

Computation of error effects in nonlinear Hamiltonian systems using Lie algebraic methods

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There exist Lie algebraic methods for obtaining transfer maps around any given trajectory of a Hamiltonian system. This paper describes an iterative procedure for finding transfer maps around the same trajectory when the Hamiltonian is perturbed by small linear terms. Such terms often result when an actual system deviates from an ideal one due to errors. Two examples from accelerator physics are worked out. Comparisons with numerical computations, and in simple cases exact analytical calculations, demonstrate the validity of the procedure.

I. INTRODUCTION

In the study of certain Hamiltonian systems, such as arise in light ray optics and charged particle optics, it is often convenient to suppose that there is some known design trajectory. What is then wanted is a description of all the trajectories in some neighborhood of the design trajectory. This is accomplished by introducing canonical deviation coordinates $(q_1 \cdots q_n, p_1 \cdots p_n)$ in such a way that these coordinates are identically zero on the design trajectory. To simplify notation, it is convenient to regard the deviation variables as the $2n$ components of a vector ζ defined by the relation

$$\zeta = (q_1, p_1, q_2, p_2, \dots, q_n, p_n). \quad (1.1)$$

Suppose the Hamiltonian H describing motion for the system in question is expanded in terms of the deviation variables in the form

$$H = H_0 + H_1 + H_2 + H_3 + \cdots. \quad (1.2)$$

Here each H_n is a homogeneous polynomial of degree n in the deviation variables ζ . Then, the constant term H_0 plays no role in the motion, and may be dropped. Moreover, the linear term H_1 vanishes by construction. (If it did not vanish, the design trajectory given by $\zeta_1 = \zeta_2 = \cdots = \zeta_{2n} = 0$ would not be a solution of Hamilton's equations.) Consequently, we need be concerned only with H_2 , which governs paraxial behavior, and with H_3 , H_4 , etc., which govern second- and higher-order aberrations.

Let ζ^{in} denote a set of initial conditions at some initial time $t = t^{\text{in}}$, and let ζ^{fin} denote the corresponding final conditions at the final time $t = t^{\text{fin}}$. Then there are Taylor series relations of the form

$$\begin{aligned} \zeta_a^{\text{fin}} = & \sum_b R_{ab} \zeta_b^{\text{in}} + \sum_{bc} T_{abc} \zeta_b^{\text{in}} \zeta_c^{\text{in}} \\ & + \sum_{bcd} U_{abcd} \zeta_b^{\text{in}} \zeta_c^{\text{in}} \zeta_d^{\text{in}} + \cdots \end{aligned} \quad (1.3)$$

The relation (1.3) between initial and final conditions may be thought of as a "transfer map" that sends ζ^{in} to ζ^{fin} . Note that the expansion (1.3) does not contain constant terms. For if there were constant terms then, contrary to the property of the design trajectory, not all components of ζ^{fin} would vanish when all components of ζ^{in} vanish.

With the above discussion in mind as background, suppose the system under study contains some errors in the sense that its Hamiltonian is not exactly the "ideal" Hamiltonian H imagined at the beginning of our discussion. Let the actual Hamiltonian be K . Then, if the actual Hamiltonian is expanded in terms of the deviation variables ζ for the ideal Hamiltonian, we find an expression of the form

$$K = K_0 + K_1 + K_2 + K_3 + \cdots. \quad (1.4)$$

As before, K_0 plays no role and can be ignored. However, unlike the previous case, K_1 no longer vanishes. Instead, it is proportional to the errors in the system and is therefore only small.

Correspondingly, the transfer map arising from K can be written in the form

$$\begin{aligned} \zeta_a^{\text{fin}} = & c_a + \sum_b R'_{ab} \zeta_b^{\text{in}} + \sum_{bc} T'_{abc} \zeta_b^{\text{in}} \zeta_c^{\text{in}} \\ & + \sum_{bcd} U'_{abcd} \zeta_b^{\text{in}} \zeta_c^{\text{in}} \zeta_d^{\text{in}} + \cdots \end{aligned} \quad (1.5)$$

Now the expansion begins with small constant terms c as a result of the small K_1 term in (1.4). Also, R' , T' , U' , etc., differ slightly from their ideal values R , T , U , etc., in (1.3).

A procedure for finding the transfer map arising from K already exists for the case where K_1 is zero.¹ The purpose of this paper is to describe a procedure for finding the transfer map in the case where K_1 is small. In particular, if K_1 is of order ϵ , we show how to compute the transfer map through terms of order ϵ^m for any (positive) integer m . Our treatment uses Lie algebraic tools, and a familiarity with these tools and their associated notation is assumed.¹⁻⁵

Section II of this paper gives a preliminary discussion of what is to be computed in a Lie algebraic context. In Sec. III we describe the algorithm developed for this purpose. In Secs. IV and V we apply this algorithm to the charged particle optics examples of a steering magnet and a mispowered bending magnet, respectively. The results of these sections are presented in a way that facilitated their incorporation into MARYLIE 3.1, a charged particle beam transport code based on Lie algebraic methods and that includes error effects.^{6,7} In Sec. VI we compare the results of Secs. IV and V for the steering magnet and bending magnet with results obtained by other means. Our conclusions are summarized in a final section.

II. PRELIMINARY DISCUSSION

Let \mathcal{U} denote the transfer map corresponding to (1.5) when the constant terms c are omitted. That is, \mathcal{U} describes the transfer map

$$\xi_a^{\text{fin}} = \sum_b R'_{ab} \xi_b^{\text{in}} + \sum_{bc} T'_{abc} \xi_b^{\text{in}} \xi_c^{\text{in}} + \dots \quad (2.1)$$

This map is symplectic since (1.5) is, and sends the origin into itself. Consequently, it can be written in the reverse factorized Lie algebraic form¹⁻³

$$\mathcal{U} = \dots e^{g_4} e^{g_3} e^{g_2}. \quad (2.2)$$

Next, let \mathcal{T} denote the translation map described by the relation

$$\xi_a^{\text{fin}} = \xi_a^{\text{in}} + c_a. \quad (2.3)$$

Evidently (2.3) is symplectic, and can be written in the Lie form

$$\mathcal{T} = e^{g_1}. \quad (2.4)$$

Finally, let \mathcal{M} denote the transfer map for the full expression (1.5). Upon combining (2.2) and (2.4), and recalling that Lie transformations act from left to right, we conclude that \mathcal{M} can be written in the form

$$\mathcal{M} = \mathcal{U} \mathcal{T} = \dots e^{g_4} e^{g_3} e^{g_2} e^{g_1}. \quad (2.5)$$

For many purposes it is useful to have a representation for \mathcal{M} in the standard factorized form

$$\mathcal{M} = e^{f_1} e^{f_2} e^{f_3} e^{f_4} \dots \quad (2.6)$$

The passage from the factorization (2.5) to the standard factorization (2.6) can be carried out in at least two ways. First, if g_1 is treated as being small of order ϵ , as it is for our discussion, then there is a standard calculus for passing back and forth between the two factorizations. This calculus involves the concept of rank, the introduction of an ideal structure within the Poisson bracket Lie algebra based on rank, and a corresponding quotient Lie algebra.⁴

Second, one can work directly with the maps. Suppose the two factorizations are written in the form

$$\mathcal{M} = \mathcal{U}_l e^{g_1} = e^{f_1} \mathcal{U}_r, \quad (2.7)$$

where \mathcal{U}_l and \mathcal{U}_r denote the left and right factors, $e^{g_4} e^{g_3} e^{g_2}$ and $e^{f_2} e^{f_3} e^{f_4} \dots$, respectively. Now invert the two representations (2.7) for \mathcal{M} , and let them act on the origin. From the first factorization we find the result

$$\mathcal{M}^{-1} 0 = e^{-g_1} \mathcal{U}_l^{-1} 0 = \mathcal{U}_l^{-1} (-c). \quad (2.8)$$

Here we have used the result

$$e^{-g_1} 0 = -c, \quad (2.9)$$

which is a consequence of (2.3) and (2.4). From the second factorization we find the result

$$\mathcal{M}^{-1} 0 = \mathcal{U}_r^{-1} e^{-f_1} 0 = e^{-f_1} 0 = -f_1 \xi. \quad (2.10)$$

Here, and above, we have again used the fact that Lie transformations act from left to right. Comparison of the results (2.8) and (2.10) gives a relation that determines f_1 ,

$$f_1 \xi = -\mathcal{U}_l^{-1} (-c). \quad (2.11)$$

The determination of \mathcal{U}_r , the remaining part of the standard factorization, is now straightforward since \mathcal{U}_r sends the origin into itself.^{2,3} We note that this second derivation shows that the standard factorization exists, provided the origin is in the domain of \mathcal{M}^{-1} or, equivalently, $-c$ is in the domain of \mathcal{U}_l^{-1} . This will be the case if c (or g_1) is sufficiently small.

We are now in a position to state the purpose of this paper more precisely: Given the Hamiltonian K as in (1.4), our goal is to find the transfer map \mathcal{M} corresponding to K in the standard factorized form (2.6). Below we

present a procedure for obtaining the polynomials f_i in a straightforward, algorithmic way directly from the Hamiltonian.

III. COMPUTATIONAL ALGORITHM

We begin by considering the general Hamiltonian, now again called H , expanded in homogeneous polynomials,

$$H(\xi, t) = H_1(\xi, t) + H_2(\xi, t) + H_3(\xi, t) + \cdots \quad (3.1)$$

The transfer map \mathcal{M} generated by H obeys the equation of motion

$$\dot{\mathcal{M}} = \mathcal{M} : -H : \quad (3.2)$$

The solution to this equation of motion is known if H_1 vanishes.¹ Our task is to find \mathcal{M} when H_1 is not zero, but its effect is small. Specifically, H_1 will be assumed to carry a factor of ϵ , and we will seek an expansion in powers of ϵ .⁸

Some problems in mathematics can be solved by iterative methods or recursion relations. Most problems in life are handled by procrastination. We will see that, in this case, the equation of motion for \mathcal{M} can be solved by an iterative procedure that has the spirit of procrastination.

Suppose the Hamiltonian (3.1) is rewritten in the form

$$H = H_1^0 + H_R^0 \quad (3.3)$$

Here $H_1^0 = H_1$ and H_R^0 denotes all the *remaining* terms in H of degree two and higher,

$$H_R^0 = H_2 + H_3 + \cdots \quad (3.4)$$

Also, suppose \mathcal{M} is written as a product of two factors in the form

$$\mathcal{M} = \mathcal{S}^0 \mathcal{R}^0 \quad (3.5)$$

Upon substituting this expression for \mathcal{M} into the equation of motion (3.2), we find the result

$$\dot{\mathcal{S}}^0 \mathcal{R}^0 + \mathcal{S}^0 \dot{\mathcal{R}}^0 = \mathcal{S}^0 \mathcal{R}^0 : -H_1^0 : + \mathcal{S}^0 \mathcal{R}^0 : -H_R^0 : \quad (3.6)$$

Next, suppose \mathcal{R}^0 is required to satisfy the equation

$$\dot{\mathcal{R}}^0 = \mathcal{R}^0 : -H_R^0 : \quad (3.7)$$

As has been pointed out earlier, this is an equation of the form we already know how to solve.¹ It now follows as a consequence of (3.6) and (3.7) that \mathcal{S}^0 must satisfy the equation

$$\dot{\mathcal{S}}^0 = \mathcal{S}^0 \mathcal{R}^0 : -H_1^0 : (\mathcal{R}^0)^{-1} \quad (3.8)$$

Moreover, from the calculus for manipulating Lie operators, we have the relation³

$$\mathcal{R}^0 : -H_1^0 : (\mathcal{R}^0)^{-1} = : -\mathcal{R}^0 H_1^0 : \quad (3.9)$$

Finally, suppose quantities H^1 , H_1^1 , and H_R^1 are defined by the relations

$$H^1 = \mathcal{R}^0 H_1^0 = H_1^1 + H_R^1 \quad (3.10)$$

(Here again the subscript R is used to denote remaining terms of degree two and higher.) Equation (3.8) now takes the final form

$$\dot{\mathcal{S}}^0 = \mathcal{S}^0 : -H^1 : \quad (3.11)$$

We see that (3.11) is of the same form as the original equation of motion (3.2). What has been gained, however, as is evident from (3.10), is that H_R^1 is now of order ϵ .

Suppose we again "procrastinate" solving (3.11) by writing \mathcal{S}^0 in the form

$$\mathcal{S}^0 = \mathcal{S}^1 \mathcal{R}^1 \quad (3.12)$$

This time \mathcal{R}^1 is required to satisfy the equation

$$\dot{\mathcal{R}}^1 = \mathcal{R}^1 : -H_R^1 : \quad (3.13)$$

Consequently, \mathcal{S}^1 satisfies the equation

$$\dot{\mathcal{S}}^1 = \mathcal{S}^1 : -H^2 : \quad (3.14)$$

where H^2 is given by the relation

$$H^2 = \mathcal{R}^1 H_1^1 = H_1^2 + H_R^2 \quad (3.15)$$

As a result of two procrastinations, H_R^2 is of order ϵ^2 . By contrast H_1^2 , like H_1^1 , is of order ϵ . This may be seen as follows: First, according to (3.13), \mathcal{R}^1 differs from the identity map only by terms of order ϵ because H_R^1 is of order ϵ . Since H_1^1 contains only linear terms in the deviation variables and H_R^2 contains quadratic and higher-order terms, only the terms in \mathcal{R}^1 that are different from the identity map contribute to H_R^2 , as defined by (3.15). Second, according to (3.10), H_1^1 is itself of order ϵ , thus making H_R^2 of order ϵ^2 . By contrast, some of the terms in H_1^1 are the same as the ones in H_1^1 (obtained by the action of the identity part of \mathcal{R}^1), and are thus of order ϵ .

What happens after $m+1$ procrastinations? Evidently, \mathcal{M} will be expressed in the form

$$\mathcal{M} = \mathcal{S}^m \mathcal{R}^m \mathcal{R}^{m-1} \cdots \mathcal{R}^0 \quad (3.16)$$

Each factor \mathcal{R}^j satisfies the soluble equation

$$\mathcal{R}^j = \mathcal{R}^j - H_R^j, \quad (3.17)$$

where each H_R^j is defined by the recursion relation

$$H_1^j + H_R^j = \mathcal{R}^{j-1} H_1^{j-1}. \quad (3.18)$$

The factor \mathcal{S}^m satisfies the equation

$$\mathcal{S}^m = \mathcal{S}^m - H^{m+1}, \quad (3.19)$$

where

$$H^{m+1} = \mathcal{R}^m H_1^m = H_1^{m+1} + H_R^{m+1}. \quad (3.20)$$

The term H_R^{m+1} is of order ϵ^{m+1} . By contrast, H_1^{m+1} is still of order ϵ , although it may contain terms of higher order in ϵ as well.

We are ready for the master stroke that reaps the benefits of procrastination. Suppose the term H_R^{m+1} is neglected in (3.20). Then, within errors of order ϵ^{m+1} , \mathcal{S}^m obeys the approximate equation of motion,

$$\mathcal{S}^m = \mathcal{S}^m - H_1^{m+1}. \quad (3.21)$$

The solution to this equation can be written in a simple closed form. Consider the Lie operators $:H_1^{m+1}:$ evaluated at two different times t' and t'' . We find that they are "self-commuting,"

$$\begin{aligned} & \{ :H_1^{m+1}(z, t'):, :H_1^{m+1}(z, t''): \} \\ & = :[H_1^{m+1}(z, t'), H_1^{m+1}(z, t'')]: = 0. \end{aligned} \quad (3.22)$$

This result holds because the Poisson bracket of any two first degree polynomials is a constant, and the Lie operator of a constant is zero. In the general case, the solution to an equation of the form (3.21) involves a time-ordered exponential. However, as a result of the self-commuting property (3.22), the solution in this case can be written in the form¹

$$\mathcal{S}^m = \exp \left(- : \int_{t^n}^{t^{fn}} H_1^{m+1} dt : \right). \quad (3.23)$$

We close this section by making two remarks about the solution we have found for \mathcal{M} . First, we observe that the solutions \mathcal{R}^j to the equations (3.17) can all be written in the standard factorized form of the type shown in Eq. (2.6) with the f_1 term missing. These maps can then be concatenated to yield again a result of the same form. Thus we have the relation

$$\mathcal{R}^m \mathcal{R}^{m-1} \dots \mathcal{R}^0 = e^{f_2} e^{f_3} e^{f_4} \dots, \quad (3.24)$$

Also, upon comparing (2.6), (3.16), (3.23), and (3.24), we conclude that f_1 is given by the relation

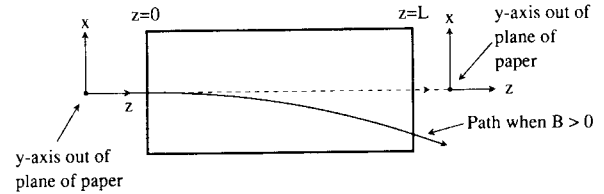


FIG. 1. Parallel face steering magnet and associated coordinate system. Also shown are the undeflected path when $B = 0$ and the deflected path for a positive particle when $B > 0$.

$$f_1 = - \int_{t^n}^{t^{fn}} H_1^{m+1} dt. \quad (3.25)$$

The second remark concerns the concept of rank. The rank of a term is defined to be the sum of its order in ϵ and its degree in the deviation variables.⁹ Suppose all the terms in the Hamiltonian expansion (3.1) are neglected beyond some H_N (with $N > 2$). Then, since H_N generally does not involve ϵ , we have, in effect, agreed to drop all terms in our computations with rank greater than N , and to retain all terms with rank less than or equal to N . It can be shown that the decision to retain or drop various terms depending on rank is equivalent to working within a quotient Lie algebra.⁴ Consequently, this approximation procedure preserves the Lie algebraic and group properties of transfer maps.

With this discussion in mind, we recall that H_R^j has been shown to be of order j in ϵ , and by construction is composed of terms of degree 2 and higher in the deviation variables. Therefore H_R^j is composed of terms of rank $j + 2$ and higher. Consequently, if we have agreed to neglect in our computations all terms with rank greater than N , we have, in effect, agreed to set

$$H_R^j = 0, \quad \text{for } j + 2 > N. \quad (3.26)$$

Thus, in this approximation scheme, all calculations terminate after $N - 1$ procrastinations.

IV. MAP FOR STEERING MAGNET

As a first application of the methods just developed, we consider the accelerator physics problem of computing the transfer map for a parallel face steering magnet.¹⁰ A steering magnet is a weak adjustable dipole inserted in a beamline to correct for the misplacement, misalignment, and mispowering of other elements. This magnet is shown in Fig. 1 along with a suitable coordinate system.¹¹ The magnet has length L and a magnetic field of the form

$$B = B_e y, \quad (4.1)$$

In this magnet, with z as the independent variable and in suitable dimensionless deviation variables, trajec-

tories of particles with charge q and rest mass m are described by the Hamiltonian,^{12,13}

$$H = - (1/l) \sqrt{1 - 2P_\tau/\beta + P_\tau^2 - (P_x^2 + P_y^2)} + \frac{1}{\beta^2 l} - \frac{P_\tau}{\beta l} + \frac{B_1 X}{l}. \quad (4.2)$$

Here l is some convenient scale length and B_1 is a dimensionless dipole strength given by the relation

$$B_1 = Bl/G. \quad (4.3)$$

The quantity G is the magnetic rigidity for the design orbit. It is defined in terms of the design momentum p_0 by the relation

$$G = p_0/q. \quad (4.4)$$

The design momentum is, in turn, given by the relation

$$p_0 = \gamma\beta mc, \quad (4.5)$$

where the quantities β and γ are the standard relativistic factors for the design orbit.

For this problem, the equations of motion generated by (4.2) are sufficiently simple that their solution can be found in closed form. After expanding this solution in the form (1.5), one can determine the factorized map representation (2.6). Rather than follow this route, we employ the methods of Sec. III. Since this problem is fairly simple, we are able to carry out all the required steps in detail, and thus see explicitly how the method works.

The Hamiltonian (4.2) has an expansion of the form (3.1), and we find for the first few terms the results¹⁴

$$H_0 = 1/(\beta^2 \gamma^2 l), \quad (4.6a)$$

$$H_1 = B_1 X/l, \quad (4.6b)$$

$$H_2 = P_x^2/(2l) + P_y^2/(2l) + P_\tau^2/(2\beta^2 \gamma^2 l), \quad (4.6c)$$

$$H_3 = P_\tau P_x^2/(2\beta l) + P_\tau P_y^2/(2\beta l) + P_\tau^3/(2\beta^3 \gamma^2 l), \quad (4.6d)$$

$$H_4 = P_x^4/(8l) + P_x^2 P_y^2/(4l) + P_y^4/(8l) + P_\tau^2 P_x^2[-1/(4l) + 3/(4\beta^2 l)] + P_\tau^2 P_y^2[-1/(4l) + 3/(4\beta^2 l)] + P_\tau^4[5/(8\beta^4 \gamma^2 l) - 1/(8\beta^2 \gamma^2 l)]. \quad (4.6e)$$

As observed earlier, the term H_0 plays no role, and will be ignored from now on. We also note that H_1 is small if B_1 is small. That is, B_1 plays the role of the ϵ factor of Sec.

III. Finally, if we agree for the sake of illustration (and simplicity) to drop all terms in H beyond H_4 , then we have agreed to retain only terms of rank $r \leq N$ with $N = 4$.

We are now ready to employ the procrastination procedure. The first step is to find \mathcal{R}^0 . This is easy because inspection of H_2 , H_3 , etc., as given by (4.6) shows that any two of them are in involution, and, in addition, their associated Lie operators are self-commuting.^{1,15} Consequently, \mathcal{R}^0 is given immediately by the relation

$$\mathcal{R}^0(z) = \exp(: - zH_2:) \exp(: - zH_3:) \exp(: - zH_4:). \quad (4.7)$$

Correspondingly, H^1 is given by the relation

$$H^1 = \mathcal{R}^0 H_1^0 = \exp(: - zH_2:) \exp(: - zH_3:) \exp(: - zH_4:) \times (B_1 X/l). \quad (4.8)$$

Upon evaluating (4.8), we find for H_1^1 , H_2^1 , and H_3^1 the results

$$H_1^1 = B_1 X/l + zB_1 P_x/l^2, \quad (4.9a)$$

$$H_2^1 = zB_1 P_\tau P_x/(\beta l^2), \quad (4.9b)$$

$$H_3^1 = zB_1 l^{-2} \left(\frac{P_x^3}{2} + \frac{P_x P_y^2}{2} + \frac{P_\tau P_x (-1 + 3/\beta^2)}{2} \right). \quad (4.9c)$$

We observe that all the terms H_2^1 , H_3^1 , etc., that comprise H_R^1 are proportional to $\epsilon (= B_1)$, as expected. Also, we see that H_2^1 and H_3^1 are of ranks 3 and 4, respectively. Had we computed H_4^1 , H_5^1 , etc., we would have found that they had ranks 5, 6, etc. Consequently, they are properly neglected.

Let us procrastinate again. The determination of \mathcal{R}^1 is again easy because H_2^1 and H_3^1 are also self-commuting and in involution. Consequently, \mathcal{R}^1 is given by the relation

$$\mathcal{R}^1 = \exp \left(- \int_0^z dz' :H_2^1(z') : \right) \times \exp \left(- \int_0^z dz' :H_3^1(z') : \right). \quad (4.10)$$

Correspondingly, H^2 is given by the relation

$$H^2 = \mathcal{R}^1 H_1^1 = \exp \left(- \int_0^z dz' :H_2^1(z') : \right) \left(- \int_0^z dz' :H_3^1(z') : \right) \times (B_1 X/l + zB_1 P_x/l^2). \quad (4.11)$$

Upon evaluating (4.11), we find for H_1^2 and H_2^2 the results

$$H_1^2 = \frac{B_1 X}{l} + \frac{z B_1 P_x}{l^2} + \frac{z^2 B_1^2 P_\tau}{2 \beta l^3}, \quad (4.12a)$$

$$H_2^2 = \frac{1}{4} (z^2 B_1^2 / l^3) [3 P_x^2 + P_y^2 + P_\tau^2 (-1 + 3/\beta^2)]. \quad (4.12b)$$

We observe that the term H_2^2 , and the terms H_3^2, H_4^2, \dots , had we computed them, all are proportional to ϵ^2 . Also, they are of ranks 4, 5, 6, ..., respectively. (Consequently, only H_2^2 is retained.) In contrast, H_1^2 remains proportional to ϵ , and also contains a term of order ϵ^2 .

Let us procrastinate yet a third time. Since H_R^2 now consists only of H_2^2 and it is self-commuting, we immediately have the result

$$\mathcal{R}^2 = \exp \left(- \int_0^z dz' : H_2^2(z') : \right). \quad (4.13)$$

Correspondingly, H^3 is given by the relation

$$\begin{aligned} H^3 &= \mathcal{R}^2 H_1^3 \\ &= \exp \left(- \int_0^z dz' : H_2^2(z') : \right) \left(\frac{B_1 X}{l} + \frac{z B_1 P_x}{l^2} \right. \\ &\quad \left. + \frac{z^2 B_1^2 P_\tau}{2 \beta l^3} \right). \end{aligned} \quad (4.14)$$

Upon evaluating (4.14), we find for H_1^3 and H_2^3 the results

$$H_1^3 = \frac{B_1 X}{l} + \left(\frac{z B_1}{l^2} + \frac{z^3 B_1^3}{2 l^4} \right) P_x + \frac{z^2 B_1^2 P_\tau}{2 \beta l^3}, \quad (4.15a)$$

$$H_2^3 = 0. \quad (4.15b)$$

Note that (4.15b) is consistent with our rank criterion (3.26) since H_2^3 is of rank 5, and we have agreed for this example to retain only terms of rank $r \leq N$ with $N = 4$. Had we originally started with $N = 5$, we would have found $H_2^3 \neq 0$, but still of rank 5. We would then have found $H_R^4 = 0$.

We are now ready to find the polynomials f_i . Use of (3.25) and (4.15a) gives for f_1 the result

$$\begin{aligned} f_1 &= \frac{-B_1 X L}{l} - \left(\frac{B_1 L^2}{2 l^2} + \frac{B_1^3 L^4}{8 l^4} \right) P_x \\ &\quad - B_1^2 P_\tau L^3 / (6 \beta l^3). \end{aligned} \quad (4.16)$$

Correspondingly, use of (3.24), (4.7), (4.10), and (4.13) gives f_2 , f_3 , and f_4 . We find the results

$$\begin{aligned} f_2 &= -L H_2 - B_1 P_\tau P_x L^2 / (2 \beta l^2) - B_1^2 [L^3 / (12 l^3)] \\ &\quad \times [3 P_x^2 + P_y^2 + P_\tau^2 (-1 + 3/\beta^2)], \end{aligned} \quad (4.17a)$$

$$\begin{aligned} f_3 &= -L H_3 - B_1 [P_x^3 + P_x P_y^2 \\ &\quad + P_\tau^2 P_x (-1 + 3/\beta^2)] L^2 / (4 l^2), \end{aligned} \quad (4.17b)$$

$$f_4 = -L H_4. \quad (4.17c)$$

There are two general observations about the procrastination procedure that can be drawn from examining this simple example. First, we observe that f_4 is of rank 4; f_3 contains terms of ranks 3 and 4; f_2 contains terms of ranks 2, 3, and 4; and f_1 contains terms of ranks 2, 3, and 4. Moreover, had we instead started with $N = 5$, the terms of ranks $r \leq 4$ in the f_i would still be unchanged. Thus, once the terms of a given rank have been computed, they remain unchanged in higher approximation.

Second, we observe that H_1^0 and H_1^1 differ by terms of rank 2; H_1^1 and H_1^2 differ by terms of rank 3; and H_1^2 and H_1^3 differ by terms of rank 4. Thus the first procrastination produces terms of rank 2 in f_1 ; the second procrastination produces terms of rank 3 in f_1 ; and the third procrastination produces terms of rank 4 in f_1 .

The reader is invited to prove to her or his own satisfaction, by examining the methods of Sec. III, that these observations hold in general.

V. MAP FOR MISPOWERED NORMAL ENTRY AND EXIT BENDING MAGNET

As a second example, again drawn from the field of charged particle optics, we treat the case of a mispowered normal entry and exit bending magnet (a sector bend).¹⁰ The magnetic field is a pure dipole, as given below in (5.1), and we compute the transfer map only for the body of the magnet, not including the leading and trailing

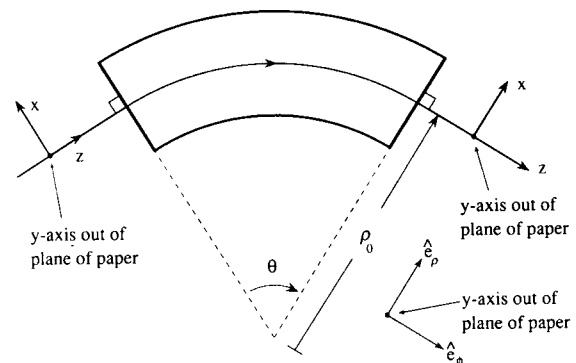


FIG. 2. Normal entry and exit bending magnet and associated cylindrical coordinate system. Shown also are local x, y, z coordinates referenced to the design orbit.

TABLE I. The polynomial f_3^0 .

$$\begin{aligned}
f_3^0 = & -\frac{\mu}{6\lambda^2} S^3 X^3 - \frac{1}{2\lambda} S^2 C X^2 P_x - \left[\frac{1}{2\beta\lambda\mu} S^3 + \frac{\epsilon}{4\beta\lambda} \left(\phi - \frac{SC}{\mu} \right) \right] X^2 P_\tau - \frac{1}{2\mu} S C^2 X P_x^2 - \frac{1}{\beta\mu^2} S^2 \left(C + \frac{\epsilon}{2} \right) X P_x P_\tau - \frac{1}{2\mu} S X P_y^2 - \left[\frac{1}{2\beta^2\mu^3} S^3 + \frac{\epsilon}{2\beta^2\mu^2} \left(\phi - \frac{SC}{\mu} \right) \right. \\
& + \left. \frac{1}{2\beta^2\gamma^2\mu} S \right] X P_\tau^2 + \frac{\lambda}{6\mu^2} (1 - C^3) P_x^3 - \left[\frac{\lambda}{2\beta\mu^3} S C^2 + \frac{\lambda}{4\beta\mu^2} \epsilon \left(\phi + \frac{SC}{\mu} \right) \right] P_x^2 P_\tau + \frac{\lambda}{2\mu^2} (1 - C) P_x P_y^2 + \left[\frac{\lambda}{2\beta^2\gamma^2\mu^2} (1 - C) - \frac{\lambda}{2\beta^2\mu^4} S^2 (C + \epsilon) \right] P_x P_\tau^2 \\
& - \left(\frac{\lambda}{2\beta\mu^3} S + \frac{\lambda}{2\beta\mu^2} \epsilon \phi \right) P_y^2 P_\tau - \left[\frac{\lambda}{2\beta^3\mu^4} \left(\frac{S^3}{3\mu} + \frac{\epsilon}{2} \left(\phi - \frac{SC}{\mu} \right) \right) + \frac{\lambda}{2\beta^3\gamma^2\mu^2} \left(\frac{S}{\mu} + \epsilon \phi \right) \right] P_\tau^3
\end{aligned}$$

fringe-field regions. The procedure is the same as that in the first example but, as will be seen, the required calculations are more formidable. Consequently, at points we do not provide detailed calculations, but only sketch the steps to be done and summarize results.¹⁶

Orbits in a normal entry and exit bending magnet are conveniently described using a triad ρ , y , ϕ of cylindrical coordinates (see Fig. 2). The magnet has bend angle Θ , radius of curvature ρ_0 , and a magnetic field of the form

$$\mathbf{B} = B\mathbf{e}_y. \quad (5.1)$$

With ϕ as the independent variable, and in suitable dimensionless deviation variables, trajectories in this magnet are described by the Hamiltonian¹⁷

$$\begin{aligned}
H = & - (X + \rho_0/l) \sqrt{1 - 2P_x/\beta + P_\tau^2 - (P_x^2 + P_y^2)} \\
& + \left(\frac{l}{2\rho_0} \right) (1 + \epsilon) \left(X + \frac{\rho_0}{l} \right)^2 - \frac{\rho_0 P_\tau}{\beta l} + \frac{\rho_0}{\beta^2 l}. \quad (5.2)
\end{aligned}$$

The quantity ϵ is defined in terms of the magnetic field strength B by the expressions

$$\epsilon = (B - B^d)/B^d \quad \text{or} \quad B = (1 + \epsilon)B^d. \quad (5.3)$$

Here B^d is the design magnetic field strength given by the relation

$$G = B^d \rho_0. \quad (5.4)$$

Thus ϵ is a measure of the mispowering of the bending magnet.

The Hamiltonian (5.2) has an expansion of the form (3.1), and we find for the first few terms the results

$$H_0 = -\lambda \left(\frac{1}{2} - 1/\beta^2 - \epsilon/2 \right), \quad (5.5a)$$

$$H_1 = \epsilon X, \quad (5.5b)$$

$$\begin{aligned}
H_2 = & \lambda P_x^2/2 + \left(\frac{1 + \epsilon}{2\lambda} \right) X^2 + \frac{X P_\tau}{\beta} + \frac{\lambda P_y^2}{2} \\
& + \lambda P_\tau^2/(2\beta^2\gamma^2), \quad (5.5c)
\end{aligned}$$

$$\begin{aligned}
H_3 = & \frac{X P_\tau^2}{2\beta^2\gamma^2} + \frac{X P_x^2}{2} + \frac{X P_y^2}{2} + \frac{\lambda P_\tau P_x^2}{2\beta} + \frac{\lambda P_\tau P_y^2}{2\beta} \\
& + \frac{\lambda P_\tau^3}{2\beta^3\gamma^2}, \quad (5.5d)
\end{aligned}$$

$$\begin{aligned}
H_4 = & \frac{P_\tau P_x^2 X}{2\beta} + \frac{P_\tau P_y^2 X}{2\beta} + \frac{P_\tau^3 X}{2\beta^3\gamma^2} + \frac{\lambda P_x^4}{8} + \frac{\lambda P_x^2 P_y^2}{4} \\
& + \frac{\lambda P_y^4}{8} + \lambda P_\tau^2 P_x^2 \left(\frac{3}{4\beta^2} - \frac{1}{4} \right) \\
& + \lambda P_\tau^2 P_y^2 \left(\frac{3}{4\beta^2} - \frac{1}{4} \right) + \lambda P_\tau^4 [5/(8\beta^4\gamma^2) \\
& - 1/(8\beta^2\gamma^2)]. \quad (5.5e)
\end{aligned}$$

Here λ denotes the ratio

$$\lambda = \rho_0/l. \quad (5.6)$$

We observe that H_1 is of order ϵ as desired, and is thus of rank 2. The polynomial H_2 contains terms both without and with a factor of ϵ . Thus H_2 contains terms of ranks 2 and 3. The polynomial H_3 is of degree 3 and rank 3, and the polynomial H_4 is of degree 4 and rank 4. If we agree to drop all terms in H beyond H_4 , then we have agreed to retain only terms of rank $r \leq N$ with $N = 4$. Correspondingly, we expect to make three procrastinations.

We are again ready to begin the procrastination procedure. However, this time the determination of \mathcal{R}^0 is not so easy because H_2 is not in involution with both H_3 and H_4 . Suppose \mathcal{R}^0 is written in the reversed factorized form

$$\mathcal{R}^0 = e^{\dot{g}_4^0} e^{\dot{g}_3^0} e^{\dot{g}_2^0}. \quad (5.7)$$

Then, since H_2 is self-commuting, g_2^0 is given by the relation

$$g_2^0 = -\phi H_2. \quad (5.8)$$

Correspondingly, we find for the matrix M^0 associated with $\exp(:g_2^0:)$ the result^{3,7,12,18}

$$M^0(\phi) = \begin{bmatrix} C & \frac{\lambda}{\mu} S & 0 & 0 & 0 & \frac{\lambda}{\beta\mu^2}[C-1] \\ -\frac{\mu}{\lambda} S & C & 0 & 0 & 0 & -\frac{1}{\beta\mu} S \\ 0 & 0 & 1 & \lambda\phi & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{1}{\beta\mu} S & -\frac{\lambda}{\beta\mu^2}[C-1] & 0 & 0 & 1 & \frac{\lambda}{\beta^2}\left(\frac{1}{\gamma^2} - \frac{1}{\mu^2}\right)\phi + \frac{\lambda}{\beta^2\mu^3} S \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \quad (5.9)$$

Here, for notational convenience, we have made the definition

$$f_2^0 = g_2^0. \quad (5.15)$$

$$\mu = \sqrt{1 + \epsilon}. \quad (5.10a)$$

Also, here and elsewhere, the symbols S and C denote the quantities

$$S = \sin(\mu\phi), \quad (5.10b)$$

$$C = \cos(\mu\phi). \quad (5.10c)$$

To find g_3^0 and g_4^0 , we use $\exp(:g_2^0:)$ to specify an interaction picture, and define interaction picture Hamiltonian elements H_j^{int} by the rule¹

$$H_j^{\text{int}} = e^{i g_2^0} H_j. \quad (5.11)$$

The quantities g_3^0 and g_4^0 are the solutions of the equations

$$g_3^0 = -H_3^{\text{int}}, \quad (5.12)$$

$$g_4^0 = -H_4^{\text{int}} + :g_3^0: (-H_3^{\text{int}}/2). \quad (5.13)$$

These equations can be solved by quadrature.¹

Now that g_2^0 , g_3^0 , and g_4^0 are known, we can write \mathcal{R}^0 in the standard factorized form using polynomials f_2^0 , f_3^0 , f_4^0 ,

$$\mathcal{R}^0 = e^{i g_4^0} e^{i g_3^0} e^{i g_2^0} = e^{i f_2^0} e^{i f_3^0} e^{i f_4^0}. \quad (5.14)$$

We immediately have for f_2^0 the result

The results for f_3^0 and f_4^0 are listed in Tables I and II. Note that if f_3^0 were to be expanded in powers of ϵ using (5.10a) and the entries of Table I, then terms of ranks 3, 4, ..., would result. Strictly speaking, only the terms of ranks 3 and 4 should be retained, since only these are within the scope of the present calculation. Similarly, in computing the solution of (5.13), all terms in H_4^{int} , g_3^0 , and H_3^{int} involving ϵ should be discarded since they would only produce terms in g_4^0 of ranks 5 and higher. Since g_2^0 contains terms of rank 3, terms of rank 5 will also be produced in passing from g_4^0 to f_4^0 using (5.14). They should also be discarded. This has been done in writing Table II.

The last task in this first round of procrastination is to find H^1 , defined by the relation

$$H^1 = \mathcal{R}^0 H_1^0 = e^{i f_2^0} e^{i f_3^0} e^{i f_4^0} (\epsilon X). \quad (5.16)$$

Use of (5.9) and the entries in Tables I and II gives through terms of rank 4 the results

$$H^1 = H_1^1 + H_2^1 + H_3^1, \quad (5.17)$$

where

$$H_1^1 = \epsilon \left[CX + \frac{\lambda}{\mu} SP_x + \frac{\lambda}{\beta\mu^2} [C-1] P_\tau \right], \quad (5.18a)$$

$$H_2^1 = -\epsilon \left[\frac{1}{2\lambda} S^2 X^2 - \frac{1}{\mu} CSXP_x + \left(\frac{1}{\beta\mu^2} S^2 + \frac{\epsilon}{2\beta\mu} \phi S \right) XP_\tau - \frac{\lambda}{2\mu^2} C(1-C) P_x^2 - \left[\frac{\lambda}{2\beta\mu^3} S(\epsilon+2C) + \frac{\epsilon\lambda}{2\beta\mu^2} \phi C \right] P_x P_\tau \right. \\ \left. + \frac{\lambda}{2\mu^2} (1-C) P_y^2 + \left[\frac{\lambda}{2\beta^2\gamma^2\mu^2} (1-C) + \frac{\lambda}{2\beta^2\mu^4} S^2 + \frac{\epsilon\lambda}{2\beta^2\mu^3} \phi S \right] P_\tau^2 \right], \quad (5.18b)$$

$$H_3^1 = -\epsilon \left[\frac{S^2}{2\beta\lambda} X^2 P_\tau + \frac{S^2}{2} X P_x^2 - \frac{SC}{\beta} X P_x P_\tau + \frac{S^2}{2} X P_y^2 + \frac{(3-\beta^2)S^2}{2\beta^2} X P_\tau^2 - \frac{\lambda SC}{2} P_x^3 - \frac{\lambda(C-1)}{2\beta} P_x^2 P_\tau - \frac{\lambda SC}{2} P_x P_y^2 \right. \\ \left. - \frac{\lambda(3-\beta^2)SC}{2\beta^2} P_x P_\tau^2 - \frac{\lambda(C-1-S^2)}{2\beta} P_y^2 P_\tau - \left[\frac{\lambda(C-1)}{2\beta^3\gamma^2} - \frac{\lambda(2-\beta^2)S^2}{2\beta^3} \right] P_\tau^3 \right]. \quad (5.18c)$$

We are ready for the second procrastination. Suppose \mathcal{R}^1 is written in the reverse factorized form,

$$\mathcal{R}^1 = e^{g_3^1} e^{g_2^1}. \quad (5.19)$$

Now the determination of g_2^1 requires some additional effort because inspection of (5.18b) shows that the Lie operators associated with H_2^1 at different ϕ values are not self-commuting. However, further inspection of (5.18b) shows that H_2^1 consists (as expected) of terms of ranks 3 and 4. This property will be put to good use in simplifying calculations.

Let \mathcal{N}^1 denote the map generated by g_2^1 ,

$$\mathcal{N}^1 = e^{g_2^1}. \quad (5.20)$$

It obeys the equation of motion

$$\dot{\mathcal{N}}^1 = \mathcal{N}^1 : -H_2^1 :. \quad (5.21)$$

Equivalently, this equation can be written in the integral form

$$\mathcal{N}^1 = \mathcal{I} + \int_0^\phi d\phi' \mathcal{N}^1(\phi') : -H_2^1(\phi') :. \quad (5.22)$$

Repeated substitution of (5.22) back into itself gives the Neumann series¹

$$\mathcal{N}^1 = \mathcal{I} + \int_0^\phi d\phi' : -H_2^1(\phi') : \\ + \int_0^\phi d\phi' \int_0^{\phi'} d\phi'' : -H_2^1(\phi'') : : -H_2^1(\phi') : + \cdots. \quad (5.23)$$

We note that successive terms on the right-hand side of (5.23) contain ever higher powers of ϵ . In particular, if we only need terms of rank 4 and lower, then the series can be terminated by keeping only the terms displayed. Correspondingly, we find for the matrix M^1 associated with \mathcal{N}^1 the results displayed in Table III.¹⁹

TABLE II. The polynomial f_4^0 .

$$f_4^0 = -\frac{1}{6\beta\lambda^2} S^3 X^3 P_\tau - \frac{1}{8\lambda} S^3 X^2 P_x^2 - \frac{1}{2\beta\lambda} S^2 CX^2 P_x P_\tau - \frac{1}{8\lambda} S^3 X^2 P_y^2 - \frac{(5-\beta^2)}{8\beta^2\lambda} S^3 X^2 P_\tau^2 - \frac{1}{4} S^2 CX P_x^3 + \left(\frac{1}{4\beta} S^3 - \frac{1}{2\beta} S \right) X P_x^2 P_\tau - \frac{1}{4} S^2 CX P_x P_y^2 \\ - \left(\frac{1}{4\beta^2\gamma^2} + \frac{1}{\beta^2} \right) S^2 CX P_x P_\tau^2 - \left(\frac{1}{4\beta} S^3 + \frac{1}{2\beta} S \right) X P_y^2 P_\tau - \left[\left(\frac{1}{4\beta^3\gamma^2} + \frac{1}{2\beta^3} \right) S^3 + \frac{1}{2\beta^3\gamma^2} S \right] X P_\tau^3 - \frac{\lambda}{8} SC^2 P_x^4 - \left[\frac{\lambda}{12\beta} S^2 C - \frac{\lambda}{6\beta} (1-C) \right] P_x^3 P_\tau + \left(\frac{\lambda}{8} S^3 \right. \\ \left. - \frac{\lambda}{4} S \right) P_x^2 P_y^2 + \left[\frac{\lambda(4-\beta^2)}{8\beta^2} S^3 - \lambda \left(\frac{1}{4\beta^2\gamma^2} + \frac{1}{2\beta^2} \right) S \right] P_x^2 P_\tau^2 - \left[\frac{\lambda}{4\beta} S^2 C - \frac{\lambda}{2\beta} (1-C) \right] P_x P_y^2 P_\tau - \left[\lambda \left(\frac{1}{4\beta^3\gamma^2} + \frac{1}{2\beta^3} \right) S^2 C - \frac{\lambda}{2\beta^3\gamma^2} (1-C) \right] P_x P_\tau^3 \\ - \frac{\lambda}{8} S P_y^4 - \left[\frac{\lambda}{8\beta^2} S^3 + \lambda \left(\frac{1}{4\beta^2\gamma^2} + \frac{1}{2\beta^2} \right) S \right] P_y^2 P_\tau^2 - \left[\lambda \left(\frac{1}{8\beta^4\gamma^2} + \frac{1}{6\beta^4} \right) S^3 + \lambda \left(\frac{1}{2\beta^4\gamma^2} + \frac{1}{8\beta^4\gamma^4} \right) S \right] P_\tau^4$$

TABLE III. Nonzero entries in the matrix M^1 associated with $\exp(\cdot g_3^1)$.

$M_{1,1}^1 = 1 + \frac{\epsilon S^2}{2} + \epsilon^2 \left(-\frac{19}{24} + \frac{C}{2} + \frac{C^2}{8} + \frac{C^3}{6} + \frac{S\phi}{2} - \frac{CS\phi}{4} - \frac{\phi^2}{8} \right)$
$M_{1,2}^1 = \epsilon \lambda \left(S - \frac{CS}{2} - \frac{\phi}{2} \right) + \epsilon^2 \lambda \left(-\frac{3S}{2} + \frac{5CS}{8} + \frac{S^3}{6} + \frac{3\phi}{8} + \frac{C^2\phi}{4} \right)$
$M_{1,6}^1 = \frac{\epsilon \lambda S^2}{2\beta} + \frac{\epsilon^2 \lambda}{\beta} \left(-\frac{31}{24} + \frac{C}{2} + \frac{5C^2}{8} + \frac{C^3}{6} + S\phi - \frac{CS\phi}{4} - \frac{\phi^2}{8} \right)$
$M_{2,1}^1 = \frac{\epsilon}{\lambda} \left(-\frac{CS}{2} + \frac{\phi}{2} \right) + \frac{\epsilon^2}{\lambda} \left(-\frac{CS}{8} + \frac{\phi}{8} + \frac{C^2\phi}{4} \right)$
$M_{2,2}^1 = 1 - \frac{\epsilon S^2}{2} + \epsilon^2 \left(\frac{25}{24} - C - \frac{3C^2}{8} + \frac{C^3}{3} + \frac{CS\phi}{4} - \frac{\phi^2}{8} \right)$
$M_{2,6}^1 = \frac{\epsilon}{\beta} \left(-\frac{CS}{2} + \frac{\phi}{2} \right) + \frac{\epsilon^2}{\beta} \left(\frac{S}{2} + \frac{3CS}{8} - \frac{3\phi}{8} - \frac{C\phi}{2} + \frac{C^2\phi}{4} \right)$
$M_{3,3}^1 = 1$
$M_{3,4}^1 = \epsilon \lambda (S - \phi) + \epsilon^2 \lambda \left(-\frac{3S}{2} + \phi \right)$
$M_{4,4}^1 = 1$
$M_{5,1}^1 = \frac{\epsilon}{\beta} \left(\frac{CS}{2} - \frac{\phi}{2} \right) + \frac{\epsilon^2}{\beta} \left(-\frac{S}{2} - \frac{3CS}{8} + \frac{3\phi}{8} + \frac{C\phi}{2} - \frac{C^2\phi}{4} \right)$
$M_{5,2}^1 = \frac{\epsilon \lambda S^2}{2\beta} + \frac{\epsilon^2 \lambda}{\beta} \left(-\frac{37}{24} + C + \frac{7C^2}{8} - \frac{C^3}{3} + \frac{S\phi}{2} - \frac{CS\phi}{4} + \frac{\phi^2}{8} \right)$
$M_{5,5}^1 = 1$
$M_{5,6}^1 = \epsilon \lambda \left[-S + \phi + \frac{1}{\beta^2} \left(S + \frac{CS}{2} - \frac{3\phi}{2} \right) \right]$
$\quad + \epsilon^2 \lambda \left[\frac{3S}{2} - \phi + \frac{1}{\beta^2} \left(-\frac{5S}{2} - \frac{7CS}{8} + \frac{15\phi}{8} + C\phi - \frac{C^2\phi}{4} \right) \right]$
$M_{6,6}^1 = 1$

The calculation of g_3^1 requires no new tools beyond those originally employed. The quantity g_3^1 obeys the equation of motion

$$\dot{g}_3^1 = -(H_3^1)^{\text{int}}, \quad (5.24)$$

where $(H_3^1)^{\text{int}}$ is defined by the relation

$$(H_3^1)^{\text{int}} = \mathcal{N}^1 H_3^1. \quad (5.25)$$

Equation (5.24) can be solved by quadrature to give g_3^1 . The results are given in Table IV.

To complete the second round of procrastination, it is necessary to find H^2 by working out the consequences of the relation

$$H^2 = \mathcal{R}^1 H_1^1. \quad (5.26)$$

Using (5.18a) and the entries in Tables III and IV, we find through terms of rank 4 the result

$$H^2 = H_1^2 + H_2^2, \quad (5.27)$$

where

 TABLE IV. The polynomial g_3^1 .

$g_3^1 = -\frac{\epsilon}{4\beta\lambda} (SC - \phi) X^2 P_\tau - \frac{\epsilon}{4} (SC - \phi) X P_x^2 - \frac{\epsilon}{2\beta} S^2 X P_x P_\tau - \frac{\epsilon}{4} (SC$
$- \phi) X P_y^2 - \frac{\epsilon}{4\beta^2} (3 - \beta^2) (SC - \phi) X P_\tau^2 - \frac{\epsilon\lambda}{4} S^2 P_x^2 - \frac{\epsilon\lambda}{2\beta} (S$
$- \phi) P_x^2 P_\tau - \frac{\epsilon\lambda}{4} S^2 P_x P_y^2 - \frac{\epsilon\lambda}{4\beta^2} (3 - \beta^2) S^2 P_x P_\tau^2 - \frac{\epsilon\lambda}{4\beta} (-3\phi + 2S$
$+ SC) P_x^2 P_\tau - \left[\frac{\epsilon\lambda}{2\beta^3 \gamma^2} (S - \phi) + \frac{\epsilon\lambda}{4\beta^3} (2 - \beta^2) (SC - \phi) \right] P_\tau^3$

$$H_2^2 = \epsilon^2 \left[-\frac{1}{2} C(\phi - CS) X P_x + \frac{1}{2\beta} \phi S X P_\tau \right. \\ \left. + \frac{\lambda}{4} (2S^2 C + \phi S) P_x^2 - \frac{\lambda}{2\beta} (S^3 + 2\phi C \right. \\ \left. - 2SC) P_x P_\tau + \frac{\lambda}{4} \phi S P_y^2 \right. \\ \left. - \frac{\lambda}{4\beta^2} (\beta^2 - 3) \phi S P_\tau^2 \right], \quad (5.28)$$

and the expression for H_1^2 is given in Table V.

To begin the third and last round of procrastination, we observe that through terms of rank 4 the map \mathcal{R}^2 is of the form

$$\mathcal{R}^2 = e^{\mathcal{F}_2^2} = \mathcal{N}^2, \quad (5.29)$$

where \mathcal{N}^2 obeys the equation of motion

$$\dot{\mathcal{N}}^2 = \mathcal{N}^2 \cdot -H_2^2. \quad (5.30)$$

 TABLE V. The polynomial H_1^2 .

$H_1^2 = \left[\epsilon C + \frac{\epsilon^2 S\phi}{2} + \epsilon^3 \left(-\frac{2C}{3} + \frac{C^2}{2} + \frac{C^4}{6} - \frac{S\phi}{8} + \frac{CS\phi}{2} - \frac{C\phi^2}{8} \right) \right] X$
$+ \left[\epsilon \lambda S + \epsilon^2 \lambda \left(-S + CS - \frac{C\phi}{2} \right) + \epsilon^3 \lambda \left(\frac{5S}{3} - \frac{9CS}{4} + \frac{C^3 S}{12} \right. \right.$
$\left. - \frac{CS^3}{12} + \frac{5C\phi}{8} - \frac{S\phi^2}{8} \right] P_x + \left[\frac{\epsilon\lambda}{\beta} (-1 + C) + \frac{\epsilon^2 \lambda}{\beta} \left(1 - C \right. \right.$
$\left. + \frac{S\phi}{2} \right) + \frac{\epsilon^3 \lambda}{\beta} \left(-\frac{1}{2} + \frac{C}{3} + \frac{C^4}{6} - \frac{5S\phi}{8} + \frac{CS\phi}{2} - \frac{C\phi^2}{8} \right) \right] P_\tau$

TABLE VI. Nonzero entries in the matrix M^2 associated with $\exp(:f_2^2:)$.

$M_{1,1}^2 = 1 + \epsilon^2 \left(\frac{2}{3} - \frac{C}{2} - \frac{C^3}{6} - \frac{S\phi}{2} \right)$
$M_{1,2}^2 = \epsilon^2 \lambda \left(\frac{S}{2} + \frac{S^3}{3} - \frac{C\phi}{2} \right)$
$M_{1,6}^2 = \frac{\epsilon^2 \lambda}{\beta} \left(\frac{7}{6} - \frac{2C}{3} - \frac{C^2}{2} + \frac{CS^2}{6} - S\phi \right)$
$M_{2,2}^2 = 1 - \epsilon^2 \left(\frac{2}{3} - \frac{C}{2} - \frac{C^3}{6} - \frac{S\phi}{2} \right)$
$M_{2,6}^2 = -\frac{\epsilon^2}{2\beta} (S - C\phi)$
$M_{3,3}^2 = 1$
$M_{3,4}^2 = \frac{\epsilon^2 \lambda}{2} (S - C\phi)$
$M_{4,4}^2 = 1$
$M_{5,1}^2 = \frac{\epsilon^2}{2\beta} (S - C\phi)$
$M_{5,2}^2 = \frac{\epsilon^2 \lambda}{\beta} \left(\frac{7}{6} - \frac{2C}{3} - \frac{C^2}{2} + \frac{CS^2}{6} - S\phi \right)$
$M_{5,5}^2 = 1$
$M_{5,6}^2 = \frac{\epsilon^2 \lambda}{2\beta^2} (3 - \beta^2) (S - C\phi)$
$M_{6,6}^2 = 1$

Since H_2^2 is of rank 4, the solution to (5.30) through terms of rank 4 is given by the first two terms in the Neumann series expansion

$$\mathcal{N}^2 = \mathcal{J} + \int_0^\phi d\phi' : -H_2^2(\phi') : + \cdots \quad (5.31)$$

Consequently, through terms of rank 4, the quantity f_2^2 is given by the relation

$$f_2^2 = - \int_0^\phi d\phi' H_2^2(\phi'). \quad (5.32)$$

The matrix M^2 associated with $\exp(:f_2^2:)$ is displayed in Table VI.

The last step is to find H_1^3 and f_1 . Through terms of rank 4 the quantity H_1^3 is given by the relation

$$H_1^3 = \mathcal{R}^2 H_1^2. \quad (5.33)$$

The quantity f_1 is, in turn, found by integrating H_1^3 according to (3.25). The results for f_1 are given in Table VII.

The calculation of \mathcal{M} for the mispowered normal entry and normal exit bending magnet is now complete through terms of rank 4. We have the result^{20,21}

$$\mathcal{M} = e^{f_1} e^{f_2^2} e^{g_3^1} e^{g_2^1} e^{f_3^0} e^{f_3^0} e^{f_4^0}. \quad (5.34)$$

TABLE VII. The polynomial f_1 .

$$f_1 = - \left[\epsilon S - \frac{\epsilon^2 C\phi}{2} - \epsilon^3 \left(-\frac{C\phi}{8} + \frac{S\phi^2}{8} \right) \right] X + \left[\epsilon \lambda (-1 + C) \right. \\ \left. + \epsilon^2 \lambda \left(\frac{1}{2} - C + \frac{C^2}{2} + \frac{S\phi}{2} \right) + \epsilon^3 \lambda \left(-\frac{1}{2} + C - \frac{C^2}{2} - \frac{5S\phi}{8} \right. \right. \\ \left. \left. + \frac{CS\phi}{2} - \frac{C\phi^2}{8} \right) \right] P_x - \left[\frac{\epsilon \lambda}{\beta} (S - \phi) - \frac{\epsilon^2 \lambda}{\beta} \left(S - \phi + \frac{C\phi}{2} \right) \right. \\ \left. - \frac{\epsilon^3 \lambda}{\beta} \left(-S - \frac{S^3}{6} + \phi - \frac{5C\phi}{8} + \frac{S\phi^2}{8} \right) \right] P_r.$$

If desired, the factors in (5.34) to the right of $\exp(:f_1:)$ could, in principle, be concatenated analytically to give a result of the form (3.24), and a final result of the form (2.6). However, for the present major use of these results in the code MARYLIE 3.1,^{6,7} the form (5.34) is satisfactory as it stands. In this code the formulas for the f s and g s are evaluated to give numerical results, and the factors are then concatenated numerically to give final results for \mathcal{M} .

VI. COMPARISONS WITH OTHER METHODS

The results of the previous section for the mispowered normal entry and exit bending magnet are remarkably complicated. In order to verify that the procrastination algorithm works as expected, and that the results presented are also free of algebraic errors, we have made comparisons with results obtained by three other methods: analytical results in the case of particularly simple initial conditions, numerical computation of the transfer map, and computation of the transfer map in terms of other known maps. We now discuss these methods in turn.

Consider the trajectory with the following simple initial conditions:

$$X^i = P_x^i = Y^i = P_y^i = \tau^i = 0; \quad P_r^i \text{ arbitrary.} \quad (6.1)$$

In this case the trajectory is confined to two dimensions (there is no motion in the y direction) and consists of a circular arc having radius ρ_0' (see Fig. 3). (For the purposes of this discussion, the words in parentheses in the figure should be ignored.) Consequently, the final conditions can be computed for arbitrary ϵ by elementary trigonometry. The results are

$$X^f = \lambda \{ (1 - r) \cos \Theta + [r^2 - (1 - r)^2 \sin^2 \Theta]^{1/2} - 1 \}, \quad (6.2a)$$

$$P_x^f = - [1 + (P_r^i)^2 - 2P_r^i/\beta]^{1/2} (1/r - 1) \sin \Theta, \quad (6.2b)$$

$$Y^f = Y^i, \quad (6.2c)$$

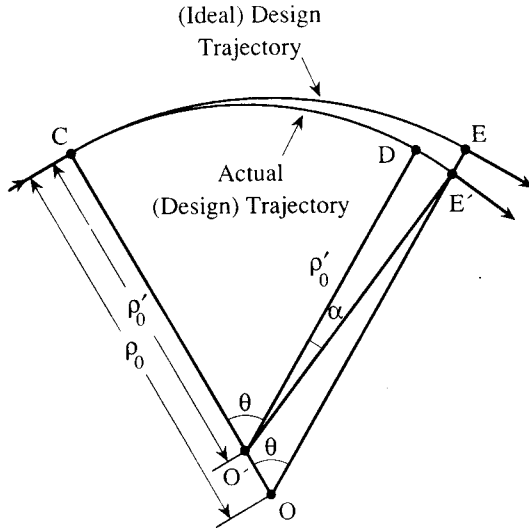


FIG. 3. Geometry of two trajectories in the mispowered normal entry and exit bending magnet. The actual trajectory is illustrated for the case $\epsilon > 0$, $P_r' > 0$.

$$P_y^f = P_y^i \quad (6.2d)$$

$$\tau^f = \lambda$$

$$\times \left[\frac{(1/\beta - P_r') \{ \Theta + \sin^{-1}[(1/r - 1)\sin \Theta] \}}{1 + \epsilon} - \frac{\Theta}{\beta} \right], \quad (6.2e)$$

$$P_r^f = P_r^i \quad (6.2f)$$

Here r denotes the ratio

$$r = \frac{\rho_0'}{\rho_0} = \frac{[1 + (P_r')^2 - 2P_r'/\beta]^{1/2}}{1 + \epsilon}. \quad (6.3)$$

Note that X^f and P_x^f vanish when $r = 1$. In this case and for these variables, the effects of mispowering and being off energy are mutually compensating.

We now compare these exact results with those obtained by applying the map (5.34) to the initial conditions (6.1). One finds that the relations (6.2c), (6.2d), and (6.2f) are satisfied exactly because the map (5.34) preserves midplane symmetry and conserves energy exactly. However, as expected, the remaining relations (6.2a), (6.2b), and (6.2e) are only satisfied approximately. Indeed, we let P_r' and ϵ be parametrized in terms of the quantities σ , n_1 , and n_2 according to the relations

$$P_r' = \sigma n_1, \quad (6.4a)$$

$$\epsilon = \sigma n_2. \quad (6.4b)$$

Since \mathcal{M} as given by (5.34) is expected to be correct through terms of rank 4, when applied to the initial conditions (6.1) it should produce results that agree with the exact results (6.2) through terms of rank 3. Correspondingly, when P_r' and ϵ are parametrized as in (6.4), the differences between the approximate and exact results for the final conditions should vanish as σ^4 when σ approaches zero. Moreover, this should be true for all values of n_1 and n_2 . This has been verified to be the case. Figure 4 shows a typical example.

As a second check on the accuracy of our results, we compare the transfer map \mathcal{M} given by (5.34) with that found by numerical methods.²² Since the Hamiltonian given by (5.2) and expanded in (5.5) has no explicit ϕ dependence, the transfer map \mathcal{M} can also be written in the form

$$\mathcal{M} = e^{i\Theta H}. \quad (6.5)$$

To make a comparison between \mathcal{M} as given by (5.34) and \mathcal{M} as given by (6.5), it is convenient to bring both expressions to the standard factorized form (2.6). As remarked earlier, \mathcal{M} as given by (5.34) can be brought to standard factorized form by simple concatenation. By contrast, somewhat more work is required to do the same for (6.5).

Suppose \mathcal{M} as given by (6.5) is allowed to act on arbitrary initial conditions ξ_b^{in} , to give a result of the form

$$\xi_b^{\text{fin}} = \mathcal{M} \xi_b^{\text{in}} = \sum_j \frac{(-\Theta)^j H^j}{j!} \xi_b^{\text{in}}. \quad (6.6)$$

For any given Hamiltonian and any desired numerical accuracy, the infinite sum can be truncated at some value $j = j_{\text{max}}$ due to the $j!$ factor in the denominator that gives ultimate convergence. The required Poisson brackets can then be carried out by a computer polynomial manipulation program to give a Taylor series result of the form (1.5). Finally the Taylor form can be brought to the standard Lie factorized form by a systematic procedure.^{2,3,22}

Note that in this approach the quantity ϵ is given a definite numerical value (as are all other quantities apart from the ξ_b^{in}) and the resulting matrix M (corresponding to e^{f_2}) and the polynomials f_1, f_3, f_4, \dots are accurate to arbitrary order in ϵ providing j_{max} is sufficiently large. By contrast, the calculations of Sec. V that led to \mathcal{M} in the form (5.34) were based on rank, and were only carried out through terms of rank 4. It follows that when (5.34) is brought to standard factorized form, the resulting f_1 will be correct only through terms of order ϵ^3 . Similarly, the matrix M corresponding to e^{f_2} will be correct through order ϵ^2 , f_3 will be correct through order ϵ and f_4 will be correct through order ϵ^0 .

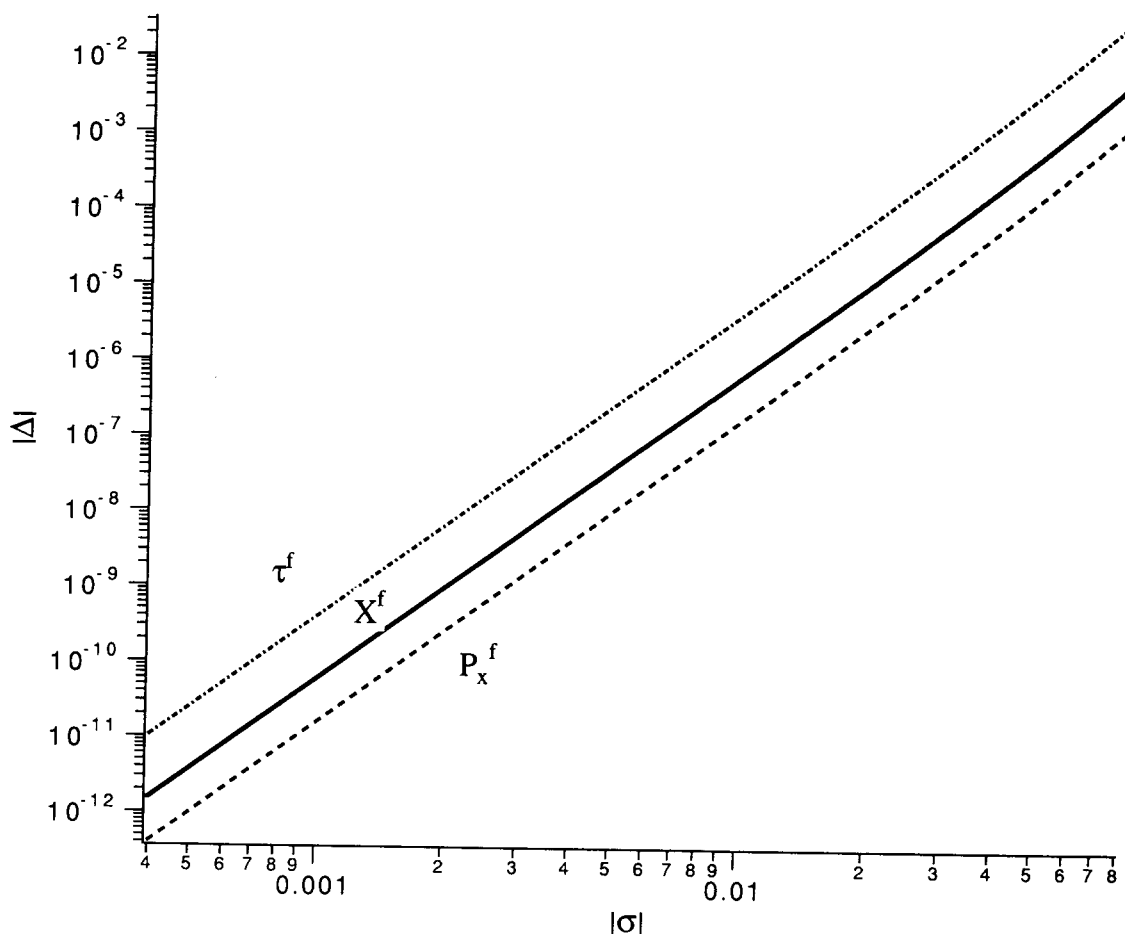


FIG. 4. A log-log plot of $|\Delta|$ vs $|\sigma|$, where Δ is the difference between the approximate and exact results for X^f , P_x^f , and τ^f . Each curve is asymptotically a straight line with slope 4, thereby indicating that the error is in fact of order σ^4 . The quantities n_1 , n_2 have the values $n_1 = n_2 = 1$. The other quantities have the values $\gamma = 1.1066$, $\lambda = 1.5$, and $\Theta = 75^\circ$. To get dimensional parameters we can, for example, set the scale length l to be 1 m. Then $\rho_0 = 1.5$ m and $G = 1.4830$ T m.

With this discussion as background, we make the following comparison with numerical methods. Suppose \mathcal{M}_e is the *exact* transfer map calculated by the numerical methods just described, and \mathcal{M}_a is the *approximate* map given by (5.34). Form the "deviation" map \mathcal{D} defined by the relation

$$\mathcal{D} = \mathcal{M}_a \mathcal{M}_e^{-1}. \quad (6.7)$$

The deviation map will have the standard factorization

$$\mathcal{D} = e^{d_1} e^{d_2} e^{d_3} e^{d_4}. \quad (6.8)$$

Moreover, if \mathcal{M}_a is indeed correctly computed, the map \mathcal{D} should be nearly the identity map. That is, the quantities d_1 , d_2 , d_3 , and d_4 should all be zero, save for ϵ -dependent terms. In particular, if \mathcal{M}_a is correctly computed through rank 4, d_1 should differ from zero only by terms of order ϵ^4 , the matrix D corresponding to e^{d_2}

should differ from the identity matrix only by terms of order ϵ^3 , d_3 should differ from zero only by terms of order ϵ^2 , and d_4 should differ from zero only by terms of order ϵ .

This has been verified to be the case. Initially we spot checked that this was true for various entries in f_1 , D , f_3 , and f_4 . Next, as a net to catch possible errors in all entries, we devised the following strategy: To examine the departure of D from the identity I we computed the matrix (maximum column sum) norm defined by the relation²³

$$\|D - I\| = \max_k \left(\sum_j |D_{jk} - \delta_{jk}| \right). \quad (6.9)$$

To examine the departures of the polynomials f_i from zero we introduced an inner product norm defined by the rule²⁴

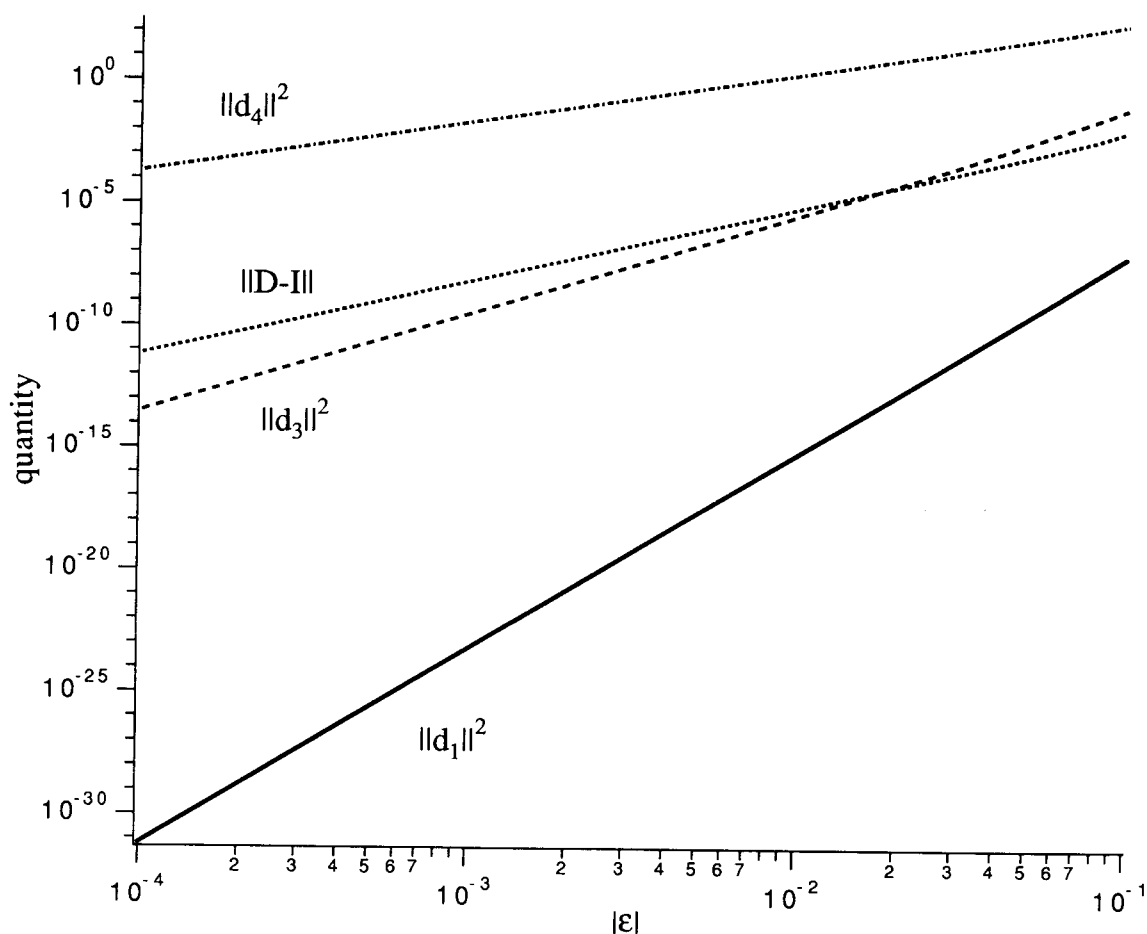


FIG. 5. A log-log plot of the quantities $\|d_1\|^2$, $\|d_3\|^2$, $\|d_4\|^2$, and $\|D-I\|$ vs $|\epsilon|$. The asymptotic slopes of the curves are 8, 4, 2, and 3, respectively. The design orbit quantities, γ , λ , etc., have the same values as in Fig. 4.

$$\|d_i\|^2 = (d_i, d_i) = \left(\frac{1}{r^2 \Omega_5} \right) \int_{\text{five-sphere}} d\Omega_5 |d_i|^2. \quad (6.10)$$

Here the Ω_5 is the solid angle of the five-sphere given by the relation^{25,26}

$$\Omega_5 = \pi^3, \quad (6.11)$$

and r is the generalized radius in six-dimensional phase space defined by the relation

$$r^2 = \sum_{a=1}^6 \xi_a^2. \quad (6.12)$$

We found that the matrix norm (6.9) indeed vanished as ϵ^3 , and the squared norms (6.10) for the polynomials vanished as ϵ^8 , ϵ^4 , and ϵ^2 for $\|d_1\|^2$, $\|d_3\|^2$, and $\|d_4\|^2$, respectively (see Fig. 5). This expected behavior verifies that all terms in \mathcal{M}_a are correctly computed through rank 4.

We conclude this section with the important observation that the transfer map for the mispowered normal entry and exit bending magnet can be written as a product of other known maps. Look again at Fig. 3, but this time ignore the part of the figure caption about $P'_\tau > 0$, and do not ignore the words in parentheses in the figure.

Consider the circular arc CDE'. It is the trajectory in the mispowered bending magnet with $\epsilon > 0$ corresponding to the initial conditions

$$X^i = P^i_x = Y^i = P^i_y = \tau^i = P^i_\tau = 0. \quad (6.13)$$

It exits the magnet with the final conditions

$$\begin{aligned} X^f &= X^0 = \lambda \left\{ \frac{\epsilon \cos \Theta + [1 - \epsilon^2 \sin^2 \Theta]^{1/2}}{1 + \epsilon} - 1 \right\}, \\ P^f_x &= P^0_x = -\epsilon \sin \Theta, \\ Y^f &= P^f_y = 0, \end{aligned} \quad (6.14)$$

$$\tau^f = \tau^0 = \{\lambda/[\beta(1+\epsilon)]\}[\sin^{-1}(\epsilon \sin \Theta) - \epsilon \Theta],$$

$$P_\tau^f = P_\tau^i = 0.$$

Here the quantities X^0 , P_x^0 , and τ^0 are obtained by evaluating equations (6.2a), (6.2b), and (6.2c) at $P_\tau^i = 0$.

We observe that the arc CD is the design trajectory for the normal entry and exit sector bend O'CD having bend angle Θ and a magnetic field B given by (5.3). Let $\mathcal{M}_s(\Theta, B)$ be the transfer map that describes trajectories around the design trajectory for this sector bend.

Next we observe that the arc DE' is the design trajectory for the *body* of the trailing half parallel face dipole magnet O'DE'O.²⁷ This magnet also has strength B , and bending angle α . The radius of curvature ρ'_0 of this "actual design trajectory" in both magnets is given by the relation

$$\rho'_0 = G/B = \rho_0/(1+\epsilon). \quad (6.15)$$

Correspondingly, by the law of sines for the triangle O'E'O, the angle α is given by the relation

$$\alpha = \sin^{-1}(\epsilon \sin \Theta). \quad (6.16)$$

Let $\mathcal{M}_b(\alpha, B)$ be the transfer map that describes trajectories around the design trajectory for the body of this trailing half parallel face dipole.

By definition, the map \mathcal{M}_s describes the relation between phase space conditions at C and D and \mathcal{M}_b describes the relation between phase space conditions at D and E'. It follows that the relation between phase space conditions at C and E' is given by the product map $\mathcal{M}_s \mathcal{M}_b$. What is wanted is the relation between phase space conditions at C and E. That is, the initial conditions are described with respect to a coordinate system at C, and the final conditions are to be described with respect to a coordinate system at E.

Observe that the exit face of the trailing half parallel face dipole at E' and the exit face of the sector bend at E lie in a common plane. Consequently, the coordinate systems at E' and E have the same orientation. It follows, after some thought, that the relation between the coordinates at E' (referenced to the actual design trajectory at E') and the coordinates at E (referenced to the ideal design trajectory at E) must consist of a simple translation in X , P_x , and τ described by the quantities X^0 , P_x^0 , and τ^0 . This line of reasoning implies that the complete transfer map from C to E can be written as the *composite* map \mathcal{M}_c given by the relation

$$\mathcal{M}_c = \mathcal{M}_s(\Theta, B) \mathcal{M}_b(\alpha, B) e^{i(-X^0 P_x + P_x^0 X - \tau^0 P_\tau)}. \quad (6.17)$$

The first two factors in (6.17) are the known maps for the sector bend and the body of a trailing half parallel face

dipole, as described previously. The last factor produces the required translation.

We have verified that \mathcal{M}_c is indeed the desired transfer map from C to E. Suppose \mathcal{M}_s and \mathcal{M}_b are computed through f_4 . Then, since the dependence on ϵ is taken into account exactly in the expressions for \mathcal{M}_s , \mathcal{M}_b , α , X^0 , P_x^0 , and τ^0 , the map \mathcal{M}_c should be correct through rank 4. Thus if \mathcal{M}_c is indeed the desired map and is computed through rank 4, as just described, the deviation map \mathcal{D} defined by the relation

$$\mathcal{D} = \mathcal{M}_a \mathcal{M}_c^{-1} \quad (6.18)$$

should be nearly the identity map in the same way as the map \mathcal{D} given by (6.7). Here, as before, \mathcal{M}_a is the approximate map given by (5.34). Alternatively, one could make a similar comparison between \mathcal{M}_c and \mathcal{M}_e . The indicated comparison between \mathcal{M}_c and \mathcal{M}_a has been made, and resulted in curves similar to those of Fig. 5.

In summary, we have verified in this section that the transfer map \mathcal{M} for the mispowered normal entry and exit bending magnet given by (5.34) is correct. We have also shown that this map can be written in the composite form (6.17). This result may be important for further applications because the maps \mathcal{M}_s and \mathcal{M}_b can be computed to high order with relative ease. Finally, we remark that the expressions (4.16) and (4.17) for the transfer map for the steering magnet have been checked in a similar manner.

VII. CONCLUDING DISCUSSION

In Sec. III we described a procedure for calculating to arbitrary accuracy the transfer map for a general Hamiltonian system providing that H_1 , the linear part of the Hamiltonian, was small. We also introduced the concept of rank, and showed that the various operations in the calculation could be organized according to rank.

In Secs. IV and V we provided examples of the use of this procedure to compute the transfer maps for a steering magnet and a mispowered normal entry and exit bending magnet.²⁸ These examples showed that this procedure can be used to treat what proved to be remarkably complicated problems. Based on that experience, we expect that the procedure may have other applications as well. For example, we hope to treat the problem of a mispowered combined function dipole.^{17,28} Finally, in Sec. VI we described checks made on the results of previous sections, and showed how the transfer map for a mispowered normal entry and exit bending magnet can be written in terms of other known maps.

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grateful to J. van Zeijts for his assistance with the program TLIE and to G. VanDerWoude for his assistance in the preparation of figures.

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- ⁷ L. M. Healy, Ph.D. thesis, Department of Physics, University of Maryland, 1986.
- ⁸ If H is an analytic function of ϵ , then so are the quantities f_i . For f_1 this can be seen as follows: By analyticity of H , ξ^{fn} as given by (1.5) is an analytic function of both ξ^{in} and ϵ . Conversely, ξ^{in} is also an analytic function of ξ^{fn} and ϵ . See, for example, F. J. Murray and K. S. Miller, *Existence Theorems for Ordinary Differential Equations* (New York Univ. Press, New York, 1954). Then (2.11) shows that f_1 is analytic in ϵ , i.e., for sufficiently small ϵ it can be expanded in a convergent power series. By examining the methods of Refs. 1–3 the reader is invited to prove analogous statements for f_i with $i \geq 2$.
- ⁹ There are other possible definitions of rank that allow for a consistent termination of the procrastination scheme after a finite number of steps (see Ref. 4). The definition employed here is appropriate in the context of accelerator physics if the machine errors described by the f_i terms are of the same order of magnitude as the phase-space extent of the beam.
- ¹⁰ Previous accelerator theory work that is closest in spirit to the present paper is described in D. C. Carey, *Nucl. Instrum. Methods Phys. Res. A* **258**, 417 (1987). For a discussion that treats error effects analytically in linear approximation, see E. D. Courant and H. S. Snyder, *Ann. Phys.* **3**, 1 (1958). For a discussion of computer simulation codes that treat error effects either in linear or in thin-lens approximation, see the following references: F. C. Iselin and E. Keil, CERN/LEP Note TH/87-37, 1987; K. Steffen and J. Kewisch, DESY Note PET-76/09, 1976; G. Guignard and Y. Marti, CERN/ISR-BOM Note TH/81-32, 1981.
- ¹¹ We remark that the map we are going to compute is really $\mathcal{M}_{\text{body}}$, the transfer map for the body of the steering magnet. The total map \mathcal{M}_{tot} for the complete steering magnet can be written in the form $\mathcal{M}_{\text{tot}} = \mathcal{M}_{\text{lf}} \mathcal{M}_{\text{body}} \mathcal{M}_{\text{rf}}$. Here \mathcal{M}_{lf} and \mathcal{M}_{rf} are the transfer maps for the leading and trailing fringe-field regions, respectively. Since the deviation variable coordinates are referenced to the design orbit (undeflected path) and this orbit is a straight line, the fringe-field maps \mathcal{M}_{lf} and \mathcal{M}_{rf} are the same as those for the normal entry and exit dipole. These maps are already known in the hard-edge approximation from previous work (see Ref. 12). Also see these theses: E. Forest, Ph.D. thesis, Department of Physics, University of Maryland, 1984; L. Sagalovsky, Ph.D. thesis, Department of Physics, University of Illinois at Urbana-Champaign, 1989. Finally, see L. Sagalovsky, *Nucl. Instrum. Methods Phys. Res. A* **298**, 205 (1990); G. E. Lee-Whiting, *ibid.* **294**, 31 (1990).
- ¹² D. Douglas, Ph.D. thesis, Department of Physics, University of Maryland, 1982.
- ¹³ L. M. Healy and A. J. Dragt (unpublished).
- ¹⁴ The calculations in this section and the next were done using Mathematica. See S. Wolfram, *Mathematica, A System for Doing Mathematics by Computer* (Addison-Wesley, Reading, MA, 1988). They were also checked independently with LISP software built on the system MAO. For a brief description of MAO, see S. Coffey, A. Deprit, E. Deprit, L. Healy, and B. Miller, in *Computer Aided Proofs in Analysis*, edited by K. R. Meyer and D. S. Schmidt, IMA Volumes in Mathematics and Its Applications (Springer-Verlag, New York, 1990), Vol. 28, pp. 97–115.
- ¹⁵ Two functions are said to be in involution if their Poisson bracket vanishes.
- ¹⁶ For details, see Ref. 7. In this connection it should be noted that there are some sign and typographical errors in Ref. 7. We believe that the results given in the present paper are correct.
- ¹⁷ A. J. Dragt, F. Neri, J. van Zeijts, and J. Diamond, University of Maryland preprint (1990).
- ¹⁸ In writing the matrix (5.9), the variables ξ are taken to be in the order $X, P_x, Y, P_y, \tau, P_\tau$.
- ¹⁹ The matrix M^1 as given in Table III is satisfactory as it stands for analytical calculations. However, if it is to be used for long-term tracking studies, it should be symplectified. The symplectification process introduces only higher-order terms in ϵ , and thus is consistent with the rank criterion (see Ref. 7).
- ²⁰ In using the f s and g s in (5.34), it is necessary to evaluate them at $\phi = \Theta$ (see Fig. 2).
- ²¹ As noted earlier, the map \mathcal{M} given by (5.34) is the transfer map for just the body of the mispowered normal entry and normal exit bending magnet. The total map for the complete bending magnet, including fringe-field effects, can again be written in the form given in Ref. 11.
- ²² J. B. J. van Zeijts, "TLIE, A Program for Charged Particle Transport Arbitrary Order Including Misalignments," University of Maryland preprint (1991).
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- ²⁷ A trailing half parallel face dipole magnet is a parallel face dipole with normal entry and non-normal exit. It can be shown that a parallel face dipole with equal entry and exit angles is composed of a leading half parallel face dipole (a parallel face dipole with non-normal entry and normal exit) followed by a trailing half parallel face dipole. Moreover, the general bending magnet with arbitrary entry and exit angles can be built from leading and trailing half parallel face dipoles and sector bends (see Refs. 7 and 13 above).
- ²⁸ Note that in both cases the magnetic field has been assumed to be that of a pure dipole. The case of a combined function dipole (a dipole that has superimposed on it normal and skew quadrupole, normal and skew sextupole, normal and skew octupole, and perhaps still higher-order multipole fields) is much harder to treat, even in the absence of errors, and at present requires numerical methods. See A. J. Dragt, F. Neri, J. B. J. van Zeijts, and J. Diamond, Numerical Third-Order Transfer Map for Combined Function Dipole (unpublished). The case of a combined function dipole with errors will also likely require numerical methods.