields a unique Q_1 , which can be shown to satisfy the requirements by following the arguments in the proof of Thm. 4. For $n \geq 2$ write

$$\hat{v} \triangleq (\hat{v}_1 \quad \hat{v}_2 \quad \dots \quad \hat{v}_n)^T$$

$$\hat{V}_2 \triangleq (\hat{v}_2 \quad \dots \quad \hat{v}_n)^T$$

$$Q_1 \triangleq (q_1 \quad q_2)$$

where without loss of generality the first element of v^* , and thus \hat{v}_1 , is assumed to be nonzero. Also q_1 is $n \times 1$ and q_2 is $n \times (n - 1)$. Then from (61) it follows that

$$Q_1 = (\hat{v}_1^{-1} (f_2^{-1} \{f_1\}_+ - q_2 \hat{V}_2) \quad q_2) \quad (62)$$

We now need to show that a proper, stable q_2 exists such that Q_1 is proper, stable and produces a stable $P^* Q_1 P^*$. Select a q_2 of the form:

¹This definition of H_2^n and $(H_2^n)^\perp$ is exactly the opposite of the one encountered in the mathematics literature, where H_2 corresponds to the L_2 -functions with analytic continuations inside the UC. Our definitions have been chosen to be consistent with the common definitions of H_2, H_2^\perp for Laplace transfer functions cite in the control literature (e. g., Francis, 1987). The transformation $\lambda = z^{-1}$ could have been employed to introduce consistency with the mathematics literature but this would unnecessarily complicate the notation.

$$q_2(z) = \hat{q}_2(z)(1 - z^{-1})^{3l_v} \prod_{i=1}^k (1 - \pi_i z^{-1})^3$$

where \hat{q}_2 is proper and stable and $\{\pi_1, \dots, \pi_k\}$ are the poles of P^* outside the UC. Then from (62) it follows that in order for $P^*Q_1P^*$ to be stable it is sufficient that $P^*\hat{v}_1^{-1}f_2^{-1}\{f_1\}_+ \{P^*\}_{1^{st} row}$ has no poles on or outside the UC. But $P^*f_2^{-1} = zP_A^*$ is stable and the only possible poles of $\hat{v}_1^{-1}\{P^*\}_{1^{st} row}$ on or outside the UC are poles of \hat{v}_1^{-1} outside the UC, because of Assns. A3 and A4. These are also the only possible unstable poles of Q_1 . Let α be such a pole (zero of \hat{v}_1). Then for stability we need to find \hat{q}_2 such that

$$\hat{q}_2(\alpha)\hat{V}_2(\alpha) = (1 - \alpha^{-1})^{-3l_v} \prod_{i=1}^k (1 - \pi_i \alpha^{-1})^{-3} f_2^{-1}(\alpha) \{f_1\}_+ \Big|_{z=\alpha} \quad (63)$$

The above equation always has a solution because the vector $\hat{V}_2(\alpha)$ is not identically zero since any common zeros in v^* outside the UC were factored out in v_0 .

We now need to examine the properness of Q_1 . Since $(P_M^*)^{-1}$ is proper and $\{f_1\}_+$ is strictly proper, $f_2^{-1}\{f_1\}_+$ is proper. Then if \hat{v}_1^{-1} is improper (\hat{v}_1 strictly proper) there exists at least one element in \hat{V}_2 that is semi-proper. Hence by solving a system of linear equations we can always select a $\hat{q}_2(z)$ such that of the first impulse response coefficients of $f_2^{-1}\{f_1\}_+ - q_2\hat{V}_2$, as many are zero as needed to make the first element of the matrix in (62) proper.

We shall now proceed to obtain an expression for $Q\hat{v}$. (7) and (62) yield

$$\begin{aligned} Q\hat{v} &= z(P_M^*)^{-1} \left[z^{-1}(P_A^*)^{-1}P^*Q_0\hat{v} - \{z^{-1}(P_A^*)^{-1}P^*Q_0\hat{v}\}_+ + \{z^{-1}(P_A^*)^{-1}\hat{v}\}_+ \right] \\ &= z(P_M^*)^{-1} \left[\{z^{-1}(P_A^*)^{-1}P^*Q_0\hat{v}\}_{0-} + \{z^{-1}(P_A^*)^{-1}\hat{v}\}_+ \right] \end{aligned} \quad (64)$$

where $\{\cdot\}_{0-}$ indicates that in the partial fraction expansion all poles on or outside the UC are retained. For (64), these poles are the poles of \hat{v} on or outside the UC; $(P_A^*)^{-1}P^*Q_0 = P_M^*Q_0$ is strictly stable and proper because of Assn. A1 and the fact that Q_0 is a stabilizing controller. The fact that $(I - P^*Q_0)V_0$ has no poles at $z = 1$ imply that $(I - P^*Q_0)$ and its derivatives up to and including the $(\ell_v - 1)^{th}$ are equal to zero at $z = 1$. Also, the fact that $(I - P^*Q_0)V_0$ is stable and that the columns of this diagonal V_0 satisfy Assn. A3, imply that $(I - P^*Q_0) = 0$ at $1, \pi_1, \dots, \pi_k$. Thus (64) simplifies to (15).

We now need to establish that a stabilizing controller Q_0 exists with the property that $(I - P^*Q_0)V_0$ is stable. The selection of a V_0 with the properties mentioned at the beginning of this proof and its use instead of V in (17) yields such a controller. \square

C Proof of Thm. 4

Again we assume $W = I$ for simplicity. From (63), (11), and (17) it follows that

$$\Phi(v^1) + \Phi(v^2) + \dots + \Phi(v^n) = \|(I - P\bar{Q})V\|_2^2 \triangleq \Phi(V)$$

The minimization of $\Phi(V)$ follows the steps in the proof of Thm. 3 up to (61), with V_M used instead of \hat{v} . In this case ℓ_v is the maximum number of poles at $z = 1$ in any element of V . From the equivalent to (61) we obtain

$$Q_1 = f_2^{-1}\{f_1\}_+ V_M^{-1} \quad (65)$$

We now have to establish that Q_1 is stable, proper and produces a stable $P^*Q_1P^*$. In $P^*Q_1P^*$ the unstable poles of the P^* on the left cancel with those of $(P_M^*)^{-1}$ in f_2^{-1} . As for the P^* on the right, cancellation follows from Assn. A5. Then in the same way that (15) follows from (64), (17) follows from (65). \square

D Proof of Thm. 5

$W = I$. A stabilizing controller that solves Obj. O2 has to solve Obj. O1 for all v^i , $i = 1, \dots, n$. Satisfying (15) for every v^i is equivalent to

$$\tilde{Q} = z(P_M^*)^{-1}\{z^{-1}(P_A^*)^{-1}\hat{V}\}_*\hat{V}^{-1} \quad (66)$$

Hence the above \tilde{Q} is the only potential solution for Obj. O2. However, it is not necessarily a stabilizing controller since not only stabilizing \tilde{Q} s satisfy (15) for some v^* . Indeed, if \hat{V} is non-minimum phase, \hat{V}^{-1} is unstable and/or improper and this results in an unstable and/or improper \tilde{Q} , which is therefore unacceptable. Hence in such a case, there exists no solution for Obj. O2, which completes the proof of part (i) of the theorem.

In the case where \hat{V}^{-1} is stable and proper (\hat{V} minimum phase), the controller given by (66) is stable and proper and therefore it is the same as the one given by (17). This fact can be explained as follows. We have

$$V = \hat{V}V_0 \quad (67)$$

where

$$V_0 = \text{diag}\{v_0^1, v_0^2, \dots, v_0^n\}$$

Since \hat{V}^{-1} is stable and proper, (67) represents a factorization of V similar to that in (16). From spectral factorization theory it follows that

$$\hat{V}(z) = V_M(z)A$$

where A is a constant matrix such that $AA^H = I$. Then (17) is not altered when \hat{V} is used instead of V_M because A cancels.

Let us now assume without loss of generality that the first j v^i s have the same v_0^i s. Consider a v^* that is a linear combination of v^1, \dots, v^j :

$$v^*(z) = \alpha_1 v^1(z) + \dots + \alpha_j v^j(z)$$

Then it follows that

$$\begin{aligned} v_0(z) &= v_0^1(z) = \dots = v_0^j(z) \\ \hat{v}(z) &= \alpha_1 \hat{v}^1(z) + \dots + \alpha_j \hat{v}^j(z) \end{aligned} \quad (68)$$

One can easily check that a \tilde{Q} that satisfies (15) for $\hat{v}^1, \dots, \hat{v}^j$, will also satisfy (15) for the \hat{v} given by (68) because of the property

$$\{\alpha_1 f_1(z) + \dots + \alpha_j f_j(z)\}_* = \alpha_1 \{f_1(z)\}_* + \dots + \alpha_j \{f_j(z)\}_*$$

But then from Thm. 3 it follows that if a stabilizing controller \tilde{Q} satisfies (15) for \hat{v} , then it minimizes the L_2 error $\Phi(v^*)$. \square

E Proof of Thm. 7

The following lemma is used in the proofs of Thms. 7 and 8.

Lemma 1 *Let $G(x) = C(xI - A)^{-1}B + xD$. Then $xG(x) = C(xI - A)^{-1}AB + CB + xD$.*

Proof. We have

$$\begin{aligned} xG(x) &= C(xI - A)^{-1}(A + xI - A)B + xD \\ &= C(xI - A)^{-1}AB + CB + xD \end{aligned}$$

Proof of Thm. 7. Since $P^*(z)$ is assumed to have no poles at $z = -1$, $\Phi + I$ is nonsingular. We can then write \square

$$\begin{aligned} \hat{G}(s) &= C\left(\frac{1+s}{1-s}I - \Phi\right)^{-1}\Gamma + D \\ &= (1-s)C(s(\Phi + I) + I - \Phi)^{-1}\Gamma + D \\ &= (1-s)C(sI - (\Phi + I)^{-1}(\Phi - I))^{-1}(\Phi + I)^{-1}\Gamma + D \end{aligned} \tag{69}$$

Use of Lem. 1 in (69) yields

$$\begin{aligned} \hat{G}(s) &= C(sI - (\Phi + I)^{-1}(\Phi - I))^{-1}(\Phi + I)^{-1}\Gamma \\ &\quad - \left[C(sI - (\Phi + I)^{-1}(\Phi - I))^{-1}(\Phi + I)^{-1}(\Phi - I)(\Phi + I)^{-1}\Gamma + C(\Phi + I)^{-1}\Gamma \right] + D \\ &= C(sI - (\Phi + I)^{-1}(\Phi - I))^{-1}(I - (\Phi + I)^{-1}(\Phi - I))(\Phi + I)^{-1}\Gamma + D - C(\Phi + I)^{-1}\Gamma \\ &= C(sI - (\Phi + I)^{-1}(\Phi - I))^{-1}2(\Phi + I)^{-2}\Gamma + D - C(\Phi + I)^{-1}\Gamma \end{aligned}$$

\square

F Proof of Thm. 8

$I - \hat{A}$ is nonsingular because $\hat{G}(s)$ is assumed to have no poles at $s = 1$. We have

$$\begin{aligned} G^*(z) &= \hat{C}\left(\frac{-1+z}{1+z}I - \hat{A}\right)^{-1}\hat{B} + \hat{D} \\ &= (1+z)\hat{C}(z(I - \hat{A}) - (I + \hat{A}))^{-1}\hat{B} + \hat{D} \\ &= (1+z)\hat{C}(zI - (I - \hat{A})^{-1}(I + \hat{A}))^{-1}(I - \hat{A})^{-1}\hat{B} + \hat{D} \end{aligned} \tag{70}$$

Application of Lem. 1 to (70) yields

$$\begin{aligned} G^*(z) &= \hat{C}(zI - (I - \hat{A})^{-1}(I + \hat{A}))^{-1}(I - \hat{A})^{-1}\hat{B} \\ &\quad + \left[\hat{C}(zI - (I - \hat{A})^{-1}(I + \hat{A}))^{-1}(I - \hat{A})^{-1}(I + \hat{A})(I - \hat{A})^{-1}\hat{B} + \hat{C}(I - \hat{A})^{-1}\hat{B} \right] + \hat{D} \\ &= \hat{C}(zI - (I - \hat{A})^{-1}(I + \hat{A}))^{-1}(I + (I - \hat{A})^{-1}(I + \hat{A}))\hat{B} + \hat{D} + \hat{C}(I - \hat{A})^{-1}\hat{B} \\ &= \hat{C}(zI - (I - \hat{A})^{-1}(I + \hat{A}))^{-1}2(I - \hat{A})^{-2}\hat{B} + \hat{D} + \hat{C}(I - \hat{A})^{-1}\hat{B} \end{aligned}$$

\square

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