ABSTRACT

Title of dissertation: Special Unipotent Arthur Packets for Real Reductive Groups

Jonathan F. Fernandes
Doctor of Philosophy, 2019

Dissertation directed by: Professor Jeffrey Adams
Department of Mathematics

Let $\mathbf{G}(\mathbb{R})$ be a real reductive group. In this thesis we study the unitary representations of $\mathbf{G}(\mathbb{R})$. In particular, we study the special Arthur unipotent parameters and the associated packets of irreducible representations of $\mathbf{G}(\mathbb{R})$. It is conjectured that these unipotent representations form the building blocks for all unitary representations of $\mathbf{G}(\mathbb{R})$.

To understand unipotent representations, we will need to compute the following invariants of irreducible representations of $\mathbf{G}(\mathbb{R})$: complex associated variety and the theta associated variety. Even though these invariants are theoretically understood, there are no known (at least to this author) results/algorithms to compute them explicitly.

The primary results of this thesis provide algorithms to compute these invariants explicitly in many cases. We then use these invariants to compute information about unipotent Arthur packets, and in favorable cases, their entire contents explicitly. In

unfavorable cases, we show how to extract more information from our results by using the stable sum formula.

We have implemented these algorithms into the Atlas of Lie Groups software, available at www.liegroups.org. We also provide some tables of data compiled using the output from Atlas.

Special Unipotent Arthur Packets for Real Reductive Groups

by

Jonathan F. Fernandes

Dissertation submitted to the Faculty of the Graduate School of the
University of Maryland, College Park in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
2019

Advisory Committee:
Professor Jeffrey Adams, Chair/Advisor
Professor Thomas Haines
Professor Xuhua He
Professor Patrick Brosnan
Professor William Gasarch (Dean's Representative)

© Copyright by
Jonathan F. Fernandes
2019

Dedication

To my grandmother - late Maria Piedade Dias

To my parents - George and Tina Fernandes

To my siblings - Laura, Leona, and Joel

Thank you and love you all!

Acknowledgments

I am very thankful for the cumulative efforts of many people that have influenced me in my journey, which has now culminated in the writing of this thesis.

I would like to thank my academic advisor, Prof. Jeffrey Adams, who has been very gracious in molding me into the mathematician that I am today. I am thankful for his encouragement and support during many critical moments of my doctoral studies.

I thank Prof. Thomas Haines, Prof. Patrick Brosnan, Prof. Xuhua He, and William Gasarch - for being on my thesis committee. Their generous feedback on an earlier draft of this thesis helped substantially improve the presentation in the final version.

I thank Prof. Michael Rapoport, Prof. Dipendra Prasad, Prof. Jonathan Rosenberg, and Prof. John Milson for their comments and feedback after a talk I gave at the Lie Groups Seminar at the University of Maryland. I am especially thankful to Dr. Rapoport's feedback, which inspired me to restructure substantial parts of this thesis.

I thank Prof. David Vogan, Prof. Dan Barbasch, Prof. Peter Trapa, Prof. Marc Van Leeuwen, Prof. Annegret Paul, and the rest of the team of Atlas of Lie Groups collaborators. The Atlas Software has helped in scaling the results of this thesis, to do computations that would otherwise be very difficult if not impossible. I am especially thankful to Prof. Leeuween for incorporating the scripting functionality in to the Atlas Software.

I would like to thank many of my mentors from past institutions that I have been affiliated with. I thank Prof. Ramesh Sreekantan and Prof. Sankaran Viswanath, who advised me on my Master's thesis at the Indian Institute of Science (IISc), which I believe influenced my leanings of doctoral research towards Lie groups. I thank Prof. Sundaram Thangavelu (IISc), Prof. Gadadhar Misra (IISc), Prof. Gautam Bharali (IISc), Prof. Kaushal Verma (IISc), Prof. Amritanshu Prasad (IMSc), Prof. S. Kumaresan (University of Hyderabad), and Prof. Amit Hogadi (TIFR) for their mentorship and mathematical inspiration.

I thank my teacher from Chowgule College of Arts and Sciences, Prof. Anand Masur and Prof. Amonkar. They were the ones to sow the seeds of mathematical curiosity, and to nourish them through my undergraduate studies.

I want to acknowledge the support and camaraderie my incredible friends and peer-mentors. I want to especially remember Balakumar (Bala) - who has taught me math for as long as I have known him, and Saptak Banerjee - with whom I have shared much time and sleepless nights working math assignments (at IISc).

I thank fellow Pandavas - Abinav Verma, Amita Malik, Prahallad Deb, and Sayan Bagchi. You made IISc and everything that followed fun! I also want to remember the other characters of the Mahabharata narrative at IISc like Prof. Thangavelu as Drona, and Shakuni, Draupadi, Eklavya, etc (who are better left un-named here). I want thank Purvi Gupta, Sandeep Bhupatiraju, Amita, and Abhinav for the many adventures together, somehow it made mathematics make sense!

I thank Ran Cui and Wan Yu, for sharing most of my mathematical journey at UMD,

in addition to sharing the office (together with Richard Rast and Srimathy) and the same academic advisor.

Marie Chau, Karamatou, Neethu George, Nicole Huang, Meghan Meros, David Russell (fun guitar times!), Stephan (what would the Holiday Parties be without you?), Reshma Nargund, Jacky Chong, Emily Morrison, Elizabeth Rudd, Phil Jones, Jake Spatz, Ramu Pyreddy, Rebecca Matthews, and Leah Scheaperow - thank you for adding a spark to my life!

I want to thank my room-mates through grad-school, I think living with them greatly enhanced my grad-school experience. I want to thank Wrick Sengupta, Ramakrishna Padnamabhan, and Sidharth Pal. I want to thank Ranchu - for his incredible friendship and the many adventures that we shared together.

Graduate school was fun because of the many communities that I could be a part of. I want to thank JTR Jujutsu International, especially late grandmaster Dong Jin Kim, for his love and mentorship at the Dojo. I want to thank the Tango community in DC and the Argentine Tango Club at UMD for the many fun times. Being a part of the choir at St. Mark the Evangelist church was a very enriching experience, thankful to Patti Sullins and rest of the choir. I want to thank the DC-Tango Orchestra and Brett Lemley for letting a guitarist share in making music together. I would also like to thank the members of the UMD Men's Chorus, especially Ianthe Marini, for adding the choral magic to my life at UMD.

I would also like to thank the staff at the Math Department at UMD, especially Bill

Schildneckt, Cristina Garcia, Ruth Yun, Rynetta Gumbs, Celeste Regaldo, and Liz Wincek.

Finally and most importantly, I want to thank my family. I wish my grandmother, late Maria Piedade Dias, would have been here to witness this moment. I am thankful for everything that my parents that have done for me to be where I am today. I am thankful for their many sacrifices and their constant support towards my aspirations. I am blessed to have wonderful siblings - Laura, Leona, Joel, and Lars Daldorff (extended sibling!) - thank you for your support and companionship all through my life. I am especially thankful to Laura for convincing me to not to drop mathematics in high school (I wanted to be a medical doctor!), and hence starting this chain of events which has forward propagated to a mathematics doctorate!

This list is surely not exhaustive, I apologize if I have forgotten to mention you here, yet am very thankful to you for being a part of this journey with me!

Table of Contents

Chapter 1: Introduction

Suppose $\mathbf{G}$ is a complex connected reductive algebraic Lie group and let $\mathfrak{g}$ be its Lie algebra. Studying the representation theory of the real forms of $\mathbf{G}$ has been a major focus over past few decades. Fix a real form $\mathbf{G}(\mathbb{R})$ of $\mathbf{G}$ and let $\mathbf{K}(\mathbb{R})$ be the maximal compact subgroup in $\mathbf{G}(\mathbb{R})$ with complexification $\mathbf{K}$. Let $\Pi(\mathfrak{g}, \mathbf{K})$ be the set of irreducible $(\mathfrak{g}, \mathbf{K})$-modules of $\mathbf{G}(\mathbb{R})$. This set has been completely classified by results of Langlands, Harish-Chandra, Vogan, Knapp, Zuckerman et al. Let $\Pi_h(\mathfrak{g}, \mathbf{K})$ be the set of irreducible $(\mathfrak{g}, \mathbf{K})$-modules of $\mathbf{G}(\mathbb{R})$ equipped with an $(\mathfrak{g}, \mathbf{K})$-invariant hermitian form. The results of Knapp and Zuckerman completely classify this set as a subset of $\Pi(\mathfrak{g}, \mathbf{K})$. Finally, let $\Pi_u(\mathfrak{g}, \mathbf{K})$ be the set irreducible $(\mathfrak{g}, \mathbf{K})$-modules in $\Pi_h(\mathfrak{g}, \mathbf{K})$ such that the invariant $(\mathfrak{g}, \mathbf{K})$-hermitian form is positive definite. The classification of $\Pi_u(\mathfrak{g}, KC)$ is a challenging and an important problem. The Atlas of Lie Groups Project - a collaboration between a wide network of mathematicians led by Jeffrey Adams, David Vogan, Marc Van Leuven, etc. - has been able to identify $\Pi_u(\mathfrak{g}, \mathbf{K})$ as a subset of $\Pi_h(\mathfrak{g}, \mathbf{K})$ using computer software computations implemented in the Atlas Software. More information about this project can be found at [Ada14].

The goal of this thesis is to understand unitary representations of $\mathbf{G}(\mathbb{R})$, that is, the set $\Pi_u(\mathfrak{g}, \mathbf{K})$. Specifically, we want to study certain representations that are conjectured to be the building blocks for the set $\Pi_u(\mathfrak{g}, \mathbf{K})$, these representations are called "unipotent representations". The notion of unipotent representations is not standard. Defining and exploring the properties of unipotent representations has been an area of active research in the field. We primarily follow the work of Arthur, Vogan, Adams, Barbasch, et al in development of these ideas.

In [Vog87], Vogan lays out a program to possibly classify $\Pi_u(\mathfrak{g}, \mathbf{K})$ along with ideas to suggest what properties might be expected of unipotent representations. He ends his introduction to [Vog87] as follows "Implicit in this discussion is the hope that the ideas described here suffice to produce all the irreducible unitary representations of any reductive group $\mathbf{G}(\mathbb{R})$. Because the constructions of complementary series and unipotent representations are still undergoing improvement, this hope is as yet not precisely defined, much less realized ...... I hope that the reader will be not disappointed by this imcompleteness, but enticed by the work still to be done." This was written in 1987, much progress has been achieved since then.

In the early 2000's the Atlas of Lie Groups project was initiated by Adams, Vogan, du Cloux, et al, to use computers "to make available information about representations of reductive Lie groups. Of particular importance is the problem of the unitary dual: classifying all of the irreducible unitary representations of a given Lie group". The Atlas Software is a powerful computational tool, that can be used to compute a variety of information about the structure of Lie groups and their representations.

The primary contribution of this thesis is in the understanding of the representation theory of $\mathbf{G}(\mathbb{R})$ and the notions of unipotent representations developed by Arthur and Vogan in [Vog87], and formalized by Adams, Barbasch, and Vogan in [ABV92], and to put these ideas in the computational context of the Atlas Software.

We now proceed to describing the results of this paper in a bit more detail. We start with the following definition of Arthur parameters, first defined in [ABV92] as follows, and can be found in Chapter 3 of this thesis (Definition 3.3):

Definition 1.1 (Arthur Parameter)**.** *An Arthur parameter for $\boldsymbol{G}$ is a homomorphism $\psi : W_{\mathbb{R}} \times SL(2, \mathbb{C}) \longrightarrow^{\Gamma} \boldsymbol{G}^{\vee}$ satisfying*

1. *the restriction of ψ to $W_{\mathbb{R}}$ is a tempered (Definition 22.3, [ABV92]) Langlands parameter,*
2. *the restriction of ψ to $SL(2, \mathbb{C})$ is holomorphic.*

We say that ψ is a *unipotent Arthur parameter* if ψ restricted to the identity component of $W_{\mathbb{R}}$ is trivial.

Given a unipotent Arthur parameter, we can attach to it two algebraic objects: 1) a "theta form" (Definition 2.4)of a complex nilpotent orbit $\mathcal{O}^{\vee}$ for the dual group $\mathbf{G}^{\vee}$, and 2) a Langlands parameter ϕ_{ψ} for $\mathbf{G}(\mathbb{R})$.

As a result, the study unipotent Arthur parameters and packets is deeply interwoven with the study of nilpotent orbits. Complex nilpotent orbits of $\mathbf{G}$ are well studied objects, when $\mathbf{G}$ is of classical type, there is well known classification in terms of certain integer partitions of $n+1$, $2n$, or $2n+1$ where n is the rank of $\mathbf{G}$.

A classification of the theta forms of $\mathcal{O}^{\vee}$ is not known. As a first step, we address this problem in Theorem **??**, under the assumption that $\mathcal{O}^{\vee}$ is even (Definition 2.2).

We can now use this classification to compute complex associated varieties and theta associated varieties (see Chapter 5 for background on associated varieties).

In Theorem 6.2 we show one can compute the complex associated variety $AV_{\mathbb{C}}(\pi)$ for any irreducible representation π of $\mathbf{G}(\mathbb{R})$. Furthermore, under certain good conditions, we can compute the theta-associated variety $AV_{\theta}(\pi)$ of $\mathbf{G}(\mathbb{R})$.

To a unipotent Arthur parameter, one can attach two types of packets as follows (Definition 7.2 and Definition 7.3 respectively):

Definition 1.2 (Weak Unipotent Arthur Packet)**.** *Let $\mathcal{O}^\vee$ be a dual even complex nilpotent orbit. Choose δ such that $\lambda(\mathcal{O}^\vee) \in \delta + X^*(\boldsymbol{H})$. The weak unipotent packet corresponding to the triple $(\xi, \eta^\vee, \mathcal{O}^\vee)$ is the set*

$$\Pi^u_{weak}(\xi, \eta^\vee, \mathcal{O}^\vee) := \{\pi \in \mathcal{B}(\lambda(\mathcal{O}^\vee)) := T_\delta^{\lambda(\mathcal{O}^\vee)}(\mathcal{B}(\delta)) \mid AV_{\mathbb{C}}(\pi^\vee) = \overline{\mathcal{O}^\vee}\}. \tag{1.1}$$

Definition 1.3 (Special Unipotent Arthur Packet)**.** *The special unipotent Arthur packet corresponding to the tuple $(\xi, \eta^\vee, \mathcal{O}^\vee_{\boldsymbol{K}^\vee})$ is the set*

$$\Pi^u(\xi, \eta^\vee, \mathcal{O}^\vee_{\boldsymbol{K}^\vee}) := \{\pi \in \Pi^u_{weak}(\xi, \eta^\vee, \mathcal{O}^\vee) \mid \overline{\mathcal{O}^\vee_{\boldsymbol{K}^\vee}} \subset AV_\theta(\pi^\vee)\}. \tag{1.2}$$

Note that these definitions rely on the formalism of strong real forms ξ and $\eta^\vee$, and the notion of blocks of representations. The basic gist of these definitions is that the ability to compute complex associated varieties and theta associated varieties, completely determines the ability to compute unipotent packets. Theorem 7.2 summarizes our main contributions to the computation of unipotent Arthur packets.

This paper is organized as follows:

1. Chapter 2 - Chapter 5: we setup the background and basic framework:
   (a) Chapter 2: basic definitions and results about nilpotent orbits that are relevant to our results. We recall the Springer correspondence and description of Weyl group representations.
   (b) Chapter 3: we describe the Langlands and Arthur parameters, following [ABV92].
   (c) Chapter 4: we introduce the Atlas setting, describe how representations are classified in this setting. We also introduce structure theory of parabolics in Atlas.
   (d) Chapter 5: we introduce associated varieties for $\mathbf{G}(\mathbb{R})$. We also explore some of their important properties.
2. Chapter 6: we give an algorithm for computing the real forms of an even nilpotent orbit.
3. Chapter 7: we introduce unipotent Arthur parameters and the corresponding special packets of unipotent representations, it ends with an algorithm to compute these packets in certain cases.
4. Chapter 8: we provide an application of results proved in Section 6 and 7.

5. Chapter 9: we provide tables of data. To work out the examples and to get the information displayed in the tables, you will need to load the script file Arthur_Packets.at (.at is the standard extension of Atlas script files) available in the atlas_ scripts folder when you install the Atlas Software.

The expert reader can safely jump right to Section 6 through 8, where lies bulk of the novelty and the main results and arguments of this thesis.

Chapter 2: Nilpotent Orbits

Let $\mathbf{G}$ be a complex reductive group, with complex Lie algebra $\mathfrak{g}$. Fix a real form $\mathbf{G}(\mathbb{R})$ of $\mathbf{G}$ and let $\mathfrak{g}_{\mathbb{R}}$ be the corresponding real Lie algebra. Let $\mathbf{K}$ be the complexfication of a maximal compact subgroup of $\mathbf{G}(\mathbb{R})$ and let θ be the Cartan involution so that $\mathbf{G}^\theta = \mathbf{K}$. Fix $\mathbf{H} \subset \mathbf{G}$, a Cartan subgroup and let

$$\begin{aligned} X^*(\mathbf{H}) &= \{\text{The lattice of rational characters (into } \mathbb{C}^\times) \text{ of } \mathbf{H}\} \qquad (2.1) \\ X_*(\mathbf{H}) &= \{\text{The lattice of one parameter subgroups of } \mathbf{H}\} \end{aligned}$$

so there is a natural pairing

$$\langle\ ,\ \rangle : X^*(\mathbf{H}) \times X_*(\mathbf{H}) \longrightarrow \mathbb{Z}. \qquad (2.2)$$

Using the following natural isomorphisms,

$$\mathfrak{h} \simeq X_*(\mathbf{H}) \otimes_{\mathbb{Z}} \mathbb{C}, \quad \mathfrak{h}^* \simeq X^*(\mathbf{H}) \otimes_{\mathbb{Z}} \mathbb{C}, \qquad (2.3)$$

where $\mathfrak{h}^*$ is the vector space dual of $\mathfrak{h}$, we can extend the pairing to

$$\langle\ ,\ \rangle : \mathfrak{h}^* \times \mathfrak{h} \longrightarrow \mathbb{C}. \tag{2.4}$$

Now, fix a set of roots $\Delta(\mathfrak{g}, \mathfrak{h})$ for $\mathfrak{g}$ and let $\Pi(\mathfrak{g}, \mathfrak{h})$ be a choice of simple positive roots. Let $\Delta^\vee(\mathfrak{g}, \mathfrak{h})$ and $\Pi^\vee(\mathfrak{g}, \mathfrak{h})$ be the corresponding set of coroots and simple coroots. The set of weights for $\mathbf{G}$ is defined as

$$P(\mathbf{G}) := \{\lambda \in X^*(\mathbf{H}) \otimes_{\mathbb{Z}} \mathbb{C} :\ \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \ \text{ for all } \ \alpha \in \Delta\}. \tag{2.5}$$

Also, the co-weights for $\mathbf{G}$ are defined as

$$P^\vee(\mathbf{G}) := \{\lambda^\vee \in X_*(\mathbf{H}) \otimes_{\mathbb{Z}} \mathbb{C} :\ \langle \alpha, \lambda^\vee \rangle \in \mathbb{Z} \ \text{ for all } \ \alpha \in \Delta\}. \tag{2.6}$$

We can identify $2\pi i\ X_*(\mathbf{H})$ with the kernel of the exponential map $\exp : \mathfrak{h} \longrightarrow \mathbf{H}$, under this identification we have

$$P^\vee(\mathbf{G}) = \{\lambda^\vee \in \mathfrak{h} :\ \exp(2\pi i\ \lambda^\vee) \in Z(\mathbf{G})\}, \tag{2.7}$$

where $Z(\mathbf{G})$ is the center of $\mathbf{G}$. Also,

$$P(\mathbf{G}) = \{\lambda \in \mathfrak{h}^* :\ \exp(2\pi i\ \lambda) \in Z(\mathbf{G}^\vee)\}, \tag{2.8}$$

where $Z(\mathbf{G}^\vee)$ is the center of the complex connected dual group $\mathbf{G}^\vee$.

We outline some facts about nilpotent adjoint and coadjoint orbits for $\mathbf{G}$, additional details can be found in [CM93]. The group $\mathbf{G}$ acts on $\mathfrak{g}$ via the adjoint action

$$\mathtt{Ad} : \mathbf{G} \longrightarrow \mathrm{End}(\mathfrak{g}), \quad g \mapsto \mathtt{Ad}(g). \tag{2.9}$$

An element $X \in \mathfrak{g}$ is called nilpotent if there exist a $k \in \mathbb{N}$ such that $\mathtt{ad}(X)^k = 0$. The set of all nilpotent elements in $\mathfrak{g}$ is called the nilpotent cone and is denoted as $\mathcal{N}$.

Definition 2.1 (Nilpotent Orbit). *A nilpotent orbit in $\mathfrak{g}$ is an orbit in $\mathcal{N}$ under the $\mathtt{Ad}$ action of $\boldsymbol{G}$.*

If $X \in \mathfrak{g}$ is a nilpotent element, then we write $\mathcal{O}_X := \mathtt{Ad}(G) \cdot X$ for the nilpotent orbit in $\mathfrak{g}$.

Theorem 2.1 (Jacobson-Morozov). *Suppose $\mathfrak{g}$ is a complex reductive Lie algebra. Let X be a non-zero nilpotent element in $\mathfrak{g}$. Then, there exist $H \in \mathfrak{h}$ (semisimple) and $Y \in \mathfrak{g}$ (nilpotent) such that*

$$[H, X] = 2X \quad , [H, Y] = -2Y \quad \textit{and} \quad [X, Y] = H, \tag{2.10}$$

where the bracket is the Lie algebra bracket in $\mathfrak{g}$.

The set $\{X, H, Y\}$ is called a standard $\mathfrak{sl}(2)$-triple, and X is called its nilpositive element. Suppose $\mathcal{A}$ is the set of $\mathtt{Ad}(\mathbf{G})$-conjugacy classes of $\mathfrak{sl}(2)$-triples in $\mathfrak{g}$, then, we can define a map

$$\Omega : \mathcal{A} \longrightarrow \{\text{nilpotent orbits}\} \ ; \ \Omega(\{X, H, Y\}) = \mathcal{O}_X. \tag{2.11}$$

The map Ω is bijective. We will conjugate the triple so that the semisimple element H of the triple is dominant with respect to $\Pi(\mathfrak{g}, \mathfrak{h})$. Furthermore, H belongs to $P^\vee(\mathbf{G})$. Using the bilinear pairing in Equation 2.4 we can label the nodes of the Dynkin diagram for $\mathfrak{g}$ by the integer $\langle \alpha, H \rangle$. Such a diagram is called a labeled Dynkin diagram and we denote it by $\mathcal{D}_H$. If H is the semisimple element of a standard $\mathfrak{sl}(2)$-triple, then, using $\mathfrak{sl}(2)$-representation theory one can show that labels for the Dynkin diagram can only be one of either $0, 1$ or 2. Let $\mathfrak{D}$ be the set of labelled Dynkin diagrams corresponding to standard $\mathfrak{sl}(2)$-triples.

Definition 2.2 (Even Nilpotent Orbits)**.** *Let $\mathcal{O}$ be a nilpotent orbit for $\boldsymbol{G}$ and let $\{X, Y, H\}$ be the corresponding Jacobson-Morozov triple. We say $\mathcal{O}$ is an even nilpotent orbit if any one of the following equivalent conditions hold.*

1. *all the nodes of the labelled Dynkin diagram $\mathcal{D}_H$ are even (i.e. either 0 or 2).*

2. *$\frac{1}{2}H \in P^\vee(\boldsymbol{G})$.*

When $\mathfrak{g}$ is reductive Lie algebra of classical type, there is a classification of nilpotent orbits in terms of partitions. We refer the reader to [[CM93], Theorem 5.1.1-5.1.4] for details of this classification.

2.1 Induction of Nilpotent Orbits

Many nilpotent orbits in $\mathfrak{g}$ can be induced from nilpotent orbits on subalgebras of $\mathfrak{g}$. We introduce some ideas (relevant to us) regarding induction of nilpotent orbits. Most of the details can be found in Chapter 7 of [CM93].

Let $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}$ be a parabolic subalgebra in $\mathfrak{g}$. Let $\mathbf{P}$ be the corresponding parabolic subgroup in $\mathbf{G}$. Suppose $\mathcal{O}_{\mathfrak{l}}$ is a nilpotent orbit in $\mathfrak{l}$. We have the following result.

Theorem 2.2 ([CM93], Theorem 7.1.1)**.** *As in the notation above, recall that, $\mathit{Ad}(\boldsymbol{P})$ is a connected subgroup of $\mathit{Ad}(\boldsymbol{G})$ with Lie algebra $\mathfrak{p}$. There is a unique nilpotent orbit $\mathcal{O}_{\mathfrak{g}}$ in $\mathfrak{g}$ meeting $\mathcal{O}_{\mathfrak{l}} + \mathfrak{n}$ in an open dense set. The intersection $\mathcal{O}_{\mathfrak{g}} \cap (\mathcal{O}_{\mathfrak{l}} + \mathfrak{n})$ consists of a single $\mathit{Ad}(\boldsymbol{P})$-orbit. The orbit $\mathcal{O}_{\mathfrak{g}}$ above will called the induced orbit from $\mathcal{O}_{\mathfrak{l}}$ and will be denoted as*

$$\mathcal{O}_{\mathfrak{g}} = Ind_{\mathfrak{p}}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{l}}).$$

The induced orbit only depends on the Levi factor $\mathfrak{l}$ of $\mathfrak{p}$:

Theorem 2.3 ([CM93], Theorem 7.1.3)**.** *Let $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}$ and $\mathfrak{p}' = \mathfrak{l} + \mathfrak{n}'$ be two parabolic subalgebras in $\mathfrak{g}$ have the same Levi subalgebra $\mathfrak{l}$ and let $\mathcal{O}_{\mathfrak{l}}$ be a nilpotent orbit in $\mathfrak{l}$. Then*

$$Ind_{\mathfrak{p}}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{l}}) = Ind_{\mathfrak{p}'}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{l}}).$$

We say a nilpotent orbit is a Richardson orbit if it is induced from the trivial orbit on some parabolic subalgebra in $\mathfrak{g}$. Suppose $\mathcal{O}$ is a nilpotent orbit with $\mathfrak{sl}(2)$-triple $\{X, H, Y\}$ and let $\mathcal{D}_H$ be the labelled Dynkin diagram for $\mathcal{O}$. Let $\Delta(\mathcal{O})$ be the complement set of vertices labelled 2 in $\mathcal{D}_H$. Let $\mathfrak{l}$ be the Levi subalgebra generated by the roots in $\Delta(\mathcal{O})$.

Theorem 2.4 ([CM93], Theorem 7.1.6)**.** *Let $\mathcal{D}'(\mathcal{O})$ be the labeled sub-diagram of $\mathcal{D}(\mathcal{O})$ consisting of vertices labeled 0 or 1. If $\mathcal{D}'$ is the labeled Dynkin diagram of a nilpotent orbit $\mathcal{O}_{\mathfrak{l}}$ in $\mathfrak{l}$, then, $\mathcal{O} = Ind_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}_{\mathfrak{l}})$.*

Recall that even nilpotent orbits have the nodes of their Dynkin diagram labelled either 0 or 2. If $\mathcal{O}$ is even, $\mathfrak{l} = \mathrm{Cent}_{\mathfrak{g}}(H)$ and, $\mathcal{D}'(\mathcal{O})$ defined in the theorem corresponds to the trivial orbit $0_{\mathfrak{l}}$ in $\mathfrak{l}$. Therefore, we have

Corollary 2.1. *Suppose $\mathcal{O}$ is an even nilpotent orbit in $\mathfrak{g}$. Then, $\mathcal{O}$ is a Richardson orbit; induced from the trivial orbit on the Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ generated by the nodes labelled 0 in $\mathcal{D}(\mathcal{O})$.*

2.2 Real Nilpotent Orbits

Recall that $\mathcal{N}$ was defined to be the cone of nilpotent elements in $\mathfrak{g}$. The real nilpotent cone is defined to be the nilpotents in $\mathfrak{g}_{\mathbb{R}}$:

$$\mathcal{N}_{\mathbb{R}} := \mathcal{N} \cap \mathfrak{g}_{\mathbb{R}}. \tag{2.12}$$

The real nilpotent cone $\mathcal{N}_{\mathbb{R}}$ is a finite union of $\mathtt{Ad}(\mathbf{G}(\mathbb{R}))$-conjugacy classes. When $\mathfrak{g}$ is of classical type, the conjugacy classes are parameterized by signed Young tableau, for more details about this classification and its explicit realization we refer the reader to ([CM93], Chapter 9).

Definition 2.3 (Real form of a complex nilpotent orbit)**.** *Let $\mathcal{O}$ be a complex nilpotent orbit for $\boldsymbol{G}$. Let $\boldsymbol{G}(\mathbb{R})$ be a real form of $\boldsymbol{G}$. By a real form of $\mathcal{O}$ we mean a $\boldsymbol{G}(\mathbb{R})$-conjugacy class of nilpotent elements in $\mathcal{O} \cap \mathfrak{g}_{\mathbb{R}}$.*

We use an alternate description of $\mathcal{N}_{\mathbb{R}}$ based on the Cartan involution θ. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ be the Cartan decomposition of $\mathfrak{g}$ with respect to θ, that is $\mathfrak{k} = \mathfrak{g}^{\theta}$ and $\mathfrak{s} = \mathfrak{g}^{-\theta}$.

Let

$$\mathcal{N}_{\theta} := \{ \text{ Nilpotent elements in } \mathfrak{s}\}. \tag{2.13}$$

Since $\mathbf{K}$ preserves $\mathfrak{s} = \mathfrak{g}/\mathfrak{k}$, $\mathbf{K}$ acts on acts on $\mathfrak{s}$ and this action partitions $\mathcal{N}_{\theta}$ into finitely many orbits.

Theorem 2.5 (Kostant-Sekiguchi)**.** *There is a natural bijective correspondence between nilpotent $\boldsymbol{G}(\mathbb{R})$-orbits in $\mathfrak{g}_{\mathbb{R}}$ and the nilpotent $\boldsymbol{K}$-orbits in $\mathfrak{s}$.*

We now define,

Definition 2.4 (θ-form of a complex nilpotent orbit)**.** *Let $\mathcal{O}$ be a complex nilpotent orbit for $\boldsymbol{G}$. Let θ be the Cartan involution defining the real form $\boldsymbol{G}(\mathbb{R})$ of $\boldsymbol{G}$. By a θ-form of $\mathcal{O}$ we mean a $\boldsymbol{K}$-conjugacy class of nilpotent elements in $\mathcal{O} \cap \mathfrak{s}$, where $\boldsymbol{K} = \boldsymbol{G}^{\theta}$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ is the Cartan decomposition of $\mathfrak{g}$ with respect to θ.*

Since θ-forms are defined using Cartan involutions, they are better suited for our applications.

2.3 Coadjoint Nilpotent Orbits

In applications, nilpotent orbits arise in the dual vector space $\mathfrak{g}^*$ of $\mathfrak{g}$. Note that $\mathfrak{g}^*$ does not have a Lie algebra structure, and as such, there is no direct way of making sense of nilpotent elements in $\mathfrak{g}^*$. If $\mathfrak{g}$ is a complex reductive Lie algebra, one can define an invariant non-degenerate symmetric bilinear form on $\mathfrak{g}$:

$$\langle \cdot, \cdot \rangle_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{C} \tag{2.14}$$

such that $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ restricted to $[\mathfrak{g}, \mathfrak{g}]$ is a nonzero constant multiple of the Killing form on $[\mathfrak{g}, \mathfrak{g}]$.

The fact that $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ is non-degenerate implies that the map $\phi : \mathfrak{g} \longrightarrow \mathfrak{g}^*$ defined by

$$X \mapsto \phi_X := \langle X, \cdot \rangle_{\mathfrak{g}} \in \mathfrak{g}^*, \tag{2.15}$$

is an isomorphism of vector spaces. Define the nilpotent cone in $\mathfrak{g}^*$ as $\mathcal{N}^* := \phi(\mathcal{N})$.

Suppose $\mathcal{O}$ is a nilpotent orbit in $\mathfrak{g}$ with nilpositive element $X \in \mathcal{O}$ (so that $\mathcal{O} = \mathtt{Ad}(G) \cdot X$), then, we define the corresponding coadjoint orbit to be

$$\mathcal{O}^* := \mathtt{Ad}(G) \cdot \phi_X \subset \mathcal{N}^*. \tag{2.16}$$

We can use the map ϕ to identify the other coadjoint cones of nilpotent elements with respect to a real form $\mathbf{G}(\mathbb{R})$ of $\mathbf{G}$ corresponding to a Cartan involution θ as follows:

$$\mathcal{N}_{\mathbb{R}}^* := \phi(\mathcal{N}_{\mathbb{R}}), \quad \mathcal{N}_{\theta}^* := \phi(\mathcal{N}_{\theta}). \tag{2.17}$$

In this setting we have bijections:

1. $\mathcal{N}/\mathbf{G}$ and $\mathcal{N}^*/\mathbf{G}$.

2. $\mathcal{N}_\theta/\mathbf{K}$ and $\mathcal{N}_\theta^*/\mathbf{K}$.

3. $\mathcal{N}_{\mathbb{R}}/\mathbf{G}(\mathbb{R})$ and $\mathcal{N}_{\mathbb{R}}^*/\mathbf{G}(\mathbb{R})$.

Therefore using the Kostant-Sekiguchi correspondence $\mathcal{N}_\theta/\mathbf{K}, \mathcal{N}_\theta^*/\mathbf{K}, \mathcal{N}_{\mathbb{R}}/\mathbf{G}(\mathbb{R})$ and $\mathcal{N}_{\mathbb{R}}^*/\mathbf{G}(\mathbb{R})$ are all in bijective correspondence.

2.4 Duality of Nilpotent Orbits

Let $\mathbf{G}$ be a complex connected reductive group and $\mathbf{G}^\vee$ be the corresponding complex connected reductive dual group.

There is a basic duality due to Spaltenstein defined as a map $d : \mathcal{N}(\mathbf{G}) \longrightarrow \mathcal{N}(\mathbf{G}^\vee)$, called the duality map. We refer the reader to Section 6.3 in [CM93] for explicit description in terms of partitions when $\mathbf{G}$ is of classical type.

2.5 The Springer Correspondence

We recall the Springer correspondence. Define $\mathcal{B}$ to be set of Borel subalgebras in $\mathfrak{g}$.Given a nilpotent element X, the variety $\mathcal{B}_X$ is the set of Borel subalgebras containing X. The group $\mathbf{G}^X := \mathrm{Cent}_{\mathbf{G}}(X)$ acts on $\mathcal{B}_X$ via the adjoint action. The induced action of this action on the cohomology $H^*(\mathcal{B}_X, \mathbb{C})$ is trivial on $\mathbf{G}_0^X$ so that

$A(\mathcal{O}_X) := \mathbf{G}^X/\mathbf{G}_0^X$ acts on $H^*(\mathcal{B}_X, \mathbb{C})$.

For an irreducible representation (π, V_π) of $A(\mathcal{O}_X)$ define

$$H^*(\mathcal{B}_X, \mathbb{C})_\pi := \mathrm{Hom}_{A(\mathcal{O}_X)}(V_\pi, H^*(\mathcal{B}_X, \mathbb{C})). \tag{2.18}$$

We are now ready to state the Springer correspondence:

Theorem 2.6 (Springer)**.** *For any nilpotent element X, there is a natural action of $\boldsymbol{W}$ on $H^*(\mathcal{B}_X, \mathbb{C})$.*

1. *The actions of $\boldsymbol{W}$ and $A(\mathcal{O}_X)$ commute; so $\boldsymbol{W}$ acts on $H^*(\mathcal{B}_X, \mathbb{C})_\pi$ for $\pi \in \widehat{A(\mathcal{O}_X)}$.*

2. *The natural maps*

   $$H^*(\mathcal{B}, \mathbb{C}) \longrightarrow H^*(\mathcal{B}_X, \mathbb{C}),$$

   induced by $H^*(\mathcal{B}_X, \mathcal{B})$, are $\boldsymbol{W}$ - equivariant.

3. *For $\pi \in \widehat{A(\mathcal{O}_X)}$, the representation $\sigma(X, \pi)$ of $\boldsymbol{W}$ on $H^{dim_{\mathbb{R}}(\mathcal{B}_X)}(\mathcal{B}_X, \mathbb{C})_\pi$ is irreducible or zero.*

4. *If π is trivial, $\sigma(X, \pi) \neq 0$.*

5. *Suppose $\sigma \in \widehat{W}$. Then there are: a nilpotent element $X \in \mathfrak{g}$, unique upto $\boldsymbol{Ad}(G)$; and a unique $\pi \in \widehat{A(\mathcal{O}_X)}$, such that*

   $$\sigma = \sigma(X, \pi).$$

The correspondence

$$(G \cdot X, \pi) \longrightarrow \sigma(X, \pi) \tag{2.19}$$

is called the Springer correspondence. We write

$$\sigma(\mathcal{O}_X) = \sigma(\mathcal{O}_X, 1). \tag{2.20}$$

The Springer correspondence provides a way of attaching to each nilpotent orbit $\mathcal{O}$ a finite set of $\mathbf{W}$-representations, having a distinguished element $\sigma(\mathcal{O})$.

2.6 Weyl Group Representations in Classical Type

We go over some facts about Weyl group representations in types B_l and C_l, details of the general situation can be found in [Car93].

Theorem 2.7 (Irreducible Weyl group representations of Type B_l and C_l)**.** *The irreducible representations of the Weyl group $\boldsymbol{W}(B_l)$ (and C_l) are in bijection with pairs of partitions (α, β) such that $|\alpha| + |\beta| = l$. We write $\sigma_{(\alpha,\beta)}$ for the $\boldsymbol{W}$-representation corresponding to (α, β).*

Suppose $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_m)$ and $\beta = (\beta_0, \beta_1, \ldots, \beta_{m-1})$ (we allow for the parts to be zero, requiring that α has one more part that β) such that

$$0 \leqslant \alpha_0 \leqslant \alpha_1 \leqslant \cdots \leqslant \alpha_m \quad \text{and} \quad 0 \leqslant \beta_0 \leqslant \beta_1 \leqslant \cdots \leqslant \beta_{m-1} \tag{2.21}$$

Lusztig attaches to (α, β) the following symbol:

$$\begin{pmatrix} \lambda \\ \\ \mu \end{pmatrix} := \begin{pmatrix} \lambda_0 & & \lambda_1 & & \dots & & \lambda_{m-1} & & \lambda_m \\ & \mu_0 & & \mu_1 & & \dots & & \mu_{m-1} & \end{pmatrix}$$

where $\lambda_i = \alpha_i + i$ and $\mu_j = \beta_i + i$ for $i = 0, 1, 2, \dots$.

Definition 2.5 (Special **W**-representation)**.** *Let $\sigma_{(\alpha,\beta)}$ be a irreducible representation of $\boldsymbol{W}$ with corresponding symbol $\begin{pmatrix} \lambda \\ \\ \mu \end{pmatrix}$. We say $\sigma_{(\alpha,\beta)}$ is special if*

$$\lambda_0 \leqslant \mu_0 \leqslant \lambda_1 \leqslant \mu_1 \leqslant \lambda_2 \leqslant \mu_2 \leqslant \cdots \leqslant \lambda_{m+1}. \tag{2.22}$$

Definition 2.6 (Special Nilpotent Orbit)**.** *Let $\mathcal{O}$ be a nilpotent orbit in $\mathfrak{g}$. We say $\mathcal{O}$ is special if the Weyl group representation $\sigma(\mathcal{O})$ associated to $\mathcal{O}$ via the Springer correspondence is a special $\boldsymbol{W}$-representation.*

The Springer correspondence is explicitly realized when **G** is of classical type, we make use of this realization in our computations and to this end has been implemented into Atlas. For the interested reader, the algorithm can be found in ([Car93], pages 419-423).

Chapter 3: An Overview of the setting

We outline the basic setting that we use for rest of this paper. Let $\mathbf{G}$ be a complex connected reductive algebraic group. Let $\mathrm{Int}(\mathbf{G}), \mathrm{Aut}(\mathbf{G})$ and $\mathrm{Out}(\mathbf{G})$ be the groups of inner automorphisms, automorphisms and outer automorphisms respectively of $\mathbf{G}$. We have the following exact sequence

$$1 \longrightarrow \mathrm{Int}(\mathbf{G}) \longrightarrow \mathrm{Aut}(\mathbf{G}) \xrightarrow{p} \mathrm{Out}(\mathbf{G}) \longrightarrow 1,$$

so that $\mathrm{Out}(\mathbf{G}) \simeq \mathrm{Aut}(\mathbf{G})/\mathrm{Int}(\mathbf{G})$.

A *splitting datum* for $\mathbf{G}$ is a tuple $(\mathbf{B}, \mathbf{H}, \{X_\alpha\})$, where $\mathbf{B}$ is a Borel subgroup of $\mathbf{G}$, $\mathbf{H}$ a Cartan subgroup and $\{X_\alpha\}$ is the set of root vectors for the of simple roots of $\mathbf{H}$ in $\mathbf{B}$. An involution of $\mathbf{G}$ is said to be *distinguished* if it preserves a splitting datum. Let $\mathbf{G}^\vee$ be the dual group of $\mathbf{G}$. There is a bijection between $\mathrm{Out}(\mathbf{G})$ and $\mathrm{Out}(\mathbf{G}^\vee)$ (Definition 2.11, [AdC09]) and we denote it by $\gamma \in \mathrm{Out}(\mathbf{G}) \mapsto \gamma^\vee \in \mathrm{Out}(\mathbf{G}^\vee)$.

Fix $\gamma \in \mathrm{Out}(\mathbf{G})$ an element of order two. An involution $\theta \in \mathrm{Aut}(\mathbf{G})$ is said to be in the *inner class* of γ if $p(\theta) = \gamma$. We will say that two involutions θ and θ' are inner to each other if they have the same image in $\mathrm{Out}(\mathbf{G})$ under the map p. We call the pair $(\mathbf{G}, \gamma)$ a basic datum, and the corresponding dual basic datum is given by $(\mathbf{G}^\vee, \gamma^\vee)$. Let $\Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{1, \sigma\}$, then in this setting,

Definition 3.1 (Extended group, L-group)**.** *The extended group for the pair* $(\boldsymbol{G}, \gamma)$ *is the semidirect product* ${}^{\Gamma}\boldsymbol{G} := \boldsymbol{G} \rtimes \Gamma$*, where* $\sigma \in \Gamma$ *acts by the distinguished involution in* $p^{-1}(\gamma)$*. The L-group for the pair* $(\boldsymbol{G}, \gamma)$ *is defined to be the extended group for the pair* $(\boldsymbol{G}^{\vee}, \gamma^{\vee})$*, often denoted as* ${}^{L}\boldsymbol{G}$ *or* ${}^{\Gamma}\boldsymbol{G}^{\vee}$*.*

A *real form* of $\mathbf{G}$ is an antiholomorphic involutive automorphism $\sigma : \mathbf{G} \longrightarrow \mathbf{G}$. Let σ_c be a compact real form of $\mathbf{G}$ chosen such that it commutes with σ, then $\theta = \sigma \circ \sigma_c$ is a holomorphic involution of $\mathbf{G}$. We prefer to work with holomorphic maps, and, to this end we need the following, (Theorem 3.2, [AdC09]),

Theorem 3.1. *The map* $\sigma \mapsto \theta$ *gives a bijection between* $\boldsymbol{G}$*-conjugacy classes of antiholomorphic involutions and* $\boldsymbol{G}$*-conjugacy classes of holomorphic involutions of* $\boldsymbol{G}$*.*

A *Cartan involution* of $\mathbf{G}$ is a holomorphic involution of $\mathbf{G}$. Henceforth, by a real form of $\mathbf{G}$ we will mean a $\mathbf{G}$-conjugacy class of Cartan involutions.

Definition 3.2 (Strong real form)**.** *A strong involution of* $\boldsymbol{G}$ *in the inner class defined by* γ *is an element* $\xi \in {}^{\Gamma}\boldsymbol{G} - \boldsymbol{G}$ *satisfying* $\xi^2 \in Z(\boldsymbol{G})$*. The set of strong involutions is denoted by* $\mathcal{I}(\boldsymbol{G}, \gamma)$*.*

A strong real form of $\boldsymbol{G}$ *in the inner class of* γ *is the* $\boldsymbol{G}$*-conjugacy class of a strong involution.*

Given a strong real form $\xi \in \mathcal{I}(\mathbf{G}, \gamma)$, we can define a Cartan involution of $\mathbf{G}$ as $\theta_\xi = \mathrm{Int}(\xi)$, $\mathbf{K}_\xi = \mathrm{Stab}_{\mathbf{G}}(\theta_\xi) = \mathbf{G}^{\theta_\xi}$. There is a surjective map from $\mathcal{I}(\mathbf{G}, \gamma)/\mathbf{G}$ onto the set of all real forms of $\mathbf{G}$ in the class defined by γ, this map is bijective if $\mathbf{G}$ is adjoint.

3.1 Langlands and Arthur Parameters

Let $W_{\mathbb{R}}$ be the Weil group of $\mathbb{R}$. A *Langlands parameter* is a homomorphism $\phi : W_{\mathbb{R}} \longrightarrow {}^{\Gamma}\mathbf{G}^{\vee}$ such that the following diagram of L-morphisms commutes

$$\begin{tikzcd} W_{\mathbb{R}} \arrow[r, "\phi"] \arrow[dr] & {}^{\Gamma}\mathbf{G}^{\vee} \arrow[d] \\ & \Gamma \end{tikzcd}$$

and, $\phi(\mathbb{C}^{\times})$ is contained in the set of semisimple elements of $\mathbf{G}^{\vee}$. The group $\mathbf{G}^{\vee}$ acts on such parameters by conjugation. To any $\mathbf{G}^{\vee}$ conjugacy class of such parameters is attached a L-packet of representations of real forms in the inner class defined by γ. Using ([ABV92], Proposition 5.6), one can identify the set of Langlands parameters with pairs $(y, \lambda^{\vee})$ satisfying the conditions

1. $y \in {}^{\Gamma}\mathbf{G}^{\vee} - \mathbf{G}^{\vee}$ and $\lambda^{\vee} \in \mathfrak{h}^{\vee}$ is a semisimple element,

2. $y^2 = \exp(2\pi i \lambda^{\vee})$, and

3. $[\lambda^{\vee}, \mathtt{Ad}(y)\lambda^{\vee}] = 0$.

Definition 3.3 (Arthur Parameter)**.** *An Arthur parameter for* $\boldsymbol{G}$ *is a homomorphism* $\psi : W_{\mathbb{R}} \times SL(2, \mathbb{C}) \longrightarrow {}^{\Gamma}\boldsymbol{G}^{\vee}$ *satisfying*

1. *the restriction of ψ to $W_{\mathbb{R}}$ is a tempered (Definition 22.3, [ABV92]) Langlands parameter,*

2. *the restriction of ψ to $SL(2, \mathbb{C})$ is holomorphic.*

The group $\mathbf{G}^{\vee}$ acts on such parameters by conjugation.

We say that ψ is a *unipotent Arthur parameter* if ψ restricted to the identity component of $W_{\mathbb{R}}$ is trivial. Given an Arthur parameter ψ, define the Langlands parameter ϕ_ψ to be

$$\phi_\psi : W_{\mathbb{R}} \longrightarrow \mathbf{G}^\vee \quad \phi_\psi(w) := \psi(w, \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix}). \tag{3.1}$$

Associated to ψ is an Arthur Packet of representations of real forms in the inner class given be γ (Definition 22.6, [ABV92]) containing the L-packet corresponding to ϕ_ψ and at most finitely many additional representations of of strong real forms in the given inner class. One of the main results of this paper is to devise and implement an algorithm to compute these packets, not always completely, when ψ is assumed to be unipotent.

Chapter 4: Atlas of Lie groups requisites

We will make use of the Atlas of Lie groups setting. More details can be found in [AdC09] and in resources available at www.liegroups.org.

We continue in the setting of the previous section. We fix a real form $\mathbf{G}(\mathbb{R})$ of $\mathbf{G}$, with corresponding Cartan involution θ, so that $\mathbf{K} = \mathbf{G}^{\theta}$. Furthermore, fix a pinning $(\mathbf{B}, \mathbf{H}, \{X_\alpha\}$ for $\mathbf{G}$. We use the Harish-Chandra homorphism to associate to $\lambda \in \mathfrak{h}^*$, an infinitesimal character which we will also denote by λ, which only depends on the Weyl group orbit $\mathbf{W} \cdot \lambda$ and is unique if we require it to be dominant with respect to a fixed choice of simple positive roots $\Pi(\mathfrak{g}, \mathfrak{h})$. We say that λ is regular (resp. integral) if $\langle \lambda, \alpha^\vee \rangle \neq 0$ (resp. $\in \mathbb{Z}$) for roots $\alpha \in \Pi(\mathfrak{g}, \mathfrak{h})$.

It is well known that irreducible admissible representations of $\mathbf{G}(\mathbb{R})$ are parameterized by irreducible admissible $(\mathfrak{g}, \mathbf{K})$-modules. We define the following sets:

$$
\begin{aligned}
\mathfrak{M}(\mathfrak{g}, \mathbf{K}) &= \text{Category of finite length } (\mathfrak{g}, \mathbf{K})\text{-modules.} \\
\mathbb{K}\mathfrak{M}(\mathfrak{g}, \mathbf{K}) &= \text{Grothendieck group of } \mathfrak{M}(\mathfrak{g}, \mathbf{K}). \\
\mathfrak{M}(\mathfrak{g}, \mathbf{K}, \lambda) &= \text{Category of finite length } (\mathfrak{g}, \mathbf{K})\text{-modules with inf char } \lambda. \\
\mathbb{K}\mathfrak{M}(\mathfrak{g}, \mathbf{K}, \lambda) &= \text{Grothendieck group of } \mathfrak{M}(\mathfrak{g}, \mathbf{K}, \lambda). \\
\Pi(\mathfrak{g}, \mathbf{K}) &= \{\text{Equiv classes of irred admissible } (\mathfrak{g}, \mathbf{K})\text{-modules.}\} \\
\Pi(\mathfrak{g}, \mathbf{K}, \lambda) &= \{J \in \Pi(\mathfrak{g}, \mathbf{K}) \text{ such that inf char of } J \text{ is } \lambda.\}
\end{aligned}
$$

By results of Harish-Chandra, the set $\Pi(\mathfrak{g}, \mathbf{K}, \lambda)$ is a finite set although there is no closed form formula for its cardinality. It is desirable to find a combinatorial description of $\Pi(\mathfrak{g}, \mathbf{K}, \lambda)$, and the results in [AdC09] do exactly that. We now outline the basic components of this description.

Recall that $\mathcal{I}(\mathbf{G}, \gamma)$ is the set of strong real forms for the basic data $(\mathbf{G}, \gamma)$, we will denote it as $\mathcal{I}$ when there is no confusion about the basic data in question. The *one-sided parameter space* is the set

$$\mathcal{X}(\mathbf{G}, \gamma) := \{\xi \in \mathcal{I} \mid \xi \in \mathrm{Norm}_{\mathbf{G}^{\Gamma}-\mathbf{G}}(\mathbf{H})\}/\mathbf{H}, \tag{4.1}$$

the equivalence is via conjugation. The set $\mathcal{X}(\mathbf{G}, \gamma)$ is finite when $\mathbf{G}$ is semisimple, and its elements are explicitly computed by the Atlas Software and are called `KGB`-elements. When there is no confusion about the basic data, we will denote $\mathcal{X}(\mathbf{G}, \gamma)$ by just $\mathcal{X}$. Given a dual basic data $(\mathbf{G}^\vee, \gamma^\vee)$, we will denote $\mathcal{X}(\mathbf{G}^\vee, \gamma^\vee)$ as just $\mathcal{X}^\vee$. Furthermore, given $x \in \mathcal{X}$, we denote the fiber of x to be

$$\mathcal{X}[x] := \{x' \in \mathcal{X} \mid q^{-1}(x) \text{ is } \mathbf{G}\text{-conjugate to } q^{-1}(x')\}, \tag{4.2}$$

where q is the natural projection map. Fix $x \in \mathcal{X}$ and suppose $\xi \in \mathcal{I}$ is such that $q(\xi) = x$, we will denote $\theta_{x,\mathbf{H}}$ to be the Cartan involution θ_ξ restricted to $\mathbf{H}$. The *two-sided parameter space* is defined as

$$\mathcal{Z}(\mathbf{G},\gamma) := \{(x,y) \in \mathcal{X} \times \mathcal{X}^\vee \mid (\theta_{x,\mathbf{H}})^t = -\theta_{y,\mathbf{H}^\vee}\} \subset \mathcal{X} \times \mathcal{X}^\vee. \tag{4.3}$$

The following theorem provides the combinatorial setup that we want,

Theorem 4.1 (Adams-DuCloux, [AdC09], Thm 10.3)**.** *Fix a set $\Lambda \subset P_{reg}(\boldsymbol{G},\boldsymbol{H})$ of representatives of $P(\boldsymbol{G})/X^*(\boldsymbol{H})$. Let $I = \mathcal{I}/\boldsymbol{G}$ be a set of representatives for the strong real forms of ${}^\Gamma\boldsymbol{G}^\vee$. For each $\xi_i \in I$ let θ_{ξ_i} be the Cartan involution corresponding to conjugation by ξ_i and let $\boldsymbol{K}_{\xi_i}$ be the fixed points of θ_{ξ_i}. There is a natural bijection*

$$\mathcal{Z}({}^\Gamma\boldsymbol{G}^\vee) \leftrightarrow \coprod_{\xi_i \in I} \coprod_{\lambda_j \in \Lambda} \Pi(\mathfrak{g}, \boldsymbol{K}_{\xi_i}, \lambda_j). \tag{4.4}$$

By (Corollary 9.9 [AdC09]), the set $\mathcal{X}$ is in bijection with the disjoint union of $\mathbf{K}_\xi$-orbits on $\mathbf{G}/\mathbf{B}$ (denoted as $\mathbf{K}_\xi\backslash\mathbf{G}/\mathbf{B}$) as ξ varies over $\mathcal{I}$. If we fix a strong real form ξ such that $q(\xi) = x$, then there is a bijection between $\mathcal{X}[x]$ and $\mathbf{K}_\xi\backslash\mathbf{G}/\mathbf{B}$. Fix an infinitesimal character λ for $\mathbf{G}$ then the set $\Pi(\mathfrak{g},\mathbf{K}_\xi,\lambda)$ satisfies

$$\Pi(\mathfrak{g},\mathbf{K}_\xi,\lambda) \subset \mathcal{Z}(\mathbf{G},\gamma) \subset \mathcal{X}[x] \times \mathcal{X}^\vee \simeq \mathbf{K}_\xi\backslash\mathbf{G}/\mathbf{B} \times \left(\coprod_{\eta_j^\vee \in \mathcal{I}^\vee} \mathbf{K}^\vee_{\eta_j^\vee}\backslash\mathbf{G}^\vee/\mathbf{B}^\vee \right), \tag{4.5}$$

where $\mathcal{I}^\vee$ is a set of representatives for the strong real forms of ${}^\Gamma\mathbf{G}^\vee$.

Now fix $\eta^\vee \in \mathcal{I}^\vee$ and let $y = q(\eta^\vee) \in \mathcal{X}^\vee$. We assume that λ satisfies $(\eta^\vee)^2 = \exp(2\pi i\lambda)$, then we recall the following definition of a block of representations:

Definition 4.1 ([DAV81], Definition 9.2.1)**.** *The block equivalence of* $(\mathfrak{g}, \boldsymbol{K})$*-modules is the equivalence relation generated by*

$$X \sim Y \quad \textit{if} \quad Ext^1_{(\mathfrak{g},\boldsymbol{K})}(X,Y) \neq 0. \tag{4.6}$$

The equivalence classes for this relation are called blocks.

Every block contains irreducible modules with a fixed infinitesimal character, hence a the cardinality of a block is finite.

Let λ satisfy $(\eta^\vee)^2 = \exp(2\pi i\lambda^\vee)$. Consider the set (denoted by $\mathcal{B}(\xi, \eta^\vee, \lambda^\vee)$) of irreducible $(\mathfrak{g}, \mathbf{K}_\xi)$-modules corresponding to the combinatorial data given by:

$$\begin{aligned} \mathcal{B}(\xi, \eta^\vee, \lambda^\vee) &= (\mathcal{X}[x] \times \mathcal{X}^\vee[y]) \cap \mathcal{Z}(\mathbf{G}, \gamma) \\ &= \left(\mathbf{K}_\xi \backslash \mathbf{G}/\mathbf{B} \times \mathbf{K}^\vee_{\eta^\vee} \backslash \mathbf{G}^\vee/\mathbf{B}^\vee\right) \cap \mathcal{Z}(\mathbf{G}, \gamma), \end{aligned} \tag{4.7}$$

In this setting, $\mathcal{B}(\xi, \eta^\vee, \lambda^\vee)$ corresponds to an equivalence class of a block of representations as defined by Definition 4.1. If we choose $\lambda^\vee \in \Lambda^\vee \subset P^\vee(\mathbf{G}, \mathbf{H})$ such that $\xi^2 = \exp(2\pi i\lambda)$, we can define the dual block of $\mathcal{B}(\xi, \eta^\vee, \lambda^\vee)$, as the block of irreducible $(\mathfrak{g}^\vee, \mathbf{K}^\vee_{\eta^\vee})$-modules at infinitesimal character $\lambda^\vee$, as follows

$$\begin{aligned}
\mathcal{B}^\vee(\xi,\eta^\vee,\lambda) &= \mathcal{B}(\eta^\vee,\xi,\lambda^\vee) \qquad (4.8)\\
&= (\mathcal{X}[y]\times\mathcal{X}^\vee[x])\cap\mathcal{Z}(\mathbf{G}\vee,\gamma^\vee)\\
&= \big(\mathbf{K}^\vee_{\eta^\vee}\backslash\mathbf{G}^\vee/\mathbf{B}^\vee\times\mathbf{K}_\xi\backslash\mathbf{G}/\mathbf{B}\big)\cap\mathcal{Z}(\mathbf{G}^\vee,\gamma^\vee),
\end{aligned}$$

with the same compatibility condition as in the definition of $\mathcal{B}$.

In this setting, we can realize Vogan duality as follows:

Definition 4.2 (Vogan Duality)**.** *Vogan duality is the natural bijection between the sets $\mathcal{B}(\xi,\eta^\vee,\lambda^\vee)$ and $\mathcal{B}^\vee(\xi,\eta^\vee,\lambda^\vee)$ obtained by the map $(x,y)\mapsto(y,x)$.*

Vogan duality provides a bijection $\pi\leftrightarrow\pi^\vee$ between irreducible representations in blocks $\mathcal{B}$ and $\mathcal{B}^\vee$, this plays a crucial role in our computations.

We end this section with a brief description of L-packets for $\mathbf{G}$. The L-packets for $\mathbf{G}$ are parameterized by KGB-elements for $\mathbf{G}^\vee$. If we fix a KGB-element $\mathtt{y}_0 = p(\eta^\vee)$ for $\mathbf{G}^\vee$, the corresponding L-packet containing representations of real forms of $\mathbf{G}$ is given by

$$\Pi(\mathbf{G},y_0) := \{(x,y)\in\mathcal{X}[x]\times\mathcal{X}^\vee[y_0] \mid (\theta_x|_{\mathfrak{h}})^T = -(\theta_{y_0}|_{\mathfrak{h}^\vee})\}. \qquad (4.9)$$

Let $x_0 := p(\xi)$, then L-packet of $(\mathfrak{g}, \mathbf{K}_\xi)$-modules corresponding to the strong real form ξ of $\mathbf{G}$ and a fixed y_0 is given as

$$\Pi(\mathfrak{g}, \mathbf{K}_\xi, y_0) := \{(x, y) \in \Pi(\mathbf{G}, y_0); \mid x \sim x_0\}. \tag{4.10}$$

4.1 Parabolic Subgroups in Atlas

The Atlas of Lie groups software computes the set $\mathcal{X}[x]$ on computer. We now explain how Atlas computes $\mathbf{K}$-conjugacy classes of Borel and parabolic subgroups.

Given $\xi \in \mathcal{I}$, a strong involution of $\mathbf{G}$ such that $q(\xi) = x$, the map $\mathbf{K}_\xi g\mathbf{B} \mapsto b\mathbf{B}g^{-1}$ is a bijection between $\mathbf{K}_\xi \backslash \mathbf{G}/\mathbf{B}$ and the $\mathbf{K}_\xi := \mathbf{G}^{\theta_\xi}$ conjugacy classes of Borel subgroups. Consider the set $\mathcal{J} := \{(\xi, \mathbf{B}') \mid q(\xi) = x, \mathbf{B}'$ a Borel subgroup of $\mathbf{G}\}$. Now fix ξ_0 such that $q(\xi_0) \in \mathcal{X}[x]$

Consider the following maps:

$$\begin{tikzcd} \mathcal{J} \arrow[r, "\phi_1"] \arrow[dr, "\phi_2"'] & \mathcal{X}[x] \arrow[d, "\psi"] \\ & \mathbf{K}_\xi \backslash \mathbf{G}/\mathbf{B} \end{tikzcd}$$

where for $(\xi, \mathbf{B}') \in \mathcal{J}$, we define ϕ_1 as follows: choose $g \in \mathbf{G}$ such that $g\mathbf{B}'g^{-1} = \mathbf{B}$ and $g\xi g^{-1} \in \mathrm{Norm}(\mathbf{H})$, and define $\phi_1(\xi, \mathbf{B}') = q(g\xi g^{-1}) \in \mathcal{X}[x]$.

Define ϕ_2 as follows: choose $g \in \mathbf{G}$ such that $g\xi g^{-1} = \xi_0$ and define $\phi_2(\xi, \mathbf{B}')$ to be the $\mathbf{K}_\xi$-conjugacy class of $g\mathbf{B}'g^{-1}$.

Both ϕ_1 and ϕ_2 are bijections and hence induce a bijection

$$\psi = \phi_2 \circ \phi_1^{-1} : \mathcal{X}[x] \longrightarrow \mathbf{K}_\xi \backslash \mathbf{G}/\mathbf{B} \tag{4.11}$$

We can generalize the above construction to parabolic subgroups. Let $S \subset \Pi(\mathfrak{g}, \mathfrak{h})$, and let $\mathbf{P}_S$ be the standard parabolic in $\mathbf{G}$ defined by S. All parabolics conjugate to $\mathbf{P}_S$ will be called parabolics of type S. Furthermore, the Weyl group $\mathbf{W}$ acts naturally on $\mathcal{X}[x]$ and so does the group $\mathbf{W}_S$ generated by the simple roots in S. In this setting, we have the following picture:

$$\begin{tikzcd} \{(\xi, \mathcal{P}) \mid q(\xi) \in \mathcal{X}[x], \mathbf{P} \text{ parabolic of type-}S\} \arrow[r, "\phi_1"] \arrow[dr, "\phi_2"'] & \mathcal{X}[x]/\mathbf{W}_S \arrow[d, "\psi"] \\ & \mathbf{K}_\xi \backslash \mathbf{G}/\mathbf{P}_S \end{tikzcd}$$

where given $(\xi, \mathbf{P})$, we define ϕ_1 as follows: choose $g \in \mathbf{G}$ such that $g\mathbf{P}g^{-1} = \mathbf{P}_S$ and $g\xi g^{-1} \in \mathrm{Norm}(\mathbf{H})$, and define $\phi_1(\xi, \mathbf{P}) = q(g\xi g^{-1}) \in \mathcal{X}[x]$.

Define ϕ_2 as follows: choose $g \in \mathbf{G}$ such that $g\xi g^{-1} = \xi_0$ and define $\phi_2(\xi, \mathbf{P})$ to be the $\mathbf{K}_\xi$-conjugacy class of $g\mathbf{P}g^{-1}$.

Both ϕ_1 and ϕ_2 are bijections and hence induce a bijection

$$\psi = \phi_2 \circ \phi_1^{-1} : \mathcal{X}[x]/\mathbf{W}_S \longrightarrow \mathbf{K}_\xi \backslash \mathbf{G}/\mathbf{P}_S \tag{4.12}$$

The main results of this paper use the explicit computation of $\mathbf{K}_\xi$ conjugacy classes of parabolic subgroups, which can now be done using $\mathcal{X}[x]/\sim_S$, the latter computation being implemented in the Atlas software.

Define a finite set $\mathcal{P}$ as follows

$$\mathcal{P} = \{(S, y) \mid S \subset \Pi(\mathfrak{g}, \mathfrak{h}), y \in \mathcal{X}[x]\}/\sim, \tag{4.13}$$

where $(S, y) \sim (S', y')$ if and only if $S = S'$ and $y \sim_S y'$.

For $(S, y) \in \mathcal{P}$, let $[Q(S, y)]$ be the $\mathbf{K}_{\xi_0}$-conjugacy class of parabolic subgroups defined by $\psi(y)$, and let $Q(S, y)$ be a representative parabolic of type S in this class.

Proposition 4.1. *The parabolic $Q(S, y)$ is θ_x-stable if and only if $\theta_x(S) = S$.*

Recall that given a semisimple element $\lambda \in \mathfrak{h}$, let $S(\lambda) \subset \Pi(\mathfrak{g}, \mathfrak{h})$ be the set of simple roots vanishing on λ. One can construct a parabolic subalgebra in $\mathfrak{g}$ as follows:

$$\begin{aligned}
\mathfrak{g}_\alpha &= \{X \in \mathfrak{g} \mid \mathtt{ad}(X)(Y) = \alpha(X)Y, \text{for all } Y \in \mathfrak{g}.\} \\
\mathfrak{n}(\lambda) &= \sum_{\alpha \in \Delta(\mathfrak{g},\mathfrak{h}), \langle \alpha, \lambda \rangle > 0} \mathfrak{g}_\alpha \\
\mathfrak{l}(\lambda) &= \mathrm{Cent}_{\mathfrak{h}}(\lambda) = \mathfrak{h} + \sum_{\alpha \in \Delta(\mathfrak{g},\mathfrak{h}), \langle \alpha, \lambda \rangle = 0} \mathfrak{g}_\alpha \\
\mathfrak{p}(\lambda) &= \mathfrak{l}(\lambda) + \mathfrak{n}(\lambda).
\end{aligned}$$

Let $\mathbf{P}(\lambda)$ be the parabolic subgroup in $\mathbf{G}$ corresponding to $\mathfrak{p}_\lambda$, then $\mathbf{P}(\lambda)$ is a parabolic subgroup of type S. Therefore there exists a $y \in \mathcal{X}[x]$ such that the parabolic $(\xi, \mathbf{P}(\lambda)) \leftrightarrow Q(S(\lambda), y) := \psi(y)$. Let $[Q(S(\lambda), y)]$ be the $\mathbf{K}_\xi$-conjugacy class of parabolic subgroups, and $Q(S, y)$ a representative parabolic of type S in this class.

Proposition 4.2. *The parabolic $Q(S(\lambda), y)$ is θ_x-stable if and only if $\theta_x(\lambda) = \lambda$.*

For more details, the interested reader can visit www.liegroups.org/Papers. In the later parts of the paper we will be in the "Atlas Setting" as follows:

Definition 4.3 (Atlas Setting)**.** *Let* $(\boldsymbol{G}, \gamma)$ *be a basic data and* $(\boldsymbol{G}^\vee, \gamma^\vee)$ *be the corresponding dual basic data. Let* $(\boldsymbol{B}, \boldsymbol{H}, \{X_\alpha\})$ *be a fixed pinning for* $\boldsymbol{G}$*. Let* ξ *be a strong real form of* $\boldsymbol{G}$ *in the inner class of* γ *and let* $\eta^\vee$ *be a strong real form for* $\boldsymbol{G}^\vee$ *in the dual inner class given by* $\gamma^\vee$*. Corresponding to* ξ *and* $\eta^\vee$*, let* $\theta_\xi = Int(\xi)$*, and* $\theta_{\eta^\vee} = Int(\eta^\vee)$ *be Cartan involutions of* $\boldsymbol{G}$ *and* $\boldsymbol{G}^\vee$ *respectively with maximal complex subgroups* $\boldsymbol{K}_\xi$ *and* $\boldsymbol{K}^\vee_{\eta^\vee}$ *respectively. Finally, let* λ *be an integral infinitesimal character for* $\boldsymbol{G}$ *and let* $\mathcal{B}(\xi, \eta^\vee, \lambda)$ *be the block of irreducible* $(\mathfrak{g}, \boldsymbol{K}_\xi)$*-modules at infinitesimal character* λ *specified by the pair of strong real forms* $(\xi, \eta^\vee)$*. Also,* $\mathcal{B}^\vee = \mathcal{B}(\eta^\vee, \xi, \lambda^\vee)$ *is the corresponding dual block of irreducible* $(\mathfrak{g}^\vee, \boldsymbol{K}^\vee_{\eta^\vee})$*-modules.*

Chapter 5: Associated Varieties

5.1 The Complex Associated Variety

Let (π, V) be an irreducible $(\mathfrak{g}, \mathbf{K})$-module. Using the universal property of $U(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$, we can think of (π, V) as a $(U(\mathfrak{g}), \mathbf{K})$ module. Let $I(\pi)$ be the annihilator of π in $U(\mathfrak{g})$, that is

$$I(\pi) = \{X \in U(\mathfrak{g}) : \pi(X)(v) = 0 \text{ for all } v \in V\}. \tag{5.1}$$

The ideal $I(\pi)$ is a primitive ideal in $U(\mathfrak{g})$, and one can construct a filtration $\{I_n(\pi) := U_n(\mathfrak{g})I(\pi)\}$, where $\{U_n(\mathfrak{g})\}$ is the standard filtration for $U(\mathfrak{g})$, and, use it to define the associated graded module:

$$\mathrm{gr}I(\pi) = \bigoplus_{n=0}^{\infty} I_n(\pi)/I_{n-1}(\pi). \tag{5.2}$$

Since $U_m(\mathfrak{g})I_n(\pi) \subset U_{m+n}(\mathfrak{g})$, $\mathrm{gr}(\pi)$ is a graded ideal in $\mathrm{gr}U(\mathfrak{g}) \simeq S(\mathfrak{g})$. Using the Poincare-Birkoff-Witt theorem, $\mathrm{gr}U(\mathfrak{g}) \simeq S(\mathfrak{g})$, and hence we can compute the support of $\mathrm{gr}I(\pi)$. We call the latter the complex associated variety, $AV_{\mathbb{C}}(\pi)$, of π, that is:

$$AV_{\mathbb{C}}(\pi) = \text{Supp}(\text{gr}I(\pi)) = \{\lambda \in \mathfrak{g}^* : X(\lambda) = 0 \ \text{ for all } X \in \text{gr}I(\pi)\}. \tag{5.3}$$

Since $\text{gr}I(\pi)$ is a graded ideal in $S(\mathfrak{g})$, $AV_{\mathbb{C}}(\pi)$ is a cone in $\mathfrak{g}^*$. Furthermore, it can be shown that $\text{gr}I(\pi)$ must contain some power of the augmentation ideal J of $S(\mathfrak{g})$, which is the collection of $\mathtt{Ad}(G)$- invariant polynomials without constant term. Let $J^k \subset \text{gr}I(\pi)$ for some $k \in \mathbb{N}$. This immediately implies that

$$AV_{\mathbb{C}} \subset \text{Supp}(J^k) = \text{Supp}(J). \tag{5.4}$$

The following theorem due to Kostant describes $\text{Supp}(J)$ in $\mathfrak{g}^*$,

Theorem 5.1 ([Vog91], Theorem 5.7)**.** *Suppose G is a reductive Lie group, and $J \subset S(\mathfrak{g})$, the augmentation ideal. Then the associated variety of J is the cone $\mathcal{N}^*$ of nilpotent elements in $\mathfrak{g}^*$.*

An application of the above theorem shows that $AV_{\mathbb{C}}(\pi) \subset \mathcal{N}^*$, the nilpotent cone in $\mathfrak{g}^*$. Therefore $AV_{\mathbb{C}}(\pi)$ must be the closure of a finite union of nilpotent orbits in $\mathfrak{g}^*$, since $\mathcal{N}$ has finite number of orbits. In fact, a much stronger statement is true,

Theorem 5.2 (Borho, Brylinski, Joseph)**.** *Let $\boldsymbol{G}$ be a complex connected reductive Lie group and let $\boldsymbol{G}(\mathbb{R})$ be a real form of $\boldsymbol{G}$. Suppose (π, V) is an irreducible $(\mathfrak{g}, \boldsymbol{K})$-module, then $AV_{\mathbb{C}}(\pi)$ is the closure of a single nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}^*$.*

Given (π, V), an admissible irreducible $(\mathfrak{g}, \mathbf{K})$-module of $\mathbf{G}(\mathbb{R})$, it is desirable to know if one can compute the invariant $AV_{\mathbb{C}}(\pi)$. In the case when $\mathbf{G}(\mathbb{R})$ is a classical connected reductive Lie group, we use an algorithm due to Noel and Jackson coupled with the Springer correspondence. These computations have been implemented in the Atlas software.

5.2 The Real and the Theta Associated Variety

We assume that (π, V) is a finite length $(\mathfrak{g}, \mathbf{K})$-module, in this case one can show that V is generated by a finite-dimensional subspace S as a $(\mathfrak{g}, \mathbf{K})$-module. Using the universal property of $U(\mathfrak{g})$ we can show that (π, V) is a $(U(\mathfrak{g}), \mathbf{K})$ - module. Furthermore, for $v \in V$, π satisfies the following conditions:

1. $\mathrm{d}\pi(Z)v = Z \cdot v \quad$ for all $\; Z \in \mathfrak{k}$.
2. $\pi(k)(X \cdot v) = (\mathtt{Ad}(k)X) \cdot \pi(k)v \quad$ for all $\; k \in \mathbf{K} \;$ and $\; X \in U(\mathfrak{g})$.

Using the local finiteness of the action of $\mathbf{K}$ we can find a finite-dimensional $\mathbf{K}$-invariant subspace V_0 of V that contains S. An easy argument shows that $V = U(\mathfrak{g})V_0$. Therefore, we can construct a filtration $\{V_n(\pi) := U_n(\mathfrak{g})V_0\}$ for V. Since, V_0 was K invariant, and the fact that the action of $\mathbf{K}$ is compatible with the action of $U(\mathfrak{g})$, we note that $V_n(\pi)$ is $\mathbf{K}$-invariant for all n.

Note that $U_n(\mathfrak{g})V_m(\pi) \subset V_{m+n}(\pi)$. Therefore, this gives us a $\mathbf{K}$-invariant graded submodule of $\mathrm{gr}U(\mathfrak{g})$ given by:

$$\mathrm{gr}\ V = \bigoplus_{n=0}^{\infty} V_n(\pi)/V_{n-1}(\pi). \tag{5.5}$$

A consequence of the PBW-Theorem is that gr$V$ is a $(S(\mathfrak{g}), \mathbf{K})$ module so that the $S(\mathfrak{g})$ and the $\mathbf{K}$ action satisfy the following:

$$\pi(k)(X \cdot v) = (\mathtt{Ad}(k)X) \cdot \pi(k)v \text{ for all } k \in \mathbf{K}, v \in \mathrm{gr}V, \text{ and } X \in S(\mathfrak{g}). \tag{5.6}$$

Differentiating the above equation and noting that $S(\mathfrak{g})$ is an abelian Lie algebra, we see that

$$Z \cdot v = 0, \quad \text{for all } X \in \mathfrak{k} \text{ and } v \in \mathrm{gr}V. \tag{5.7}$$

As a result, the action $(S(\mathfrak{g}), \mathbf{K})$ action on gr$V$ descends to a $(S(\mathfrak{g}/\mathfrak{k}), \mathbf{K})$ action. We define

$$\begin{aligned} AV_\theta(\pi) &= \mathrm{Supp}(\mathrm{gr}V) \\ &= \{\lambda \in (\mathfrak{g}/\mathfrak{k})^* : v \cdot (\lambda) = 0 \text{ for all } v \in \mathrm{gr}V\} \subset (\mathfrak{g}/\mathfrak{k})^*. \end{aligned} \tag{5.8}$$

As in the case of $AV_{\mathbb{C}}(\pi)$, we can show that $AV_\theta(\pi)$ is closed under dilations, which implies that $AV_\theta(\pi)$ lies in a cone in $(\mathfrak{g}/\mathfrak{k})^*$. Furthermore, the fact that the module (π, V) is quasisimple implies that gr$V$ contains some power of the augmentation ideal J of $S(\mathfrak{g}/\mathfrak{k})$, therefore

$$AV_\theta(\pi) \subset \mathrm{Supp}(J). \tag{5.9}$$

As in the case of $AV_{\mathbb{C}}(\pi)$, we can show that Supp(J) is in fact the nilpotent cone $\mathcal{N}_\theta^*$ in $(\mathfrak{g}/\mathfrak{k})^*$. Hence $AV_\theta(\pi)$ must be a union of finitely many $\mathbf{K}$-orbits in $(\mathfrak{g}/\mathfrak{k})^*$. Using the Kostant-Sekiguchi correspondence, to the $\mathbf{K}$-orbits in $AV_\theta(\pi)$ one can find the corresponding $\mathbf{G}(\mathbb{R})$ orbits in $\mathcal{N}_{\mathbb{R}}^*$ and the union of these orbits is called the real associated variety of π, denoted as $AV_{\mathbb{R}}(\pi)$.

To summarize the above discussion, $AV_\theta(\pi)$ satisfies:

$$AV_\theta(\pi) = \overline{\mathcal{O}_K^1 \cup \mathcal{O}_K^2 \cup \cdots \cup \mathcal{O}_K^r}, \tag{5.10}$$

where $\mathcal{O}_K^i$ are nilpotent $\mathbf{K}$-orbits on $(\mathfrak{g}/\mathfrak{k})^*$. For $i = 1, 2, \ldots, r$, if $\mathcal{O}_{\mathbb{R}}^i$ is the $\mathbf{G}(\mathbb{R})$ orbit corresponding to $\mathcal{O}_K^i$ under the Kostant-Sekiguchi correspondence then we define the real associated variety to be

$$AV_{\mathbb{R}}(\pi) = \overline{\mathcal{O}_{\mathbb{R}}^1 \cup \mathcal{O}_{\mathbb{R}}^2 \cup \cdots \cup \mathcal{O}_{\mathbb{R}}^r}. \tag{5.11}$$

The two invariants $AV_{\mathbb{C}}(\pi)$ and $AV_\theta(\pi)$ attached to (π, V) are related as follows:

1. $AV_{\mathbb{C}}(\pi) \subset \mathcal{N}^*$.

2. $AV_\theta(\pi) \subset \mathcal{N}_\theta^*$.

3. $AV_{\mathbb{R}}(\pi) \subset \mathcal{N}_{\mathbb{R}}^*$.

4. If $AV_\theta(\pi) = \overline{\mathcal{O}_K^1 \cup \mathcal{O}_K^2 \cup \cdots \cup \mathcal{O}_K^r}$, then

$$AV_{\mathbb{C}}(\pi) = \overline{\mathtt{Ad}(\mathbf{G}) \cdot \mathcal{O}_K^i}.$$

5. If $AV_{\mathbb{R}}(\pi) = \overline{\mathcal{O}^1_{\mathbb{R}} \cup \mathcal{O}^2_{\mathbb{R}} \cup \cdots \cup \mathcal{O}^r_{\mathbb{R}}}$, then

$$AV_{\mathbb{C}}(\pi) = \overline{\mathtt{Ad}(\mathbf{G}) \cdot \mathcal{O}^i_{\mathbb{R}}}.$$

We end this section with a brief description of cohomologically induced modules and their associated varieties. Fix a Cartan involution θ for $\mathbf{G}$. Let $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}$ be a theta-stable parabolic subalgebra of $\mathfrak{g}$, so that $\mathfrak{p}$, $\mathfrak{l}$ and $\mathfrak{n}$ are preserved by θ. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$, be the Cartan decomposition of $\mathfrak{g}$ and $\mathbf{s} = \dim(\mathfrak{s} \cap \mathfrak{n})$. We start with a $(\mathfrak{l}, \mathbf{L} \cap \mathbf{K})$- module and construct a $(\mathfrak{g}, \mathbf{K})$-module using Zuckerman's cohomological induction functor.

Suppose Z is a one dimensional $(\mathfrak{l}, (\mathbf{L} \cap \mathbf{K}))$-module with infinitesimal character γ_L. We can extend Z to a $(\mathfrak{p}, \mathbf{L} \cap \mathbf{K})$-module by making $\mathfrak{n}$ act trivially. Then Zuckerman defines the following produced module

$$X = \mathrm{pro}^{(\mathfrak{g}, \mathbf{L} \cap \mathbf{K})}_{(\mathfrak{p}, \mathbf{L} \cap \mathbf{K})}(Z), \tag{5.12}$$

and a functor

$$(\mathcal{R}_{\mathfrak{p}, \mathbf{L}})^0(Z) = \Gamma^{(\mathfrak{g}, \mathbf{K})}_{(\mathfrak{g}, \mathbf{L} \cap \mathbf{K})}(X). \tag{5.13}$$

$\mathcal{R}^0$ is a left exact functor and because the category of $(\mathfrak{l}, (\mathbf{L} \cap \mathbf{K}))$-modules has enough injectives, one can define

$$(\mathcal{R}_{\mathfrak{p}, \mathbf{L}})^i = i\text{th right derived functor of } (\mathcal{R}_{\mathfrak{p}, \mathbf{L}})^0. \tag{5.14}$$

In this setting,

Theorem 5.3 (Zuckerman, Vogan, Theorem 6.8, [Vog87])**.** *Suppose $\boldsymbol{L}(\mathbb{R})$ is a Levi subgroup of $\boldsymbol{G}(\mathbb{R})$ attached to the θ-stable parabolic subalgebra $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}$. Let $\boldsymbol{L}$ be the complexification of $\boldsymbol{L}(\mathbb{R})$, and $\boldsymbol{s}$ the dimension of $\mathfrak{n} \cap \mathfrak{k}$. Let $2\rho(\mathfrak{n})$ be the sum of roots positive on $\mathfrak{n}$. Consider the functors*

$$\mathcal{R}^j = (\mathcal{R}_{\mathfrak{p},\boldsymbol{L}})^j \quad j \in \{0, 1, 2, \ldots, \boldsymbol{s}\} \tag{5.15}$$

from the category of $(\mathfrak{l}, \boldsymbol{L} \cap \boldsymbol{K})$-modules, to the category of $(\mathfrak{g}, \boldsymbol{K})$-modules. Let Z be a $(\mathfrak{l}, \boldsymbol{L} \cap \boldsymbol{K})$-module and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{l}$. Assume that Z has $\boldsymbol{L}$-infinitesimal character $\gamma_L \in \mathfrak{h}^*$ then

1. *$\mathcal{R}^j(Z)$ has $\boldsymbol{G}$-infinitesimal character $\gamma_L + \rho(\mathfrak{n})$.*

2. *Assume that for each root α of $\mathfrak{h}$ in $\mathfrak{n}$,*

   $$Re\langle \gamma_L + \rho(n), \alpha \rangle \geqslant 0.$$

   Then $\mathcal{R}^j(Z)$ is zero for j not equal to $\boldsymbol{s}$.

3. *Under the above hypothesis, if Z is unitary, then so is $\mathcal{R}^{\boldsymbol{s}}(Z)$.*

4. *If we assume that for each root α of $\mathfrak{h}$ in $\mathfrak{n}$,*

   $$Re\langle \gamma + \rho(\mathfrak{n}), \alpha \rangle > 0.$$

Then, if Z is non-zero, so is $\mathcal{R}^{s}(Z)$.

For most of our applications, we will take Z to be a one dimensional $(\mathfrak{l}, \mathbf{L} \cap \mathbf{K})$-module. Let $\lambda = dZ \in \mathfrak{h}^{*}$. Let $\mathfrak{p} := \mathfrak{p}(\lambda) = \mathfrak{l}(\lambda) + \mathfrak{n}(\lambda)$, and by $\mathcal{A}_{\mathfrak{p}}(Z)$, we will really mean $\mathcal{R}^{\mathbf{s}}(Z)$. The modules $\mathcal{A}_{\mathfrak{p}}(Z)$, often denoted as $\mathcal{A}_{\mathfrak{p}}(\lambda)$ are defined using the θ-stable data $(\mathfrak{p}, \lambda := dZ)$ and have infinitesimal character $\lambda + \rho(\mathfrak{n}(\lambda))$.

The following theorem shows that even though cohomological induction functor depends on Z, the associated variety of the cohomologically induced module $\mathcal{A}_{\mathfrak{p}}(Z)$ depends only on $\mathfrak{n}(\lambda)$, which in turn depends only on dZ.

Theorem 5.4. *[Yam94] Let $\boldsymbol{G}(\mathbb{R})$ be a real group corresponding to the Cartan involution θ. Let $\pi = \mathcal{A}_{\mathfrak{p}}(Z)$, where the θ-stable data is given by $\mathfrak{p} = \mathfrak{p}(\lambda)$ and Z is a one dimensional representation of $\mathfrak{l}$ satisfying $dZ = \lambda \in \mathfrak{l}^{*}$. Suppose $\langle \lambda + \rho(\mathfrak{n}), \alpha \rangle > 0$ for all $\alpha \in \Delta(\mathfrak{n})$, where $\Delta(\mathfrak{n})$ is the set of roots on $\mathfrak{n}$ and $2\rho(\mathfrak{n})$ is the sum of roots in $\Delta(\mathfrak{n})$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ be the Cartan decomposition of $\mathfrak{g}$. Then, $AV_{\theta}(\pi) = \overline{\boldsymbol{K} \cdot (\mathfrak{n} \cap \mathfrak{s})}$, is the closure of a single $\boldsymbol{K}$-orbit in $\mathcal{N}_{\theta}$.*

5.3 Coherent Continuation and Translation Functors

We know that $X^{*}(\mathbf{H})$ is the lattice of weights of finite dimensional representations for $\mathbf{G}$. So that given a finite dimensional representation F of $\mathbf{G}$, the set $\Delta(F)$ of weights of F, is a subset of $X^{*}(\mathbf{H})$. We begin with the definition of a coherent family:

Definition 5.1 (Coherent Family)**.** *A coherent family of virtual modules is a map*

$$\Theta : X^*(\boldsymbol{H}) \longrightarrow \mathbb{K}\mathfrak{M}(\mathfrak{g}, \boldsymbol{K}), \tag{5.16}$$

such that:

1. $\Theta(\lambda)$ *has infinitesimal character* $\lambda \in X^*(\boldsymbol{H})$*; and*

2. *For every finite-dimensional representation* F *of* $\boldsymbol{G}$*,*

$$F \otimes \Theta(\lambda) = \sum_{\mu \in \Delta(F)} \Theta(\lambda + \mu). \tag{5.17}$$

Now, fix $\gamma \in X^*(\mathbf{H})$. Given $M \in \mathfrak{M}(\mathfrak{g}, \mathbf{K}, \gamma)$, we say $\Theta : X^*(\mathbf{H}) \longrightarrow \mathbb{K}\mathfrak{M}(\mathfrak{g}, \mathbf{K})$ is a coherent family through M if $\Theta(\gamma) = M$. The set of all coherent families on X^* is a finite rank, free $\mathbb{Z}$-module. If γ is assumed to be regular then we have a basis for coherent families on $X^*(\mathbf{H})$ given by $\{\Theta_M\}$, where Θ_M is a coherent family through M, and $M \in \Pi(\mathfrak{g}, \mathbf{K}, \gamma)$.

Suppose $w \in \mathbf{W}$ and $\Theta : X^* \longrightarrow \mathbb{K}\mathfrak{M}(\mathfrak{g}, \mathbf{K})$ is a coherent family. We can construct a new coherent family $w \cdot \Theta$ defined by

$$w \cdot \Theta(\lambda) = \Theta(w^{-1}\lambda) \tag{5.18}$$

Since the infinitesimal character is equivalent up to the action of $\mathbf{W}$, $w \cdot \Theta(\lambda)$ has infinitesimal character λ. Since the weights of a finite dimensional representation of $\mathbf{G}$ are invariant under the action of $\mathbf{W}$, the second condition for the definition of coherent family is also true for $w \cdot \Theta$. To summarize the above discussion:

Theorem 5.5 (Coherent Continuation Action)**.** *Suppose γ is a fixed regular integral infinitesimal character. Then there is an action of $\boldsymbol{W}$ on the set of all coherent families*

$$\Theta : X^* \longrightarrow \mathbb{K}\mathfrak{M}(\mathfrak{g}, \boldsymbol{K})$$

defined by

$$w \cdot \Theta(\lambda) = \Theta(w^{-1}\lambda). \tag{5.19}$$

We can use the coherent continuation action to define a $\mathbf{W}$ action on $\mathbb{K}\mathfrak{M}(\mathfrak{g}, \mathbf{K}, \gamma)$. Since $\Pi(\mathfrak{g}, \mathbf{K}, \gamma)$ is a basis for $\mathbb{K}\mathfrak{M}(\mathfrak{g}, \mathbf{K}, \gamma)$, we only need to define the action of $\mathbf{W}$ on this basis and then linearly extend this action.

Suppose $J \in \Pi(\mathfrak{g}, \mathbf{K}, \gamma)$. Choose a coherent family $\Theta : X^* \longrightarrow \mathbb{K}\mathfrak{M}(\mathfrak{g}, \mathbf{K})$ such that $\Theta(\gamma) = J$. Then,

$$w \cdot J = (w \cdot \Theta)(\gamma). \tag{5.20}$$

We can use the action of $\mathbf{W}$ on $\mathbb{K}\mathfrak{M}(\mathfrak{g}, \mathbf{K}, \gamma)$ to define a partial order on representations in $\Pi(\mathfrak{g}, \mathbf{K}, \gamma)$ as follows:

Definition 5.2. *Suppose $X, Y \in \mathfrak{M}(\mathfrak{g}, \boldsymbol{K}, \gamma)$.*

1. *We say $X <_\gamma Y$ if Y appears in $w \cdot X$ for some $w \in \boldsymbol{W}_\gamma$*

2. *We say $X \sim_\gamma Y$ if $X <_\gamma Y$ and $Y <_\gamma X$.*

The relation $\sim_\gamma$ is an equivalence relation on $\Pi(\mathfrak{g}, \mathbf{K}, \gamma)$, and, the equivalence classes are called Harish-Chandra cells.

The following result provides a relation between coherent continuation action and the operation of computing associated varieties.

Proposition 5.1. *Let $J \in \Pi(\mathfrak{g}, \boldsymbol{K}, \gamma)$. Suppose $\Theta : X^* \longrightarrow \mathbb{K}\mathfrak{M}(\mathfrak{g}, \boldsymbol{K})$ such that $\Theta(\gamma) = J$. Let $w \in \boldsymbol{W}$ be an arbitrary element in $\boldsymbol{W}_\gamma$, then*

1. $AV_{\mathbb{C}}(J) = AV_{\mathbb{C}}(w \cdot J)$.

2. $AV_\theta(J) = AV_\theta(w \cdot J)$.

3. $AV_{\mathbb{R}}(J) = AV_{\mathbb{R}}(w \cdot J)$.

Proof. This results comes down to checking that the graded algebras involved in the computations of the associated varieties for J and $w \cdot J$ are all isomorphic, since J and $w \cdot J$ differ only up to tensoring with finite dimensional representations of $\mathbf{G}$. The conclusion about associated varieties then follows. $\square$

Recall that $\Pi(\mathfrak{g}, \mathbf{K}, \gamma)$ is the set of irreducible representations with infinitesimal character γ. Zuckerman's ideas of tensoring representations with finite-dimensional representations lead to the the theory of translation functors, which is a way of studying the representation theory at an infinitesimal character δ (possibly different from γ) in terms of the representation theory at γ. These ideas will be used extensively in computing unipotent representations.

Let $\gamma \in \mathfrak{h}^*$ be a fixed infinitesimal character. Fix a weight $\phi \in X^*(\mathbf{H})$ and let F_ϕ be the finite dimensional representation of $\mathbf{G}$ with extremal weight ϕ. Let $\pi \in \Pi(\mathfrak{g}, \mathbf{K}, \gamma)$ be an irreducible $(\mathfrak{g}, \mathbf{K})$-module. For, $\gamma \in X^*$, let $\xi_\gamma : \mathfrak{z}(U(\mathfrak{g})) \longrightarrow \mathbb{C}$ be the character on $\mathfrak{z}(U(\mathfrak{g}))$ given by Harish Chandra's isomorphism. Define the projection map:

$$P_\gamma : \mathbb{K}\mathfrak{M}(\mathfrak{g}, \mathbf{K}) \longrightarrow \mathbb{K}\mathfrak{M}(\mathfrak{g}, \mathbf{K}, \gamma), \tag{5.21}$$

where the map takes $\pi \in \mathfrak{M}(\mathfrak{g}, \mathbf{K})$ to the largest submodule of π annihilated by $(I - \xi_\gamma)|_{\mathfrak{z}(U(\mathfrak{g}))}$. In other words, P_γ takes π to the largest submodule with infinitesimal character γ.

Definition 5.3 (Translation (to the Wall) Functor)**.** *Suppose F_ϕ is a finite dimensional representation of $\boldsymbol{G}$ with highest weight ϕ. Let $\gamma \in \mathfrak{h}^*$ be regular and integral, and, let $\pi \in \mathbb{K}\mathfrak{M}(\mathfrak{g}, \boldsymbol{K}, \gamma)$. Assume that $\gamma + \phi$ is dominant (possibly singular). The translation functor is the functor*

$$T_\gamma^{\gamma+\phi} : \mathfrak{M}(\mathfrak{g}, \boldsymbol{K}, \gamma) \longrightarrow \mathfrak{M}(\mathfrak{g}, \boldsymbol{K}, \gamma + \phi), \quad \pi \mapsto P_{\gamma+\phi}(\pi \otimes F_\phi). \tag{5.22}$$

Alternately, we can define translation functors using coherent families as follows: suppose $J \in \Pi(\mathfrak{g}, \mathbf{K}, \gamma)$, choose a coherent family Θ such that $\Theta(\gamma) = J$, then

$$T_\gamma^{\gamma+\phi}(J) = \Theta(\gamma + \phi). \tag{5.23}$$

Since $J \in \Pi(\mathfrak{g}, \mathbf{K}, \gamma)$ is a basis for $\mathfrak{M}(\mathfrak{g}, \mathbf{K}, \gamma)$, we can then linearly extend this definition. Using the relationship of coherent families and associated varieties, we have

Proposition 5.2. *Let* $\gamma \in \mathfrak{h}^*$ *be a regular integral infinitesimal character and let* $\pi \in \mathfrak{M}(\mathfrak{g}, \boldsymbol{K}, \gamma)$. *Let* $\phi \in X^*$ *be an extremal weight of* F_ϕ, *a finite dimensional representation of* $\boldsymbol{G}$. *Then,*

1. $AV_{\mathbb{C}}(\pi) = AV_{\mathbb{C}}(T_\gamma^{\gamma+\phi}(\pi))$.

2. $AV_\theta(\pi) = AV_\theta(T_\gamma^{\gamma+\phi}(\pi))$.

3. $AV_{\mathbb{R}}(\pi) = AV_{\mathbb{R}}(T_\gamma^{\gamma+\phi}(\pi))$.

Proof. Since translation functors are nothing but evaluation of coherent families the result follows from the fact that asociated varieties are constant for a fixed coherent family. □

We package this information about associated varieties being constant on coherent families into the following result,

Proposition 5.3. *Suppose*

$$\mathfrak{M}(\mathfrak{g}, K, \gamma) = \coprod_{\text{HC-Cells}} \mathcal{C},$$

then $AV_{\mathbb{C}}(\pi), AV_\theta(\pi)$, *and* $AV_{\mathbb{R}}(\pi)$ *remain constant as one varies* π *over a fixed cell* $\mathcal{C}$.

Proposition 5.3 allows us to define the notion of associated variety of a cell, that is, if $\mathcal{C}$ is a HC-cell, we can define $AV_{\mathbb{C}}(\mathcal{C})$, $AV_\theta(\mathcal{C})$, $AV_{\mathbb{R}}(\mathcal{C})$ to be the respective associated varieties of a fixed $\pi \in \mathcal{C}$.

Suppose $\mathcal{C}$ is a HC-cell. Taking the irreducible representations in $\mathcal{C}$ as a basis, we can linearly extend the coherent continuation action to a $\#\mathcal{C} := c$-dimensional complex representation of $\mathbf{W}$. Understanding this Weyl group representation on the cell $\mathcal{C}$ will be the main goal of the following section.

The coherent continuation action on $\mathcal{C}$ contains a unique special Weyl group representation. We can then use the Springer-correspondence to attach a complex nilpotent orbit of $\mathfrak{g}$ to $\mathcal{C}$. This complex nilpotent orbit turns out to be the complex associated variety of representations in this cell. There are at least two approaches to computing the special $\mathbf{W}$ -representation of the cell $\mathcal{C}$ - one due to Noel and Jackson, and the other due to Binegar. In the case when the group $\mathbf{G}(\mathbb{R})$ is of classical type, the algorithm due to Noel and Jackson is very amenable to implementation in Atlas.

5.4 The Noel-Jackson Algorithm

Let $\mathbf{G}(\mathbb{R})$ be the real form a complex classical connect reductive algebraic group $\mathbf{G}$. The special $\mathbf{W}$-representation attached to a HC-cell $\mathcal{C}$ can be studied using the sign representation. More precisely, suppose π is a representation of $\mathbf{W}$ and let $\mathbf{L}(\pi)$ be the set of all parabolic subgroups $\mathcal{P}$ of $\mathbf{W}$, such that $\mathrm{Res}_{\mathcal{P}}^{\mathbf{W}}(\pi)$ contains the sign representation of $\mathcal{P}$, $\mathbf{L}(\pi)$ is called the Levi-set of π.

Theorem 5.6 (Noel-Jackson, [FJMN18]). *Suppose* $\boldsymbol{W}$ *is a Weyl group of classical* *type. Let π be an irreducible representation of $\boldsymbol{W}$. Then, π is determined by its Levi set $\boldsymbol{L}(\pi)$.*

Alternately, starting with a Levi set $\boldsymbol{L}$, it is possible to construct a unique $\pi \in \widehat{\boldsymbol{W}}$ such that $\boldsymbol{L}(\pi) = \boldsymbol{L}$.

We need the following definition:

Definition 5.4 (Tau-invariant). *Let γ be a regular integral infinitesimal character. Suppose $J \in \Pi(\mathfrak{g}, K, \gamma)$. Fix a set of positive simple roots $\Pi^+(\mathfrak{g}, \mathfrak{h})$. We say that a simple root α is in the tau invariant of J if and only if $s_\alpha \cdot J = -J$ (in the Grothendieck group $\mathbb{K}\mathfrak{M}_\gamma(\mathfrak{g}, K)$). We denote this set by $\tau(J)$.*

Fix a HC-cell $\mathcal{C}$, let $J \in \mathcal{C}$ and let π be the special $\mathbf{W}$-representation attached to $\mathcal{C}$. We can construct a parabolic subgroup $\mathcal{P}_J$ of W using the s_α for $\alpha \in \tau(J)$ as generators. Furthermore, by definition of the tau-invariant, we see that $\mathrm{Res}^{\mathbf{W}}_{\mathcal{P}_J}(\pi)$ contains the sign representation of $\mathcal{P}_J$. Therefore, using tau-invariants of representations in $\mathcal{C}$, we can extract a Levi-set $\mathbf{L}(\mathcal{C})$ for the cell $\mathcal{C}$. This is the Levi-set for the coherent continuation representation on $\mathcal{C}$. We can now use the Noel-Jackson algorithm (in [FJMN18]) to compute the special cell representation on $\mathcal{C}$. We have implemented this algorithm into the Atlas software, so as to use it's functionality in computing associated varieties.

Chapter 6: Parameterizing Theta Forms of Even Complex Nilpotent Orbits

Let $\mathbf{G}$ be a complex connected reductive algebraic group. Let ${}^{\Gamma}\mathbf{G}^{\vee}$ be a L-group for $\mathbf{G}$. We will be in the Atlas Setting (refer 4.3) for the rest of this paper. In this section we outline an algorithm to parameterize real forms of an even complex nilpotent orbit.

6.1 Unipotent Arthur Parameters

Fix a unipotent Arthur parameter (Definition 3.3), say ψ. Using the restriction of ψ to $SL(2,\mathbb{C})$, and under a "integrality" assumption, we get a nilpotent orbit $\mathcal{O}^{\vee}$ of $\mathbf{G}^{\vee}$ on $\mathfrak{g}^{\vee}$. Furthermore, the restriction of ψ to $W_{\mathbb{R}}$ is determined once we specify $\psi(j)$, which must be an element of order two in ${}^{\Gamma}\mathbf{G}^{\vee}$ satisfying:

1. $\psi(j) \in \mathrm{Cent}_{\mathbf{G}^{\vee}}(\psi|_{SL(2,\mathbb{C})})$,
2. $\psi(j) \in {}^{\Gamma}\mathbf{G}^{\vee} - \mathbf{G}^{\vee}$.

Corresponding to ψ, let

$$\psi_1 := \psi|_{SL(2,\mathbb{C})} : SL(2,\mathbb{C}) \longrightarrow \mathbf{G}^{\vee}, \tag{6.1}$$

and let

$$\lambda = \lambda_1 = d\psi_1 \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \in \mathfrak{h}^\vee, \quad E_\psi = d\psi_1 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{6.2}$$

If λ is integral, then $\mathcal{O}^\vee$ is an even nilpotent orbit for $\mathbf{G}^\vee$ (else, it is a nilpotent orbit of a proper subgroup of $\mathbf{G}^\vee$). We can construct the parabolic subalgebra $\mathfrak{p}(\lambda)^\vee = \mathfrak{l}(\lambda)^\vee + \mathfrak{n}(\lambda)^\vee \subset \mathfrak{g}^\vee$. Let $\mathcal{P}^\vee$ be the $\mathbf{G}^\vee$-conjugacy class of parabolic subalgebras conjugate to $\mathfrak{p}(\lambda)^\vee$.

Let $y \in \mathcal{I}(\mathbf{G}^\vee, \gamma^\vee)$ be a representative for a strong real form of $\mathbf{G}^\vee$ and let $\theta^\vee = \mathrm{Int}(y)$. Let $\mathfrak{g}^\vee = \mathfrak{k}^\vee \oplus \mathfrak{s}^\vee$ be the Cartan decomposition of $\mathfrak{g}^\vee$ with respect to $\theta^\vee$. In this setting, $E_\psi \in \mathfrak{n}(\lambda)^\vee \cap \mathfrak{s}^\vee$, and using ([ABV92], Lemma 27.8), it belongs to the Richardson class ([ABV92], Proposition 20.4) corresponding to $\mathcal{P}^\vee$, denoted as $\mathcal{Z}_{\mathcal{P}^\vee}$.

Given a unipotent Arthur parameter ψ, let $\mathcal{O}^\vee_\lambda = \mathbf{G}^\vee \cdot \lambda$ be the semisimple orbit containing $\lambda = d\psi_1 \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \in \mathfrak{h}^\vee$. Let $X(\mathcal{O}^\vee_\lambda, {}^\Gamma\mathbf{G}^\vee) := \{(y', \lambda') \mid y' \sim y \ \text{ and } \ \lambda' \in \mathcal{O}^\vee_\lambda\}$. We say that a unipotent parameter ψ' is *supported* on $X(\mathcal{O}^\vee_\lambda, {}^\Gamma\mathbf{G}^\vee)$ if $\lambda' \in \mathcal{O}^\vee_\lambda$, where $\lambda' = d\psi'_1 \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$.

6.2 Parameterizing Theta Forms of an even complex nilpotent orbit.

Let $\mathcal{O}^\vee \subset \mathfrak{g}^\vee$ be a complex even $\mathbf{G}^\vee$-nilpotent orbit. The goal of this section is to find a "good" parameterization for the theta-forms of $\mathcal{O}^\vee$ defined in (2.4). Using the Kostant-Sekiguchi correspondence, we get a parameterization of the real forms of

$\mathcal{O}^\vee$ defined in (2.3).

Let $\{X^\vee, Y^\vee, H^\vee\}$ be the Jacobson-Morozov triple for $\mathcal{O}^\vee$, so that $\mathcal{O}^\vee = \mathbf{G}^\vee \cdot X^\vee$. We recall a special case/corollary of ([ABV92], Theorem 27.10),

Corollary 6.1. *Let $\theta^\vee$ be the Cartan involution of $\boldsymbol{G}^\vee$ satisfying $(\boldsymbol{G}^\vee)^{\theta^\vee} = \boldsymbol{K}^\vee$ and let $\mathfrak{g}^\vee = \mathfrak{k}^\vee \oplus \mathfrak{s}^\vee$ be the Cartan decomposition. Let $\boldsymbol{G}(\mathbb{R})^\vee$ be the real form of $\boldsymbol{G}^\vee$ corresponding to the Cartan involution $\theta^\vee$. Furthermore, assume that the semisimple orbit $\mathcal{O}^\vee_\lambda$ corresponding to λ comes from a homomorphism $\psi_1 : SL(2,\mathbb{C}) \longrightarrow \boldsymbol{G}^\vee$ attached to the even nilpotent $\boldsymbol{G}^\vee$ orbit $\mathcal{O}^\vee$. Then there is a correspondence between the following sets:*

1. *The equivalence classes of unipotent Arthur parameters supported on $X(\mathcal{O}^\vee_\lambda, {}^\Gamma\boldsymbol{G}^\vee)$.*
2. *$\boldsymbol{K}^\vee$ conjugacy classes of of parabolic subgroups $\boldsymbol{Q}^\vee \in \mathcal{P}^\vee = \boldsymbol{G}^\vee/\boldsymbol{P}^\vee$, where $\boldsymbol{P}^\vee$ is a fixed parabolic subgroup of $\boldsymbol{G}^\vee$ such that its Levi factor $\boldsymbol{L}^\vee$ has Lie algebra $\mathfrak{l}^\vee = Cent_{\mathfrak{g}^\vee}(\lambda)$ and $\boldsymbol{Q}^\vee$ satisfies:*
   - *(a) $\theta^\vee(\boldsymbol{Q}) = \boldsymbol{Q}^\vee$.*
   - *(b) Let $\mathfrak{q}^\vee = \mathfrak{l}^\vee + \mathfrak{n}^\vee$ be the Langlands decomposition of $\mathfrak{q}^\vee = Lie(\boldsymbol{Q}^\vee)$, then*

$$\mathfrak{n}^\vee \cap \mathfrak{s}^\vee \cap \mathcal{O}^\vee \neq \varnothing$$

3. *$\boldsymbol{K}^\vee$ orbits on $\mathfrak{s}^\vee \cap \mathcal{O}^\vee$.*

Proof. Let $\{X^\vee, Y^\vee, H^\vee\}$ be the Jacobson-Morozov triple for $\mathcal{O}^\vee$ and let $\mathfrak{l}^\vee = \mathrm{Cent}_{\mathfrak{g}^\vee}(H^\vee)$. Using ([CM93], Corollary 7.1.7), we note that the even nilpotent orbit $\mathcal{O}^\vee$ is a

Richardson orbit, in fact, it is induced from the trivial obit on $\mathfrak{l}^\vee$. This implies that $\mathcal{Z}_{\mathcal{P}^\vee} = \mathcal{O}^\vee$. Using ([ABV92], Theorem 27.10) in the light of these observations we get the three correspondences. □

The conditions (1) and (3) in the Corollary above are most intuitive to the reader, yet checking them is not easy. We will use the Atlas Setting and software to work with condition (2): (2a) is elementary and has been implemented in Atlas, (2b) is the more difficult one to test, and our method uses representation theory to arrive at an algorithm to test it successfully in many cases.

We continue to be in the Atlas Setting as follows: $\mathbf{G}$ a complex connected reductive algebraic group, $\mathbf{G}^\vee$ the dual group. Fix a strong real form $\eta^\vee$ of $\mathbf{G}^\vee$ and let $\theta^\vee_{\eta^\vee}$ be the corresponding Cartan involution of $\mathbf{G}^\vee$. Let $\mathbf{K}^\vee = \mathrm{Cent}_{\mathbf{G}^\vee}(\eta^\vee)$. Let $\mathfrak{g}^\vee = \mathfrak{k}^\vee \oplus \mathfrak{s}^\vee$ be the Cartan decomposition of $\mathfrak{g}^\vee$ with respect to $\theta^\vee_{\eta^\vee}$. Furthermore, we choose ξ to be the dual quasisplit strong real form of $\mathbf{G}$, in the dual inner class of ${}^\Gamma\mathbf{G}^\vee$.

Fix a regular integral infinitesimal character $\gamma \in \mathfrak{h}^* \simeq \mathfrak{h}^\vee$. We are fixing ξ, $\eta^\vee$ and γ, so we will suppress them from the notation. In this setting, we have a block of irreducible representations of $\mathbf{G}(\mathbb{R})$ at infinitesimal character γ, $\mathcal{B} = \mathcal{B}(\xi, \eta^\vee, \gamma) \subset \Pi(\mathfrak{g}, \mathbf{K}, \gamma)$. Corresponding to $\mathcal{B}$, we have the dual block $\mathcal{B}^\vee$ of irreducible $(\mathfrak{g}^\vee, \mathbf{K}^\vee)$-modules at infinitesimal character $\gamma^\vee$. Since $\gamma^\vee$ is integral, the full Weyl group $\mathbf{W}^\vee$ acts on $\mathcal{B}^\vee$. Using the coherent continuation action, $\mathcal{B}^\vee$ decomposes into into disjoint

HC-cells:

$$\mathcal{B}^\vee = \coprod_{HC-\text{cells}} \mathcal{C}^\vee. \tag{6.3}$$

Recall that $AV_{\mathbb{C}}(\mathcal{C}^\vee)$ makes sense, since the associated variety remains constant for a fixed cell $\mathcal{C}^\vee$.

Now fix an even nilpotent orbit $\mathcal{O}^\vee \subset \mathfrak{g}^\vee$ with Jacobson-Morozov triple $\{X^\vee, Y^\vee, H^\vee\}$ and fix $\lambda = \frac{1}{2}H^\vee$ and let $\mathfrak{l}^\vee = \text{Cent}_{\mathfrak{g}^\vee}(\lambda)$. Let

$$\mathcal{P}^\vee(\mathfrak{l}^\vee) := \{\theta^\vee \text{ - stable parabolic subalgebras of } \mathfrak{g}^\vee \text{ with Levi-factor } \mathfrak{l}^\vee\} \subset \mathcal{P}^\vee. \tag{6.4}$$

Then, every $\mathfrak{p}^\vee \in \mathcal{P}^\vee(\mathfrak{l}^\vee)$ is conjugate to a parabolic subalgebra the form $\mathfrak{p}(\lambda')^\vee$ for some semisimple $\lambda' \in \mathfrak{l}^\vee$ and the $\theta^\vee$-stable condition comes down to checking that $\theta^\vee(\lambda') = \lambda'$.

Definition 6.1 (Parameter set for theta forms of $\mathcal{O}^\vee$)**.** *Suppose $\eta^\vee$ is a strong real form of $\boldsymbol{G}^\vee$ and $\theta^\vee = Int(\eta^\vee)$ is a corresponding Cartan involution of $\boldsymbol{G}^\vee$. Associated to the pair $(\mathcal{O}^\vee, \eta^\vee)$ is the set*

$$\mathcal{S}(\mathcal{O}^\vee, \eta^\vee) := \{\mathfrak{p}^\vee \in \boldsymbol{K}^\vee \backslash \mathcal{P}^\vee(\mathfrak{l}^\vee) \mid \theta^\vee(\mathfrak{p}^\vee) = \mathfrak{p}^\vee, \ \ \mathfrak{n}^\vee \cap \mathfrak{s}^\vee \cap \mathcal{O}^\vee \neq \varnothing\}, \tag{6.5}$$

where $\mathfrak{p}^\vee = \mathfrak{l}^\vee + \mathfrak{n}^\vee$ is Langlands decomposition of $\mathfrak{p}^\vee$ and $\mathfrak{g}^\vee = \mathfrak{k}^\vee \oplus \mathfrak{s}^\vee$ is the Cartan decomposition of $\mathfrak{g}^\vee$. In our setting, $\eta^\vee$ wil be fixed, so we drop it from the notation, i.e. the parameter set will be denoted as $\mathcal{S}(\mathcal{O}^\vee)$.

As noted in Corollary 6.1, we know that $\mathcal{S}(\mathcal{O}^\vee)$ parameterizes the theta forms of $\mathcal{O}^\vee$. Even though the conditions defining $\mathcal{S}(\mathcal{O}^\vee)$ are explicit, it is not clear how one could check the last condition - $\mathfrak{n}^\vee \cap \mathfrak{s}^\vee \cap \mathcal{O}^\vee \neq \varnothing$ - to explicitly compute $\mathcal{S}(\mathcal{O}^\vee)$. The first result of this paper addresses this problem:

Theorem 6.1. *Suppose $\eta^\vee$ a strong real form of $\boldsymbol{G}^\vee$. Choose ξ to be the dual quasisplit strong real form of $\boldsymbol{G}$ corresponding to $\eta^\vee$. Let $\mathcal{B}(\xi, \eta^\vee, \gamma)$ and $\mathcal{B}^\vee(\xi, \eta^\vee, \gamma^\vee)$ be blocks of representations at regular integral infinitesimal characters γ, $\gamma^\vee$ repectively. Let $\theta = Int(\xi)$ and $\theta^\vee = Int(\eta^\vee)$ be Cartan involutions of $\boldsymbol{G}$ and $\boldsymbol{G}^\vee$ corresponding to ξ and $\eta^\vee$.*

Let $\mathcal{O}^\vee$ be an even nilpotent orbit in the complexified Lie algebra $\mathfrak{g}^\vee$. Let $\{X^\vee, Y^\vee, H^\vee\}$ be the Jacobson-Morosov triple for $\mathcal{O}^\vee$ and let $\mathfrak{l}^\vee = Cent_{\mathfrak{g}^\vee}(H^\vee)$. Let $\mathfrak{g}^\vee = \mathfrak{k}^\vee \oplus \mathfrak{s}^\vee$ be the Cartan decomposition of $\mathfrak{g}^\vee$ with respect to $\theta^\vee$. Let $\mathcal{S}(\mathcal{O}^\vee)$ be set in Definition 6.1

Then,

$$\mathcal{S}(\mathcal{O}^\vee) \leftrightarrow \{\mathfrak{p}^\vee \in \boldsymbol{K}^\vee \backslash \mathcal{P}^\vee(\mathfrak{l}^\vee) \mid \ AV_{\mathbb{C}}(\mathcal{R}_{\mathfrak{p}^\vee}(\chi_{triv})) = \overline{\mathcal{O}^\vee}\}. \tag{6.6}$$

There is an algorithm to explicitly compute the latter set which is implementable in the Atlas of Lie Groups software.

Proof. The computation of the set $\mathcal{S}(\mathcal{O}^\vee)$ involves checking for two conditions:

1. We need to know how to check if a given parabolic is theta stable, which is elementary.

2. We need to find a method to check the condition $\mathfrak{n}^\vee \cap \mathfrak{s}^\vee \cap \mathcal{O}^\vee \neq \varnothing$, which is the difficult part. The main idea of this theorem is to replace this condition with something more amenable to computation, in this case to reduce it to computing the complex associated variety.

Let $\lambda = H^\vee$ be the semisimple element in the Jacobson-Morozov triple for $\mathcal{O}^\vee$. Recall that $\mathcal{P}^\vee(\mathfrak{l}^\vee)$ is the set of theta-stable parabolics having Levi factor $\mathfrak{l}^\vee = \mathrm{Cent}_{\mathfrak{g}^\vee}(\lambda)$. As a first step we find a description of $\mathbf{K}^\vee\backslash\mathcal{P}^\vee(\mathfrak{l}^\vee)$, the $\mathbf{K}^\vee$-conjugacy classes of parabolics in $\mathcal{P}^\vee(\mathfrak{l}^\vee)$.

Let $S(\lambda)$ be the set of simple roots of $\mathfrak{g}^\vee$ which are singular on λ. Then, a $\mathbf{K}^\vee$-conjugacy class of parabolic subgroups is determined by specifying a parabolic $\mathbf{Q}^\vee$ corresponding to the data $(y, S(\lambda))$, where y is a representative for a $\mathbf{K}^\vee$-orbit of $\mathbf{G}^\vee/\mathbf{B}^\vee$. The parabolic $\mathbf{Q}^\vee(y, S(\lambda))$ is $\theta^\vee$-stable if and only if $\theta^\vee_y(\lambda) = \lambda$, where $\theta^\vee_y$ is the Cartan involution on $\mathbf{G}^\vee$ corresponding to the `KGB`-element y.

We can use this description of parabolics to compute the set $\mathbf{K}^\vee\backslash\mathcal{P}^\vee(\mathfrak{l}^\vee)$ in Atlas.

Since by definition, $\mathcal{S}(\mathcal{O}^\vee) \subset \mathbf{K}^\vee\backslash\mathcal{P}^\vee(\mathfrak{l}^\vee)$, our goal will be to pare down $\mathcal{S}'(\mathcal{O}^\vee) := \mathbf{K}^\vee\backslash\mathcal{P}^\vee(\mathfrak{l}^\vee)$ to $\mathcal{S}(\mathcal{O}^\vee)$. To achieve this, we will use the second condition defining $\mathcal{S}(\mathcal{O}^\vee)$: $\mathfrak{n}^\vee \cap \mathfrak{s}^\vee \cap \mathcal{O}^\vee \neq \varnothing$, for a $\mathfrak{n}^\vee$ arising as the nilpotent part of the the Langlands decomposition of $\mathfrak{p}^\vee \in \mathcal{S}'(\mathcal{O}^\vee)$.

Define the map:

$$\Xi : \mathbf{K}^\vee \backslash \mathcal{P}^\vee(\mathfrak{l}^\vee) \longrightarrow \{\mathbf{K}^\vee \text{ - orbits on } \mathfrak{s}^\vee\}, \quad \mathfrak{p}^\vee \mapsto \mathbf{K}^\vee \cdot (\mathfrak{n}^\vee \cap \mathfrak{s}^\vee). \tag{6.7}$$

Using Theorem 5.4, we have $\mathfrak{n}^\vee \cap \mathfrak{s}^\vee$ is open and dense in a single $\mathbf{K}^\vee$-orbit of $\mathfrak{s}^\vee$, as a result, the map Ξ is well defined.

We note the following consequences of Theorem 6.1:

1. The image of Ξ contains all the $\mathbf{K}^\vee$ - orbits on $\mathfrak{s}^\vee \cap \mathcal{O}^\vee$.
2. The restriction of Ξ to $\mathcal{S}(\mathcal{O}^\vee)$ is a bijective correspondence between $\mathcal{S}(\mathcal{O}^\vee)$ and $\mathbf{K}^\vee$-orbits on $\mathfrak{s}^\vee \cap \mathcal{O}^\vee$.

Therefore, to compute $\mathcal{S}(\mathcal{O}^\vee)$, it comes down to checking if $\Xi(\mathfrak{p}^\vee) \subset \mathfrak{s}^\vee \cap \mathcal{O}^\vee$.

Proposition 6.1. *Suppose we are in the above setting and let $\mathfrak{p}^\vee \in \boldsymbol{K}^\vee \backslash \mathcal{P}^\vee(\mathfrak{l}^\vee)$. Then $\Xi(\mathfrak{p}^\vee) \subset \mathfrak{s}^\vee \cap \mathcal{O}^\vee$ (that is $\mathfrak{p}^\vee \in \mathcal{S}(\mathcal{O}^\vee)$) if and only if $AV_{\mathbb{C}}(\mathcal{R}_{\mathfrak{p}^\vee}(\chi_{triv})) = \overline{\mathcal{O}^\vee}$, where χ_{triv} is the trivial character on $\mathfrak{p}^\vee$.*

Proof. Suppose $\Xi(\mathfrak{p}^\vee)$ is a $\mathbf{K}^\vee$-orbit on $\mathfrak{s}^\vee \cap \mathcal{O}^\vee$, then $\mathfrak{n}^\vee \cap \mathfrak{s}^\vee \cap \mathcal{O}^\vee \neq \varnothing$. Let $X^\vee \in \mathfrak{n}^\vee \cap \mathfrak{s}^\vee \cap \mathcal{O}^\vee$ be a generic element, then using Theorem 5.4, we see that

$$AV_\theta(\mathcal{R}_{\mathfrak{p}^\vee}(\chi_{\text{triv}})) = \overline{\mathbf{K}^\vee \cdot (\mathfrak{n}^\vee \cap \mathfrak{s}^\vee)} = \overline{\mathbf{K}^\vee \cdot X^\vee},$$

therefore $\Xi(\mathfrak{p}^\vee)$ is by definition $AV_\theta(\mathcal{R}_{\mathfrak{p}^\vee}(\chi_{\text{triv}}))$.

Furthermore, using the relationship between $AV_{\mathbb{C}}$ and AV_θ for a fixed module, we have

$$
\begin{aligned}
AV_{\mathbb{C}}(\mathcal{R}_{\mathfrak{p}^\vee}(\chi_{\text{triv}})) &= \overline{\mathbf{G}^\vee \cdot (\mathbf{K}^\vee \cdot (\mathfrak{n}^\vee \cap \mathfrak{s}^\vee))} \\
&= \overline{\mathbf{G}^\vee \cdot (\mathbf{K}^\vee \cdot X^\vee)} \\
&= \overline{\mathbf{G}^\vee \cdot X^\vee} \\
&= \overline{\mathcal{O}^\vee}
\end{aligned}
$$

This implies that $AV_{\mathbb{C}}(\mathcal{R}_{\mathfrak{p}^\vee}(\chi_{\text{triv}})) = \overline{\mathcal{O}^\vee}$ if $\mathfrak{n}^\vee \cap \mathfrak{s}^\vee \cap \mathcal{O}^\vee \neq \emptyset$.

Now, if $\Xi(\mathfrak{p}^\vee) \not\subset \mathfrak{s}^\vee \cap \mathcal{O}^\vee$, then by Corollary 6.1, $\Xi(\mathfrak{p}^\vee) = AV_\theta(\mathcal{R}_{\mathfrak{p}^\vee}(\chi_{\text{triv}}))$ cannot be a theta-form of the complex nilpotent orbit $\mathcal{O}^\vee$, so that $AV_{\mathbb{C}}(\mathcal{R}_{\mathfrak{p}^\vee}(\chi_{\text{triv}})) = \mathbf{G}^\vee \cdot AV_\theta(\mathcal{R}_{\mathfrak{p}^\vee}(\chi_{\text{triv}})) \neq \overline{\mathcal{O}^\vee}$. This completes the proof of the proposition. $\square$

Given the fact that we can compute $\mathbf{K}^\vee \backslash \mathcal{P}^\vee(\mathfrak{l}^\vee)$ using Atlas, Proposition 6.1 reduces the computation of $\mathcal{S}(\mathcal{O}^\vee)$ to the computation of complex associated varieties of all representations in the given block $\mathcal{B}^\vee$. It turns out that there are algorithms to take care of this latter step, it is dealt in two cases:

1. When $\mathbf{G}^\vee$ is of classical type.

2. When $\mathbf{G}^\vee$ is of exceptional type.

For Case 1, an algorithm by Noel-Jackson computes the special $\mathbf{W}^\vee$-representation attached to a representation $\pi^\vee$ of $\mathbf{G}(\mathbb{R})^\vee$ by computing the special $\mathbf{W}^\vee$-representation $\sigma(\mathcal{C}^\vee)$ attached to the HC-cell $\mathcal{C}^\vee$ containing $\pi^\vee$, using the tau-invariants of the representations in $\mathcal{C}^\vee$.

We can then apply the Springer correspondence to compute the special nilpotent orbit attached to $\sigma(\mathcal{C}^\vee)$, the closure of this special nilpotent orbit is $AV_{\mathbb{C}}(\pi^\vee))$. Both, the Noel-Jackson algorithm and the Springer correspondences can be implemented as functions in the Atlas software, so that given the block $\mathcal{B}^\vee$, one gets an output specifying $AV_{\mathbb{C}}(\mathcal{C}^\vee)$ for every HC-cell in $\mathcal{B}^\vee$.

For Case 2, we use tables computed by Binegar to find out what the $AV_{\mathbb{C}}(\mathcal{C}^\vee)$, for a given HC-cell in $\mathcal{B}^\vee$. There is an algorithm due to Vogan that would compute the special Weyl group representation of an irreducible representation of a group of exceptional type, work is in progress to write it down in a way that could be implemented in the Atlas software.

This completes the proof of the theorem. For the reader's convenience, we summarize the algorithm to compute $\mathcal{S}(\mathcal{O}^\vee)$:

1. Given $\mathcal{O}^\vee$, compute the neutral element $H^\vee$ in the Jacobson-Morozov triple for $\mathcal{O}^\vee$ and let $\lambda = H^\vee$. Let $\mathfrak{l}^\vee = \mathrm{Cent}_{\mathfrak{g}^\vee}(\lambda)$.

2. Compute the set $\mathcal{S}'(\mathcal{O}^\vee) := \mathbf{K}^\vee \backslash \mathcal{P}^\vee(\mathfrak{l}^\vee)$, which is possible in Atlas.

3. Using the Noel-Jackson algorithm or the tables by Binegar, compute the $AV_{\mathbb{C}}(\pi^\vee)$ for every $\pi^\vee \in \mathcal{B}^\vee$.

4. To pare down $\mathcal{S}'(\mathcal{O}^\vee)$ to $\mathcal{S}(\mathcal{O}^\vee)$, for every $\mathfrak{p}^\vee \in \mathcal{S}'(\mathcal{O}^\vee)$ compute $AV_{\mathbb{C}}(\mathcal{R}_{\mathfrak{p}^\vee}(\chi_{\mathrm{triv}}))$ using previous step. If $AV_{\mathbb{C}}(\mathcal{R}_{\mathfrak{p}^\vee}(\chi_{\mathrm{triv}})) = \overline{\mathcal{O}^\vee}$ put $\mathfrak{p}^\vee$ into $\mathcal{S}(\mathcal{O}^\vee)$, or else discard it from the list.

5. Since $\mathcal{S}'(\mathcal{O}^\vee)$ was a finite set, this algorithm will terminate in finite number of steps and at the end we will be left with exactly $\mathcal{S}(\mathcal{O}^\vee)$.

$\square$

6.3 Computing Theta Associated Variety

Continuing in the setting of the last section, now we describe an algorithm to compute the theta-real-associated variety of an irreducible $(\mathfrak{g}^\vee, \mathbf{K}^\vee)$-module, $\pi^\vee \in \mathcal{B}^\vee$, at regular integral infinitesimal character $\gamma^\vee$. Let $AV_{\mathbb{C}}(\pi^\vee) = \overline{\mathcal{O}^\vee}$. We specify certain "good" conditions on $\pi^\vee$, which when satisfied, $AV_\theta(\pi^\vee)$ can be explicitly computed.

To begin, fix the nilpotent orbit $\mathcal{O}^\vee$ in $\mathfrak{g}^\vee$ and let $\lambda(\mathcal{O}^\vee) = \frac{1}{2}H^\vee$ infinitesimal character coming from the semisimple element in the Jacobson-Morozov triple corresponding to $\mathcal{O}^\vee$. Let $\mathfrak{p}^\vee(\lambda) = \mathfrak{l}^\vee + \mathfrak{n}(\lambda)^\vee$ be the parabolic subalgebra corresponding to λ. If we assume that $\mathcal{O}^\vee$ is even, the set $\mathcal{S}(\mathcal{O}^\vee)$ corresponding to the real forms of $\mathcal{O}^\vee$ in terms of representatives of $\mathbf{K}^\vee$-conjugacy classes of $\theta^\vee$-stable parabolics, is computable and the algorithm is described in Section 6.4. The first "good" condition to compute $AV_\theta(\pi^\vee)$ is as follows:

Condition 1

$AV_{\mathbb{C}}(\pi^\vee)$ is the closure of an even nilpotent orbit $\mathcal{O}^\vee$.

Fix a parabolic $\mathfrak{q}^\vee \in \mathcal{S}(\mathcal{O}^\vee)$ and let $\mathfrak{q}^\vee = \mathfrak{l}^\vee + \mathfrak{n}^\vee$ be its Langlands decomposition.

Let $\mathbf{Q}^\vee = \mathbf{L}^\vee\mathbf{N}^\vee$ be the corresponding parabolic of $\mathbf{G}^\vee$, with $\mathbf{L}^\vee$ the complexification of the real Levi $\mathbf{L}(\mathbb{R})^\vee$.

Let λ be a one dimensional $(\mathfrak{l}^\vee, \mathbf{L}^\vee \cap \mathbf{K}^\vee)$-module, we then cohomologically induce it up to a $(\mathfrak{g}^\vee, \mathbf{K}^\vee)$-module, denoting this final representation as $\mathcal{R}_{\mathfrak{q}^\vee}(\lambda)$.

Proposition 6.2. *Suppose $\mathfrak{q}^\vee$ is as above and χ_1 and χ_2 are two one dimensional representations of $\boldsymbol{L}^\vee \cap \boldsymbol{K}^\vee$, such that $\chi_1|_{(\boldsymbol{L}^\vee\cap\boldsymbol{K}^\vee)_0} = \chi_2|_{(\boldsymbol{L}^\vee\cap\boldsymbol{K}^\vee)_0}$, then*

$$AV_\theta(\mathcal{R}_{\mathfrak{q}^\vee}(\chi_1)) = AV_\theta(\mathcal{R}_{\mathfrak{q}^\vee}(\chi_2)). \tag{6.8}$$

Proof. Since $\chi_1|_{(\mathbf{L}^\vee\cap\mathbf{K}^\vee)_0} = \chi_2|_{(\mathbf{L}^\vee\cap\mathbf{K}^\vee)_0}$, we have an equality of derivatives $d\chi_1 = d\chi_2$, let's call this λ_0. Using Theorem 5.4 and the discussion preceding it, we know that the theta-associated variety $AV_\theta(\mathcal{R}_{\mathfrak{q}^\vee}(\chi_1))$ and $AV_\theta(\mathcal{R}_{\mathfrak{q}^\vee}(\chi_2))$ depend only on $\mathfrak{n}(\lambda_0)$, and hence the equality of the two theta-associated varieties follows. $\square$

We know how to compute all the real forms of $\mathcal{O}^\vee$ in terms of $\theta^\vee$-stable parabolics, using Theorem 6.1 denoted as $\mathcal{S}(\mathcal{O}^\vee)$. Suppose $\mathcal{S}(\mathcal{O}^\vee) = \{\mathfrak{p}_1^\vee, \mathfrak{p}_2^\vee, \ldots, \mathfrak{p}_r^\vee\}$ and let $\{\mathbf{L}_1^\vee, \mathbf{L}_2^\vee, \ldots, \mathbf{L}_r^\vee\}$ be the corresponding the Levi subgroups. Choose one dimensional $(\mathfrak{l}_i^\vee, \mathbf{L}_i^\vee \cap \mathbf{K}^\vee)$-modules χ_{ij} for $j = 0, 1, 2, \cdots, s-1$ where $s = |\mathbf{L}^\vee/\mathbf{L}_0^\vee|$ such that the infinitesimal character of $\chi_{ij} = \rho(\mathbf{L}_i) = \gamma_L$ (in fact, any regular integral infinitesimal character for $\mathbf{L}_i$ would work, we make this choice so that our induced modules have infinitesimal character $\rho(\mathbf{G})$). For a fixed i and for all j, $AV_\theta(\mathcal{R}_{\mathfrak{p}_i^\vee}(\chi_{ij}))$ corresponds to the $\mathbf{K}^\vee$-orbit corresponding to the same $\mathfrak{p}_i$. That is, the real associated variety remains the same as we vary j but keep i fixed.

Recall that $\mathcal{B}^\vee$ is partitioned into HC-cells, $\mathcal{C}^\vee$, given by

$$\mathcal{B}^\vee = \coprod \mathcal{C}^\vee.$$

Definition 6.2 ("Good" Cells)**.** *In the above setting we say that a HC-cell $\mathcal{C}^\vee$ is good if it contains a representation of the form $\mathcal{R}_{\mathfrak{p}_i^\vee}(\chi_{ij})$ for some choice of i and j.*

Let $\mathcal{C}_{ij}^\vee$ be the set of good cells corresponding to the representations $\mathcal{R}_{\mathfrak{p}_i^\vee}(\chi_{ij})$ for $i \in \{1, 2, \ldots, r\}$ and $j \in \{1, 2, \ldots, s\}$. Note that $AV_\theta(\mathcal{C}_{i,j}^\vee)$ is the closure of a single $\mathbf{K}^\vee$-orbit that corresponds to the $\theta^\vee$-stable parabolic $\mathfrak{p}_i^\vee \in \mathcal{S}(\mathcal{O}^\vee)$.

We can now state the second good condition:

Condition 2

$\pi^\vee$ lies in a good cell.

Definition 6.3 (Good Condition)**.** *Suppose $\pi^\vee \in \mathcal{B}^\vee$. We say that $\pi^\vee$ satisfies the good condition if Conditions 1 and 2 (above) are both satisfied.*

We are led to the following theorem:

Theorem 6.2. *Let ξ, $\eta^\vee$ be strong real forms of $\boldsymbol{G}$, $\boldsymbol{G}^\vee$ in the Atlas Setting, Definition 4.3. Let $\mathcal{B}$ and $\mathcal{B}^\vee$ be blocks of representations at regular integral infinitesimal* *characters γ, $\gamma^\vee$ respectively. Let $\mathcal{O}^\vee$ be a fixed even nilpotent orbit and let $\pi^\vee \in \mathcal{B}^\vee$. Then,*

1. *$AV_{\mathbb{C}}(\pi^\vee)$ can be explicitly computed for all $\pi^\vee \in \mathcal{B}^\vee$.*

2. *if $\pi^\vee$ satisfies the "good condition" (that is when Condition 1 and 2 are both satisfied), then $AV_\theta(\pi^\vee)$ can be computed as the closure of a single $\boldsymbol{K}^\vee$-orbit in $\mathfrak{s}^\vee \cap \mathcal{O}^\vee$.*

Proof. We describe the two algorithms mentioned in the theorem. The algorithm to compute the complex associated variety is as follows:

1. Suppose $\pi^\vee \in \mathcal{B}^\vee$. By the decomposition of $\mathcal{B}^\vee$ into HC-Cells, there must be a cell $\mathcal{C}^\vee$, such that $\pi^\vee \in \mathcal{C}^\vee$.

2. When $\mathbf{G}$ is of classical type, we use the Noel-Jackson algorithm to compute the special Weyl group representation $\sigma(\mathcal{C}^\vee)$ attached to $\mathcal{C}^\vee$, this algorithm has been implemented in Atlas. If $\mathbf{G}$ is of Exceptional type, there are tables for the special Weyl groups representations attached to cells, by Binegar for example.

3. We apply the Springer correspondence (again implemented in Atlas) to $\sigma(\mathcal{C}^\vee)$ to get the special nilpotent orbit attached to $\mathcal{C}^\vee$, by construction, this is exactly $AV_{\mathbb{C}}(\mathcal{C}^\vee)$.

4. Since the associated variety remains constant on the cell, we have hence computed $AV_{\mathbb{C}}(\pi^\vee)$.

When $\mathbf{G}$ is of exceptional type, these computations have already been tabulated in literature. We mention that case here only for the sake of completeness.

1. We compute $AV_{\mathbb{C}}(\pi^\vee)$ using the previous algorithm, this will be a closure of a single nilpotent orbit. It is possible to check if this complex nilpotent orbit is even, if it is even, we have Condition 1 satisfied and denote this nilpotent orbit as $\mathcal{O}^\vee$.

2. Since $\mathcal{O}^\vee$ is even, we can compute $\mathcal{S}(\mathcal{O}^\vee)$ corresponding to the block $\mathcal{B}^\vee$ as in Theorem 6.1.

3. Now, suppose $\pi^\vee$ is in a good cell, say $\mathcal{C}^\vee$. By definition of a good cell, $\mathcal{C}^\vee$ must contain a $\mathcal{R}_{\mathfrak{p}_i^\vee}(\chi_{ij})$ for a choice of i and j. The theta-associated variety of $\mathcal{R}_{\mathfrak{p}_i^\vee}(\chi_{ij})$ and hence of $\mathcal{C}^\vee$ is the closure of a single theta-form parameterized by $\mathfrak{p}_i \in \mathcal{S}(\mathcal{O}^\vee)$.

Therefore if the good condition is satisfied, we can compute the theta-associated variety of $\pi^\vee$ as the closure of the theta-form corresponding to the parabolic $\mathfrak{p}_i \in \mathcal{S}(\mathcal{O}^\vee)$. This algorithm has been implemented in the Atlas software, so that if you input a representation into the software, we can check if the good condition holds, and if it does, we output the theta-associated variety in terms a parabolic corresponding to a theta-form of $\mathcal{O}^\vee$. $\square$

Chapter 7: Special Unipotent Packets for Real Reductive Groups

We now return to the main goal of this paper, to compute unipotent Arthur packets in the "good" case, and when things are not "good", to provide a list of representations that can be used to complete these packets. We provide an algorithm that explicitly computes Atlas/Langlands parameters of representations in these packets. We have implemented this algorithm into the Atlas of Lie groups software.

7.1 Special Unipotent Parameters and Packets

We will be in the Atlas Setting of Definition 4.3. That is:

Let $(\mathbf{G}, \gamma)$ be a basic data and $(\mathbf{G}^\vee, \gamma^\vee)$ be the corresponding dual basic data. Let $(\mathbf{B}, \mathbf{H}, \{X_\alpha\})$ be a fixed pinning for $\mathbf{G}$. Let ξ be a strong real form of $\mathbf{G}$ in the inner class of γ and let $\eta^\vee$ be a strong real form for $\mathbf{G}^\vee$ in the dual inner class given by $\gamma^\vee$. Corresponding to ξ and $\eta^\vee$, let $\theta_\xi = \mathrm{Int}(\xi)$, and $\theta_{\eta^\vee} = \mathrm{Int}(\eta^\vee)$ be Cartan involutions of $\mathbf{G}$ and $\mathbf{G}^\vee$ respectively with maximal complex subgroups $\mathbf{K}_\xi$ and $\mathbf{K}^\vee_{\eta^\vee}$ respectively.

Let δ be a regular integral infinitesimal character for $\mathbf{G}$ and let $\mathcal{B}(\xi, \eta^\vee, \delta)$ be the block of irreducible $(\mathfrak{g}, \mathbf{K}_\xi)$-modules at infinitesimal character δ specified by the pair of strong real forms $(\xi, \eta^\vee)$. Also, $\mathcal{B}^\vee = \mathcal{B}(\eta^\vee, \xi, \delta^\vee)$ is the corresponding dual block of irreducible $(\mathfrak{g}^\vee, \mathbf{K}^\vee_{\eta^\vee})$-modules.

Definition 7.1 (Block at Singular infinitesimal character)**.** *Suppose $\lambda \in \delta + X^*(\boldsymbol{H})$, then by a block at (possibly singular infinitesimal character) λ we will mean the translation of the block at regular integral infinitesimal character at δ to the infinitesimal character λ, that is*

$$\mathcal{B}(\lambda) = \mathcal{B}(\xi, \eta^\vee, \lambda) := T_\delta^\lambda(\mathcal{B}(\xi, \eta^\vee, \delta)). \tag{7.1}$$

The $\mathcal{B}(\lambda)$ does not depend on the choice of a regular integral $\delta \in X^*(\mathbf{H})$.

Fix a unipotent Arthur parameter (Definition: 3.3)ψ, and let ϕ_ψ be the corresponding Langlands parameter with data (y, λ). We recall that the pair (y, λ) satisfies:

1. Let ψ_0 be the tempered Langlands parameter corresponding to the restriction of ψ to $W_\mathbb{R}$. Let (y_0, λ_0) be the data corresponding to the parameter ψ_0.

2. Let ψ_1 be the restriction of ψ to $SL(2, \mathbb{C})$. Define:

$$y_1 = \psi \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \lambda_1 = d\psi_1 \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}.$$

   Then the Langlands parameter ϕ_ψ corresponding to ψ is given by (y, λ) where, $y = y_0 y_1$ and $\lambda = \lambda_0 + \lambda_1$.

Recall that y must satisfy: $y^2 = \exp(2\pi i\lambda)$. We can attach the nilpotent element $E_\psi := d\psi_1 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ to ψ . The element $E_\psi \in \mathfrak{n}(\lambda)^\vee \cap \mathfrak{s}^\vee$. If $\mathcal{O}^\vee = \mathbf{G}^\vee \cdot E_\psi$, then $\mathcal{O}^\vee$ is a even nilpotent orbit for $\mathbf{G}^\vee$ if and only if λ is integral. Let $\{X^\vee, H^\vee, Y^\vee\}$ be the Jacobson-Morozov triple corresponding to $\mathcal{O}^\vee$, and now define

$$\lambda(\mathcal{O}^\vee) := \frac{1}{2}H^\vee. \tag{7.2}$$

Definition 7.2 (Weak Unipotent Arthur Packet). *Let $\mathcal{O}^\vee$ be a dual even complex nilpotent orbit. Choose δ such that $\lambda(\mathcal{O}^\vee) \in \delta + X^*(\boldsymbol{H})$. The weak unipotent packet corresponding to the triple $(\xi, \eta^\vee, \mathcal{O}^\vee)$ is the set*

$$\Pi^u_{weak}(\xi, \eta^\vee, \mathcal{O}^\vee) := \{\pi \in \mathcal{B}(\lambda(\mathcal{O}^\vee)) := T_\delta^{\lambda(\mathcal{O}^\vee)}(\mathcal{B}(\delta)) \mid AV_{\mathbb{C}}(\pi^\vee) = \overline{\mathcal{O}^\vee}\}. \tag{7.3}$$

An easy consequence of the definition of weak unipotent packets is the fact that two weak unipotent Arthur packets are either equal or disjoint.

We can construct the parabolic subalgebra $\mathfrak{p}^\vee = \mathfrak{l}(\lambda)^\vee + \mathfrak{n}(\lambda)^\vee$ and define $\mathcal{P}^\vee$ to be the conjugacy class of parabolic subalgebras conjugate to $\mathfrak{p}^\vee$. In this setting $E_\psi \in \mathcal{Z}_{\mathcal{P}^\vee}$, the Richardson orbit corresponding to $\mathcal{P}^\vee$. Since $\mathcal{O}^\vee$ is even, $\mathcal{Z}_{\mathcal{P}^\vee} = \mathcal{O}^\vee$. This implies that $E_\psi \in \mathfrak{n}(\lambda)^\vee \cap \mathfrak{s}^\vee \cap \mathcal{O}^\vee$, as a result, we can find a $\mathbf{K}^\vee$ orbit on $\mathfrak{s}^\vee$ (this is exactly $\mathbf{K}^\vee \cdot E_\psi$) corresponding to the Arthur parameter ψ, call this orbit $\mathcal{O}^\vee_{\mathbf{K}^\vee}$. The map

$$\psi \mapsto \mathbf{K}^\vee \cdot E_\psi, \tag{7.4}$$

defines a bijection between unipotent Arthur packets corresponding to unipotent Arthur parameters supported on $X_j(\mathcal{O}^\vee_\lambda, {}^\Gamma\mathbf{G}^\vee)$ and the theta real forms of $\mathcal{O}^\vee$ in the block $\mathcal{B}^\vee(\lambda(\mathcal{O}^\vee))$. Note that $\mathbf{G}^\vee \cdot E_\psi = \mathcal{O}^\vee$. We now define a special unipotent Arthur packet.

Definition 7.3 (Special Unipotent Arthur Packet)**.** *The special unipotent Arthur packet corresponding to the tuple $(\xi, \eta^\vee, \mathcal{O}^\vee_{\boldsymbol{K}^\vee})$ is the set*

$$\Pi^u(\xi, \eta^\vee, \mathcal{O}^\vee_{\boldsymbol{K}^\vee}) := \{\pi \in \Pi^u_{weak}(\xi, \eta^\vee, \mathcal{O}^\vee) \mid \overline{\mathcal{O}^\vee_{\boldsymbol{K}^\vee}} \subset AV_\theta(\pi^\vee)\}. \tag{7.5}$$

The theta associated variety of an irreducible representation need not necessarily be the closure of a single orbit, as a result, we can only hope for an inclusion of $\mathcal{O}^\vee_{\mathbf{K}^\vee}$ inside $AV_\theta(\pi^\vee)$ as a result two distinct unipotent Arthur packets need not necessarily be disjoint.

7.2 Computing Special Unipotent Packets

Continuing with the definitions of unipotent packets, we now proceed to compute them. Even though the packets are explicitly defined, the computation of its actual contents is difficult. The difficulty lies in the computation of the invariants $AV_{\mathbb{C}}(\pi)$ and $AV_\theta(\pi)$.

There is no algorithm that computes the contents of a general unipotent packets. The results relating to the computation of the invariants $AV_{\mathbb{C}}(\pi)$ and $AV_\theta(\pi)$ (Theorem 6.2) in the earlier sections provide us with the tools to study these packets and to compute some of these packets in special cases.

Since we can compute $AV_{\mathbb{C}}(\pi)$ for any π, we will first show how to completely compute weak unipotent Arthur packets.

The computation of $AV_{\theta}(\pi)$ depends on a couple assumptions. Even under these assumptions, we cannot always compute the special unipotent Arthur packets completely, however, we can always identify a non-empty set of representations inside the packet and also provide a list of representations that could possibly complete this set to the full packet.

Fix a dual even complex nilpotent orbit $\mathcal{O}^\vee$. Let $\lambda := \lambda(\mathcal{O}^\vee) = \frac{1}{2}H^\vee$, where $H^\vee$ is the semisimple element in the Jacobson-Morozov triple. We want to compute

$$\Pi^u_{\mathrm{weak}}(\xi, \eta^\vee, \mathcal{O}^\vee) := \{\pi \in \mathcal{B}(\lambda) \mid AV_{\mathbb{C}}(\pi^\vee) = \overline{\mathcal{O}^\vee}\}. \tag{7.6}$$

Fix a regular and integral $\delta \in X^*(\mathbf{H})$, such that $\lambda \in \delta + X^*(\mathbf{H})$. Let $\mathcal{B}(\xi, \eta^\vee, \delta)$ be a block of irreducible $(\mathfrak{g}, \mathbf{K})$-modules at regular integral infinitesimal character δ, note that $\eta^\vee$ satisfies $(\eta^\vee)^2 = e^{2\pi i\lambda}$. Let $\mathcal{B}^\vee$ be the dual block, a block of irreducible $(\mathfrak{g}^\vee, \mathbf{K}^\vee)$-modules at infnitesimal character $\delta^\vee$. Using Vogan duality, we note that there is a bijection between $\mathcal{B}(\gamma)$ and $\mathcal{B}^\vee(\gamma^\vee)$. If $\pi \in \mathcal{B}(\gamma)$, its Vogan dual will be denoted as $\pi^\vee \in \mathcal{B}^\vee(\gamma^\vee)$.

This leads us to the first main result of this section:

Theorem 7.1. *Let $\mathcal{O}^\vee$ be an even nilpotent orbit in $\mathfrak{g}^\vee$. Suppose we are in the setting described above, then $\Pi^u_{weak}(\xi, \eta^\vee, \mathcal{O}^\vee)$ can be completely and explicitly computed.*

Proof. We will prove this result in a series of steps as follows:

1. Recall that the dual block is a disjoint union of HC-cells

$$\mathcal{B}^\vee(\delta^\vee) = \coprod \mathcal{C}^\vee.$$

2. Given a $\mathbf{W}^\vee$-cell $\mathcal{C}^\vee$ and for any $\pi^\vee \in \mathcal{C}^\vee$, by Theorem 6.2 part (a), we know how to compute $AV_{\mathbb{C}}(\pi^\vee)$. Since the associated variety remains constant on $\mathcal{C}^\vee$, this lets us compute $AV_{\mathbb{C}}(\mathcal{C}^\vee)$.

3. Let $\mathbf{C}^\vee(\mathcal{O}^\vee)$ be the set of all cells $\mathcal{C}^\vee$ satisfying $AV_{\mathbb{C}}(\mathcal{C}^\vee) = \overline{\mathcal{O}^\vee}$. For every cell $\mathcal{C}^\vee$, we use Vogan-duality to compute the dual cell $\mathcal{C}$ and put this cell into the set $\mathbf{C}(\mathcal{O}^\vee)$, so that $\mathcal{C}(\mathcal{O}^\vee)$ is set of HC-cells for $\mathcal{B}(\delta)$ such that the dual cell $\mathcal{C}^\vee$ has complex associated variety $\overline{\mathcal{O}^\vee}$.

4. The representations in the cells $\mathcal{C} \in \mathbf{C}(\mathcal{O}^\vee)$ all have dual complex associated variety $\overline{\mathcal{O}^\vee}$, and, have infinitesimal character δ. To get representations at infinitesimal character $\lambda := \lambda(\mathcal{O}^\vee)$, we apply the translation functor T_δ^λ. Since we chose δ such that $\lambda \in \delta + X^*(\mathbf{H})$, the application of the translation functor is valid.

5. Therefore, the computation of the weak unipotent Arthur packets is the set:

$$\Pi^u_{\text{weak}}(\xi, \eta^\vee, \mathcal{O}^\vee) = \coprod_{\mathcal{C} \in \mathbf{C}(\mathcal{O}^\vee)} T_\delta^\lambda(\mathcal{C}). \tag{7.7}$$

To implement this algorithm in the Atlas software, in addition to using in built functions (for induction and translation functors), we have:

1. implemented the algorithm to compute $H^\vee$, the semisimple element in the Jacobson-Morozov triple corresponding to $\mathcal{O}^\vee$.

2. implemented the Noel-Jackson algorithm to compute the special Weyl group representation when $\mathbf{G}$ is of classical type. In the case when $\mathbf{G}$ is of exceptional type we hard code the special nilpotent orbit attached to a cell.

3. implemented the Springer correspondence to compute the special nilpotent orbit given the special $\mathbf{W}$-representation, when $\mathbf{G}$ is of classical type.

4. implemented Vogan-duality to compute a dual cell.

5. each of these functions have combined so that if one inputs the pair $(\mathcal{B}, \mathcal{O}^\vee)$ we output the set $\Pi^u_{\text{weak}}(\xi, \eta^\vee, \mathcal{O}^\vee)$ in terms of explicit Langlands parameters.

$\square$

We move to computing special unipotent Arthur packets. Let $\mathcal{O}^\vee$ be a fixed dual even complex nilpotent orbit for $\mathbf{G}$. Let $\lambda := \lambda(\mathcal{O}^\vee)$ be the infinitesimal character attached to $\mathcal{O}^\vee$. Let ξ be a strong real form of $\mathbf{G}$. Choose $\delta \in X^*(\mathbf{H})$ so that $\lambda \in \delta + X^*(\mathbf{H})$. Let $\eta^\vee$ be a strong real form for $\mathbf{G}^\vee$ such that $(\eta^\vee)^2 = e^{2\pi i\lambda}$ and $\eta^\vee$ is in the dual quasisplit inner class for $\mathbf{G}$. Let $\mathcal{B}(\xi, \eta^\vee, \delta)$ be a block for the strong real form ξ of $\mathbf{G}$ at infinitesimal character δ. Let $\mathcal{B}^\vee(\delta^\vee)$ be the corresponding dual block.

Given the complex nilpotent orbit $\mathcal{O}^\vee$, we have a set of $\mathbf{K}^\vee$-conjugacy classes of $\theta^\vee$-stable ($\theta^\vee = \mathrm{Int}(\eta^\vee)$) parabolic subgalgebras in $\mathfrak{g}^\vee$ parameterizing the theta forms of $\mathcal{O}^\vee$ in the block $\mathcal{B}^\vee(\delta^\vee)$, denoted as $\mathcal{S}(\mathcal{O}^\vee)$ and computed in Theorem 6.1. Let $\mathcal{S}(\mathcal{O}^\vee) = \{\mathfrak{p}_1^\vee, \mathfrak{p}_2^\vee, \ldots, \mathfrak{p}_r^\vee\}$ and suppose for $i = 1, 2, \ldots, r$, let $\mathfrak{p}_i^\vee = \mathfrak{l}_i^\vee + \mathfrak{n}_i^\vee$ be the Langlands decomposition. Let $\{\mathcal{O}^\vee_{\mathbf{K}^\vee,1}, \mathcal{O}^\vee_{\mathbf{K}^\vee,2}, \ldots, \mathcal{O}^\vee_{\mathbf{K}^\vee,r}\}$ be the set of theta real forms of $\mathcal{O}^\vee$ corresponding to the ordered set $\mathcal{S}(\mathcal{O}^\vee)$.

For a fixed i, let s_i be the number of connected components of the real Levi subgroup, $\mathbf{L}_i^\vee$, corresponding to $\mathfrak{l}_i$. For $j = 1, 2, \ldots, s_i$, let χ_{ij} be a character on $\mathbf{L}_i^\vee$ such that $d\chi_{ij} = \delta^\vee_{\mathbf{L}^\vee}$ for all $j = 1, 2, \ldots, s_i$.
Corresponding to each χ_{ij}, let $\mathcal{C}^\vee_{i,j}$ be the HC-cell in $\mathcal{B}^\vee(\delta^\vee)$ containing $\mathcal{R}_{\mathfrak{p}_i^\vee}(\chi_{ij})$. Following the proof of Theorem 6.2, recall that for a fixed i, $AV_\theta(\mathcal{C}^\vee_{ij}) = \overline{\mathcal{O}^\vee_{\mathbf{K}^\vee,i}}$. We define

$$\mathbf{C}^\vee(\mathcal{O}^\vee_{\mathbf{K}^\vee,i}) := \{\mathcal{C}^\vee_{ij} \mid j = 1, 2, \ldots s_i\} \subset \mathbf{C}^\vee(\mathcal{O}^\vee). \tag{7.8}$$

Note that every cell $\mathcal{C}^\vee \in \mathbf{C}^\vee(\mathcal{O}^\vee_{\mathbf{K}^\vee,i})$ satisfies $AV_\theta(\mathcal{C}^\vee) = \overline{\mathcal{O}^\vee_{\mathbf{K}^\vee,i}}$, and let

$$\mathbf{C}(\mathcal{O}^\vee_{\mathbf{K}^\vee,i}) := \{\text{the dual cell of } \mathcal{C}^\vee, \text{ for every } \mathcal{C}^\vee \in \mathbf{C}^\vee(\mathcal{O}^\vee_{\mathbf{K}^\vee,i})\} \subset \mathbf{C}(\mathcal{O}^\vee). \tag{7.9}$$

Let

$$\mathbf{C}^\vee(\mathcal{O}^\vee_{\mathbf{K}^\vee}) := \bigcup_i \mathbf{C}^\vee(\mathcal{O}^\vee_{\mathbf{K}^\vee,i}), \tag{7.10}$$

and let

$$\mathbf{C}(\mathcal{O}^\vee_{\mathbf{K}^\vee}) := \bigcup_i \mathbf{C}(\mathcal{O}^\vee_{\mathbf{K}^\vee,i}). \tag{7.11}$$

Now, let

$$\Pi^u_{\text{icp}}(\xi, \eta^\vee, \mathcal{O}^\vee_{\mathbf{K}^\vee,i}) := \coprod_{\mathcal{C} \in \mathbf{C}(\mathcal{O}^\vee_{\mathbf{K}^\vee,i})} T^\lambda_\delta(\mathcal{C}). \tag{7.12}$$

Note that $\Pi^u_{\text{icp}}(\xi, \eta^\vee, \mathcal{O}^\vee_{\mathbf{K}^\vee}) \subset \Pi^u(\xi, \eta^\vee, \mathcal{O}^\vee_{\mathbf{K}^\vee})$. Every representation $\pi \in \Pi^u_{\text{icp}}(\xi, \eta^\vee, \mathcal{O}^\vee_{\mathbf{K}^\vee,i})$ is such that $AV_\theta(\pi^\vee)$ is the closure of a single theta form $\mathcal{O}^\vee_{\mathbf{K}^\vee,i}$ of $\mathcal{O}^\vee$, and hence irreducible.

It can be the case that there is a representation $\pi \in \mathcal{B}(\lambda)$ such that $AV_\theta(\pi^\vee)$ is reducible and that $\mathcal{O}^\vee_{\mathbf{K}^\vee,i}$ is just one of the components, then π must belong to $\Pi^u(\xi, \eta^\vee, \mathcal{O}^\vee_{\mathbf{K}^\vee,i})$, however such a π cannot belong to $\Pi^u_{\text{icp}}(\xi, \eta^\vee, \mathcal{O}^\vee_{\mathbf{K}^\vee,i})$. That is why we use the subscript "icp" which stands for "incomplete packet".

Now, recall that $\mathbf{C}(\mathcal{O}^\vee)$ is the set of all cells $\mathcal{C} \in \mathcal{B}(\delta)$ such that $AV_{\mathbb{C}}(\mathcal{C}^\vee) = \overline{\mathcal{O}^\vee}$. Let

$$\mathbf{C}_{\mathbf{K}^\vee}(\mathcal{O}^\vee) = \coprod_{i=1}^r \mathbf{C}(\mathcal{O}^\vee_{\mathbf{K}^\vee,i}) \subset \mathbf{C}(\mathcal{O}^\vee). \tag{7.13}$$

Definition 7.4 (Good Condition for Unipotence)**.** *We will say that the good condition for unipotence is satisfied if $\boldsymbol{C}(\mathcal{O}^\vee) = \boldsymbol{C}_{\boldsymbol{K}^\vee}(\mathcal{O}^\vee)$.*

When the good condition for unipotence is satisfied, all the unipotent Arthur packets $\Pi^u_{\text{icp}}(\xi, \eta^\vee, \mathcal{O}^\vee_{\mathbf{K}^\vee,i}) = \Pi^u(\xi, \eta^\vee, \mathcal{O}^\vee_{\mathbf{K}^\vee,i})$ for all $i = 1, 2, \dots, r$. Furthermore, in this case, two unipotent Arthur packets are either disjoint or equal.

There are cases when the good condition for unipotence is not satisfied. This mostly has to do with the fact that there is no clear understanding about the real associated variety of some HC-cell $\mathcal{C}^\vee$ in $\mathcal{B}^\vee$, in this case $AV_\theta(\pi^\vee)$ is likely reducible or if it is irreducible $\pi^\vee$ does not belong to any of cells $\mathcal{C}^\vee \in \mathbf{C}^\vee(\mathcal{O}^\vee_{\mathbf{K}^\vee,i})$ for any i. Let

$$\mathcal{C}_{\mathrm{mis}}(\mathcal{O}^\vee) = \mathbf{C}(\mathcal{O}^\vee) - \mathbf{C}_{\mathbf{K}^\vee}(\mathcal{O}^\vee). \tag{7.14}$$

Then testing for the good condition for unipotence is equivalent to checking if $\mathbf{C}_{\mathrm{mis}}(\mathcal{O}^\vee)$ is empty. Finally let

$$\Pi^u_{\mathrm{mis}}(\xi,\eta^\vee,\mathcal{O}^\vee) := \coprod_{\mathcal{C}\in\mathbf{C}_{\mathrm{mis}}(\mathcal{O}^\vee)} T^\lambda_\gamma(\mathcal{C}) \subset \Pi^u_{\mathrm{weak}}(\xi,\eta^\vee,\mathcal{O}^\vee). \tag{7.15}$$

The set $\Pi^u_{\mathrm{mis}}(\xi,\eta^\vee,\mathcal{O}^\vee)$ is exactly the set of representations, a subset of which when added to $\Pi^u_{\mathrm{ic}}(\xi,\eta^\vee,\mathcal{O}^\vee_{\mathbf{K}^\vee,i})$, one gets the complete unipotent Arthur packet $\Pi^u(\xi,\eta^\vee,\mathcal{O}^\vee_{\mathbf{K}^\vee,i})$. For this reason we use the subscript "mis" which stands for "missing representations". It is not immediately clear what subset of $\Pi^u_{\mathrm{miss}}(\xi,\eta^\vee,\mathcal{O}^\vee)$ can be added to $\Pi^u_{\mathrm{ic}}(\xi,\eta^\vee,\mathcal{O}^\vee_{\mathbf{K}^\vee,i})$ to get a complete unipotent Arthur packet. In ongoing work with Jeffrey Adams, we explore some ideas about stable characters to achieve this completion in some cases. We summarize the above discussion in the following theorem

Theorem 7.2. *Let $\mathcal{O}^\vee$ be an even nilpotent orbit in $\mathfrak{g}^\vee$. Let ξ be a strong real form of $\boldsymbol{G}$ and let $\eta^\vee$ a strong real form of $\boldsymbol{G}^\vee$, and δ a integral regular infinitesimal character for $\boldsymbol{G}$ be chosen such that $(\eta^\vee)^2 = exp(2\pi i\lambda(\mathcal{O}^\vee))$ and $\lambda(\mathcal{O}^\vee) \in \gamma + X^*(\boldsymbol{H})$. Let $\mathcal{B}(\delta) := \mathcal{B}(\xi,\eta^\vee,\delta)$ be the block of $(\mathfrak{g},\boldsymbol{K}_\xi)$-modules and $\mathcal{B}^\vee(\delta^\vee)$ the corresponding* *dual block. Let* $\{\mathcal{O}^\vee_{\boldsymbol{K}^\vee,1}, \mathcal{O}^\vee_{\boldsymbol{K}^\vee,2}, \ldots, \mathcal{O}^\vee_{\boldsymbol{K}^\vee,r}\}$ *be the theta real forms of* $\mathcal{O}^\vee$ *in the block* $\mathcal{B}^\vee(\delta^\vee)$.

1. *Suppose the good condition for unipotence is satisfied, then* $\Pi^u(\xi, \eta^\vee, \mathcal{O}^\vee_{\boldsymbol{K}^\vee,i})$ *can be computed for all* $i = 1, 2, \ldots r$. *This computation can be implemented in Atlas to compute the explicit Langlands parameters of representations in these complete unipotent packets.*

2. *Suppose the good condition for unipotence is not satisfied, then for each* $i = 1, 2, \ldots, r$, *we can compute a set*

$$\Pi^u_{icp}(\xi, \eta^\vee, \mathcal{O}^\vee_{\boldsymbol{K}^\vee,i}) \subset \Pi^u(\xi, \eta^\vee, \mathcal{O}^\vee_{\boldsymbol{K}^\vee,i}),$$

and, a set

$$\Pi^u_{mis}(\xi, \eta^\vee, \mathcal{O}^\vee) \subset \Pi^u_{weak}(\xi, \eta^\vee, \mathcal{O}^\vee)$$

such that

$$\Pi^u(\xi, \eta^\vee \mathcal{O}^\vee_{\boldsymbol{K}^\vee,i}) - \Pi^u_{icp}(\xi, \eta^\vee, \mathcal{O}^\vee_{\boldsymbol{K}^\vee}) \subset \Pi^u_{mis}(\xi, \eta^\vee, \mathcal{O}^\vee),$$

for each $i = 1, 2, \ldots, r$. *For* $i = 1, 2, \ldots, r$, *we have the following inclusions:*

$$\Pi^u_{icp}(\mathcal{O}^\vee_{\boldsymbol{K}^\vee,i}) \subset \Pi^u(\mathcal{O}^\vee_{\boldsymbol{K}^\vee,i}) \subset \Pi^u_{icp}(\mathcal{O}^\vee_{\boldsymbol{K}^\vee,i}) \bigsqcup \Pi^u_{mis}(\mathcal{O}^\vee) = \Pi^u_{weak}(\mathcal{O}^\vee_{\mathbb{R}}),$$

such that, except $\Pi^u(\xi, \eta^\vee, \mathcal{O}^\vee_{\boldsymbol{K}^\vee,i})$, *all the other sets are completely and explicitly computable in Atlas.*

Chapter 8: An application and some examples

Recall that Theorem 6.2 computes the real associated variety only when the 'good condition' given in Definition 6.3 is satisfied.

Here are two possiblities of how the good condition might fail to be true:

1. The cell $\mathcal{C}^\vee$ contains a $\mathcal{R}_{\mathfrak{q}}(\lambda)$, but $\mathfrak{q}$ is not conjugate to any of the parabolics in $\mathcal{S}(\mathcal{O}^\vee)$.

2. The cell $\mathcal{C}^\vee$ does not contain a cohomologically induced module of the type $\mathcal{R}_{\mathfrak{q}}(\lambda)$ for any choice of theta-stable data $(\mathfrak{q}, \lambda)$.

In case of (1), we know that the associated variety is definitely irreducible and therefore has to be one of the theta-forms corresponding to a parabolic in $\mathcal{S}(\mathcal{O}^\vee)$. It is possible that such a scenario does not arise, but at this point we don't know how to prove otherwise.

In case of (2), we will use Theorem 7.2 to try to figure out the the associated variety. The main tool in this application is the stable sum formula for unipotent packets which we now state:

Theorem 8.1 (Theorem 22.7, [ABV92])**.** *Suppose we are in setting of previous sec-*

tion, that is the Atlas Setting, let $\mathcal{O}^\vee$ be a dual nilpotent orbit and let $\mathcal{O}^\vee_{\boldsymbol{K}^\vee,i}$ be one of its theta-forms. Let $\Pi^u(\xi,\eta^\vee,\mathcal{O}^\vee_{\boldsymbol{K}^\vee,i}) := \Pi_u(\mathcal{O}^\vee_{\boldsymbol{K},i})$. Then corresponding to $\mathcal{O}^\vee_{\boldsymbol{K},i}$, is a strongly stable virtual character given by

$$\eta(\mathcal{O}^\vee_{\boldsymbol{K},i}) = \sum_{\pi\in\Pi_u(\mathcal{O}^\vee_{\boldsymbol{K},i})} a(\pi)\pi, \tag{8.1}$$

the coefficients $a(\pi)$ are explicitly determined, and are non-zero.

Since all the coefficients $a(\pi)$ are non-zero, given a complete Arthur packet, we should be able to compute $\eta(\mathcal{O}^\vee_{\mathbf{K},i})$.

Alternately, if we start with a subset of an Arthur packet which does not have a stable sum of virtual characters and we inductively add a representation, from finite set, to this subset checking for stable sums at each step, then, in this scheme, suppose we did not find a stable sum at stage n, and we find a stable sum with all non-zero coefficients at stage $n+1$. This would imply that adding these $n+1$ representations to the subset we started with gives us the complete Arthur packet or a better approximation to the Arthur packet than the original subset. In this setup, $\Pi^u_{\text{icp}}(\mathcal{O}^\vee_{\mathbf{K}^\vee,i})$ is the subset we want to start with and $\Pi^u_{\text{mis}}(\mathcal{O}^\vee)$ is the set from which we add representations.

Suppose we started out with $\Pi^u_{\text{icp}}(\mathcal{O}^\vee_{\mathbf{K}^\vee,j})$, $i \neq j$ and repeated the same process as above to compute the unipotent packet $\Pi^u_{\text{icp}}(\mathcal{O}^\vee_{\mathbf{K}^\vee,j})$, then the representations of $\Pi^u_{\text{mis}}(\mathcal{O}^\vee)$ that are in both $\Pi^u_{\text{icp}}(\mathcal{O}^\vee_{\mathbf{K}^\vee,i})$ and $\Pi^u_{\text{icp}}(\mathcal{O}^\vee_{\mathbf{K}^\vee,j})$ would contain $\mathcal{O}^\vee_{\mathbf{K}^\vee,i}$ and $\mathcal{O}^\vee_{\mathbf{K}^\vee,j})$ in their theta-associated variety, proving that the associated variety is reducible. As we vary over all the theta-forms, we end up computing the associated varieties of all the representations in $\Pi^u_{\text{mis}}(\mathcal{O}^\vee)$.

Work on this is still in progress, a crucial component is the implementation of the computations of stable sum formulas that have been implemented into the Atlas software by Adams.

8.1 Some Examples

In this section we work out some examples of applications of the results of this paper. We will be repeatedly applying Corollary 6.1 to compute the real forms of a complex nilpotent orbit. We will then compare our answers with the output from Atlas.

We briefly outline the framework of Corollary 6.1. A unipotent Arthur parameter is a homomorphism:

$$\psi : W_{\mathbb{R}} SL(2,\mathbb{C}) \longrightarrow^{\Gamma} \mathbf{G}^{\vee}.$$

Suppose $\psi|_{SL(2,\mathbb{C})}$ corresponds to $\mathcal{O}^{\vee}$ a complex nilpotent orbit fpr $\mathbf{G}^{\vee}$. Suppose $W_{\mathbb{R}} = \langle \mathbb{C}^{\times}, j \rangle$, then we know that $\psi|_{\mathbb{C}^{\times}}$ is trivial. Therefore, the value of $\psi(j)$ will then determine ψ, to this end we note that $\psi(j)$ must be an element of order 2 and that

$$\psi(j) \in \mathrm{Cent}_{G^{\vee}}(\psi(SL(2,\mathbb{C}))) \cap ({}^{\Gamma}\mathbf{G}^{\vee} - \mathbf{G}^{\vee}). \tag{8.2}$$

Let $y_0 = \psi \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ then $y = y_0\psi(j)$ defines and element of order 2 in ${}^{\Gamma}\mathbf{G}^{\vee} - \mathbf{G}^{\vee}$, so that $\theta_y^{\vee}$ defines a Cartan involution of $\mathbf{G}^{\vee}$. So every choice of $\psi(j)$ will give us a real form of $\mathcal{O}^{\vee}$ for the real group of $\mathbf{G}^{\vee}$ defined by $y = y_0\psi(j)$.

We will be concerned with the neutral element $H \in SL(2,\mathbb{C})$, where $H = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. Given an Arthur parameter ψ, we will denote $y_0 = \psi(H)$. Recall that $y = y_0 \cdot \psi(j)$ gives a strong real form of $\mathbf{G}^\vee$.

8.1.1 $\mathbf{G}(\mathbb{R}) = SL(2,\mathbb{R})$

In this case the dual group $\mathbf{G}^\vee$ is $SO(3,\mathbb{C})$. Recall that $SL(2,\mathbb{C}) \simeq SO(3,\mathbb{C})$ has two complex nilpotent orbits:

1. the principal orbit parameterized by the partition [2].
2. the trivial orbit parameterized by the partition $[1,1]$.

Let $\mathcal{O}^\vee = [1,1]$. We now compute unipotent Arthur parameters ψ. We have have the following:

1. by unipotence of ψ, $\psi|_{\mathbb{C}^\times} \equiv 1$.
2. since $\mathcal{O}^\vee$ is $[1,1]$, $\psi(SL(2,\mathbb{C})) \equiv 1$, so that $\mathrm{Cent}_{G^\vee}(\psi(SL(2,\mathbb{C}))) = \mathbf{G}^\vee$.

Since elements of ${}^\Gamma\mathbf{G}^\vee$ are just pairs (x,σ) where $x \in SO(3,\mathbb{C})$ and $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$, for convenience we will drop off σ from the notation.

For the condition

$$\psi(SL(2,\mathbb{C})) \equiv 1 \quad \text{and} \quad \mathrm{Cent}_{G^\vee}(\psi(SL(2,\mathbb{C}))) =^\Gamma \mathbf{G}^\vee,$$

we note that $\psi(H) = 1$ in this case and that we have two possibilities (upto conjugation by $\mathbf{G}^\vee$) for $\psi(j)$:

1. $\psi_1(j) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, so that corresponding strong real form is $y = y_0 \cdot \psi_1(j) = \psi_1(j)$. The strong real form y corresponds to $SO(3)$, so that the Arthur parameter ψ_1 captures the trivial nilpotent orbit in $SO(3)$.
2. $\psi_2(j) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ so that corresponding strong real form is $y = y_0 \cdot \psi(j) = \psi_2(j)$. The strong real form y corresponds to $SO(2,1)$, so that the Arthur parameter ψ_2 caputures the trivial nilpotent orbit in $SO(2,1)$.

To compute the Arthur packets, we need to know the elements of the blocks $\mathcal{B}_1 = \mathcal{B}(SL(2,\mathbb{R}), SO(3))$ and $\mathcal{B}_2 = \mathcal{B}(SL(2,\mathbb{R}), SO(2,1))$ which are given by

1. $\mathcal{B}_1 = \{PS(2\rho)\}$ where $PS(2\rho)$ is the irreducible principal series at infinitesimal character 2ρ.
2. $\mathcal{B}_2 = \{DS(+,\rho), DS(-,\rho), \chi_{\text{triv}}\}$, where $DS(+,\rho)$ is the discrete series at ρ with positive K-types, $DS(-,\rho)$ is the discrete series at ρ with negative K-types and, χ_{triv} is the trivial representation.

In this setting, the Arthur packets corresponding to two Arthur parameters ψ_1 and ψ_2 are given by

1. $\Pi(SL(2,\mathbb{R}), \psi_1) = \{PS(0)\}$, where $PS(0)$ is the irreducible principal series with infinitesimal character 0.

2. $\Pi(SL(2,\mathbb{R}),\psi_2) = \{LDS(+,0), LDS(-,0)\}$, where $LDS(+,0)$ is the limit of discrete series with positive K-types (respectively $LDS(-,0)$).

In the case with $\mathcal{O}^\vee = [2]$ the Arthur parameter ψ restricted to the Cartan subgroup of SL satisfies (upto conjugation by $\mathbf{G}^\vee$)

$$\psi \begin{pmatrix} z & 0 \\ 0 & 1/z \end{pmatrix} = \begin{pmatrix} z^2 & 0 & 0 \\ 0 & 1/z^2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

so that $y_0 = \psi \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Furthermore, $\mathrm{Cent}_{G^\vee}(\psi(SL(2,\mathbb{C}))) := \{1\}$ so that $\psi(j)$ is forced to be 1 (actually the identity matrix in $SO(3,\mathbb{C})$). So that the strong real form $y = y_0\psi(j) = y_0$ corresponds to the real group $SO(2,1)$. The Arthur parameter ψ identifies the unique real form of $[2]$ in $SO(2,1)$. The corresponding Arthur packet is given as

$$\Pi(SL(2,\mathbb{R}),\psi) = \{\chi_{\text{triv}}\}.$$

This completes the computation of all the Arthur packets for $SL(2,\mathbb{R})$.

Chapter 9: Tables of Data

In this section we present some tables of data corresponding to output from the Atlas Software. In the earlier sections we simplified a lot of notation, so as to have cleaner presentation.

We want to be in the Atlas setting as in Definition 4.3. For all of the tables below, we will fix a real form $\mathbf{G}(\mathbb{R})$ (equivalently ξ) and vary over all dual real forms $\eta^\vee$ in the quasisplit inner class corresponding to ξ. For each such $\eta^\vee$ we have a dual real form $\mathbf{G}(\mathbb{R})^\vee_{\eta^\vee}$ of $\mathbf{G}(\mathbb{R})$.

We fix a regular integral infinitesimal character λ for $\mathbf{G}(\mathbb{R})$ determined by $\eta^\vee$ satisfying $(\eta^\vee)^2 = \exp(2\pi i \lambda^\vee)$. Recall that the triple $(\xi, \eta^\vee, \lambda)$ defines a block of irreducible $(\mathfrak{g}, \mathbf{K}^\vee_{\eta^\vee})$-modules for $\mathbf{G}(\mathbb{R})$ at infinitesimal character λ, where $\mathbf{K}^\vee_{\eta^\vee} = \mathbf{G}^{\eta^\vee}$. Atlas blocks only depend on the images of ξ and $\eta^\vee$ in real forms, there is a process to get parameters for blocks corresponding to strong real forms which we will show in an example soon.

Since ξ and λ are fixed, the block of representation will only depend on $\eta^\vee$. We will only be concerned with real forms when dealing with blocks in Atlas and hence if $\mathbf{G}(\mathbb{R})^\vee$ is the real form corresponding to $\eta^\vee$ we denote the block corresponding to the triple $(\xi, \eta^\vee, \lambda)$ by $\mathcal{B}(\mathbf{G}(\mathbb{R})^\vee)$. Let $\mathcal{B}^\vee(\mathbf{G}(\mathbb{R})^\vee)$ be the dual block corresponding to $\mathcal{B}(\mathbf{G}(\mathbb{R})^\vee)$.

Furthermore, recall that the block $\mathcal{B}(\mathbf{G}(\mathbb{R})^\vee)$ is partitioned into HC-cells, these cells are parameterized by integers in Atlas, and the set of cells for $\mathcal{B}(\mathbf{G}(\mathbb{R})^\vee)$ will be denoted by $\mathbf{C}(\mathbf{G}(\mathbb{R})^\vee)$. Using Vogan-duality we can compute the corresponding dual cells in $\mathcal{B}^\vee(\mathbf{G}(\mathbb{R})^\vee)$ and we denote the set by $\mathbf{C}^\vee(\mathbf{G}(\mathbb{R})^\vee)$. Note that in this setting:

$$\mathcal{B}(\mathbf{G}(\mathbb{R})^\vee) = \bigcup_{\mathcal{C}\in\mathbf{C}(\mathbf{G}(\mathbb{R})^\vee)} \mathcal{C} \quad \mathcal{B}^\vee(\mathbf{G}(\mathbb{R})^\vee) = \bigcup_{\mathcal{C}^\vee\in\mathbf{C}^\vee(\mathbf{G}(\mathbb{R})^\vee)} \mathcal{C}^\vee. \tag{9.1}$$

We will also use the fact that when $\mathbf{G}$ is of classical type, nilpotent orbits $\mathcal{O}$ (resp. $\mathcal{O}^\vee$) for $\mathbf{G}(\mathbb{R})$ (resp. $\mathbf{G}(\mathbb{R})^\vee$) are parameterized by certain integer partitions. Fix a dual complex nilpotent orbit $\mathcal{O}^\vee$ for $\mathbf{G}(\mathbb{R})$, using Theorem 6.1, we know how to parameterize the set $\mathcal{S}(\mathcal{O}^\vee, \mathbf{G}(\mathbb{R})^\vee)$. Suppose s is the cardinality of $\mathcal{S}(\mathcal{O}^\vee, \mathbf{G}(\mathbb{R})^\vee)$, in Atlas we compute this set to be in correspondence with a set of integers $\{0, 1, 2, \ldots, s-1\}$, so that the pair $(\mathbf{K}^\vee, i)$ (or equivalently $(\mathbf{G}(\mathbb{R})^\vee, i)$) determines the ith theta-form $\mathcal{O}^\vee(\mathbf{K}^\vee, i)$ of $\mathcal{O}^\vee$ for $\mathbf{G}(\mathbb{R})^\vee$. So that

$$\mathcal{O}^\vee \cap (\mathfrak{g}^\vee)^{-\theta_{\eta^\vee}} = \bigcup_{i=0}^{s-1} \mathcal{O}^\vee(\mathbf{K}^\vee, i). \tag{9.2}$$

Note that $\mathbf{K}^\vee = (\mathbf{G}^\vee)^{\theta_{\eta^\vee}}$.

We now consider the following sets:

$$\mathbf{C}^\vee(\mathbf{G}(\mathbb{R})^\vee,\mathcal{O}^\vee) := \{\mathcal{C}^\vee \in \mathbf{C}^\vee(\mathbf{G}(\mathbb{R})^\vee) \mid AV_{\mathbb{C}}(\mathcal{C}^\vee) = \overline{\mathcal{O}^\vee}\}. \tag{9.3}$$

$$\mathbf{C}(\mathbf{G}(\mathbb{R})^\vee,\mathcal{O}^\vee) := \{d(\mathcal{C}^\vee) \in \mathbf{C}(\mathbf{G}(\mathbb{R})^\vee) \mid \mathcal{C}^\vee \in \mathbf{C}^\vee(\mathbf{G}(\mathbb{R})^\vee,\mathcal{O}^\vee)\} := d(\mathbf{C}^\vee(\mathbf{G}(\mathbb{R})^\vee,\mathcal{O}^\vee)), \tag{9.4}$$

where d is the Vogan-duality map.

$$\mathbf{C}^\vee_{\mathrm{irr}}(\mathbf{G}(\mathbb{R})^\vee,\mathcal{O}^\vee,i) := \{\mathcal{C}^\vee \in \mathbf{C}^\vee(\mathbf{G}(\mathbb{R})^\vee,\mathcal{O}^\vee) \mid AV_\theta(\mathcal{C}^\vee) = \overline{\mathcal{O}^\vee(\mathbf{G}(\mathbb{R})^\vee,i)}\}. \tag{9.5}$$

$$\mathbf{C}^\vee_{\mathrm{irr},0}(\mathbf{G}(\mathbb{R})^\vee,\mathcal{O}^\vee,i) := \{\mathcal{C}^\vee \in \mathbf{C}^\vee_{\mathrm{irr}}(\mathbf{G}(\mathbb{R})^\vee,\mathcal{O}^\vee,i) \mid \text{there exists } j \text{ such that } \mathcal{R}_{\mathfrak{p}_i^\vee}(\chi_{ij}) \in \mathcal{C}^\vee\}. \tag{9.6}$$

$$\mathbf{C}^\vee_{\mathrm{irr},0}(\mathbf{G}(\mathbb{R})^\vee,\mathcal{O}^\vee) := \bigcup_{i=0}^{s-1} \mathbf{C}^\vee_{\mathrm{irr},0}(\mathbf{G}(\mathbb{R})^\vee,\mathcal{O}^\vee,i). \tag{9.7}$$

$$\mathbf{C}^\vee_{\mathrm{mis}}(\mathbf{G}(\mathbb{R})^\vee,\mathcal{O}^\vee) = \mathbf{C}^\vee(\mathbf{G}(\mathbb{R})^\vee,\mathcal{O}^\vee) - \mathbf{C}^\vee_{\mathrm{irr},0}(\mathbf{G}(\mathbb{R})^\vee,\mathcal{O}^\vee). \tag{9.8}$$

So that

$$\mathbf{C}_{\mathrm{irr},0}(\mathbf{G}(\mathbb{R})^\vee,\mathcal{O}^\vee,i) := d(\mathbf{C}^\vee_{\mathrm{irr},0}(\mathbf{G}(\mathbb{R})^\vee,\mathcal{O}^\vee,i)) \quad \mathbf{C}_{\mathrm{mis}}(\mathbf{G}(\mathbb{R})^\vee,\mathcal{O}^\vee) := d(\mathbf{C}^\vee_{\mathrm{mis}}(\mathbf{G}(\mathbb{R})^\vee,\mathcal{O}^\vee)). \tag{9.9}$$

Recall that we know how to explicitly compute $\mathbf{C}^\vee_{\mathrm{irr},0}(\mathbf{G}(\mathbb{R})^\vee,\mathcal{O}^\vee,i)$ for all i, and hence can compute $\mathbf{C}^\vee_{\mathrm{irr},0}(\mathbf{G}(\mathbb{R})^\vee,\mathcal{O}^\vee)$. We also know how to compute complex associated varieties using Atlas, hence we can compute $\mathbf{C}^\vee(\mathbf{G}(\mathbb{R})^\vee,\mathcal{O}^\vee)$.
Therefore if $|\mathbf{C}^\vee_{\mathrm{mis}}(\mathbf{G}(\mathbb{R})^\vee,\mathcal{O}^\vee)| = 0$, we know how to compute the theta-associated varieties of all representations in the block $\mathcal{B}(\mathbf{G}(\mathbb{R})^\vee)$ whose complex associated variety is $\mathcal{O}^\vee$, in particular we can compute the complete unipotent packets corresponding to the theta-forms of $\mathcal{O}^\vee$. Alternately, the size of $|\mathbf{C}^\vee_{\mathrm{mis}}(\mathbf{G}(\mathbb{R})^\vee,\mathcal{O}^\vee)|$ determines the how far we are from computing a complete packet.
Also, if $|\mathbf{C}^\vee_{\mathrm{mis}}(\mathbf{G}(\mathbb{R})^\vee,\mathcal{O}^\vee)| = 0$, all the unipotent packets will be disjoint.

Let $\{X^\vee, H^\vee, Y^\vee\}$ be the Jacobson-Morozov triple for $\mathcal{O}^\vee$, where $H^\vee$ is the neutral element. Let $\gamma = \frac{1}{2}H^\vee$ and let T^γ_λ be the translation functor taking irreducible $(\mathfrak{g},\mathbf{K})$-modules with infinitesimal character λ to those with infinitesimal character γ. In this setting we get the following sets of unipotent representations constructed in Theorem 7.2:

$$\Pi^u_{\mathrm{weak}}(\mathbf{G}(\mathbb{R}),\mathbf{G}(\mathbb{R})^\vee,\mathcal{O}^\vee) = \bigcup_{\mathcal{C}\in\mathbf{C}(\mathbf{G}(\mathbb{R})^\vee,\mathcal{O}^\vee)} \{T^\gamma_\lambda(\pi) \mid \pi\in\mathcal{C}\} := T^\gamma_\lambda(\mathcal{C}), \tag{9.10}$$

$$\Pi^u_{\mathrm{icp}}(\mathbf{G}(\mathbb{R}),\mathbf{G}(\mathbb{R})^\vee,\mathcal{O}^\vee,i) = \bigcup_{\mathcal{C}\in\mathbf{C}_{\mathrm{irr},0}(\mathbf{G}(\mathbb{R})^\vee,\mathcal{O}^\vee,i)} \{T^\gamma_\lambda(\pi) \mid \pi\in\mathcal{C}\} := T^\gamma_\lambda(\mathcal{C}), \tag{9.11}$$

$$\Pi^u_{\text{mis}}(\mathbf{G}(\mathbb{R}), \mathbf{G}(\mathbb{R})^\vee, \mathcal{O}^\vee) = \bigcup_{\mathcal{C} \in \mathbf{C}_{\text{mis}}(\mathbf{G}(\mathbb{R})^\vee, \mathcal{O}^\vee)} \{T^\gamma_\lambda(\pi) \mid \pi \in \mathcal{C}\} := T^\gamma_\lambda(\mathcal{C}). \tag{9.12}$$

Recall that the set of special unipotent representations corresponding to the theta-form $\mathcal{O}^\vee(\mathbf{G}(\mathbb{R})^\vee, i)$ was denoted as $\Pi^u(\mathbf{G}(\mathbb{R}), \mathcal{O}(\mathbf{G}(\mathbb{R})^\vee, i))$ and satisfies the following inclusions:

$$\Pi^u_{\text{icp}}(\mathbf{G}(\mathbb{R}), \mathbf{G}(\mathbb{R})^\vee, \mathcal{O}^\vee, i) \subset \Pi^u(\mathbf{G}(\mathbb{R}), \mathcal{O}(\mathbf{G}(\mathbb{R})^\vee, i)),$$

$$\Pi^u(\mathbf{G}(\mathbb{R}), \mathcal{O}(\mathbf{G}(\mathbb{R})^\vee, i)) \subset \Pi^u_{\text{icp}}(\mathbf{G}(\mathbb{R}), \mathbf{G}(\mathbb{R})^\vee, \mathcal{O}^\vee, i) \bigcup \Pi^u_{\text{mis}}(\mathbf{G}(\mathbb{R}), \mathbf{G}(\mathbb{R})^\vee, \mathcal{O}^\vee),$$

$$\Pi^u_{\text{icp}}(\mathbf{G}(\mathbb{R}), \mathbf{G}(\mathbb{R})^\vee, \mathcal{O}^\vee, i) \bigcup \Pi^u_{\text{mis}}(\mathbf{G}(\mathbb{R}), \mathbf{G}(\mathbb{R})^\vee, \mathcal{O}^\vee) = \Pi^u_{\text{weak}}(\mathbf{G}(\mathbb{R}), \mathbf{G}(\mathbb{R})^\vee, \mathcal{O}^\vee).$$

In the tables below, we will compute $\mathbf{C}(\mathbf{G}(\mathbb{R})^\vee, \mathcal{O}^\vee)$, $\mathbf{C}_{\text{irr},0}(\mathbf{G}(\mathbb{R})^\vee, \mathcal{O}^\vee, i)$, $\mathbf{C}_{\text{mis}}(\mathbf{G}(\mathbb{R})^\vee, \mathcal{O}^\vee)$. For space constraints, will only compute the cardinalities of $\Pi^u_{\text{icp}}(\mathbf{G}(\mathbb{R}), \mathbf{G}(\mathbb{R})^\vee, \mathcal{O}^\vee, i)$, $\Pi^u_{\text{weak}}(\mathbf{G}(\mathbb{R}), \mathbf{G}(\mathbb{R})^\vee, \mathcal{O}^\vee)$, $\Pi^u_{\text{mis}}(\mathbf{G}(\mathbb{R}), \mathbf{G}(\mathbb{R})^\vee, \mathcal{O}^\vee)$ and we invite the interested reader to use this information to compute the actual parameters in Atlas.

9.1 $\mathbf{G}(\mathbb{R}) = SL(2, \mathbb{R})$.

In this case $\mathbf{G}^\vee = PGL(2, \mathbb{C})$ with real forms $PGL(2, \mathbb{R})$ and $PSU(2)$. Here is some basic information about blocks and cells:

1. $|\mathcal{B}(PGL(2, \mathbb{R}))| = 3$, $|\mathbf{C}(PGL(2, \mathbb{R}))| = 3$.
2. $|\mathcal{B}(PSU(2)| = 1$, $|\mathbf{C}(PSU(2))| = 1$.

Here is the basic information about associated varieties for cells.

$\mathcal{O}^\vee$	$\mathbf{G}(\mathbb{R})^\vee$	$\mathbf{C}^\vee(\mathbf{G}(\mathbb{R})^\vee, \mathcal{O}^\vee)$	$\mathbf{C}(\mathcal{O}^\vee)$	$\mathbf{C}^\vee_{\text{irr},0}(\mathcal{O}^\vee)$	$\mathbf{C}_{\text{irr},0}(\mathcal{O}^\vee)$	$\mathbf{C}^\vee_{\text{mis}}(\mathcal{O}^\vee)$
$[2]$	$PGL(2, \mathbb{R})$	$\{0\}$	$\{2\}$	$\{0\}$	$\{2\}$	$\varnothing$
$[1^2]$	$PGL(2, \mathbb{R})$	$\{1, 2\}$	$\{0, 1\}$	$\{1, 2\}$	$\{0, 1\}$	$\varnothing$
	$PSU(2)$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\varnothing$

Here is the information about the real forms of even complex dual nilpotent orbits:

$\mathcal{O}^\vee$	$\mathbf{G}(\mathbb{R})^\vee$	$\mathcal{S}(\mathbf{G}(\mathbb{R})^\vee, \mathcal{O}^\vee)$ indices	$\mathcal{R}_{\mathfrak{p}_i^\vee}(\chi_{ij})$ in dual block	$\mathbf{C}^\vee_{\mathrm{irr},0}(\mathbf{G}(\mathbb{R})^\vee, \mathcal{O}^\vee, i)$	$\mathbf{C}_{\mathrm{irr},0}(\mathbf{G}(\mathbb{R})^\vee, \mathcal{O}^\vee, i)$
$[2]$	$PGL(2,\mathbb{R})$	0	$\{0\}$	$\{0\}$	$\{2\}$
$[1^2]$	$PGL(2,\mathbb{R})$	0	$\{1,2\}$	$\{1,2\}$	$\{0,1\}$
	$PSU(2)$	0	$\{0\}$	$\{0\}$	$\{0\}$

Here is the information about the cardinalities of the special unipotent packets:

$\mathcal{O}^\vee$	$\mathbf{G}(\mathbb{R})^\vee$	$\mathcal{S}(\mathbf{G}(\mathbb{R})^\vee, \mathcal{O}^\vee)$	$\frac{1}{2}H^\vee$	$\lvert\Pi^u_{\mathrm{weak}}(\mathcal{O}^\vee)\rvert$	$\lvert\Pi^u_{\mathrm{icp}}(\mathcal{O}^\vee, i)\rvert$	$\lvert\Pi^u_{\mathrm{mis}}(\mathcal{O}^\vee)\rvert$
$[2]$	$PGL(2,\mathbb{R})$	0	$[1]$	1	1	0
$[1^2]$	$PGL(2,\mathbb{R})$	0	$[0]$	2	2	0
$[1^2]$	$PSU(2)$	0	$[0]$	1	1	0

9.2 $\mathbf{G}(\mathbb{R}) = PGL(2,\mathbb{R}) \simeq SO(2,1)$.

In this case $\mathbf{G}^\vee = SL(2,\mathbb{C})$ with strong real forms $SL(2,\mathbb{R})$ and $SU(2,0)$ and $SU(0,2)$. Here is some basic information about blocks and cells:

1. $|\mathcal{B}(SL(2,\mathbb{R}))| = 3$, $|\mathbf{C}(SL(2,\mathbb{R}))| = 3$.
2. $|\mathcal{B}(SU(2,0)| = 1$, $|\mathbf{C}(SU(2,0))| = 1$.

Since Atlas computes blocks only at the level of real-forms, we are missing the block corresponding to the strong real form $SU(0,2)$. However, even though we do not have the block corresponding to $SU(0,2)$, it is still possible to find the parameters corresponding to that block in Atlas.

Here is the basic information about associated varieties for cells.

$\mathcal{O}^\vee$	$\mathbf{G}(\mathbb{R})^\vee$	$\mathbf{C}^\vee(\mathbf{G}(\mathbb{R})^\vee, \mathcal{O}^\vee)$	$\mathbf{C}(\mathcal{O}^\vee)$	$\mathbf{C}^\vee_{\mathrm{irr},0}(\mathcal{O}^\vee)$	$\mathbf{C}_{\mathrm{irr},0}(\mathcal{O}^\vee)$	$\mathbf{C}^\vee_{\mathrm{mis}}(\mathcal{O}^\vee)$
$[2]$	$SL(2,\mathbb{R})$	$\{0,1\}$	$\{1,2\}$	$\{0,1\}$	$\{1,2\}$	$\varnothing$
$[1^2]$	$SL(2,\mathbb{R})$	$\{2\}$	$\{0\}$	$\{2\}$	$\{0\}$	$\varnothing$
	$SU(2,0)$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\varnothing$

Here is the information about the real forms of even complex dual nilpotent orbits:

$\mathcal{O}^\vee$	$\mathbf{G}(\mathbb{R})^\vee$	$\mathcal{S}(\mathbf{G}(\mathbb{R})^\vee, \mathcal{O}^\vee)$ indices	$\mathcal{R}_{\mathfrak{p}_i^\vee}(\chi_{ij})$ in dual block	$\mathbf{C}^\vee_{\mathrm{irr},0}(\mathbf{G}(\mathbb{R})^\vee, \mathcal{O}^\vee, i)$	$\mathbf{C}_{\mathrm{irr},0}(\mathbf{G}(\mathbb{R})^\vee, \mathcal{O}^\vee, i)$
$[2]$	$SL(2,\mathbb{R})$	0	$\{0\}$	$\{0\}$	$\{1\}$
		1	$\{1\}$	$\{1\}$	$\{2\}$
$[1^2]$	$SL(2,\mathbb{R})$	0	$\{2\}$	$\{2\}$	$\{0\}$
	$SU(2,0)$	0	$\{0\}$	$\{0\}$	$\{0\}$

Here is the information about the cardinalities of the special unipotent packets:

$\mathcal{O}^\vee$	$\mathbf{G}(\mathbb{R})^\vee$	$\mathcal{S}(\mathbf{G}(\mathbb{R})^\vee, \mathcal{O}^\vee)$	$\frac{1}{2}H^\vee$	$\lvert\Pi^u_{\mathrm{weak}}(\mathcal{O}^\vee)\rvert$	$\lvert\Pi^u_{\mathrm{icp}}(\mathcal{O}^\vee, i)\rvert$	$\lvert\Pi^u_{\mathrm{mis}}(\mathcal{O}^\vee)\rvert$
$[2]$	$SL(2,\mathbb{R})$	0 1	$\frac{1}{2}[1]$	2	1 1	0
$[1^2]$	$SL(2,\mathbb{R})$	0	$[0]$	0	0	0
$[1^2]$	$SU(2,0)$	0	$[0]$	1	1	0
$[1^2]$	$SU(0,2)$	0	$[0]$	1	1	0

9.3 $\mathbf{G}(\mathbb{R}) = SO(3,2)$.

In this case $\mathbf{G}^\vee = Sp(4,\mathbb{C})$ with strong real forms $Sp(4,\mathbb{R})$, $Sp(1,1)$, $Sp(2,0)$, and $Sp(0,2)$. Here is some basic information about blocks and cells:

1. $|\mathcal{B}(Sp(4,\mathbb{R}))| = 12$, $|\mathbf{C}(Sp(4,\mathbb{R}))| = 6$.
2. $|\mathcal{B}(Sp(1,1))| = 4$, $|\mathbf{C}(Sp(1,1))| = 2$.
3. $|\mathcal{B}(Sp(2,0)| = 1$, $|\mathbf{C}(Sp(2,0))| = 1$.
4. $|\mathcal{B}(Sp(0,2)| = 1$, $|\mathbf{C}(Sp(0,2))| = 1$, we will use the parameters from the block coming from $Sp(2,0)$ to compute the unipotent representations in this block.

Here is the basic information about associated varieties for cells.

$\mathcal{O}^\vee$	$\mathbf{G}(\mathbb{R})^\vee$	$\mathbf{C}^\vee(\mathbf{G}(\mathbb{R})^\vee,\mathcal{O}^\vee)$	$\mathbf{C}(\mathcal{O}^\vee)$	$\mathbf{C}^\vee_{\mathrm{irr},0}(\mathcal{O}^\vee)$	$\mathbf{C}_{\mathrm{irr},0}(\mathcal{O}^\vee)$	$\mathbf{C}^\vee_{\mathrm{mis}}(\mathcal{O}^\vee)$
$[4]$	$Sp(4,\mathbb{R})$	$\{0,1\}$	$\{4,5\}$	$\{0,1\}$	$\{4,5\}$	$\varnothing$
$[2^2]$	$Sp(4,\mathbb{R})$	$\{2,3,4\}$	$\{2,3,1\}$	$\{2,3,4\}$	$\{2,3,4\}$	$\varnothing$
	$Sp(1,1)$	$\{0\}$	$\{1\}$	$\{0\}$	$\{1\}$	$\varnothing$
$[1^4]$	$Sp(4,\mathbb{R})$	$\{5\}$	$\{0\}$	$\{5\}$	$\{0\}$	$\varnothing$
	$Sp(1,1)$	$\{1\}$	$\{0\}$	$\{1\}$	$\{0\}$	$\varnothing$
	$Sp(2,0)$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\varnothing$

Here is the information about the real forms of even complex dual nilpotent orbits:

$\mathcal{O}^\vee$	$\mathbf{G}(\mathbb{R})^\vee$	#real-forms in $\mathbf{G}(\mathbb{R})^\vee$
$[4]$	$Sp(4,\mathbb{R})$	2
$[2^2]$	$Sp(4,\mathbb{R})$	3
	$Sp(1,1)$	1
$[1^4]$	$Sp(4,\mathbb{R})$	1
	$Sp(1,1)$	1
	$Sp(2,0)$	1

The following is the information about real forms of even nilpotent orbits in terms of Atlas output:

$\mathcal{O}^\vee$	$\mathbf{G}(\mathbb{R})^\vee$	$\mathcal{S}(\mathbf{G}(\mathbb{R})^\vee,\mathcal{O}^\vee)$ indices	$\mathcal{R}_{\mathfrak{p}_i^\vee}(\chi_{ij})$ in dual block	$\mathbf{C}^\vee_{\mathrm{irr},0}(\mathbf{G}(\mathbb{R})^\vee,\mathcal{O}^\vee,i)$	$\mathbf{C}_{\mathrm{irr},0}(\mathbf{G}(\mathbb{R})^\vee,\mathcal{O}^\vee,i)$
$[4]$	$Sp(4,\mathbb{R})$	0	$\{0\}$	$\{0\}$	$\{4\}$
		1	$\{1\}$	$\{1\}$	$\{5\}$
$[2^2]$	$Sp(4,\mathbb{R})$	0	$\{2\}$	$\{2\}$	$\{2\}$
		1	$\{3\}$	$\{3\}$	$\{3\}$
		2	$\{4\}$	$\{4\}$	$\{1\}$
	$Sp(1,1)$	0	$\{2\}$	$\{0\}$	$\{1\}$
$[1^4]$	$Sp(4,\mathbb{R})$	0	$\{10\}$	$\{5\}$	$\{0\}$
	$Sp(1,1)$	0	$\{3\}$	$\{1\}$	$\{0\}$
	$Sp(2,0)$	0	$\{0\}$	$\{0\}$	$\{0\}$

Here is the information about the cardinalities of the special unipotent packets:

$\mathcal{O}^\vee$	$\mathbf{G}(\mathbb{R})^\vee$	$\mathcal{S}(\mathbf{G}(\mathbb{R})^\vee, \mathcal{O}^\vee)$	$\frac{1}{2}H^\vee$	$\lvert\Pi^u_{\text{weak}}(\mathcal{O}^\vee)\rvert$	$\lvert\Pi^u_{\text{icp}}(\mathcal{O}^\vee, i)\rvert$	$\lvert\Pi^u_{\text{mis}}(\mathcal{O}^\vee)\rvert$
$[4]$	$Sp(4,\mathbb{R})$	0	$\frac{1}{2}[3,1]$	2	1	0
		1			1	
$[2^2]$	$Sp(4,\mathbb{R})$	0	$\frac{1}{2}[1,1]$	6	2	0
		1			2	
		2			2	
	$Sp(1,1)$	0		1	1	0
$[1^4]$	$Sp(4,\mathbb{R})$	0	$[0,0]$	0	0	0
	$Sp(1,1)$	0		1	1	0
	$Sp(2,0)$	0		1	1	0
	$Sp(0,2)$	0		1	1	0

We now show how this works in Atlas:

```
atlas> set G=SO(3,2)
Variable G: RealForm (overriding previous instance, which had type RealForm)
atlas> G
Value: disconnected split real group with Lie algebra 'so(3,2)'
atlas> set B=all_blocks (G)
Variable B: [Block] (overriding previous instance, which had type [Block])
atlas> B
Value: [Block of 1 elements,Block of 4 elements,Block of 12 elements]
atlas> dual_real_forms (G)
Value: [compact connected real group with Lie algebra 'sp(2)',
connected real group with Lie algebra 'sp(1,1)',
connected split real group with Lie algebra 'sp(4,R)']
```

Atlas output for the orbit $[4]$:

```
atlas> get_packets_from_cells ([4], [3,1]/2, B[2])
Value: (1,[final parameter(x=6,lambda=[3,1]/2,nu=[3,1]/2)])

atlas> get_packets_from_cells ([5], [3,1]/2, B[2])
Value: (1,[final parameter(x=6,lambda=[5,3]/2,nu=[3,1]/2)])
```

Atlas output for the orbit $[2^2]$:

```
atlas> get_packets_from_cells ([2], [1,1]/2, B[2])
Value: (2,[final parameter(x=3,lambda=[1,1]/2,nu=[0,1]/2),
final parameter(x=6,lambda=[5,3]/2,nu=[1,1]/2)])

atlas> get_packets_from_cells ([3], [1,1]/2, B[2])
Value: (2,[final parameter(x=3,lambda=[1,3]/2,nu=[0,1]/2),
```

```
final parameter(x=6,lambda=[3,1]/2,nu=[1,1]/2)])

atlas> get_packets_from_cells ([1], [1,1]/2, B[2])
Value: (2,[final parameter(x=1,lambda=[1,1]/2,nu=[0,0]/1),
final parameter(x=5,lambda=[3,3]/2,nu=[1,1]/2)])

atlas> get_packets_from_cells ([1], [1,1]/2, B[1])
Value: (1,[final parameter(x=5,lambda=[3,3]/2,nu=[1,1]/1)])
```

Atlas output for $[1^4]$:

```
atlas> get_packets_from_cells ([0], [0,0], B[0])
Value: (1,[final parameter(x=6,lambda=[5,3]/2,nu=[0,0]/1)])

atlas> get_packets_from_cells ([0], [0,0], B[1])
Value: (1,[final parameter(x=2,lambda=[1,-1]/2,nu=[0,0]/1)])

atlas> all_parameters_gamma (G, [0,0])
Value: [final parameter(x=6,lambda=[3,1]/2,nu=[0,0]/1),
final parameter(x=6,lambda=[5,3]/2,nu=[0,0]/1),
final parameter(x=2,lambda=[1,-1]/2,nu=[0,0]/1)]
```

In the last piece of output, we want to point out the following:

1. using the *get_packets_from_cells* command on the input $([0],[0,0],B[2])$ leads to an error (error message not printed here), and that is because the discrete series for $SO(3,2)$ at infinitesimal character $[3/2,1/2]$ cannot be translated to the infinitesimal character $[0,0]$.

2. using the *all_parameters_gamma* command, we can compute the unipotent packet corresponding to the strong real form $Sp(0,2)$ as

   ```
   final parameter(x=6,lambda=[3,1]/2,nu=[0,0]/1).
   ```

9.4 $\mathbf{G}(\mathbb{R}) = SO(4,3)$.

In this case $\mathbf{G}^\vee = Sp(6,\mathbb{C})$ with strong real forms $Sp(6,\mathbb{R})$, $Sp(2,1)$, $Sp(1,2)$, $Sp(0,3)$, and $Sp(3,0)$. Here is some basic information about blocks and cells from Atlas.

```
atlas> set G=SO(4,3)
Variable G: RealForm (overriding previous instance, which had type RealForm)
atlas> G
Value: disconnected split real group with Lie algebra 'so(4,3)'

atlas> dual_real_forms(G)
Value: [compact connected real group with Lie algebra 'sp(3)',
connected real group with Lie algebra 'sp(2,1)',
connected split real group with Lie algebra 'sp(6,R)']

atlas> block_cell_info (G)
Value: ([Block of 1 elements,Block of 9 elements,Block of 53 elements],[1,3,16])
```

so that

1. $|\mathcal{B}(Sp(6,\mathbb{R}))| = 53$, $|\mathbf{C}(Sp(6,\mathbb{R}))| = 16$.
2. $|\mathcal{B}(Sp(2,1))| = 9$, $|\mathbf{C}(Sp(1,1))| = 3$.
3. $|\mathcal{B}(Sp(3,0)| = 1$, $|\mathbf{C}(Sp(3,0))| = 1$.
4. $|\mathcal{B}(Sp(0,3)| = 1$, $|\mathbf{C}(Sp(0,3))| = 1$, we will use the parameters from the block coming from $Sp(3,0)$ to compute the unipotent representations in this block.

Following is the basic information about associated varieties for cells:

$\mathcal{O}^\vee$	$\mathbf{G}(\mathbb{R})^\vee$	$\mathbf{C}^\vee(\mathbf{G}(\mathbb{R})^\vee, \mathcal{O}^\vee)$	$\mathbf{C}(\mathcal{O}^\vee)$	$\mathbf{C}^\vee_{\text{irr},0}(\mathcal{O}^\vee)$	$\mathbf{C}_{\text{irr},0}(\mathcal{O}^\vee)$	$\mathbf{C}^\vee_{\text{mis}}(\mathcal{O}^\vee)$
$[6]$	$Sp(6,\mathbb{R})$	$\{0,1\}$	$\{14,15\}$	$\{0,1\}$	$\{14,15\}$	$\varnothing$
$[4,2]$	$Sp(6,\mathbb{R})$	$\{2,3,4,6\}$	$\{13,9,12,11\}$	$\{2,3,4,6\}$	$\{13,9,12,11\}$	$\varnothing$
$[3^2]$	$Sp(6,\mathbb{R})$	$\{8\}$	$\{7\}$	$\{8\}$	$\{7\}$	$\varnothing$
	$Sp(2,1)$	$\{0\}$	$\{2\}$	$\{0\}$	$\{2\}$	$\varnothing$
$[2^3]$	$Sp(6,\mathbb{R})$	$\{5,7,9,10,11\}$	$\{8,10,5,6,4\}$	$\{5,7,9,10\}$	$\{8,10,5,6\}$	$\{4\}$
$[1^6]$	$Sp(6,\mathbb{R})$	$\{15\}$	$\{0\}$	$\{15\}$	$\{0\}$	$\varnothing$
	$Sp(2,1)$	$\{2\}$	$\{0\}$	$\{2\}$	$\{0\}$	$\varnothing$
	$Sp(3,0)$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\varnothing$

The following is the information about real forms of even nilpotent orbits of $Sp(6,\mathbb{C})$:

$\mathcal{O}^\vee$	$\mathbf{G}(\mathbb{R})^\vee$	#real-forms in $\mathbf{G}(\mathbb{R})^\vee$
$[6]$	$Sp(6,\mathbb{R})$	2
$[4,2]$	$Sp(6,\mathbb{R})$	4
$[3^2]$	$Sp(6,\mathbb{R})$	1
	$Sp(2,1)$	1
$[2^3]$	$Sp(6,\mathbb{R})$	4
$[1^6]$	$Sp(6,\mathbb{R})$	1
	$Sp(2,1)$	1
	$Sp(3,0)$	1

The following is the information about real forms of even nilpotent orbits and associated varieties of cells in terms of Atlas output:

$\mathcal{O}^\vee$	$\mathbf{G}(\mathbb{R})^\vee$	$\mathcal{S}(\mathbf{G}(\mathbb{R})^\vee, \mathcal{O}^\vee)$ indices	$\mathcal{R}_{\mathfrak{p}_i^\vee}(\chi_{ij})$ in dual block	$\mathbf{C}^\vee_{\mathrm{irr},0}(\mathbf{G}(\mathbb{R})^\vee, \mathcal{O}^\vee, i)$	$\mathbf{C}_{\mathrm{irr},0}(\mathbf{G}(\mathbb{R})^\vee, \mathcal{O}^\vee, i)$
$[6]$	$Sp(6,\mathbb{R})$	0	$\{0\}$	$\{0\}$	$\{14\}$
		1	$\{4\}$	$\{1\}$	$\{15\}$
$[4,2]$	$Sp(6,\mathbb{R})$	0	$\{1\}$	$\{2\}$	$\{13\}$
		1	$\{3\}$	$\{4\}$	$\{12\}$
		2	$\{10\}$	$\{3\}$	$\{9\}$
		3	$\{11\}$	$\{6\}$	$\{11\}$
$[3^2]$	$Sp(6,\mathbb{R})$	0	$\{16\}$	$\{8\}$	$\{7\}$
	$Sp(2,1)$	0	$\{2\}$	$\{0\}$	$\{2\}$
$[2^3]$	$Sp(6,\mathbb{R})$	0	$\{5\}$	$\{5\}$	$\{8\}$
		1	$\{7\}$	$\{7\}$	$\{10\}$
		2	$\{17\}$	$\{9\}$	$\{5\}$
		3	$\{18\}$	$\{10\}$	$\{6\}$
$[1^6]$	$Sp(6,\mathbb{R})$	0	$\{50\}$	$\{15\}$	$\{0\}$
	$Sp(2,1)$	0	$\{8\}$	$\{2\}$	$\{0\}$
	$Sp(3,0)$	0	$\{0\}$	$\{0\}$	$\{0\}$

Here is the information about the cardinalities of the special unipotent packets:

$\mathcal{O}^\vee$	$\mathbf{G}(\mathbb{R})^\vee$	$\mathcal{S}(\mathbf{G}(\mathbb{R})^\vee, \mathcal{O}^\vee)$	$\frac{1}{2}H^\vee$	$\lvert\Pi^u_{\text{weak}}(\mathcal{O}^\vee)\rvert$	$\lvert\Pi^u_{\text{icp}}(\mathcal{O}^\vee, i)\rvert$	$\lvert\Pi^u_{\text{mis}}(\mathcal{O}^\vee)\rvert$
$[6]$	$Sp(6,\mathbb{R})$	0	$\frac{1}{2}[5,3,1]$	2	1	0
		1			1	
$[4,2]$	$Sp(6,\mathbb{R})$	0	$\frac{1}{2}[3,1,1]$	8	2	0
		1			2	
		2			2	
		3			2	
$[3^2]$	$Sp(6,\mathbb{R})$	0	$[1,1,0]$	0	0	0
	$Sp(2,1)$	0		1	1	0
	$Sp(1,2)$	0		1	1	0
$[2^3]$	$Sp(6,\mathbb{R})$	0	$\frac{1}{2}[1,1,1]$	5	1	1
		1			1	
		2			1	
		3			1	
$[1^6]$	$Sp(6,\mathbb{R})$	0	$[0,0,0]$	0	0	0
	$Sp(2,1)$	0		1	1	1
	$Sp(1,2)$	0		1	1	1
	$Sp(3,0)$	0		1	1	1
	$Sp(0,3)$	0		1	1	1

Following is the Atlas output with the unipotent parameters that we are interested in:

```
atlas> get_packets_from_cells ([14], [5,3,1]/2, B[2])
Value: (1,[final parameter(x=24,lambda=[5,3,1]/2,nu=[5,3,1]/2)])
atlas> get_packets_from_cells ([15], [5,3,1]/2, B[2])
Value: (1,[final parameter(x=24,lambda=[7,5,3]/2,nu=[5,3,1]/2)])


atlas> get_packets_from_cells ([13], [3,1,1]/2, B[2])
Value: (2,[final parameter(x=18,lambda=[5,1,3]/2,nu=[3,0,1]/2),
final parameter(x=24,lambda=[5,3,1]/2,nu=[3,1,1]/2)])

atlas> get_packets_from_cells ([12], [3,1,1]/2, B[2])
Value: (2,[final parameter(x=18,lambda=[7,1,1]/2,nu=[3,0,1]/2),
final parameter(x=24,lambda=[7,5,3]/2,nu=[3,1,1]/2)])

atlas> get_packets_from_cells ([9], [3,1,1]/2, B[2])
Value: (2,[final parameter(x=15,lambda=[7,1,1]/2,nu=[3,0,0]/2),
final parameter(x=22,lambda=[7,3,3]/2,nu=[3,1,1]/2)])

atlas> get_packets_from_cells ([11], [3,1,1]/2, B[2])
Value: (2,[final parameter(x=15,lambda=[5,1,1]/2,nu=[3,0,0]/2),
final parameter(x=22,lambda=[5,3,3]/2,nu=[3,1,1]/2)])
```

```
atlas> get_packets_from_cells ([2], [1,1,0], B[1])
Value: (1,[final parameter(x=24,lambda=[5,3,1]/2,nu=[1,1,0]/1)])


atlas> get_packets_from_cells ([8], [1,1,1]/2, B[2])
Value: (1,[final parameter(x=24,lambda=[5,3,1]/2,nu=[1,1,1]/2)])
atlas> get_packets_from_cells ([10], [1,1,1]/2, B[2])
Value: (1,[final parameter(x=24,lambda=[7,5,3]/2,nu=[1,1,1]/2)])
atlas> get_packets_from_cells ([5], [1,1,1]/2, B[2])
Value: (1,[final parameter(x=19,lambda=[3,5,3]/2,nu=[1,1,1]/2)])
atlas> get_packets_from_cells ([6], [1,1,1]/2, B[2])
Value: (1,[final parameter(x=19,lambda=[3,3,3]/2,nu=[1,1,1]/2)])

atlas> get_packets_from_cells ([4], [1,1,1]/2, B[2])
Value: (1,[final parameter(x=11,lambda=[1,3,3]/2,nu=[0,1,1]/2)])


atlas> get_packets_from_cells ([0], [0,0,0], B[1])
Value: (1,[final parameter(x=7,lambda=[1,-1,3]/2,nu=[0,0,0]/1)])
atlas> get_packets_from_cells ([0], [0,0,0], B[0])
Value: (1,[final parameter(x=24,lambda=[7,5,3]/2,nu=[0,0,0]/1)])
```

9.5 $\mathbf{G}(\mathbb{R}) = SO(5,4)$.

In this case $\mathbf{G}^\vee = Sp(8,\mathbb{C})$ with strong real forms $Sp(8,\mathbb{R})$, $Sp(2,2)$, $Sp(3,1)$, $Sp(1,3)$, $Sp(4,0)$ and $Sp(0,4)$. Here is some basic information about blocks and cells from Atlas.

```
atlas> set G=SO(5,4)
Variable G: RealForm (overriding previous instance, which had type RealForm)
atlas> G
Value: disconnected split real group with Lie algebra 'so(5,4)'
atlas> set B=all_blocks(G)
Variable B: [Block] (overriding previous instance, which had type [Block])
atlas> B
Value: [Block of 1 elements,Block of 16 elements,
Block of 42 elements,Block of 258 elements]
atlas> dual_real_forms(G)
Value: [compact connected real group with Lie algebra 'sp(4)',
connected real group with Lie algebra 'sp(3,1)',
connected real group with Lie algebra 'sp(2,2)',
connected split real group with Lie algebra 'sp(8,R)']

atlas> block_cell_info(G)
Value: ([Block of 1 elements,Block of 16 elements,
Block of 42 elements,Block of 258 elements],[1,3,6,35])
```

so that

1. $|\mathcal{B}(Sp(8,\mathbb{R}))| = 258$, $|\mathbf{C}(Sp(8,\mathbb{R}))| = 35$.
2. $|\mathcal{B}(Sp(2,2))| = 42$, $|\mathbf{C}(Sp(1,1))| = 6$.
3. $|\mathcal{B}(Sp(3,1)| = 16$, $|\mathbf{C}(Sp(3,0))| = 3$.
4. $|\mathcal{B}(Sp(4,0)| = 1$, $|\mathbf{C}(Sp(0,3))| = 1$.

We will need to compute unipotent representations in blocks corresponding to strong real forms $Sp(1,3)$ and $Sp(0,4)$, to do this we will use parameters from blocks corresponding to $Sp(3,1)$ and $Sp(4,0)$ respectively.

Following is the basic information about associated varieties for cells:

$\mathcal{O}^\vee$	$\mathbf{G}(\mathbb{R})^\vee$	$\mathbf{C}^\vee(\mathbf{G}(\mathbb{R})^\vee, \mathcal{O}^\vee)$	$\mathbf{C}(\mathcal{O}^\vee)$	$\mathbf{C}^\vee_{\text{irr},0}(\mathcal{O}^\vee)$	$\mathbf{C}_{\text{irr},0}(\mathcal{O}^\vee)$	$\mathbf{C}^\vee_{\text{mis}}(\mathcal{O}^\vee)$
$[8]$	$Sp(8,\mathbb{R})$	$\{0,1\}$	$\{33,34\}$	$\{0,1\}$	$\{33,34\}$	$\varnothing$
$[6,2]$	$Sp(8,\mathbb{R})$	$\{2,3,5,8\}$	$\{26,30,25,32\}$	$\{3,8,5,2\}$	$\{26,30,25,32\}$	$\varnothing$
$[4,4]$	$Sp(8,\mathbb{R})$	$\{4,6,10\}$	$\{29,24,31\}$	$\{4,10,6\}$	$\{29,24,31\}$	$\varnothing$
	$Sp(2,2)$	$\{0\}$	$\{5\}$	$\{0\}$	$\{5\}$	
$[4,2^2]$	$Sp(8,\mathbb{R})$	$\{7,9,11,12,14,$ $15,17,18,23\}$	$\{22,23,27,15,28,$ $19,20,21,10\}$	$\{7,9,11,14,$ $17,18\}$	$\{22,23,27,28,$ $20,21\}$	$\{15,19,10\}$
$[3^2,1^2]$	$Sp(8,\mathbb{R})$	$\{28\}$	$\{11\}$	$\{28\}$	$\{11\}$	$\varnothing$
	$Sp(2,2)$	$\{1,2\}$	$\{3,4\}$	$\{1,2\}$	$\{3,4\}$	$\varnothing$
	$Sp(3,1)$	$\{0\}$	$\{2\}$	$\{0\}$	$\{2\}$	$\varnothing$
$[2^4]$	$Sp(8,\mathbb{R})$	$\{13,16,25,$ $27,29,30\}$	$\{13,17,6,$ $7,5,4\}$	$\{13,16,25,$ $27,29\}$	$\{13,17,6$ $7,5\}$	$\{4\}$
	$Sp(2,2)$	$\{3\}$	$\{2\}$	$\{3\}$	$\{2\}$	$\varnothing$
$[1^8]$	$Sp(8,\mathbb{R})$	$\{34\}$	$\{0\}$	$\{34\}$	$\{0\}$	$\varnothing$
	$Sp(2,2)$	$\{5\}$	$\{0\}$	$\{5\}$	$\{0\}$	$\varnothing$
	$Sp(3,1)$	$\{2\}$	$\{0\}$	$\{2\}$	$\{0\}$	$\varnothing$
	$Sp(4,0)$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\varnothing$

Here are the tables outlining the number of real forms of a given even complex nilpotent orbit:

$\mathcal{O}^\vee$	$\mathbf{G}(\mathbb{R})^\vee$	#real-forms in $\mathbf{G}(\mathbb{R})^\vee$
$[8]$	$Sp(8,\mathbb{R})$	2
$[6,2]$	$Sp(8,\mathbb{R})$	4
$[4,4]$	$Sp(8,\mathbb{R})$	3
	$Sp(2,2)$	1
$[4,2^2]$	$Sp(8,\mathbb{R})$	6
$[3^2,1^2]$	$Sp(8,\mathbb{R})$	1
	$Sp(2,2)$	2
	$Sp(3,1)$	1
$[2^4]$	$Sp(8,\mathbb{R})$	5
	$Sp(2,2)$	1
$[1^8]$	$Sp(8,\mathbb{R})$	1
	$Sp(2,2)$	1
	$Sp(3,1)$	1
	$Sp(4,0)$	1

The following is the information about real forms of even nilpotent orbits and associated varieties of cells in terms of Atlas output:

$\mathcal{O}^\vee$	$\mathbf{G}(\mathbb{R})^\vee$	$\mathcal{S}(\mathbf{G}(\mathbb{R})^\vee,\mathcal{O}^\vee)$ indices	$\mathcal{R}_{\mathfrak{p}_i^\vee}(\chi_{ij})$ in dual block	$\mathbf{C}^\vee_{\mathrm{irr},0}(\mathbf{G}(\mathbb{R})^\vee,\mathcal{O}^\vee,i)$	$\mathbf{C}_{\mathrm{irr},0}(\mathbf{G}(\mathbb{R})^\vee,\mathcal{O}^\vee,i)$
$[8]$	$Sp(8,\mathbb{R})$	0	$\{0\}$	$\{0\}$	$\{33\}$
		1	$\{5\}$	$\{1\}$	$\{34\}$
$[6,2]$	$Sp(8,\mathbb{R})$	0	$\{4\}$	$\{3\}$	$\{30\}$
		1	$\{10\}$	$\{8\}$	$\{32\}$
		2	$\{24\}$	$\{5\}$	$\{25\}$
		3	$\{25\}$	$\{2\}$	$\{26\}$
$[4^2]$	$Sp(8,\mathbb{R})$	0	$\{2\}$	$\{4\}$	$\{29\}$
		1	$\{9\}$	$\{10\}$	$\{31\}$
		2	$\{36\}$	$\{6\}$	$\{24\}$
	$Sp(2,2)$	0	$\{12\}$	$\{0\}$	$\{5\}$
$[4,2^2]$	$Sp(8,\mathbb{R})$	0	$\{6\}$	$\{7\}$	$\{22\}$
		1	$\{8\}$	$\{9\}$	$\{23\}$
		2	$\{45\}$	$\{17\}$	$\{20\}$
		3	$\{46\}$	$\{18\}$	$\{21\}$
		4	$\{47\}$	$\{11\}$	$\{27\}$
		5	$\{48\}$	$\{14\}$	$\{28\}$
$[3^2,1^2]$	$Sp(8,\mathbb{R})$	0	$\{93\}$	$\{28\}$	$\{11\}$
	$Sp(2,2)$	0	$\{2\}$	$\{1\}$	$\{3\}$
		1	$\{5\}$	$\{2\}$	$\{4\}$
	$Sp(3,1)$	0	$\{9\}$	$\{0\}$	$\{2\}$
$[2^4]$	$Sp(8,\mathbb{R})$	0	$\{13\}$	$\{13\}$	$\{13\}$
		1	$\{15\}$	$\{16\}$	$\{17\}$
		2	$\{87\}$	$\{25\}$	$\{6\}$
		3	$\{88\}$	$\{27\}$	$\{7\}$
		4	$\{97\}$	$\{29\}$	$\{5\}$
	$Sp(2,2)$	0	$\{25\}$	$\{3\}$	$\{2\}$
$[1^8]$	$Sp(8,\mathbb{R})$	0	$\{252\}$	$\{34\}$	$\{0\}$
	$Sp(2,2)$	0	$\{41\}$	$\{5\}$	$\{0\}$
	$Sp(3,1)$	0	$\{15\}$	$\{2\}$	$\{0\}$
	$Sp(4,0)$	0	$\{0\}$	$\{0\}$	$\{0\}$

Here is the information of the cardinalities of unipotents packets we are interested in:

$\mathcal{O}^\vee$	$\mathbf{G}(\mathbb{R})^\vee$	$\mathcal{S}(\mathbf{G}(\mathbb{R})^\vee, \mathcal{O}^\vee)$	$\frac{1}{2}H^\vee$	$\lvert\Pi^u_{\text{weak}}(\mathcal{O}^\vee)\rvert$	$\lvert\Pi^u_{\text{icp}}(\mathcal{O}^\vee, i)\rvert$	$\lvert\Pi^u_{\text{mis}}(\mathcal{O}^\vee)\rvert$
$[8]$	$Sp(8,\mathbb{R})$	0	$\frac{1}{2}[7,5,3,1]$	2	1	0
		1			1	
$[6,2]$	$Sp(8,\mathbb{R})$	0	$\frac{1}{2}[5,3,1,1]$	8	2	0
		1			2	
		2			2	0
		3			2	0
$[4^2]$	$Sp(8,\mathbb{R})$	0	$\frac{1}{2}[3,3,1,1]$	6	2	0
		1			2	
		2			2	
	$Sp(2,2)$	0		1	1	0
$[4,2^2]$	$Sp(8,\mathbb{R})$	0	$\frac{1}{2}3,1,1,1]$	9	1	3
		1			1	
		2			1	
		3			1	
		4			1	
		5			1	
$[3^2,1^2]$	$Sp(8,\mathbb{R})$	0	$[2,2,0,0]$	0	0	0
	$Sp(2,2)$	0		2	1	0
		1			1	
	$Sp(3,1)$	0		1	1	0
	$Sp(1,3)$	0		1	1	0
$[2^4]$	$Sp(8,\mathbb{R})$	0	$\frac{1}{2}[1,1,1,1]$	12	2	2
		1			2	
		2			2	
		3			2	
		4			2	
$[1^6]$	$Sp(8,\mathbb{R})$	0	$[0,0,0]$	0	0	0
	$Sp(2,2)$	0		1	1	0
	$Sp(3,1)$	0		1	1	0
	$Sp(1,3)$	0		1	1	0
	$Sp(4,0)$	0		1	1	0
	$Sp(0,4)$	0		1	1	0

9.6 $\mathbf{G}(\mathbb{R}) = Sp(4, \mathbb{R})$.

In this case $\mathbf{G}^\vee = SO(5, \mathbb{C})$, with dual real forms $SO(3,2)$, $SO(4,1)$ and $SO(5)$. Following is the output from Atlas about blocks and cells:

```
atlas> set G=Sp(4,R)
Variable G: RealForm (overriding previous instance, which had type RealForm)
atlas> set B=all_blocks(G)
Variable B: [Block] (overriding previous instance, which had type [Block])
atlas> G
Value: connected split real group with Lie algebra 'sp(4,R)'
atlas> B
Value: [Block of 1 elements,Block of 5 elements,Block of 12 elements]
atlas> block_cell_info (G)
Value: ([Block of 1 elements,Block of 5 elements,Block of 12 elements],[1,3,6])
```

so that

1. $|\mathcal{B}(SO(3,2))| = 12$, $|\mathbf{C}(SO(3,2))| = 6$.
2. $|\mathcal{B}(SO(4,1))| = 5$, $|\mathbf{C}(SO(4,1))| = 3$.
3. $|\mathcal{B}(SO(5))| = 1$, $|\mathbf{C}(SO(5))| = 1$.

Following is the basic information about associated varieties of cells:

$\mathcal{O}^\vee$	$\mathbf{G}(\mathbb{R})^\vee$	$\mathbf{C}^\vee(\mathbf{G}(\mathbb{R})^\vee, \mathcal{O}^\vee)$	$\mathbf{C}(\mathcal{O}^\vee)$	$\mathbf{C}^\vee_{\text{irr},0}(\mathcal{O}^\vee)$	$\mathbf{C}_{\text{irr},0}(\mathcal{O}^\vee)$	$\mathbf{C}^\vee_{\text{mis}}(\mathcal{O}^\vee)$
$[5]$	$SO(3,2)$	$\{0\}$	$\{5\}$	$\{0\}$	$\{5\}$	$\varnothing$
$[3,1^2]$	$SO(3,2)$	$\{1,2,3\}$	$\{4,2,3\}$	$\{1,2,3\}$	$\{4,2,3\}$	$\varnothing$
	$SO(4,1)$	$\{0\}$	$\{2\}$	$\{0\}$	$\{2\}$	$\varnothing$
$[1^5]$	$SO(3,2)$	$\{4,5\}$	$\{0,1\}$	$\{4,5\}$	$\{0,1\}$	$\varnothing$
	$SO(4,1)$	$\{1,2\}$	$\{0,1\}$	$\{1,2\}$	$\{0,1\}$	$\varnothing$
	$SO(5)$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\varnothing$

Here is information about the number of real forms of even nilpotent orbits for $SO(5, \mathbb{C})$.

$\mathcal{O}^\vee$	$\mathbf{G}(\mathbb{R})^\vee$	#real-forms in $\mathbf{G}(\mathbb{R})^\vee$
$[5]$	$SO(3,2)$	1
$[3,1^2]$	$SO(3,2)$	2
	$SO(4,1)$	1
$[1^5]$	$SO(3,2)$	1
	$SO(4,1))$	1
	$SO(5)$	1

Here is information of real forms of even nilpotent orbit of dual group in terms the results from this paper:

$\mathcal{O}^\vee$	$\mathbf{G}(\mathbb{R})^\vee$	$\mathcal{S}(\mathbf{G}(\mathbb{R})^\vee, \mathcal{O}^\vee)$ indices	$\mathcal{R}_{\mathfrak{p}_i^\vee}(\chi_{ij})$ in dual block	$\mathbf{C}^\vee_{\mathrm{irr},0}(\mathbf{G}(\mathbb{R})^\vee, \mathcal{O}^\vee, i)$	$\mathbf{C}_{\mathrm{irr},0}(\mathbf{G}(\mathbb{R})^\vee, \mathcal{O}^\vee, i)$
$[5]$	$SO(3,2)$	0	$\{0\}$	$\{0\}$	$\{5\}$
$[3,1^2]$	$SO(3,2)$	0	$\{1\}$	$\{1\}$	$\{4\}$
		1	$\{3,4\}$	$\{2,3\}$	$\{2,3\}$
	$SO(4,1)$	0	$\{1\}$	$\{0\}$	$\{2\}$
$[1^5]$	$SO(3,2)$	0	$\{8,9\}$	$\{4,5\}$	$\{0,1\}$
	$SO(4,1)$	0	$\{3,4\}$	$\{1,2\}$	$\{0,1\}$
	$SO(5)$	0	$\{0\}$	$\{0\}$	$\{0\}$

The cardinalities of unipotent sets that we are interested in:

$\mathcal{O}^\vee$	$\mathbf{G}(\mathbb{R})^\vee$	$\mathcal{S}(\mathbf{G}(\mathbb{R})^\vee, \mathcal{O}^\vee)$	$\frac{1}{2}H^\vee$	$\lvert\Pi^u_{\mathrm{weak}}(\mathcal{O}^\vee)\rvert$	$\lvert\Pi^u_{\mathrm{icp}}(\mathcal{O}^\vee, i)\rvert$	$\lvert\Pi^u_{\mathrm{mis}}(\mathcal{O}^\vee)\rvert$
$[5]$	$SO(3,2)$	0	$[2,1]$	1	1	0
$[3,1^2]$	$SO(3,2)$	0	$[1,0]$	6	2	0
		1			4	
	$SO(4,1)$	0		2	2	0
$[1^5]$	$SO(3,2)$	0	$[0,0]$	2	2	0
	$SO(4,1))$	0		2	2	0
	$SO(5)$	0		1	1	0

9.7 $\mathbf{G}(\mathbb{R}) = Sp(6,\mathbb{R})$.

In this case $\mathbf{G}^\vee = SO(7,\mathbb{C})$, with dual real forms $SO(4,3)$, $SO(5,2)$, $SO(6,1)$, and $SO(7)$. Following is the output from Atlas about blocks and cells:

```
atlas> set G=Sp(6,R)
Variable G: RealForm (overriding previous instance, which had type RealForm)
atlas> G
Value: connected split real group with Lie algebra 'sp(6,R)'
atlas> set B=all_blocks(G)
Variable B: [Block] (overriding previous instance, which had type [Block])
atlas> B
Value: [Block of 1 elements,Block of 7 elements,
Block of 27 elements,Block of 53 elements]
atlas> block_cell_info (G)
Value: ([Block of 1 elements,Block of 7 elements,
Block of 27 elements,Block of 53 elements],[1,3,8,16])
```

so that

1. $|\mathcal{B}(SO(4,3))| = 53$, $|\mathbf{C}(SO(4,3))| = 16$.
2. $|\mathcal{B}(SO(5,2))| = 27$, $|\mathbf{C}(SO(4,1))| = 8$.
3. $|\mathcal{B}(SO(6,1))| = 7$, $|\mathbf{C}(SO(6,1))| = 3$.
4. $|\mathcal{B}(SO(7))| = 1$, $|\mathbf{C}(SO(7))| = 1$.

Following is the basic information about associated varieties of cells:

$\mathcal{O}^\vee$	$\mathbf{G}(\mathbb{R})^\vee$	$\mathbf{C}^\vee(\mathbf{G}(\mathbb{R})^\vee, \mathcal{O}^\vee)$	$\mathbf{C}(\mathcal{O}^\vee)$	$\mathbf{C}^\vee_{\text{irr},0}(\mathcal{O}^\vee)$	$\mathbf{C}_{\text{irr},0}(\mathcal{O}^\vee)$	$\mathbf{C}^\vee_{\text{mis}}(\mathcal{O}^\vee)$
$[7]$	$SO(4,3)$	$\{0\}$	$\{15\}$	$\{0\}$	$\{15\}$	$\varnothing$
$[5,1^2]$	$SO(4,3)$	$\{1,2,3\}$	$\{14,12,13\}$	$\{2,3,1\}$	$\{14,12,13\}$	$\varnothing$
	$SO(5,2)$	$\{0\}$	$\{7\}$	$\{0\}$	$\{7\}$	$\varnothing$
$[3^2,1]$	$SO(4,3)$	$\{4,5,6,8,10\}$	$\{11,9,10,5,7\}$	$\{4,5,6\}$	$\{11,9,10\}$	$\{5,7\}$
	$SO(5,2)$	$\{2,3\}$	$\{5,6\}$	$\{2,3\}$	$\{5,6\}$	$\varnothing$
$[3,1^4]$	$SO(4,3)$	$\{9,11,12,13\}$	$\{3,6,4,2\}$	$\{9,11,12,13\}$	$\{3,6,4,2\}$	$\varnothing$
	$SO(5,2)$	$\{1,4,5\}$	$\{4,2,3\}$	$\{1,4,5\}$	$\{4,2,3\}$	$\varnothing$
	$SO(6,1)$	$\{0\}$	$\{2\}$	$\{0\}$	$\{2\}$	$\varnothing$
$[1^7]$	$SO(4,3)$	$\{14,15\}$	$\{0,1\}$	$\{14,15\}$	$\{0,1\}$	$\varnothing$
	$SO(5,2)$	$\{6,7\}$	$\{0,1\}$	$\{6,7\}$	$\{0,1\}$	$\varnothing$
	$SO(6,1)$	$\{1,2\}$	$\{0,1\}$	$\{1,2\}$	$\{0,1\}$	$\varnothing$
	$SO(5)$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\varnothing$

Here is information of real forms of even nilpotent orbit of dual group in terms the results from this paper:

$Oo^\vee$	$\mathbf{G}(\mathbb{R})^\vee$	$\mathcal{S}(\mathbf{G}(\mathbb{R})^\vee, \mathcal{O}^\vee)$ indices	$\mathcal{R}_{\mathfrak{p}_i^\vee}(\chi_{ij})$ in dual block	$\mathbf{C}^\vee_{\text{irr},0}(\mathbf{G}(\mathbb{R})^\vee, \mathcal{O}^\vee, i)$	$\mathbf{C}_{\text{irr},0}(\mathbf{G}(\mathbb{R})^\vee, \mathcal{O}^\vee, i)$
$[7]$	$SO(4,3)$	0	$\{0\}$	$\{0\}$	$\{15\}$
$[5,1^2]$	$SO(4,3)$	0	$\{5,6\}$	$\{2,3\}$	$\{12,13\}$
		1	$\{7\}$	$\{1\}$	$\{14\}$
	$SO(5,2)$	0	$\{1\}$	$\{0\}$	$\{7\}$
$[3^2,1]$	$SO(4,3)$	0	$\{2\}$	$\{4\}$	$\{11\}$
		1	$\{9,10\}$	$\{5,6\}$	$\{9,10\}$
	$SO(5,2)$	0	$\{5,6\}$	$\{2,3\}$	$\{5,6\}$
$[3,1^4]$	$SO(4,3)$	0	$\{14,15\}$	$\{9,11\}$	$\{3,6\}$
		1	$\{19,20\}$	$\{12,13\}$	$\{4,2\}$
	$SO(5,2)$	0	$\{2\}$	$\{1\}$	$\{4\}$
		1	$\{11,12\}$	$\{4,5\}$	$\{2,3\}$
	$SO(6,1)$	0	$\{3\}$	$\{0\}$	$\{2\}$
$[1^7]$	$SO(4,3)$	0	$\{45,49\}$	$\{14,15\}$	$\{0,1\}$
	$SO(5,2)$	0	$\{23,24\}$	$\{6,7\}$	$\{0,1\}$
	$SO(6,1)$	0	$\{5,6\}$	$\{1,2\}$	$\{0,1\}$
	$SO(7)$	0	$\{0\}$	$\{0\}$	$\{0\}$

The following is information about the cardinalities of unipotent sets computed in this paper:

$\mathcal{O}^\vee$	$\mathbf{G}(\mathbb{R})^\vee$	$\mathcal{S}(\mathbf{G}(\mathbb{R})^\vee, \mathcal{O}^\vee)$	$\frac{1}{2}H^\vee$	$\lvert\Pi^u_{\mathrm{weak}}(\mathcal{O}^\vee)\rvert$	$\lvert\Pi^u_{\mathrm{icp}}(\mathcal{O}^\vee, i)\rvert$	$\lvert\Pi^u_{\mathrm{mis}}(\mathcal{O}^\vee)\rvert$
$[7]$	$SO(4,3)$	0	$[3,2,1]$	1	1	0
$[5,1^2]$	$SO(4,3)$	0	$[2,1,0]$	6	4	0
		1			2	0
	$SO(5,2)$	0		2	2	0
$[3^2,1]$	$SO(4,3)$	0	$[1,1,0]$	5	1	2
		1			2	
	$SO(5,2)$	0		2	2	0
$[3,1^4]$	$SO(4,3)$	0	$[1,0,0]$	8	4	0
		1			4	
	$SO(5,2)$	0		6	2	0
		1			4	
	$SO(6,1)$	0		2	2	0
$[1^7]$	$SO(4,3)$	0	$[0,0,0]$	2	2	0
	$SO(5,2)$	0		2	2	0
	$SO(6,1)$	0		2	2	0
	$SO(7)$	0		1	1	0

9.8 $\mathbf{G}(\mathbb{R}) = Sp(8,\mathbb{R})$.

In this case $\mathbf{G}^\vee = SO(9,\mathbb{C})$, with dual real forms $SO(5,4)$, $SO(6,3)$, $SO(7,2)$, $SO(8,1)$, and $SO(9)$. Following is the output from Atlas about blocks and cells:

```
atlas> set G=Sp(8,R)
Variable G: RealForm (overriding previous instance, which had type RealForm)
atlas> G
Value: connected split real group with Lie algebra 'sp(8,R)'
atlas> dual_real_forms(G)
Value: [compact connected real group with Lie algebra 'so(9)',
disconnected real group with Lie algebra 'so(8,1)',
disconnected real group with Lie algebra 'so(7,2)',
disconnected real group with Lie algebra 'so(6,3)',
disconnected split real group with Lie algebra 'so(5,4)']
atlas> set B=all_blocks(G)
Variable B: [Block] (overriding previous instance, which had type [Block])
atlas> block_cell_info (G)
Value: ([Block of 1 elements,Block of 9 elements,
Block of 48 elements,Block of 144 elements,
Block of 258 elements],[1,3,8,20,35])
```

so that

1. $|\mathcal{B}(SO(5,4))| = 258$, $|\mathbf{C}(SO(5,4))| = 35$.
2. $|\mathcal{B}(SO(6,3))| = 144$, $|\mathbf{C}(SO(6,3))| = 20$.
3. $|\mathcal{B}(SO(7,2))| = 48$, $|\mathbf{C}(SO(7,2))| = 8$.
4. $|\mathcal{B}(SO(8,1))| = 9$, $|\mathbf{C}(SO(8,1))| = 3$.
5. $|\mathcal{B}(SO(9))| = 1$, $|\mathbf{C}(SO(9))| = 1$.

Following is the basic information about associated varieties of cells:

$\mathcal{O}^\vee$	$\mathbf{G}(\mathbb{R})^\vee$	$\mathbf{C}^\vee(\mathbf{G}(\mathbb{R})^\vee, \mathcal{O}^\vee)$	$\mathbf{C}(\mathcal{O}^\vee)$	$\mathbf{C}^\vee_{\mathrm{irr},0}(\mathcal{O}^\vee)$	$\mathbf{C}_{\mathrm{irr},0}(\mathcal{O}^\vee)$	$\mathbf{C}^\vee_{\mathrm{mis}}(\mathcal{O}^\vee)$
$[9]$	$SO(5,4)$	$\{0\}$	$\{34\}$	$\{0\}$	$\{34\}$	$\varnothing$
$[7,1^2]$	$SO(5,4)$	$\{1,2,3\}$	$\{33,31,32\}$	$\{1,2,3\}$	$\{33,31,32\}$	$\varnothing$
	$SO(6,3)$	$\{0\}$	$\{19\}$	$\{0\}$	$\{19\}\varnothing$	
$[5,3,1]$	$SO(5,4)$	$\{4,5,6,7,$ $13,17\}$	$\{30,29,25,27,$ $13,16\}$	$\{5,6,7,4\}$	$\{29,25,27,30\}$	$\{13,16\}$
	$SO(6,3)$	$\{1,2\}$	$\{16,17\}$	$\{1,2\}$	$\{16,17\}$	$\varnothing$
$[5,1^4]$	$SO(5,4)$	$\{8,9,12,16\}$	$\{24,26,21,22\}$	$\{12,16,8,9\}$	$\{21,22,24,26\}$	$\varnothing$
	$SO(6,3)$	$\{4,5,6\}$	$\{18,13,14\}$	$\{5,6,4\}$	$\{13,14,18\}$	$\varnothing$
	$SO(7,2)$	$\{0\}$	$\{7\}$	$\{0\}$	$\{7\}$	$\varnothing$
$[3^3]$	$SO(5,4)$	$\{14,18\}$	$\{19,20\}$	$\{14,18\}$	$\varnothing$	
	$SO(6,3)$	$\{3\}$	$\{15\}$	$\{3\}$	$\{15\}$	$\varnothing$
$[3^2,1^3]$	$SO(5,4)$	$\{10,15,19,20,21,$ $22,23,27,28\}$	$\{23,12,15,17,18,$ $7,9,11,14\}$	$\{10,15,19,$ $20,21\}$	$\{23,12,15,$ $17,18\}$	$\{7,9,$ $11,14\}$
	$SO(6,3)$	$\{7,8,9,10,$ $11,13\}$	$\{9,10,12,11,$ $6,7\}$	$\{7,8,9,10\}$	$\{9,10,12,11\}$	$\{6,7\}$
	$SO(7,2)$	$\{2,3\}$	$\{5,6\}$	$\{2,3\}$	$\{5,6\}$	$\varnothing$
$[3,1^6]$	$SO(5,4)$	$\{25,26,30,32\}$	$\{5,2,3,8\}$	$\{25,26,30,32\}$	$\{5,2,3,8\}$	$\varnothing$
	$SO(6,3)$	$\{12,14,16,17\}$	$\{3,5,4,2\}$	$\{12,14,16,17\}$	$\{3,5,4,2\}$	$\varnothing$
	$SO(7,2)$	$\{1,4,5\}$	$\{4,2,3\}$	$\{1,4,5\}$	$\{4,2,3\}$	$\varnothing$
	$SO(8,1)$	$\{0\}$	$\{2\}$	$\{0\}$	$\{2\}$	$\varnothing$
$[1^9]$	$SO(4,3)$	$\{34,33\}$	$\{0,1\}$	$\{33,34\}$	$\{0,1\}$	$\varnothing$
	$SO(6,3)$	$\{18,19\}$	$\{0,1\}$	$\{18,19\}$	$\{0,1\}$	$\varnothing$
	$SO(7,2)$	$\{6,7\}$	$\{0,1\}$	$\{6,7\}$	$\{0,1\}$	$\varnothing$
	$SO(8,1)$	$\{1,2\}$	$\{0,1\}$	$\{1,2\}$	$\{0,1\}$	$\varnothing$
	$SO(9)$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\varnothing$

Here is information of real forms of even nilpotent orbit of dual group in terms the results from this paper:

$Oo^\vee$	$\mathbf{G}(\mathbb{R})^\vee$	$\mathcal{S}(\mathbf{G}(\mathbb{R})^\vee, \mathcal{O}^\vee)$ indices	$\mathcal{R}_{\mathfrak{p}_i^\vee}(\chi_{ij})$ in dual block	$\mathbf{C}^\vee_{\mathrm{irr},0}(\mathbf{G}(\mathbb{R})^\vee, \mathcal{O}^\vee, i)$	$\mathbf{C}_{\mathrm{irr},0}(\mathbf{G}(\mathbb{R})^\vee, \mathcal{O}^\vee, i)$
$[9]$	$SO(5,4)$	0	$\{0\}$	$\{0\}$	$\{34\}$
$[7,1^2]$	$SO(5,4)$	0	$\{4\}$	$\{1\}$	$\{33\}$
		1	$\{12,13\}$	$\{2,3\}$	$\{31,32\}$
	$SO(6,3)$	0	$\{9\}$	$\{0\}$	$\{19\}$
$[5,3,1]$	$SO(5,4)$	0	$\{3\}$	$\{5\}$	$\{29\}$
		1	$\{14,15\}$	$\{6,7\}$	$\{25,27\}$
		2	$\{21\}$	$\{4\}$	$\{30\}$
	$SO(6,3)$	0	$\{15,16\}$	$\{1,2\}$	$\{16,17\}$
$[5,1^4]$	$SO(5,4)$	0	$\{43,44\}$	$\{12,16\}$	$\{21,22\}$
		1	$\{47,48\}$	$\{8,9\}$	$\{24,26\}$
	$SO(6,3)$	0	$\{21,22\}$	$\{5,6\}$	$13,14\}$
		1	$\{23\}$	$\{4\}$	$\{18\}$
	$SO(7,2)$	0	$\{2\}$	$\{0\}$	$\{7\}$
$[3^3]$	$SO(5,4)$	0	$\{40,41\}$	$\{14,18\}$	$\{19,20\}$
	$SO(6,3)$	0	$\{2\}$	$\{3\}$	$\{15\}$
$[3^2,1^3]$	$SO(5,4)$	0	$\{5\}$	$\{10\}$	$\{23\}$
		1	$\{31,32\}$	$\{15,19\}$	$\{12,15\}$
		2	$\{61,62\}$	$\{20,21\}$	$\{17,18\}$
	$SO(6,3)$	0	$\{28,29\}$	$\{7,8\}$	$\{9,10\}$
		1	$\{32,33\}$	$\{9,10\}$	$\{12,11\}$
	$SO(7,2)$	0	$\{14,15\}$	$\{2,3\}$	$\{5,6\}$
$[3,1^6]$	$SO(5,4)$	0	$\{108,109\}$	$\{25,26\}$	$\{5,2\}$
		1	$\{137,141\}$	$\{30,32\}$	$\{3,8\}$
	$SO(6,3)$	0	$\{40,41\}$	$\{12,14\}$	$\{3,5\}$
		1	$\{85,89\}$	$\{16,17\}$	$\{4,2\}$
	$SO(7,2)$	0	$\{3\}$	$\{1\}$	$\{4\}$
		1	$\{30,31\}$	$\{4,5\}$	$\{2,3\}$
	$SO(8,1)$	0	$\{5\}$	$\{0\}$	$\{2\}$
$[1^9]$	$SO(4,3)$	0	$\{242,247\}$	$\{33,34\}$	$\{0,1\}$
	$SO(6,3)$	0	$\{136,140\}$	$\{18,19\}$	$\{0,1\}$
	$SO(7,2)$	0	$\{44,45\}$	$\{6,7\}$	$\{0,1\}$
	$SO(8,1)$	0	$\{7,8\}$	$\{1,2\}$	$\{0,1\}$
	$SO(9)$	0	$\{0\}$	$\{0\}$	$\{0\}$

The following is information about the cardinalities of unipotent sets computed in this paper:

$\mathcal{O}^\vee$	$\mathbf{G}(\mathbb{R})^\vee$	$\mathcal{S}(\mathbf{G}(\mathbb{R})^\vee, \mathcal{O}^\vee)$	$\frac{1}{2}H^\vee$	$\lvert\Pi^u_{\text{weak}}(\mathcal{O}^\vee)\rvert$	$\lvert\Pi^u_{\text{icp}}(\mathcal{O}^\vee, i)\rvert$	$\lvert\Pi^u_{\text{mis}}(\mathcal{O}^\vee)\rvert$
$[9]$	$SO(5,4)$	0	$[4,3,2,1]$	1	1	0
$[7,1^2]$	$SO(5,4)$	0 1	$[3,2,1,0]$	6	2 4	0
	$SO(6,3)$	0		2	2	0
$[5,3,1]$	$SO(5,4)$	0 1 2	$[2,1,1,0]$	12	2 4 2	4
	$SO(6,3)$	0		4	4	0
$[5,1^4]$	$SO(5,4)$	0 1	$[2,1,0,0]$	8	4 4	0
	$SO(6,3)$	0 1		6	4 2	0
	$SO(7,2)$	0		2	2	0
$[3^3]$	$SO(5,4)$	0	$[1,1,1,0]$	2	2	0
	$SO(6,3)$	0		1	1	0
$[3^2,1^3]$	$SO(5,4)$	0 1 2	$[1,1,0,0]$	9	1 2 2	4
	$SO(6,3)$	0 1		6	2 2	2
	$SO(7,2)$	0		2	2	0
$[3,1^6]$	$SO(5,4)$	0 1	$[1,0,0,0]$	8	4 4	0
	$SO(6,3)$	0 1		8	4 4	0
	$SO(7,2)$	0 1		6	2 4	0
	$SO(8,1)$	0			2	0
$[1^9]$	$SO(5,4)$	0	$[0,0,0,0]$	2	2	0
	$SO(6,3)$	0		2	2	0
	$SO(7,2)$	0		2	2	0
	$SO(8,1)$	0		2	2	0
	$SO(9)$	0		1	1	0

9.9 $\mathbf{G}(\mathbb{R}) = Sp(10, \mathbb{R})$.

In this case $\mathbf{G}^\vee = SO(11, \mathbb{C})$, with dual real forms $SO(6,5)$, $SO(7,4)$, $SO(8,3)$, $SO(9,2)$, $SO(10,1)$, and $SO(11)$. Following is the output from Atlas about blocks and cells:

```
atlas> set G=Sp(10,R)
Variable G: RealForm (overriding previous instance, which had type RealForm)
atlas> G
Value: connected split real group with Lie algebra 'sp(10,R)'
atlas> dual_real_forms(G)
Value: [compact connected real group with Lie algebra 'so(11)',
disconnected real group with Lie algebra 'so(10,1)',
disconnected real group with Lie algebra 'so(9,2)',
disconnected real group with Lie algebra 'so(8,3)',
disconnected real group with Lie algebra 'so(7,4)',
disconnected split real group with Lie algebra 'so(6,5)']

atlas> block_cell_info (G)
Value: ([Block of 1 elements,Block of 11 elements,
Block of 75 elements,Block of 305 elements,
Block of 810 elements,Block of 1342 elements],[1,3,8,20,44,72])
```

so that

1. $|\mathcal{B}(SO(6,5))| = 1342$, $|\mathbf{C}(SO(6,5))| = 72$.
2. $|\mathcal{B}(SO(7,4))| = 810$, $|\mathbf{C}(SO(7,4))| = 44$.
3. $|\mathcal{B}(SO(8,3))| = 305$, $|\mathbf{C}(SO(8,3))| = 20$.
4. $|\mathcal{B}(SO(9,2))| = 75$, $|\mathbf{C}(SO(9,2))| = 8$.
5. $|\mathcal{B}(SO(10,1))| = 11$, $|\mathbf{C}(SO(10,1))| = 3$.
6. $|\mathcal{B}(SO(11))| = 1$, $|\mathbf{C}(SO(11))| = 1$.

Following is the basic information about associated varieties of cells:

$\mathcal{O}^\vee$	$\mathbf{G}(\mathbb{R})^\vee$	$\mathbf{C}^\vee(\mathbf{G}(\mathbb{R})^\vee, \mathcal{O}^\vee)$	$\mathbf{C}(\mathcal{O}^\vee)$	$\mathbf{C}^\vee_{\mathrm{irr},0}(\mathcal{O}^\vee)$	$\mathbf{C}_{\mathrm{irr},0}(\mathcal{O}^\vee)$	$\mathbf{C}^\vee_{\mathrm{mis}}(\mathcal{O}^\vee)$
$[11]$	$SO(6,5)$	$\{0\}$	$\{71\}$	$\{0\}$	$\{71\}$	$\varnothing$
$[9,1^2]$	$SO(6,5)$	$\{1,2,3\}$	$\{70,68,69\}$	$\{1,2,3\}$	$\{70,68,69\}$	$\varnothing$
	$SO(7,4)$	$\{0\}$	$\{43\}$	$\{0\}$	$\{43\}$	$\varnothing$
$[7,3,1]$	$SO(6,5)$	$\{4,5,6,7, 15,17\}$	$\{64,67,65,66, 53,55\}$	$\{4,5,6,7\}$	$\{64,67,65,66\}$	$\{53,55\}$
	$SO(7,4)$	$\{1,2\}$	$\{39,41\}$	$\{1,2\}$	$\{39,41\}$	$\varnothing$
$[7,1^4]$	$SO(6,5)$	$\{12,13,19,23\}$	$\{62,63,56,57\}$	$\{12,13,19,23\}$	$\{62,63,56,57\}$	$\varnothing$
	$SO(7,4)$	$\{5,6,7\}$	$\{42,37,40\}$	$\{5,6,7\}$	$\{42,37,40\}$	$\varnothing$
	$SO(8,3)$	$\{0\}$	$\{19\}$	$\{0\}$	$\{19\}$	$\varnothing$
$[5^2,1]$	$SO(6,5)$	$\{8,9,16,18, 20,24,36,41\}$	$\{59,60,52,54, 49,50,33,35\}$	$\{8,9,16,18\}$	$\{59,60,52,54\}$	$\{49,50,33,35\}$
	$SO(7,4)$	$\{3,15,19\}$	$\{38,34,35\}$	$\{3\}$	$\{38\}$	$\{34,35\}$
$[5,3^2]$	$SO(6,5)$	$\{11,21,25\}$	$\{51,46,48\}$	$\{11,21,25\}$	$\{51,46,48\}$	$\varnothing$
	$SO(7,4)$	$\{4\}$	$\{36\}$	$\{4\}$	$\{36\}$	$\varnothing$
$[5,3,1^3]$	$SO(6,5)$	$\{14,22,26,27, 28,30,31,32, 33,37,42\}$	$\{58,45,47,23, 30,38,39,44, 43,31,34\}$	$\{14,30,31,32, 33,22,26\}$	$\{58,38,39,44, 43,45,47\}$	$\{23,30,31, 34\}$
	$SO(7,4)$	$\{8,9,11,14, 16,18,20\}$	$\{33,27,29,32, 14,31,18\}$	$\{8,9,11, 14,18\}$	$\{33,27,29, 32,31\}$	$\{14,18\}$
	$SO(8,3)$	$\{1,2\}$	$\{16,17\}$	$\{1,2\}$	$\{16,17\}$	$\varnothing$
$[5,1^6]$	$SO(6,5)$	$\{38,43,49,51\}$	$\{40,41,19,27\}$	$\{38,43,49,51\}$	$\{40,41,19,27\}$	$\varnothing$
	$SO(7,4)$	$\{10,12,25,26\}$	$\{26,28,23,24\}$	$\{10,12,25,26\}$	$\{26,28,23,24\}$	$\varnothing$
	$SO(8,3)$	$\{4,5,6\}$	$\{18,13,14\}$	$\{4,5,6\}$	$\{18,13,14\}$	$\varnothing$
	$SO(9,2)$	$\{0\}$	$\{7\}$	$\{0\}$	$\{7\}$	$\varnothing$
$[3^3,1^2]$	$SO(6,5)$	$\{29,35,40,46, 48,56,57\}$	$\{22,21,29,12, 18,24,32\}$	$\{29,35,40, 46,48\}$	$\{22,21,29, 12,18\}$	$\{24,32\}$
	$SO(7,4)$	$\{17,21,23,24\}$	$\{13,17,21,22\}$	$\{17,21,23,24\}$	$\{13,17,21,22\}$	$\varnothing$
	$SO(8,3)$	$\{3\}$	$\{15\}$	$\{3\}$	$\{15\}$	$\varnothing$
$[3^2,1^5]$	$SO(6,5)$	$\{39,44,50,52, 54,54,58,60, 62,64\}$	$\{20,28,11,17, 9,16,13,26, 8,25\}$	$\{39,44,50,52, 54,55\}$	$\{20,28,11,17, 9,16\}$	$\{13,26, 8,25\}$
	$SO(7,4)$	$\{13,27,28,29, 30,31,32, 36,37\}$	$\{25,19,20,12, 16,7,11, 10,15\}$	$\{13,27,28, 29,30\}$	$\{25,19,20, 12,16\}$	$\{7,11,10, 15\}$
	$SO(8,3)$	$\{7,8,9,11, 13,14\}$	$\{9,10,6,7, 11,12\}$	$\{7,8,13,14\}$	$\{9,10,11,12\}$	$\{6,7\}$
	$SO(9,2)$	$\{2,3\}$	$\{5,6\}$	$\{2,3\}$	$\{5,6\}$	$\varnothing$
$[3,1^8]$	$SO(6,5)$	$\{63,65,68,69\}$	$\{3,14,5,2\}$	$\{63,65,68,69\}$	$\{3,14,5,2\}$	$\varnothing$
	$SO(7,4)$	$\{34,35,40,41\}$	$\{5,2,3,8\}$	$\{34,35,40,41\}$	$\{5,2,3,8\}$	$\varnothing$
	$SO(8,3)$	$\{10,12,16,17\}$	$\{3,5,4,2\}$	$\{10,12,16,17\}$	$\{3,5,4,2\}$	$\varnothing$
	$SO(9,2)$	$\{1,4,5\}$	$\{4,2,3\}$	$\{1,4,5\}$	$\{4,2,3\}$	$\varnothing$
	$SO(10,1)$	$\{0\}$	$\{2\}$	$\{0\}$	$\{2\}$	$\varnothing$
$[1^11]$	$SO(6,5)$	$\{70,71\}$	$\{0,1\}$	$\{70,71\}$	$\{0,1\}$	$\varnothing$
	$SO(7,4)$	$\{42,43\}$	$\{0,1\}$	$\{42,43\}$	$\{0,1\}$	$\varnothing$
	$SO(8,3)$	$\{18,19\}$	$\{0,1\}$	$\{18,19\}$	$\{0,1\}$	$\varnothing$
	$SO(9,2)$	$\{6,7\}$	$\{0,1\}$	$\{6,7\}$	$\{0,1\}$	$\varnothing$
	$SO(10,1)$	$\{1,2\}$	$\{0,1\}$	$\{1,2\}$	$\{0,1\}$	$\varnothing$
	$SO(11)$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\varnothing$

Here is information of real forms of even nilpotent orbit of dual group in terms the results from this paper:

$Oo^\vee$	$\mathbf{G}(\mathbb{R})^\vee$	$\mathcal{S}(\mathbf{G}(\mathbb{R})^\vee, \mathcal{O}^\vee)$ indices	$\mathcal{R}_{\mathfrak{p}_i^\vee}(\chi_{ij})$ in dual block	$\mathbf{C}^\vee_{\mathrm{irr},0}(\mathbf{G}(\mathbb{R})^\vee, \mathcal{O}^\vee, i)$	$\mathbf{C}_{\mathrm{irr},0}(\mathbf{G}(\mathbb{R})^\vee, \mathcal{O}^\vee, i)$
$[11]$	$SO(6,5)$	0	$\{0\}$	$\{0\}$	$\{71\}$
$[9,1^2]$	$SO(6,5)$	0	$\{22,23\}$	$\{2,3\}$	$\{68,69\}$
		1	$\{30\}$	$\{1\}$	$\{70\}$
	$SO(7,4)$	0	$\{5\}$	$\{0\}$	$\{43\}$
$[7,3,1]$	$SO(6,5)$	0	$\{4\}$	$\{4\}$	$\{64\}$
		1	$\{45\}$	$\{5\}$	$\{67\}$
		2	$\{47,48\}$	$\{6,7\}$	$\{65,66\}$
	$SO(7,4)$	0	$\{24,25\}$	$\{1,2\}$	$\{39,41\}$
$[7,1^4]$	$SO(6,5)$	0	$\{66,67\}$	$\{12,13\}$	$\{62,63\}$
		1	$\{101,102\}$	$\{19,23\}$	$\{56,57\}$
	$SO(7,4)$	0	$\{8\}$	$\{5\}$	$\{42\}$
		1	$\{79,80\}$	$\{6,7\}$	$\{37,40\}$
	$SO(8,3)$	0	$\{30\}$	$\{0\}$	$\{19\}$
$[5^2,1]$	$SO(6,5)$	0	$\{26,27\}$	$\{16,18\}$	$\{52,54\}$
		1	$\{73,74\}$	$\{8,9\}$	$\{59,60\}$
	$SO(7,4)$	0	$\{3\}$	$\{3\}$	$\{38\}$
$[5,3^2]$	$SO(6,5)$	0	$\{6\}$	$\{11\}$	$\{51\}$
		1	$\{95,96\}$	$\{21,25\}$	$\{46,48\}$
	$SO(7,4)$	0	$\{71\}$	$\{4\}$	$\{36\}$
$[5,3,1^3]$	$SO(6,5)$	0	$\{64,65\}$	$\{30,31\}$	$\{38,39\}$
		1	$\{89\}$	$\{14\}$	$\{58\}$
		2	$\{105,106\}$	$\{32,33\}$	$\{44,43\}$
		3	$\{142,143\}$	$\{22,26\}$	$45,47\}$
	$SO(7,4)$	0	$\{7\}$	$\{8\}$	$\{33\}$
		1	$\{83,84\}$	$\{9,11\}$	$\{27,29\}$
		2	$\{108,109\}$	$\{14,18\}$	$\{32,31\}$
	$SO(8,3)$	0	$\{39,40\}$	$\{1,2\}$	$\{16,17\}$
$[5,1^6]$	$SO(6,5)$	0	$\{352,356\}$	$\{49,51\}$	$\{19,27\}$
		1	$\{360,364\}$	$\{38,43\}$	$\{40,41\}$
	$SO(7,4)$	0	$\{195,196\}$	$\{25,26\}$	$\{23,24\}$
		1	$\{199,200\}$	$\{10,12\}$	$\{26,28\}$
	$SO(8,3)$	0	$\{54,55\}$	$\{5,6\}$	$\{13,14\}$
		1	$\{56\}$	$\{4\}$	$\{18\}$
	$SO(9,2)$	0	$\{3\}$	$\{0\}$	$\{7\}$
$[3^3,1^2]$	$SO(6,5)$	0	$\{9\}$	$\{29\}$	$\{22\}$
		1	$\{168,169\}$	$\{35,40\}$	$\{21,29\}$
		2	$\{230,231\}$	$\{46,48\}$	$\{12,18\}$
	$SO(7,4)$	0	$\{52,53\}$	$\{17,21\}$	$\{13,17\}$
		1	$\{170,171\}$	$\{23,24\}$	$\{21,22\}$
	$SO(8,3)$	0	$\{45\}$	$\{3\}$	$\{15\}$
$[3^2,1^5]$	$SO(6,5)$	0	$\{126,127\}$	$\{39,44\}$	$\{20,28\}$
		1	$\{271,272\}$	$\{50,52\}$	$\{11,17\}$
		2	$\{446,450\}$	$\{54,55\}$	$\{9,16\}$
	$SO(7,4)$	0	$\{9\}$	$\{13\}$	$\{25\}$
		1	$\{242,243\}$	$\{27,28\}$	$\{19,20\}$
		2	$\{258,262\}$	$\{29,30\}$	$\{12,16\}$
	$SO(8,3)$	0	$\{62,63\}$	$\{7,8\}$	$\{9,10\}$
		1	$\{125,129\}$	$\{14,13\}$	$\{12,11\}$
	$SO(9,2)$	0	$\{35,36\}$	$\{2,3\}$	$\{5,6\}$
$[3,1^8]$	$SO(6,5)$	0	$\{773,777\}$	$\{63,65\}$	$\{3,14\}$
		1	$\{883,888\}$	$\{68,69\}$	$\{5,2\}$
	$SO(7,4)$	0	$\{371,372\}$	$\{34,35\}$	$\{5,2\}$
		1	$\{585,590\}$	$\{40,41\}$	$\{3,8\}$
	$SO(8,3)$	0	$\{84,85\}$	$\{10,12\}$	$\{3,5\}$
		1	$\{233,237\}$	$\{16,17\}$	$\{4,2\}$
	$SO(9,2)$	0	$\{4\}$	$\{1\}$	$\{4\}$
		1	$\{57,58\}$	$\{4,5\}$	$\{2,3\}$
	$SO(10,1)$	0	$\{7\}$	$\{0\}$	$\{2\}$
$[1^11]$	$SO(6,5)$	0	$\{1310,1324\}$	$\{70,71\}$	$\{0,1\}$
	$SO(7,4)$	0	$\{749,799\}$	$\{42,43\}$	$\{0,1\}$
	$SO(8,3)$	0	$\{297,301\}$	$\{18,19\}$	$\{0,1\}$
	$SO(9,2)$	0	$\{71,72\}$	$\{6,7\}$	$\{0,1\}$
	$SO(10,1)$	0	$\{9,10\}$	$\{1,2\}$	$\{0,1\}$
	$SO(11)$	0	$\{0\}$	$\{0\}$	$\{0\}$

The following is information about the cardinalities of unipotent sets computed in this paper:

$\mathcal{O}^\vee$	$\mathbf{G}(\mathbb{R})^\vee$	$\mathcal{S}(\mathbf{G}(\mathbb{R})^\vee, \mathcal{O}^\vee)$	$\frac{1}{2}H^\vee$	$\lvert\Pi^u_{\mathrm{weak}}(\mathcal{O}^\vee)\rvert$	$\lvert\Pi^u_{\mathrm{icp}}(\mathcal{O}^\vee, i)\rvert$	$\lvert\Pi^u_{\mathrm{mis}}(\mathcal{O}^\vee)\rvert$
$[11]$	$SO(6,5)$	0	$[5,4,3,2,1]$	1	1	0
$[9,1^2]$	$SO(6,5)$	0 1	$[4,3,2,1,0]$	6	4 2	0
	$SO(7,4)$	0		2	2	0
$[7,3,1]$	$SO(6,5)$	0 1 2	$[3,2,1,1,0]$	8	2 2 4	0
	$SO(7,4)$	0		4	4	0
$[7,1^4]$	$SO(6,5)$	0 1	$[3,2,1,0,0]$	8	4 4	0
	$SO(7,4)$	0 1		6	2 4	0
	$SO(8,3)$	0		2	2	0
$[5^2,1]$	$SO(6,5)$	0 1	$[2,2,1,1,0]$	8	2 2	4
	$SO(7,4)$	0		1	1	0
$[5,3^2]$	$SO(6,5)$	0 1	$[2,1,1,1,0]$	6	2 4	0
	$SO(7,4)$	0		2	2	0
$[5,3,1^3]$	$SO(6,5)$	0 1 2 3	$[2,1,1,0,0]$	22	4 2 4 4	8
	$SO(7,4)$	0 1 2		14	2 4 4	4
	$SO(8,3)$	0		4	4	0
$[5,1^6]$	$SO(6,5)$	0 1	$[2,1,0,0,0]$	8	4 4	0
	$SO(7,4)$	0 1		8	4 4	0
	$SO(8,3)$	0 1		6	2 4	0
	$SO(9,2)$	0		2	2	0
$[3^3,1^2]$	$SO(6,5)$	0 1 2	$[1,1,1,0,0]$	14	2 4 4	4
	$SO(7,4)$	0 1		8	4 4	0
	$SO(8,3)$	0		2	2	0
$[3^2,1^5]$	$SO(6,5)$	0 1 2	$[1,1,0,0,0]$	10	2 2 2	4
	$SO(7,4)$	0 1 2		9	1 2 2	4
	$SO(8,3)$	0 1		6	2 2	2
	$SO(9,2)$	0		2	2	0
$[3,1^8]$	$SO(6,5)$	0 1	$[1,0,0,0,0]$	8	4 4	0
	$SO(7,4)$	0 1		8	4 4	0
	$SO(8,3)$	0 1		8	4 4	0
	$SO(9,2)$	0 1		6	2 4	0
	$SO(10,1)$	0		2	2	0
$[1^9]$	$SO(6,5)$	0	$[0,0,0,0,0]$	2	2	0
	$SO(7,4)$	0		2	2	0
	$SO(8,3)$	0		2	2	0
	$SO(9,2)$	0		2	2	0
	$SO(10,1)$	0		2	2	0
	$SO(11)$	0		1	1	0

Bibliography

[ABV92] Jeffrey Adams, Dan Barbasch, and David A. Vogan, Jr. *The Langlands classification and irreducible characters for real reductive groups*, volume 104 of *Progress in Mathematics*. Birkhuser Boston, Inc., Boston, MA, 1992.

[Ada14] Jeffrey Adams. Atlas of lie groups and representations (web site). 2014. www.liegroups.org.

[AdC09] Jeffrey Adams and Fokko du Cloux. Algorithms for representation theory of real reductive groups. *J. Inst. Math. Jussieu*, 8(2):209–259, 2009.

[Car93] Roger W. Carter. *Finite groups of Lie type*. Wiley Classics Library. John Wiley & Sons, Ltd., Chichester, 1993. Conjugacy classes and complex characters, Reprint of the 1985 original, A Wiley-Interscience Publication.

[CM93] David H. Collingwood and William M. McGovern. *Nilpotent orbits in semisimple Lie algebras*. Van Nostrand Reinhold Mathematics Series. Van Nostrand Reinhold Co., New York, 1993.

[DAV81] Jr. David A. Vogan. *Representations of Real Reductive Lie Groups*. Progress in Mathematics. Birkhauser, 1 edition, 1981.

[FJMN18] Thomas Folz-Donahue, Steven Glenn Jackson, Todor Milev, and Alfred G. Noël. Tau Signatures and Characters of Weyl Groups. *arXiv e-prints*, page arXiv:1810.05556, Oct 2018.

[Vog87] David A. Vogan, Jr. *Unitary representations of reductive Lie groups*, volume 118 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1987.

[Vog91] David A. Vogan, Jr. Associated varieties and unipotent representations. In *Harmonic analysis on reductive groups (Brunswick, ME, 1989)*, volume 101 of *Progr. Math.*, pages 315–388. Birkhäuser Boston, Boston, MA, 1991.

[Yam94] Hiroshi Yamashita. Associated varieties and gelfand-kirillov dimensions for the discrete series of a semisimple lie group. *Proc. Japan Acad. Ser. A Math. Sci.*, 70(2):50–55, 1994.