ABSTRACT

Title of Dissertation: QUADRATIC DIFFERENTIALS AND THE MONODROMY OF THE HITCHIN FIBRATIONS

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The purpose of this dissertation is to understand the monodromy action on the Hitchin fibration for the group $SO(2,2)$. We denote by $\mathcal{M}_{SO(2,2)}$ the moduli space of $SO(2,2)$-Higgs bundles; $\mathcal{A}_{SO_4}$ the Hitchin base for SO_4. Since $SO(2,2)$ is the split real form of $SO_4(\mathbb{C})$, the generic fibers of the Hitchin map $h : \mathcal{M}_{SO(2,2)} \to \mathcal{A}_{SO_4}$ correspond to the 2-torsion points of the $SO_4(\mathbb{C})$-Hitchin fibers.

By using the braid group as a tool, the present research is to study the fundamental group of the regular Hitchin bases involving quadratic differentials. For example, the Hitchin fibration for SL_2 is determined by a single quadratic differential. Therefore, a natural step further is to examine the Hitchin fibration for $SO(2,2)$. This Hitchin base involves two quadratic differentials. We consider the subset $\mathcal{A}^{reg}_{SO_4} \subset \mathcal{A}_{SO_4}$ over which the fibers are isomorphic to abelian varieties. In the present study, the mixed braid group is introduced to study the fundamental group of the regular $SO(2,2)$-Hitchin base $\mathcal{A}^{reg}_{SO_4}$, which has a rich structure already.

In addition, by using the data from $\pi_1(\mathcal{A}^{reg}_{SO_4})$, one can study the monodromy action on the Hitchin fibration for the group $SO(2,2)$. The study is able to develop the monodromy representation for $SO(2,2)$-Hitchin fibration directly without using the method involving low-rank exceptional isogeny $\mathcal{M}_{SL_2(\mathbb{R})\times SL_2(\mathbb{R})} \to \mathcal{M}_{SO(2,2)}$.

A direct application of this monodromy representation is to count the number of orbits under the monodromy actions. Then the number of components of the $SO(2,2)$-character variety is calculable. The orbit count based on this study is smaller than Baraglia and Schaposnik's result because this study takes an extra type of generators into consideration.

QUADRATIC DIFFERENTIALS AND THE MONODROMY OF THE HITCHIN FIBRATIONS

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Chapter 1: Overviews and the statements

Higgs bundles were introduced by Hitchin in [1]. Since then, links have been discovered between theoretical physics and complex geometry, algebraic geometry and many other areas in mathematics.

First, for a compact Riemann surface Σ of genus $g \geq 2$, Higgs bundles are defined as pairs (E, Φ) where

- E is a holomorphic vector bundle over E and
- the Higgs field $\Phi : E \to E \otimes K$ can be considered as a section in $H^0(\Sigma, End(E) \otimes K)$, where K is the canonical bundle on Σ.

For example, based on Corlette and Simpson's study [2], the nonabelian Hodge theory gives a correspondence between Higgs bundles and representations of the fundamental group of a smooth compact Riemann surface. In fact, those can be applied to a compact Kähler manifold. This gives a link between algebraic geometry and differential geometry.

Hitchin examines the correspondence between $Sp(2n)$-Higgs bundles and $SO(2n+1)$-Higgs bundles moduli spaces. He gives additional work on G_2-Higgs bundles moduli space. Later, the work of Hausel, Thaddeus [3], provides a bridge from mirror symmetry to the duality theory of Lie groups and, more broadly, to the geometric Langlands program. From [4], Donagi

and Pantev show that the Hitchin integrable system for a simple complex Lie group G is dual to the Hitchin system for the Langlands dual group of G. Recently, due to Mukhopadhyay and Wentworth, $Spin(2n)$-Higgs bundles and $PSO(2m)$ moduli spaces and $Spin(2n+1)$-Higgs bundles and $PSp(2m)$-Higgs bundles moduli spaces are also established as examples of Langlands duality [5] .

Let G be a real form of a complex semisimple Lie group G^c and Σ a compact Riemann surface. We denote by $\mathcal{M}_{G^c}$ the moduli space of G^c-Higgs bundles and $\mathcal{A}_{G^c} = \bigoplus_{i=1}^{k} H^0(\Sigma, K^{d_i})$ the Hitchin base for the group G^c. Through nonabelian Hodge theory, the moduli space $\mathcal{M}_G$ of stable G-Higgs bundles over Σ and the moduli space of representations of the fundamental group of Σ to the group G, or the character variety $Rep(G)$ of the group G.

Hitchin considered the real forms of $SL(n, \mathbb{C})$ and $GL(n, \mathbb{C})$ and did an extensive study on $SL(2, \mathbb{R})$-Higgs bundles. In addition, through an extensive study by Bradlow, Gothen, Garcia-Prada in [6] and [7], the connectivity of the moduli space of G-Higgs bundles is studied.

Furthermore, for different real forms of $SO(2n, \mathbb{C})$ and $SO(2n+1, \mathbb{C})$, the connectivity of the moduli spaces of those Higgs bundles was studied by Aparicio and Garcia-Prada.

Now we turn to the special cases related to Lie groups $SO(4, \mathbb{C})$ and $SL(2, \mathbb{C})$. The split real forms of Lie groups $SO(4, \mathbb{C})$ and $SL(2, \mathbb{C})$ are $SO(2, 2)$ and $SL(2, \mathbb{R})$.

In this thesis we consider the Hitchin fibrations of these two real forms of Lie groups $SO(2, 2)$ and $SL(2, \mathbb{R})$. Under the $SO(2, 2)$ or $SL(2, \mathbb{R})$ Hitchin fibrations, spectral curves depend only on the quadratic differentials, which are sections in $H^0(\Sigma, K^2)$. In particular, spectral curves for $SL(2, \mathbb{R})$ depend on a single quadratic differential; and spectral curves for $SO(2, 2)$ depend on a pair of qradratic differentials.

For the full group $SL(2, \mathbb{C})$, Copeland studied the monodromy of the Hitchin fibrations

for $SL(2,\mathbb{C})$ over hyperelliptic curves. Copeland considered the dual curve of a graph model of a hyperelliptic curve Σ. He considered a subspace $\mathcal{A}^{reg}$, the space of quadratic differentials with simple zeros, then $\mathcal{M}_{SL_2}|_{\mathcal{A}^{reg}}$ is a fiber bundle of abelian varieties. Then he studied the monodromy of this bundle by means of combinatoric methods. By considering the $4g-4$ simple zeros of a quadratic differential, one can construct a configuration space based on those simple zeros. The fundamental group of this configuration space is called the surface braid group. In particular, one type of generator in this surface braid group is called *swap* which consists of two strings interchanging a pair of points. By means of this combinatoric method, he studied the monodromy of this SL_2-Hitchin fibrations through a generalization of the classical Burau representation to the surface braid group. He concluded that the monodromy representation of $\pi_1(\mathcal{A}^{reg})$ is generated by the edges of his constructed graph model. In particular, if we translate his language, the monodromy representation is generated by swaps of pairs and an involution action induced by the spectral curves over Σ branched over $4g-4$ simple zeros [8].

Since $SL(2,\mathbb{R})$ is the split real form of $SL(2,\mathbb{C})$, we know what $SL(2,\mathbb{R})$-Higgs bundles look like. Based on Copeland's graph model [8] and Walker's generalization [9], Schaposnik gave an explicit monodromy representation by using matrix calculations. What is more, the study of the connectivity of the moduli space of $SL(2,\mathbb{R})$-Higgs bundles is furthered. The number of the connected components of this moduli space is calculated as well.

This thesis will focus on the study of the monodromy representation of the $SO(2,2)$-Hitchin fibrations. First, we know that $SO(2,2)$ is the split real form of the Lie group of $SO(4,\mathbb{C})$. For a compact Riemann surface Σ, the $SO(2,2)$-Hitchin fibration is

$$\mathcal{M}_{SO(2,2)} \to \mathcal{A}_{SO(4)}.$$

Based on [10], the spectral curves are generically singular. Therefore, through normalization and taking a quotient by an involution action, we get branched double $\overline{S}$ covers over Σ. Since the elements of the Hitchin base are pairs of quadratic differentials, we impose some conditions so that the double covers are smooth with ramifications points above the zeros of those two quadratic differentials.

We start from Chapter 2 to introduce the definition of Higgs bundles and G^c-Higgs bundles for some classical complex reductive Lie group G^c based on [1] and [10]. Then the concepts of the Hitchin fibration for G^c, the spectral curve, Prym variety are subsequently developed. Then through Hitchin fibrations, the spectral data is introduced by means of [11]. In addition, Hitchin fibrations for several classical Lie groups are also introduced by means of [12].

We also introduce principal G-Higgs bundles where G is a real form of a complex reductive Lie group G^c. We describe G-Higgs bundles as fixed points of the involution on the moduli space of G^c-Higgs bundles [13]. We also examine what $SL(2,\mathbb{R})$ and $SO(2,2)$-Higgs bundles look like.

In Chapter 3, we introduce the configuration space and generalize this concept to construct a mixed configuration space as a tool to study for later study. In addition, by taking surface braid group as the fundamental groups of the configuration space, we generalize this concept to the mixed surface braid group, the fundamental group of the mixed configuration space, in order to understand the fundamental group $\pi_1(\mathcal{A}^{reg})$ of the regular points in the Hitchin base. This fundamental group is important to study the monodromy actions in later chapters.

Here we examine the structure of the braid groups by means of [8] and [9]. We expand Copeland and Walker's methods to examine the structure of the mixed braid groups. Since the spectral curves depend on quadratic differentials, we give a correspondence between the genera-

tors of the mixed braid groups and closed paths in $\mathcal{A}^{reg}_{SO_4}$, which will be defined.

The spectral curve is always based on a pair of quadratic differentials. We impose the condition that they have only simple zeros with no zeros in common. We divide those $8g-8$ distinct zeros into two groups, one for each quadratic differential. We found out that the monodromy action is based on two different generators:

- a generator representing a *swap* that switches two simple zeros in one group
- a generator representing a *double swap* that traces a path from a simple zero of a differential and winds around a different simple zero of *the other differential* without touching any other zero and then goes back to the original position.

In Chapter 4 we start to give a monodromy representation of $SO(2,2)$-Hitchin fibrations. Observing that it is necessary to study how $\pi_1(\mathcal{A}^{reg}_{SO_4})$ acts on $\Lambda_{\overline{S}}[2]$, where $\overline{S}$ is a branched double cover over Σ and $\Lambda_{\overline{S}}$ can be identified with Jacobian $Jac_{\overline{S}}$ of $\overline{S}$, we generalize ideas from [14] to give a decomposition of this space $\Lambda_{\overline{S}}[2]$ into $\Lambda_{\Sigma}[2] \oplus W \oplus \Sigma[2]$. This vector space W in the middle will be studied carefully in Chapter 5. Then the monodromy representations by using matrices representing each generator from the previous paragraph can be studied.

Theorem 5.3 *The monodromy action on $\Lambda_{\overline{S}}[2]$ is generated by the groups of matrices of the form*

- $A^x_{ij} = \begin{bmatrix} I_{2g} & L^a_{ij} & S^a \\ 0 & I & (L^a_{ij})^t \\ 0 & 0 & I_{2g} \end{bmatrix}$

- $s_{ij} = \begin{bmatrix} I_{2g} & 0 & 0 \\ 0 & \sigma_{ij} & 0 \\ 0 & 0 & I_{2g} \end{bmatrix}$ and
- $M^r = \begin{bmatrix} I_{2g} & 0 & B^r \\ 0 & I & 0 \\ 0 & 0 & I_{2g} \end{bmatrix}$

In Chapter 5, we give some applications of this study. Based on the calculation from previous chapters, we can give a geometric meaning to the decomposition of $\Lambda_{\overline{S}}[2]$. In particular, we give a description of space W. Eventually, we use the previous calculation to study the connectivity of the moduli space of $SO(2,2)$-Higgs bundles. This is equivalent to calculating the number of connected components of the $SO(2,2)$-character variety.

Throrem 5.4. *The $SO(2,2)$-character variety has $2 \times 2^{2g} + 2g^2 - 2g + 1$ connected components.*

This result is different from the conclusion from [14, Corollary 7.1] in which the orbit count is $6 \cdot 2^{2g} + 4g^2 - 6g - 3$. Since we take an extra generator from the mixed braid group, the result from this thesis reduces the orbits based on the monodromy actions. Hence, the number of the connected components of the $SO(2,2)$-character variety is smaller based on our analysis.

Chapter 2: Background

In this chapter, we give basic concepts of the moduli space of the holomorphic vector bundles on a smooth Riemann surface. Hitchin studied the solution to the self-duality equation, and this leads to the concept of the Higgs pairs: holomorphic vector bundles and the corresponding Higgs fields [1]. We also introduce the concept of the moduli space of the Higgs bundles.

It is also natural to consider the moduli space of the G^c-Higgs bundles, where G^c is a complex reductive Lie group. We denote by $\mathcal{M}_{G^c}$ the moduli space of G^c-Higgs bundles and $\mathcal{A}_{G^c} = \bigoplus_{i=1}^{k} H^0(\Sigma, K^{d_i})$ the Hitchin base for the group G^c.

We will briefly describe the nonabelian Hodge correspondence, which gives a one-to-one correspondence between the moduli space $\mathcal{M}_{G^c}$ and the moduli space of the surface group representations for G^c. Detailed construction can be found in the works by Donaldson [20] and Corlette [21]. In addition, for a split real form G of G^c, one can also study the moduli space of G-Higgs bundles. In addition, based on [28], we understand that the intersection of the subspace of the Higgs bundle moduli space $\mathcal{M}_{G^c}$ corresponding to the split real form $\mathfrak{g}$ of $\mathfrak{g}^c$ with the smooth fibers of the Hitchin fibration $h : \mathcal{M}_{G^c} \to \mathcal{A}_{G^c}$ is given by the elements of order two in those fibers. Due to this result, in the later chapters, we might focus on the monodromy actions on the space of 2-torsion points of the Hitchin fibers.

Since we tend to study the monodromy action on Hitchin fibration for some certain group,

the definition of the Hitchin fibration and the concept of the spectral curves are subsequently developed. We also give the definition of SL_n and $SO(n,\mathbb{C})$-spectral data based on [11] and [10].

Since $SL_2(\mathbb{R})$ and $SO(2,2)$ are the real forms of $SL(2,\mathbb{C})$ and $SO(4,\mathbb{C})$, we can also view $SL_2(\mathbb{R})$ and $SO(2,2)$-Higgs bundles as bundles invariant under the involution action on the moduli space of the Higgs bundles. Finally, due to [15], we will get the intersection of the subspace of the Higgs bundle moduli space $\mathcal{M}_{G^c}$ which corresponds to the split real form of $\mathfrak{g}^c$ with the smooth fibers of the Hitchin fibration

$$h : \mathcal{M}_{G^c} \to \mathcal{A}_{G^c},$$

is given by the elements of order two in the fibers. In particular, the Higgs bundles for the groups $SL_2(\mathbb{R})$ and $SO(2,2)$ are described as examples. Therefore, $\mathcal{M}_{SL_2(\mathbb{R})}$ and $\mathcal{M}_{SO(2,2)}$ make sense.

2.1 Moduli spaces of Higgs bundles

Remark 2.1. For the standard background materials on Riemann surfaces, holomorphic bundles, divisors and etc. See [16].

Consider a compact Riemann surface Σ of genus $g \geq 2$ topological vector bundles over Σ can be classified by rank n and degree d.

There are several concepts related to the moduli space of vector bundles.

2.1.1 Some important definitions

Definition 2.2. The slope of a vector bundle E over Σ is

$$\mu(E) := \frac{\deg E}{\text{rank}E}.$$

Definition 2.3.

- A holomorphic vector bundle is stable if for any proper holomorphic subbundle $F \subset E$

$$\mu(F) < \mu(E).$$

- A holomorphic vector bundle E is semistable if for any proper holomorphic subbundle $F \subset E$

$$\mu(F) \leq \mu(E).$$

- A holomorphic vector bundle E is polystable if E is a direct sum of stable vector bundles with same slopes.

Proven by Narasimhan and Seshadri [17], a theorem can be translated into the following [18].

Theorem 2.4 (Narasimhan-Seshadri)**.** *Let Σ be a compact Riemann surface of genus $g \geq 2$. A holomorphic vector bundle E on Σ of degree 0 is stable if and only if E arises from an irreducible unitary representation of the fundamental group of Σ.*

In fact, there is another statement in [17] for a holomorphic vector bundle of any given degree.

This gives a natural correspondence between the representations of the fundamental groups and stable vector bundles. In fact, considering Donaldson's proof in [19] and mentioned in [18], stable vector bundles correspond to Yang-Mills connections, whose holonomy gives the representations of the fundamental group.

Definition 2.5. A Higgs bundle is a pair (E, Φ) for a holomorphic vector bundle E over Σ, and $\Phi : E \to E \otimes K$ a section in $H^0(\Sigma, \mathrm{End}(E) \otimes K)$ where K is the canonical bundle over Σ. Φ is called the Higgs field.

Definition 2.6. If subbundle F of E, $F \subset E$, satisfies the condition $\Phi(F) \subset F \otimes K$, then F is called Φ-invariant. This terminology is essential to define the stable (semistable) Higgs bundles.

Definition 2.7. A Higgs bundle (E, Φ) is stable if, for a Φ-invariant proper subbundle $F \subset E$, we have $\mu(F) < \mu(E)$.

A Higgs bundle (E, Φ) is semistable if, for a Φ-invariant subbundle $F \subset E$, we have $\mu(F) \leq \mu(E)$.

A Higgs bundle (E, Φ) is polystable if it can be considered as a direct sum $(E_1, \Phi_1) \oplus \cdots \oplus (E_l, \Phi_l)$ and each component (E_i, Φ_i) is stable with $\mu(E_1) = \cdots = \mu(E_l)$.

To each concept we introduced before: Higgs bundles and representations of the fundamental group, we can define a moduli space.

- The group of complex gauge transformations $\mathcal{G}^c$ acts on the set $\mathcal{H}$ of Higgs bundles by formula

$$g \cdot (E, \Phi) = (g \cdot E, g\Phi g^{-1}).$$

Let $\mathcal{H}^{ss}$ and $\mathcal{H}^s$ denote the subsets of semistable and stable Higgs bundles, we obtain

moduli spaces

$$\mathcal{M}^{ss} = \mathcal{H}^{ss}//\mathcal{G}^c \text{ and}$$

$$\mathcal{M}^{s} = \mathcal{H}^{s}/\mathcal{G}^c$$

in the sense of Atiyah and Bott [18].

In particular, if we let the Higgs fields $\Phi = 0$, we obtain the moduli spaces of semistable and stable holomorphic vector bundles $\mathcal{N}^{ss} \subset \mathcal{M}^{ss}$ and $\mathcal{N}^{s} \subset \mathcal{M}^{s}$.

- The set of representations $Hom(\pi_1(\Sigma), GL(n,\mathbb{C}))$ of the fundamental group of Σ is acted on by general linear group by conjugation. Similarly, by means of geometric invariant theory, we take the subsets semisimple representations $Hom^{+}(\pi_1(\Sigma), GL(n,\mathbb{C}))$ and irreducible representations $Hom^{*}(\pi_1(\Sigma), GL(n,\mathbb{C}))$ of $Hom(\pi_1(\Sigma), GL(n,\mathbb{C}))$.

  Then we define moduli spaces

$$Rep^{+} = Hom^{+}(\pi_1(\Sigma), GL(n,\mathbb{C}))//GL(n,\mathbb{C}) \text{ and}$$

$$Rep^{*} = Hom^{*}(\pi_1(\Sigma), GL(n,\mathbb{C}))/GL(n,\mathbb{C}).$$

Due to work by Donaldson [20] and later generalized by Corlette [21], we have the following nonabelian Hodge theorem, which is considered as a generalization of Narasimhan-Seshadri theorem.

Theorem 2.8. *Nonabelian Hodge theorem:*

- *A Higgs bundle can be generated by a semistable representation of the fundamental group if and only if it is polystable.*

- *A Higgs bundle can be generated by a stable representation of the fundamental group if and only if it is stable.*

This theorem can be recognized as the homeomorphism of moduli spaces.

Theorem 2.9 (Nonabelian Hodge theorem)**.** *There are homeomorphisms of moduli spaces*

$$Rep^{+} \cong \mathcal{M}^{ss} \text{ and}$$

$$Rep^{*} \cong \mathcal{M}^{s}.$$

In nonabelian Hodge theorem, the Narasimhan-Seshadri theorem can be obtained as a special case if we set the Higgs fields $\Phi = 0$. In fact, the nonabelian Hodge correspondence is crucial for our study in this thesis.

2.2 Hitchin fibration and Hitchin base

We introduce the concepts of Hitchin fibrations and also discuss the Hitchin fibrations for complex reductive Lie group G^c in this section.

For a complex reductive group G^c and the Lie algebra $\mathfrak{g}^c$ of G^c, we have a homogeneous basis $\{p_1, \ldots, p_k\}$ of the algebra of the invariant polynomials on $\mathfrak{g}^c$, where d_i is the degree of p_i. In addition, k is the rank of G^c.

Here, based on the previous section, we denote by $\mathcal{M}_{G^c}$ the moduli space of polystable G^c-Higgs bundles.

Remark 2.10. For a complex reductive group G^c, we may also add the condition of the stability

for principal G^c-bundles. The similar can be applied to stable, semistable and polystable G^c-Higgs bundles.

Based on Hitchin's construction [10], Hitchin fibration is defined as follows.

Definition 2.11. Hitchin fibration is:

$$\begin{aligned} h : \mathcal{M}_{G^c} &\to \mathcal{A}_{G^c} = \bigoplus_{i=1}^{k} H^0(\Sigma, K^{d_i}) \\ (E, \Phi) &\mapsto (p_1(\Phi), \ldots, p_k(\Phi)). \end{aligned} \tag{2.1}$$

Once again, k is the rank of G^c, and $\{p_1, \ldots, p_k\}$ is a homogeneous basis for the algebra of invariant polynomials on the Lie algebra $\mathfrak{g}^c$. Hence, the Hitchin fibration for G^c is defined.

This is called the Hitchin map, and this map is a proper map [10] for any basis.

Remark 2.12. By [10], the dimension of $\mathcal{A}_{G^c}$ and the dimension of $\mathcal{M}_{G^c}$ have the following relation:

$$\dim \mathcal{A}_{G^c} = \frac{1}{2} \dim \mathcal{M}_{G^c}.$$

Furthermore, by [10], the Hitchin fibration makes the moduli space of the Higgs bundles into a completely algebraically Hamiltonian integrable system.

In this thesis, we are specially interested in the cases when $SL_2(\mathbb{C})$ and $G^c = SO_4(\mathbb{C})$. We also discuss their corresponding split real forms $G = SL_2(\mathbb{R})$ and $SO(2,2)$.

2.3 Spectral curves

Having introduced the concept of spectral curves based on the Hitchin fibration, we describe the materials based on the construction done in [11].

We consider K the canonical bundle of Σ and $|K|$ the total space of K. We have the following projection $\pi : |K| \to \Sigma$. Consider the tautological section η of the pullback π^*K on Σ, then, η^i is a section of ith power of π^*K. For $a_i \in H^0(\Sigma, K^i)$, we consider the curve S with the equation

$$\eta^n + a_1\eta^{n-1} + a_2\eta^{n-2} + \cdots + a_n = 0 \tag{2.2}$$

in the total space $|K|$. With the appropriate conditions on a_i's, the curve S is smooth [11].

Based on [11], we have an equation $\eta^n + a_1\eta^{n-1} + a_2\eta^{n-2} + \cdots + a_n = 0$. Suppose that S is the corresponding smooth curve. Then we have the following map $\pi : S \to \Sigma$ which is a n-folded branched cover. Then we consider a line bundle L on the smooth curve $\pi : S \to \Sigma$. By the definition of the direct image, for an open set $U \subset \Sigma$, we have

$$H^0(\pi^{-1}(U), L) = H^0(U, \pi_*L).$$

Thus, a rank n bundle is obtained by taking the direct image π_*L as in [11].

In addition, since η is a section of π^*K, then multiplication by η gives a map

$$\eta : H^0(\pi^{-1}(U), L) \to H^0(\pi^{-1}(U), L \otimes \pi^*K).$$

By taking the pushforward of a line bundle on the spectral curve, we recover the vector bundle

over the original Riemann surface. Then by taking the pushforward of a section of π^*K, we recover the Higgs field

$$\Phi : \pi_*L \to \pi_*L \otimes K.$$

This is $\Phi \in H^0(\Sigma, \mathrm{End}E \otimes K)$, where $E = \pi_*L$. In addition, the characteristic equation of the Higgs field is

$$\eta^n + a_1\eta^{n-1} + a_2\eta^{n-2} + \cdots + a_n = 0.$$

And the smooth curve S is called the spectral curve. The Higgs bundle is recovered by considering this spectral data.

For the cover $\pi : S \to \Sigma$, we can define the norm map on divisor classes as follows.

Definition 2.13. For the cover $\pi : S \to \Sigma$, the norm map associated to π is

$$\mathrm{Nm} : Pic(S) \to Pic(\Sigma),$$

is defined on divisor classes. In particular,

$$\mathrm{Nm}(\sum n_i p_i) = \sum (n_i \pi(p_i)).$$

From [11, Section 4], there is a formula for the norm map.

Theorem 2.14 ([11])**.** *For a branched cover $\pi : S \to \Sigma$ and a line bundle over S,*

$$\begin{aligned} Nm(L) &\cong \det(\pi_*L) \otimes (\det(\pi_*\mathcal{O}))^{-1} \\ &\cong \det(\pi_*L) \otimes K^{n(n-1)/2}. \end{aligned} \tag{2.3}$$

Definition 2.15. The kernel of the norm map is called the Prym variety. We use the notation $Prym(S, \Sigma)$.

$$Prym(S, \Sigma) := \ker(\mathrm{Nm}).$$

According to [11, Proposition 3.6], we have the following theorem.

Theorem 2.16 ([11])**.** *Let Σ be a smooth compact Riemann surface and $a = (a_i) \in \bigoplus_{i=1}^{n} H^0(\Sigma, K^i)$ in the Hitchin base. We assume that the corresponding spectral curve S is smooth. Then there is a bijection correspondence between isomorphism classes of line bundles on S and isomorphism classes of Higgs pairs (E, Φ) where E is a vector bundle of rank n and $\Phi : E \to E \otimes K$ a homomorphism with characteristic coefficients (a_i).*

This correspondence from [11] identifies the fiber of the Hitchin map with Picard variety. But we can shift the degree of the given degree of the Picard variety to make it degree 0. Hence, it becomes a point in the Jacobian of S. We use the notation $Jac(S)$. This is an abelian variety of S. Thus, the Hitchin fiber of

$$h : \mathcal{M} \to \mathcal{A}$$

can be isomorphic to the Jacobian of the spectral curve S. Therefore, the Hitchin fibration makes the moduli space of Higgs bundles a completely algebraically integrable system.

Definition 2.17. Here, we call the points in the Hitchin base with fibers isomorphic to abelian varieties *regular points*. They form a subset of the Hitchin base. We denote by $\mathcal{A}^{reg} \subset \mathcal{A}$ the regular points of the Hitchin base.

The non-regular points in the Hitchin base will cause singular situations. Therefore, it is interesting to study the monodromy action of $\mathcal{A}^{reg}$ on the Hitchin fibration.

Now, we will consider Higgs bundles for some classical complex Lie groups G^c as well. One can observe that those special Higgs bundles for groups G^c consist of the regular Higgs pairs (E, Φ). But in addition, the underlying bundles and the Higgs fields should satisfy some additional conditions, for both underlying bundles and Higgs fields.

Now we discuss Hitchin fibrations for these two groups $G^c = SL(n, \mathbb{C})$ and $SO(n, \mathbb{C})$.

2.3.1 $G^c = SL(n, \mathbb{C})$

When $G^c = SL(n, \mathbb{C})$, the invariant polynomials of the Lie alebra $\mathfrak{sl}(n, \mathbb{C})$ is the invariant polynomial of a matrix $A \in \mathfrak{sl}(n, \mathbb{C})$ with $\text{tr}(A) = 0$. That means $a_1 = 0$.

Therefore, according to [10], an $SL(n, \mathbb{C})$-Higgs bundle is a Higgs pair (E, Φ) where E has trivial determinant $\det(E) \cong \mathcal{O}$ and the trace of the Higgs field is zero. Then the corresponding spectral curve $\pi : S \to \Sigma$ as defined above has an equation

$$\eta^n + a_2\eta^{n-2} + a_3\eta^{n-3} + \cdots + a_n = 0,$$

where $a_i \in H^0(\Sigma, K^i)$. The equation above is the characteristic equation of the Higgs field.

Remark 2.18. By [22] and [10], we can use the adjunction formula to show that

$$g_S = 1 + n^2(g - 1),$$

where g is the genus of the curve Σ.

Then the coefficients of all these $a_i \in H^0(\Sigma, K^i)$ form an $SL(n, \mathbb{C})$-Hitchin base. We have

the following

$$h : \mathcal{M}_{SL(n,\mathbb{C})} \to \mathcal{A}_{SL(n,\mathbb{C})},$$

where $\mathcal{A}_{SL(n,\mathbb{C})} = \bigoplus_{i=2}^{n} H^0(\Sigma, K^i)$.

Based on [11], given a line bundle L of a spectral curve S, we can recover the Higgs bundle $E = \pi_* L$. In addition, since E has trivial determinant, we can calculate that

$$\mathrm{Nm}(L) \cong \det(\pi_* L) K^{n(n-1)/2}.$$

Therefore, $\mathrm{Nm}(L) \cong K^{n(n-1)/2}$. In addition, by shifting L by tensoring $K^{-n(n-1)/2}$, we have a line bundle $L \otimes \pi^* K^{-n(n-1)/2}$ with $\mathrm{Nm}(L \otimes \pi^* K^{-n(n-1)/2}) = \mathcal{O}$. The analysis above picks a square root of K.

Theorem 2.19 ([10]). *The generic fiber of SL_2-Hitchin fibration is equivalent to the Prym variety $Prym(S, \Sigma)$ of the corresponding spectral curve S.*

Example 2.20. For $G^c = SL(2, \mathbb{C})$, it is a special case of the previous analysis.

For a quadratic differential $a \in H^0(\Sigma, K^2)$ and a tautological section λ of $\pi^* K$, the spectral curve comes from the polynomial

$$\lambda^2 + a = 0.$$

The solution to this polynomial gives SL_2-spectral curves S [1].

It is easy to see that the smoothness of S depends on the condition that a has only simple zeros, i.e., there are no repeated zeros of a. Then we define $\mathcal{A}^{reg}_{SL_2}$ to be the regular locus of the Hitchin fibration $\mathcal{M}_{SL_2} \to \mathcal{A}_{SL_2}$ [10]. In addition, based on the previous theorem, the generic fibers of the map $h : \mathcal{M}_{SL_2} \to \mathcal{A}_{SL_2}$ are equivalent to the Prym variety $Prym(S, \Sigma)$ associated

to $\pi : S \to \Sigma$.

2.3.2 $G^c = SO(2n+1, \mathbb{C})$

Now we consider the case when $G^c = SO(2n+1, \mathbb{C})$. We let $\mathfrak{so}(2n+1, \mathbb{C})$ be the Lie algebra of the complex Lie group $SO(2n+1, \mathbb{C})$. Based on [10], for a matrix $A \in \mathfrak{so}(2n+1, \mathbb{C})$, there are $2n+1$ distinct eigenvalues $\pm\lambda_1, \ldots, \pm\lambda_n$ and 0. So the invariant polynomial of the Higgs field Φ is

$$\det(\eta I - A) = \eta(\eta^{2n} + a_1\eta^{2n-2} + \cdots + a_n), \tag{2.4}$$

where $\{a_1, \ldots, a_n\}$ is a basis for the polynomial on $\mathfrak{so}(n+1, \mathbb{C})$.

The zero eigenspace links $\mathfrak{so}(2n+1)$ and $\mathfrak{sp}(2n)$ in the duality [12].

Since $SO(2n+1) \subset SL(2n+1)$, we consider a holomorphic vector bundle E of rank $2n+1$, with trivial determinant and a non-degenerate symmetric bilinear form $g(v, w)$ such that Φ satisfies $g(\Phi v, w) + g(v, \Phi w) = 0$.

Remark 2.21. In particular, the moduli space of the $SO(2n+1)$-Higgs bundles consists of two components, classified by a class called the Stiefel-Whitney class $w_2 \in H^2(\Sigma, \mathbb{Z}_2) \cong \mathbb{Z}_2$. The two choices depend on whether the bundle E can have a lift to a spin bundle or not.

From (2.4), it determines a spectral curve whose equation is

$$\eta^{2n} + a_1\eta^{2n-2} + \cdots + a_n = 0,$$

where $a_i \in H^0(\Sigma, K^{2i})$. A spectral curve S with the equation above is $2n$-folded branched cover over Σ.

Therefore, this gives the desired Hitchin map

$$h : \mathcal{M}_{SO_{2n+1}} \to \mathcal{A}_{SO_{2n+1}}.$$

Based on the work in [12], the curve S has an involution action σ such that $\sigma : \eta \mapsto -\eta$. Then we have a quotient curve $\overline{S} := S/\sigma$ and $\overline{\pi} : S \to \overline{S}$ is a double cover.

It can be shown that the generic Hitchin fiber can be identified with the Prym variety $Prym(S, \overline{S})$ associated to this double cover $\overline{\pi} : S \to \overline{S}$.

Detailed description of spectral data for odd orthogonal group $SO(2n+1, \mathbb{C})$ can be found both in [10] and [12].

2.3.3 $G^c = SO(2n, \mathbb{C})$

We also consider the even orthogonal group $G^c = SO(2n, \mathbb{C})$. This has a similar situation: a matrix $A \in \mathfrak{so}(2n, \mathbb{C})$ has the characteristic polynomial

$$\det(\eta I - A) = \eta^{2n} + a_1\eta^{2n-2} + \cdots + a_n = 0.$$

This invariant polynomial has distinct eigenvalues $\pm\lambda_1, \ldots, \pm\lambda_n$. In this collection, a_n is always a square p_n^2 of some p_n. We call this $p_n \in H^0(\Sigma, K^n)$ the Pfaffian [10]. $(a_1, \ldots, p_n)$ is in the $SO(2n)$-Hitchin base $\mathcal{A}_{SO(2n,\mathbb{C})} = \bigoplus_{i=1}^{n-1} H^0(\Sigma, K^{2i}) \oplus H^0(\Sigma, K^n)$.

In particular, (E, Φ) is an $SO(2n, \mathbb{C})$-Higgs pair, with E a holomorphic vector bundle with

a non-degenerate symmetric bilinear form g and $\Phi \in H^0(\Sigma, \mathrm{End}(E) \otimes K)$ the Higgs field with

$$g(\Phi v, w) + g(v, \Phi w) = 0.$$

Then the character polynomial $\det(\eta I - \Phi)$ of a Higgs bundle (E, Φ) is given by the equation

$$\det(\eta I - \Phi) = \eta^{2n} + a_1 \eta^{2n-2} + \cdots + p_n^2,$$

for $(a_1, \ldots, p_n) \in \bigoplus_{i=1}^{n-1} H^0(\Sigma, K^{2i}) \oplus H^0(\Sigma, K^n)$. We use the equation above to obtain a spectral curve S, which is a $2n$- folded cover S over Σ. We can also consider the following Hitchin map for $SO(2n, \mathbb{C})$:

$$h : \mathcal{M}_{SO_{2n}} \to \mathcal{A}_{SO_{2n}} = H^0(\Sigma, K^{2i}) \oplus H^0(\Sigma, K^n).$$

Since p_n^2 is always a square, this curve S is always singular, with singularities zeros of the Pfaffian p_n. The curve has a natural involution $\sigma(\eta) = -\eta$ as well. According to [10], we take the normalization $\tilde{S}$ of S. Then by adjunction formula again, we have $g_S = 1 + 4n^2(g-1)$. In addition, by formula from [22], we get

$$g_{\tilde{S}} = g_S - \deg(p_n)$$

$$= 1 + 4n^2(g-1) - 2n(g-1) = 1 + 2n(2n-1)(g-1).$$

In addition, the involution action σ extends to an involution $\tilde{\sigma}$ on $\tilde{S}$. This $\tilde{\sigma}$ has no fixed points. Therefore, this action $\tilde{\sigma}$ is a free action. We define $\overline{S} := \tilde{S}/\tilde{\sigma}$. Since the involution action

has no fixed points, this $\overline{S}$ is generically a smooth curve over Σ.

In the even orthogonal case, we need to use line bundles on $\tilde{S}$, the normalization of S, to recover Higgs bundles on Σ. Detailed discussion can also be found in [10]. Here, we state the results.

For a Higgs field Φ, we can obtain a line bundle M on $\tilde{S}$ based on $\ker(\eta I - \Phi)$. In particular, this bundle satisfies the condition

$$\tilde{\sigma}(M) \cong M^* \otimes (K_{\tilde{S}} K^*)^{-1}.$$

Once again, by picking a square root $K^{1/2}$ of K, we define $L := M \otimes (K_{\tilde{S}} K^*)^{1/2}$. Thus, by this operation, L is in $Prym(\tilde{S}, \overline{S})$.

This Prym variety $Prym(\tilde{S}, \overline{S})$ of $\tilde{S}$ is a generic fiber of the Hichin fibration for $SO_{2n}(\mathbb{C})$. We can also check the dimension of this Prym variety.

Since $\tilde{\pi} : \tilde{S} \to \overline{S}$ is an unramified cover, then $2 - 2g_{\tilde{S}} = 2(2 - 2g_{\overline{S}})$. Thus, we have the following.

$$\begin{aligned} \dim(Prym(\tilde{S}, \overline{S})) &= \dim(Jac(\tilde{S})) - \dim(Jac(\overline{S})) \\ &= g_{\tilde{S}} - g \\ &= g_{\tilde{S}} - \frac{1 - g_{\overline{S}}}{2} \\ &= n(2n-1)(g-1). \end{aligned} \tag{2.5}$$

2.4 Higgs bundles for real forms

2.4.1 Real forms

Based on [10], we have the following property of stable Higgs bundles on a compact Riemann surface Σ of genus $g \geq 2$.

Theorem 2.22. *If a Higgs bundle (E, Φ) is stable and deg(E)=0, then there is a unique unitary connection A on E such that*

$$F_A + [\Phi, \Phi^*] = 0 \in \Omega^{1,1}(\Sigma, End E), \tag{2.6}$$

where F_A is the curvature of the connection.

In addition, 2.6 combined with the condition $d''_A \Phi = 0$, we get the Hitchin equations.

Based on [15], Hitchin equations can be considered when A is a connection on a principal G-bundles P, where G is a real form of a complex Lie algebra G^c with a real structure. Therefore, it is necessary to give definitions of real forms.

Definition 2.23. A real form $\mathfrak{g}$ of $\mathfrak{g}^c$ is a Lie algebra satisfying:

$$\mathfrak{g}^c = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}.$$

In fact, it can be considered as the set of fixed points of an antilinear involution τ on $\mathfrak{g}^c$.

For the adjoint operation ad() of $\mathfrak{g}^c$ given by Lie bracket

$$\mathrm{ad}X(Y) = [X, Y] \text{ for } X, Y \in \mathfrak{g}^c.$$

We define the bilinear form

$$B(X,Y) = \text{tr}(\text{ad}X\text{ad}Y)$$

the Killing form on $\mathfrak{g}^c$.

Definition 2.24. A real Lie algebra $\mathfrak{g}$ is called compact if the Killing form is negative definite on $\mathfrak{g}$.

Definition 2.25. A real form $\mathfrak{g}$ of a finite-dimensional complex semisimple Lie algebra $\mathfrak{g}^c$ is said to be split if in each Cartan decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, the space $\mathfrak{h}$ contains a maximal abelian subalgebra of $\mathfrak{g}$, i.e. its Cartan subalgebra. Cartan proved that every complex semisimple Lie algebra $\mathfrak{g}^c$ has a split real form, which is unique up to isomorphism [23, P426].

In fact, all Cartan subalgebras $\mathfrak{h}$ of $\mathfrak{g}$ have the same dimension. We call the rank of $\mathfrak{g}$ to be this dimension.

Example 2.26. $\mathfrak{sl}(n,\mathbb{R})$ is the split real form of $\mathfrak{sl}(n,\mathbb{C})$.

Example 2.27. $\mathfrak{so}(n,n)$ is the split real form of $\mathfrak{so}(2n,\mathbb{C})$.

Example 2.28. In particular, $\mathfrak{sl}(2,\mathbb{R})$ is the split real form of $\mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{so}(2,2)$ is the split real form of $\mathfrak{so}(4,\mathbb{C})$.

We recall that nonabelian Hodge theory on the compact Riemann surface Σ corresponds the moduli space of reductive representations of $\pi_1(G^c)$ to the moduli space of G^c-Higgs bundles.

The antilinear involution in G^c in the moduli space of the representations can be considered as a holomorphic involution Θ of the moduli space of G^c-Higgs bundles.

We try to use this construction to get G-Higgs bundle. For a connection A which solves equation (3) and a flat connection

$$\nabla = \nabla_A + \Phi + \Phi^*,$$

it requires ∇ to have holonomy in G. Here τ is the real structure of G and the corresponding Lie algebra is $\mathfrak{g}$.

Based on [13, 3.1.2], ∇ has holonomy in G if ∇_A is invariant under σ and Φ is anti-invariant.

For G a real form of a complex semisimple Lie group G^c, we can construct G-Higgs bundles. In fact, let H be the maximal compact subgroup of G, and the corresponding Cartan decomposition is

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m},$$

where $\mathfrak{h}$ is the Lie algebra of H. Based on this construction, we restrict the adjoint representation to H^c to get

$$\mathrm{Ad}|_{H^c} : H^c \to GL(\mathfrak{m}^c).$$

Then based on [7, 2.2] and [13, Definition 3.12], we have the following.

Definition 2.29. A principal G-Higgs pair is a pair (P, Φ) such that

1. P is a holomorphic principal H^c-bundle over Σ and
2. Φ is a section in $H^0(\Sigma, P \times_{\mathrm{Ad}} \mathfrak{m}^c \otimes K)$.

Now we examine the case when $G = SL(n, \mathbb{R})$ and $G = SO(n, n)$.

2.4.2 Real forms of $SO(n, \mathbb{C})$

$G = SO(p, q)$

According to [23], $\mathfrak{so}(n)$ is the compact form of $\mathfrak{g}^c = \mathfrak{so}(n, \mathbb{C})$ with $n = p + q$. We take a map

$$\theta(X) = I_{p,q} X I_{p,q} \text{ for } p \geq q.$$

.

Since we consider the action θ, there is a decomposition $\mathfrak{u} = \mathfrak{h} + i\mathfrak{m}$ where

$$\mathfrak{h} = \{ \begin{bmatrix} X_1 & 0 \\ 0 & X_3 \end{bmatrix} | X_1 \in \mathfrak{so}(p), X_2 \in \mathfrak{so}(q) \},$$

$$\mathfrak{m} = \{ \begin{bmatrix} 0 & X_2 \\ X_2^t & 0 \end{bmatrix} | X_2 \text{ a } p \times q \text{ matrix} \}.$$

.

Therefore, the real form $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ of the $\mathfrak{so}(p, q)$ is

$$\mathfrak{so}(p, q) = \{ \begin{bmatrix} X_1 & X_2 \\ X_2^t & X_3 \end{bmatrix} | X_1 \in \mathfrak{so}(p), X_2 \in \mathfrak{so}(q) \}.$$

Then the induced non-compact real forms $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ of $\mathfrak{so}(n, \mathbb{C})$ are the following.

- If $n = p + q$ is even, $\mathfrak{g}$ is split real form if and only if $p = q$,
- If $n = p + q$ is odd, $\mathfrak{g}$ is split real form if and only if $p = q + 1$.

Proposition 2.30. *There exists an involution of the Higgs pair* $\Theta : (E, \Phi) \to (E, -\Phi)$ *induced by the involution of the Lie algebra* $\mathfrak{so}(n, \mathbb{C})$.

Then the $SO(p, q)$-Higgs bundles are fixed points of the involution of the Higgs bundles E which sends Φ to $-\Phi$.

Therefore, $SO(p, q)$-Higgs bundles can be formally classified as follows.

The maximal compact subalgebra of $\mathfrak{so}(p, q)$ is $\mathfrak{so}(p) \oplus \mathfrak{so}(q)$ and the Cartan decomposition of the Lie algebra $\mathfrak{so}(p + q, \mathbb{C}) = (\mathfrak{so}(p) \oplus \mathfrak{so}(q)) \oplus \mathfrak{m}$, where the orthogonal complement

$$\mathfrak{m} = \{ \begin{bmatrix} 0 & \beta \\ -\beta^T & 0 \end{bmatrix} | X_2 \text{ is a real } p \times q \text{ matrix} \}.$$

Definition 2.31. An $SO(p, q)$-Higgs bundle is a Higgs pair (E, Φ) where E is of rank $n = p + q$. In addition, $E = V \oplus W$ for V is a complex vector bundle of rank p with orthogonal structure and W is a complex vector bundle of rank q with orthogonal structure. In particular, the Higgs field Φ is of the form

$$\Phi = \begin{bmatrix} 0 & \beta \\ -\beta^T & 0 \end{bmatrix}$$

where β is a section in $H^0(\Sigma, Hom(W, V) \otimes K)$. Therefore, the Higgs field Φ is a seciton in $H^0(\Sigma, Hom(W, V) \oplus Hom(V, W) \otimes K)$.

2.4.3 Real forms of $SL(n, \mathbb{C})$

$G = SL(n, \mathbb{R})$

The compact form $\mathfrak{u} = \mathfrak{su}(n)$ of $\mathfrak{g}^c = \mathfrak{sl}(n, \mathbb{C})$. There is the involution by taking the

conjugation. Thus, the decomposition $\mathfrak{u}$ is given by

$$\mathfrak{h} = \mathfrak{so}(n) \text{ and}$$

$$\mathfrak{m} = \text{ real } n \times n \text{ matrices of trace } 0.$$

And $\mathfrak{u} = \mathfrak{h} + i\mathfrak{m}$.

In addition, the split real form of $\mathfrak{sl}(n, \mathbb{C})$ is $\mathfrak{sl}(n, \mathbb{R})$.

According to [13], we have the following.

Proposition 2.32. *The involution action*

$$\Theta : (E, \Phi) \to (E, \Phi^T)$$

acts on the moduli space of $SL(n, \mathbb{C})$ Higgs bundles. The $SL(n, \mathbb{R})$-Higgs bundles are the fixed points of this involution action Θ.

Since the real form of $\mathfrak{g}^c$ induces an involution on the moduli space $\mathcal{M}_{G^c}$ of G^c-Higgs bundles. The involution action is :

$$\Theta : (E, \Phi) \to (\sigma(E), -\sigma(\Phi)).$$

Based on Schaposnik's analysis [13, Theorem 4.12], we have the following theorem.

Theorem 2.33 ([13])**.** *The intersection of the subspace of the Higgs bundle moduli space $\mathcal{M}_{G^c}$*

corresponding to the split real form $\mathfrak{g}$ *of* $\mathfrak{g}^c$ *with the smooth fibers of the Hitchin fibration*

$$h : \mathcal{M}_{G^c} \to \mathcal{A}_{G^c},$$

is given by the elements of order two in those fibers.

Example 2.34. $SL_2(\mathbb{R})$ is the split real form of $SL_2(\mathbb{C})$. Therefore, one can consider the Hitchin map

$$h : \mathcal{M}_{SL_2(\mathbb{R})} \to \mathcal{A}_{SL_2(\mathbb{R})}.$$

In fact, there is no difference between $\mathcal{A}_{SL_2(\mathbb{R})}$ and $\mathcal{A}_{SL_2(\mathbb{C})}$, since they are both space of quadratic differentials $H^0(\Sigma, K^2)$.

According to the theorem above, the generic Hitchin fibers of $h : \mathcal{M}_{SL_2(\mathbb{R})} \to \mathcal{A}_{SL_2(\mathbb{R})}$ correspond to 2-torsion points in the Prym variety $Prym(S, \Sigma)$ associated to the cover $\pi : S \to \Sigma$.

Example 2.35. $SO(2,2)$ is the split real form of $SO_4(\mathbb{C})$. Therefore, one can consider the Hitchin map

$$h : \mathcal{M}_{SO(2,2)} \to \mathcal{A}_{SO_4(\mathbb{C})}.$$

In fact, there is no difference between $\mathcal{A}_{SO_4(\mathbb{C})}$ and $\mathcal{A}_{SO(2,2)}$, since they are both space of quadratic differential pairs $H^0(\Sigma, K^2) \oplus H^0(\Sigma, K^2)$.

Once again, the generic Hitchin fibers of $h : \mathcal{M}_{SO(2,2)} \to \mathcal{A}_{SO_4(\mathbb{C})}$ correspond to 2-torsion points in the Prym variety $Prym(\tilde{S}, \overline{S})$ associated to the unramified cover $\tilde{\pi} : \tilde{S} \to \overline{S}$.

In fact, given a complex reductive Lie group G^c and a Lie group involution σ defining the split real form of G^c, we have the involution action $\Theta : (E, \Phi) \to (\sigma(E), -\sigma(\Phi))$ on the moduli

space $\mathcal{M}_{G^c}$. In particular, the fixed points of the involution $\Theta : (E, \Phi) \to (\sigma(E), -\sigma(\Phi))$ on $\mathcal{M}_{G^c}$ give a covering space of the regular locus of A^{reg}.

In addition, given the covering space described above, the actions of $\pi_1(\mathcal{A}^{reg})$ yield a permutation on the fibers. In the case of the split real form $SL(n, \mathbb{R})$ of $SL(n, \mathbb{C})$, the fixed points of Θ in $\mathcal{M}_{SL(n,\mathbb{C})}$ are given by points of order two in the Prym varieties. In the following chapters we shall study the particular case of $SL(2, \mathbb{R})$-Higgs bundles and $SO(2, 2)$-Higgs bundles via the action of $\pi_1(\mathcal{A}^{reg})$.

Chapter 3: Braid group and the quadratic differentials

As we have seen from the previous chapter, we intend to understand the monodromy action on Hitchin fibrations

$$h : \mathcal{M}_G \to \mathcal{A}_G$$

for group $SO(2,2)$. Therefore, it is essential to understand the fundamental group of the subset $\mathcal{A}^{reg}_{SO_4(\mathbb{C})}$ of the Hitchin base. Recall that $\mathcal{A}^{reg}$ consists of the regular points over which the fibers are equivalent to abelian varieties.

We are specially interested in two groups $G = SL_2(\mathbb{R})$ and $SO(2,2)$, since the Hitchin bases for those groups involve only quadratic differentials. We will examine the regular points in the Hitchin bases for $G = SL_2(\mathbb{R})$ and $SO(2,2)$. We call them $\mathcal{A}^{reg}_{SL_2}$ and $\mathcal{A}^{reg}_{SO_4}$.

In order to examine the monodromy action on Hitchin fibrations for groups $SL_2(\mathbb{R})$ and $SO(2,2)$, we need to understand the fundamental groups of $\mathcal{A}^{reg}_G$ for $G = SL_2(\mathbb{R})$ and $SO(2,2)$. We will see that those fundamental groups have a close relationship with the braid groups.

We will define the braid group by defining the configuration space, which is nth Cartesian product of the Riemann surface Σ with distinct tuples. Then the braid group is defined to be the fundamental group of the so-called unordered configuration space $\Sigma^{[n]}$, which is the quotient of the configuration space by the symmetric group on n elements. Then the Braid group $B_n(\Sigma) := \pi_1(\Sigma^{[n]})$.

We describe the construction done by [8] and show that there is a natural map $\rho : \mathcal{A}^{reg}_{SL_2} \to \Sigma^{[4g-4]}$. Followed by the Abel-Jacobi map to $Jac(\Sigma)$, Copeland used the combinatorial method to show that $\rho_*\pi_1(\mathcal{A}^{reg}_{SL_2(\mathbb{R})}) = \ker(B_{4g-4}(\Sigma) \to H_1(\Sigma, \mathbb{Z}))$. In addition, we know that $\pi_1(\mathcal{A}^{reg}_{SL_2})$ consists of so-called *swaps* interchanging pairs of points. Although Copeland's method [8] works for hyperelliptic curves, Walker's work in [9] extended the result to any smooth compact Riemann surface Σ.

This study intends to extend this method to the Hitchin fibration for $SO(2,2)$. Since $\mathcal{A}^{reg}_{SO_4}$ consists of pairs of quadratic differentials, we construct the so-called mixed braid group $B_{n,m}(\Sigma)$. We introduce the concept of mixed configuration space $\Sigma^{[n]+[m]}$, which consists of $n+m$ distinct tuples of points in Σ in two groups:a group of m points and a group of n points. We denote by $B_{n,m}(\Sigma)$ the fundamental group of the mixed configuration space. By using results in [8] and [25], we know the generators of the mixed braid group.

By doing the similar analysis, we show that there is a natural map $\rho : \mathcal{A}^{reg}_{SO_4} \to \Sigma^{[4g-4]+[4g-4]}$. In addition, we can show that $\rho_*\pi_1(\mathcal{A}^{reg}_{SO_4}) = \ker(B_{4g-4,4g-4}(\Sigma) \to H_1(\Sigma, \mathbb{Z}) \times H_1(\Sigma, \mathbb{Z}))$. Those correspond to the aforementioned *swaps* and so-called *double swaps*. Furthermore, we can lift those generators in $\mathcal{A}^{reg}_{SO_4}$ to find paths corresponding to those *swaps* and *double swaps* by examining the nature of the quadratic differentials according to the study done by [9].

3.1 Hitchin bases for $SL_2(\mathbb{R})$ and $SO(2,2)$

Since we are using some arguments from the previous results for the $SL_2(\mathbb{R})$-Hitchin fibration, we give all the results we are using in the later chapters here.

For a quadratic differential $a \in H^0(\Sigma, K^2)$ and a tautological section λ of π^*K, the

$SL_2(\mathbb{R})$-Hitchin fibration comes from polynomial

$$\lambda^2 + a = 0.$$

The solution to this polynomial gives $SL_2(\mathbb{R})$-spectral curves S [10].

It is easy to see that the smoothness of S depends on the condition that a has only simple zeros, i.e., there is no repeated zeros of a. Then we define $\mathcal{A}_{SL_2}^{reg}$ to be the regular locus of the Hitchin fibration $\mathcal{M}_{SL_2} \to \mathcal{A}_{SL_2}$ [10].

We would like to give a description of the fundamental group of the Hitchin base $\mathcal{A}_{SO_4}^{reg}$ as well. By examining $\mathcal{A}_{SO_4}^{reg}$, we will be able to study the monodromy action of the $SO(2,2)$-Hitchin base on the corresponding Hitchin fibers. At first, it is reasonable to study the fundamental group of the $SO(2,2)$-Hitchin base. The moduli space has two connected components which correspond to the value of the second Stiefel-Whitney class of the underlying $SO(2,2)$-Higgs bundle.

Recall that we have a pair $a_2, p \in H^0(\Sigma, K^2)$ from the equation

$$\lambda^4 + a_2\lambda^2 + p^2 = 0.$$

The corresponding curve $S \subset |K|$ defined by the equation is always singular [10]. Therefore we take the normalization $\tilde{S}$. Thus, the involution action $\sigma : \lambda \to -\lambda$ lifts to a free action $\tilde{\nu} : \tilde{S} \to S$ on $\tilde{S}$. Then we let $\overline{S}$ to be the quotient of $\tilde{S}$ by the action $\tilde{\nu}$. Then we have $\tilde{\pi} : \tilde{S} \to \overline{S}$. This quotient $\overline{S}$ can be identified with S/ν. In addition, we consider the projection $\overline{\pi} : K^2 \to \Sigma$ and a

tautological section y of $\overline{\pi}^* K^2$. Then $\overline{S} \subset |K^2|$ is from the equation

$$y^2 + a_2 y + p^2 = 0.$$

The surface $\overline{S}$ is smooth if and only if the discriminant $\Delta = a_2^2 - 4p^2$ has only simple zeros. In addition, it always factors as a product $\Delta = c_1 c_2$, where $c_1 = a_2 + 2p$ and $c_2 = a_2 - 2p$. So we have a pair of quadratic differentials c_1 and c_2 and each one of them has only simple zeros and no zeros in common [24].

3.2 Configuration space

Let Σ be any compact Riemann surface. Let Σ^n be the n-th Cartesian product of the space Σ. In addition, we can also define an unordered space of Σ. It is the n-fold symmetric product of the space Σ, which is the the quotient of Σ^n by S_n, the symmetric group on n elements [25].

The nth Cartesian product Σ^n of the space Σ has a disadvantage. The paths of the above space can generate arbitrary strings. Since we are interested in the case when the strings never cross each other, it is necessary to consider the subspace of the configuration space consisting of only n distinct points. This can certainly achieved by eliminating any collapsing situation in which $p_i = p_j$ for some $1 \leq i < j \leq n$, i.e., p_i and p_j collapse. We call this subset the "fat diagonal Δ". Now we focus on the subspace Y of Σ^n with the fat diagonal Δ removed. So $Y = \Sigma^n \setminus \Delta$. Since the space has dimension more than 2, this space is connected.

Definition 3.1. For a topological space Σ, the nth configuration space of Σ is the set of n-tuples

of distinct points in Σ.

$$\text{Conf}_n(\Sigma) := \Sigma^n \setminus \Delta.$$

This space is endowed with subspace topology based on the inclusion of $\text{Conf}_n(\Sigma)$ into Σ^n.

Since we consider the n distinct points, there is an action of symmetric group S_n on n elements on this configuration space $\text{Conf}_n(\Sigma)$:

$$\begin{aligned} S_n \times \text{Conf}_n(\Sigma) &\to \text{Conf}_n(\Sigma) \\ (\sigma, x) &\mapsto \sigma(x) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)}). \end{aligned} \tag{3.1}$$

Definition 3.2. This natural action gives us the following topological space

$$\Sigma^{[n]} := \text{Conf}_n(\Sigma)/S_n.$$

We call it nth unordered configuration space of Σ.

From now on, we are using the notation $\Sigma^{[n]}$ to denote the unordered configuration space of Σ.

Definition 3.3. The braid group:

- The nth braid group of a connected topological space Σ is

$$B_n(\Sigma) := \pi_1(\Sigma^{[n]}),$$

the fundamental group of the nth unordered configuration space of Σ.

- The nth pure braid group of a connected topological space Σ is

$$PB_n(\Sigma) := \pi_1(\text{Conf}_n(\Sigma)),$$

the fundamental group of the nth configuration space of Σ.

Since we focus on the unordered configuration space and the braid group, the fundamental group of the unordered configuration space, then the element of the braid group consists of n strings. They orient vertically from the bottom to the top.

A general n-braid group can be constructed by iteratively applying the σ_i, $1 \leq i \leq n$ operator, which interchanges the ith and the $(i+1)$th strings. In fact, those operators keep all the endpoints fixed while tracing paths which switch ith and $(i+1)$th endpoints. Since it takes place in the unordered configuration space, it will be an element of the braid group.

We consider an open disk U in Σ and $X \in U^{[n]} \subset \Sigma^{[n]}$. This means we pick n distinct points in Σ, and by the classic braid group theory, the fundamental group $\pi_1(U^{[n]}, X)$ is generated by transpositions $\sigma_1, \ldots, \sigma_{n-1}$, where σ_i is the transposition of the pair x_i and x_{i+1}.

$\pi_1(\Sigma^{[n]})$ is well-known as the braid group of n elements on a genus g surface Σ. In particular, let $l_1, \ldots, l_{2g}$ be $2g$ standard generators of $\pi_1(\Sigma)$, let $\rho_{ij}, 1 \leq i \leq n, 1 \leq j \leq 2g$, denote an element of B_n representing a path starting from p_i following along l_j. Let D be any disk containing $p_k, p_l, 1 \leq k < l \leq n$ and let κ_{kl} be a generator of $\pi_1(D^{[2]}, (p_k, p_l))$. We have an induced element of B_n by letting all other points move along constant paths. Similarly, let σ_{st} be a generator of $\pi_1(D^{[2]}/S_2, (p_s, p_t))$. Then we have the following theorem.

Theorem 3.4. *$B_n(\Sigma)$ is generated by $\rho_{ij}, 1 \leq i \leq n, 1 \leq j \leq 2g$ and $\sigma_{s,s+1}, 1 \leq s < n$, induced by the σ described from the previous paragraph.*

In fact, this induces the generators called *swaps* between ith and $(i+1)$th endpoints. In addition, we can multiply those swaps in order to get a swap between any two endpoints. Proved by E. Artin [26, Thoerem 16], we have the following.

Theorem 3.5 ([26])**.** *The braid group has a presentation*

$$\sigma_i\sigma_j = \sigma_j\sigma_i \textit{ for } |i-j| \geq 2,$$

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \textit{ for all } i.$$

3.3 Generalization of the configuration space

We modify the previous configuration spaces.

Definition 3.6. We define the space $\Sigma_m := \Sigma \setminus X$ for some $X \subset \Sigma$ with cardinality m. As long as X consists of m distinct points, for different choices of points in X, Σ_m are homeomorphic to each other.

By using the similar notation, we let Σ_m^n be nth Cartesian product of Σ_m. Similarly, we can also define the configuration space $\mathrm{Conf}_n(\Sigma_m)$ of Σ_m by taking out the fat diagonal Δ of Σ_m. Correspondingly, we denote by $\Sigma_m^{[n]} := \dfrac{\mathrm{Conf}_n(\Sigma_m)}{S_n}$ the unordered configuration space of Σ_m.

Definition 3.7.

- We denote by Σ_m^n the nth Cartesian product of the space Σ_m. It is actually $(\Sigma_m)^n$.
- We denote by $\Sigma_m^{[n]}$ the nth unordered configuration space of Σ_m. It is actually $(\Sigma_m)^{[n]}$. To achieve this space, we need to eliminate the fat diagonal Δ of Σ_m^n to get the nth distinct tuples and then take the quotient of $(\Sigma_m^n \setminus \Delta)/S_n$.

In order to study the mixed braid group, we need to use a classic result from Fadell and Neuwirth [25, Theorem 3].

Theorem 3.8 (Fadell-Neuwirth, 1962)**.** *The map $Conf_n(\Sigma_m) \to Conf_r(\Sigma_m)$ is a fibration, with fiber $Conf_{n-r}(\Sigma_{m+r})$.*

Then we get the following fibration:

$$Conf_n(\Sigma_m) \to Conf_{n+m}(\Sigma) \to Conf_m(\Sigma). \tag{3.2}$$

Consider $m = 1$, and by taking the quotient by S_n, we have a modification of the previous theorem:

Corollary 3.9.

$$\frac{Conf_n(\Sigma_m)}{S_n} \to \frac{Conf_{n+m}(\Sigma)}{S_n \times S_m} \to \frac{Conf_m(\Sigma)}{S_m}. \tag{3.3}$$

Definition 3.10. We denote by $\Sigma^{[n]+[m]}$ the mixed configuration space of two groups of points $X, Y \subset \Sigma$ with cardinality $|X| = n, |Y| = m$, where $X \cap Y = \emptyset$. This space marks two groups of points with cardinality m and n. In fact, $\Sigma^{[n]+[m]}$ is the quotient of $\mathrm{Conf}_{n+m}(\Sigma)$ by $S_n \times S_m$. We intend to call the fundamental group of this mixed configuration space the mixed braid group.

Then the previous fibration 3.3 becomes:

$$(\Sigma_m)^{[n]} \to \Sigma^{[m]+[n]} \to \Sigma^{[m]}.$$

Then we immediately get:

Corollary 3.11. $\pi_1(\Sigma_1^{[n]}) \trianglelefteq \pi_1(\Sigma^{[n]+[1]})$.

In addition, by the quotient of 3.2,we will have the following corollaries.

Corollary 3.12. $\pi_1(\Sigma_n^{[m]}) \trianglelefteq \pi_1(\Sigma^{[n]+[m]})$.

It is obvious that we can reverse the role of m and n to get the similar result $\pi_1(\Sigma_m^{[n]}) \trianglelefteq \pi_1(\Sigma^{[n]+[m]})$.

Then we can see that the generators of $\pi_1(\Sigma^{[n]+[m]})$ consists of the generators of $\pi_1(\Sigma_m^{[n]})$ and $\pi_1(\Sigma_n^{[m]})$.

We consider an open disk U in Σ and $X \in U^{[n]} \subset \Sigma^{[n]}$. This means we pick n distinct points in Σ, and by the classic braid group theory, the fundamental group $\pi_1(U^{[n]}, X)$ is generated by transpositions $\sigma_1, \ldots, \sigma_{n-1}$, where σ_i is the transposition of the pair x_i and x_{i+1}.

In addition, according to [8, Theorem 3.4 (2)], for any point $x_i \in U$ and a point $x \in \Sigma \setminus U$, $\pi_1(\Sigma \setminus x, x_i)$ is generated by η induced by a basis of $H_1(\Sigma \setminus x, \mathbb{Z})$, where x_i traces a path in $\Sigma \setminus (U \cup \{x\})$ but homotopic to a contractible path in $\Sigma \setminus U$.

Furthermore, $\pi_1((\Sigma \setminus x)^{[n]}, X)$ is generated by $\sigma_1, \ldots, \sigma_{n-1}$ and η.

Theorem 3.13. *$\pi_1(\Sigma_m^{[n]})$ is generated by $\rho_{ij}, 1 \leq i \leq n, 1 \leq j \leq 2g$, $\sigma_{s,s+1}, 1 \leq s < n$, and $\eta_{kl}, 1 \leq k \leq n, n < l \leq n+m$. Here η_{kl} denote an path starting from p_k following a nontrivial path around p_l.*

Corollary 3.14. *For X and Y with $|X| = n, |Y| = m$, we can get the similar result that*

$$\pi_1((\Sigma \setminus X)^{[m]}) \times \pi_1((\Sigma \setminus Y)^{[n]}) \to \pi_1(\Sigma^{[m]+[n]}).$$

The morphism is surjective.

Remark 3.15. For $8g-8$ elements, we give an even partition of those $8g-8$ elements. We also

consider $S_{4g-4} \times S_{4g-4}$.

We can construct the mixed configuration space

$$\pi_1(\Sigma^{[4g-4]+[4g-4]}) = \pi_1\left(\frac{\Sigma^{8g-8}\backslash\{(x_1,\ldots,x_{8g-8})|x_i = x_j \text{ for some pairs}\}}{S_{4g-4}\times S_{4g-4}}\right).$$

Then based on previous analysis, we have the following.

Corollary 3.16. *$\pi_1(\Sigma^{[4g-4]+[4g-4]})$ is generated by $\rho_{ij}, 1 \leq i \leq 8g-8, 1 \leq j \leq 2g$ and for $1 \leq k < l \leq 4g-4, 4g-3 \leq k' < l' \leq 8g-8$, we have generators $\sigma_{kl}, \sigma_{k'l'}$, and $\eta_{kl'}$, $\eta_{k'l}$.*

3.4 Configuration space SL_2-case

We have introduced the unordered configuration space and the corresponding braid group, the fundamental group of the configuration space. Now we use them as tools to study the monodromy actions.

Based on the previous chapter, $\mathcal{M}_{SL_2}$ denotes the moduli space of semistable SL_2-Higgs bundles and $h : \mathcal{M}_{SL_2(\mathbb{R})} \to \mathcal{A}_{SL_2}$ the Hitchin fibration. The target space $\mathcal{A}_{SL_2} = H^0(\Sigma, K^2)$. In order to study the monodromy action, we focus on the subspace $\mathcal{A}^{reg}_{SL_2}$ be the space of quadratic differentials with on simple zeros. Since the degree of K is $2g-2$, for each quadratic differential, we get $4g-4$ distinct simple zeros.

We pick a quadratic differential $q \in \mathcal{A}^{reg}_{SL_2}$ with only simple zeros, there is a natural map to the configuration space

$$\Sigma^{[4g-4]} = (\Sigma^{4g-4} \setminus \Delta)/S_{4g-4},$$

where Δ is the fat diagonal with $\Delta = \{x_i = x_j, \text{for some } 1 \leq i < j \leq 4g-4\}$. This map send a

given quadratic differential to its $4g-4$ distinct simple zeros.

Since we focus on the regular locus $\mathcal{A}_{SL_2}^{reg}$, the image will not be in the fat diagonal Δ because there is no repeated zeros.

Then we apply the functor $\pi_1(\ )$ to the map $\rho : \mathcal{A}_{SL_2}^{reg}$ to get $\rho_* : \pi_1(\mathcal{A}_{SL_2}^{reg}) \to \pi_1(\Sigma^{[4g-4]}) = B_{4g-4}$.

Then we apply the induced Abel-Jacobi map on the last space to get [8, Proposition 2.1]:

Proposition 3.17 ([8]). $\rho_*\pi_1(\mathcal{A}_{SL_2}^{reg}) < \ker(\pi_1(\Sigma^{[4g-4]}) \to H_1(\Sigma, \mathbb{Z})$.

Proof. This follows from a long exact sequence. We use the Abel-Jacobi map

$$AJ : \Sigma^{4g-4} \to Jac(\Sigma) \cong Pic^{\circ}(\Sigma).$$

$AJ(D) = |D| \otimes K^{-2}$. Actually, $AJ^{-1}(0) = \overline{\mathcal{A}_{SL_2}^{reg}} := \mathcal{A}_{SL_2}^{reg}/\mathbb{C}^*$. Since we have

$$Jac(\Sigma) = H^0(\Sigma, K)^*/H_1(\Sigma, \mathbb{Z}),$$

there is a natural isomorphism $H_1(Jac(\Sigma)) = H_1(\Sigma, \mathbb{Z})$. Then we take π_1, after Hurewicz homomorphism, we have

$$\pi_1(\Sigma^{[4g-4]}) \to \pi_1(Jac(\Sigma)) \to H_1(Jac(\Sigma)) \to H_1(\Sigma).$$

Then by the long exact sequence, we have the desired result. □

It is natural to pose the question if

$$\rho_*\pi_1(\mathcal{A}^{reg}_{SL_2}) = \ker(\pi_1(\Sigma^{[4g-4]}) \to H_1(\Sigma, \mathbb{Z})).$$

We will give two proofs of the reverse direction of the proposition 3.17.

3.4.1 A special case for the hyperelliptic curve

Copeland worked on the hyperelliptic curves [8]. Now we assume that Σ is hyperelliptic. Here, we list some facts related to the hyperelliptic curves in [8]. Let f be some polynomial of degree $2g+2$ with distinct roots such that Σ is the nonsingular compactification of the zero set $y^2 = f(x)$. In addition, we assume $f(0) \neq 0$. We can consider $\Sigma \to \mathbb{P}^1$ as a double branched cover ramified at $2g+2$ roots of f. In fact, ∞ is not a ramification point, so there are 2 points $\infty^{\pm} \in \Sigma$ above $\infty \in \mathbb{P}^1$. This Σ is equipped with a hyperelliptic involution $\iota : (x, y) \to (x, -y)$. In addition, ι gives a decomposition of the space of quadratic differentials:

$$H^0(\Sigma, K^2) = H^0(\Sigma, K^2)^+ \oplus H^0(\Sigma, K^2)^-.$$

We can calculate the divisor of dx on the hyperelliptic curve $\Sigma : y^2 = f(x) = (x - \alpha_1)\dots(x - \alpha_{2g+2})$, where $\alpha_1, \dots, \alpha_{2g+2}$ are simple zeros of $f(x)$. Because $2g+2$ is even, by choosing different local parameters around different points, we have

$$div(dx) = \sum_{i=1}^{2g+2}(\alpha_i, 0) - 2(\infty_+ + \infty_-).$$

In addition, we have

$$div(y) = \sum_{i=1}^{2g+2} (\alpha_i, 0) - (g+1)(\infty_+ + \infty_-).$$

Then we actually have a quadratic differential $(dx/y)^2$, which is holomorphic with divisor $< (dx/y)^2 >= (2g-2) < \infty^+ > + < 2g-2 >< \infty^- >$. This is in fact fixed under the involution. Then any quadratic differential decomposes as

$$\omega = \omega^+ + \omega^- = p(x) \left(\frac{dx}{y}\right)^2 + q(x)y \left(\frac{dx}{y}\right)^2.$$

Then by counting, the degree of $p(x)$ is at most $2g-2$ and the degree of $q(x)$ is at most $g-3$. Then we have $\dim H^0(\Sigma, K^2)^+ = 2g-1$ and $\dim H^0(\Sigma, K^2)^- = g-2$. So together we have $\dim H^0(\Sigma, K^2) = (2g-1) + (g-2) = 3g-3$. This is exactly the result we had from Riemann-Roch Theorem.

3.4.2 Graph Model of the hyperelliptic curve

For simplicity we consider the curve $f(x) = x^{2g+2} - 1$, then by Copeland and Walker, this can be generalized to any compact Riemann surface. On $\mathbb{P}_1$, M branches over the set

$$\{x | x^{2g+2} \in \mathbb{R}, 0 \leq x^{2g+2} \leq 1\}.$$

Then we draw the set

$$\{x | x^{2g+2} \in \mathbb{R}, -\infty \leq x^{2g+2} \leq 1\},$$

we have $2g+2$ short edges and $2g+2$ rays which close up at infinity. Since neither 0 nor ∞ is a branch point, then the curve we are considering has $0^{\pm}$ and $\infty^{\pm}$. Then in the 2-folded branched cover over $\mathbb{P}_1$, there are $2g+2$ short edges connecting 0^+ with 0^-. Actually $0^{\pm}$ and $\infty^{\pm}$ are the only vertices of the curve. This gives a triangulation of M into $4g+4$ faces.

Now we dualize this triangulation. By doing this costruction and collapsing 8 edges, we create a graph with no loops or double edges.

Here the vertices of the dualized graph can be any points in the faces of the original triangulation. Since $\dim(H^0(M,K^2))>0$ if $g\geq 3$, we can consider $x\mapsto x^{2g+2}-2^{2g+2}$. the map has a root in each triangle of the fundamental triangulation of M. Now we consider the quadratic differential

$$\omega=\frac{x^{2g+2}-2^{2g+2}}{(x-2\zeta^1)(x-2\zeta^3)(x-2\zeta^5)(x-2\zeta^7)}\left(\frac{dx}{y}\right)^2.$$

This quadratic differential has $4g-4$ simple zeros, and each face contains exactly one zero. Then the edges of the dualized graph can represent the elements in the braid group which swap the simple zeros of ω.

Now give a label for each vertex in the dualized graph, $1,2,\ldots,4g-4$.

By giving a 2-cell embedding of this curve, Copeland was able to give an explicit loop of a quadratic differential based on the swaps. Then he used the result to study the monodromy action on SL_2-Hitchin fibration. Then he generalized the result to any hyperelliptic curves.

Here we consider the same hyperelliptic curve. Based on Copeland's method, we can develop the corresponding elements in $\rho_*\pi_1(\overline{\mathcal{A}^{reg}_{SL_2}})$. We mainly use the pre-chosen quadratic differentials ω . We have the following situation.

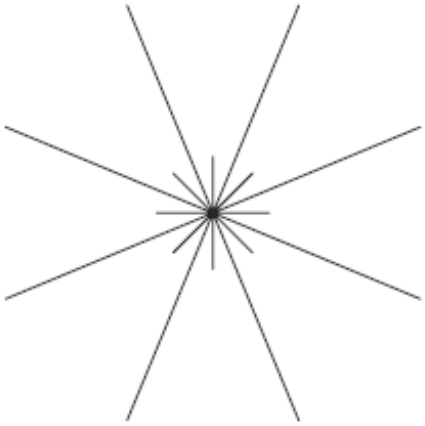

Figure 3.1: Edges

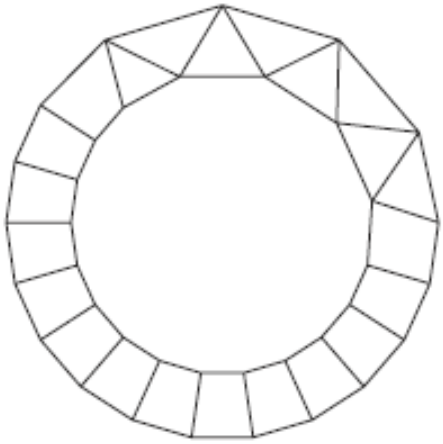

Figure 3.2: Dual Graph

Based on the chosen quadratic differential

$$\omega = \frac{x^{2g-2} - 2^{2g-2}}{(x-2\zeta^1)(x-2\zeta^3)(x-2\zeta^5)(x-2\zeta^7)} \left(\frac{dx}{y}\right)^2,$$

the pair of points $(x, y) = (2\zeta^j, \pm\kappa)$ are neighbors across some edge in the graph. The transposition is an element of $\rho_*\pi_1(\overline{\mathcal{A}^{reg}_{SL_2}})$.

Copeland considered a path $\omega(t) = \frac{\omega}{x-2\zeta^j}(x - \alpha(t))$, where $\alpha(t)$ is an arc connecting $y = +\kappa$ to $y = -\kappa$ across $y = 0$. We may choose $\alpha(t)$ explicitly as

$$\alpha(t) = \zeta^j \left(\frac{3-\epsilon}{2} + \frac{1+\epsilon}{2}\cos(t) + \epsilon i \sin(t)\right) \text{ for } \epsilon \text{ small enough.}$$

This gives $[\omega] \in \rho_*\pi_1(\overline{\mathcal{A}^{reg}_{SL_2}})$.

3.4.3 Quadratic differentials revisited

Definition 3.18. A meromorphic quadratic differential a, on a Riemann surface Σ, is a meromorphic section of the square of the canonical bundle, K of Σ. Hence, in local coordinates , we associate a meromorphic function f_α to each neighborhood such that it satisfies the transition property

$$f_\beta(z_\beta)\left(\frac{dz_\beta}{dz_\alpha}\right)^2 = f_\alpha(z_\alpha)$$

on the intersection $U_\alpha \cap U_\beta$.

According to Walker [9], we are using the concept horizontal trajectory and vertical trajectory to construct the loop in the Hitchin base. Then the argument does not depend on the ellipticity of the Riemann surface Σ.

Definition 3.19. A θ-trajectory is a smooth curve $\gamma : [0,1] \to \Sigma$ such that $f(\gamma(t))(\gamma'(t)dt)^2$ has argument 2θ. Therefore, a horizontal trajectory of a quadratic differential a on Σ is a smooth curve $\gamma : [0,1] \to \Sigma$ such that $f(\gamma(t))(\gamma'(t)dt)^2$ is real and positive for all t. A θ-trajectory connecting two critical points of a is called a saddle connection. Near a zero of a of order n, p_0, there exists $n+2$ curves $\gamma(t) = te^{i(2\pi/(n+2))k}$ for $k = 0, 1, \ldots, n+1$, are trajectories which contract to p_0 [27].

3.4.4 Degeneration of a Hitchin base

We observe that $\ker(AJ_* : \pi_1(\Sigma^{[4g-4]}) \to H_1(\Sigma, \mathbb{Z})$ is generated by $\sigma_{kl}$where $1 \leq k < l \leq 4g-4$,. Here σ means swaps of two distinct points within the same collection of $4g-4$ zeros. We would like to show that $\ker(AJ_*) < \iota_*\pi_1(\mathcal{A}_{SL_2})$.

We consider $cl(\mathcal{A}_{SL_2}^{reg})$ to be the closure of $\mathcal{A}_{SL_2}^{reg}$. In the closure, there are two special situations: a single quadratic differential might have repeated double zeros; two quadratic differentials might share a common zero. For a certain simple quadratic differential $q \in \mathcal{A}_{SL_2}^{reg}$ sufficiently near to an element of $cl(\mathcal{A}_{SL_2}^{reg})$, we might construct some transposition or square transposition of zeros of q in $\pi_1(\mathcal{A}_{SL_2}^{reg})$. Based on [9, Section 5], we have the following.

Remark 3.20. A zero p of $q \in H^0(K^2)$ of order 2 can be broken into two simple zeros, if there exists ϵ_0 such that for every $\epsilon < \epsilon_0$ there exists a quadratic differential q' and a disk $D' \subset \Sigma$ of radius ϵ in the q' metric such that $\Sigma \backslash D_\epsilon(p)$ is isometric to $\Sigma \backslash D'$ and D' contains two simple zeros.

Theorem 3.21 ([9])**.** *Let p be a double zero of a quadratic differential $q \in cl(H^0(\Sigma, K^2)^{reg})$, then p can be broken into 2 simple zeros.*

Proof. Consider a disk D_ϵ centered at p, of radius ϵ. Here the metric is based on the quadratic differential q. In addition, the radius ϵ is small enough such that there is no other zero inside the disk. Since q is a quadratic differential with a double zero at p, there are $2+2=4$ horizontal trajectories that are originated from p. It will divide the disk D_ϵ into 4 sectors (half disks). Now we pick a smaller δ with $0<\delta<\epsilon$. We can construct two half disks by splitting the double point p into two simple zeros p_1 and p_2 with distance 2δ. The newly constructed edge joining p_1 and p_2 is a saddle connection. This connection depends on the choice of θ. Suppose one of the original horizontal trajectory has angle 0°, then the newly constructed saddle connection can be in any direction $\theta\in[0,2\pi)$. Thus we have a newly formed quadratic differential q' which has two simple zeros p_1 and p_2 coming from p by decoupling. This newly formed quadratic differential depends on the choice of θ and δ. And the two disks can be glued back together. □

3.4.5 Swaps viewed as a path in $\mathcal{A}^{reg}_{SL_2}$

Theorem 3.22 ([9]). *Suppose we have $q'\in H^0(\Sigma,K^2)^{reg}$ with p_1 and p_2 as two simple zeros such that their saddle connection has length δ. In addition, all other saddle connections have length far greater than δ. Then there exists an element of $\pi_1(H^0(\Sigma,K^2)^{reg})$ representing the swap.*

Proof. Suppose q' is obtained by decoupling a double zero of a quadratic differential $q\in H^0(\Sigma,K^2)\subset cl(H^0(\Sigma,K^2))$. Consider a curve $\gamma:[0,\delta]\to cl(H^0(\Sigma,K^2))$ by deforming quadratic differential q through time $t\in[0,\delta]$ such that the distance between the decoupled two points is t. We have a disk around the double zero p with radius r small enough so that it contains only p. Then we pick $\delta_0<\min\{\delta,r\}$ and obtain a quadratic differential $\gamma(\delta_0)\in$

$H^0(\Sigma, K^2)^{reg}$ such that the decoupled 2 simple zeros have distance δ_0. Since the angle of the saddle connection θ is pre-chosen, we vary θ by adding 2π continuously to get the same quadratic differential (same simple zeros after rotation). Then this is a loop of quadratic differentials $\gamma' \subset H^0(\Sigma, K^2)^{reg}$ with center q. Here two paths γ and γ' intersect at $\gamma(\delta_0)$. Then we consider the subcurve $\overline{\gamma} = \gamma([\delta_0, \delta])$ of γ. Then the composition $\overline{\gamma} \circ \gamma' \circ \overline{\gamma}^{-1}$ gives us the corresponding element in $H^0(\Sigma, K^2)^{reg}$ representing the swaps of simple zeros. □

In fact, this gave us the induced element in $\pi_1(\mathcal{A}_{SL_2}^{reg})$ which represents the swaps of simple zeros affiliated to a single quadratic differential.

Clearly, the argument above doesn't depend on the hyperellipticity of the Riemann surface Σ [8].

3.5 $SO(2,2)$-Hitchin fibrations

Based on the previous development, we are generalizing this ideas to the $SO(2,2)$-Hitchin fibrations.

3.5.1 Regular Loci of the Hitchin Fibration for $G = SO(2,2)$

Recall that Hitchin fibration for the group $SO(2,2)$ based on works in [10] and [11, Section 3]. The spectral curves for the group $SO(2,2)$ come from the polynomial $\lambda^4 + a\lambda^2 + p^2$, where λ is a tautological section of $\overline{\pi}^* K$. The last term p^2 is always a square of p, the Pfaffian [10, 5.14]. Therefore, we have a pair a and $p \in H^0(\Sigma, K^2)$ from the equation

$$\lambda^4 + a\lambda^2 + p^2 = 0.$$

Instead of considering the original spectral curve S, we consider the double cover $\overline{S}$ over Σ branched at zeros of the pair of quadratic differentials a and p as described in 3.1.

In addition, we recall that the newly defined $\overline{S}$ is lying inside the total space $|K^2|$. A tautological section y of $\overline{\pi}^* K^2$ and $\overline{S} \subset |K^2|$ is from the equation

$$y^2 + ay + p^2 = 0.$$

Then $\overline{S}$ is smooth if and only if the discriminant $\Delta = a^2 - 4p^2$ has only simple zeros. In fact, Δ factors as a product $\Delta = c_1 c_2$, where $c_1 = a_2 + 2p$ and $c_2 = a_2 - 2p$. Therefore, we have a pair of quadratic differentials c_1 and $c_2 \in H^0(\Sigma, K^2)$ and each one of them has only simple zeros and no zeros are in common [14, Section 7].

Then, for Hitchin fibration for $SO(2,2)$:

$$h : \mathcal{M}_{SO(2,2)} \to \mathcal{A}_{SO_4}$$

the Hitchin fibration, where

$$\mathcal{A}_{SO_4} = H^0(\Sigma, K^2) \oplus H^0(\Sigma, K^2).$$

Here we focus on a subspace $\mathcal{A}^{reg}_{SO_4}$ such that it contains pairs of quadratic differentials with simple zeros and no zeros in common. We recall that this $\mathcal{A}^{reg}_{SO_4}$ is the space of *regular points* of $SO(2,2)$-Hitchin base.

3.5.2 Configuration space $SO(2,2)$ case

In order to study the monodromy action of $SO(2,2)$-Hitchin system, we need to understand the fundamental group $\pi_1(\mathcal{A}^{reg}_{SO_4})$ of the Hitchin base $\mathcal{A}^{reg}_{SO_4}$. Since the degree of K is $2g-2$, we have a pair of quadratic differentials $(c_1, c_2) \in \mathcal{A}^{reg}_{SO_4}$. Similar to the work done by Copeland [8], we generalize it a different configuration space. Here in this case we have in total $8g-8$ number of simple zeros and we divide them into two groups of $4g-4$ zeros. Hence we construct a mixed configuration space as follows. Consider $8g-8$ copies of the base curve Σ, then we take out the fat diagonal $\{(x_1, \ldots, x_{8g-8}) | x_i = x_j \text{ for some } i, j\}$ in order to make sure our tuples have no common coordinators.

Let us consider

$$\Sigma^{[4g-4]+[4g-4]} = \frac{\Sigma^{8g-8} \backslash \{(x_1, \ldots, x_{8g-8}) | x_i = x_j, \text{ for some } i, j\}}{S_{4g-4} \times S_{4g-4}}$$

to be the quotient of the unordered $8g-8$ configuration space by two symmetric groups on $4g-4$ elements. Hence, we get a mixed configuration space of two groups of mixed unordered distinct $4g-4$ points.

Then we define

$$B_{4g-4,4g-4} = \pi_1 \left(\frac{\Sigma^{8g-8} \backslash \{(x_1, \ldots, x_{8g-8}) | x_i = x_j, \text{ for some } i, j\}}{S_{4g-4} \times S_{4g-4}} \right)$$

to be the mixed surface braid group of the $(4g-4, 4g-4)$-mixed configuration space.

We introduced the the regular locus of the Hitchin space $\mathcal{A}^{reg}_{SO_4}$, pairs of quadratic differen-

tials with simple zeros and no zeros in common. Then this Hitchin space $\mathcal{A}_{SO_4}^{reg}$ maps naturally to the order mixed configuration space. Actually, it passes to the unordered mixed configuration space $\Sigma^{[4g-4]+[4g-4]}$ by sending a pair of quadratic differentials to their corresponding simple zeros. In particular, we have $8g-8$ of them due to the degree of K.

Let $\rho : \mathcal{A}_{SO_4}^{reg} \to \Sigma^{[4g-4]+[4g-4]}$ be the map which sends a pair of quadratic differentials to their zero sets.

The pairs of quadratic differentials (c_1, c_2) are linear transformation of (a, p) with $c_1 = a+2p, c_2 = a-2p$. By rescaling by a nonzero complex scalers, (c_1, c_2) will yield the same simple zeros, with no points changed. Therefore, the previous map actually factor through $\mathcal{A}_{SO_4}^{reg}/(\mathbb{C}^* \times \mathbb{C}^*) = \overline{\mathcal{A}_{SO_4}^{reg}} \hookrightarrow \Sigma^{[4g-4]+[4g-4]}$.

This quotient space is the space of pairs of quadratic differentials up to multiplication by nonzero complex numbers.

3.5.3 Hyperelliptic curves special case

Once again, we consider a hyperelliptic curve Σ for a moment. Then the basic setup will be the same as done in 3.4.1.

For simplicity we consider the curve $f(x) = x^{2g+2} - 1$, then by Copeland [8] and Walker [9], this can be generalized to any compact Riemann surface. Once again, on $\mathbb{P}_1$, M branches over the set

$$\{x | x^{2g+2} \in \mathbb{R}, 0 \leq x^{2g+2} \leq 1\}.$$

Then we draw the set

$$\{x | x^{2g+2} \in \mathbb{R}, -\infty \leq x^{2g+2} \leq 1\},$$

we have $2g+2$ short edges and $2g+2$ rays which close up at infinity. Since neither 0 nor ∞ is a branch point, then the curve we are considering has $0^{\pm}$ and $\infty^{\pm}$. Then in the 2-folded branched cover over $\mathbb{P}_1$, there are $2g+2$ short edges connecting 0^+ with 0^-. Actually $0^{\pm}$ and $\infty^{\pm}$ are the only vertices of the curve. This gives a triangulation of M into $4g+4$ faces.

Here the vertices of the dualized graph can be any points in the faces of the original triangulation. Since $\dim(H^0(M, K^2)) > 0$ if $g \geq 3$, we can consider $x \mapsto x^{2g+2} - 2^{2g+2}$. the map has a root in each triangle of the fundamental triangulation of M. Now we consider the quadratic differential

$$\omega = \frac{x^{2g+2} - 2^{2g+2}}{(x - 2\zeta^1)(x - 2\zeta^3)(x - 2\zeta^5)(x - 2\zeta^7)} \left(\frac{dx}{y}\right)^2.$$

This quadratic differential has $4g-4$ simple zeros, and each face contains exactly one zero. Then the edges of the dualized graph can represent the elements in the braid group which swap the simple zeros of ω.

Similarly, we also consider another quadratic differential

$$\overline{\omega} = \frac{x^{2g+2} - 3^{2g+2}}{(x - 3\zeta^1)(x - 3\zeta^3)(x - 3\zeta^5)(x - 3\zeta^7)} \left(\frac{dx}{y}\right)^2,$$

then the edges of the dualized graph can represent the elements in the braid group which swap the simple zeros of $\overline{\omega}$.

Now give a label for each vertex in the dualized graph, $1, 2, \ldots, 4g-4$ and $1', 2', \ldots, 4g-4$. The convoluted generators in the mixed braid group can be represented by the edges which connect k-th vertex of the first(inner) dualized graph with k'-th vertex of the second(outer) dualized graph.

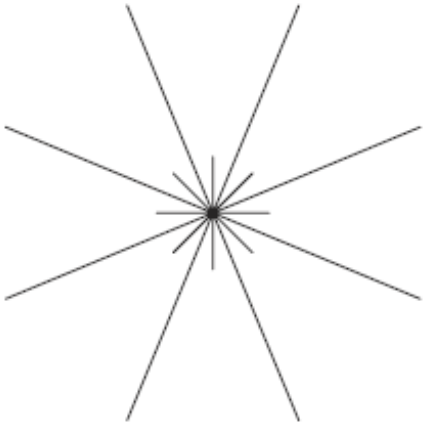

Figure 3.3: Edges

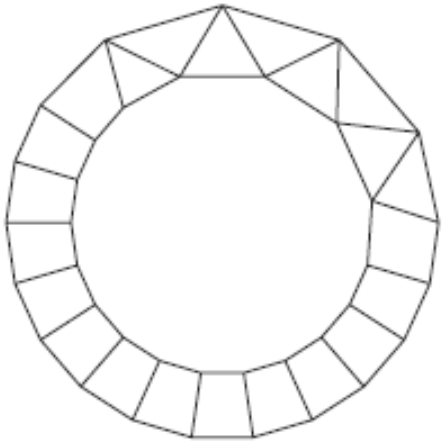

Figure 3.4: Dual Graph

By giving a 2-cell embedding of this curve, Copeland was able to give an explicit loop of a quadratic differential based on the swaps. Then he used the result to study the monodromy action on SL_2-Hitchin fibration. Then he generalized the result to any hyperelliptic curves. Later, Walker generalized the result to any compact Riemann surfaces [9].

Here we consider the same hyperelliptic curve. Based on Copeland's method [8, 7], we can develop the corresponding elements in $\rho_*\pi_1(\overline{\mathcal{A}^{reg}_{SO_4}})$. We mainly use the pre-chosen quadratic differentials ω and $\overline{\omega}$. Here we have three different situations.

- Case 1. Based on the chosen quadratic differential

$$\omega = \frac{x^{2g-2} - 2^{2g-2}}{(x-2\zeta^1)(x-2\zeta^3)(x-2\zeta^5)(x-2\zeta^7)} \left(\frac{dx}{y}\right)^2,$$

  the pair of points $(x, y) = (2\zeta^j, \pm\kappa)$ are neighbors across some edge in the graph. Their transposition is an element of $\rho_*\pi_1(\overline{\mathcal{A}^{reg}_{SO_4}})$.

  Copeland consider a path $\omega(t) = \frac{\omega}{x-2\zeta^j}(x - \alpha(t))$, where $\alpha(t)$ is an arc connecting $y = +\kappa$ to $y = -\kappa$ across $y = 0$. We may choose $\alpha(t)$ explicitly as

$$\alpha(t) = \zeta^j \left(\frac{3-\epsilon}{2} + \frac{1+\epsilon}{2}\cos(t) + \epsilon i \sin(t)\right) \text{ for } \epsilon \text{ small enough.}$$

  Since we did not change anything in $\overline{\omega}$ This gives $[\omega\overline{\omega}] \in \rho_*\pi_1(\overline{\mathcal{A}^{reg}_{SO_4}})$.

- Case 2. Copeland uses the similar argument to get the following. If $a = (2\zeta^j, \kappa)$ and $b = (2\zeta^{j+1}, \kappa)$ are roots of ω, then the transposition of a and b is in $\rho_*\pi_1(\overline{\mathcal{A}^{reg}_{SO_4}})$.

- Case 3. If $a = (2\zeta^j, \kappa)$ is a zero of ω and $b = (3\zeta^j, \lambda)$ is a zero of $\overline{\omega}$, then the class

representing a closed loop around b is in $\rho_*\pi_1(\overline{\mathcal{A}}^{reg}_{SO_4})$.

Here we consider an arc

$$\alpha(t) = \frac{\omega}{(x-2\zeta^j)}(x-(2+(1-\epsilon)\epsilon)\zeta^j) \text{ for } 0 \leq t \leq 1 \text{ and } \epsilon \text{ small enough.}$$

$$\alpha(t) = \frac{\omega}{x-2\zeta^j}(x-(3-\epsilon)\zeta^j(\cos t + i\sin t)) \text{ for } 1 \leq t \leq 2 \text{ and } \epsilon \text{ small enough.}$$

$$\alpha(t) = \frac{\omega}{(x-2\zeta^j)}(x-((3-\epsilon)-(1-\epsilon)(t-2))\zeta^j) \text{ for } 1 \leq t \leq 2 \text{ and } \epsilon \text{ small enough.}$$

This gives a loop around $b \in Z(\overline{\omega})$ and leave all over zeros untouched. Hence we get a class $[\omega\overline{\omega}] \in \rho_*\pi_1(\overline{\mathcal{A}}^{reg}_{SO_4})$.

3.5.4 Degeneration of a Hitchin base

Again, copeland's work can be generalized to any compact Riemann surface due to Walker's work. Now we will adopt Walker's method [9].

First we observe that $\ker(AJ_* \times AJ_* : \pi_1(\Sigma^{[4g-4,4g-4]}) \to H_1(\Sigma,\mathbb{Z}) \times H_1(\Sigma,\mathbb{Z}))$ is generated by $\sigma_{kl}, \sigma_{k'l'}$, and $\eta_{kl'}$, $\eta_{k'l}$, where $1 \leq k < l \leq 4g-4, 4g-3 \leq k' < l' \leq 8g-8$. Here σ's mean swaps of two distinct points within the same collection of $4g-4$ zeros and η's mean that a simple zero in one collection of $4g-4$ points go around a point affiliated to another collection

of zeros. We would like to show that $\ker(AJ_* \times AJ_*) < \iota_*\pi_1(\mathcal{A}_{SO_4})$

We consider $cl(\mathcal{A}^{reg}_{SO_4})$ to be the closure of $\mathcal{A}^{reg}_{SO_4}$. In the closure, there are two special situations: a single quadratic differential might have repeated double zeros; two quadratic differentials might share a common zero. For a certain $(c_1, c_2) \in \mathcal{A}^{reg}_{SO_4}$ sufficiently near to an element of $cl(\mathcal{A}^{reg}_{SO_4})$, we might construct some transposition or square transposition of zeros of (c_1, c_2) in $\pi_1(\mathcal{A}^{reg}_{SO_4})$.

Definition 3.23. A zero p of $q \in H^0(K^2)$ of order 2 can be broken into two simple zeros, if there exists ϵ_0 such that for every $\epsilon < \epsilon_0$ there exists a quadratic differential q' and a disk $D' \subset \Sigma$ of radius ϵ in the q' metric such that $\Sigma \backslash D_\epsilon(p)$ is isometric to $\Sigma \backslash D'$ and D' contains two simple zeros.

Theorem 3.24. *Let p be a double zero of a quadratic differential $q \in cl(H^0(\Sigma, K^2)^{reg})$, then p can be broken into 2 simple zeros.*

Proof. Consider a disk D_ϵ centered at p, of radius ϵ. Here the metric is based on the quadratic differential q. In addition, the radius ϵ is small enough such that there is no other zero inside the disk. Since q is a quadratic differential with a double zero at p, there are $2 + 2 = 4$ horizontal trajectories that are originated from p. It will divide the disk D_ϵ into 4 sectors (half disks). Now we pick a smaller δ with $0 < \delta < \epsilon$. We can construct two half disks by splitting the double point p into two simple zeros p_1 and p_2 with distance 2δ. The newly constructed edge joining p_1 and p_2 is a saddle connection. This connection depends on the choice of θ. Suppose one of the original horizontal trajectory has angle 0°, then the newly constructed saddle connection can be in any direction $\theta \in [0, 2\pi)$. Thus we have a newly formed quadratic differential q' which has two simple zeros p_1 and p_2 coming from p by decoupling. This newly formed quadratic differential

depends on the choice of θ and δ. And the two disks can be glued back together. $\square$

Here we introduce two differential methods (from Walker and Copeland) to show

$$\rho_*\pi_1(\mathcal{A}_{SO_4}^{reg}) < \ker(\pi_1(\Sigma^{4g-4,4g-4}) \to H_1(\Sigma,\mathbb{Z}) \times H_1(\Sigma,\mathbb{Z})).$$

3.5.5 Swaps

This is exactly the same *swaps* we discussed in 3.4.5 because the *swaps* involve only a single quadratic differential. Here, we just repeat the argument.

Theorem 3.25. *Suppose we have $(q',\tilde{q}) \in \mathcal{A}_{SO_4}^{reg}$ with p_1 and p_2 as two simple zeros such that their saddle connection has length δ. In addition, all other saddle connections have length far greater than δ. Then there exists an element of $\pi_1(\mathcal{A}_{SO_4}^{reg})$ representing the swap. In particular, in this case, we only operate one quadratic differential q' and leave the other quadratic differential $\tilde{q}$ alone.*

Proof. Suppose q' is obtained by decoupling a double zero of a quadratic differential $q \in H^0(\Sigma,K^2) \subset cl(H^0(\Sigma,K^2))$. Consider a curve $\gamma : [0,\delta] \to cl(H^0(\Sigma,K^2))$ by deforming quadratic differential q through time $t \in [0,\delta]$ such that the distance between the decoupled two points is t. We have a disk around the double zero p with radius r small enough so that it contains only p. Then we pick $\delta_0 < \min\{\delta, r\}$ and obtain a quadratic differential $\gamma(\delta_0) \in H^0(\Sigma,K^2)^{reg}$ so that the decoupled 2 simple zeros have distance δ_0. Since the angle of the saddle connection θ is pre-chosen, we vary θ by adding 2π continuously to get the same quadratic differential (same simple zeros after rotation). Then this is a loop of quadratic differentials $\gamma' \subset H^0(\Sigma,K^2)^{reg}$ with center q. Here two paths γ and γ' intersect at $\gamma(\delta_0)$. Then we consider the subcurve $\overline{\gamma} = \gamma([\delta_0,\delta])$

of γ. Then the composition $\overline{\gamma} \circ \gamma' \circ \overline{\gamma}^{-1}$ gives us the corresponding element in $H^0(\Sigma, K^2)^{reg}$ representing the swaps of simple zeros. $\square$

In fact, this gave us the induced element in $\pi_1(\mathcal{A}^{reg}_{SO_4})$ which represents the swaps of simple zeros affiliated to a single quadratic differential.

3.5.6 Double swaps

We need to construct an element of $\pi_1(\mathcal{A}^{reg}_{SO_4})$ which can represents the other generators $\iota_{kl'}$ other than swaps $\sigma_{s,s+1}$ in $\ker(AJ_* \times AJ_*)$. Here we still use the similar technique, but this time we try to decouple a double zero of a quartic differential.

Here we have a pair of quadratic differentials (q_1, q_2) such that p is the only simple zero of both q_1 and q_2. We need a method to separate the double zero to 2 simple zeros of the product quartic differential. We pick a ball around p with radius ϵ small enough such that it contains no other zeros. Then for a quartic differential, p has $2+4=6$ horizontal trajectories originated from p. Therefore there exists 6 half-disks. For the factor q_2 we add a family of quadratic differentials $q_2'(t)$ with same simple zeros except for p. Then $(q+q')_2(t)$ will give a path of the simple zero $p(t)$ such that this $p(t)$ is moving away from p and all other zeros remain untouched. In fact, by decoupling, we get two critical points p and $p(t)$, and each one of them is equipped with 3 horizontal trajectories with respect to each quadratic differential factor q_1 and $(q+q')_2$, but the vertical trajectories would link the decoupled two simple zeros together.

Theorem 3.26. *Suppose we have $(c_1, c_2) \in \mathcal{A}^{reg}_{SO_4}$ with $p \in Z(c_1)$ and $p' \in Z(c_2)$ such that they share a common vertical trajectory with length δ. In addition, all other saddle connections and common vertical trajectories have length far greater than δ. Then there exists an element of*

$\pi_1(\mathcal{A}^{reg}_{SO_4})$ *representing the double swap of a zero around another.*

Proof. Suppose p and p' are obtained by decoupling a common zero p of $c_1c_2' = C \in H^0(\Sigma, K^4) \subset cl(\mathcal{A}^{reg}_{SO_4})$. Here p is a simple zero of both c_1 and c_2'. Consider a curve $\gamma : [0, \delta] \to cl(\mathcal{A}^{reg}_{SO_4})$ by deforming the quadratic differential c_2' through time $t \in [0, \delta]$ as decribed in the previous paragraph. We have a disk around the p of radius r small enough so that it contains only p. Then we pick $\delta_0 < \min\{\delta, r\}$ and obtain a quartic differential $c_1c_2 = \gamma(\delta_0) \in \mathcal{A}^{reg}_{SO_4}$ such that the splitted zeros p and $p(\delta_0)$ have distance δ_0. Since the vertical trajectory that connects 2 zeros are pre-chosen. We would vary the angle of the trajectory from p (which is now a simple zero of c_1) by adding 3π. At the same time, by modifying c_2, the simple zero $p(\delta_0)$ move around p in a full circle and maintain the vertical trajectory connection with p. Then we have a loop $\gamma' \subset \mathcal{A}^{reg}_{SO_4}$ of quartic differentials as a product of two distinct quadratic differentials with center C. Here, the two paths γ and γ' intersect at $\gamma(\delta_0)$. Then we consider the subcurve $\overline{\gamma} = \gamma([\delta_0, \delta])$ of γ. The composition $\overline{\gamma} \circ \gamma' \circ \overline{\gamma}^{-1}$ gives us the corresponding element of $\mathcal{A}^{reg}_{SO_4}$ representing the other generator in $\ker(AJ_* \times AJ_*)$. □

Chapter 4: Monodromy actions on Hitchin fibrations

In this chapter, we study the monodromy action on Hitchin fibration for the group $SO(2,2)$. Since we use a similar method from works done by [28] and [14], we will describe the construction in details.

First, we give a construction of the monodromy representation of the Hitchin fibration for $SL_2(\mathbb{C})$ based on [8]. Since a generic fiber of Hitchin fibration for SL_2 is equivalent to the Prym variety $Prym(S,\Sigma)$ associated to $\pi : S \to \Sigma$. The monodromy action on SL_2-Hitchin fibration is equivalent to finding the quotient action on the edge set of graph model 4.1 from the previous chapter.

Copeland's method is for the monodromy action on Hitchin fibration for $SL_2(\mathbb{C})$. In [28], Schaposnik gave a matrix representation of the monodromy action on Hitchin fibration for $SL_2(\mathbb{R})$. Recall that a generic fiber of $SL_2(\mathbb{R})$ Hitchin map is isomorphic to the space of 2-torsion points $Pyrm(S,\Sigma)[2]$ in $Pyrm(S,\Sigma)$. It is shown in [28] that $Prym(S,\Sigma)[2]$ has a decomposition into $\mathbb{Z}_2^{2g} \oplus \mathbb{Z}_2^{4g-6}$. Then the monodromy matrix can be written in a matrix form

$$\begin{pmatrix} I_{2g} & A \\ 0 & \pi \end{pmatrix}, \tag{4.1}$$

where π represents the *swaps* of two different simple zeros of a quadratic differential.

Then we start considering the monodromy action on Hitchin fibration for $SO(2,2)$. First, we conclude that there is a so-called sheet-swapping action induced by an involution action $\tau : \overline{S} \to \overline{S}$. Then, we examine that the monodromy action on Hitchin fibration for $SO(2,2)$ is equivalent to finding how the group $\pi_1(\mathcal{A}^{reg}_{SO_4})$ acts on the Hitchin fiber, which is a space of the 2-torsion points $\Lambda_{\overline{S}}[2]$ of the vector space $\Lambda_{\overline{S}}$ through some identification. This space $\Lambda_{\overline{S}}$ can be identified by the 2-torsion points of the Jacobian $Jac_{\overline{S}}[2]$ of $\overline{S}$. Similar to [8] and [14], the monodromy action is the automorphism of $\Lambda_{\overline{S}}[2]$ induced by a Dehn twist of $\overline{S}$ around the embedded loops γ and δ corresponding to *swaps* and *double swaps*. Then the monodromy action acts on $\Lambda_{\overline{S}}[2]$ as a Picard-lefschetz transformation:

$$\gamma \cdot x = x + \langle c_\gamma, x \rangle c_\gamma,$$

$$\delta \cdot x = x + \langle c_\delta, x \rangle c_\delta.$$

Thus, this vector space $\Lambda_{\overline{S}}[2]$ can be decomposed into a direct sum of $\mathbb{Z}_2$-vector spaces $\Lambda_\Sigma[2] \oplus W \oplus \Lambda_\Sigma[2]$. By splitting $\Lambda_{\overline{S}}[2]$ this way, the monodromy action is equivalent to finding the monodromy action on this direct sum. In addition, we find a matrix representation of those monodromy actions.

Theorem 4.17 *The monodromy action of Hitchin fibration for $SO(2,2)$ is generated by the following matrices.*

$$A^x_{ij} = \begin{bmatrix} I_{2g} & L^a_{ij} & S^a \\ 0 & I & (L^a_{ij})^t \\ 0 & 0 & I_{2g} \end{bmatrix}.$$

and

$$s_{ij} = \begin{bmatrix} I_{2g} & 0 & 0 \\ 0 & \sigma_{ij} & 0 \\ 0 & 0 & I_{2g} \end{bmatrix}.$$

where

- $L^a_{ij} : W \to \Lambda_\Sigma[2]$
- $L^a_{ij} : W \to \Lambda_\Sigma[2]$ *and*
- $S^a : \Lambda_\Sigma[2] \to \Lambda_\Sigma[2]$.

4.0.1 Monodromy for $SL_2(\mathbb{R})$-Hitchin fibrations

The monodromy action on the $SL_2(\mathbb{C})$ Hitchin fibration was studied by using a combinatorial approach by Copeland [8]. The results were generalized by [9] to any compact Riemann surface. The results can be summarized as follows.

Theorem 4.1 ([8], [9])**.** *Given a compact Riemann surface Σ of genus $g \geq 2$, we can associate a graph 3.2 with edge set E and a skew bilinear pairing $\langle e, e' \rangle$ on $e, e' \in \mathbb{Z}[E]$ such that*

- *1. the monodromy representation of $\pi_1(\mathcal{A}^{reg})$ acting on $H_1(Prym(S, \Sigma), \mathbb{Z})$ is generated by elements σ_e labeled by the edges $e \in E$.*
- *2. there exist an action of $\pi_1(\mathcal{A}_{reg})$ on $e' \in \mathbb{Z}[E]$ by means of*

$$\sigma_e(e') = e' - \langle e', e \rangle \, e,$$

- *3. the monodromy representation of the action of $\pi_1(\mathcal{A}_{reg})$ on $H_1(Prym(S,\Sigma),\mathbb{Z})$ is a quotient of this module $\mathbb{Z}[E]$.*

For the group $SL_2(\mathbb{C})$, the fiber of $SL_2(\mathbb{C})$ is isomorphic to the Prym variety $Prym(S,\Sigma)$. In [8, Theorem 16 and Proposition 21], Copeland shows the following results through combinatorial method.

Proposition 4.2 ([8]). *If E and F are the edge and the face sets of the dual graph 4.1, then there is an induced homeomorphism*

$$Prym(S,\Sigma) \cong \frac{\mathbb{R}[E]}{(\mathbb{Z}[F] \cap \frac{1}{2}\mathbb{Z}[E])}, \tag{4.2}$$

where the inclusion $\mathbb{R}[F] \subset \mathbb{R}[E]$ is defined by some relations between edges and the boundaries $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4$ of the faces:

$$\begin{aligned}
\tilde{x}_1 &= \sum_{i=1}^{2g-2} l_i; \\
\tilde{x}_2 &= \sum_{i=1}^{2g-2} u_i; \\
\tilde{x}_3 &= \sum_{even \geq 6} u_i - \sum_{odd \geq 5} l_i + \sum_{i=1}^{2g+2} b_i; \\
\tilde{x}_4 &= l_1 + l_3 - u_2 - u_4 + \sum_{odd} u_i - \sum_{even} l_i + \sum_{i=1}^{2g+2} b_i.
\end{aligned} \tag{4.3}$$

Remark 4.3. Here, the quotient space can be realized by the following relation:

$$\tilde{x}_3 + \tilde{x}_4 + \tilde{x}_1 - \tilde{x}_2 = 2\left(l_1 + l_3 - u_2 - u_4 + \sum_{i=1}^{2g+2} b_i\right) = 2\tilde{x}_5. \tag{4.4}$$

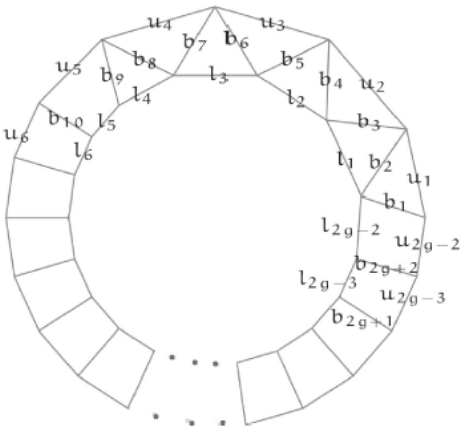

Figure 4.1: Labels

Therefore, we can consider $\tilde{x}_5$ as an element in $\mathbb{Z}[F] \cap \frac{1}{2}\mathbb{Z}[E]$.

Corollary 4.4. *From the previous proposition 4.2, the fibers of the Hitchin fibration for SL_2 are the quotient space of $\mathbb{R}[E]$ by the relation 4.4.*

Using the same construction and our description of Hitchin fibration for $SL_2(\mathbb{R})$, the fibers of the Hitchin fibration for $SL_2(\mathbb{R})$ can be identified by $Prym(S, \Sigma)[2]$, the 2-torsion points of $Prym(S, \Sigma)$. Then the monodromy action on $SL_2(\mathbb{R})$-Hitchin fibrations is a quotient of a representation of the action on the space $\mathbb{Z}_2[E]$.

Now, the space $\mathbb{Z}_2[E]$ is a $\mathbb{Z}_2$-vector space. Then a representation of this action of $\pi_1(\mathcal{A}_{reg})$ on $\mathbb{Z}_2[E]$ can be studied by examining the image B_0 and the kernel Z_1 of $\partial : C_1 \to C_0$. Detailed analysis can be found in [28, 7].

From [28, Proposition 7.2], we have the following.

Proposition 4.5 ([28])**.** *The dimensions of the image B_0 and the kernel Z_1 of the derivative $\partial : C_1 \to C_0$ are, respectively, $5g - 5$ and $2g + 3$.*

Proposition 4.6. *Then there is splitting of the vector space* $\mathbb{Z}_2[E]$*:*

$$0 \to Z_1 \to \mathbb{Z}_2[E] \to B_0 \to 0. \tag{4.5}$$

Through combinatorial methods, Schaposnik got the following [28, Proposition 7.3].

Theorem 4.7 ([28])**.** *There is a matrix representation of the monodromy action on* $\mathbb{Z}_2[E] \cong Z_1 \oplus B_0$*. The action is generated by the following two matrices:*

$$\begin{aligned} [h_e] &= \begin{pmatrix} I_{2g+3} & A_e \\ 0 & I_{4g-5} \end{pmatrix} \\ [s] &= \begin{pmatrix} I_{2g+3} & 0 \\ 0 & \pi_s \end{pmatrix}. \end{aligned} \tag{4.6}$$

In addition, there is a conjugation relationship between $[h_e]$ *and* $[s]$*:*

$$\begin{aligned} [s][h_e][s]^{-1} &= \begin{pmatrix} I_{2g+3} & 0 \\ 0 & \pi_s \end{pmatrix} \begin{pmatrix} I_{2g+3} & A_e \\ 0 & I_{4g-5} \end{pmatrix} \begin{pmatrix} I_{2g+3} & 0 \\ 0 & \pi_s^{-1} \end{pmatrix} \\ &= \begin{pmatrix} I_{2g+3} & A_e\pi_s^{-1} \\ 0 & I_{4g-5} \end{pmatrix}. \end{aligned} \tag{4.7}$$

In addition, as we have seen previously, $P[2] := Prym(S, \Sigma)[2]$ can be considered as a quotient of $\mathbb{Z}_2[E]$. By calculation done by [28, Proposition 8.1], we have the following quotient of the previous splitting 4.5.

Remark 4.8.

$$\mathbb{Z}_2^{2g} \cong \frac{Z_1}{\langle x_1, x_2, x_4 \rangle} \to P[2] \to \frac{B_0}{\langle (1, 1, \ldots, 1) \rangle} \cong \mathbb{Z}_2^{4g-6}. \tag{4.8}$$

Then we get the following results from [28, Theorem 8.1].

Theorem 4.9 ([28])**.** *The monodromy group $G \subset GL(6g-6, \mathbb{Z}_2)$ on the first $\mathbb{Z}_2$-cohomology of the fibers of the Hitchin fibration is given by the group of matrices of the form*

$$\begin{pmatrix} I_{2g} & A \\ 0 & \pi \end{pmatrix}, \tag{4.9}$$

where

- *π is the quotient action of permutation representation on B_0 by $(1, \ldots, 1)$*
- *the off-diagonal block A can be any $2g \times (4g-6)$ matrix with binary entries.*

4.1 Monodromy for $SO(2,2)$-Hitchin fibrations

4.1.1 Sheets swapping actions

Here, we would like to discuss one simple action of order 2 for the Hitchin fibration for $SO(2,2)$.

We already observed that the kernel of the map $\pi_1(\mathcal{A}_{SL_2}^{reg}) \to \pi_1(\overline{\mathcal{A}_{SL_2}^{reg}})$ is $\pi_1(\mathbb{C}^*) = \mathbb{Z}$. [28]

Based on Schaposnik's analysis, the kernel $\mathbb{Z}$ of the map acts as swapping of the two branches of the spectral curve S over the original compact Riemann surface Σ for the $SL_2(\mathbb{R})$-Hitchin fibrations [28].

Now we generalize this to the $SO(2,2)$-Hitchin fibrations.

From the last chaper, we observe that the kernel of the map $\pi_1(\mathcal{A}^{reg}_{SO_4}) \to \pi_1(\overline{\mathcal{A}^{reg}_{SO_4}})$ is $\pi_1(\mathbb{C}^*) \times \pi_1(\mathbb{C}^*) = \mathbb{Z} \times \mathbb{Z}$. We would show that this represents the sheet swapping generator τ based on the two-folded branched cover $\overline{S} \to \Sigma$.

Theorem 4.10. *The kernel of the map $\pi_1(\mathcal{A}^{reg}_{SO_4}) \to \pi_1(\overline{\mathcal{A}^{reg}_{SO_4}})$ can be viewed as a group action on the Hitchin fibrations with stabilizer isomorphic to $\mathbb{Z}$.*

Proof. We have the the fibration $\mathcal{A}^{reg}_{SO_4} \to \overline{\mathcal{A}^{reg}_{SO_4}}$ with fiber $\mathbb{C}^* \times \mathbb{C}^*$ So

$$\mathbb{Z} \times \mathbb{Z} = \pi_1(\mathbb{C}^*) \times \pi_1(\mathbb{C}^*) \to \pi_1(\mathcal{A}^{reg}_{SO_4}) \to \pi_1(\overline{\mathcal{A}^{reg}_{SO_4}})$$

is exact. In fact, the $\mathbb{C}^*$-action on quadratic differentials is rescaling by a nonzero number. Now we consider a loop representing a generator of $\pi_1(\mathbb{C}^*) = \mathbb{Z}$. Then from the quadratic formula

$$y = \frac{a_2 \pm \sqrt{\Delta}}{2}$$

we consider the $(e^{2\pi it}, id)$ action on $\omega = (c_1, c_2) \in \mathcal{A}^{reg}_{SO_4}$ by multiplication component-wise. Since there is a square root over Δ from the quadratic formula, the $(e^{2\pi it}, id)$ action will transfer $y = \frac{a_2 + \sqrt{\Delta}}{2}$ to $y = \frac{a_2 - \sqrt{\Delta}}{2}$ or the other way around. Actually, the other generator $(id, e^{2\pi is})$ will yield the same result on the spectral curve. In fact, if we rescale a and p simultaneusly, then we rescale c_1 and c_2 simultaneously because of the linear transformation, then y will remain the same. We have this $(\mathbb{Z} \times \mathbb{Z})$-aciton on $\overline{S}$ with kernel $\mathbb{Z}$, hence a $\mathbb{Z}$-action. Since $\overline{S} \to \Sigma$ is a 2-folded branch cover. Then the generator represents the sheet-swapping. □

4.1.2 The monodromy representation

In fact, not only can we pick the canonical bundle $K = T^*\Sigma$, but we can pick the bundle over Σ with greater degree $l \geq 2g-2$. Then we can generalize the situation when the degree of the line bundle L can have degree more than $2g-2$, which is the degree of the canonical bundle. Then the canonical bundle K case can be considered as a special case of this general construction.

We recall that for line bundles with degree $l \geq 2g-2$, we have the following similar mixed configuration space. For $n = 2l$,

$$\Sigma^{[n]+[n]} = \frac{\Sigma^{2n} \backslash \{(a_1, \ldots, a_n, b_1, \ldots b_n) | \text{some } a_i \text{ and } b_j \text{ collapse}\}}{S^n \times S^n}.$$

Theorem 4.11. *We consider a line bundle of degree big enough and let $(q_1, q_2) \in \mathcal{A}^{reg}_{SO_4}$, and let $a_1, \ldots, a_n$ be simple zeros of q_1 and $b_1, \ldots, b_n$ be simple zeros of q_2. Then, we have an isomorphism*

$$\pi_1(\overline{\mathcal{A}^{reg}_{SO_4}}, a_1 + \cdots + a_n + b_1 + \cdots + b_n) \cong \ker(B_{n,n}(\Sigma, (a_1 + \cdots + a_n + b_1 + \cdots + b_n)) \to H_1(\Sigma, \mathbb{Z})) \times H_1(\Sigma, \mathbb{Z}).$$

Proof. Similar to [14, Proposition 4.1],

for $(a_1, \ldots, a_n, b_1, \ldots, b_n) \in \Sigma^{[n]+[n]}(\Sigma)$, we let $d_0 = a_1 + \cdots + a_n + b_1 + \ldots b_n \in \Sigma^{[n]+[n]}(\Sigma)$ consist of two groups of points such that all terms are distinct in Σ. A path $\gamma : [0,1] \to \Sigma$ is joining two points affiliated with only one group, say $b_i, b_j \in \{b_1, b_2, \ldots, b_n\}$, i.e., $\gamma(0) = b_i$ and $\gamma(1) = b_j$. For D^2 a disk $D^2 \subset \mathbb{R}^2$, we consider an embedding $e : D^2 \to \Sigma$ such

that $\gamma(t) = e(t - 1/2, 0)$ and the image of e consists of none other points of d_0. We can have a pair γ^+ and γ^- such that

$$\gamma^+ := e(t - 1/2, \sin(\pi t) \text{ and } \gamma^- := e(1/2 - t, -\sin(\pi t)).$$

Together we define a loop $p_\gamma(t) = a_1(t) + \cdots + a_n(t) + b_1(t) + \cdots + b_n(t)$ where $b_i(t) = \gamma^+(t)$, $b_j(t) = \gamma^-(t)$ and $a_k(t) = a_k$ for all k and $b_k(t) = b_k$ for $k \neq i, j$. The homotopy class $[p_\gamma] \in B_{n,n}(\Sigma, d_0)$ depends only on γ. We call this a swap of b_i and b_j and both b_i and b_j are zeros of $\omega \in H^0(\Sigma, N)$, where N has degree n.

Similarly, we consider an embedding $f : D^2 \to \Sigma$ such that $a_i = f(1/2, 0)$ and $b_i = f(0, 0)$. We consider a path $\delta : [0, 1] \to \Sigma$ such that

$$\delta(t) := f\left(\frac{1}{2}\cos(2\pi t), \frac{1}{2}\sin(2\pi t)\right), \delta(0) = \delta(1) = a_i \text{ and } b_i \text{ is the center of } \delta.$$

This defines a loop $p_\delta(t) = a_1(t) + \cdots + a_n(t) + b_1(t) + \cdots + b_n(t)$ where $a_i(t) = \delta(t)$, $b_i(t) = b_i$ and $a_k(t) = a_k$ for $k \neq i$. The homotopy class $[p_\delta] \in B_{n,n}(\Sigma, d_0)$ depends only on the choice of δ. We call this a double swap of a_i around b_i, where $a_i \in Z(\omega)$, $b_i \in Z(\overline{\omega})$ and ω,$\overline{\omega} \in H^0(\Sigma, N)$. □

Now we consider $n = 2l$, and $N = L^2$, where $deg(L) \geq deg(K)$, so that $(c_1, c_2) \in \mathcal{A}^{reg}_{SO_4}$.

We lift all those generators to $\mathcal{A}^{reg}_{SO_4}$.

Inspired by the work done by Schaposnik and Baraglia [14], here we let $\mathcal{L} \to \Sigma \times Jac_n(\Sigma)$ be the Poincare bundle of degree k when we fix a point $p \in \Sigma$. This is the bundle satisfying:

- 1. for all $N \in Jac_n(\Sigma)$, $\mathcal{L}|_{\Sigma \times \{N\}} \cong N$.

- 2. $\mathcal{L}|_{\{p\}\times Jac_n(\Sigma)}$ is trivial.

For degree n big enough ($n > 2g-2$), we obtain a vector bundle $b : V_n \to Jac_n(\Sigma)$ with fiber over N being $H^0(\Sigma \times \{N\}, \mathcal{L}|_{\Sigma\times\{N\}}) \cong H^0(\Sigma, N)$. By taking the divisors we have an isomorphism $\mathbb{P}(V_n) \cong S^n\Sigma$. So under the Abel-Jacobi map we have the projection $\mathbb{P}(V_n) \to Jac_n(\Sigma)$. We actually have a projective bundle $\mathbb{P}(V_n) \to Jac_n(\Sigma)$ that can lift to a vector bundle $V_n \to Jac_n(\Sigma)$. Correspondingly, we have $\tilde{S_n}(\Sigma) = \mathbb{P}(\tilde{V_n}) \to Jac_n(\Sigma)$ and $\tilde{V_n} \to Jac_n(\Sigma)$.

Now we can generalize this to the case we need. We consider the aforementioned mixed configuration space $\Sigma^{[n]+[n]}$ and consider double Abel-Jacobi maps. Then we have a principal $(\mathbb{C}^*\times\mathbb{C}^*)$-bundle $\tilde{b} : \Sigma^{[n]+[n]} \to Jac_n(\Sigma)\times Jac_n(\Sigma)$, and the fiber over $\{N\}\times\{N\} \in Jac_n(\Sigma)\times Jac_n(\Sigma)$ is $\mathcal{A}^{reg}_{SO_4} = (H^0(\Sigma, N) \times H^0(\Sigma, N))^{simp}$.

Proposition 4.12. *Let $(c_1, c_2) \in \mathcal{A}^{reg}_{SO_4}$, where $N \in Jac_n(\Sigma)$ and $n > 2g-2$. Then we have an exact sequence*

$$\pi_1(\mathcal{A}^{reg}_{SO_4}, (c_1, c_2)) \xrightarrow{i_*} \pi_1(\tilde{V}^{n,n}, (c_1, c_2)) \xrightarrow{\alpha_*\circ b_*} \pi_1(Jac_n(\Sigma) \times Jac_n(\Sigma), (N, N)) \longrightarrow 1. \tag{4.10}$$

Proof. Similar to [14, Proposition 4.2], we have the commutative diagram

$$\begin{array}{ccccccc}
 & & \mathbb{C}^*\times\mathbb{C}^* & = & \mathbb{C}^*\times\mathbb{C}^* & & \\
 & & \downarrow & & \downarrow & & \\
 & & \mathcal{A}^{reg}_{SO_4} & \longrightarrow & \tilde{V}_{n,n} & \xrightarrow{\tilde{\alpha}\circ\tilde{b}} & Jac_n(\Sigma)\times Jac_n(\Sigma) \\
 & & \downarrow & & \downarrow{\scriptstyle \tilde{b}} & & \| \\
\overline{\mathcal{A}^{reg}_{SO_4}} & = & \frac{\mathcal{A}^{reg}_{SO_4}}{\mathbb{C}^*\times\mathbb{C}^*} & \xrightarrow{i} & \tilde{S}^{n,n}\Sigma & \xrightarrow{\tilde{\alpha}} & Jac_n(\Sigma)\times Jac_n(\Sigma)
\end{array} \tag{4.11}$$

By considering the fundamental groups, we have another commutative diagram

$$\begin{array}{ccccc}
\mathbb{Z}\times\mathbb{Z} & = & \mathbb{Z}\times\mathbb{Z} & & \\
\downarrow & & \downarrow & & \\
\pi_1(\mathcal{A}^{reg}_{SO_4},(c_1',c_2')) & \longrightarrow & \pi_1(\tilde{V}_{n,n},(c_1',c_2')) & \xrightarrow{\tilde{\alpha}_*\circ\tilde{b}_*} & \pi_1(Jac_n(\Sigma)\times Jac_n(\Sigma),(N\times N)) \\
\downarrow & & \downarrow{\scriptstyle \tilde{b}} & & \| \\
\pi_1\left(\overline{\mathcal{A}^{reg}_{SO_4}},(c_1,c_2)\right) & \xrightarrow{i_*} & \pi_1(\tilde{S}^{n,n}(\Sigma),(c_1,c_2)) & \xrightarrow{\tilde{\alpha}} & \pi_1(Jac_n(\Sigma)\times Jac_n(\Sigma),(N\times N))
\end{array} \tag{4.12}$$

Based on the discussion in the previous sections, we have the exactness of the third row. Then the exactness on the second row will follow.

Based on the previous discussion, swaps and double swaps give elements in $\pi_1\left(\overline{\mathcal{A}^{reg}_{SO_4}},(c_1,c_2)\right)$. Now we are lifting each one of them to a loop in $\tilde{V}^{n,n}$. First we deal with the swaps. Consider a path γ from b_i to b_j, let $e : D^2 \to \Sigma$ be the aforementioned oriented embedding such that $\gamma(t) = e(t-1/2, 0)$ and the image of e contains no other simple zeros. Since our choice of two points b_i and b_j are from a single q_1, we can consider a well-defined map $i : S^2(D^2) \to S^{n,n}(\Sigma)$, where $S^2(D^2)$ is the symmetric product of the disk D^2. i sends two points u, v to the divisor $e(u) + e(v) + \sum_{k\neq i,j} a_k + \sum b_k$. Here a_k and b_k represent zeroes affiliated with different quadratic differentials. Since we have the principal $\mathbb{C}^* \times \mathbb{C}^*$-bundle $q : \tilde{V}^{n,n} \to \tilde{S}^{n,n}(\Sigma)$. The restriction $i^*(\tilde{V}^{n,n})$ is a bundle over the contractible space $S^2(D^2)$. This admits a section $s : S^2(D^2) \to \tilde{V}^{n,n}$ such that $q \circ s = id$. Therefore we pick a section s such that $s((-1/2, 0), (1/2, 0)) = (c_1', c_2')$. By using the section s we lift the loop defined by γ to a loop p_γ in $\tilde{V}^{n,n}$ based at (c_1', c_2'). The homotopy class $[p_\gamma] \in \pi_1(\tilde{V}^{n,n}, (c_1', c_2'))$ is independent of the embedding e and section s. From the previous proposition, the class $[p_\gamma]$ is in the image of $i_* : \pi_1(\mathcal{A}^{reg}_{SO_4}, (c_1', c_2')) \to \pi_1(\tilde{V}^{n,n}, (c_1', c_2'))$. Then by exactness, we get an (maybe not a unique)

element in $\pi_1(\mathcal{A}^{reg}_{SO_4})$. The same argument will work for the case when we choose two points from the other quadratic differential.

Now we consider the two points a_i and b_i from differential quadratic differentials q_1 and q_2, say a_i a zero of q_1 and b_i a zero of q_2. By nature of our mixed configuration space $\Sigma^{[n]+[n]}$ and by using the embedding f in the previous section, instead of considering the symmetric product $S^2(D^2)$, we consider a map $i : D^2 \to \tilde{S}^{n,n}(\Sigma)$. i sends a pair $(0,0)$ and $(1/2,0)$ to the divisor $f(0,0) + \sum_{k\neq i} a_k + f(1/2,0) + \sum_{k\neq i} b_k$. Since D^2 is contractible, then the restriction $i^*(\tilde{V}^{n,n})$ to D^2 admits a section s. Now we pick a section $s : D^2 \to \tilde{V}^{n,n}$ such that $\tilde{p}_\delta(t) := s(1/2\cos(2\pi t), 1/2\sin(2\pi t))$. This is a lift of p_δ to a loop in $\tilde{V}_{n,n}$ based at (c_1', c_2').

In fact, all the lifted loops in $\mathcal{A}^{reg}_{SO_4}$ may not be unique. □

By a modification of the construction in [14, Page 5505,5506],

for $n = 2l$, we determine the monodromy of swaps and double swaps. Like the works done by Schaposnik and Baraglia, we map $\mathcal{A}^{reg}_{SO_4}$ to a larger family of branched covers of Σ.

Let $sq \times sq : Jac_l(\Sigma) \times Jac_l(\Sigma) \to Jac_{2l}(\Sigma) \times Jac_{2l}(\Sigma)$ be the diagonal squaring map $sq(L, L) = (L^2, L^2)$. We consider two pullback diagrams:

$$\begin{array}{ccc} Y_{l,l} & \xrightarrow{\alpha} & Jac_l(\Sigma) \times Jac_l(\Sigma) \\ \downarrow p & & \downarrow sq\times sq \\ S^{n,n}(\Sigma) & \xrightarrow{\alpha} & Jac_n(\Sigma) \times Jac_n(\Sigma) \end{array} \tag{4.13}$$

and

$$\begin{array}{ccc} Z_{l,l} & \xrightarrow{q} & Y_{l,l} \\ \downarrow p & & \downarrow p \\ V_{n,n}(\Sigma) & \xrightarrow{q} & S^{n,n}(\Sigma) \end{array} \tag{4.14}$$

Together, we can restrict the pullback diagrams to $\tilde{S}^{n,n}(\Sigma)$ and $\tilde{V}_{n,n}(\Sigma)$.

$$\begin{array}{ccccc} \tilde{Z}_{l,l} & \xrightarrow{\tilde{q}} & \tilde{Y}_{l,l} & \xrightarrow{\tilde{\alpha}} & Jac_l(\Sigma) \times Jac_l(\Sigma) \\ \downarrow{\scriptstyle \tilde{p}} & & \downarrow{\scriptstyle \tilde{p}} & & \downarrow{\scriptstyle sq\times sq} \\ \tilde{V}_{n,n}(\Sigma) & \xrightarrow{\tilde{q}} & \tilde{S}^{n,n}(\Sigma) & \xrightarrow{\tilde{\alpha}} & Jac_n(\Sigma) \times Jac_n(\Sigma) \end{array} \tag{4.15}$$

When we pick a degree l line bundle L and an tuple $(c_1, c_2) \in \mathcal{A}^{reg}_{SO_4}$. This represents a point $z \in \tilde{Z}_{l,l}$. Then this gives a natural inclusion $\iota : \mathcal{A}^{reg}_{SO_4} \to \tilde{Z}_{l,l}(\Sigma)$. Therefore, any point z defines a branched double cover

$$S_z := \{y \in L^2 | y^2 + ay + p^2 = 0, a = \frac{c_1 + c_2}{2} \text{ and } p = \frac{c_1 - c_2}{4}\}.$$

Then we vary our choices of z to get a family $\tilde{\mathbb{S}}$ of double branched covers over Σ.

$$\begin{array}{ccc} \tilde{\mathbb{S}} & & \\ {\scriptstyle v}\downarrow & \searrow^{w} & \\ \Sigma \times \tilde{Z}_{l,l} & \xrightarrow{p_2} & \tilde{Z}_{l,l} \end{array} \tag{4.16}$$

Moreover, the restriction $v|_{S_z}$ is the branched double covering $S_z \to \Sigma$. Here we use the identification $\text{Aut}(H^1(S, \mathbb{Z})) = \text{Aut}(Jac(S))$, we have a representation $\rho_{\tilde{Z}_{l,l}} : \pi_1(\tilde{Z}_{n,n}, z) \to \text{Aut}(Jac(S))$. Then the family of spectral covering over $\mathcal{A}^{reg}_{SO_4}$ is the pullback $\iota^*(w)$ of $w : \tilde{\mathbb{S}} \to \tilde{Z}_{l,l}$, where $\iota : \mathcal{A}^{reg}_{SO_4} \to \tilde{Z}_{l,l}$.

Here $\tilde{Z}_{l,l} \to \tilde{V}_{n,n}(\Sigma)$ is a covering space, therefore we natually have a injective map $\tilde{p}_* : \pi_1(\tilde{Z}_{l,l}, z) \to \pi_1(\tilde{V}_{n,n}, (c_1, c_2))$. Hence we have another commutative diagram:

$$\begin{array}{ccc} & & \mathrm{Aut}(Jac(S)) \\ & \overset{\rho}{\nearrow} & \uparrow{\scriptstyle \rho_{\tilde{Z}_{l,l}}} \\ \pi_1(\mathcal{A}^{reg}_{SO_4}, (c_1, c_2)) & \xrightarrow{\iota_*} & \pi_1(\tilde{Z}_{l,l}, z) \\ & \underset{i_*}{\searrow} & \downarrow{\scriptstyle \tilde{p}_*} \\ & & \pi_1(\tilde{V}_{n,n}, (c_1, c_2)) \end{array} \tag{4.17}$$

Since by the choices of γ and δ in our previous discussion, we have the lifted swaps and double swaps $[p_\gamma]$, $[p_\delta]$. Those classes lie in the image of $\tilde{p}_*$. Therefore injectivity of $\tilde{p}_*$ we have a unique class in $\pi_1(\tilde{Z}_{l,l}, z)$. Then we have a well-defined monodromy action $\rho_{\tilde{Z}_{l,l}}([p_\gamma])$ or $\rho_{\tilde{Z}_{l,l}}([p_\delta])$.

In order to get the commutativity of the upper triangle, we need to show that $\rho_{\tilde{Z}_{l,l}}([p_\gamma])$ and $\rho_{\tilde{Z}_{l,l}}([p_\delta])$ are associated with γ and δ.

For the γ, i.e, the swap of two points, we use l_γ to represent the loop in S associated to γ. In addition, the homology class $[l_\gamma] \in H_1(S, \mathbb{Z})$ satisfies $\pi_*(l_\gamma) = 0$ in $H_1(\Sigma, \mathbb{Z})$. Then we have the poincare dual c_γ. Then it acts on $H^1(S, \mathbb{Z})$ as a Picard-Lefschetz transformation.

Similarly, for the path defined by δ in Σ, a small deformation δ' of the loop δ is around b_1 from a_1. The pre-image of $\pi : S \to \Sigma$ is also a loop around $u_1 = \pi^{-1}(a_1)$ and $v_1 = \pi^{-1}(b_1)$. This also forms a homology class l_δ associated to δ such that $\pi_*([l_\delta]) = 0$ in $H^1(\Sigma, \mathbb{Z})$. Hence it is also zero in $H^1(\Sigma, \mathbb{Z}) \times H^1(\Sigma, \mathbb{Z})$. By taking the poincare dual, we have a class c_δ in the cohomology. It acts on $H^1(S, \mathbb{Z})$ as a Picard-Lefschetz transformation as well.

Thus the monodromy action of the loop associated to γ or δ is

$$\gamma \cdot x := \rho_{\tilde{Z}_{n,n}}(\tilde{s}_\gamma)x = x + \langle c_\gamma, x \rangle c_\gamma \text{ and}$$

$$\delta \cdot x := \rho_{\tilde{Z}_{n,n}}(\tilde{s}_\delta)x = x + \langle c_\delta, x\rangle c_\delta.$$

4.1.2.1 Monodromy action on the 2-torsion points.

Let $B = \{a_i, \ldots, a_{2l}, b_1, \ldots, b_{2l}\}$ be the two groups of branched points and let $\mathbb{Z}_2[B]$ be the vector space over the finite field $\mathbb{Z}_2$ with basis in B.

Based on [14, 4.4], we have the following.

Given the double branched covering $\overline{S} \to \Sigma$ with branched points B, we are using the identification $\Lambda_\Sigma = H^1(Jac(\Sigma, \mathbb{Z})) = H^1(\Sigma, \mathbb{Z})$ and $\Lambda_{\overline{S}} = H^1(Jac(\overline{S}, \mathbb{Z})) = H^1(\overline{S}, \mathbb{Z})$.

In addition, the pullback and the norm maps induce maps in cohomology $\pi_* : \Lambda_{\overline{S}} \to \Lambda_\Sigma$ and $\pi^* : \Lambda_\Sigma \to \Lambda_{\overline{S}}$. Furthermore, we have $\pi_*\pi^*(x) = 2x$.

Now we consider $\Lambda_P = H^1(Prym(\overline{S}, \Sigma))$, then by using the homotopy long exact sequence of the short exact sequence and Hurewicz theorem, we have

$$1 \longrightarrow Prym(\overline{S}, \Sigma) \longrightarrow Jac(\overline{S}) \xrightarrow{Nm} Jac(\Sigma) \longrightarrow 1. \tag{4.18}$$

Followed by Hurewicz theorem, we get $H_1(Prym(\overline{S}, \Sigma), \mathbb{Z}) = \ker(\pi_* : H_1(\overline{S}, \mathbb{Z}) \to H_1(\Sigma, \mathbb{Z}))$. Then by using Poincare duality on each term, we get $\Lambda_P = \ker(\pi_* : \Lambda_{\overline{S}} \to \Lambda_\Sigma)$.

No we consider a counting map $s : \mathbb{Z}_2[B] \to \mathbb{Z}_2$ with $s(a_i) = s(b_j) = 1$ for all i, j. First we observe that $\mathbb{Z}_2[B]^{even}$ the even number of points in B is the kernel of this counting map s, i.e., $\mathbb{Z}_2[B]^{even} = \ker s$.

Proposition 4.13 ([14], Proposition 4.7)**.** *There is a short exact sequence of the vector spaces.*

$$0 \longrightarrow \Lambda_\Sigma[2] \xrightarrow{\pi^*} \Lambda_P[2] \xrightarrow{\epsilon} \mathbb{Z}_2[B]^{even}/(d) \longrightarrow 0 \tag{4.19}$$

Proof. Here we use the result and the argument from [14, Proposition 4.7]. By using the universal coefficient theorem, we have $\Lambda_\Sigma[2] = \Lambda_\Sigma \otimes_{\mathbb{Z}} \mathbb{Z}_2$ and similarly $\Lambda_{\overline{S}}[2]$, $\Lambda_P[2]$. An typical element of $\Lambda_P[2]$ is a line bundle L over $\overline{S}$ such that $L^2 = \mathcal{O}$ with $\sigma^*(L) = L^*$ where σ is the involution. Therefore L is invariant under this σ^*-action, i.e., $\sigma^*(L) = L$.

Therefore, the line bundle L can be viewed as a $\mathbb{Z}_2$-local system with an isomorphism $\tilde{\sigma} : L \to L$ related to the involution σ. For ramification points $u_1, \ldots, u_{2l}$ lying over $a_1, \ldots, a_{2l}$ and $v_1, \ldots, v_{2l}$ over $b_1, \ldots, b_{2l}$, $\tilde{\sigma}$ acts on the fibers L_{w_i} as $(-1)^{\epsilon_i}$, where $w_i = u_i$ or v_i and $\epsilon_i \in \mathbb{Z}_2$. Therefore, we have a well-defined map ϵ such that $\epsilon(L) = \epsilon_1 a_1 + \cdots + \epsilon_{2l} a_{2l} + \epsilon'_1 b_1 + \cdots + \epsilon'_{2l} b_{2l} \in \mathbb{Z}_2[B]$. In fact, the image of ϵ is always even, i.e., $\mathbb{Z}_2[B]^{even}$. We also have $\epsilon(L, -\tilde{\sigma}) = \epsilon(L, \tilde{\sigma}) + d$, where $d = a_1 + \cdots + a_{2l} + b_1 + \cdots + b_{2l}$. In fact, when we restrict the bundle L to $\overline{S} \setminus \{u_1, \ldots, u_{2l}, v_1, \ldots, v_{2l}\}$, the bundle descends to $\Sigma \setminus \{a_1, \ldots, a_{2l}, b_1, \ldots, b_{2l}\}$. Hence we have the desired short exact sequence. □

Proposition 4.14. *Let γ be a path joining a distinct pair of zeros a_i, a_j (or b_i, b_j) of a section in $H^0(\Sigma, L^2)$ and let $c_\gamma \in \Lambda_P[2]$ be the corresponding cycle in $\overline{S}$. Then we have*

$$\epsilon(c_\gamma) = a_i + a_j \text{ (or } b_i + b_j).$$

Proof. This is a necessary step. Without loss of generality, we can assume $L = K$. With a bundle of greater degree, the same argument will follow. Since γ corresponds to a swap between two zeros of a quadratic differential. We repeat the argument in [14, Proposition 4.8]. Since zeros a_i, a_j or b_i, b_j yield the same result, we focus only on zeros of one quadratic differential q. Consider a single quadratic differential $q \in H^0(K^2)$ with only simple zeros, and consider b_i and b_j are two distinct zeros. Suppose γ is a path joining b_i and b_j without passing through any other

simple zeros, we obtain a cycle $l_\gamma \in H_1(\overline{S}, \mathbb{Z})$. Then we take the Poincare dual c_γ of l_γ and this $c_\gamma \in H^1(\overline{S}, \mathbb{Z}_2)$. Thus we view c_γ as a certain $\mathbb{Z}_2$-local ststem on $\overline{S}$. Then the holonomy of c_γ around a cycle $l \in H_1(\overline{S}, \mathbb{Z})$ coincides with the intersection pairing of l and l_γ. Now we choose l to be a cycle obtained from a path γ_{km} joining b_k and b_m, two simple zeros of the same quadratic differential. Then intersection number of l_γ and $l_{\gamma_{km}}$ is the number of common elements in $\{i, j\}$ and $\{k, m\}$ mod 2. Since the involution $\tilde{\sigma}$ acts on the fiber over u_i as $(-1)^{\epsilon_i}$. In addition, the pre-image of γ_{km} consists of two path $\tilde{\gamma_1}$ and $\tilde{\gamma_1}$ joining u_k and u_m. So the holonomy around $l_{\gamma_{km}}$ is $(-1)^{\epsilon_k+\epsilon_m}$. Thus $\epsilon(c_\gamma) = b_i + b_j$.

If $c \in \Lambda_P[2]$ such that $\epsilon(c) = b_i + b_j$, then there exists a path γ joining b_i and b_j for $c = c_\gamma$.

Conversely, when $0 \neq l \in \Lambda_\Sigma[2]$. It corresponds to a cycle $l \in H_1(\Sigma, \mathbb{Z}_2)$. This cycle can be viewed as an embedded loop in Σ. This corresponds to path joining b_i and b_j along an embedded loop represented by l followed by a simple path from b_i and b_j without touching other marked points. In addition, since the mapping class group acts on $H^1(\Sigma, \mathbb{Z}_2)$ transitively in the nontrivial subset $H^1(\Sigma, \mathbb{Z}_2)\backslash\{0\}$. In particular, we are always able to find a class $a \in H^1(\Sigma, \mathbb{Z}_2)$.

□

Now we consider a non-degenerate symmetric bilinear form $((,)) : \mathbb{Z}_2[B] \otimes \mathbb{Z}_2[B] \to \mathbb{Z}_2$ by letting $((b_i, b_j)) = \delta_{ij}$. It is easy to see that the orthogonal complement of d in $\mathbb{Z}_2$ is $(\mathbb{Z}_2 B)^{even}$ since d is even, i.e., $(d)^\perp = (\mathbb{Z}_2 B)^{even}$ and $(d) \subset (\mathbb{Z}_2 B)^{even}$. In addition, since d includes all branch points of the double covering, then $((x, d)) = ((x, x))$ for $x \in \mathbb{Z}_2 B$. Then it is clear that by taking modulo (d) the induced bilinear form $((,))$ is 0 for any $x \in W = (\mathbb{Z}_2 B)^{even}/(d)$, i.e., $((x, x)) = 0$ for all $x \in W$. Based on the previous short exact sequence, it is easy to see that the subspace $\Lambda_\Sigma[2]$ is null, thus we have the following:

Proposition 4.15. *For $\epsilon : \Lambda_P[2] \to (\mathbb{Z}_2 B)^{even}/(d)$, a bilinear form $\langle , \rangle$ of $\Lambda_P[2]$ is given by the pullback of $((,))$ under the map ϵ. To be more specific,*

$$\langle (x_1, y_1), (x_2, y_2) \rangle = ((y_1, y_2))$$

for all $(x_1, y_1), (x_2, y_2) \in \Lambda_P[2] \cong \Lambda_\Sigma[2] \oplus (\mathbb{Z}_2 B)^{even}/(d)$.

Proof. Based on the previous short exact sequence, $\Lambda_P[2]$ is spanned by the image of $\Lambda_\Sigma[2]$ through pullback and elements of the form c_γ and c_δ, where γ is joining two branched points and δ is an contractible loop including one branch point and enclose one other branch point. $W = (\mathbb{Z}_2 B)^{even}/(d)$. Since $\Lambda_\Sigma[2]$ is null with respect to $\langle , \rangle$, we just need to verify for the pair of different c_γ and $c_{\gamma'}$ or c_δ. By previous discussion, it remains to find the intersection number of c_γ and c'_γ or c_δ. But this is exact the number of common points we choose from 4 branch points affiliated with γ and γ' or δ mod 2. This is $((y_1, y_2)) = ((\epsilon(c_\gamma), \epsilon(c_{\gamma'}))) = ((a_i + a_j, a_k + a_m))$ (or $= ((b_i + b_j, b_k + b_m))$). □

4.1.2.2 Splitting

Theorem 4.16. *Based on Schaposnik and Baraglia [14], we can choose a specific splitting of the vector space $\Lambda_{\overline{S}}[2]$.*

Recall that by choosing a splitting $\Lambda_P[2] = \Lambda_\Sigma[2] \oplus (\mathbb{Z}_2[B])^{even}/(d)$ of the previous short exact sequence, there exists a splitting of the short exact sequence

$$0 \longrightarrow \Lambda_P[2] \longrightarrow \Lambda_{\overline{S}}[2] \xrightarrow{\pi_*} \Lambda_\Sigma[2] \longrightarrow 0 \tag{4.20}$$

such that

$$\Lambda_{\overline{S}}[2] \cong \Lambda_P[2] \oplus \Lambda_\Sigma[2] \cong \Lambda_\Sigma[2] \oplus (\mathbb{Z}_2 B)^{even}/(d) \oplus \Lambda_\Sigma[2].$$

And the intersection form on $\Lambda_{\overline{S}}[2]$ *is*

$$\langle (a,b,c),(a',b',c') \rangle = \langle a, c' \rangle + ((b,b')) + \langle c, a' \rangle$$

for $a, a', c, c' \in \Lambda_\Sigma[2]$ *and* $b, b' \in (\mathbb{Z}_2 B)^{even}/(d)$.

Proof. $W = (\mathbb{Z}_2 B)^{even}/(d)$ we choose a splitting $\Lambda_P[2] \cong \Lambda_\Sigma[2] \oplus W$ as we mentioned previously. $\Lambda_{\overline{S}}[2]$. The restriction of the intersection form to W is the degenerate bilinear form $((,))$. W is non-degenerate, we now have an orthogonal splitting $\Lambda_{\overline{S}}[2] = W \oplus W^\perp$. It is easy to see that the restriction $\pi_*|_{W^\perp} : W^\perp \to \Lambda_\Sigma[2]$ is surjective and we have $\ker(\pi_*|_{W^\perp}) = \ker(\pi_*) \cap W^\perp = \Lambda_P[2] \cap W^\perp = \pi^*(\Lambda_\Sigma[2])$. So we have a corresponding short exact sequence:

$$0 \longrightarrow \Lambda_\Sigma[2] \xrightarrow{\pi^*} W^\perp \xrightarrow{\pi_*|_{W^\perp}} \Lambda_\Sigma[2] \longrightarrow 0. \tag{4.21}$$

We choose a symmetric bilinear form $\beta : \Lambda_\Sigma[2] \otimes \Lambda_\Sigma[2] \to \mathbb{Z}_2$ where $\iota : \Lambda_\Sigma[2] \to W^\perp$ is a splitting of the short exact sequence. In fact, we can modify the splitting ι to construct an isotropic splitting. Choose an $F \in \mathrm{End}(\Lambda_\Sigma[2])$, we obtain an splitting $\iota' : \Lambda_\Sigma[2] \to W^\perp$ by setting $\iota' a = \iota a + \pi^*(Fa)$. We have the following

$$\begin{aligned}\langle \iota' a, \iota' b\rangle &= \langle \iota a + \pi^*(Fa), \iota b + \pi^*(Fb)\rangle \\ &= \beta(a,b) + \langle \iota a, \pi^*(Fa)\rangle + \langle \pi^*(Fa), \iota b\rangle \\ &= \beta(a,b) + \langle \pi_*\iota a, Fa\rangle + \langle Fa, \pi_*\iota b\rangle \\ &= \beta(a,b) + \langle a, Fb\rangle + \langle Fa, b\rangle.\end{aligned} \tag{4.22}$$

Since β is even and symmetric, we can find an F such that $\langle \iota' a, \iota' b\rangle$ vanishes for all $a, b \in \Lambda_\Sigma[2]$. So we have an isotropic splitting.

So given an isotropic splitting $\iota : \Lambda_\Sigma[2] \to W^\perp$ and the inclusion $W^\perp \to \Lambda_{\overline{S}}[2]$, just need to check the intersection pairing $\langle \pi^*(a), \iota(c')\rangle = \langle a, \pi_*\iota(c')\rangle = \langle a, c'\rangle$, then the rest will follow. Thus we have the desired intersection pairing $\langle , \rangle$ for $\Lambda_{\overline{S}}[2]$. □

4.1.3 Matrix representation of the monodromy action

Given all the information above, we choose the splitting from above, then

$$\Lambda_{\overline{S}}[2] = \Lambda_\Sigma[2] \oplus (\mathbb{Z}_2 B)^{even}/(d) \oplus \Lambda_\Sigma[2]. \tag{4.23}$$

We observe that W is spanned by $a_{12}, a_{23}, \ldots, a_{2l-1,2l}, b_{12}, \ldots, b_{2l-1,2l}$, where $a_{i,i+1} = a_i + a_{i+1}, b_{j,j+1} = b_j + b_{j+1}$, with the relation $a_{12} + a_{34} + \cdots + a_{2l-1,2l} + b_{12} + b_{34} + \cdots + b_{2l-1,2l} = 0$ We also have $\pi^*(a) = (a, 0, 0)$, $\pi_*(a, b, c) = c$ and the given intersection pairing. We now find out the monodromy action on $\Lambda_{\overline{S}}[2]$.

Given an element $c \in \Lambda_{\overline{S}}[2]$, we let s_c denote the corresponding Picard-Lefschetz transfor-

mation $s_c(x) = x + \langle c, x\rangle c$.

By previous discussion, the monodromy action on $\Lambda_{\overline{S}}[2]$ is generated by the involution σ and Picard-Lefschetz transformation s_c, c is any element in $\Lambda_P[2]$. By previous splitting 4.23, we can consider $c = (a, b, 0)$, b can be a_{ij} or $b_{i'j'}$. If we consider the term $c = (0, b, 0)$, b represents a permutation $\sigma \in S_{2l} \times S_{2l}$. This means a permutation ω of branch points associated with a single quadratic differential. So σ acts on B by $\sigma(a_i) = a_{\sigma(i)}$ (similarly for the second group of branched points). This action preserves $\mathbb{Z}_2 B^{even}$ then it descends to W. This has the form

$$s_{ij} = \begin{bmatrix} I_{2g} & 0 & 0 \\ 0 & \sigma_{ij} & 0 \\ 0 & 0 & I_{2g} \end{bmatrix}$$

where $\sigma_{ij} \in S_{2l} \times S_{2l}$.

Given a line bundle L over $\overline{S}$, for the involution action σ, we have the following $\sigma^*(L) \otimes L = \pi^*(Nm(L))$. Then correspondingly, for an element $x = (r, s, t) \in \Lambda_\Sigma[2]$, we have $s_\sigma(x) + x = \pi^*(\pi_*(x))$.. Then over the finite field $\mathbb{Z}_2$, we have $s_\sigma(x) = x + \pi^*(\pi_*(x))$. Then for $x = (r, s, t)$, we have

$$\begin{aligned} s_\sigma(r, s, t) &= (r, s, t) + \pi^*(\pi_*((r, s, t))) \\ &= (r, s, t) + \pi^*(t) \\ &= (r, s, t) + (t, 0, 0) \\ &= (r + t, s, t). \end{aligned} \tag{4.24}$$

Then the matrix representation of this involution action is

$$\sigma = \begin{bmatrix} I_{2g} & 0 & I \\ 0 & I & 0 \\ 0 & 0 & I_{2g} \end{bmatrix}$$

In order to get the matrix representations of other generators, we need to dissect the Picard Lefschetz formula

$$s_c(x) = x + \langle c, x \rangle c.$$

Then based on the monodromy representation $\Lambda_{\overline{S}}[2] = \Lambda_\Sigma[2] \oplus W \oplus \Lambda_\Sigma[2]$, we will have a matrix representation of the monodromy actions.

Consider $x = (r, s, t)$ and $c = (a, b_{ij}, 0) \in \Lambda_\Sigma[2] \oplus W \oplus \Lambda_\Sigma[2]$ where $b_{ij} = b_i + b_j \in W$, now we have

$$\begin{aligned} s_c(x) &= x + \langle c, x \rangle c \\ &= (r, s, t) + \langle (a, b_{ij}, 0), (r, s, t) \rangle (a, b_{ij}, 0) \\ &= (r, s, t) + (\langle a, t \rangle + ((b_{ij}, s))) (a, b_{ij}, 0). \end{aligned} \tag{4.25}$$

Then we can break the monodromy action into several linear actions. The monodromy matrix $S_c(x)$ of $s_c(x)$ can be written as

$$S_c = \begin{bmatrix} I_{2g} & L^a_{ij} & S^a \\ 0 & \sigma_{ij} & (L^a_{ij})^t \\ 0 & 0 & I_{2g} \end{bmatrix}.$$

where $L_{ij}^a : W \to \Lambda_\Sigma[2]$ is given by $L_{ij}^a(s) = ((b_{ij}, s))a$ and $(L_{ij}^a)^t : \Lambda_\Sigma[2] \to W$ is given by $(L_{ij}^a)^t(t) = \langle a, t\rangle b_{ij}$ and $S^a : \Lambda_\Sigma[2] \to \Lambda_\Sigma[2]$ is given by $S^a(t) = \langle a, t\rangle a$. In addition, $\sigma_{ij}(s) = s + ((b_{ij}, s))b_{ij}$.

Therefore, we have the following theorem.

Theorem 4.17. *We have two generators for the monodromy action:*

$$A_{ij}^x = \begin{bmatrix} I_{2g} & L_{ij}^a & S^a \\ 0 & I & (L_{ij}^a)^t \\ 0 & 0 & I_{2g} \end{bmatrix}.$$

and

$$s_{ij} = \begin{bmatrix} I_{2g} & 0 & 0 \\ 0 & \sigma_{ij} & 0 \\ 0 & 0 & I_{2g} \end{bmatrix}.$$

or the $\sigma_{ij}(s)$, if both b_i and b_j are in s, then after the σ_{ij}-action, no zeros are switched because the intersection number is 0. On the other hand, when $((s, b_{ij})) = 1$, $\sigma_{ij}(s)$ will interchange b_i and b_j in s. In addition, the i and j should be both zeros of a single quadratic differential. Then the matrix representation of s_{ij} is

$$s_{ij} = \begin{bmatrix} I_{2g} & 0 & 0 \\ 0 & \sigma_{ij} & 0 \\ 0 & 0 & I_{2g} \end{bmatrix}.$$

where $\sigma_{ij} \in S_{4g-4} \times S_{4g-4}$.

Chapter 5: Applications

As we have seen from the previous chapter, we have a matrix representation of the monodromy action of the $SO(2,2)$-Hitchin fibration. We gave a splitting of the vector space $\Lambda_{\overline{S}}[2]$ as a direct sum of the vector spaces $\Lambda_{\Sigma}[2] \oplus W \oplus \Lambda_{\Sigma}[2]$. In this chapter, we are achieving two goals. We try to understand the geometric meaning of the vector space W from the splitting and count the orbits of the monodromy action. By using the results from [29] and [30], we get the following theorem.

Theorem 5.2. *The $SO(2,2)$-Higgs bundle E can be decomposed as a direct sum $E = V \oplus W$. In addition, $V = N_1N_2 \oplus (N_1N_2)^*$ and $W = N_1N_2^* \oplus (N_1N_2^*)^*$. Hence the degree of each component is fully understood.*

$$deg(N_1N_2) = 2g - 2 - \frac{\tilde{m_1} + \tilde{m_2}}{2} \text{ and } deg(N_1N_2^*) = \frac{\tilde{m_1} - \tilde{m_2}}{2}.$$

In addition, by using the matrix representation from the previous chapter, we found out:

Theorem 5.3. *The monodromy action on $\Lambda_{\overline{S}}[2]$ is generated by the groups of matrices of the form*

- $A^x_{ij} = \begin{bmatrix} I_{2g} & L^a_{ij} & S^a \\ 0 & I & (L^a_{ij})^t \\ 0 & 0 & I_{2g} \end{bmatrix}$
- $s_{ij} = \begin{bmatrix} I_{2g} & 0 & 0 \\ 0 & \sigma_{ij} & 0 \\ 0 & 0 & I_{2g} \end{bmatrix}$ and
- $M^r = \begin{bmatrix} I_{2g} & 0 & B^r \\ 0 & I & 0 \\ 0 & 0 & I_{2g} \end{bmatrix}$.

In addition, by using those matrices, we count the number of the connected components in $SO(2,2)$-character variety.

Theorem 5.4 *The $SO(2,2)$-character variety has $2 \times 2^{2g} + 2g^2 - 2g + 1$ connected components.*

5.1 Re-examining the space $W = \mathbb{Z}_2[B]^{\text{even}}/(d)$

5.1.1 $SL_2(\mathbb{R})$-Case

To make things simpler, we consider the $SL_2(\mathbb{R})$-Hitchin fibration. For $G = SL_2$, the Hitchin fiber is isomorphic to 2-torsion points of the Prym variety $P(S,\Sigma)[2]$. Based on Schaposnik's argument, for simple zeros of a quadratic differential q, this space can be splitted into direct sum $P(S,\Sigma) = \Lambda_S[2] \oplus W$, where $W = \mathbb{Z}_2[B]^{\text{even}}/(d)$ and $(d) = a_1 + \cdots + a_{4g-4}$ [28]. In addition, based on Hitchin's original paper [1], any SL_2-Higgs bundle E can be decomposed

as a direct sum $E = V \oplus V^*$.

Now we examine what is the relationship between the number of the zeros invariant under the action induced by the involution action and the degree of the bundle L which appears in the direct sum.

Consider a line bundle $M \in P(S, \Sigma)$ on S, for the double branched cover $\pi : S \to \Sigma$, we have $\pi_* M = V \oplus V^*$. The involution σ preserves M, and it acts as ± 1 on those $4g - 4$ fixed points. Suppose $\tilde{m}$ is the number of the fixed points on which σ acts as -1, for an open set $\mathcal{U} \subset \Sigma$, we have

$$H^0(\mathcal{U}, \pi_*(M)) = H^0(\pi^{-1}(\mathcal{U}), M) = H^0(\pi^{-1}(\mathcal{U}), M)^+ \oplus H^0(\pi^{-1}(\mathcal{U}), M)^-.$$

We also have

$$H^0(\pi^{-1}(\mathcal{U}), M)^+ \cong H^0(\mathcal{U}, V)$$

$$H^0(\pi^{-1}(\mathcal{U}), M)^- \cong H^0(\mathcal{U}, V^*)$$

Here we set $h^+ = \dim(H^0(\mathcal{U}, V))$ and $h^- = \dim(H^0(\mathcal{U}, V^*))$ [13].

We use a trick here. Given L a line bundle over S, the involution σ acts trivially on the pull-back $\pi^* L$. We also have the following result from [22]

$$\pi_*(M \otimes \pi^* L) = \pi_* M \otimes L.$$

We manually pick some L with degree large enough, then $H^1(S, M \otimes \pi^* L) = 0$. Therefore we can use Riemann-Roch to calculate the dimension of $H^0(S, M \otimes \pi^* L)$. Furthermore, over the

fixed points of σ, σ acts still as $+1$ on $\pi_* L$. So the involution acts as -1 on $M \otimes \pi^* L$ for exactly $\tilde{m}$ points. Then without loss of generality, we can assume that $H^1(S, M) = 0$.

Now we use the holomorphic Lefschetz fixed point theorem in [29, 4.11], we have

$$\begin{aligned} L(\sigma) &= \Sigma(-1)^q \text{ trace } \sigma|_{H^{0,q}(M)} \\ &= \text{trace } \sigma|_{H^0(M)} \\ &= \dim H^0(M)^+ + \dim H^0(M)^- \\ &= h^+ - h^-. \end{aligned} \tag{5.1}$$

In addition, based on a theorem in [29, Theorem 4.12], we can compute the Lefschetz number as follows. For the involution action σ and a bundle morphism $\phi : \sigma^* M \to M$,

$$L(\sigma) = \Sigma_P \frac{\text{trace}_{\mathbb{C}} \phi_P}{\det_{\mathbb{C}}(1 - d\sigma_P)}.$$

Since we can write out the involution action explicitly, i.e., by negating the tautological section λ, then for each fixed point P, $d\sigma = -1$. Then the denominator $\det_{\mathbb{C}}(1 - d\sigma_P) = 2$ for all P. In addition, at fixed points P, ϕ_P acts as a permutation of M_P. So $\text{trace}_{\mathbb{C}} \phi_P = -1$ when σ^* acts as -1 on M_P and similarly, $\text{trace}_{\mathbb{C}} \phi_P = +1$ when σ^* acts as $+1$ on M_P. Then we have

$$\begin{aligned} L(\sigma) &= \frac{-\tilde{m} + 4g - 4 - \tilde{m}}{2} \\ &= 2g - 2 - \tilde{m}. \end{aligned} \tag{5.2}$$

Now we use Riemann-Roch on both h^+ and h^- and by using the twisted line bundle

$$\pi_*(M \otimes \pi^* L) = W \oplus W',$$

this yields

$$2 \deg V = \deg V - \deg V^* = \deg W - \deg W' = h^+ - h^-.$$

Then we have the following result

$$\deg V = g - 1 - \frac{\tilde{m}}{2}.$$

So this shows that there is a linear relationship between the number of simple zeros in W and the degree of the line bundles L after the push-forward.

5.1.2 $SO(2,2)$-Case

Here we are using Bradlow and Schaposnik's results [30] in order to generalize the above argument to the $SO(2,2)$ case.

Consider two quadratic differentials q_1 and q_2 with only simple zeros and no zeros in common. This means that the product $q_1 q_2$ has only simple zeros. We can consider two spectral curves based on

$$\lambda_1^2 + q_1 = 0 \text{ and } \lambda_2^2 + q_2 = 0.$$

We can use these two spectral curves to produce a smooth $SO(2,2)$-spectral curve by doing the following operation.

Theorem 5.1. *For a compact Riemann surface Σ, given two spectral curves based on two distinct quadratic differential q_1 and q_2 with distinct zeros and no zeros in common. An $SO(2,2)$-spectral curve can be generated from this data.*

We define $\lambda = \lambda_1 + \lambda_2$, we have

$$\lambda_1^2 = -q_1 \text{ and } \lambda_2^2 = -q_2.$$

Therefore

$$\lambda^2 = \lambda_1^2 + \lambda_2^2 + 2\lambda_1\lambda_2.$$

Hence

$$\lambda^2 + q_1 + q_2 = 2\lambda_1\lambda_2.$$

Now, we compute

$$\begin{aligned}
\lambda^4 &= \lambda_1^4 + 4\lambda_1^3\lambda_2 + 6\lambda_1^2\lambda_2^2 + 4\lambda_1\lambda_2^3 + \lambda_2^4 \\
&= q_1^2 + 6q_1q_2 + q_2^2 + 4\lambda_1\lambda_2(\lambda_1^2 + \lambda_2^2) \\
&= q_1^2 + 6q_1q_2 + q_2^2 + 2(\lambda^2 + q_1 + q_2)(-q_1 - q_2) \\
&= q_1^2 + 6q_1q_2 + q_2^2 - 2q_1^2 - 4q_1q_2 - 2q_2^2 - (2q_1 + 2q_2)\lambda^2 \\
&= -q_1^2 + 2q_1q_2 - q_2^2 - (2q_1^2 + 2q_2^2)\lambda^2
\end{aligned} \tag{5.3}$$

Therefore, we have a quartic equation

$$\lambda^4 + 2(q_1 + q_2)\lambda^2 + (q_1 - q_2)^2 = 0.$$

This is exactly how we use two branched covers spectral curves S_1 and S_2 to produce a fiber product $S = S_1 \times_\Sigma S_2$. This fiber product curve is automatically smooth due to the construction. Since each one of these two spectral curves adopts an involution by sending λ_1 or λ_2 to their negatives, namely σ_1 and σ_2, then the fiber product S adopts a similar involution $\sigma = (\sigma_1, \sigma_2)$. But since generically q_1 and q_2 have different simple zeros, then σ has no fixed points. Therefore, we define a smooth curve $\overline{S} = S/\sigma$.

We can consider $a = 2(q_1 + q_2)$ and $p = q_1 - q_2$. Then our $\overline{S}$ is defined by using the equation $y^2 + ay + p^2 = 0$ where y is a section of $\pi^* K^2$. Since a and p are a linear transformation of q_1 and q_2, then the simple zeros of a and p are the same as those of q_1 and q_2. Then B is the combination of the zeros of q_1 and q_2. Now we are using a result from Bradlow and Schaposnik [30].

5.1.3 Relationship between the degrees of the line bundle components and $\tilde{m}$.

Here, we only consider the underlying bundles for the moment, we will deal with Higgs fields later. Detailed construction can be found in [30]. Let p_1 and p_2 be a projection from $S = S_1 \times_\Sigma S_2$ onto each factor. Consider $((S_1, L_1), (S_2, L_2))$ the spectral data for a point in $\mathcal{M}_{SL_2(\mathbb{R})\times SL_2(\mathbb{R})}$ represented by

$$\left((N_1 \oplus N_1^*), (N_2 \oplus N_2^*)\right),$$

and let (S, L) be the corresponding spectral data for the point in $\mathcal{M}_{SO(2,2)}$ induced by $((S_1, L_1), (S_2, L_2))$ spectral data. Then $S = S_1 \times_\Sigma S_2$ is defined as above and $L = p_1^* L_1 \otimes p_2^* L_2$. In addition, the

point in $\mathcal{M}_{SO(2,2)}$ is represented by

$$((N_1N_2 \oplus (N_1N_2)^*) \oplus N_1N_2^* \oplus (N_1N_2^*)^*)\,.$$

By using this proposition and the computation from the first section, we have the formulas for the degrees of both N_1 and N_2 [30]:

$$deg(N_1) = g - 1 - \frac{\tilde{m_1}}{2} \text{ and } deg(N_2) = g - 1 - \frac{\tilde{m_2}}{2}.$$

Theorem 5.2. *The $SO(2,2)$-Higgs bundle E can be decomposed as a direct sum $E = V \oplus W$. In addition, $V = N_1N_2 \oplus (N_1N_2)^*$ and $W = N_1N_2^* \oplus (N_1N_2^*)^*$. Hence the degree of each component is fully understood.*

$$deg(N_1N_2) = 2g - 2 - \frac{\tilde{m_1} + \tilde{m_2}}{2} \text{ and } deg(N_1N_2^*) = \frac{\tilde{m_1} - \tilde{m_2}}{2}.$$

5.2 Orbits of the Monodromy Action

First, by examining the matrix products, we intend to calculate the orbits of the monodromy actions on $\Lambda_{\overline{S}}[2]$. Based on [**?** , Section 6.1], the number of the orbits is equal to the number of the connected components of $SO(2,2)$-character variety.

Suppose we fix a $b_{ij} = b_i + b_j$, then for $a, b \in \Lambda_\Sigma[2]$, we have

$$\begin{aligned} A^a_{ij} A^b_{ij} &= \begin{bmatrix} I_{2g} & L^a_{ij} & B^a \\ 0 & I & (L^a_{ij})^t \\ 0 & 0 & I_{2g} \end{bmatrix} \begin{bmatrix} I_{2g} & L^b_{ij} & B^b \\ 0 & I & (L^b_{ij})^t \\ 0 & 0 & I_{2g} \end{bmatrix} \\ &= \begin{bmatrix} I_{2g} & L^a_{ij} + L^b_{ij} & B^a + L^a_{ij}(L^b_{ij})^t + B^b \\ 0 & I & (L^a_{ij})^t + (L^b_{ij})^t \\ 0 & 0 & I_{2g} \end{bmatrix} \\ &= \begin{bmatrix} I_{2g} & L^a_{ij} + L^b_{ij} & B^a + B^b \\ 0 & I & (L^a_{ij})^t + (L^b_{ij})^t \\ 0 & 0 & I_{2g} \end{bmatrix} \end{aligned} \tag{5.4}$$

because $((b_{ij}, b_{ij})) = 0$.

Therefore,

$$
\begin{aligned}
A^a_{ij}A^b_{ij}A^{a+b}_{ij} &= \begin{bmatrix} I_{2g} & L^a_{ij}+L^b_{ij} & B^a+B^b \\ 0 & I & (L^a_{ij})^t+(L^b_{ij})^t \\ 0 & 0 & I_{2g} \end{bmatrix} \begin{bmatrix} I_{2g} & L^{a+b}_{ij} & B^{a+b} \\ 0 & I & (L^{a+b}_{ij})^t \\ 0 & 0 & I_{2g} \end{bmatrix} \\
&= \begin{bmatrix} I_{2g} & L^a_{ij}+L^b_{ij}+L^{a+b}_{ij} & B^{a+b}+(L^a_{ij}+L^b_{ij})(L^{a+b}_{ij})^t+B^a+B^b \\ 0 & I & (L^a_{ij})^t+(L^b_{ij})^t+(L^{a+b}_{ij})^t \\ 0 & 0 & I_{2g} \end{bmatrix} \\
&= \begin{bmatrix} I_{2g} & 0 & B^a+B^b+B^{a+b} \\ 0 & I & 0 \\ 0 & 0 & I_{2g} \end{bmatrix} \\
&= \begin{bmatrix} I_{2g} & 0 & B^{a,b} \\ 0 & I & 0 \\ 0 & 0 & I_{2g}, \end{bmatrix}
\end{aligned}
\tag{5.5}
$$

where $B^{a,b}(t) = \langle a,t\rangle b + \langle b,t\rangle$. To show more detail, $B^a(t) + B^b(t) + B^{a+b}(t) = \langle a,t\rangle a + \langle b,t\rangle b + \langle a+b,t\rangle(a+b) = \langle a,t\rangle b + \langle b,t\rangle a = B^{a,b}(t)$.

Suppose we pick a pair b_{ij} and b_{kl}, we examine how the matrix products go. Since

$$\begin{aligned} A^a_{ij}A^b_{kl} &= \begin{bmatrix} I_{2g} & L^a_{ij} & B^a \\ 0 & I & (L^a_{ij})^t \\ 0 & 0 & I_{2g} \end{bmatrix} \begin{bmatrix} I_{2g} & L^b_{kl} & B^b \\ 0 & I & (L^b_{kl})^t \\ 0 & 0 & I_{2g} \end{bmatrix} \\ &= \begin{bmatrix} I_{2g} & L^a_{ij} + L^b_{kl} & B^a + L^a_{ij}(L^b_{kl})^t + B^b \\ 0 & I & (L^a_{ij})^t + (L^b_{kl})^t \\ 0 & 0 & I_{2g} \end{bmatrix} \end{aligned} \tag{5.6}$$

Here $L^a_{ij}(L^b_{kl})^t(t) = L^a_{ij}(\langle b,t\rangle b_{kl}) = \langle b,t\rangle((b_{ij},b_{kl}))a$. Then it is obvious that the result depends on the intersection number $((b_{ij},b_{kl}))$.

Suppose $((b_{ij},b_{kl})) = 0$, then $A^a_{ij}A^b_{kl} = \begin{bmatrix} I_{2g} & L^a_{ij} + L^b_{kl} & B^a + B^b \\ 0 & I & (L^a_{ij})^t + (L^b_{kl})^t \\ 0 & 0 & I_{2g} \end{bmatrix}$. By using the block matrix inverse formula $\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}BD^{-1} \\ 0 & D^{-1} \end{bmatrix}$ repeatly, we can get $[A^a_{ij}, A^b_{kl}] = I$.

On the other hand, suppose $((b_{ij},b_{kl})) = 1$, then $A^a_{ij}A^b_{kl} = \begin{bmatrix} I_{2g} & L^a_{ij} + L^b_{kl} & B^a + \langle b,\rangle a + B^b \\ 0 & I & (L^a_{ij})^t + (L^b_{kl})^t \\ 0 & 0 & I_{2g} \end{bmatrix}$

by using the inverse matrix formula, $(A^a_{ij}A^b_{kl})^{-1} = \begin{bmatrix} I_{2g} & L^a_{ij} + L^b_{kl} & B^a + \langle a,\rangle b + B^b \\ 0 & I & (L^a_{ij})^t + (L^b_{kl})^t \\ 0 & 0 & I_{2g} \end{bmatrix} =$

$A^b_{kl}A^a_{ij}$. Then we have the following

$$
\begin{aligned}
[A^a_{ij}, A^b_{kl}] &= (A^a_{ij})^{-1}(A^b_{kl})^{-1}A^a_{ij}A^b_{kl} = (A^b_{kl}A^a_{ij})^{-1}A^a_{ij}A^b_{kl} \\
&= A^a_{ij}A^b_{kl}A^a_{ij}A^b_{kl} = \begin{bmatrix} I_{2g} & L^a_{ij} + L^b_{kl} & B^a + \langle b, \rangle a + B^b \\ 0 & I & (L^a_{ij})^t + (L^b_{kl})^t \\ 0 & 0 & I_{2g} \end{bmatrix}^2 \\
&= \begin{bmatrix} I_{2g} & 0 & B^{a,b} \\ 0 & I & 0 \\ 0 & 0 & I_{2g} \end{bmatrix}
\end{aligned} \tag{5.7}
$$

In addition, we also observe that by fixing $a \in \Lambda_\Sigma[2]$, $A^a_{12}A^a_{34}\dots A^a_{4g-5,4g-4}A^a_{1'2'}\dots A^a_{4g-5',4g-4'} = I$. Since we need to do an even number of matrix multiplications.

Finally, there exists an extra matrix generator

$$
\begin{bmatrix} I_{2g} & 0 & B^r \\ 0 & I & 0 \\ 0 & 0 & I_{2g} \end{bmatrix}
$$

and this matrix cannot be written as a product of any aforementioned generators. This corresponds to the embedded loop in Σ from a zero a_i of one quadratic differential q around another zero b_j of the other quadratic differential q'.

Together with 4.17, we have the following theorem.

Theorem 5.3. *The monodromy action on $\Lambda_{\overline{S}}[2]$ is generated by the groups of matrices of the form*

- $A^x_{ij} = \begin{bmatrix} I_{2g} & L^a_{ij} & S^a \\ 0 & I & (L^a_{ij})^t \\ 0 & 0 & I_{2g} \end{bmatrix}$
- $s_{ij} = \begin{bmatrix} I_{2g} & 0 & 0 \\ 0 & \sigma_{ij} & 0 \\ 0 & 0 & I_{2g} \end{bmatrix}$ *and*
- $M^r = \begin{bmatrix} I_{2g} & 0 & B^r \\ 0 & I & 0 \\ 0 & 0 & I_{2g} \end{bmatrix}$

When these are understood, we can use the matrix representation to count the orbits of the monodromy actions.

For $(r, s, t) \in \Lambda_{\overline{S}}[2]$, we consider the case when $t = 0$. If we assume further that $s = 0$, then $(r, 0, 0)$ is invariant under monodromy actions. There exists 2^{2g} choices of them. If we consider $(r, s, 0)$ when $s \neq 0$, then the monodromy actions give

$$\begin{bmatrix} I_{2g} & L^a_{ij} & B^a \\ 0 & \sigma_{ij} & (L^a_{ij})^t \\ 0 & 0 & I_{2g} \end{bmatrix} \begin{bmatrix} r \\ s \\ 0 \end{bmatrix} = \begin{bmatrix} r + L^a_{ij}(s) \\ \sigma_{ij}(s) \\ 0 \end{bmatrix}.$$

σ_{ij} permutate zeros in s but doesn't change the number of zeros in s, but the first row can be any element in $J_\Sigma[2]$. Then the orbits depend on the pairs of zeros in s. Since we also have $W = (\mathbb{Z}_2)^{\text{even}}/(d)$, then the orbits of this monodromy action are $8g - 8/4 = 2g - 2$.

Now in (r, s, t), we suppose $t \neq 0$. Without loss of generality, we can always use the

monodromy generators to eliminate s to produce $(r, 0, t)$. Since we cannot touch the middle term 0 again, we can only use the symmetric generators to act on $(r, 0, t)$.

- Case 1: Suppose $\langle r, t\rangle = 0$, then we take t' such that $\langle t, t'\rangle = 1$, then we consider the linear transformation $B^{r,t'}$. Then $B^{r,t'}(t) = \langle r, t\rangle t' + \langle t', t\rangle r = r$. Then we use the matrix

$$M^{r,t'} = \begin{bmatrix} I_{2g} & 0 & B^{r,t'} \\ 0 & I & 0 \\ 0 & 0 & I_{2g} \end{bmatrix}$$

  to act on $(r, 0, t)$. Then $M(r, 0, t) = (r, 0, t) + B^{r,t'}((r, 0, t)) = (0, 0, t)$. Since for any t there is a t' such that $\langle t, t'\rangle = 1$, then given t, all $(r, 0, t)$ and $(0, 0, t)$ are in the same orbit.

- Case 2: Suppose on the other hand $\langle r, t\rangle = 1$, for any r' such that $\langle r', t\rangle = 1$, we consider a corresponding linear transformation $B^{r,r'}$. Then $B^{r,r'}(t) = \langle r, t\rangle r' + \langle r', t\rangle r = r + r'$. Then the matrix

$$M^{r,r'} = \begin{bmatrix} I_{2g} & 0 & B^{r,r'} \\ 0 & I & 0 \\ 0 & 0 & I_{2g} \end{bmatrix}$$

  acts on $(r, 0, t)$, then we get $M(r, 0, t) = (r, 0, t) + B^{r,r'}((r, 0, t)) = (r, 0, t) + (r + r', 0, t) = (r', 0, t)$. That means when we fix a $t \in \Lambda_\Sigma[2]$, all $r \in \Lambda_\Sigma[2]$ that pair with t nontrivially are in the same orbit.

- Case 3. Finally, given $r \in \Lambda_\Sigma[2]$, we have new generators

$$M^r = \begin{bmatrix} I_{2g} & 0 & B^r \\ 0 & I & 0 \\ 0 & 0 & I_{2g} \end{bmatrix}.$$

This monodromy action generator cannot be products of previous generators. Given t, $r \in \Lambda_\Sigma[2]$ such that $\langle r, t\rangle = 0$, we consider $r' \in \Lambda_\Sigma[2]$ and $B^{r'}$. Then $B^{r'}(r) = r + \langle r', t\rangle r' = r + r'$ will pair with t as $\langle r + r', t\rangle = 1$. Then our aforementioned case 1 and case 2 are in the same orbit.

5.3 Counting the Number of the Orbits

Based on the monodromy actions described above, here we are giving a method to count the orbits.

Our strategy is to investigate how the monodromy matrices act on different types of triples representing elements in $\Lambda_\Sigma[2]$. Therefore, we need to analyze them case by case.

First, we consider the triple $(a, b, c) \in \Lambda_\Sigma[2] \oplus W \oplus \Lambda_\Sigma[2] = \Lambda_{\overline{S}}[2]$ with $b = c = 0$. Since the monodromy matrices are all of the upper triangular form, they will be invariant under the monodromy actions. Therefore, the number of the orbits in this situation depends of the choices of a. Since $a \in \Lambda_\Sigma[2]$, we have 2^{2g} choices of a.

We observe that W consists of $8g - 8$ entries, and each is binary. In addition, W has a relation that $(1, 1, \ldots, 1) = 0$. Since those $8g-8$ entries come with pairs, with the above relation, W is of dimension $8g-10$. Now we consider the case when $(a, b, c) \in \Lambda_\Sigma[2] \oplus W \oplus \Lambda_\Sigma[2]$ where

$c \neq 0$.

Then we consider the number of possibilities when we permutate 1's in W. Since we have two groups of points related to zeros of two different quadratic differentials, we need to break this further into cases.

Let m_1 and m_2 mark the numbers of pairs of 1 in W, it means we have m_1 1's and m_2 1's in W. The generators of monodromy action will not interchange two zeros from differential quadratic differentials. Therefore, after applying a generator, m_1 and m_2 will be preserved. In addition, since we have a relation $(1, 1, \ldots, 1) = 0$, we let $\tilde{m_1} = 2g-2-m_1$ and $\tilde{m_2} = 2g-2-m_2$ mark the numbers of 1's which are equivalent to m_1 and m_2. Now we break this into several cases.

- Case 1. When $0 < m_1 < g-1$ and $0 < m_2 < g-1$, this is equivalent to the case when $g-1 < \tilde{m_1} < 2g-2$ and $g-1 < \tilde{m_2} < 2g-2$. Once again, the numbers of 1's within a group are preserved under monodromy actions. The choices of m_1 and m_2 are independent. We have $(g-2)^2$ options.

- Case 2. When $0 < m_1 < g-1$ and $g-1 < m_2 < 2g-2$, this is equivalent to the case when $g-1 < \tilde{m_1} < 2g-2$ and $0 < \tilde{m_2} < g-1$. In this case, the choices of m_1 and m_2 is $(g-2)^2$.

- Case 3. When $m_1 = 0$ and $0 < m_2 < g-1$, this is equivalent to the case when $\tilde{m_1} = 2g-2$ and $g-1 < \tilde{m_2} < 2g-2$. Therefore, in this case, the number is based on the choice of m_2, which is $g-2$.

- Case 4. When $m_1 = 0$ and $g-1 < m_2 < 2g-2$, this is equivalent to the case when $\tilde{m_1} = 2g-2$ and $0 < \tilde{m_2} < g-1$. Therefore, in this case, the number is based on the choice of m_2, which is $g-2$.

- Case 5. When $m_1 = g-1$ and $0 < m_2 < g-1$, this is equivalent to the case when $\tilde{m_1} = g-1$ and $g-1 < \tilde{m_2} < 2g-2$. There is another $g-2$ choices.

- Case 6,7,8. Similar methods when m_1 and m_2 are reversed. So those duplicate the cases 3,4 and 5. Therefore, $3g-6$ more choices.

  Now, we are considering the boundaries.

- Case 9. When $m_1 = m_2 = 0$, this is equivalent to $\tilde{m_1} = \tilde{m_2} = 2g-2$. This is one more.

- Case 10. When $m_1 = 0$ and $m_2 = g-1$, this is equivalent to $\tilde{m_1} = 2g-2$ and $\tilde{m_2} = g-1$. This is one more.

- Case 11. When $m_1 = 0$ and $m_2 = 2g-2$, this is equivalent to $\tilde{m_1} = 2g-2$ and $\tilde{m_2} = 0$. This is one more.

- Case 12. When $m_1 = g-1$ and $m_2 = 0$, this is equivalent to $\tilde{m_1} = g-1$ and $\tilde{m_2} = 2g-2$. This is one more.

- Case 13. When $m_1 = m_2 = g-1$, this is equivalent to themselves, so one more.

Theorem 5.4. *The $SO(2,2)$-character variety has $2 \times 2^{2g} + 2g^2 - 2g + 1$ connected components.*

Proof. Based on this analysis and a simple counting, we have $2(g-2)^2+6(g-2)+5 = 2g^2-2g+1$ choices. By adding two copies of 2^{2g}, we have $2 \times 2^{2g} + 2g^2 - 2g + 1$ orbits.

□

This result in [14, Corollary 7.1] is presented with no details. Since we count the orbits of the monodromy action directly based on our construction in the previous chapter, we have extra

generators which correspond to *double swaps*. We count the orbits without using the methods involving

$$\mathcal{M}_{SL_2(\mathbb{R})\times SL_2(\mathbb{R})} \to \mathcal{M}_{SO(2,2)}.$$

from [30]. Therefore, we get smaller number of orbits.

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