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Non-blocking Supervisory Control of Nondeterministic Systems via Prioritized Synchronization

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Abstract

In a previous paper [15], we showed that supervisory control of nondeterministic discrete event systems, in the presence of driven events, can be achieved using prioritized synchronous composition as a mechanism of control, and trajectory models as a modeling formalism. The specifications considered in [15] were given by *prefix-closed* languages. In this paper, we extend the theory of trajectory models and prioritized synchronous composition to include markings so that non-closed specifications and issues such as blocking can be addressed. It is shown that the usual notion of non-blocking, called *language model non-blocking*, is inadequate in the setting of nondeterministic systems, and a stronger notion, called *trajectory model non-blocking*, is introduced. Necessary and sufficient conditions for the existence of non-marking and language model non-blocking as well as trajectory model non-blocking supervisors is obtained for nondeterministic systems in the presence of driven events. We also show that our approach is also suitable for modular supervisory control.

Keywords: discrete event systems, supervisory control, nondeterministic automata, driven events, prioritized synchronization, trajectory models, blocking

AMS (MOS) subject classifications: 68Q75, 93B25, 93C83

1 Introduction

Discrete event systems are systems that involve quantities which take on a discrete set of values and which are constant except at discrete times when events occur in the system. Examples include communication networks, intelligent vehicle highway systems, manufacturing systems and computer programs. Supervisory control theory was developed to provide a mathematical framework for the design of controllers for such systems in order to meet various *qualitative* constraints. A survey of this area (up to 1989) with extensive references may be found in [14].

The majority of the research effort in this area has focused on the supervisory control of *deterministic* systems, and relatively little progress has been made towards supervisory control of *nondeterministic* systems–systems in which knowledge of the current state and next event is insufficient to uniquely determine the next state. Nondeterminism in the context of supervisory control arises due to unmodeled system dynamics such as suppression of "internal" events, suppression of "ticks of the clock" to obtain the untimed model from a timed model [2], partial observation of the system behavior due to lack of enough sensors or presence of faulty sensors, etc.

In the Ramadge-Wonham approach to supervisory control, every event is generated by the plant and synchronously executed by the supervisor [13] which acts passively by disabling certain controllable events possible in the open-loop plant. The disablement action is accomplished by a control-input map which specifies a set of disabled events based on the current state of the supervisor. Alternatively, in the work of Kumar-Garg-Marcus [10], the disablement action is accomplished by removing certain transitions from the structure of the supervisor while continuing to require that the plant and supervisor be connected by strict synchronous composition (SSC). In the work of Golaszewski-Ramadge [3] and, in the real-time setting, the work of Brandin-Wonham [2], the supervisor is able to initiate certain so-called *forcible* events that the plant synchronously executes. In the work of Balemi and coworkers [1], events can originate in the supervisor (so-called *command events*) or in the plant (so-called *response events*). The assumption is made that the plant and supervisor are *mutually receptive*, meaning that neither the plant nor the supervisor can refuse to execute an event initiated by the other.

Common to all of the above approaches is the assumption that there are never events which may occur in the supervisor without the participation of the plant. However, this assumption may be unreasonably restrictive for nondeterministic systems. When the plant is nondeterministic, there is generally no way to know a priori whether a command issued by the supervisor can be executed by the plant in its current state. For example, it may be impossible to know that a device is in a faulted state until after it fails to respond to a command from the controller.

Heymann has introduced an interconnection operator called *prioritized synchronous composition* (PSC) [4], which relaxes the synchronization requirements between the plant and supervisor. Each process in a PSC-interconnection is assigned a *priority set* of events. For an event to be enabled in the interconnected system, it must be enabled in all processes whose priority sets contain that event. Also, when an enabled event occurs, it occurs in each subsystem in which the event is enabled. In the context of supervisory control, the priority set of the plant contains the controllable and uncontrollable events, while the priority set of the supervisor contains the controllable and driven events. Thus, controllable events require the participation of both plant and supervisor; uncontrollable events require the participation of the plant and will occur synchronously in the supervisor whenever possible; driven events require the participation of the supervisor and will occur synchronously in the plant whenever possible.

It is important to distinguish between PSC and other types of parallel composition in the literature. For example, Hoare [6] defines a concurrent composition operator in which each process has its own alphabet and the processes synchronize on the events in the intersection of their alphabets. This is generalized to trace-dependent alphabets, called event-control sets, by Inan-Varaiya [9]. The key difference between concurrent composition and PSC is that in PSC, although a process cannot block events which are outside its priority set, it may be able to execute these events–and, whenever possible, will execute these events synchronously when they occur in the other process[1].

Language models identify processes that have the same set of traces. The failures model of Hoare [6] identifies processes that have the same set of so-called *failures.* Failure equivalence refines language equivalence. Heymann showed that failure equivalence is too coarse to support the PSC operator [4]. In other words, there exist plants $\mathcal{P}_1$, $\mathcal{P}_2$ with the same failures model (and hence with the same language model) such that their PSC's $\mathcal{P}_1 \,{}_A\|_B\, \mathcal{S}$, $\mathcal{P}_2 \,{}_A\|_B\, \mathcal{S}$ with a common supervisor have different language models. Thus, neither the language model nor even the failures model retains enough information about a process to do control design using the operation of PSC.

This has led Heymann to introduce the *trajectory model*, a refinement of the failures model [4, 5]. The trajectory model is similar to the *failure-trace model* (also called the refusal-testing model) in concurrency theory [12], but differs from this model in its treatment of hidden transitions. The trajectory model treats hidden transitions in a way that is consistent with the failures model. In a previous paper [15], we proved that if $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{S}_1$, $\mathcal{S}_2$ are nondeterministic state machines (with ϵ-transitions), and if the pairs $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{S}_1$, $\mathcal{S}_2$ each have the same trajectory models, then $\mathcal{P}_1 \,{}_A\|_B\, \mathcal{S}_1$ and $\mathcal{P}_2 \,{}_A\|_B\, \mathcal{S}_2$ have the same trajectory model–and hence the same language model as well. Thus, the trajectory model retains sufficient process detail to permit PSC-based controller design.

In [15], we showed that supervisory control of nondeterministic discrete event systems, in the presence of driven events, can be achieved using prioritized synchronous composition as a mechanism of control, and trajectory models as a modeling formalism. The specifications considered in [15] were given by *prefix-closed* languages. In this paper, we extend our previous work on supervisory control of nondeterministic systems using trajectory models

[1] If applied to so-called *improper* processes, the parallel operator defined by Inan [8] can be viewed as a generalized form of PSC, but only in the deterministic setting. However, when supervisory control is considered in this reference, the assumption is made that the plant is proper and has a constant event control set. This assumption excludes driven events.

and prioritized synchronous composition to include the notion of markings, by introducing the notion of recognized and generated trajectory sets, so that non-closed specifications and issues such as blocking can be addressed.

The usual notion of non-blocking, referred to as *language model non-blocking* in this paper, requires that each trace belonging to the generated language of a controlled system be extendable to a trace belonging to the recognized language. This property adequately captures the notion of non-blocking in a deterministic setting. However, the execution of a trace belonging to the generated behavior of a controlled nondeterministic system may lead to more than one state. Language model non-blocking only requires that every such trace be extendable to a trace in the recognized behavior from *at least one* such state–as opposed to *all* such states. Thus, a language model non-blocking nondeterministic system can deadlock, as illustrated by the example in the next section. Consequently, there is a need for a stronger type of non-blocking for nondeterministic systems. This leads us to introduce the property of *trajectory model non-blocking*, which requires that each refusal-trace belonging to the generated trajectory set of a controlled nondeterministic system be extendable to a refusal-trace belonging to the recognized trajectory set.

Another desirable property of a supervisor is that it should be *non-marking*, i.e., given a trace (respectively, a refusal-trace) of the controlled system, it should belong to the recognized language (respectively, the recognized trajectory set) of the controlled system if and only if a marked state of the uncontrolled system is reached due to its execution (regardless of the type of state reached in the supervisor). We first obtain a necessary and sufficient condition for the existence of a non-marking and language model non-blocking supervisor for a given nondeterministic system in the presence of driven events. This result is then used to obtain a necessary and sufficient condition for the existence of a non-marking and trajectory model non-blocking supervisor in that setting. We also demonstrate that our approach is suitable for modular supervisory control.

2 A Motivating Example

In this section, we describe an example that illustrates some of the issues to be addressed in this paper. Figure 1(a) gives a deterministic model for a plant in which parts arrive at a machine from a conveyor and are then processed. The incoming parts are of two types that differ slightly in their widths. The standard width is the wider one. Events a_1 and a_2 denote the arrival at the machine of wide and narrow parts respectively. Events b_1 and b_2 denote the input into the machine of a part with the guides set to wide and narrow respectively. The default setting of the guides is wide, but intervention by a controller can reset them to narrow. A wide part can only be input with the guides set to wide. A narrow part can be input with either guide setting. However, input of a narrow part with the guides set to wide leads to the machine jamming–event d. If a part is input with the correct guide setting, then it can be successfully processed and output–event c. It is assumed that a_1, a_2, c, d are uncontrollable events and that there is no sensor that can distinguish between the two widths of incoming parts–i.e., the observation mask $M(\cdot)$ identifies a_1 and a_2–say

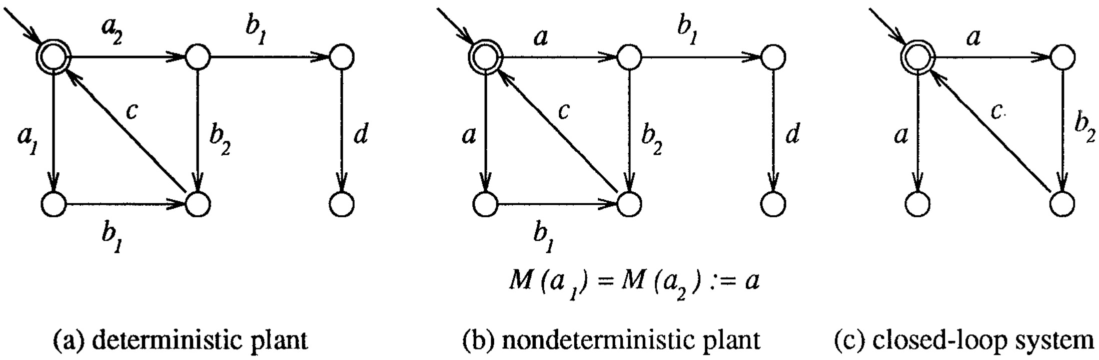


Figure 1: Diagram illustrating the example of Section 2

$M(a_1) = M(a_2) := a$. A natural control specification is that the supervised plant be non-blocking since this guarantees that continuous operation is possible.

It is clear that the performance specification cannot be met by any supervisor S of the Ramadge-Wonham type that is consistent with the observation mask. To prevent blocking arising from the uncontrollable jamming event d, S would need to disable b_1 following any occurrence of a_2. However, since the mask cannot distinguish between a_1 and a_2, S would also disable b_1 following any occurrence of a_1. But this would give a controlled plant that would deadlock with the arrival of the first wide part.

Suppose we replace the event labels a_1 and a_2 by their common mask value a, thereby obtaining the nondeterministic system shown in Figure 1(b). By so identifying a_1 and a_2, the events are made indistinguishable from the viewpoints of specification, control and observation–whereas in the partially observed deterministic model, they are indistinguishable only from the viewpoint of observation. For this system, however, the nondeterministic model is essentially equivalent to the partially observed one from the viewpoint of control since a_1, a_2 are uncontrollable and hence could not be distinguished in a supervisory control law. However, the non-blocking specification implicitly distinguishes between a_1 and a_2 and consequently forces the definition of a new type of non-blocking appropriate for nondeterministic systems.

Let $\mathcal{P}$ denote the nondeterministic state machine (NSM) depicted in Figure 1(b), and let $L(\mathcal{P})$, $L^m(\mathcal{P})$ denote its generated and recognized languages respectively. Then

$$L^m(\mathcal{P}) = [a(b_1 + b_2)c]^*, \qquad L(\mathcal{P}) = pr[[a(b_1 + b_2)c]^* ab_1 d],$$

where $pr(\cdot)$ denotes the prefix-closure operation. Having replaced the original partially observed deterministic model with a completely observed nondeterministic model, let us consider whether the specification can be met by a supervisor of the Ramadge-Wonham type. The closed-loop nondeterministic system $\mathcal{Q}$ obtained by disabling b_1 following any occurrence of a is depicted in Figure 1(c). Since $L(\mathcal{Q}) = pr((ab_2c)^*) = pr(L^m(\mathcal{Q}))$, the supervisor is non-blocking from the language model point of view. However, this control design is clearly unsatisfactory since the closed-loop system can deadlock. After all, the nondeterministic plant model is derived from the partially observed deterministic plant model, and there is no non-blocking Ramadge-Wonham type supervisor for that model.

The problem is that the usual language model definition of non-blocking given by $L(\mathcal{P}) = pr(L^m(\mathcal{P}))$ is not suitable for control specifications in a nondeterministic setting. This motivates us to consider a stronger non-blocking requirement which we refer to as *trajectory model non-blocking* to distinguish it from the usual language model non-blocking condition. Using trajectory models for the plant and supervisor, and PSC as the mode of interconnection, it is possible to design a supervisor so that the closed-loop system meets the stronger non-blocking requirement. The details are given in Section 5, Example 3.

3 Notation and Preliminaries

Given a finite event set Σ, Σ^* is used to denote the collection of all *traces*, i.e., finite sequences of events, including the zero length sequence, denoted by ϵ. A subset of Σ^* is called a language. Symbols H, K, etc. are used to denote languages. The set $2^\Sigma \times (\Sigma \times 2^\Sigma)^*$ is used to denote the collection of all *refusal-traces*, i.e., finite sequences of alternating *refusals* and events [5, 15] of the type:

$$\Sigma_0(\sigma_1, \Sigma_1)\ldots(\sigma_n, \Sigma_n),$$

where $n \in \mathcal{N}$. The sequence $\sigma_1 \ldots \sigma_n \in \Sigma^*$ is the trace, and for each $i \leq n$, $\Sigma_i \subseteq \Sigma$ is the set of events refused (if offered) at the indicated point. Symbols P, Q, R, S, etc. are used to denote sets of refusal-traces. Refusal-traces are also referred to as *trajectories*.

Given $s \in \Sigma^*$, we use $|s|$ to denote the length of s, and for each $k \leq |s|$, $\sigma_k(s) \in \Sigma$ is used to denote the kth event in s. If $t \in \Sigma^*$ is another trace such that $|t| \leq |s|$ and for each $k \leq |t|$, $\sigma_k(t) = \sigma_k(s)$, then t is said to be a prefix of s, denoted $t \leq s$. For each $k \leq |s|$, s^k denotes the prefix of length k of s. The prefix-closure of $s \in \Sigma^*$, denoted $pr(s) \subseteq \Sigma^*$, is defined as $pr(s) := \{t \in \Sigma^* \mid t \leq s\}$. The prefix-closure map can be defined for a set of traces in a natural way. Given $e \in 2^\Sigma \times (\Sigma \times 2^\Sigma)^*$, we use $|e|$ to denote the length of e, and for each $k \leq |e|$, $\Sigma_k(e) \subseteq \Sigma$ is used to denote the kth refusal in e and $\sigma_k(e) \in \Sigma$ is used to denote the kth event in e, i.e.,

$$e = \Sigma_0(e)(\sigma_1(e), \Sigma_1(e))\ldots(\sigma_k(e), \Sigma_k(e))\ldots(\sigma_{|e|}(e), \Sigma_{|e|}(e)).$$

If $f \in 2^\Sigma \times (\Sigma \times 2^\Sigma)^*$ is another refusal-trace such that $|f| \leq |e|$ and for each $k \leq |f|$, $\Sigma_k(f) = \Sigma_k(e)$ and $\sigma_k(f) = \sigma_k(e)$, then f is said to be a prefix of e, denoted $f \leq e$. For each $k \leq |e|$, e^k is used to denote the prefix of length k of e. If $f \in 2^\Sigma \times (\Sigma \times 2^\Sigma)^*$ is such that $|f| = |e|$ and for each $k \leq |f|$, $\Sigma_k(f) \subseteq \Sigma_k(e)$ and $\sigma_k(f) = \sigma_k(e)$, then f is said to be dominated by e, denoted $f \sqsubseteq e$. The prefix-closure of $e \in 2^\Sigma \times (\Sigma \times 2^\Sigma)^*$, denoted $pr(e) \subseteq 2^\Sigma \times (\Sigma \times 2^\Sigma)^*$, is defined as $pr(e) := \{f \in 2^\Sigma \times (\Sigma \times 2^\Sigma)^* \mid f \leq e\}$, and the dominance-closure of e, denoted $dom(e) \subseteq 2^\Sigma \times (\Sigma \times 2^\Sigma)^*$, is defined as $dom(e) := \{f \in 2^\Sigma \times (\Sigma \times 2^\Sigma)^* \mid f \sqsubseteq e\}$. The prefix-closure and dominance-closure maps can be defined for a set of refusal-traces in a natural way. Given a refusal-trace $e \in 2^\Sigma \times (\Sigma \times 2^\Sigma)^*$, the trace of e, denoted $tr(e) \in \Sigma^*$, is defined as $tr(e) := \sigma_1(e)\ldots\sigma_{|e|}(e)$. The trace map can be extended to a set of refusal-traces in a natural way. Given a set of refusal-traces $P \subseteq 2^\Sigma \times (\Sigma \times 2^\Sigma)^*$, we use $L(P) := tr(P)$ to denote its set of traces.

Symbols $\mathcal{P}, \mathcal{Q}, \mathcal{R}$, etc. are used to denote NSM's (with ϵ-moves). Let the 5-tuple

$$\mathcal{P} := (X_{\mathcal{P}}, \Sigma, \delta_{\mathcal{P}}, x^0_{\mathcal{P}}, X^m_{\mathcal{P}})$$

represent a discrete event system modeled as an NSM, where $X_{\mathcal{P}}$ is the state set, Σ is the finite event set, $\delta_{\mathcal{P}} : X_{\mathcal{P}} \times (\Sigma \cup \{\epsilon\}) \rightarrow 2^{X_{\mathcal{P}}}$ denotes the nondeterministic transition function[2], $x^0_{\mathcal{P}} \in X_{\mathcal{P}}$ is the initial state, and $X^m_{\mathcal{P}} \subseteq X_{\mathcal{P}}$ is the set of accepting or marked states. A triple $(x_1, \sigma, x_2) \in X_{\mathcal{P}} \times (\Sigma \cup \{\epsilon\}) \times X_{\mathcal{P}}$ is said to be a transition if $x_2 \in \delta_{\mathcal{P}}(x_1, \sigma)$. A transition (x_1, ϵ, x_2) is referred to as a *silent* or *hidden* transition. We assume that the plant cannot undergo an unbounded number of silent transitions, i.e., $\mathcal{P}$ does not contain any cycle of silent transitions. The *ϵ-closure* of $x \in X_{\mathcal{P}}$, denoted $\epsilon^*_{\mathcal{P}}(x) \subseteq X_{\mathcal{P}}$, is defined recursively as

$$x \in \epsilon^*_{\mathcal{P}}(x), \text{ and } x' \in \epsilon^*_{\mathcal{P}}(x) \Rightarrow \delta_{\mathcal{P}}(x', \epsilon) \subseteq \epsilon^*(x),$$

and the set of *refusal events* at $x \in X_{\mathcal{P}}$, denoted $\Re_{\mathcal{P}}(x) \subseteq \Sigma$, is defined as

$$\Re_{\mathcal{P}}(x) := \{\sigma \in \Sigma \mid \delta_{\mathcal{P}}(x', \sigma) = \emptyset, \forall x' \in \epsilon^*_{\mathcal{P}}(x)\}.$$

In other words, given $x \in X_{\mathcal{P}}$, $\epsilon^*_{\mathcal{P}}(x)$ is the set of states that can be reached from x on zero or more ϵ-moves, and $\Re_{\mathcal{P}}(x)$ is the set of events that are undefined at each state in the ϵ-closure of x.

The transition function $\delta_{\mathcal{P}} : X \times (\Sigma \cup \{\epsilon\}) \rightarrow 2^{X_{\mathcal{P}}}$ is extended to the set of *traces* as $\delta^*_{\mathcal{P}} : X \times \Sigma^* \rightarrow 2^{X_{\mathcal{P}}}$, which is defined inductively as:

$$\forall x \in X_{\mathcal{P}} : \begin{cases} \delta^*_{\mathcal{P}}(x, \epsilon) := \epsilon^*_{\mathcal{P}}(x), \\ \forall s \in \Sigma^*, \sigma \in \Sigma : \delta^*_{\mathcal{P}}(x, s\sigma) := \epsilon^*_{\mathcal{P}}(\delta_{\mathcal{P}}(\delta^*_{\mathcal{P}}(x, s), \sigma)), \end{cases}$$

where in the last inequality, the transition function $\delta_{\mathcal{P}} : X \times (\Sigma \cup \{\epsilon\}) \rightarrow 2^{X_{\mathcal{P}}}$ has been extended to $\delta_{\mathcal{P}} : 2^{X_{\mathcal{P}}} \times (\Sigma \cup \{\epsilon\}) \rightarrow 2^{X_{\mathcal{P}}}$ in a natural way. The transition function is also extended to the set of *refusal-traces* as $\delta^T_{\mathcal{P}} : X \times (2^\Sigma \times (\Sigma \times 2^\Sigma)^*) \rightarrow 2^{X_{\mathcal{P}}}$, which is defined inductively as:

$$\forall x \in X_{\mathcal{P}} : \begin{cases} \forall \Sigma' \subseteq \Sigma : \delta^T_{\mathcal{P}}(x, \Sigma') := \{x' \in \epsilon^*_{\mathcal{P}}(x) \mid \Sigma' \subseteq \Re_{\mathcal{P}}(x')\}, \\ \forall e \in 2^\Sigma \times (\Sigma \times 2^\Sigma)^*, \sigma \in \Sigma, \Sigma' \subseteq \Sigma : \\ \delta^T_{\mathcal{P}}(x, e(\sigma, \Sigma')) := \{x' \in \epsilon^*_{\mathcal{P}}(\delta_{\mathcal{P}}(\delta^T_{\mathcal{P}}(x, e), \sigma)) \mid \Sigma' \subseteq \Re_{\mathcal{P}}(x')\}. \end{cases}$$

These maps are then used to obtain the language models and the trajectory models of $\mathcal{P}$ as follows:

$$L(\mathcal{P}) := \{s \in \Sigma^* \mid \delta^*_{\mathcal{P}}(x^0_{\mathcal{P}}, s) \neq \emptyset\}, \;\; L^m(\mathcal{P}) := \{s \in L(\mathcal{P}) \mid \delta^*_{\mathcal{P}}(x^0_{\mathcal{P}}, s) \cap X^m_{\mathcal{P}} \neq \emptyset\},$$

$$T(\mathcal{P}) := \{e \in 2^\Sigma \times (\Sigma \times 2^\Sigma)^* \mid \delta^T_{\mathcal{P}}(x^0_{\mathcal{P}}, e) \neq \emptyset\}, T^m(\mathcal{P}) := \{e \in T(\mathcal{P}) \mid \delta^T_{\mathcal{P}}(x^0_{\mathcal{P}}, e) \cap X^m_{\mathcal{P}} \neq \emptyset\}.$$

$L(\mathcal{P}), L^m(\mathcal{P}), T(\mathcal{P}), T^m(\mathcal{P})$ are called the *generated language, recognized language, generated trajectory set, recognized trajectory set*, respectively, of $\mathcal{P}$. It is easily seen that $L(T^m(\mathcal{P})) = L^m(\mathcal{P})$ and $L(T(P)) = L(\mathcal{P})$. The pairs $(L^m(\mathcal{P}), L(\mathcal{P}))$ and $(T^m(\mathcal{P}), T(\mathcal{P}))$ are called the language model and the trajectory model, respectively, of $\mathcal{P}$. Two language models $(K^m_1, K_1), (K^m_2, K_2)$ are said to be equal, written $(K^m_1, K_1) = (K^m_2, K_2)$, if $K^m_1 = K^m_2, K_1 = K_2$; equality of two trajectory models is defined analogously.

[2] ϵ represents both an *internal* or *unobservable* event and an *internal* or *nondeterministic* choice [6, 11].

4 Trajectory Models and Prioritized Synchronization

It is clear that a language pair (K^m, K) with $K^m, K \subseteq \Sigma^*$ is a language model if and only if

- $K \neq \emptyset$, and $K^m \subseteq K = pr(K)$.

Next we obtain a necessary and sufficient condition for a given refusal-trace set pair to be a trajectory model. This requires the definition of saturated refusal-traces. Given a refusal-trace set $P \subseteq 2^\Sigma \times (\Sigma \times 2^\Sigma)^*$, we define the *saturation map* on P by $sat_P : P \to 2^\Sigma \times (\Sigma \times 2^\Sigma)^*$ where

$$\begin{aligned} sat_P(e) &:= \Sigma_0(\sigma_1(e), \Sigma_1) \ldots (\sigma_{|e|}, \Sigma_{|e|}), \\ \Sigma_k &:= \Sigma_k(e) \cup \{\sigma \in \Sigma \mid e^k(\sigma, \emptyset) \notin dom(pr(P))\}, \forall k \leq |e|. \end{aligned}$$

The *saturated refusal-traces* of P, denoted $P_{sat} \subseteq P$, is defined to be the set of fixed points of $sat_P(\cdot)$. The following characterization of generated trajectory sets was given in [15, Theorem 1].

Theorem 1 [15] Given a refusal-trace set $P \subseteq 2^\Sigma \times (\Sigma \times 2^\Sigma)^*$, there exists an NSM $\mathcal{P}$ with generated trajectory set P, i.e., $P = T(\mathcal{P})$, if and only if

1. $P \neq \emptyset$
2. $P = pr(P)$
3. $P = dom(P)$
4. $sat_P(P) \subseteq P$
5. $\forall e \in P : \sigma_{k+1}(e) \notin \Sigma_k(e), \quad k \leq |e| - 1$

Corollary 1 If P is a generated trajectory set, then

1. $sat_P(\cdot)$ is idempotent,
2. $P_{sat} = sat_P(P)$.

Proof: Let $e \in P$, $\hat{e} := sat_P(e)$, and $k \leq |e| = |\hat{e}|$. We need to show that if $\sigma \notin \Sigma_k(\hat{e})$, then $g := \hat{e}^k(\sigma, \emptyset) \in P$. Since $\sigma \notin \Sigma_k(\hat{e})$, $f := e^k(\sigma, \emptyset) \in P$. Since $(sat_P(f))^k = sat_P(e^k) = \hat{e}^k$, it follows that $g \sqsubseteq sat_P(f) \in P$ by Theorem 1, Property 4. By Property 3, it follows that $g \in P$, so $sat_P(\hat{e}) = \hat{e}$. The second claim is an immediate consequence of the first. ■

Remark 1 It follows easily from Corollary 1 that Properties 3 and 4 in Theorem 1 can be replaced by the single property $P = dom(P_{sat})$.

The following result generalizes Theorem 1 to characterize those refusal-trace set pairs that are trajectory models of NSM's. If $\Sigma_1, \ldots, \Sigma_n$ are subsets of Σ, then $min(\Sigma_1, \Sigma_2, \ldots, \Sigma_n)$ denotes the set of minimal sets from among the given subsets with respect to the inclusion partial order.

Theorem 2 Given a pair of refusal-trace sets (P^m, P) with $P^m, P \subseteq 2^\Sigma \times (\Sigma \times 2^\Sigma)^*$, it is a trajectory model if and only if

> **T1:** $P \neq \emptyset$
> **T2:** $P = pr(P)$
> **T3:** $P = dom(P_{sat})$
> **T4:** $\forall e \in P : \sigma_{k+1}(e) \not\in \Sigma_k(e), \quad k \leq |e| - 1$
> **T5:** $P^m = dom(P_{sat} \cap P^m)$

Proof: First suppose that (P^m, P) is a trajectory model, i.e., there exists an NSM $\mathcal{P} := (X_\mathcal{P}, \Sigma, \delta_\mathcal{P}, x^0_\mathcal{P}, X^m_\mathcal{P})$ such that $T^m(\mathcal{P}) = P^m$ and $T(\mathcal{P}) = P$. Then it follows from Theorem 1 and Remark 1 that T1 through T4 hold. In order to show that T5 also holds, we first show that $P^m = dom(P^m)$, i.e., $dom(P^m) \subseteq P^m$. Pick $e \in dom(P^m)$, then there exists $f \in P^m$ such that $e \sqsubseteq f$. Hence $\delta^T_\mathcal{P}(x^0_\mathcal{P}, e) \supseteq \delta^T_\mathcal{P}(x^0_\mathcal{P}, f)$, which implies that $\delta^T_\mathcal{P}(x^0_\mathcal{P}, e) \cap X^m_\mathcal{P} \supseteq \delta^T_\mathcal{P}(x^0_\mathcal{P}, f) \cap X^m_\mathcal{P}$, which is nonempty since $f \in P^m$. Thus $e \in P^m$, so $P^m = dom(P^m)$. Hence, $dom(P_{sat} \cap P^m) \subseteq dom(P^m) = P^m$. It remains to show that $P^m \subseteq dom(P_{sat} \cap P^m)$. We show using induction on length of refusal-traces that for each $e \in P$ and $x \in \delta^T_\mathcal{P}(x^0_\mathcal{P}, e)$, there exists $f \in P_{sat}$ such that $e \sqsubseteq f$ and $x \in \delta^T_\mathcal{P}(x^0_\mathcal{P}, f)$. If $|e| = 0$, then $e = \Sigma' \subseteq \Sigma$. If $x \in \delta^T_\mathcal{P}(x^0_\mathcal{P}, \Sigma')$, then $x \in \epsilon^*_\mathcal{P}(x^0_\mathcal{P})$ and $\Sigma' \subseteq \Re_\mathcal{P}(x)$. Set $f := \Re_\mathcal{P}(x)$; then clearly, $f \in P_{sat}, e \sqsubseteq f$, and $x \in \delta^T_\mathcal{P}(x^0_\mathcal{P}, f)$. This proves the base step of induction. In order to prove the induction step, let $e = \bar{e}(\sigma, \Sigma') \in P$, $x \in \delta^T_\mathcal{P}(x^0_\mathcal{P}, e)$. Then $x \in \epsilon^*_\mathcal{P}(\delta_\mathcal{P}(\bar{x}, \sigma))$, where $\bar{x} \in \delta^T_\mathcal{P}(x^0_\mathcal{P}, \bar{e})$ and $\Sigma' \subseteq \Re_\mathcal{P}(x)$. Since T2 holds, $e \in P$ implies $\bar{e} \in P$. Hence from induction hypothesis, there exists $\bar{f} \in P_{sat}$ such that $\bar{e} \sqsubseteq \bar{f}$ and $\bar{x} \in \delta^T_\mathcal{P}(x^0_\mathcal{P}, \bar{f})$. Set $f := \bar{f}(\sigma, \Re_\mathcal{P}(x))$; then $f \in P_{sat}, e \sqsubseteq f$, and $x \in \delta^T_\mathcal{P}(x^0_\mathcal{P}, f)$. This proves the induction step. Hence it follows that given $e \in P^m$, so that there exists $x \in X^m_\mathcal{P}$ with $x \in \delta^T_\mathcal{P}(x^0_\mathcal{P}, e)$, we can select $f \in P_{sat}$ such that $e \sqsubseteq f$ and $x \in \delta^T_\mathcal{P}(x^0_\mathcal{P}, f)$, i.e., $f \in P_{sat} \cap P^m$. Since $e \sqsubseteq f$, this implies that $e \in dom(P_{sat} \cap P^m)$ as desired.

Next assume that T1 through T5 hold. We need to show that (P^m, P) is a trajectory model, i.e., there exists an NSM $\mathcal{P}$ such that $T^m(\mathcal{P}) = P^m$ and $T(\mathcal{P}) = P$. Consider the NSM $\mathcal{P} := (X_\mathcal{P}, \Sigma, \delta_\mathcal{P}, x^0_\mathcal{P}, X^m_\mathcal{P})$, where

- $X_\mathcal{P} := P_{sat}$,
- $x^0_\mathcal{P} := \{\sigma \in \Sigma \mid \emptyset(\sigma, \emptyset) \not\in P\}$,
- $X^m_\mathcal{P} := P_{sat} \cap P^m$,
- $\delta_\mathcal{P} : X_\mathcal{P} \times (\Sigma \cup \{\epsilon\}) \to 2^{X_\mathcal{P}}$ is defined as:

  1. $\forall e \in P_{sat}, \sigma \in \Sigma :$

$$\delta_\mathcal{P}(e, \sigma) := \begin{cases} e(\sigma, \{\sigma' \in \Sigma \mid e(\sigma, \emptyset)(\sigma', \emptyset) \not\in P\}) & \text{if } e(\sigma, \emptyset) \in P \\ \emptyset & \text{otherwise,} \end{cases}$$

  2 (a). $\forall \Sigma' \subseteq \Sigma$ such that $\Sigma' \in P_{sat}$:

$$\delta_\mathcal{P}(\Sigma', \epsilon) := min(\{\Sigma'' \subseteq \Sigma \mid \Sigma'' \in P_{sat}, \Sigma' \subset \Sigma''\}),$$

  2 (b). $\forall e \in 2^\Sigma \times (\Sigma \times 2^\Sigma)^*, \sigma \in \Sigma, \Sigma' \subseteq \Sigma$ such that $e(\sigma, \Sigma') \in P_{sat}$:

$$\delta_\mathcal{P}(e(\sigma, \Sigma'), \epsilon) := \{e(\sigma, \Sigma'') \mid \Sigma'' \in min(\{\hat{\Sigma} \subseteq \Sigma \mid e(\sigma, \hat{\Sigma}) \in P_{sat}, \Sigma' \subset \hat{\Sigma}\})\}.$$

The NSM construction given above is the same as that of a canonical NSM given in [15, Algorithm 1] except that accepting states are also defined. From [15, Lemma 1], it follows that $x^0_{\mathcal{P}} \in P_{sat}$ and for each $e \in P_{sat}, \sigma \in \Sigma,$ $\delta_{\mathcal{P}}(e,\sigma) \in P_{sat}$ whenever it is nonempty. Thus NSM $\mathcal{P}$ is well-defined. It follows from [15, Proposition 2] that $T(\mathcal{P}) = P$. It remains to show that $T^m(\mathcal{P}) = P^m$. By definition we have $T^m(\mathcal{P}) = \{e \in T(\mathcal{P}) \mid \delta^T_{\mathcal{P}}(x^0_{\mathcal{P}}, e) \cap X^m_{\mathcal{P}} \neq \emptyset\}$. Since $T(\mathcal{P}) = P, X^m_{\mathcal{P}} = P_{sat} \cap P^m$, and for each $e \in P$, $\delta^T_{\mathcal{P}}(x^0_{\mathcal{P}}, e) = \{f \in P_{sat} \mid e \sqsubseteq f\}$ [15, Corollary 1], we have

$$\begin{aligned} T^m(\mathcal{P}) &= \{e \in P \mid \{f \in P_{sat} \mid e \sqsubseteq f\} \cap (P_{sat} \cap P^m) \neq \emptyset\} \\ &= \{e \in P \mid \{f \in P_{sat} \cap P^m \mid e \sqsubseteq f\} \neq \emptyset\} \\ &= dom(P_{sat} \cap P^m) \\ &= P^m, \end{aligned}$$

where the last equality follows from T5. ■

The following result was obtained in the course of the proof of Theorem 2.

Corollary 2 Let $\mathcal{P} := (X_{\mathcal{P}}, \Sigma, \delta_{\mathcal{P}}, x^0_{\mathcal{P}}, X^m_{\mathcal{P}})$ be an NSM. Then for each $e \in T(\mathcal{P})$ and $x \in \delta^T_{\mathcal{P}}(x^0_{\mathcal{P}}, e)$, there exists $f \in (T(\mathcal{P}))_{sat}$ such that $e \sqsubseteq f$, $x \in \delta^T_{\mathcal{P}}(x^0_{\mathcal{P}}, f)$ and $\Sigma_{|f|}(f) = \Re_{\mathcal{P}}(x)$.

For a language model (K^m, K), the prefix-closure of the recognized language K^m is the generated language of an appropriate state machine provided K^m is nonempty. The situation is different for a trajectory model (P^m, P). If P^m is nonempty, then its prefix-closure, $pr(P^m)$, satisfies properties T1,T2,T4. However, $pr(P^m)$ need not satisfy T3 in which case it cannot be the generated trajectory set of any NSM. (See example 1 below.) The following result shows that a generated trajectory set can be obtained by taking *saturation closure*.

Proposition 1 Let Q be a nonempty refusal-trace set satisfying $pr(dom(Q)) = Q$ and T4. Then $R := dom(sat_Q(Q))$ is a generated trajectory set.

Proof: R is trivially nonempty and $R = dom(R)$. Since Q is prefix-closed and for each $e \in Q, k \leq |e|$, $sat_Q(e^k) = (sat_Q(e))^k$, R is prefix-closed. Also, from the definition of $sat_Q(\cdot)$, T4 holds for R since it holds for Q. It remains only to show that $sat_R(R) \subseteq R$. Since $sat_R(\cdot)$ is monotone (with respect to $\sqsubseteq$) and R is dominance-closed, it suffices to show that $sat_R(sat_Q(Q)) \subseteq R$.

Let $e \in Q$ and let $\hat{e} := sat_Q(e)$. Suppose there exist k, σ such that $\sigma \notin \Sigma_k(\hat{e})$. Since $\sigma \notin \Sigma_k(\hat{e})$, it follows that $f := e^k(\sigma, \emptyset) \in Q$. Since the map $sat_Q(\cdot)$ commutes with the operation of taking the length-k prefix, we have

$$\hat{e}^k(\sigma, \emptyset) = (sat_Q(e))^k(\sigma, \emptyset) = sat_Q(e^k)(\sigma, \emptyset) \sqsubseteq sat_Q(f) \in R.$$

This shows that $sat_R(\hat{e}) = \hat{e}$, completing the proof. ■

Corollary 3 Let P^m be a recognized trajectory set. Then $dom(sat_{pr(P^m)}(pr(P^m))$ is a generated trajectory set.

Example 1 Let $\mathcal{P}$ denote an NSM with $\Sigma = \{a, b\}$ and transitions

$$(x_{\mathcal{P}}^0, \epsilon, x_{\mathcal{P}}^1), (x_{\mathcal{P}}^0, \epsilon, x_{\mathcal{P}}^2), (x_{\mathcal{P}}^1, a, x_{\mathcal{P}}^3), (x_{\mathcal{P}}^2, b, x_{\mathcal{P}}^4),$$

and unspecified marking. Thus, $\mathcal{P}$ is obtained from the nondeterministic choice between two deterministic subsystems–one that executes a and deadlocks, the other that executes b and deadlocks. Then $P := T(\mathcal{P}) = pr(dom(\{e_1, e_2\}))$, where $e_1 = \{b\}(a, \{a, b\})$, $e_2 = \{a\}(b, \{a, b\})$. e_1, e_2 are saturated refusal-traces of P. The refusal-trace $e_3 = \emptyset(a, \{a, b\})$ is also a saturated refusal-trace of P. Let $P^m = dom(\{e_3\})$. Then the refusal-trace set pair (P^m, P) satisfies T1-T5, so it follows from Theorem 2 that there exists an NSM $\mathcal{Q}$ such that $T(\mathcal{Q}) = P$, $T^m(\mathcal{Q}) = P^m$. (One choice for $\mathcal{Q}$ is the canonical NSM described in the proof of Theorem 2, which has 7 states.) However, it is easy to check that there is no marking of the states of $\mathcal{P}$ for which $T^m(\mathcal{P}) = P^m$. It can be seen that $pr(P^m) = pr(dom(\{e_3\}))$ does not satisfy T3 and hence cannot be the generated trajectory set of any NSM. Note that $sat_{pr(P^m)}(e_3) = \{b\}(a, \{a, b\}) = e_1$. Hence $dom(sat_{pr(P^m)}(pr(P^m))) = dom(pr(e_1))$, which is the generated trajectory set of the NSM $\mathcal{R}$ with transition $(x_{\mathcal{R}}^0, a, x_{\mathcal{R}}^1)$.

Note that given a trajectory model, the trace map can be used to obtain the associated language model. On the other hand, given a language model (K^m, K), it is possible to obtain a *deterministic* trajectory model having (K^m, K) as the language model.

Definition 1 A trajectory model (P^m, P) is said to be *deterministic* if there exists a deterministic state machine $\mathcal{P} := (X_{\mathcal{P}}, \Sigma, \delta_{\mathcal{P}}, x_{\mathcal{P}}^0, X_{\mathcal{P}}^m)$ such that $T^m(\mathcal{P}) = P^m$ and $T(\mathcal{P}) = P$.

Given a language model (K^m, K), the trajectory map $trj_K : K \to 2^{\Sigma} \times (\Sigma \times 2^{\Sigma})^*$ can be used to obtain the associated deterministic trajectory model:

$$\begin{aligned} trj_K(s) &:= \Sigma_0(s)(\sigma_1(s), \Sigma_1(s)) \dots (\sigma_{|s|}(s), \Sigma_{|s|}(s)) \in 2^{\Sigma} \times (\Sigma \times 2^{\Sigma})^*, \text{ where} \\ \Sigma_k(s) &:= \{\sigma \in \Sigma \mid s^k\sigma \notin K\}, \forall k \le |s|. \end{aligned}$$

Let $det(K) := dom(trj_K(K))$ and $det^m(K^m, K) := dom(trj_K(K^m))$.

Proposition 2 Given a language model (K^m, K), $(det^m(K^m, K), det(K))$ is the unique deterministic trajectory model with language model (K^m, K).

Proof: From a standard result, there exists a deterministic state machine $\mathcal{P}$ such that $L^m(\mathcal{P}) = K^m$ and $L(\mathcal{P}) = K$. Let (P^m, P) be the trajectory model of $\mathcal{P}$. By [15, Proposition 3], $det(K)$ is the unique deterministic generated trajectory model with generated language K, so $P = det(K)$. It is clear from the definition of trj_K that $P_{sat} = trj_K(K)$. By T5, it follows that $P^m = dom(trj_K(K) \cap P^m)$. Thus, $K^m = L(P^m) = L(trj_K(K) \cap P^m)$, which implies that $trj_K(K) \cap P^m = trj_K(K^m)$, so $P^m = det^m(K^m, K)$. ■

In [4, 5, 15], prioritized synchronous composition (PSC) of systems is used as the mechanism of control. In this setting, each system is assigned a priority set of events. When systems are interconnected via PSC, an event can occur in the composite system only if it can occur in each subsystem which has that event in its priority set. In this way, a subsystem can prevent the occurrence of certain events, thereby implementing a type of supervisory control. The formal definition of the PSC of two NSM's is as follows:

Definition 2 Let $\mathcal{P} := (X_{\mathcal{P}}, \Sigma, \delta_{\mathcal{P}}, x^0_{\mathcal{P}}, X^m_{\mathcal{P}})$, $\mathcal{Q} := (X_{\mathcal{Q}}, \Sigma, \delta_{\mathcal{Q}}, x^0_{\mathcal{Q}}, X^m_{\mathcal{Q}})$ be two NSM's having priority sets $A, B \subseteq \Sigma$ respectively. The PSC of $\mathcal{P}$ and $\mathcal{Q}$ is another NSM which is denoted by

$$\mathcal{P}\ {}_A\|_B\ \mathcal{Q} := \mathcal{R} := (X_{\mathcal{R}}, \Sigma, \delta_{\mathcal{R}}, x^0_{\mathcal{R}}, X^m_{\mathcal{R}}),$$

where $X_{\mathcal{R}} = X_{\mathcal{P}} \times X_{\mathcal{Q}}, x^0_{\mathcal{R}} = (x^0_{\mathcal{P}}, x^0_{\mathcal{Q}}), X^m_{\mathcal{R}} = X^m_{\mathcal{P}} \times X^m_{\mathcal{Q}}$, and the state transition function $\delta_{\mathcal{R}} : X_{\mathcal{R}} \times (\Sigma \cup \{\epsilon\}) \rightarrow 2^{X_{\mathcal{R}}}$ is defined as:

$\forall x_r = (x_p, x_q) \in X_{\mathcal{R}}$:

$$\begin{aligned}
\forall \sigma \in \Sigma : \delta_{\mathcal{R}}(x_r, \sigma) &:= \begin{cases} \delta_{\mathcal{P}}(x_p, \sigma) \times \delta_{\mathcal{Q}}(x_q, \sigma) & \text{if } \delta_{\mathcal{P}}(x_p, \sigma), \delta_{\mathcal{Q}}(x_q, \sigma) \neq \emptyset \\ \delta_{\mathcal{P}}(x_p, \sigma) \times \{x_q\} & \text{if } \delta_{\mathcal{P}}(x_p, \sigma) \neq \emptyset, \sigma \in \Re_{\mathcal{Q}}(x_q), \sigma \notin B \\ \{x_p\} \times \delta_{\mathcal{Q}}(x_q, \sigma) & \text{if } \delta_{\mathcal{Q}}(x_q, \sigma) \neq \emptyset, \sigma \in \Re_{\mathcal{P}}(x_p), \sigma \notin A \\ \emptyset & \text{otherwise,} \end{cases} \\
\delta_{\mathcal{R}}(x_r, \epsilon) &:= [\delta_{\mathcal{P}}(x_p, \epsilon) \cup \{x_p\}] \times [\delta_{\mathcal{Q}}(x_q, \epsilon) \cup \{x_q\}] - \{(x_p, x_q)\}.
\end{aligned}$$

This definition of the PSC of two NSM's extends the one given in [15, Definition 9] to take into account the sets of accepting states.

For notational convenience, given $\Sigma', \Sigma_1, \Sigma_2, \Sigma'' \subseteq \Sigma$, we define

$$\Sigma'\ {}_{\Sigma_1}\!\otimes_{\Sigma_2}\ \Sigma'' := (\Sigma' \cap \Sigma'') \cup (\Sigma' \cap \Sigma_1) \cup (\Sigma'' \cap \Sigma_2).$$

Given *generated* trajectory sets P, Q, the PSC of a pair of trajectories $e_p \in P$, $e_q \in Q$ was defined in [15, Definition 10] as follows:

Definition 3 Let P, Q be generated trajectory sets with $e_p \in P$, $e_q \in Q$. Then the PSC of e_p and e_q (with respect to P and Q), denoted $e_p\ {}_A\|_B\ e_q$, is defined inductively on $|e_p| + |e_q|$ as follows:

$\forall \Sigma_p, \Sigma_q \subseteq \Sigma$ such that $\Sigma_p \in P, \Sigma_q \in Q$:

$$\Sigma_p\ {}_A\|_B\ \Sigma_q := \{\Sigma' \subseteq \Sigma_p\ {}_A\!\otimes_B\ \Sigma_q\},$$

$\forall e_p \in P; e_q \in Q; \sigma_p, \sigma_q \in \Sigma; \Sigma_p, \Sigma_q \subseteq \Sigma$ such that $e_p(\sigma_p, \Sigma_p) \in P, e_q(\sigma_q, \Sigma_q) \in Q$:

$$e_p(\sigma_p, \Sigma_p)\ {}_A\|_B\ e_q(\sigma_q, \Sigma_q) := T_1 \cup T_2 \cup T_3,$$

where

$$T_1 := \begin{cases} \{e(\sigma_p, \Sigma') \mid e \in e_p\ {}_A\|_B\ e_q(\sigma_q, \Sigma_q); \Sigma' \subseteq \Sigma_p\ {}_A\!\otimes_B\ \Sigma_q\} & \text{if } \sigma_p \notin B \text{ and} \\ & e_q(\sigma_q, \Sigma_q)(\sigma_p, \emptyset) \notin Q \\ \emptyset & \text{otherwise} \end{cases}$$

$$T_2 := \begin{cases} \{e(\sigma_q, \Sigma') \mid e \in e_p(\sigma_p, \Sigma_p)\ {}_A\|_B\ e_q; \Sigma' \subseteq \Sigma_p\ {}_A\!\otimes_B\ \Sigma_q\} & \text{if } \sigma_q \notin A \text{ and} \\ & e_p(\sigma_p, \Sigma_p)(\sigma_q, \emptyset) \notin P \\ \emptyset & \text{otherwise} \end{cases}$$

$$T_3 := \begin{cases} \{e(\sigma, \Sigma') \mid e \in e_p\ {}_A\|_B\ e_q; \Sigma' \subseteq \Sigma_p\ {}_A\!\otimes_B\ \Sigma_q\} & \text{if } \sigma_p = \sigma_q := \sigma \\ \emptyset & \text{otherwise} \end{cases}$$

It should be noted that $e_p\, {}_A\|_B\, e_q$ is a set of refusal-traces that depends on the generated trajectory sets P, Q as well as on the particular trajectories e_p, e_q. The dependence on P, Q is not explicitly indicated in the notation.

Using the definition of PSC of refusal-traces, we next define the PSC of trajectory models.

Definition 4 Let (P^m, P), (Q^m, Q) be trajectory models with priority sets $A, B \subseteq \Sigma$ respectively. The PSC of (P^m, P) and (Q^m, Q), denoted $(P^m, P)\, {}_A\|_B\, (Q^m, Q)$, is the pair of refusal-trace sets (R^m, R), where

$$R^m := \bigcup_{e_p \in P^m, e_q \in Q^m} e_p\, {}_A\|_B\, e_q, \quad R := \bigcup_{e_p \in P, e_q \in Q} e_p\, {}_A\|_B\, e_q,$$

where $e_p\, {}_A\|_B\, e_q$ is with respect to P, Q in the definitions of both R^m and R.

We will use the notation $(P^m\, {}_A\|_B\, Q^m, P\, {}_A\|_B\, Q)$ for $(P^m, P)\, {}_A\|_B\, (Q^m, Q)$. However, it must be kept in mind that $P^m\, {}_A\|_B\, Q^m$ implicitly depends on P and Q.

Theorem 3 $T^m(\mathcal{P})\, {}_A\|_B\, T^m(\mathcal{Q}) = T^m(\mathcal{P}\, {}_A\|_B\, \mathcal{Q})$ and $T(\mathcal{P})\, {}_A\|_B\, T(\mathcal{Q}) = T(\mathcal{P}\, {}_A\|_B\, \mathcal{Q})$.

Proof: The fact that $T(\mathcal{P})\, {}_A\|_B\, T(\mathcal{Q}) = T(\mathcal{P}\, {}_A\|_B\, \mathcal{Q})$ is proved in [15, Theorem 2]. For notational convenience, let $\mathcal{R} := \mathcal{P}\, {}_A\|_B\, \mathcal{Q}$; we first prove that $T^m(\mathcal{R}) \subseteq T^m(\mathcal{P})\, {}_A\|_B\, T^m(\mathcal{Q})$. Pick $e \in T^m(\mathcal{R})$. Then $e \in T(\mathcal{R})$, and there exists $x_r = (x_p, x_q) \in \delta^T_{\mathcal{R}}(x^0_{\mathcal{R}}, e) \cap X^m_{\mathcal{R}} = \delta^T_{\mathcal{R}}(x^0_{\mathcal{R}}, e) \cap (X^m_{\mathcal{P}} \times X^m_{\mathcal{Q}})$. It follows from [15, Corollary 5] that given e and x_r, there exists $e_p \in T(\mathcal{P})$ and $e_q \in T(\mathcal{Q})$ such that

$$e \in e_p\, {}_A\|_B\, e_q \text{ and } x_r \in \delta^T_{\mathcal{P}}(x^0_{\mathcal{P}}, e_p) \times \delta^T_{\mathcal{Q}}(x^0_{\mathcal{Q}}, e_q). \tag{1}$$

It follows that $\delta^T_{\mathcal{P}}(x^0_{\mathcal{P}}, e_p) \cap X^m_{\mathcal{P}} \neq \emptyset$ and $\delta^T_{\mathcal{Q}}(x^0_{\mathcal{Q}}, e_q) \cap X^m_{\mathcal{Q}} \neq \emptyset$. This implies that $e_p \in T^m(\mathcal{P})$ and $e_q \in T^m(\mathcal{Q})$, proving that $e \in T^m(\mathcal{P})\, {}_A\|_B\, T^m(\mathcal{Q})$.

Next we prove that $T^m(\mathcal{P})\, {}_A\|_B\, T^m(\mathcal{Q}) \subseteq T^m(\mathcal{R})$. Pick $e \in T^m(\mathcal{P})\, {}_A\|_B\, T^m(\mathcal{Q})$. Then there exist $e_p \in T^m(\mathcal{P})$ and $e_q \in T^m(\mathcal{Q})$ such that $e \in e_p\, {}_A\|_B\, e_q$. Since $e_p \in T(\mathcal{P})$ and $e_q \in T(\mathcal{Q})$, it follows from [15, Corollary 5] that $\delta^T_{\mathcal{P}}(x^0_{\mathcal{P}}, e_p) \times \delta^T_{\mathcal{Q}}(x^0_{\mathcal{Q}}, e_q) \subseteq \delta_{\mathcal{R}}(x^0_{\mathcal{R}}, e)$. This implies that

$$[\delta^T_{\mathcal{P}}(x^0_{\mathcal{P}}, e_p) \cap X^m_{\mathcal{P}}] \times [\delta^T_{\mathcal{Q}}(x^0_{\mathcal{Q}}, e_q) \cap X^m_{\mathcal{Q}}] \subseteq \delta^T_{\mathcal{R}}(x^0_{\mathcal{R}}, e) \cap X^m_{\mathcal{R}}. \tag{2}$$

Since $e_p \in T^m(\mathcal{P})$ and $e_q \in T^m(\mathcal{Q})$, we have $\delta^T_{\mathcal{P}}(x^0_{\mathcal{P}}, e_p) \cap X^m_{\mathcal{P}} \neq \emptyset$ and $\delta^T_{\mathcal{Q}}(x^0_{\mathcal{Q}}, e_q) \cap X^m_{\mathcal{Q}} \neq \emptyset$. Thus, Equation 2 implies that $\delta^T_{\mathcal{R}}(x^0_{\mathcal{R}}, e) \cap X^m_{\mathcal{R}} \neq \emptyset$, so $e \in T^m(\mathcal{R})$. ■

Corollary 4 Let $\mathcal{P}_1, \mathcal{P}_2, \mathcal{Q}_1, \mathcal{Q}_2$ be NSM's such that $T^m(\mathcal{P}_1) = T^m(\mathcal{P}_2), T(\mathcal{P}_1) = T(\mathcal{P}_2)$, $T^m(\mathcal{Q}_1) = T^m(\mathcal{Q}_2), T(\mathcal{Q}_1) = T(\mathcal{Q}_2)$. Then for any $A, B \subseteq \Sigma$:

1. $T^m(\mathcal{P}_1\, {}_A\|_B\, \mathcal{Q}_1) = T^m(\mathcal{P}_2\, {}_A\|_B\, \mathcal{Q}_2)$ and $T(\mathcal{P}_1\, {}_A\|_B\, \mathcal{Q}_1) = T(\mathcal{P}_2\, {}_A\|_B\, \mathcal{Q}_2)$,

2. $L^m(\mathcal{P}_1\, {}_A\|_B\, \mathcal{Q}_1) = L^m(\mathcal{P}_2\, {}_A\|_B\, \mathcal{Q}_2)$ and $L(\mathcal{P}_1\, {}_A\|_B\, \mathcal{Q}_1) = L(\mathcal{P}_2\, {}_A\|_B\, \mathcal{Q}_2)$.

Proof: The first part follows immediately from Theorem 3. The second part follows from the first part since $L^m(\mathcal{P}_1\,{}_A\|_B\,\mathcal{Q}_1) = L(T^m(\mathcal{P}_1\,{}_A\|_B\,\mathcal{Q}_1))$ etc. ∎

The first part of Corollary 4 states that the trajectory model contains enough detail to support the prioritized synchronous composition of NSM's. The second part of Corollary 4 states that the trajectory model serves as a *language congruence* [4] with respect to the operation of prioritized synchronous composition.

Corollary 5 Let (P^m, P) and (Q^m, Q) be two trajectory models, and $A, B \subseteq \Sigma$. Then $(P^m\,{}_A\|_B\,Q^m, P\,{}_A\|_B\,Q)$ is a trajectory model.

Proof: By hypothesis, there exist NSM's $\mathcal{P}$ and $\mathcal{Q}$ such that $(T^m(\mathcal{P}), T(\mathcal{P})) = (P^m, P)$ and $(T^m(\mathcal{Q}), T(\mathcal{Q})) = (Q^m, Q)$. It follows from Theorem 3 that $(P^m\,{}_A\|_B\,Q^m, P\,{}_A\|_B\,Q) = (T^m(\mathcal{P}\,{}_A\|_B\,\mathcal{Q}), T(\mathcal{P}\,{}_A\|_B\,\mathcal{Q}))$, which completes the proof. ∎

The following corollary states that when the priority sets of two systems are each equal to the entire event set, then the language model of the composed system can be obtained as the intersection of the individual language models.

Corollary 6 Let $\mathcal{P} := (X_{\mathcal{P}}, \Sigma, \delta_{\mathcal{P}}, x^0_{\mathcal{P}}, X^m_{\mathcal{P}}), \mathcal{Q} := (X_{\mathcal{Q}}, \Sigma, \delta_{\mathcal{Q}}, x^0_{\mathcal{Q}}, X^m_{\mathcal{Q}})$ be NSM's. Then

1. $L^m(\mathcal{P}\,{}_\Sigma\|_\Sigma\,\mathcal{Q}) = L^m(\mathcal{P}) \cap L^m(\mathcal{Q})$,
2. $L(\mathcal{P}\,{}_\Sigma\|_\Sigma\,\mathcal{Q}) = L(\mathcal{P}) \cap L(\mathcal{Q})$.

Proof: We only prove the first part; the second part can be proved analogously. Let $\mathcal{R} := \mathcal{P}\,{}_\Sigma\|_\Sigma\,\mathcal{Q}$. We have $L^m(\mathcal{P}) = L(T^m(\mathcal{P})), L^m(\mathcal{Q}) = L(T^m(\mathcal{Q})), L^m(\mathcal{R}) = L(T^m(\mathcal{R}))$, and it follows from Theorem 3 that $T^m(\mathcal{R}) = T^m(\mathcal{P})\,{}_\Sigma\|_\Sigma\,T^m(\mathcal{Q})$. Hence it suffices to show that $L(T^m(\mathcal{P})\,{}_\Sigma\|_\Sigma\,T^m(\mathcal{Q})) = L(T^m(\mathcal{P})) \cap L(T^m(\mathcal{Q}))$. Pick $s \in L(T^m(\mathcal{P})\,{}_\Sigma\|_\Sigma\,T^m(\mathcal{Q}))$. Then there exists $e \in T^m(\mathcal{P})\,{}_\Sigma\|_\Sigma\,T^m(\mathcal{Q})$ such that $tr(e) = s$. Since $e \in T^m(\mathcal{P})\,{}_\Sigma\|_\Sigma\,T^m(\mathcal{Q})$, it follows that there exist $e_p \in T^m(\mathcal{P}), e_q \in T^m(\mathcal{Q})$ such that $e \in e_p\,{}_A\|_B\,e_q$ and $tr(e_p) = tr(e_q) = tr(e) = s$. (Refer to [15, Remark 6].) I.e., $s \in L(T^m(\mathcal{P})) \cap L(T^m(\mathcal{Q}))$. This proves that $L(T^m(\mathcal{P})\,{}_\Sigma\|_\Sigma\,T^m(\mathcal{Q})) \subseteq L(T^m(\mathcal{P})) \cap L(T^m(\mathcal{Q}))$.

Similarly, given $s \in L(T^m(\mathcal{P})) \cap L(T^m(\mathcal{Q}))$, there exist $e_p \in T^m(\mathcal{P}), e_q \in T^m(\mathcal{Q})$ such that $tr(e_p) = tr(e_q) = s$. Choose $e \in e_p\,{}_\Sigma\|_\Sigma\,e_q$, which is clearly nonempty. Then $e \in T^m(\mathcal{P})\,{}_\Sigma\|_\Sigma\,T^m(\mathcal{Q})$ and $tr(e) = s$. (Refer to [15, Remark 6].) Consequently, we have $s \in L(T^m(\mathcal{P})\,{}_\Sigma\|_\Sigma\,T^m(\mathcal{Q}))$, which proves that $L(T^m(\mathcal{P})) \cap L(T^m(\mathcal{Q})) \subseteq L(T^m(\mathcal{P})\,{}_\Sigma\|_\Sigma\,T^m(\mathcal{Q}))$.∎

Corollary 7 Let (P^m, P) and (Q^m, Q) be trajectory models. Then

1. $L(P^m\,{}_\Sigma\|_\Sigma\,Q^m) = L(P^m) \cap L(Q^m)$,
2. $L(P\,{}_\Sigma\|_\Sigma\,Q) = L(P) \cap L(Q)$.

Proof: From the hypothesis there exist NSM's $\mathcal{P}, \mathcal{Q}$ such that $(T^m(\mathcal{P}), T(\mathcal{P})) = (P^m, P)$ and $(T^m(\mathcal{Q}), T(\mathcal{Q})) = (Q^m, Q)$. Hence the result follows from Corollary 6. ∎

Next we prove the *associative* property of PSC. It is immediate from Definition 2 that given NSM's $\mathcal{P}, \mathcal{Q}$ with priority sets $A, B \subseteq \Sigma$ respectively, the following holds for for each $x_r = (x_p, x_q) \in X_{\mathcal{R}}$, where $\mathcal{R} := \mathcal{P}\,{}_A\|_B\,\mathcal{Q}$:

$$\Re_{\mathcal{R}}(x_r) = \Re_{\mathcal{P}}(x_p)\,{}_A\!\otimes_B\,\Re_{\mathcal{Q}}(x_q) = [\Re_{\mathcal{P}}(x_p) \cap \Re_{\mathcal{Q}}(x_q)] \cup [\Re_{\mathcal{P}}(x_p) \cap A] \cup [\Re_{\mathcal{Q}}(x_q) \cap B].$$

Theorem 4 Let $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ be NSM's and $A, B, C \subseteq \Sigma$. Then

$$(\mathcal{P}\ {}_{A}\|_{B}\ \mathcal{Q})\ {}_{A\cup B}\|_{C}\ \mathcal{R} = \mathcal{P}\ {}_{A}\|_{B\cup C}\ (\mathcal{Q}\ {}_{B}\|_{C}\ \mathcal{R}).$$

Proof: Let $\mathcal{S}_1 := (\mathcal{P}\ {}_{A}\|_{B}\ \mathcal{Q})\ {}_{A\cup B}\|_{C}\ \mathcal{R}$, and $\mathcal{S}_2 := \mathcal{P}\ {}_{A}\|_{B\cup C}\ (\mathcal{Q}\ {}_{B}\|_{C}\ \mathcal{R})$. Then it follows from Definition 2 that $X_{\mathcal{S}_1} = X_{\mathcal{S}_2} = X_{\mathcal{P}} \times X_{\mathcal{Q}} \times X_{\mathcal{R}} := X_{\mathcal{S}}$, $x^0_{\mathcal{S}_1} = x^0_{\mathcal{S}_2} = (x^0_{\mathcal{P}}, x^0_{\mathcal{Q}}, x^0_{\mathcal{R}})$, $X^m_{\mathcal{S}_1} = X^m_{\mathcal{S}_2} = X^m_{\mathcal{P}} \times X^m_{\mathcal{Q}} \times X^m_{\mathcal{R}}$, and for each $x_s = (x_p, x_q, x_r) \in X_{\mathcal{S}}$:

$$\delta_{\mathcal{S}_1}(x_s, \epsilon) = \delta_{\mathcal{S}_2}(x_s, \epsilon) = [\delta_{\mathcal{P}}(x_p, \epsilon) \cup \{x_p\}] \times [\delta_{\mathcal{Q}}(x_q, \epsilon) \cup \{x_q\}] \times [\delta_{\mathcal{R}}(x_r, \epsilon) \cup \{x_q\}] - \{x_s\}.$$

It remains to show that for each $x_s = (x_p, x_q, x_r) \in X_{\mathcal{S}}$ and $\sigma \in \Sigma$:

$$\delta_{\mathcal{S}_1}(x_s, \sigma) = \delta_{\mathcal{S}_2}(x_s, \sigma). \tag{3}$$

It follows from Definition 2 that

$$\delta_{\mathcal{S}_1}(x_s, \sigma) = \begin{cases} \delta_{\mathcal{P}}(x_p, \sigma) \times \delta_{\mathcal{Q}}(x_q, \sigma) \times \delta_{\mathcal{R}}(x_r, \sigma) & \text{if } \delta_{\mathcal{P}}(x_p, \sigma), \delta_{\mathcal{Q}}(x_q, \sigma), \delta_{\mathcal{R}}(x_r, \sigma) \neq \emptyset \\ \delta_{\mathcal{P}}(x_p, \sigma) \times \{x_q\} \times \delta_{\mathcal{R}}(x_r, \sigma) & \text{if } \delta_{\mathcal{P}}(x_p, \sigma), \delta_{\mathcal{R}}(x_r, \sigma) \neq \emptyset, \sigma \in \Re_{\mathcal{Q}}(x_q), \\ & \qquad \sigma \notin B \\ \{x_p\} \times \delta_{\mathcal{Q}}(x_q, \sigma) \times \delta_{\mathcal{R}}(x_r, \sigma) & \text{if } \delta_{\mathcal{Q}}(x_q, \sigma), \delta_{\mathcal{R}}(x_r, \sigma) \neq \emptyset, \sigma \in \Re_{\mathcal{P}}(x_p), \\ & \qquad \sigma \notin A \\ \delta_{\mathcal{P}}(x_p, \sigma) \times \delta_{\mathcal{Q}}(x_q, \sigma) \times \{x_r\} & \text{if } \delta_{\mathcal{P}}(x_p, \sigma), \delta_{\mathcal{Q}}(x_q, \sigma) \neq \emptyset, \sigma \in \Re_{\mathcal{R}}(x_r), \\ & \qquad \sigma \notin C \\ \delta_{\mathcal{P}}(x_p, \sigma) \times \{x_q\} \times \{x_r\} & \text{if } \delta_{\mathcal{P}}(x_p, \sigma) \neq \emptyset, \sigma \in \Re_{\mathcal{Q}}(x_q) \cap \Re_{\mathcal{R}}(x_r), \\ & \qquad \sigma \notin B \cup C \\ \{x_p\} \times \delta_{\mathcal{Q}}(x_q, \sigma) \times \{x_r\} & \text{if } \delta_{\mathcal{Q}}(x_q, \sigma) \neq \emptyset, \sigma \in \Re_{\mathcal{P}}(x_p) \cap \Re_{\mathcal{R}}(x_r), \\ & \qquad \sigma \notin A \cup C \\ \{x_p\} \times \{x_q\} \times \delta_{\mathcal{R}}(x_r, \sigma) & \text{if } \delta_{\mathcal{R}}(x_r, \sigma) \neq \emptyset, \sigma \in \Re_{\mathcal{P}}(x_p)\ {}_{A}\!\otimes_{B}\ \Re_{\mathcal{Q}}(x_q), \\ & \qquad \sigma \notin A \cup B \\ \emptyset & \text{otherwise} \end{cases}$$

An expression for $\delta_{\mathcal{S}_2}(x_s, \sigma)$ can be analogously obtained. Note that in the fifth clause of the expression for $\delta_{\mathcal{S}_1}(x_s, \sigma)$, the condition $\sigma \in \Re_{\mathcal{Q}}(x_q) \cap \Re_{\mathcal{R}}(x_r)$ and $\sigma \notin B \cup C$ is equivalent to $\sigma \in \Re_{\mathcal{Q}}(x_q)\ {}_{B}\!\otimes_{C}\ \Re_{\mathcal{R}}(x_r)$ and $\sigma \notin B \cup C$. Similarly, in the sixth clause of the expression for $\delta_{\mathcal{S}_1}(x_s, \sigma)$, the condition $\sigma \in \Re_{\mathcal{P}}(x_p) \cap \Re_{\mathcal{R}}(x_r)$ and $\sigma \notin A \cup C$ is equivalent to $\sigma \in \Re_{\mathcal{P}}(x_p)\ {}_{A}\!\otimes_{C}\ \Re_{\mathcal{R}}(x_r)$ and $\sigma \notin A \cup C$. If similar simplifications in the clauses of the expression for $\delta_{\mathcal{S}_2}(x_s, \sigma)$ are performed, then Equation 3 is easily proved. ■

Corollary 8 Let (P^m, P), (Q^m, Q) and (R^m, R) be trajectory models and $A, B, C \subseteq \Sigma$. Then

1. $(P^m\ {}_{A}\|_{B}\ Q^m)\ {}_{A\cup B}\|_{C}\ R^m = P^m\ {}_{A}\|_{B\cup C}\ (Q^m\ {}_{B}\|_{C}\ R^m)$,
2. $(P\ {}_{A}\|_{B}\ Q)\ {}_{A\cup B}\|_{C}\ R = P\ {}_{A}\|_{B\cup C}\ (Q\ {}_{B}\|_{C}\ R)$.

Proof: From the hypothesis, there exist NSM's $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ such that $(T^m(\mathcal{P}), T(\mathcal{P})) = (P^m, P)$, $(T^m(\mathcal{Q}), T(\mathcal{Q})) = (Q^m, Q)$, and $(T^m(\mathcal{R}), T(\mathcal{R})) = (R^m, R)$. It is easily shown using the result of Theorem 3 that

$$\begin{aligned}
(P^m\, {}_A\|_B\, Q^m)\, {}_{A\cup B}\|_C\, R^m &= T^m((\mathcal{P}\, {}_A\|_B\, \mathcal{Q})\, {}_{A\cup B}\|_C\, \mathcal{R}),\\
P^m\, {}_A\|_{B\cup C}\, (Q^m\, {}_B\|_C\, R^m) &= T^m(\mathcal{P}\, {}_A\|_{B\cup C}\, (\mathcal{Q}\, {}_B\|_C\, \mathcal{R})),\\
(P\, {}_A\|_B\, Q)\, {}_{A\cup B}\|_C\, R &= T((\mathcal{P}\, {}_A\|_B\, \mathcal{Q})\, {}_{A\cup B}\|_C\, \mathcal{R}),\\
P\, {}_A\|_{B\cup C}\, (Q\, {}_B\|_C\, R) &= T(\mathcal{P}\, {}_A\|_{B\cup C}\, (\mathcal{Q}\, {}_B\|_C\, \mathcal{R})).
\end{aligned}$$

Thus the result follows from Theorem 4. ■

Part 2 of Corollary 8 is stated without proof in [5]; a direct proof of part 2 that does not depend on the corresponding result for NSM's is given in [15, Theorem 3].

Next we define *augmentation*, which is the PSC of a given NSM with an NSM of the type $\mathcal{D} := (\{x^0_{\mathcal{D}}\}, \Sigma, \delta_{\mathcal{D}}, x^0_{\mathcal{D}}, \{x^0_{\mathcal{D}}\})$, the transition function of which is defined as:

$$\forall \sigma \in \Sigma \cup \{\epsilon\}, \quad \delta_{\mathcal{D}}(x^0_{\mathcal{D}}, \sigma) := \begin{cases} \{x^0_{\mathcal{D}}\} & \text{if } \sigma \in D \\ \emptyset & \text{otherwise,} \end{cases}$$

where $D \subseteq \Sigma$ is a given set of events. I.e., $\mathcal{D}$ is a deterministic state machine consisting of a single state having self-loops on events in $D \subseteq \Sigma$. It is clear that $L^m(\mathcal{D}) = L(\mathcal{D}) = D^*$ and $T^m(\mathcal{D}) = T(\mathcal{D}) = det(D^*)$.

Definition 5 Given an NSM $\mathcal{P} := (X_{\mathcal{P}}, \Sigma, \delta_{\mathcal{P}}, x^0_{\mathcal{P}}, X^m_{\mathcal{P}})$ and $D \subseteq \Sigma$, the *augmented NSM with respect to D*, denoted $\mathcal{P}^D$, is defined to be $\mathcal{P}^D := \mathcal{P}\, {}_\emptyset\|_\emptyset\, \mathcal{D}$. Given a trajectory model (P^m, P), *the augmented trajectory model with respect to D*, denoted $(P^m, P)^D$, is defined to be $(P^m, P)^D := (P^m, P)\, {}_\emptyset\|_\emptyset\, (det(D^*), det(D^*))$.

Using $((P^m)^D, P^D)$ to denote $(P^m, P)^D$, we have $(P^m)^D = P^m\, {}_\emptyset\|_\emptyset\, det(D^*)$ and $P^D = P\, {}_\emptyset\|_\emptyset\, det(D^*)$. As is always the case with the PSC of *recognized* trajectory sets, the expression for $(P^m)^D$ implicitly depends on the corresponding *generated* trajectory sets P and $det(D^*)$. Clearly the trajectory model $(T^m(\mathcal{P}^D), T(\mathcal{P}^D))$ of the augmented NSM $\mathcal{P}^D$ is equal to $((T^m(\mathcal{P}))^D, (T(\mathcal{P}))^D)$. Also, since $det(D^*)$ can always execute every event in D and can never execute any event in $\Sigma - D$, it follows that given any trajectory model (P^m, P), any $A \subseteq \Sigma - D$, and any $B \subseteq D$

$$\begin{aligned}
(P^m)^D &:= P^m\, {}_\emptyset\|_\emptyset\, det(D^*) = P^m\, {}_A\|_B\, det(D^*),\\
P^D &:= P\, {}_\emptyset\|_\emptyset\, det(D^*) = P\, {}_A\|_B\, det(D^*).
\end{aligned}$$

Next we show that the PSC of two systems is equivalent to the strict synchronous composition (over the union of the two priority sets) of the associated augmented systems.

Proposition 3 Let $(P^m, P), (Q^m, Q)$ be trajectory models, and $A, B \subseteq \Sigma$. Then

1. $P^m\, {}_A\|_B\, Q^m = (P^m)^{B-A}\, {}_{A\cup B}\|_B\, Q^m = (P^m)^{B-A}\, {}_{A\cup B}\|_{A\cup B}\, (Q^m)^{A-B}$,

2. $P\, {}_A\|_B\, Q = P^{B-A}\, {}_{A\cup B}\|_B\, Q = P^{B-A}\, {}_{A\cup B}\|_{A\cup B}\, Q^{A-B}$.

Proof: We only prove the first part; the second part can be proved analogously. Again, we only prove the first equality as the second equality follows from symmetry and a second application of the first equality. From the definition of augmentation and associativity of PSC (Corollary 8), we have

$$\begin{aligned}(P^m)^{B-A}\,{}_{A\cup B}\|_B\, Q^m &= (P^m\,{}_A\|_{B-A}\, det((B-A)^*))\,{}_{A\cup B}\|_B\, Q^m \\ &= det((B-A)^*)\,{}_{B-A}\|_{A\cup B}\,(P^m\,{}_A\|_B\, Q^m) \\ &= P^m\,{}_A\|_B\, Q^m,\end{aligned}$$

where the last equality follows from the two facts: (i) The priority set of $P^m\,{}_A\|_B\, Q^m$ is $A \cup B$, and $B - A \subseteq A \cup B$, so $det((B-A)^*)$ cannot execute an event that $P^m\,{}_A\|_B\, Q^m$ does not execute. (ii) $det((B-A)^*)$ can always execute each event in its priority set, so that it cannot block any event in $P^m\,{}_A\|_B\, Q^m$. ■

5 Supervisory Control Using PSC

In [15], we studied the supervisory control of nondeterministic systems in the setting of trajectory models and prioritized synchronization under the assumption that all refusal-traces of the plant are marked and the desired behavior is specified by a prefix-closed language; thus the issue of deadlock/blocking was not investigated. In this section, we generalize the results in [15] to include non-closed specifications and arbitrary marking of the plant refusal-traces. In addition, we consider *modular synthesis* of supervisors.

Since trajectory models contain sufficient detail to support the operation of prioritized synchronization, we use trajectory models, rather than NSM's, to represent a discrete event system. Unless otherwise specified, the trajectory model of the plant and that of the supervisor are denoted by (P^m, P) and (S^m, S), and the priority set of the plant and that of the supervisor are denoted by A and B respectively. In the setting of supervisory control, $A = \Sigma_u \cup \Sigma_c$, where $\Sigma_u, \Sigma_c \subseteq \Sigma$ denote the sets of *uncontrollable* and *controllable* events respectively [13, 14]; and $\Sigma_d \subseteq B \subseteq \Sigma_c \cup \Sigma_d$, where $\Sigma_d \subseteq \Sigma$ denotes the set of so-called *driven* [4] or *forcible* [3, 2] or *command* events [1]. The sets Σ_u, Σ_c and Σ_d are pairwise disjoint and exhaust the entire event set. Note that since $A = \Sigma_u \cup \Sigma_c$ and $\Sigma_d \subseteq B$, $A \cup B = \Sigma$.

The controlled (or closed-loop) system is $(P^m, P)\,{}_A\|_B\,(S^m, S)$. Since A contains every event that is either controllable or uncontrollable, such events cannot occur solely in the supervisor. This is consistent with the modeling assumption made in the Ramadge-Wonham theory [13] that these events originate in the plant. Since B contains every driven event, these events cannot occur solely in the plant. This corresponds to the modeling assumption that driven events originate in the supervisor and are executed synchronously by the plant whenever possible. If there is only one supervisor–i.e., the non-modular case–then the assumption that $\Sigma_d \subseteq B \subseteq \Sigma_c \cup \Sigma_d$ should be strengthened to $B = \Sigma_c \cup \Sigma_d$ to reflect the modeling assumption that the occurrence of controllable events requires the participation of the supervisor. However, the weaker assumption is useful when modular synthesis is considered, since it is not necessary to require that every subsystem in a modular supervisor participate in each controllable event.

In certain situations, a plant may consist of several "sub-plants" operating under prioritized synchrony. In such a case, if (P_i^m, P_i) is the trajectory model of the ith sub-plant, and $A_i \subseteq \Sigma$ is its priority set, then the trajectory model of the composed plant is given by $(P^m, P) := (\|_{A_i} P_i^m, \|_{A_i} P_i)$, where the notation $(\|_{A_i} P_i^m, \|_{A_i} P_i)$ is used to denote the PSC of sub-plants $\{(P_i^m, P_i)\}$. (Since PSC is associative (Corollary 8), $(\|_{A_i} P_i^m, \|_{A_i} P_i)$ is well defined, and it follows from Corollary 5 that it is a trajectory model.) The priority set of the composed plant is given by $A := \cup_i A_i$.

Definition 6 Given a plant (P^m, P) with priority set $A \subseteq \Sigma$, a supervisor, with trajectory model (S^m, S) and priority set $B \subseteq \Sigma$, is said to be *non-marking* if $S^m = S$; it is said to be *language model non-blocking* if $pr(L(P^m {}_A\|_B S^m)) = L(P {}_A\|_B S)$; and it is said to be *trajectory model non-blocking* if $pr(P^m {}_A\|_B S^m) = P {}_A\|_B S$.

Note that if a supervisor is trajectory model non-blocking, then it is also language model non-blocking. On the other hand, if a plant as well as a supervisor are deterministic, and the supervisor is language model non-blocking, then it is also trajectory model non-blocking.

Next we obtain a necessary and sufficient condition for the existence of a non-marking and language model non-blocking supervisor. This result is then used to obtain a necessary and sufficient condition for the existence of a non-marking and trajectory model non-blocking supervisor.

Theorem 5 Let (P^m, P) be the trajectory model of a plant, $A, B \subseteq \Sigma$ with $A \cup B = \Sigma$, and $K^m \subseteq L((P^m)^{\Sigma - A})$ with $K^m \neq \emptyset$. Then there exists a non-marking and language model non-blocking supervisor with trajectory model (S, S) such that $L(P^m {}_A\|_B S) = K^m$ if and only if

Relative-closure: $pr(K^m) \cap L((P^m)^{\Sigma - A}) = K^m$
Controllability: $pr(K^m)(A - B) \cap L(P^{\Sigma - A}) \subseteq pr(K^m)$.

In this case, S can be chosen to be $det(pr(K^m))$.

Proof: We begin with the proof for necessity. Suppose there exists a non-marking and language model non-blocking supervisor with trajectory model (S, S) such that $L(P^m {}_A\|_B S) = K^m$. Then

$$pr(K^m) = pr(L(P^m {}_A\|_B S)) = L(P {}_A\|_B S), \tag{4}$$

where the last equality follows from the supervisor being language model non-blocking. It follows from Definitions 4 and 5 that $(P^m)^{\Sigma - A} \subseteq P^{\Sigma - A}$, which implies that $pr(K^m) \subseteq pr(L((P^m)^{\Sigma - A})) \subseteq L(P^{\Sigma - A})$. Hence it follows from the necessity part of [15, Theorem 4] that $pr(K^m)(A - B) \cap L(P^{\Sigma - A}) \subseteq pr(K^m)$-i.e., the controllability property is satisfied. It remains to show that the relative-closure property is also satisfied. We have the following series of equalities:

$$\begin{aligned} pr(K^m) \cap L((P^m)^{\Sigma - A}) &= L(P {}_A\|_B S) \cap L((P^m)^{\Sigma - A}) \\ &= L((P {}_A\|_B S) {}_\Sigma\|_A P^m) \\ &= L((P {}_A\|_A P^m) {}_A\|_B S) \\ &= L((P {}_A\|_A P^m)^{\Sigma - A}) \cap L(S^{\Sigma - B}), \end{aligned} \tag{5}$$

where the first equality follows from Equation 4, the second equality follows from Corollary 7 and Proposition 3, the third equality follows from associativity of PSC (Corollary 8), and the final equality follows from Corollary 7. On the other hand, we have

$$K^m = L(P^m {}_A\|_B\, S) = L((P^m)^{\Sigma-A}) \cap L(S^{\Sigma-B}), \tag{6}$$

where the last equality follows from Proposition 3 and Corollary 7. It follows from Equations 5 and 6 that in order to show that the relative-closure property is satisfied, it suffices to show $L((P\,{}_A\|_A\, P^m)^{\Sigma-A}) = L((P^m)^{\Sigma-A})$. We have

$$\begin{aligned} L((P\,{}_A\|_A\, P^m)^{\Sigma-A}) &= L((P\,{}_A\|_A\, P^m)\,{}_A\|_{\Sigma-A}\, det((\Sigma-A)^*)) \\ &= L(P\,{}_A\|_\Sigma\, (P^m\,{}_A\|_{\Sigma-A}\, det((\Sigma-A)^*))) \\ &= L(P\,{}_A\|_\Sigma\, (P^m)^{\Sigma-A}) \\ &= L(P^{\Sigma-A}) \cap L((P^m)^{\Sigma-A}) \\ &= L((P^m)^{\Sigma-A}), \end{aligned}$$

where the first and third equalities follow from the definition of augmentation, the second equality follows from the associativity of PSC (Corollary 8), and the fourth equality follows from Proposition 3. This completes the proof of necessity.

Next we prove sufficiency. Suppose the relative-closure and controllability properties are satisfied. Then it follows from the sufficiency part of [15, Theorem 4] that the non-marking deterministic supervisor with trajectory model (S, S), where $S := det(pr(K^m))$, yields

$$L(P\,{}_A\|_B\, S) = pr(K^m), \tag{7}$$

and S^m is arbitrary. We select $S^m = S$, so the supervisor is non-marking. It follows from Equation 7 that

$$pr(K^m) = L(P^{\Sigma-A}) \cap L(S^{\Sigma-B}). \tag{8}$$

Using the relative-closure property gives the following series of equalities:

$$\begin{aligned} K^m &= pr(K^m) \cap L((P^m)^{\Sigma-A}) \\ &= [L(P^{\Sigma-A}) \cap L(S^{\Sigma-B})] \cap L((P^m)^{\Sigma-A}) \\ &= L((P^m)^{\Sigma-A}) \cap L(S^{\Sigma-B}) \\ &= L(P^m\,{}_A\|_B\, S). \end{aligned}$$

Since $pr(K^m) = L(P\,{}_A\|_B\, S)$ and $K^m = L(P^m\,{}_A\|_B\, S)$, the supervisor is language model non-blocking. ■

Remark 2 In contrast to the standard conditions [13] for the existence of a non-marking and language model non-blocking supervisor in the absence of driven events, the controllability and relative-closure conditions given in Theorem 5 refer to the language model of the *augmented* plant. It is easily demonstrated [15, Example 3] that this language model depends on the *trajectory model* of the plant and generally cannot be determined if only the language model of the plant is known.

We will obtain necessary and sufficient conditions for the existence of a deterministic non-marking and *trajectory model non-blocking* supervisor that imposes a desired closed-loop recognized language. The following preliminary results are needed.

Lemma 1 [15, Lemma 6] Let P, Q be generated trajectory sets, $A, B \subseteq \Sigma$, $f_p, e_p \in P$, $f_q, e_q \in Q$, with $f_p \sqsubseteq e_p$, $f_q \sqsubseteq e_q$. Then

$$f_p\, {}_A\|_B\, f_q \subseteq e_p\, {}_A\|_B\, e_q.$$

Lemma 2 Let (P^m, P) be a trajectory model, $\emptyset \neq H = pr(H) \subseteq \Sigma^*$, and $K := L(P) \cap H$. Then

1. $P\, {}_\Sigma\|_\Sigma\, det(H) = P\, {}_\Sigma\|_\Sigma\, det(K)$,
2. $P^m\, {}_\Sigma\|_\Sigma\, det(H) = P^m\, {}_\Sigma\|_\Sigma\, det(K)$.

Proof: By T3, every refusal-trace in a generated trajectory set is dominated by a saturated trajectory. By T5, every refusal-trace in a recognized trajectory set is dominated by a saturated marked trajectory. By Lemma 1, it suffices to show that for any saturated trajectories $e \in P$ (respectively, $e \in P^m$), $f \in det(H)$, $g \in det(K)$, we have

$$\begin{aligned} e\, {}_\Sigma\|_\Sigma\, f &\subseteq P\, {}_\Sigma\|_\Sigma\, det(K) \quad (\text{respectively, } P^m\, {}_\Sigma\|_\Sigma\, det(K)) \qquad (9)\\ e\, {}_\Sigma\|_\Sigma\, g &\subseteq P\, {}_\Sigma\|_\Sigma\, det(H) \quad (\text{respectively, } P^m\, {}_\Sigma\|_\Sigma\, det(H)). \qquad (10)\end{aligned}$$

The left side of Equation 9 is empty unless $tr(e) = tr(f) \in K$, while the left side of Equation 10 is empty unless $tr(e) = tr(g) \in K$. Thus, to establish Equations 9 and 10, it suffices to show that for $e \in P_{sat}$ with $tr(e) := t \in K$, $f := trj_H(t)$, $g := trj_K(t)$, we have

$$e\, {}_\Sigma\|_\Sigma\, f = e\, {}_\Sigma\|_\Sigma\, g. \qquad (11)$$

We have for each $k \leq |t|$, $\sigma \in \Sigma_k(f)$ (respectively, $\sigma \in \Sigma_k(g)$) if and only if $t^k\sigma \notin H$ (respectively, $t^k\sigma \notin K$). It follows from Definition 3 that

$$\begin{aligned} e\, {}_\Sigma\|_\Sigma\, f &= \{h \in 2^\Sigma \times (\Sigma \times 2^\Sigma)^* \mid tr(h) = t, \Sigma_k(h) \subseteq \Sigma_k(e) \cup \Sigma_k(f),\ k = 0, \ldots, |t|\},\\ e\, {}_\Sigma\|_\Sigma\, g &= \{h \in 2^\Sigma \times (\Sigma \times 2^\Sigma)^* \mid tr(h) = t, \Sigma_k(h) \subseteq \Sigma_k(e) \cup \Sigma_k(g),\ k = 0, \ldots, |t|\}.\end{aligned}$$

Thus, to establish Equation 11, it suffices to show that

$$\Sigma_k(e) \cup \Sigma_k(f) = \Sigma_k(e) \cup \Sigma_k(g), \quad k = 0, \ldots, |t|. \qquad (12)$$

Since $K \subseteq H$, it follows that $\Sigma_k(f) \subseteq \Sigma_k(g)$. On the other hand, if $\sigma \in \Sigma_k(g) - \Sigma_k(e)$, then $t^k\sigma \notin K$ and, since $e \in P_{sat}$, it follows that $e^k(\sigma, \emptyset) \in P$, so $t^k\sigma \in L(P)$. Hence, $t^k\sigma \notin H$, so $\sigma \in \Sigma_k(f)$ proving Equation 12. ■

Proposition 4 Let (P^m, P) be the trajectory model of a plant, $A, B \subseteq \Sigma$ with $A \cup B = \Sigma$, and $K^m \subseteq L((P^m)^{\Sigma-A})$ with $K^m \neq \emptyset$. If (S, S) is any non-marking language model non-blocking deterministic supervisor with $L(P^m\,{}_A\|_B\, S) = K^m$, then

$$\begin{aligned} P\,{}_A\|_B\, det(pr(K^m)) &= P\,{}_A\|_B\, S, && (13) \\ P^m\,{}_A\|_B\, det(pr(K^m)) &= P^m\,{}_A\|_B\, S. && (14) \end{aligned}$$

Proof: It follows that the relative-closure and controllability conditions in Theorem 3 hold, and hence that $(det(pr(K^m)), det(pr(K^m)))$ is a non-marking and language model non-blocking supervisor with $L(P^m\,{}_A\|_B\, det(pr(K^m))) = K^m$. Thus,

$$pr(K^m) = L(P\,{}_A\|_B\, det(pr(K^m))) = L(P^{\Sigma-A}) \cap L((det(pr(K^m)))^{\Sigma-B}).$$

This together with Lemma 2 gives

$$\begin{aligned} P^{\Sigma-A}\,{}_\Sigma\|_\Sigma\, (det(pr(K^m)))^{\Sigma-B} &= P^{\Sigma-A}\,{}_\Sigma\|_\Sigma\, det(pr(K^m)), && (15) \\ (P^m)^{\Sigma-A}\,{}_\Sigma\|_\Sigma\, (det(pr(K^m)))^{\Sigma-B} &= (P^m)^{\Sigma-A}\,{}_\Sigma\|_\Sigma\, det(pr(K^m)). && (16) \end{aligned}$$

Since (S, S) is language model non-blocking,

$$pr(K^m) = L(P\,{}_A\|_B\, S) = L(P^{\Sigma-A}) \cap L(S^{\Sigma-B}).$$

Hence it follows from Lemma 2 and the fact that $S^{\Sigma-B}$ is deterministic that

$$\begin{aligned} P^{\Sigma-A}\,{}_\Sigma\|_\Sigma\, S^{\Sigma-B} &= P^{\Sigma-A}\,{}_\Sigma\|_\Sigma\, det(pr(K^m)), && (17) \\ (P^m)^{\Sigma-A}\,{}_\Sigma\|_\Sigma\, S^{\Sigma-B} &= (P^m)^{\Sigma-A}\,{}_\Sigma\|_\Sigma\, det(pr(K^m)). && (18) \end{aligned}$$

From Equations 15-18 we have

$$\begin{aligned} P^{\Sigma-A}\,{}_\Sigma\|_\Sigma\, (det(pr(K^m)))^{\Sigma-B} &= P^{\Sigma-A}\,{}_\Sigma\|_\Sigma\, S^{\Sigma-B}, \\ (P^m)^{\Sigma-A}\,{}_\Sigma\|_\Sigma\, (det(pr(K^m)))^{\Sigma-B} &= (P^m)^{\Sigma-A}\,{}_\Sigma\|_\Sigma\, S^{\Sigma-B}. \end{aligned}$$

This implies Equations 13-14. ∎

Theorem 6 Let (P^m, P) be the trajectory model of a plant, $A, B \subseteq \Sigma$ with $A \cup B = \Sigma$, and $K^m \subseteq L((P^m)^{\Sigma-A})$ with $K^m \neq \emptyset$. Then there exists a non-marking and trajectory model non-blocking deterministic supervisor with trajectory model (S, S) such that $L(P^m\,{}_A\|_B\, S) = K^m$ if and only if

Relative-closure: $pr(K^m) \cap L((P^m)^{\Sigma-A}) = K^m$
Controllability: $pr(K^m)(A - B) \cap L(P^{\Sigma-A}) \subseteq pr(K^m)$
Trajectory-closure: $P\,{}_A\|_B\, det(pr(K^m)) = pr[P^m\,{}_A\|_B\, det(pr(K^m))]$.

In this case, S can be chosen to be $det(pr(K^m))$.

Proof: We first prove the sufficiency. Since the relative-closure and controllability conditions hold, it follows from the sufficiency part of Theorem 5 that the non-marking deterministic supervisor (S, S), where $S := det(pr(K^m))$, yields $L(P^m\ {}_A\|_B\ S) = K^m$ and $L(P\ {}_A\|_B\ S) = pr(K^m)$. It follows from the trajectory-closure condition that the supervisor is also trajectory model non-blocking.

Next we prove the necessity. Suppose there exists a non-marking and trajectory model non-blocking deterministic supervisor with trajectory model (S, S) such that $L(P^m\ {}_A\|_B\ S) = K^m$. Since a trajectory model non-blocking supervisor is also language model non-blocking, it follows from Theorem 5 that the relative-closure and controllability conditions hold, and from Proposition 4 that

$$\begin{aligned} P\ {}_A\|_B\ det(pr(K^m)) &= P\ {}_A\|_B\ S, \qquad (19)\\ P^m\ {}_A\|_B\ det(pr(K^m)) &= P^m\ {}_A\|_B\ S. \qquad (20)\end{aligned}$$

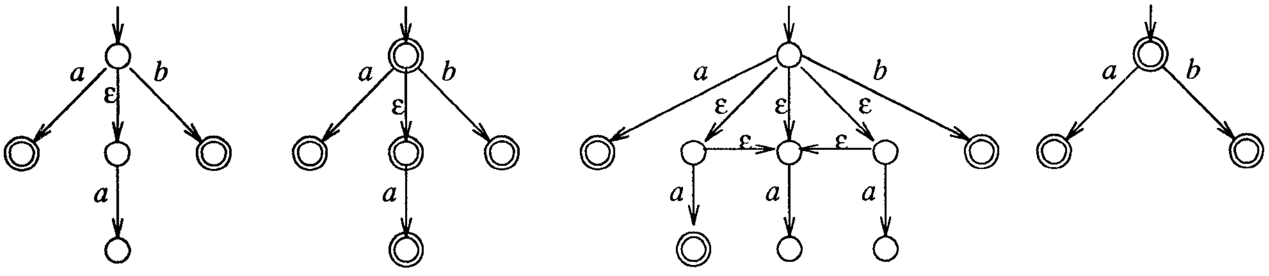
Since (S, S) is trajectory model non-blocking, we have $P\ {}_A\|_B\ S = pr(P^m\ {}_A\|_B\ S)$. Hence, the trajectory-closure condition is implied by Equations 19-20. ■

Remark 3 The controllability condition of Theorem 5 is needed for the existence of a supervisor such that the closed-loop generated language equals the closure of the language to be recognized by the closed-loop system, i.e., $L(P\ {}_A\|_B\ S) = pr(K^m)$. The relative-closure condition is needed so that the corresponding non-marking supervisor (S, S) yields equality of closed-loop recognized language and the desired language, i.e., $L(P^m\ {}_A\|_B\ S) = K^m$. Clearly, this design automatically yields that the non-marking supervisor (S, S) is also language model non-blocking. The extra trajectory-closure condition of Theorem 6 is needed so that the non-marking supervisor (S, S) is also trajectory model non-blocking. Thus all three conditions of Theorem 6 are needed for the existence of a non-marking and trajectory model non-blocking supervisor so that the closed-loop recognized language is the desired one.

In Theorem 6 it is proved that if there exists a non-marking and trajectory model non-blocking *deterministic* supervisor, then the trajectory-closure condition holds. The following example illustrates that the requirement of determinism of supervisor cannot be relaxed. In other words, if there exists a non-marking and trajectory model non-blocking *nondeterministic* supervisor, then the trajectory-closure condition need not hold.

Example 2 Consider the plant NSM defined on the event set $\Sigma = \{a, b\}$, shown in Figure 2(a). Since the refusal-trace $\{b\}(a, \{a, b\})$ belongs to the generated trajectory set of the plant, and there exists no refusal-trace in the recognized trajectory set of the plant with $\{b\}(a, \{a, b\})$ as its prefix, the plant is trajectory model *blocking*.

Consider the nondeterministic supervisor depicted in Figure 2(b). The supervisor has the same structure as the plant except that all its states are marked, so that it is non-marking. Let the priority set of both plant and supervisor be the entire event set, so that the set of uncontrollable events as well as the set of driven events is empty. The closed-loop system under the nondeterministic supervision is shown in Figure 2(c). The generated trajectory set



(a) plant/closed-loop system under deterministic supervisor (b) nondeterministic-supervisor (c) closed-loop syatem under nondeterministic supervisor (d) deterministic-supervisor

Figure 2: Diagram illustrating Example 2

of the closed-loop system is same as that of the plant. However, the recognized trajectory set is larger than that of the plant. In particular, the refusal-trace $\{b\}(a, \{a, b\})$ belongs to the recognized trajectory set of the closed-loop system, and the closed-loop system is trajectory model *non-blocking.*

The generated language of the closed-loop system is $pr(a + b)$. The non-marking deterministic supervisor shown in Figure 2(d) generates this language. The closed-loop system under the supervision of the deterministic supervisor has the same NSM as the plant, and is hence trajectory model blocking. Thus the trajectory-closure condition of Theorem 6 does not hold even though a non-marking and trajectory model non-blocking supervisor exists for the given plant.

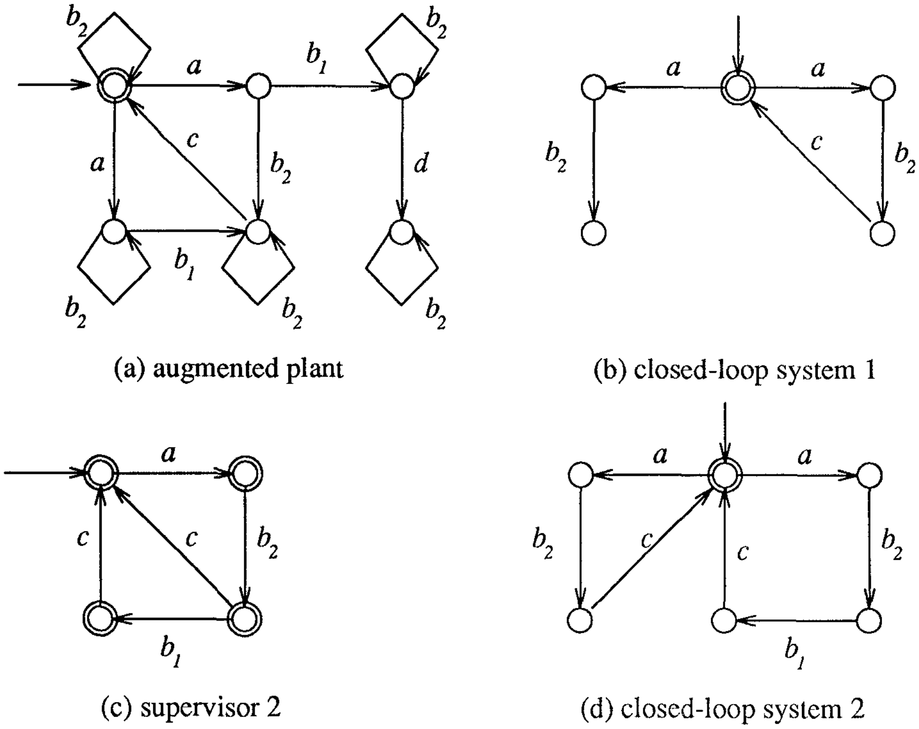
With Theorem 6 in hand, we revisit the example of Section 2.

Example 3 Consider the open-loop NSM $\mathcal{P}$ described in Section 2 and depicted in Figure 1(b). It is given that a, c, d are uncontrollable. We regard b_1 as a controllable event–i.e., requiring the participation of both the plant and supervisor. However, we regard b_2 as a driven event. This models the possibility that the supervisor may request the input of a part with the guides set to narrow, but that this request may be refused by the plant if the arriving part is wide. Thus, for PSC-based design, the priority sets of the plant and supervisor are $A = \{a, b_1, c, d\}$ and $B = \{b_1, b_2\}$ respectively.

In Section 2, we indicated that a language model non-blocking Ramadge-Wonham type supervisor could be constructed that gives $K_1^m := (ab_2c)^*$ as the closed-loop recognized language. We also noted that the closed-loop system is unsatisfactory since deadlock can occur. Let us determine whether a PSC-based supervisor can impose the specification K_1^m without permitting deadlock. The augmented plant $\mathcal{P}^{\Sigma-A}$ is shown in Figure 3(a) and has generated language given by

$$L(\mathcal{P}^{\Sigma-A}) = pr[[b_2^*a(b_2^*b_1 + b_2)b_2^*c]^*b_2^*ab_1b_2^*db_2^*].$$

It is straightforward to verify that K_1^m satisfies the controllability and relative-closure conditions of Theorem 5. Thus, it follows from Theorem 5 that the non-marking supervisor



(a) augmented plant

(b) closed-loop system 1

(c) supervisor 2

(d) closed-loop system 2

Figure 3: Diagram illustrating Example 3

(S_1, S_1), $S_1 := det(pr(K_1^m))$, is language model non-blocking and imposes K_1^m as the closed-loop recognized language.

To determine whether there is any deterministic non-marking and *trajectory model non-blocking* supervisor that can impose K_1^m, we must check the trajectory-closure condition in Theorem 6. Let $\mathcal{S}_1$ denote the minimal deterministic state machine (with 3 states) with $L^m(\mathcal{S}_1) = L(\mathcal{S}_1) = pr(K_1^m)$. Let $\mathcal{Q}_1 := \mathcal{P}\,_A\|_B\, \mathcal{S}_1$ depicted in Figure 2(b). Letting (P^m, P) denote the trajectory model of $\mathcal{P}$, it follows from Theorem 3 that

$$P\,_A\|_B\, det(pr(K_1^m)) = T(\mathcal{Q}_1), \quad P^m\,_A\|_B\, det(pr(K_1^m)) = T^m(\mathcal{Q}_1).$$

Since $e := \emptyset(a, \emptyset)(b_2, \{c\}) \in T(\mathcal{Q}_1) - pr(T^m(\mathcal{Q}_1))$, the trajectory-closure condition fails to hold. Thus, K_1^m cannot be imposed without trajectory model blocking.

Consider the alternative specification $K_2^m := (ab_2(\epsilon + b_1)c)^*$, which also satisfies the controllability and relative-closure conditions of Theorem 5. Let $\mathcal{S}_2$ denote the deterministic state machine with $L^m(\mathcal{S}_2) = L(\mathcal{S}_2) = pr(K_2^m)$ depicted in Figure 2(c). The resulting closed-loop system $\mathcal{Q}_2 := P\,_A\|_B\, \mathcal{S}_2$ is shown in Figure 2(d). Since $T(\mathcal{Q}_2) = pr(T^m(\mathcal{Q}_2))$, the trajectory-closure condition holds. Thus, the non-marking deterministic supervisor (S_2, S_2), $S_2 := det(pr(K_2^m))$, is trajectory model non-blocking and imposes K_2^m as the closed-loop recognized language.

The supervisor implements the following control strategy: When a part arrives, the supervisor requests that it be input with the guides set to narrow. If the part happens to be narrow, the plant accepts this request and executes the event b_2. However, if the part is wide, the plant refuses to execute the event and the transition on b_2 occurs only in the supervisor. The part is then input with the guides set to wide. By following this strategy, a narrow part is never input with the guides set to wide, so jamming does not occur. Allowing for the possibility that a supervisor-initiated event may be refused by the plant is an essential feature of this control design.

Next we show that our approach is also suitable for modular supervisory control. It was shown in [16] that such a design results in significant computational advantage provided the specification languages are *non-conflicting.*

Definition 7 Given $K_1^m, K_2^m \subseteq \Sigma^*$, K_1^m and K_2^m are said to be *non-conflicting* if $pr(K_1^m) \cap pr(K_2^m) = pr(K_1^m \cap K_2^m)$.

Since $pr(K_1^m \cap K_2^m) \subseteq pr(K_1^m) \cap pr(K_2^m)$ for any $K_1^m, K_2^m \subseteq \Sigma^*$, K_1^m and K_2^m being non-conflicting is equivalent to $pr(K_1^m) \cap pr(K_2^m) \subseteq pr(K_1^m \cap K_2^m)$.

Theorem 7 Let (P^m, P) be the trajectory model of a plant, $A, B_1, B_2 \subseteq \Sigma$ be such that $A \cup B_1 = A \cup B_2 = \Sigma$, $K_1^m, K_2^m \subseteq L((P^m)^{\Sigma - A})$, and (S_1, S_1), (S_2, S_2) be the trajectory models of two non-marking and language model non-blocking supervisors such that $L(P^m\ {}_A\|_{B_1}\ S_1) = K_1^m$ and $L(P^m\ {}_A\|_{B_2}\ S_2) = K_2^m$.

1. The non-marking supervisor (S, S), where $S := S_1\ {}_{B_1}\|_{B_2}\ S_2$, is such that
$$L(P^m\ {}_A\|_{B_1 \cup B_2}\ S) = K_1^m \cap K_2^m.$$
2. (S, S) is language model non-blocking if and only if K_1^m and K_2^m are non-conflicting.
3. If (S_1, S_1) and (S_2, S_2) are deterministic and trajectory model non-blocking, then (S, S) is deterministic, and is trajectory model non-blocking if and only if
$$\begin{aligned} & pr[P^m\ {}_A\|_{B_1 \cup B_2}\ [det(pr(K_1^m))\ {}_{B_1}\|_{B_2}\ det(pr(K_2^m))]] \\ & \qquad = pr(P^m)\ {}_A\|_{B_1 \cup B_2}\ [det(pr(K_1^m))\ {}_{B_1}\|_{B_2}\ det(pr(K_2^m))]. \end{aligned} \tag{21}$$

Proof: We have the following series of equalities, which are obtained from one or more applications of one or more of these results: Corollaries 7 and 8, and Proposition 3.

$$\begin{aligned} K_1^m \cap K_2^m &= L(P^m\ {}_A\|_{B_1}\ S_1) \cap L(P^m\ {}_A\|_{B_2}\ S_2) \\ &= [L((P^m)^{\Sigma - A} \cap L(S_1^{\Sigma - B_1})] \cap [L((P^m)^{\Sigma - A}) \cap L(S_2^{\Sigma - B_2})] \\ &= [L((P^m)^{\Sigma - A}) \cap L(S_1^{\Sigma - B_1})] \cap L(S_2^{\Sigma - B_2}) \\ &= L(P^m\ {}_A\|_{B_1}\ S_1) \cap L(S_2^{\Sigma - B_2}) \\ &= L((P^m\ {}_A\|_{B_1}\ S_1)\ {}_{A \cup B_1}\|_{B_2}\ S_2) \\ &= L(P^m\ {}_A\|_{B_1 \cup B_2}\ (S_1\ {}_{B_1}\|_{B_2}\ S_2)) \\ &= L(P^m\ {}_A\|_{B_1 \cup B_2}\ S). \end{aligned} \tag{22}$$

Next, note that since (S_1, S_1) and (S_2, S_2) are both language model non-blocking, we have $pr(K_1^m) = L(P\,{}_A\|_{B_1}\, S_1)$ and $pr(K_2^m) = L(P\,{}_A\|_{B_2}\, S_2)$. Thus, it can be shown as above that

$$pr(K_1^m) \cap pr(K_2^m) = L(P\,{}_A\|_{B_1 \cup B_2}\, S). \tag{23}$$

Therefore, it follows from Equations 22 and 23 that (S, S) is language model non-blocking if and only if $pr(K_1^m \cap K_2^m) = pr(K_1^m) \cap pr(K_2^m)$, i.e., if and only if K_1^m and K_2^m are non-conflicting.

In order to show the final part, first note that since (S_1, S_1) and (S_2, S_2) are deterministic, (S, S) is deterministic. Next note that repeated applications of Proposition 4 and Corollary 8 yields

$$\begin{aligned} P\,{}_A\|_{B_1 \cup B_2}\, S &= (P\,{}_A\|_{B_1}\, S_1)\,{}_\Sigma\|_{B_2}\, S_2 \\ &= [P\,{}_A\|_{B_1}\, det(pr(K_1^m))]\,{}_\Sigma\|_{B_2}\, S_2 \\ &= (P\,{}_A\|_{B_2}\, S_2)\,{}_\Sigma\|_{B_1}\, det(pr(K_1^m)) \\ &= [P\,{}_A\|_{B_2}\, det(pr(K_2^m))]\,{}_\Sigma\|_{B_1}\, det(pr(K_1^m)) \\ &= P\,{}_A\|_{B_1 \cup B_2}\, [det(pr(K_1^m))\,{}_{B_1}\|_{B_2}\, det(pr(K_2^m))]. \end{aligned} \tag{24}$$

Similarly, we have

$$P^m\,{}_A\|_{B_1 \cup B_2}\, S = P^m\,{}_A\|_{B_1 \cup B_2}\, [det(pr(K_1^m))\,{}_{B_1}\|_{B_2}\, det(pr(K_2^m))]. \tag{25}$$

It follows from Equations 24 and 25 that (S, S) is trajectory model non-blocking if and only if

$$\begin{aligned} &P\,{}_A\|_{B_1 \cup B_2}\, [det(pr(K_1^m))\,{}_{B_1}\|_{B_2}\, det(pr(K_2^m))] \\ &\qquad = pr[P^m\,{}_A\|_{B_1 \cup B_2}\, [det(pr(K_1^m))\,{}_{B_1}\|_{B_2}\, det(pr(K_2^m))]]. \end{aligned} \tag{26}$$

Since (S_1, S_1) is non-marking, trajectory model non-blocking, and deterministic, it follows from Theorem 6 that the following trajectory-closure condition holds:

$$P\,{}_A\|_{B_1}\, det(pr(K_1^m)) = pr[P^m\,{}_A\|_{B_1}\, det(pr(K_1^m))].$$

Hence we have

$$\begin{aligned} &P\,{}_A\|_{B_1 \cup B_2}\, [det(pr(K_1^m))\,{}_{B_1}\|_{B_2}\, det(pr(K_2^m))] \\ &\qquad = [P\,{}_A\|_{B_1}\, det(pr(K_1^m))]\,{}_\Sigma\|_{B_2}\, det(pr(K_2^m)) \\ &\qquad = pr[P^m\,{}_A\|_{B_1}\, det(pr(K_1^m))]\,{}_\Sigma\|_{B_2}\, det(pr(K_2^m)) \\ &\qquad \subseteq [pr(P^m)\,{}_A\|_{B_1}\, det(pr(K_1^m))]\,{}_\Sigma\|_{B_2}\, det(pr(K_2^m)) \\ &\qquad = pr(P^m)\,{}_A\|_{B_1 \cup B_2}\, [det(pr(K_1^m))\,{}_{B_1}\|_{B_2}\, det(pr(K_2^m))]. \end{aligned} \tag{27}$$

The containment follows from facts that $P^m \subseteq pr(P^m)$ and $pr(P^m)\,{}_A\|_{B_1}\, det(pr(K_1^m))$ is prefix-closed. The final equality follows from Corollary 8 using the fact that $(pr(P^m), P)$ is a trajectory model. Since $pr(P^m) \subseteq P$, the reverse containment of Equation 27 always holds. Therefore, we have

$$\begin{aligned} P\ {}_A\|_{B_1 \cup B_2}\ &[det(pr(K_1^m))\ {}_{B_1}\|_{B_2}\ det(pr(K_2^m))] \\ &= pr(P^m)\ {}_A\|_{B_1 \cup B_2}\ [det(pr(K_1^m))\ {}_{B_1}\|_{B_2}\ det(pr(K_2^m))]. \end{aligned}$$

Hence Equation 26 holds if and only if Equation 21 holds. ∎

Remark 4 Since it is assumed that $A \cup B_1 = A \cup B_2 = \Sigma$, it follows that $\Sigma_d \subseteq B_1 \cap B_2$– i.e., every driven event belongs to the priority sets of both of the modular supervisors. Consequently, for a driven event to occur in the closed-loop system, it must occur in both S_1, S_2, and will occur in P if possible. Although we have not explicitly made the assumption that $\Sigma_c \subseteq B_1 \cup B_2$, this is a natural requirement, since events in $\Sigma - (B_1 \cup B_2)$ cannot be prevented by S. However, there is certainly no need to require that $\Sigma_c \subseteq B_1 \cap B_2$. Thus, for a controllable event to occur in the closed-loop system, it must occur in P and in at least one of the supervisors.

6 Conclusion

In this paper, we have extended our earlier work on supervisory control of nondeterministic systems to include markings so that the issue of blocking can be investigated. Since language model non-blocking is inadequate for certain nondeterministic systems, the stronger requirement of trajectory model non-blocking is introduced. Necessary and sufficient conditions are obtained for the existence of language model non-blocking as well as trajectory model non-blocking supervisors that meet given language specifications.

These necessary and sufficient conditions require that the tests of controllability, relative-closure and trajectory-closure be performed. In the case where the languages involved are regular, one way to perform the test of controllability/relative-closure of the desired language with respect to the generated language of the augmented (nondeterministic) plant is to construct a language-equivalent deterministic system and apply a known test for controllability/relative-closure. However, it is possible to perform these tests without having to do such a nondeterministic to deterministic conversion. Hence algorithms of polynomial complexity (polynomial in the number of states of the plant NSM and number of states in the recognizer of the desired language) for testing controllability/relative-closure can be obtained. A test for the trajectory-closure condition can be obtained in terms of *coaccessibility* [7] of the NSM obtained by SSC of the plant NSM and the deterministic generator of the closure of the desired language specification. Thus the computational complexity of testing the trajectory-closure condition is also polynomial.

We have also addressed the issue of modular supervisory control and shown that our approach is suitable for such designs. It can be shown that the conditions for the existence of such designs are also polynomially testable.

References

[1] S. Balemi, G. J. Hoffmann, P. Gyugyi, H. Wong-Toi, and G. F. Franklin. Supervisory control of a rapid thermal multiprocessor. Technical Report ISL/GFF/91-1, Department of Electrical Engineering, Stanford University, Stanford, CA 94305, November 1991.

[2] B. A. Brandin and W. M. Wonham. Supervisory control of timed discrete event systems. Technical Report CSGR 9210, University of Toronto, Toronto, Canada, 1992.

[3] C. H. Golaszewski and P. J. Ramadge. Control of discrete event processes with forced events. In *Proceedings of 26th IEEE Conference on Decision and Control*, pages 247–251, Los Angeles, CA, 1987.

[4] M. Heymann. Concurrency and discrete event control. *IEEE Control Systems Magazine*, 10(4):103–112, 1990.

[5] M. Heymann and G. Meyer. An algebra of discrete event processes. Technical Report NASA 102848, NASA Ames Research Center, Moffett Field, CA, June 1991.

[6] C. A. R. Hoare. *Communicating Sequential Processes.* Prentice Hall, Inc., Englewood Cliffs, NJ, 1985.

[7] J. E. Hopcroft and J. D. Ullman. *Introduction to Automata Theory, Languages and Computation.* Addison-Wesley, Reading, MA, 1979.

[8] K. M. Inan. An algebraic approach to supervisory control. *Mathematics of Control, Signals and Systems*, 5:151–164, 1992.

[9] K. M. Inan and P. P. Varaiya. Algebras of discrete event models. *Proceedings of the IEEE*, 77(1):24–38, 1989.

[10] R. Kumar, V. K. Garg, and S. I. Marcus. On controllability and normality of discrete event dynamical systems. *Systems and Control Letters*, 17(3):157–168, 1991.

[11] R. Milner. *A Calculus of Communicating Systems.* Springer Verlag, Berlin, 1980.

[12] I. Phillips. Refusal testing. *Theoretical Computer Science*, 50:241–284, 1987.

[13] P. J. Ramadge and W. M. Wonham. Supervisory control of a class of discrete event processes. *SIAM Journal of Control and Optimization*, 25(1):206–230, 1987.

[14] P. J. Ramadge and W. M. Wonham. The control of discrete event systems. *Proceedings of IEEE: Special Issue on Discrete Event Systems*, 77:81–98, 1989.

[15] M. A. Shayman and R. Kumar. Supervisory control of nondeterministic systems with driven events via prioritized synchronization and trajectory models. *SIAM Journal of Control and Optimization*, 1992. Accepted.

[16] W. M. Wonham and P. J. Ramadge. Modular supervisory control of discrete event systems. *Mathematics of Control Signals and Systems*, 1(1):13–30, 1988.