THESIS REPORT
Master's Degree

Institute for Systems Research

Systematic Methodologies for the Automatic Enumeration of the Topological Structures of Mechanisms

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The Institute for Systems Research is supported by the National Science Foundation Engineering Research Center Program (NSFD CD 8803012), Industry and the University

M.S. 93-1

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by

Hsin-I Hsieh

Thesis submitted to the Faculty of the Graduate School
of the University of Maryland in partial fulfillment
of the requirements for the degree of
Master of Science
1992

Advisory Committee:

Professor Lung-Wen Tsai, Chairman/Advisor
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Abstract

Title of Thesis: Systematic Methodologies for the Automatic Enumeration of the Topological Structures of Mechanisms

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Degree and year: Master of Science, 1992

Thesis directed by: Professor Lung-Wen Tsai
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This thesis proposes new algorithms for the enumeration of the topological structure of mechanisms. The definitions of dual graph and dual of a contracted graph are modified to provide a one-to-one correspondence between graphs. In this study, three efficient algorithms have been developed for automatic enumeration and structural representation of graphs.

The first method enumerates conventional graphs by deriving the vertex-to-vertex incident matrix directly. The second method derives conventional graphs from contracted graph families by the arrangement of binary chains. The row vector formed by listing of binary vertex chains is used instead of the vertex-to-vertex incident matrix. The third method uses the edge-to-vertex incident matrix as the expression of graphs instead of the vertex-to-vertex incident matrix. The

dual of a conventional graph is derived from the dual of a contracted graph by the arrangement of parallel edges. A conventional graph is formed from the dual graph by the following definition of a dual graph.

Two tables of conventional graphs with seven and eight vertices, and with up to eleven edges have been developed. We believe that the results of these conventional graphs are new. Mechanisms of higher pair joints with six loops and seven or eight links can now be synthesized from these tables.

Acknowledgements

This thesis has been the result of a joyful association with the Department of Mechanical Engineering and Institute of System Research at the University of Maryland. It is also dedicated to all those who have stimulated and encouraged me to undertake this work. First of all, I would like to express my sincere appreciation to Professor Lung-Wen Tsai, my thesis advisor, for his guidence, patience, and inspiration throughout this work. Special thanks are due to Drs. Berger and Magrab for their serving on my advisory committee, providing informative comments regarding my thesis. Appreciation is also given to Dr. Jyh-Jone Lee's initial effort in the development of the numerical method used in this thesis.

Most of all, I would like to dedicate this work to my family for guiding me through my studies. Without their immeasurable love and encourangement, this thesis would not have been possible.

Financial support for this work was provided by a U.S. Department of Energy grant (No. DEFG 05-88ER 13977).

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Nomenclature

F	Degrees of freedom of a mechanism
L	Number of loops in a mechanism, including the peripheral loop
L^c	Number of loops in a contracted graph, including the peripheral loop
L^d	Number of loops in a dual graph, including the peripheral loop
L^{dc}	Number of loops in the dual of a contracted graph
b_i	Number of binary-vertex chains with i vertices and $i+1$ edges
e	Number of edges in a conventional graph
e^c	Number of edges in a contracted graph
e^d	Number of edges in a dual graph
e^{dc}	Number of edges in the dual of a contracted graph
j	Number of joints in a mechanism
n	Number of links in a mechanism
v	Number of vertices in a conventional graph
v^c	Number of vertices in a contracted graph
v^d	Number of vertices in a dual graph
v^{dc}	Number of vertices in the dual of a contracted graph
v_i	Number of vertices of degree i

λ Motion parameter,

$\lambda = 3$ for planar or spherical motion, and

$\lambda = 6$ for spatial motion

Chapter 1

Introduction

1.1 Historical review

In 1964 graph theory was first applied for structural synthesis of mechanisms (Crossley, 1964a; Crossley, 1964b). The graph of a kinematic chain is obtained by representing each link by a vertex and each joint by an edge. Each edge connection between two vertices corresponds to a joint connection between two links. The advantages using graph representation are :

1. Network propertities of graphs are directly applicable.
2. It leads to automatic kinematic and dynamic analysis of mechanisms.
3. A single atlas of graphs can be used to enumerate a large class of mechanisms.
4. The structural topology of a mechanism can be uniquely identified.

Techniques for the identification and classification of the kinematic structures of mechanisms have attracted much attention for nearly three decades. (Crossley, 1964b; Crossley, 1964a; Crossley, 1965; Freudenstein, 1967; Buchsbaum and

Freudenstein, 1970; Freudenstein and Maki, 1979; Mruthyunjaya, 1979; Yan, 1980; Tuttle and Peterson, 1987; Tuttle and Titus, 1989a; Tuttle and Titus, 1989b; Hwang and Liu, 1991; Alhakim and Shrivastava, 1991). Some of the existing methods employed for the number synthesis of kinematic chains with simple joints are based on graph theory (Freudenstein, 1967; Woo, 1967; Huang and Soni, 1973; Sohn and Freudenstein, 1986; Tsai, 1987; Hsu, 1989).

Woo (1967) defined the contracted graph for the classification and enumeration of conventional graphs with ten vertices and thirteen edges. Mruthyunjaya (Mruthyunjaya, 1984a; Mruthyunjaya, 1984b; Mruthyunjaya, 1984c) developed a computer program, which is based on the method of the transformation of binary chains, for structural synthesis of kinematic chains. Sohn and Freudenstein (1986) introduced the concept of dual graphs, which were used to establish large categories of mechanism structures.

1.2 Motivation

Graphs are applied to represent the structural topologies of mechanisms. Conventional graphs, dual graphs and contracted graphs have been used for the enumeration of the topological structures of kinematic chains. However, the correspondences between these graphs are not unique. This study modifies the definition of dual graph and establishes a one-to-one correpsondence between the conventional graph and the dual graphs, and between a contracted graph and the dual of a contracted graph. The relationships between the graphs are also studied.

Although a few approaches for the enumeration of kinematic chains have been developed, existing methodologies tend to be complicated and inefficient. In this study, three systematic algorithms are suggested to improve computational efficiency.

1.3 Outline

In Chapter Two basic definitions of a conventional graph, a contracted graph, dual graph, and the dual of a contracted graph will be established. The correspondences between these graph representations will also be described.

In Chapter Three a systematic procedure for the enumeration of contracted graphs will be presented. Using contracted graphs as the data base, three systematic procedures for the enumeration of conventional graphs will be given in Chapter Four.

The first method in Chapter Four directly enumerates the vertex-to-vertex incident matrices of conventional graphs. The second method derives conventional graphs by the arrangement of binary chains on contracted graphs. The third method constructs conventional graphs from the duals of contracted graphs. An edge-to-vertex incident matrix is introduced for the graph representation instead of a vertex-to-vertex incident matrix. No permutations are needed in this method. These procedures use the one-to-one correspondence between the contracted graph and the dual of a contracted graph, and between the conventional graph and its dual defined in Chapter Two. These methodologies identify all the admissible graphs of a given specification. It is very straightforward and involves no guess work.

In Appendices D and E two tables of conventional graphs with seven and eight vertices are developed and tabulated. These tables can be used to generate mechanisms with up to six loops and eight links.

Chapter 2

Definitions

In this chapter the basic definitions of a conventional graph, a contracted graph, a dual graph, and the dual of a contracted graph are defined. In addition, the correspondences between these graphs are stated. Using these definitions new procedures for the enumeration of contracted graphs and conventional graphs will be outlined in Chapters Three and Four, respectively.

2.1 Conventional Graphs

A graph is a collection of vertices joined by edges. In the conventional graph representation of a mechanism, vertices represent links and the edge-connection between vertices corresponds to the pair connection between links. From this definition, it is clear that the number of vertices is equal to the number of links, and the number of edges is equal to the number of joints, i.e.,

$$v = n \tag{2.1}$$

$$e = j \tag{2.2}$$

The relationship between the number of loops, the number of vertices and the number of edges is given by Euler's theorem (Harary, 1969):

$$L = e - v + 2 \tag{2.3}$$

The *degree* of a vertex in a graph is defined as the number of edges incident with that vertex (Harary, 1969). Let v_i be the number of vertices with degree i. Then v_i is related to v and e as (Woo, 1967):

$$\sum_{i=1}^{n} v_i = v \tag{2.4}$$

$$\sum_{i=1}^{n} i\ v_i = 2e \tag{2.5}$$

The planar representation of a graph divides the plane into several connected regions called *faces* or *loops.* Each loop is bounded by several edges of the graph and it shall not contain any other vertices and/or edges within the region. For example, shown Fig. 2.1 is a mechanism with ten links and thirteen joints. Fig. 2.2 shows the graph representation of the mechanism. In the graph of Fig. 2.2, the region bounded by the path 3-9-4-10-3 is a loop. However, the region bounded by the path 1-5-2-8-7-1 will not be considered a loop, since it contains vertex 6 and two edges within the region. The region external to the graph is called the *peripheral loop.* One may think of the peripheral loop as a loop that encloses no other vertices outside of the loop. The peripheral loop shown in Fig. 2.2 is formed by the path 1-3-9-4-2-8-7-1. Including the peripheral loop, the graph shown in Fig. 2.2 contains five loops designated L_1 to L_5. According to graph theory, any loop can be transformed into the peripheral loop using a stereographic projection (Gibbsons, 1985).

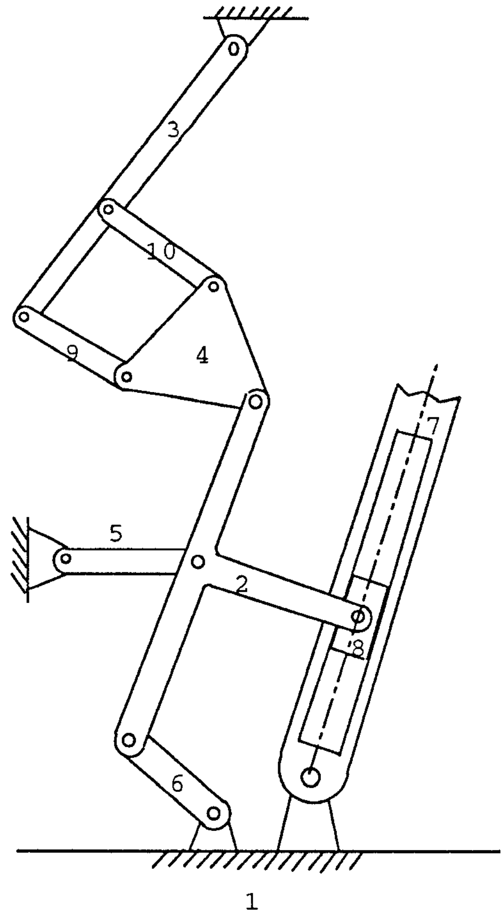


Figure 2.1: A mechanism with ten links and thirteen joints.

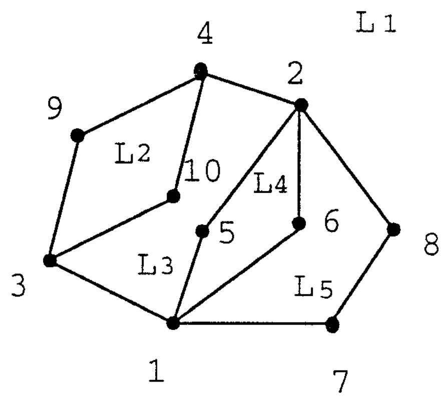


Figure 2.2: (10,13) Conventional graph of Fig. 2.1

The *V-V (vertex-to-vertex) incident matrix* is defined as an n by n matrix with elements $A_{i,j}$ defined as follows:

$$A_{i,j} = \begin{cases} 1, & \text{if vertex } i \text{ is connected to vertex } j \\ 0, & \text{otherwise (including } i = j) \end{cases}$$

Therefore the V-V incident matrix is symmetric with all the diagonal elements set to zero. The V-V incident matrix of a conventional graph is called the "A" matrix. For example, the V-V incident matrix A of the graph shown in Fig. 2.2 is given by

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.6}$$

A conventional graph with v vertices and e edges will be denoted as a (v, e) graph. A *self-loop* is an edge that connects a vertex to itself. Two edges are said to be *parallel* if they connect the same end-points. In a conventional graph representation of mechanism, self-loops and parallel edges are not permitted. A graph is planar if it can be drawn on a plane with no two edges crossing each other and with the edges drawn by straight lines. In this thesis, we shall concern ourselves with only those graphs which are planar with no articulation points or cut edges.

2.2 Contracted Graph

A *binary-vertex chain* in a conventional graph is defined as the maximum possible sequence of alternating edges and vertices of degree two, starting and ending with an edge. Let S_i, S_{k1}, S_{k2}, ..., S_{km}, S_j, be a series of vertices in a conventional graph in which vertex S_i is connected to S_{k1}, S_{k1} to S_{k2}, ..., and S_{km} to S_j by one and only one edge, respectively. If S_{k1}, S_{k2}, ..., S_{km} are all binary vertices, but S_i and S_j are not binary, then the sequence of alternating edges and vertices, $-S_{k1}-S_{k2}-\cdots-S_{km}-$, is called a *binary-vertex chain.* The beginning and ending edges of a binary-vertex chain must be connected to vertices of degree greater than two. The 4–9 and 9–3 edges together with vertex 9 shown in Fig. 2.2 is a binary-vertex chain. The 2–8, 8–7 and 7–1 edges together with vertices 8 and 7 form another binary-vertex chain. Let b_m denote the number of binary-vertex chains with m vertices and $m+1$ edges and let b_0 denote the number of single edges connecting vertices of degree greater than two. Then, in the graph shown in Fig. 2.2, b_0=2, b_1=4, and b_2=1.

A *contracted graph* is a graph derived from a conventional graph by replacing each of the binary-vertex chains with a single edge. Parallel edges may exist in a contracted graph.

From the above definition it is clear that the number of vertices, edges and loops in a contracted graph are related to that of the corresponding conventional graph by the following equations.

$$v_2 = \sum_{i=1}^{m} i\ b_i \tag{2.7}$$

$$v^c = v - v_2 \tag{2.8}$$

$$e^c = e - v_2 \tag{2.9}$$

$$L^c = e^c - v^c + 2 = L \tag{2.10}$$

For example, the contracted graph of Fig. 2.2 has four vertices and seven edges as shown in Fig. 2.3. In the contracted graph the binary-vertex chains (–9–), (–10–), (–5–), (–6–), and (–8–7–) are replaced with single edges shown in Fig. 2.3. By contracting, the loops L_2, L_4 and L_5 become two-edged loops. There are two parallel edges connecting vertices 3 and 4, and three parallel edges connecting vertices 1 and 2.

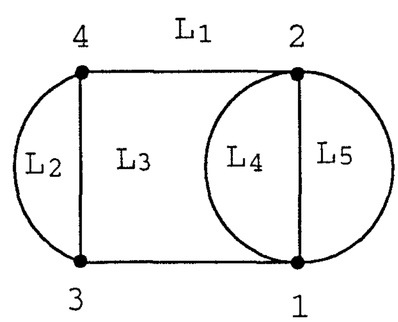


Figure 2.3: Contracted graph of Fig. 2.2

The V-V incident matrix of a contracted graph, called the "C" matrix, is a symmetric matrix of order v^c in which all the diagonal elements are set to zero, and a nonzero element $C_{i,j}$ denotes the number of parallel edges connecting vertices i and j. The V-V incident matrix C for the contracted graph shown in Fig. 2.3 is

$$C = \begin{pmatrix} 0 & 3 & 1 & 0 \\ 3 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 \end{pmatrix} \tag{2.11}$$

The minimal degree of a vertex in a contracted graph is three. Hence, the sum of each row (or column) in matrix C must be equal to or greater than three.

2.3 Dual Graphs

The dual graph of a conventional graph is a graph whose vertices represent loops of a conventional graph. The number of parallel edges connecting any pair of vertices in the dual graph represents the number of edges dividing the two corresponding loops in the conventional graph. The loops in a dual graph correspond to the vertices of the conventional graph. From the above definition it is clear that the number of vertices, edges and loops in a dual graph are related to those in the corresponding conventional graph as follows:

$$v^d = L \tag{2.12}$$

$$e^d = e \tag{2.13}$$

$$L^d = v \tag{2.14}$$

The degree of a vertex in a conventional graph corresponds to the number of edges in the corresponding loop of the dual graph. Therefore, the positions of binary-vertex chains in the dual graph cannot be interchanged arbitrarily. Using the above definition, a one-to-one correspondence between the dual graph and the conventional graph is established.

Define L_i as the ith vertex in a dual graph corresponding to the ith loop in the conventional graph. Following the definition of a dual graph, if loops i and j in a conventional graph have a binary-vertex chain with k edges as their boundary, then vertices L_i and L_j in the corresponding dual graph are connected by k parallel edges, which are adjacent to each other; if the edges dividing loops i and j in a conventional graph are not connected in a series, then the parallel edges in the corresponding dual graph will not be drawn adjacent to each other, i.e., they must be divided by other interconnected loops; and if vertex i in a conventional graph is the common vertex of loops L_a, L_b and L_c, then the loop formed by vertices L_a-L_b-L_c-L_a in the dual graph shall be labeled as loop i.

For example, Fig. 2.4 shows the dual graph of the conventional graph shown in Fig. 2.2. The edges dividing loops 1 and 3 in the conventional graph are not connected to each other. In the corresponding dual graph (Fig. 2.4) the two parallel edges connecting vertices L_1 and L_3 are not adjacent to each other and are separated by vertex L_2. Vertices L_1 and L_5 in the graph shown in Fig. 2.4 have three parallel edges, which correspond to the three common edges between loops 1 and 5 in Fig. 2.2. Vertex 1 in Fig. 2.2 is the common vertex of loops 1, 3, 4 and 5. Therefore the loop formed by vertices L_1-L_3-L_4-L_5-L_1 in Fig. 2.4 is labeled as 1.

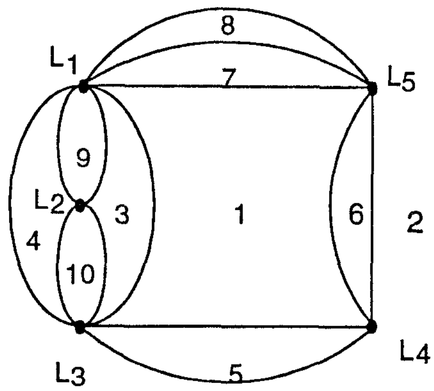


Figure 2.4: Dual graph of Fig. 2.2

$$D = \begin{pmatrix} 0 & 2 & 2 & 0 & 3 \\ 2 & 0 & 2 & 0 & 0 \\ 2 & 2 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 2 \\ 3 & 0 & 0 & 2 & 0 \end{pmatrix} \tag{2.15}$$

The V-V incident matrix of a dual graph, called the "D" matrix, is a symmetric matrix of order L in which all the diagonal elements are zero. The nonzero element $D_{i,j}$ denotes the number of edges connecting vertices i and j. The incident matrix of the dual graph shown in Fig. 2.4 is given by Eq. (2.15).

This definition of dual graph is different from that reported in Sohn and Freudenstein (1986). In this definition, parallel edges are permitted. A binary-vertex chain is transformed into a set of parallel edges which forms several two-edged loops in the dual graph.

2.4 Duals of Contracted Graphs

The *dual of a contracted graph* is obtained by taking the dual graph of a contracted graph. Following the definition of a dual graph, it can be shown that the number of vertices, edges and loops are related to that of the corresponding contracted graph by the following equations.

$$v^{dc} = L \tag{2.16}$$

$$e^{dc} = e^{c} \tag{2.17}$$

$$L^{dc} = v^{c} = e^{c} - L + 2 \tag{2.18}$$

For example, the contracted graph shown in Fig. 2.3 has four vertices, seven edges and five loops. The dual of this contracted graph shown in Fig. 2.5 has five vertices, seven edges and four loops. The number of vertices in Fig. 2.5 is equal to the number of loops in Fig. 2.3 . The number of edges of both graphs is the same.

The dual of a contracted graph can also be obtained by contracting the dual graph, i.e., by replacing all the adjacent parallel edges with a single edge. However, when two parallel edges are separated by other vertex-chains, it shall not be replaced with a single edge in order to retain the loop information in a contracted graph.

For example, the three parallel edges connecting vertices L_1 and L_5 in Fig. 2.4 are replaced by a single edge, while the two parallel edges connecting vertices L_1 and L_3 are not replaced by a single edge as shown in Fig. 2.5.

Using this definition, the degree of a vertex in the conventional graph becomes the number of edges in the corresponding loop of the dual of a contracted graph.

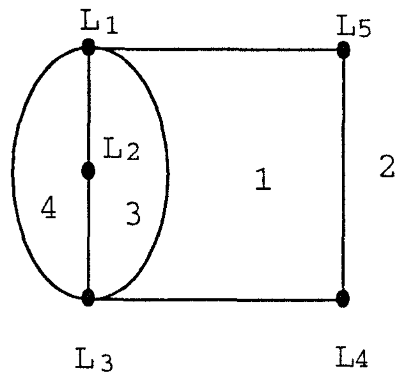


Figure 2.5: Dual of contracted graph of Fig. 2.3

The arrangement of edges and binary-vertex chains for loop boundaries is crucial and cannot be arbitrarily exchanged. In what follows we denote the *vertex-degree listing* as a listing of the degree associated with the vertices of a graph, and *the loop-edge listing* as a listing of the numbers of edges associated with the loops in descending order.

Fig. 2.6 shows three completely different contracted graphs and their corresponding duals. However, the three duals have identical connectivity between their vertices and edges, i.e., they all share the same vertex degree listings, $(L_1,L_3,L_4,L_5,L_2,L_6)$=(5,3,3,3,2,2), and the same V-V incident matrix.

As a matter of fact, the three duals are isomorphic with one another under the definition of graph isomorphism. Two graphs are *isomorphic* if there exists a one-to-one correspondence between their point sets which preserves adjacency (Harary, 1969). Two isomorphic graphs may have different matrices which represents the adjacency of vertices. However, they will be the same after rearranging the columns and rows of one of the adjacency matrices. The dual of a contracted graph shown in Fig. 2.6(b) can be obtained by placing the binary-vertex chain associated with vertex L_2 in Fig. 2.6(a) inside loop 3. The

Figure 2.6	Vertex-degree listing of the contracted graphs	Loop-edge listing in the duals of contracted graphs
a	V: 6/3/3/3/3	L: 6/3/3/3/3
b	V: 5/4/3/3/3	L: 5/4/3/3/3
c	V: 4/4/4/3/3	L: 4/4/4/3/3

Table 2.1: The vertex-degree listing and loop-edge listing

dual of a contracted graph shown in Fig. 2.6(c) can be obtained by placing the binary-vertex chain associated with vertex L_6 in Fig. 2.6(b) inside loop 4. Although these three duals have identical vertex-to-vertex connectivity, their planar embeddings are different from one another which results in different loop-edge listings. As defined in Section 2.1, the loop-edge listings associated with the dual graphs of Fig. 2.6(a)-(c) are (6,3,3,3,3), (5,4,3,3,3), and (4,4,4,3,3), respectively. In this thesis these three duals are considered as non-isomorphic graphs. Note that the loop-edge listing of the dual graph corresponds to the vertex-degree listing of the contracted graph. See Table 2.1.

The V-V incident matrix T of the dual of a contracted graph is a symmetric matrix of order L^{dc} in which all diagonal elements are set to zero. The off diagonal element $T_{i,j}$ is defined as follows.

$$T_{i,j} = \begin{cases} 1, & \text{if vertex } i \text{ is connected to vertex } j \text{ by a single edge;} \\ m \ (m > 1), & \text{if vertex } i \text{ is connected to vertex } j \text{ by m non-adjacent} \\ & \text{parallel edges; and} \\ 0, & \text{otherwise} \end{cases}$$

For example, the incident matrix for the dual of a contracted graph shown

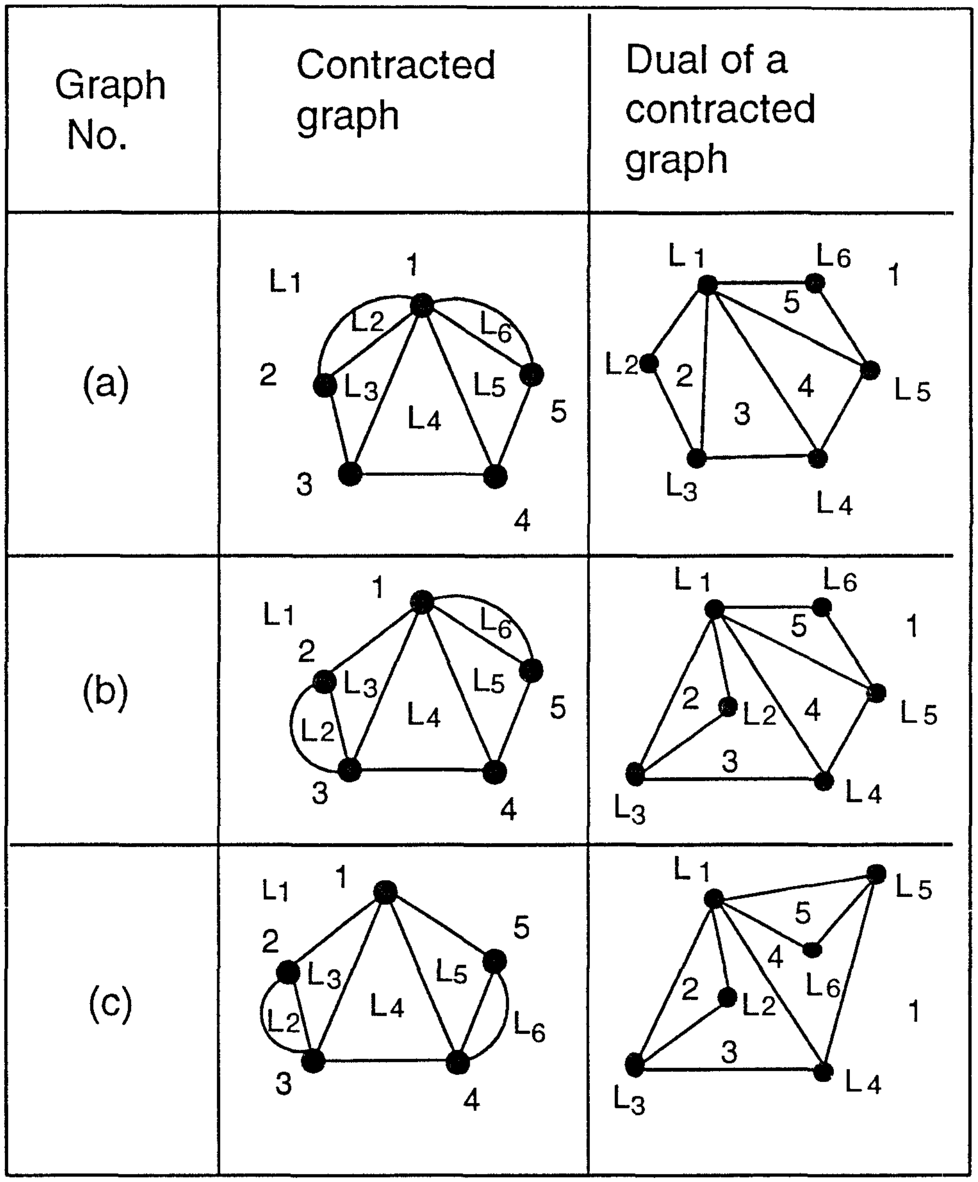


Figure 2.6: Three non-isomorphic contracted graphs and their duals

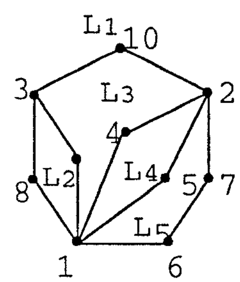
in Fig. 2.5 is

$$T = \begin{pmatrix} 0 & 1 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{2.19}$$

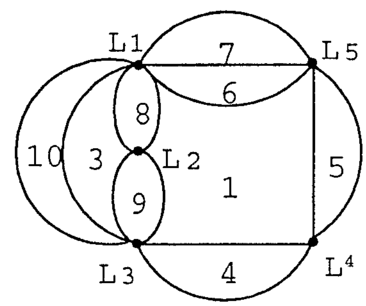
2.5 Correspondences Between Graphs

It is denoted that the definitions of a dual graph and the dual of a contracted graph are different from those defined by Sohn and Freudenstein (1986). In the dual graph of a mechanism defined by Sohn and Freudenstein (1986), the vertices represent the loops of the conventional graph, and an edge connecting each pair of vertices in a dual graph corresponds to the one or more common edges between two loops. These new definitions lead to one-to-one correspondence between the conventional graph representation of a mechanism and its dual, and between the contracted graph and its dual. In what follows, the correspondences between these graphs will be summarized. Using the new definition of dual graph, not only are the vertex-to-vertex and edge-to-vertex incident informations preserved, but also the loop-to-edge and loop-to-loop incident informations are preserved.

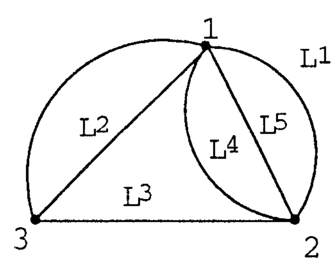
There is a one-to-one correspondence between a conventional graph and its dual graph. Fig 2.7(a) shows another conventional graph that is non-isomorphic with that of Fig. 2.2. The dual graphs shown in Figs. 2.4 and 2.7(b) have identical vertex-to-vertex and edge-to-vertex incident matrices. However, the loop-to-loop and edge-to-loop incident matrices are different. Note that the loops in the dual graphs shown in Figs. 2.2 and 2.7(b) are denoted by numerals from 1 to 10, which



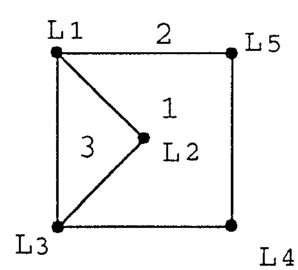
(a) Conventional graph



(b) Dual graph



(c) Contracted graph



(d) Dual of contracted graph

Figure 2.7: The conventional graph and its corresponding graphs

correspond to the vertices in their corresponding conventional graphs shown in Figs. 2.4 and 2.7(a), respectively. The two edges dividing loops 1 and 3 in the conventional graph shown in Fig 2.7(a) are connected in series. Thus, the two parallel edges connecting vertices L_1 and L_3 in Fig. 2.7(b) form a two-edged loop, 10. On the other hand, the two edges dividing loops 1 and 3 in the conventional graph shown in Fig. 2.2 are disconnected. Therefore the two parallel edges connecting vertices L_1 and L_3 in the corresponding dual graph shown in Fig. 2.4 do not form a two-edged loop. Instead, they are separated by other vertex-edge chains. When the loop-to-loop and loop-to-edge information are also preserved, there exists a one-to-one correspondence between a conventional graph and its dual graph.

The contracted graph of Fig. 2.7(a) and its dual graph are shown in Figs. 2.7(c) and 2.7(d), respectively. Note that the contracted graph and the dual of a contracted graph shown in Figs. 2.7(c) and 2.7(d) are clearly different from those shown in Figs. 2.3 and 2.5, respectively. There is also a one-to-one correspondence between the contracted graph and its dual.

Using Sohn and Freudenstein's definition of dual graph (1986), the two graphs shown in Fig. 2.2 and 2.7(a) yield the identical dual graph shown in Fig. 2.8, where the numerals on the edges identify the number of common edges between two adjacent loops in the conventional graph. Hence, there is no one-to-one correspondence between the conventional graph and its dual graph.

The dual of a contracted graph can also be formed by replacing every set of adjacent parallel-edges in the dual graph by a single edge. It is similar to the definition of a contracted graph, which is formed by replacing every binary-vertex chain in a conventional graph by a single edge. The non-adjacent parallel edgess

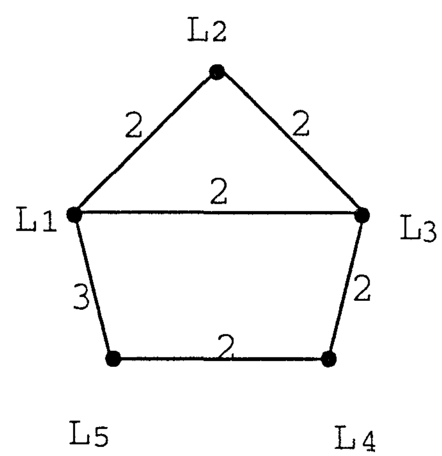


Figure 2.8: Dual graph of Fig 2.2 and 2.7(a) as defined in Sohn and Freudenstein (1986)

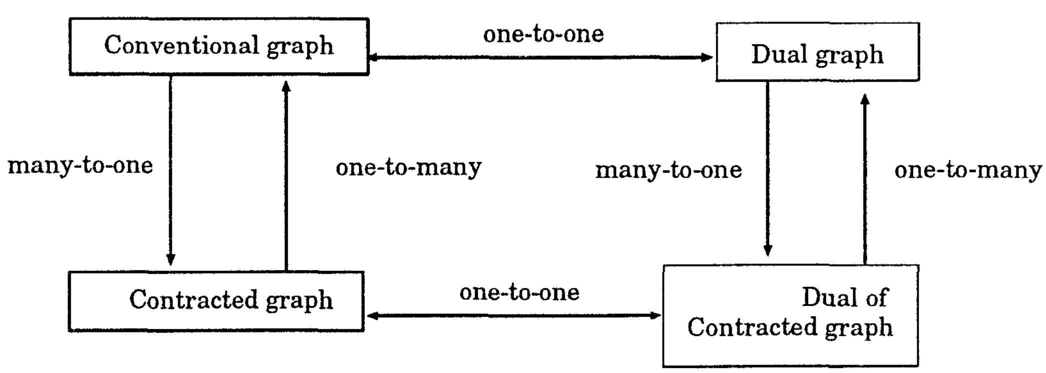


Figure 2.9: Correspondences between graphs

Type of graph	Number of vertices	Number of edges	Number of loops
Conventional graph	v	e	L
Dual graph	L	e	v
Contracted graph	$v - v_2$	$e - v_2$	L
Dual of contracted graph	L	$e - v_2$	$v - v_2$

Table 2.2: The number of vertices and edges of graphs

in a dual graph are preserved. Many different dual graphs can be contracted into the same dual of a contracted graph. On the other hand, one dual of contracted graph can produce many dual graphs by expansion of the edges. Hence, there is no one-to-one correspondence between a dual graph and the dual of a contracted graph. One may consider the dual of a conventional graph as a labeled dual of contracted graph. Replacing an edge in a dual of contracted graph by parallel edgess results in different dual graphs.

Replacing all binary-vertex chains in a conventional graph by single edges forms a contracted graph. Therefore, several non-isomorphic conventional graphs can be reduced to one contracted graph. On the other hand, a contracted graph can be expanded into several conventional non-isomorphic graphs by replacing its single edges with binary-vertex chains. Consequently, there is no one-to-one correspondence between a contracted graph and a conventional graph. One important fact is that conventional graphs with different corresponding contracted graphs will not be isomorphic. This will be discussed later in Section 4.4 describing isomorphism between graphs. The correspondences between these graphs are summarized in Fig. 2.9. The correspondences among the numbers of vertices, edges and loops are summarized in Table 2.2.

Chapter 3

Procedure for the Enumeration of Contracted Graphs and Their Duals

In this chapter a procedure for the enumeration of contracted graphs will be presented. Although several methodologies have been proposed previously (Woo, 1967; Hsu, 1989), the procedure described in this chapter represents a new and efficient method of achieving the same purpose.

3.1 Procedure for Enumeration of Contracted Graphs

Recall that the V-V incident matrix of a contracted graph is a symmetric matrix in which all the diagonal elements are set to zero. The off-diagonal element $C_{i,j}$ denotes the number of parallel edges connecting vertices i and j. Therefore, the summation of the elements in the ith row represents the degrees of the ith vertex. Let $n = v^c$ be the number of vertices, e^c be the number of edges, and D_i be the degree of the ith vertex in a contracted graph. Then there exist v^c equations relating the elements of the incident matrix and the degree of vertices as follows:

$$\begin{array}{ccccccccccccc} 0 & + & C_{1,2} & + & C_{1,3} & + & \cdots & + & C_{1,n} & = & D_1 \\ C_{1,2} & + & 0 & + & C_{2,3} & + & \cdots & + & C_{2,n} & = & D_2 \\ \vdots & & & \ddots & & & & & & \vdots & \\ C_{1,n-2} & + & \cdots & + & 0 & + & C_{n-2,n-1} & + & C_{n-2,n} & = & D_{n-2} \\ C_{1,n-1} & + & \cdots & + & \cdots & + & 0 & + & C_{n-1,n} & = & D_{n-1} \\ C_{1,n} & + & \cdots & + & \cdots & + & C_{n-1,n} & + & 0 & = & D_n \end{array} \tag{3.1}$$

where $C_{i,j}$ and D_i are non-negative integers. In a contracted graph, $D_i \geq 3$ for all the vertices. Since each edge has two end vertices, the summation of all elements of the matrix is equal to two times the number of edges, i.e.,

$$\sum_{i=1}^{v^c} \sum_{j=1}^{v^c} C_{i,j} = 2\ e^c \tag{3.2}$$

which means that

$$\sum_{i=1}^{v^c} D_i = 2\ e^c \tag{3.3}$$

Given v^c and e^c, all admissible non-isomorphic contracted graphs can be derived by solving Eqs. (3.1) and (3.3). First Eq. (3.3) is used to obtain D_i. Solving Eq. (3.3) amounts to partitioning the $2e^c$ number of ones into the corresponding D_i. Using the method described in Appendix A, all partitions are obtained. For each set of D_i, Eq. (3.1) is used to obtain $C_{i,j}$. It is more convenient to arrange Eq. (3.1) in the ascending (or descending) order of D_i to simplify the programming. Due to symmetry and the null diagonal elements of the matrix, there are only $n(n-1)/2$ unknown variables.

A procedure for solving Eq. (3.1) will now be described. Note that the first equation in Eq. (3.1) contains $(n-1)$ unknowns. Hence, these $(n-1)$ unknowns can be solved for using the procedure outlined in Appendix A. Once $C_{1,j}$, j=2,3,...,n, are obtained, one substitutes $C_{1,2}$ into the second equation in Eq. (3.1) and solves for $C_{2,j}$, j=3,4,...,n. The procedure continues until only three unknown variables $C_{n-2,n-1}$, $C_{n-2,n}$, and $C_{n-1,n}$ are left in the last three equations in Eq. (3.1). These last three equations can be solved by using Gauss elimination. The procedure is summarized as follows:

1. Given v^c and e^c, solve Eq. (3.3) for D_i using the method outlined in Appendix A.

2. For each set of D_i, solve the first equation in Eq. (3.1) for $C_{1,j}$, j=2,3,..., n.

3. Similarly, substitute the solution of $C_{i,j}$, i=2 initially and j=1,...,$i+1$, into the $i+1$th equation and solve the resulting equation for the variables, $C_{i+1,j}$, $j = i+2,\ i+3,\ldots,n$.

4. Continue this procedure until $i = n-3$.

5. Substitute all the known $C_{i,j}$ into the last three equations in Eq. (3.1), and solve them for the remaining three variables.

6. Check if all the $C_{i,j}$ are non-negative integers.

7. Check for the existence of articulation points.

8. Check for graph isomorphism.

9. Repeat steps 2 to 9 until all the possible partitions of D_i are exhausted.

3.2 Examples

Problem statement: Derive all sets of contracted graphs with five loops.

Solution: There are five sets of contracted graphs, (v^c, e^c)= (2,5), (3,6), (4,7), (5,8) and (6,9), as a result of the following two restrictions.

1. The number of vertices and the number of edges are related to the number of loops by Euler's formula: $L = e^c - v^c + 2$.

2. Given the number of loops, the maximum number of edges is given by $e^c_{max} = 3(L-2)$, which occurs when all the loops are triangular loops.

For the purpose of illustration some (4,7) and (5,8) contracted graphs will be derived.

3.2.1 Enumeration of (4,7) Contracted Graphs

For $(v^c, e^c) = (4,7)$, Eq. (3.1) reduces to:

$$
\begin{aligned}
0 + C_{1,2} + C_{1,3} + C_{1,4} &= D_1 && (3.4)\\
C_{1,2} + 0 + C_{2,3} + C_{2,4} &= D_2 && (3.5)\\
C_{1,3} + C_{2,3} + 0 + C_{3,4} &= D_3 && (3.6)\\
C_{1,4} + C_{2,4} + C_{3,4} + 0 &= D_4 && (3.7)
\end{aligned}
$$

and Eq. (3.3) reduces to:

$$D_1 + D_2 + D_3 + D_4 = 2 \times 7 = 14 \qquad (3.8)$$

But $D_i \geq 3$, for i=1 to 4. Therefore, without loss of generality one may let $D_{i+1} \geq D_i$, for i=1,2, and 3. Using the method outlined in Appendix A, two solutions are obtained:

D_1	D_2	D_3	D_4
3	3	3	5
3	3	4	4

Case (a) For the first set, $(D_1, D_2, D_3, D_4) = (3,3,3,5)$, and Eqs. (3.4)-(3.7) become

$$0 + C_{1,2} + C_{1,3} + C_{1,4} = 3 \tag{3.9}$$

$$C_{1,2} + 0 + C_{2,3} + C_{2,4} = 3 \tag{3.10}$$

$$C_{1,3} + C_{2,3} + 0 + C_{3,4} = 3 \tag{3.11}$$

$$C_{1,4} + C_{2,4} + C_{3,4} + 0 = 5 \tag{3.12}$$

Solving Eq. (3.9) we obtain the following solution sets for $C_{1,2}$, $C_{1,3}$ and $C_{1,4}$.

$C_{1,2}$	$C_{1,3}$	$C_{1,4}$
0	0	3
0	1	2
0	3	0
1	0	1
1	1	2
1	2	0
3	0	0

Case (a.1) For $(C_{1,2}, C_{1,3}, C_{2,3}) = (0,0,3)$, Eqs. (3.10)-(3.12) reduce to

$$C_{2,3} + C_{2,4} = 3 \tag{3.13}$$

$$C_{2,3} + C_{3,4} = 3 \tag{3.14}$$

$$C_{2,4} + C_{3,4} = 2 \tag{3.15}$$

Solving Eqs. (3.13)-(3.15), we obtain $(C_{2,3}, C_{2,4}, C_{3,4}) = (2, 1, 1)$. Therefore, the V-V incident matrix for the contracted graph is

$$C = \begin{bmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 2 & 1 \\ 0 & 2 & 0 & 1 \\ 3 & 1 & 1 & 0 \end{bmatrix} \tag{3.16}$$

The resulting contracted graph is shown in Fig. 3.1. In this case, vertex 4 is an articulation point, i.e., the removal of vertex 4 results in a disconnected graph. Therefore, it is rejected from further consideration.

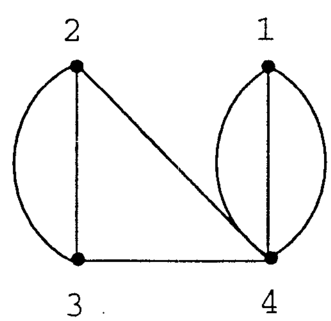


Figure 3.1: A (4,7) contracted graph with an articulation point

Case (a.2) For $(C_{1,2}, C_{1,3}, C_{1,4}) = (0, 1, 2)$, Eqs. (3.10)-(3.12) reduce to

$$C_{2,3} + C_{2,4} = 3 \tag{3.17}$$

$$C_{2,3} + C_{3,4} \quad = 2 \tag{3.18}$$

$$C_{2,4} + C_{3,4} \quad = 3 \tag{3.19}$$

Solving Eqs. (3.17)-(3.19), we obtain $(C_{2,3}, C_{2,4}, C_{3,4}) = (1, 2, 1)$. Hence, the V-V incident matrix is given by

$$C = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 2 & 2 & 1 & 0 \end{bmatrix} \tag{3.20}$$

The resulting contracted graph is shown in Fig. 3.2.

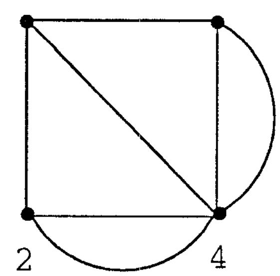


Figure 3.2: A (4,7) contracted graph without articulation point

Following the same process, the contracted graphs for all the other sets of $(C_{1,2}, C_{1,3}, C_{1,4})$ can be obtained.

Case (b) For the second set $(D_1, D_2, D_3, D_4) = (3, 3, 4, 4)$, Eqs. (3.4)-(3.7) become

$$0 + C_{1,2} + C_{1,3} + C_{1,4} \quad = 3 \tag{3.21}$$

$$C_{1,2} + 0 + C_{2,3} + C_{2,4} \quad = 3 \tag{3.22}$$

$$C_{1,3} + C_{2,3} + 0 + C_{3,4} \quad = 4 \tag{3.23}$$

$$C_{1,4} + C_{2,4} + C_{3,4} + 0 \quad = 4 \tag{3.24}$$

Again, we can solve Eq. (3.21) for $C_{1,2}$, $C_{1,3}$, and $C_{1,4}$ first and then solve Eqs. (3.22)-(3.24) for the remaining $C_{i,j}$.

3.2.2 Enumeration of (5,8) Contracted Graphs

For $(v^c, e^c) = (5, 8)$, Eq. (3.1) reduces to

$$
\begin{aligned}
0 + C_{1,2} + C_{1,3} + C_{1,4} + C_{15} &= D_1 & (3.25)\\
C_{1,2} + 0 + C_{2,3} + C_{2,4} + C_{,2,5} &= D_2 & (3.26)\\
C_{1,3} + C_{2,3} + 0 + C_{3,4} + C_{3,5} &= D_3 & (3.27)\\
C_{1,4} + C_{2,4} + C_{3,4} + 0 + C_{4,5} &= D_4 & (3.28)\\
C_{1,5} + C_{2,5} + C_{3,5} + C_{4,5} + 0 &= D_5 & (3.29)
\end{aligned}
$$

and Eq. (3.3) reduces to:

$$D_1 + D_2 + D_3 + D_4 + D_5 = 2 \times 8 = 16 \qquad (3.30)$$

Again, let $D_{i+1} \geq D_i \geq 3$ for i=1,2,3,4. Solving Eq. (3.30) yields $(D_1, D_2, D_3, D_4, D_5) = (3, 3, 3, 3, 4)$ as the only solution. Substituting D_1=3 into Eq. (3.25) yields $(C_{1,2}, C_{1,3}, C_{1,4}, C_{1,5}) = (2, 0, 1, 0)$ as one of the many possible solutions.

For $(C_{1,2}, C_{1,3}, C_{1,4}, C_{1,5}) = (2, 0, 1, 0)$, Eq. (3.26) becomes

$$C_{2,3} + C_{2,4} + C_{2,5} = D_2 - C_{1,2} = 1 \qquad (3.31)$$

Solving Eq. (3.31) results in three sets of solutions. One such solution is $(C_{2,3}, C_{2,4}, C_{2,5}) = (1, 0, 0)$. Substituting $(C_{1,2}, C_{1,3}, C_{1,4}, C_{1,5}, C_{2,3}, C_{2,4}, C_{2,5}) = (2, 0, 1, 0, 1, 0, 0)$ into Eqs. (3.27)-(3.29), yields

$$C_{3,4} + C_{3,5} = D_3 - C_{1,3} - C_{2,3} = 2 \tag{3.32}$$

$$C_{3,4} + C_{4,5} = D_4 - C_{14} - C_{2,4} = 2 \tag{3.33}$$

$$C_{3,5} + C_{4,5} = D_5 - C_{1,5} - C_{2,5} = 4 \tag{3.34}$$

Solving Eqs. (3.32)-(3.34) yields $(C_{3,4}, C_{3,5}, C_{4,5}) = (0, 2, 2)$. Therefore, the V-V incident matrix for the contracted graph is

$$C = \begin{bmatrix} 0 & 2 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 2 & 0 \end{bmatrix}$$

The resulting contracted graph is shown in Fig. 3.3. To derive all the (5,8) contracted graphs, steps 2 to 9 outlined in Section 3.1 are repeated.

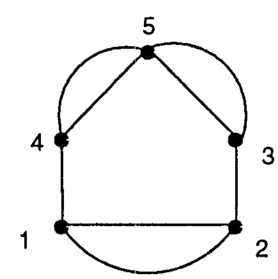


Figure 3.3: A (5,8) contracted graph

3.2.3 Enumeration of the Duals of Contracted Graphs

Applying the procedure outlined in Section 3.1, a computer program can be written for the systematic enumeration of contracted graphs. Contracted graphs with up to six loops have been previously enumerated by other researchers (Hsu, 1989). Although we believe this method presented in this study is more efficient and does not need any guess work, no attempt was made to recreate the contracted graphs. For the convenience of the reader, a set of contracted graphs with up to five loops is listed in Tables 3.1 and 3.2. Using the definition of dual graph, duals of the contracted graphs have also been sketched and listed in Tables 3.1 and 3.2. Appendix B lists contracted graphs with six loops and their duals.

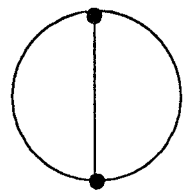
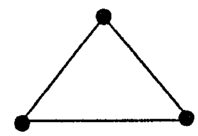
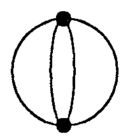
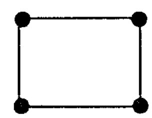
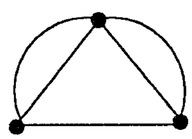
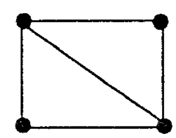
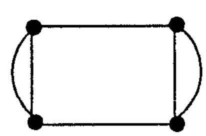
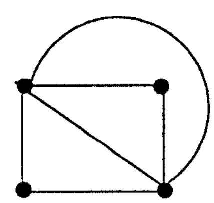
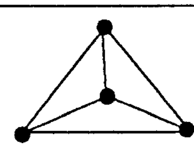
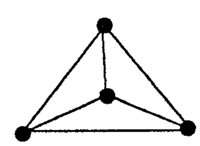
	Contracted graph	Dual of contracted graph
1	 (2,3)	 (3,3)
1	 (2,4)	 (4,4)
2	 (3,5)	 (4,5)
3	 (4,6)	 (4,5)
4	 (4,6)	 (4,6)

Table 3.1: Contracted graphs with three and four loops

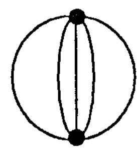
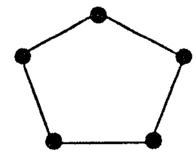
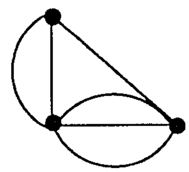
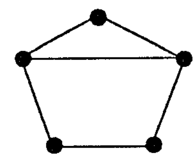
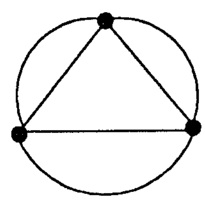
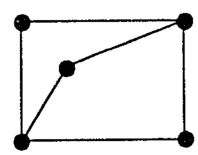
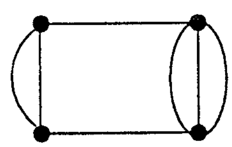
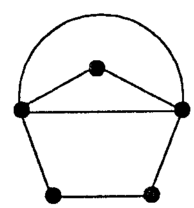
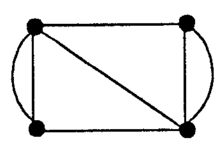
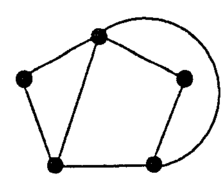
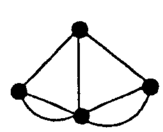
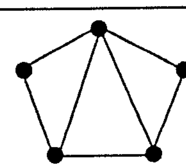
	Contracted graph	Dual of contracted graph
1	 (2,5)	 (5,5)
2	 (3,6)	 (5,6)
3	 (3,6)	 (5,6)
4	 (4,7)	 (5,7)
5	 (4,7)	 (5,7)
6	 (4,7)	 (5,7)

Table 3.2: Contracted graphs with five loops

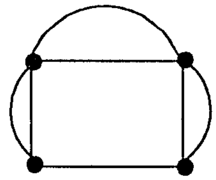
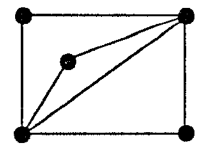
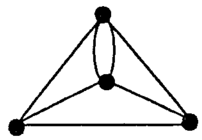
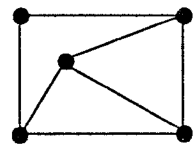
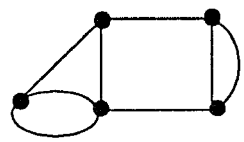
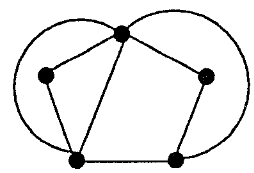
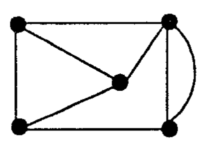
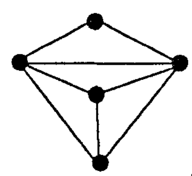
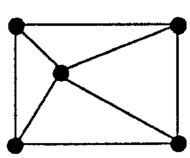
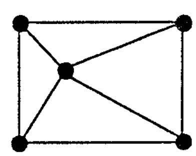
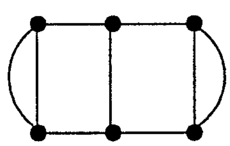
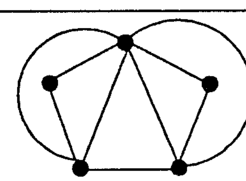
	Contracted graph	Dual of contracted graph
7	 (4,7)	 (5,7)
8	 (4,7)	 (5,7)
9	 (5,8)	 (5,8)
10	 (5,8)	 (5,8)
11	 (5,8)	 (5,8)
12	 (6,9)	 (5,9)

Table 3.2: Continued

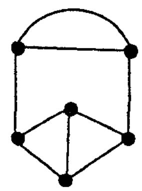
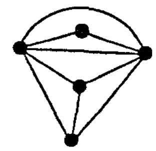
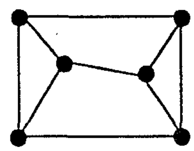
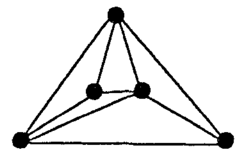
Contracted graph		Dual of contracted graph
13	 (6,9)	 (5,9)
14	 (6,9)	 (5,9)

Table 3.2: Continued

Chapter 4

Procedures for the Enumeration of Conventional Graphs

In this chapter three methods for the enumeration of conventional graphs will be discussed. In the first method the V-V incident matrix of a conventional graph is derived using the procedure described in Chapter Three.

In the second method the insertion of binary vertices on edges of a contracted graph are used to derive conventional graphs. The edges of a contracted graph are labeled. A row vector is used to represent the edges and a 2 by v^c matrix is used to store the end vertices of the edges. This makes efficient computation possible. Graph isomorphism due to symmetry in parallel binary-vertex chains in a conventional graph can be detected before conventional graphs are generated.

In the third method the edges of the dual of a contracted graph are labeled. The E-V incident matrix expresses the edge-vertex relationships of the dual of a contracted graph. The E-V incident matrices of dual graphs are generated directly by an expansion of E-V incident matrix of the dual of contracted graphs. The random number technique of the characteristic polynomial (Tsai, 1987) is used to detect graph isomorphism.

4.1 Direct Enumeration of Conventional Graphs

The V-V incident matrix of a conventional graph is a symmetric matrix with zero daigonal elements. The summation of elements in the ith row represents the degree of the ith vertex. Let $n = v$ be the number of vertices, e be the number of edges, and D_i be the degree of the ith vertex in a conventional graph. Then there exist v equations relating the elements of the incident matrix and the degree of vertices as given below.

$$
\begin{array}{ccccccccccccc}
0 & + & A_{1,2} & + & A_{1,3} & + & \cdots & + & A_{1,n} & = & D_1 \\
A_{1,2} & + & 0 & + & A_{2,3} & + & \cdots & + & A_{2,n} & = & D_2 \\
\vdots & & & \ddots & & & & & & \vdots & \\
A_{1,n-2} & + & \cdots & + & 0 & + & A_{n-2,n-1} & + & A_{n-2,n} & = & D_{n-2} \\
A_{1,n-1} & + & \cdots & + & \cdots & + & 0 & + & A_{n-1,n} & = & D_{n-1} \\
A_{1,n} & + & \cdots & + & \cdots & + & A_{n-1,n} & + & 0 & = & D_n
\end{array}
\tag{4.1}
$$

For a conventional graph without articulation points, $D_i > 1$ are integers and $A_{i,j}$ take the values of 1 or 0.

Since each edge has two end vertices which appears as two ones in the A matrix, the summation of all the elements in the matrix is equal to two times the number of edges, i.e.,

$$\sum_{i=1}^{v}\sum_{j=1}^{v} A_{i,j} = 2\ e \tag{4.2}$$

which means that

$$\sum_{i=1}^{v} D_i = 2\ e \tag{4.3}$$

Given v and e, all admissible non-isomorphic conventional graphs can be derived by solving Eqs. (4.1) and (4.3). We first solve Eq.(4.3) for D_i, i=1,...,n, and then solve Eq. (4.1) for $A_{i,j}$. This procedure is identical to that described in Chapter Three.

4.1.1 Example

Problem statement: Derive all sets of conventional graphs with six vertices and eight edges.

Solution

For $(v, e) = (6, 8)$, Eq. (4.1) reduces to:

$$0 + A_{1,2} + A_{1,3} + A_{1,4} + A_{1,5} + A_{1,6} = D_1 \tag{4.4}$$

$$A_{1,2} + 0 + A_{2,3} + A_{2,4} + A_{2,5} + A_{2,6} = D_2 \tag{4.5}$$

$$A_{1,3} + A_{2,3} + 0 + A_{3,4} + A_{3,5} + A_{3,6} = D_3 \tag{4.6}$$

$$A_{1,4} + A_{2,4} + A_{3,4} + 0 + A_{4,5} + A_{4,6} = D_4 \tag{4.7}$$

$$A_{1,5} + A_{2,5} + A_{3,5} + A_{4,5} + 0 + A_{5,6} = D_5 \tag{4.8}$$

$$A_{1,6} + A_{2,6} + A_{3,6} + A_{4,6} + A_{5,6} + 0 = D_6 \tag{4.9}$$

and Eq. (4.3) reduces to:

$$D_1 + D_2 + D_3 + D_4 + D_5 + D_6 = 2 \times 8 = 16 \tag{4.10}$$

Since $D_i \geq 2$, for i=1 to 6, without loss of generality we may let $D_{i+1} \geq D_i$, for i=1,2,...,5. Using the method outlined in Appendix A, the solutions of $(D_1, D_2, D_3, D_4, D_5, D_6)$ are obtained as follows.

D_1	D_2	D_3	D_4	D_5	D_6
2	2	2	2	2	6
2	2	2	2	3	5
2	2	2	3	3	4
2	2	3	3	3	3

For the fourth set, $(D_1, D_2, D_3, D_4, D_5, D_6) = (2, 2, 3, 3, 3, 3)$, Eqs. (4.4)-(4.9) become

$$0 + A_{1,2} + A_{1,3} + A_{1,4} + A_{1,5} + A_{1,6} = 2 \tag{4.11}$$

$$A_{1,2} + 0 + A_{2,3} + A_{2,4} + A_{2,5} + A_{2,6} = 2 \tag{4.12}$$

$$A_{1,3} + A_{2,3} + 0 + A_{3,4} + A_{3,5} + A_{3,6} = 3 \tag{4.13}$$

$$A_{1,4} + A_{2,4} + A_{3,4} + 0 + A_{4,5} + A_{4,6} = 3 \tag{4.14}$$

$$A_{1,5} + A_{2,5} + A_{3,5} + A_{4,5} + 0 + A_{5,6} = 3 \tag{4.15}$$

$$A_{1,6} + A_{2,6} + A_{3,6} + A_{4,6} + A_{5,6} + 0 = 3 \tag{4.16}$$

Since $A_{i,j}$ take the values of 1 or 0, solving Eq. (4.11) yields ten solution sets. For the solution $(A_{1,2}, A_{1,3}, A_{1,4}, A_{1,5}, A_{1,6}) = (1, 0, 0, 1, 0)$, Eq. (4.12) becomes

$$A_{2,3} + A_{2,4} + A_{2,5} + A_{2,6} = 1 \tag{4.17}$$

Solving Eq. (4.17) yields four solution sets for $A_{2,3}$, $A_{2,4}$, $A_{2,5}$, and $A_{2,6}$.

For $(A_{2,3}, A_{2,4}, A_{2,5}, A_{2,6}) = (1, 0, 0, 0)$, Eq. (4.13) becomes

$$A_{3,4} + A_{3,5} + A_{3,6} = 2 \tag{4.18}$$

There are three solution sets for $A_{3,4}$, $A_{3,5}$, and $A_{3,6}$.

For $(A_{1,2}, A_{1,3}, A_{1,4}, A_{1,5}, A_{1,6}, A_{2,3}, A_{2,4}, A_{2,5}, A_{2,6}, A_{3,4}, A_{3,5}, A_{3,6})$ $= (1,0,0,1,0,1,0,0,0,1,0,1)$, Eqs. (4.14)-(4.16) reduce to

$$A_{4,5} + A_{4,6} = V_4 - A_{1,4} - A_{2,4} - A_{3,4} = 2 \tag{4.19}$$

$$A_{4,5} + A_{5,6} = V_5 - A_{1,5} - A_{2,5} - A_{3,5} = 2 \tag{4.20}$$

$$A_{4,6} + A_{5,6} = V_6 - A_{1,6} - A_{2,6} - A_{3,6} = 2 \tag{4.21}$$

Solving Eqs. (4.19)-(4.21) yields $(A_{4,5}, A_{4,6}, A_{5,6}) = (1,1,1)$. Hence, the V-V incident matrix is given as:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \tag{4.22}$$

The resulting conventional graph is shown in Fig. 4.1.

A computer program can be written to enumerate the conventional graphs systematically. Graphs enumerated in this manner may contain articulation points and/or isomorphic graphs, and they must be identified and screened out.

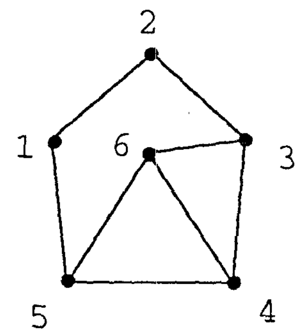


Figure 4.1: The (6,8) conventional graph

4.2 Enumeration of Conventional Graphs from Contracted Graphs

Conventional graphs can be derived from contracted graphs by using a technique called "expansion" first introduced by Woo (1967).

Given v and e for the desired conventional graphs, Woo's approach starts with solving the following two equations:

$$\sum_{i=2}^{\beta} v_i = v \tag{4.23}$$

$$\sum_{i=2}^{\beta} i \; v_i = 2e \tag{4.24}$$

where $\beta \leq e - v + 2$.

Using Eqs. (4.23) and (4.24), the degree listing of all possible (v, e) graphs can be obtained. For example, the solutions for v=10, e=13, one edgree-of-freedom, planar linkages are shown below:

V_2	V_3	V_4	V_5
4	6	0	0
5	4	1	0
6	2	2	0
6	3	0	1
7	0	3	0
7	1	1	1
8	0	0	2

For each of the solution sets found in the previous step, the corresponding contracted graphs are identified. Then the v_2 binary vertices are partitioned into all possible partitions satisfying the kinematic requirement on partial graphs. For example, the contracted graphs corresponding to v_2=5, v_3=4, and v_4=1 are the (5,8) graphs identified as Nos. 9, 10, and 11 in Table 3.2.

In each of these three contracted graphs there are eight edges. The partition of binary vertices can be obtained by solving the following two equations:

$$\sum_{i=0}^{m} b_i = e^c \tag{4.25}$$

$$\sum_{i=0}^{m} (i+1)\ b_i = e \tag{4.26}$$

where b_i denotes the number of binary-vertex chains with i vertices connected in series with $i+1$ edges as shown in Fig. 4.2. The maximum number of vertices in the binary-vertex chain is limited by $m \leq\ F + \lambda - 2$ (Sohn and Freudenstein, 1986).

For a one degree-of-freedom planar mechanism with simple joints, we have F=1, $\lambda = 3$. Hence, m=2. For the above example, we have that v_2=5, v_3=4,

v_4=1, e=13 and e^c=8. Writing Eqs. (4.25) and (4.26) for $e^c = 8$, $e = 13$ and $m = 2$ yields

$$b_0 + b_1 + b_2 = 8 \tag{4.27}$$

$$b_0 + 2b_1 + 3b_2 = 13 \tag{4.28}$$

Solving Eqs. (4.27) and (4.28) yields the following partitions of binary-vertex chain

b_0	b_1	b_2
5	1	2
4	3	1
3	5	0

The last step in Woo's approach is to find the permutation group for each contracted graph and to enumerate conventional graphs for each partition. The procedure is systematic. However, it is not straightforward to implement on a computer program. In particular, the algorithm for finding the permutation group for a contracted graph can be very complicated. In what follows, we present a more straightforward procedure which can be easily implemented on a computer.

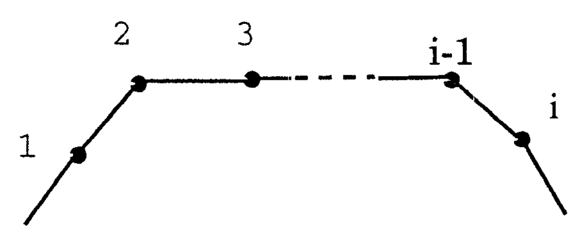


Figure 4.2: A binary-vertex chain b_i

4.2.1 Modified Woo's Approach

Given v and e for the desired conventional graphs, the enumeration proceeds in the following sequence:

Step 1. Identify of the admissible contracted graphs.

Step 2. Expand the single edges into binary-vertex chains.

Step 3. Identify the admissible graphs and graph isomorphism.

In the first step Euler's formula is used to determine the number of loops required of the contracted graphs. Thus,

$$L = e - v + 2 \tag{4.29}$$

For example, L=5 for the (10,13) graphs. Therefore, only those contracted graphs listed in Table 3.2 with five loops are permissible for the enumeration of (10,13) graphs.

In the second step edges in a contracted graph are labeled and arranged as the row vector

$$K = (k_1, k_2, \ldots, k_p) \tag{4.30}$$

where p=e^c is equal to the number of edges in a contracted graph.

The k_i, i=1,2,...,p, initially take the values of one. The end vertices of all edges in a contracted graph are arranged in a 2 by e^c matrix, called the "E" matrix, to establish a data base for the enumeration of conventional graphs. Each row in the E matrix represents an edge in a contracted graph. The two elements in a row represent the node numbers associated with the end vertices of an edge. For example, edges of the No. 2 contracted graph listed in Table 3.2 can be labeled as shown in Fig. 4.3.

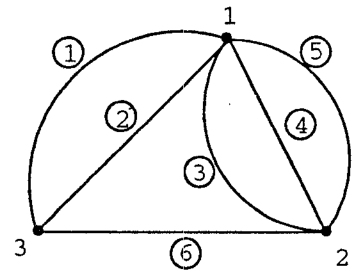


Figure 4.3: A (3,6) contracted graph

The E matrix for the contracted graph shown in Fig. 4.3 is given by

$$E = \begin{bmatrix} 1 & 3 \\ 1 & 3 \\ 1 & 2 \\ 1 & 2 \\ 1 & 2 \\ 2 & 3 \end{bmatrix} \tag{4.31}$$

For convenience, the end vertices in each row of the E matrix are arranged in an ascending order.

During the process of expansion the values of k_i's are replaced by positive integers. The value of k_i represents the number of edges in a binary-vertex chain. Thus, if k_i=1, the ith edge in a contracted graph is not altered; if k_i=2, the ith edge in a contracted graph is replaced by a binary-vertex chain with one vertex and two incident edges, etc. Hence,

$$k_1 + k_2 + \ldots + k_p = e \tag{4.32}$$

where $(F + \lambda - 1) \geq k_i \geq 1$, i=1,2,...,p, are all integers.

The expansion of a contracted graph is equivalent to solving Eq. (4.32) for all possible positive integer solutions. Fortunately, this can be easily accomplished by either the solution method outlined in Appendix A or by a computer program to vary the values of k_i from 1 to (F+λ-1).

For example, using the (3,6) contracted graph shown in Fig. 4.3, (10,13) conventional graphs can be derived by solving

$$k_1 + k_2 + \ldots + k_6 = 13 \tag{4.33}$$

For planar, one degree-of-freedom mechanisms with simple joints, the upper and lower limits on k_i are

$$3 \geq k_i \geq 1 \tag{4.34}$$

Solving Eq. (4.33) subject to constraints imposed by Eq. (4.34) yields all possible solutions. Parallel edges in a contracted graph are identified in order to reduce the number of loops in the computer program.

End vertices in the E matrix are compared, and edges with identical end vertices are recorded and rearranged in such a way that each set of parallel edges forms a group, which are arranged in a descending order according to the number of edges in its group. If $k_s, k_{s+1}, \ldots, k_{s+r}$ belong to a group of parallel edges, then the constraints, $k_s \leq k_{s+1} \leq \ldots \leq k_{s+r}$, are imposed on k_i. This reduces greatly the chance of producing isomorphic graphs and the number of loops in the computer program.

For example, the E matrix of the contracted graph shown in Fig. 4.3 is rearranged as

$$E = \left[\begin{array}{cc} 1 & 2 \\ 1 & 2 \\ 1 & 2 \\ 1 & 3 \\ 1 & 3 \\ 2 & 3 \end{array}\right] \begin{array}{l} \\ Group\ 1 \\ \\ \\ Group\ 2 \\ Group\ 3 \end{array} \tag{4.35}$$

The rearrangement of E is equivalent to renumbering the edges in a contracted graph. For this eaxmple, the edges of the graph shown in Fig. 4.3 are renumbered as shown in Fig. 4.4.

The nested do-loop can be written as :

Group 1

$Do \;\; i1 = \;\; 1, \; 3$

$K(1) = \;\; i1;$

$Do \;\; i2 = \;\; i1, \; 3$

$K(2) = \;\; i2;$

$Do \;\; i3 = \;\; i2, \; 3$

$K(3) = \;\; i3;$

Group 2

$Do \;\; i4 = 1, \; 3$

$K(4) = i4;$

$Do \;\; i5 = i4, \; 3$

$K(5) = i5;$

Group 3

$K(6) = 13 - K(1) - K(2) - K(3) - K(4) - K(5)$

$If \; K(6) < 1, try \; next \; K(5)$

$If \; K(6) \geq 1, \; save \; K(i), \; i = 1, \ldots, 6$

$Continue$

$Continue$

$Continue$

$Continue$

4.2.2 Construction of the V-V Incident Matrices

Once the values of k_i, i=1,2,...,p, are found, the V-V incident matrix of the corresponding conventional graph can be constructed. The value of $(k_i - 1)$ denotes the number of vertices to be inserted on the ith edge of a contracted graph. Since the pair of end vertices of every edge in a contracted graph is contained in a row of the E matrix, insertion of $(k_i - 1)$ binary vertices on each edge will increase the number of vertices and the number of edges by $(k_i - 1)$ each in the E matrix of the corresponding conventional graph. The number of rows in the E matrix will also increase by $(k_i - 1)$. The process of incrementing the numbers of vertices and edges is called "expansion." The V-V incident matrix can be constructed as follows:

We start with the first edge. If the first element is $k_1 = 1$ and the first row of the E matrix is (a, b), then let

$$A_{a,b} = 1$$

If the first element $k_1 = x$, then $x - 1$ vertices are inserted onto the first edge as follows:

$$A_{a,v^c+1} = 1$$

$$A_{v^c+1,v^c+2} = 1$$

$$\vdots$$

$$A_{v^c+x-2,v^c+x-1} = 1$$

$$A_{v^c+x-1,b} = 1$$

where v^c denotes the number of vertices in the contracted graph.

If the second element is k_2=1 and the second row of end vertices pair matrix is (c, d), then let

$$A_{c,d} = 1$$

If the second element $k_2 = y$, then $y - 1$ vertices are inserted onto the second edge as follows:

$$A_{c,v^c+x} = 1$$

$$A_{v^c+x,v^c+x+1} = 1$$

$$\vdots$$

$$A_{v^c+x+y-3,v^c+x+y-2} = 1$$

$$A_{v^c+x+y-2,d} = 1$$

This process is repeated until all binary-vertex chains are added to the contracted graph. Finally, let

$$A_{i,j} = A_{j,i}, \quad i \neq j$$

and

$$A_{i,j} = 0, \qquad i = j$$

for all i and j needed to complete the matrix.

Example

The number of the vertices in the contracted graph shown in Fig. 4.4 is 3. The matrix E for this contracted graph is given by Eq. (4.36). Let us assume that one of the solutions to Eq. (4.33), $K = (2, 2, 2, 2, 2, 3)$, to illustrate the concept.

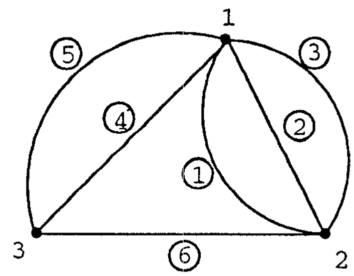


Figure 4.4: A (3,6) contracted graph

$$E = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \\ 1 & 3 \\ 1 & 3 \\ 2 & 3 \end{bmatrix} \tag{4.36}$$

Since K(1)=2 and the first row of E is (1,2), let

$$A_{1,4} = 1$$

$$A_{4,2} = 1$$

Since K(2)=2 and the second row of E is (1,2), let

$$A_{1,5} = 1$$

$$A_{5,2} = 1$$

Since K(3)=2 and the third row of E is (1,2), let

$$A_{1,6} = 1$$

$$A_{6,2} = 1$$

Since K(4)=2 and the fourth row of E is (1,3), let

$$A_{1,7} = 1$$

$$A_{7,3} = 1$$

Since K(5)=2 and the fifth row of E is (1,3), let

$$A_{1,8} = 1$$

$$A_{8,3} = 1$$

Since K(6)=3 and the sixth row of E is (2,3), let

$$A_{2,9} = 1$$

$$A_{9,10} = 1$$

$$A_{10,3} = 1$$

The complete V-V incident matrix A is as shown in Eq. (4.37).

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{4.37}$$

The corresponding conventional graph is shown in Fig. 4.5.

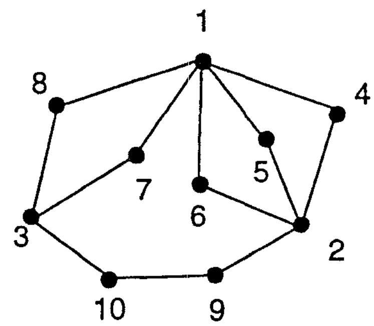


Figure 4.5: The resulting (10,13) conventional graph

4.3 Enumeration of Conventional Graphs from the Dual of Contracted Graphs

In this section the enumeration of conventional graphs from the dual of contracted graphs is presented. The edges of a graph are labeled such that an E-V (Edge-to-Vertex) incident matrix of a graph can be introduced to represent the graph. The E-V incident matrix of the duals of contracted graphs are used as the data base. The E-V incident matrix of the dual of a conventional graph is derived by replacing the edges in the dual of a contracted graph with parallel edges. The procedure of using the permutataion of binary chains is not necessary here, since the E-V incident matrix completely describes the topology of a conventional graph.

The procedure to derive conventional graphs from the dual of contracted graphs is as follows:

1. Construct the E-V incident matrices of the duals of contracted graphs as a data base.
2. Identify the admissible duals of contracted graphs.
3. Insert of parallel edge sets into the duals of contracted graphs to form the E-V incident matrices of the duals of conventional graphs.
4. Construct the duals of conventional graphs from the E-V incident matrices.
5. Sketch the conventional graphs from the dual graphs.

4.3.1 Construction of E-V Incident Matrices for the Duals of Contracted Graphs

Let e^{dc} be the number of edges and v^{dc} be the number of vertices in the dual of a contracted graph. Label each edge in the dual of a contracted graph with lower case letters. The E-V incident matrix E^* is a e^{dc} by v^{dc} matrix. The element of the E-V incident matrix is defined as follows:

$$E^*_{i,j} = \begin{cases} 1, & \text{if vertex } i \text{ is the end vertex of edge } j \\ 0, & \text{otherwise} \end{cases}$$

The non-zero elements in a column represent the pair of end vertices of each edge. Therefore the sum of each column is equal to two. The sum of each row denotes the number of edges incident with a vertex in a contracted graph. For example, Fig. 4.6 is the dual of a contracted graph with five vertices and seven edges.

The E-V incident matrix of Fig. 4.6 is

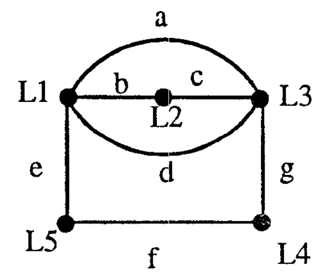


Figure 4.6: The labeled dual of contracted graph

$$E^* = \begin{array}{c} \begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\ \left[\begin{array}{ccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right] \end{array} \begin{array}{c} \\ L_1 \\ L_2 \\ L_3 \\ L_4 \\ L_5 \end{array} \tag{4.38}$$

For each contracted graph, the E-V incident matrix for the corresponding dual graph is built as a data base. This E-V incident matrix can be expanded as the E-V incident matrix of the dual of a conventional graph by replacing each labeled edge with parallel edges. The way to find the definite sets of parallel edges is described in the next section.

4.3.2 Identification of the Duals of Contracted Graphs

Given v and e, the number of loops L is computed from the Euler formula

$$L = e - v + 2$$

All the duals of contracted graphs having L vertices are potential candidates for the enumeration of conventional graphs.

4.3.3 Expansion of the E-V incident Matrices for the Duals of Conventional Graphs

Let $e = v^{dc}$ be the number of edges in a dual of a contracted graph. The derivation of a dual graph from the dual of a contracted graph requires the replacement of each edge in the dual of a contracted graph by a set of parallel edges. Let k_i denote the number of parallel edges used to replace the ith edge in the dual of a contracted graph. Then, the sum of all the sets of parallel edges is equal to the total numebr of edges in the dual of a conventional graph, i.e.,

$$k_1 + k_2 + \ldots + k_p = e \tag{4.39}$$

where $k_i \geq 1$, i=1,...,p, are positive integers.

Using the method outlined in Appendix A, all k_i, i=1,...,p, can be determined. Alternatively, Eq. (4.39) can also be solved by using the nested loops outlined in Section 4.2.1. Once the k_i are found, the ith edge in the dual of a contracted graph is replaced by k_i parallel edges. If k_i=1, then the original ith edge will be retained as a single edge. The E-V incident matrix of a dual graph can, therefore, be obtained by an expansion of the number of columns in accordance with the values of k_i. The procedure is similar to that outlined in Section 4.2.2.

Because parallel edges share the same end vertices, parallel edges form a set of adjacent columns in the E-V incident matrix of a dual graph. The columns in each parallel edge set are identical to one another.

4.3.4 Construction of Dual Graphs

Once the E-V incident matrix of a dual graph is obtained, the dual graph can be constructed according to the definition of an E-V incident matrix. Note that a set of identical adjacent columns in the E-V incident matrix represents a set of adjacent parallel edges in the dual graph, and should be sketched as such.

4.3.5 Construction of Conventional Graphs

A number, which corresponds to a vertex in the conventional graph, is assigned to each loop in the dual graph. The vertex i and vertex j of the conventional graph are connected with an edge if the loops i and j in the dual graph are adjacent to each other. This process is continued until all the edges are constructed.

4.3.6 Example

<u>Problem statement</u>: Derive a (10,13) conventional graph from the (5,7) dual of contracted graph as shown in Fig. 4.7.

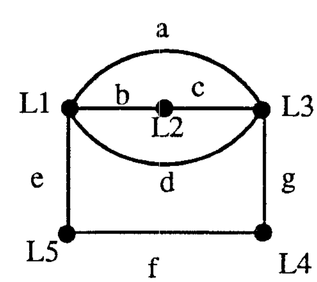


Figure 4.7: The labeled dual of contracted graph

The E-V incident matrix of Fig. 4.7 is

$$
E = \begin{array}{c} \begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\ \left[\begin{array}{ccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right] \end{array} \begin{array}{c} \\ L_1 \\ L_2 \\ L_3 \\ L_4 \\ L_5 \end{array} \tag{4.40}
$$

Since p=7 and e=13, Eq. 4.39 reduces to

$$
k_1 + k_2 + k_3 + k_4 + k_5 + k_6 + k_7 = 13 \tag{4.41}
$$

Using the method outlined in Appendix A yields $(k_1, k_2, k_3, \cdots, k_7)$ =(1,2,2,1,3,2,2) as one of the solution sets. For $(k_1, k_2, k_3, k_4, k_5, k_6, k_7) = (1, 2, 2, 1, 3, 2, 2)$, the procedure to expand the E-V incident matrix for the corresponding dual graph is as follows:

Since k_1=1, the first column is not altered.

Since k_2=2, the second column is written twice as the second and third columns in the E^* matrix as shown in Eq. (4.42).

Since k_3=2, the third column in Eq. (4.40) is written twice as the fourth and fifth columns in Eq. (4.42).

Since k_4=1, the fourth column in Eq. (4.40) is written once as the sixth column in Eq. (4.42).

Since k_5=3, the fifth column in Eq. (4.40) is written three times as the seventh, eighth and ninth columns in Eq. (4.42).

Since k_6=k_7=2, the sixth and seventh columns in Eq. (4.40) are both written twice as the 10th-13th columns in Eq. (4.42).

$$E^* = \begin{array}{c} \begin{array}{ccccccccccccc} a & b & b' & c & c' & d & e & e' & e'' & f & f' & g & g' \end{array} \\ \left[\begin{array}{ccccccccccccc} 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{array}\right] \end{array} \begin{array}{c} \\ L_1 \\ L_2 \\ L_3 \\ L_4 \\ L_5 \end{array} \tag{4.42}$$

The corresponding dual graph is shown in Fig. 4.8.

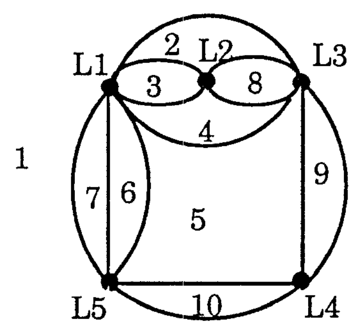


Figure 4.8: The dual graph

Following the procedure outlined in Section 4.3.5, the corresponding conventional graph is sketched as shown in Fig. 4.9.

4.3.7 Conversion of E-V Incident Matrices to V-V Incident Matrices

In Section 4.3.3, we have obtained the E-V incident matrix of a dual graph. In order to check for graph isomorphism, one needs to convert the E-V incident

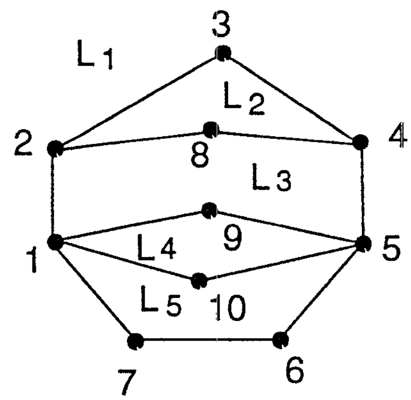


Figure 4.9: The corresponding conventional graph

matrix of a dual graph to the V-V incident matrix of a conventional graph. The computer algorithm for automatically converting an E-V incident matrix of a dual graph to the corresponding V-V incident matrix of a conventional graph is very involved. It is not developed in this thesis since the second method is more effective and is chosen for the enumeration of conventional graphs.

4.4 Identification of Graph Isomorphism

In Sections 4.1, 4.2 and 4.3, procedures for deriving conventional graphs have been described. However, detection of graph isomorphism is also necessary. In this thesis, the random number technique of the characteristic polynomial is used to check graph isomorphism (Tsai, 1987).

The linkage characteristic polynomial $p(x)$ is defined as the determinant of the matrix $(xI - A)$, where x is a dummy variable and I is an identity matrix of the same order as the V-V incident matrix A. Thus,

$$p(x) = det(xI - A)$$

Instead of symbolically developing the polynomial, Tsai replaces x in $p(x)$ by a random number and then computes the value of $p(x)$ numerically. If two graphs are isomorphic, then their characteristic polynomials are necessarily identical to one another (Uicker and Raicu, 1975).

However, the characteristic polynomial is only a necessary, but not sufficient condition for checking graph isomorphism. It has been shown that when the number of verticess becomes sufficiently large (Sohn and Freudenstein, 1986), two non-isomorphic graphs may possess identical characteristic polynomials. This problem can be avoided by classifying conventional graphs into different contracted graph families before the values of characteristic polynomials are compared.

The probability for two non-isomorphic graphs that belong to the same contracted graph and, at the same time, possess identical characteristic polynomials is negligibly small. However, the probability for two non-isomorphic graphs to possess identical characteristic polynomials is not negligible. Hence, conventional graphs are classified according to their corresponding contracted graphs. Only those graphs that belong to the same contracted graph family are checked for graph isomorphism.

Using the method developed by Tsai (1987), we have not found two non-isomorphic graphs that, both, have the same characteristic polynomial and belong to the same contracted graph. For example, the three pairs of graphs shown in Fig. 4.10 have the same characteristic polynomials but they belong to different contracted graphs.

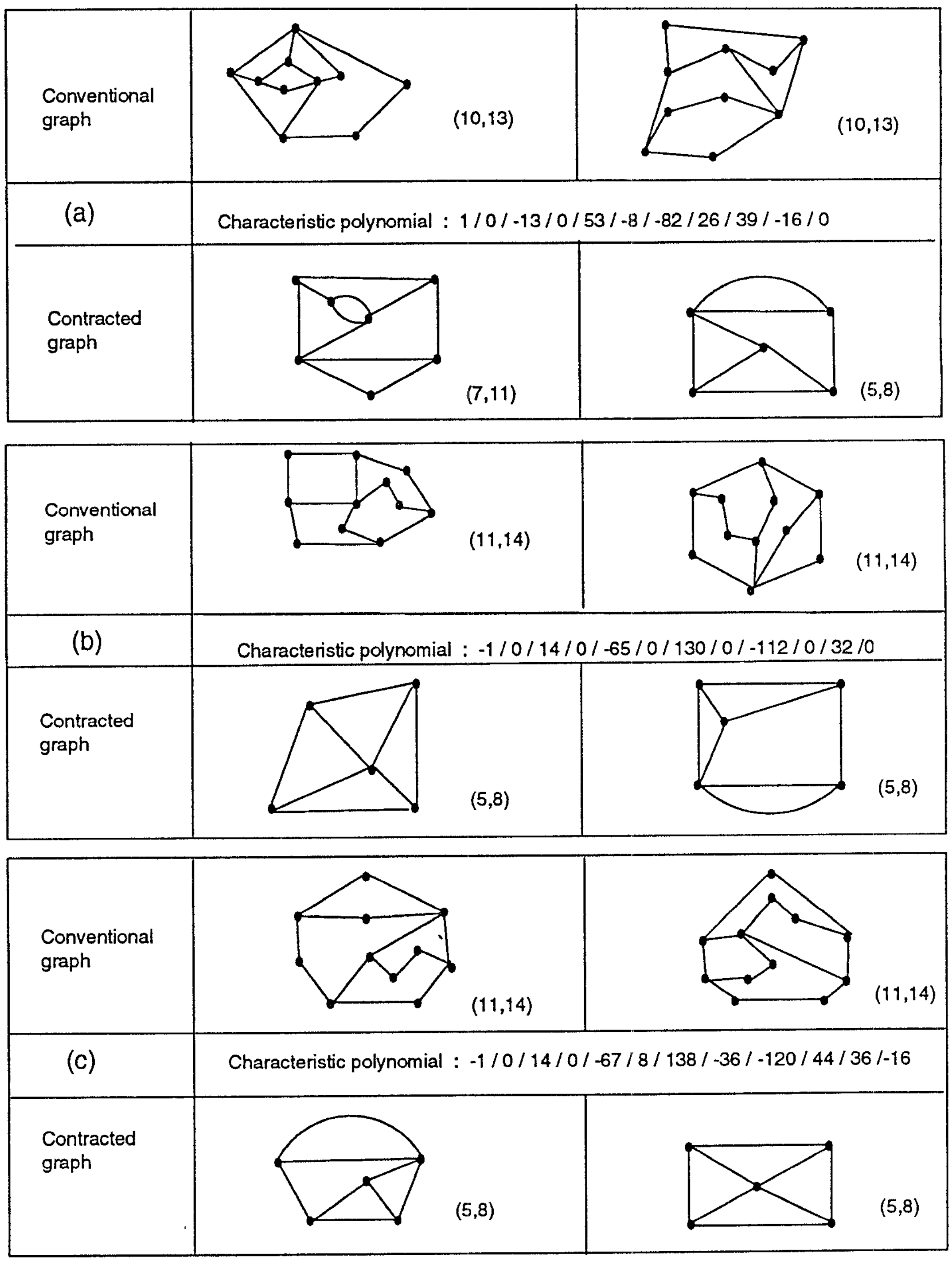


Figure 4.10: Conventional graphs with the same characteristic polynomials

4.5 Discussion

In this chapter, three methodologies for the enumeration of conventional graphs are described. An algorithm is developed to solve for m integer variables in n equations, where $m > n$. The advantage of these procedures is that it provides an efficient computation technique to solve for the variables. The method for solving this problem is outlined in Appendix A.

The first method directly derives the vertex-to-vertex incident matrix of a conventional graph. The procedure is similar to that outlined in Chapter 3 for the derivation of contracted graphs. Both procedures are typical for solving for m integer variables with n equations, where $m > n$. The detection of articulation points for conventional graphs is necessary.

The second method enumerates conventional graphs from contracted graph families by the arrangement of binary-vertex chains on a contracted graph. Two data bases are necessary. One data base contains the families of contracted graphs with the same number of loops. The other contains the end vertices for each contracted graph in the families. No permutations of binary chains are necessary.

The third method uses an E-V incident matrix to represent graphs instead of a V-V incident matrix. It combines two data bases used in the second method. However, the parallel edges are arranged on the dual of a contracted graph to form a dual graph. The E-V incident matrix of a dual graph is expanded correspondingly. The E-V incident matrix of a dual graph is the edge-to-loop incident matrix of a conventional graph. The conventional graph can be formed from a dual graph by the inspection method discussed in Section 4.3.5. However, the conversion from E-V incident matrix to V-V incident matrix is required for the detection of isomorphism between conventional graphs.

The first method is very efficient when the number of vertices is reasonably small. However, the required nested loops in the computer program become unmanageable when the number of vertices is large. The third method requires the conversion of a dual graph to a conventional graph. Although a computer program can be developed to accomplish this conversion, the algorithm can be quite involved. Therefore the second method is judged to be the most straightforward and efficient method of all. Based on the second method, a computer program was written for the enumeration of conventional graphs with up to eight vertices and six loops.

4.6 Results

A comprehensive set of 219 planar, five-loop, single degree of freedom mechanisms with simple joints was developed by Woo (1967). Planar mechanisms with up to five loops and three degrees-of-freedom were determined by Sohn and Freudenstein (1986). An atlas of conventional graphs with up to six vertices, which can be used to create mechanisms of any type, was developed by Buchsbaum and Freudenstein (1970).

In this thesis, more than 700 new conventional graphs with up to eight vertices and six loops have been enumerated. Table C in Appendix C lists the number of enumerated graphs as a function of the various combinations of the number of vertices, edges and loops. The tables in Appendices D and E contain 318 of these new conventional graphs with seven and eight vertices, and with up to eleven edges. The remaining 387 graphs, which have eight vertices and twelve edges, are not included in this thesis due to the complexity of the graphs.

Chapter 5

Conclusion

This thesis presents new definitions of a dual graph, a dual of a contracted graph, and the correspondences between various graph representations of mechanisms. A method for solving one linear equation with more than one integer variable is shown to be effetive for graph enumerations. Using these definitions three methodologies for the enumeration of the kinematic structures are suggested.

The first method is a direct enumeration of conventional graphs. It is similar to the enumeration procedure developed for the enumeration of contracted graphs. The second method derives conventional graphs from contracted graphs. An efficient methodology for the arrangement of binary-vertex chains in contracted graphs is also described. The third method derives conventional graphs from the duals of contracted graphs. It uses the E-V incident matrices of the dual of a contracted graph instead of the V-V incident matrix of a contracted graph. Duals of contracted graphs are used as the data base for the enumeration. Dual graphs are obtained from the dual of contracted graphs. Conventional graphs are then derived from dual graphs. No permutations are needed in these methods.

The random number techniques of characteristic polynomial is used to detect graph isomorphism. Since different contracted graphs have different structural topologies, conventional graphs can be isomorphic with each other only if they are derived from the same contracted graph. Since conventional graphs are derived from contracted graphs or the duals of contracted graphs, it is only necessary to check graph isomorphism of conventional graphs within the same contracted graph family. The existing problem of two non-isomorphic graphs yielding identical characteristic polynomials is avoided.

Conventional graphs with up to eight vertices have been derived. The resulting graphs with four to six vertices are in complete agreement with the findings by Bushsbum and Freudenstein (1970). The atlas of graphs with seven and eight vertices are believed to be new.

The second method of enumeration is judged to be the most effective method of all, and is recommended for further development, including automatic sketching of graphs and their corresponding mechanisms.

The principal contributions of this thesis are:

1. The definition of a dual graph has been modified. This definition has been applied to both conventional graphs and contracted graphs to create the dual graphs of a conventional graph, as well as the dual graphs of a contracted graph. Conventional graphs can now be generated using these definitions of dual graphs. The correspondences among conventional graphs, dual graphs, contracted graphs, and the dual of a contracted graph have been established.

2. Three algorithms for the systematic and automatic enumeration of the graphs of kinematic chains have been developed. The algorithms are very

simple and straightforward, and can be easily implemented in a computer program.

3. Two tables of conventional graphs with seven and eight vertices, respectively, have been built. These tables can be used to create mechanisms with up to six loops and eight links.

Appendices

A simple procedure for solving n linear equations in m unknown integers will be described. First, the procedure will be applied to solve one linear equation with m unknowns, and then to n linear equations with m unknowns.

A.1 One equation in m integer variables

Equation (A.1) shows a linear equation in m integers, k_i, $i = 1, \ldots, m$, where p is a known positive integer. Solving Eq. (A.1) for these m integers, all of which are non-negative, is called m partitions of an integer p.

$$k_1 + k_2 + \ldots + k_m = p \tag{A.1}$$

Since there are m unknowns in one equation, we may choose $m-1$ unknowns arbitrarily and then solve Eq. (A.1) for the remaining unknown, provided all the k_i's are non-negative. In order to systematically choose these $m-1$ integers, we let

$$l_1 = \sum_{i=2}^{m} k_i \tag{A.2}$$

Substituting Eq. (A.2) into Eq. (A.1), yields

$$k_1 + l_1 = p \tag{A.3}$$

Equation (A.3) contains only two unknowns. Hence, we may choose one of the two variables arbitrarily and solve for the other variable. This can be easily accomplished by letting k_1 assume the values of 0 to p, and then solving Eq. (A.3) for l_1. The results are as shown below:

Variables	Solutions				
k_1	0	1	2	$\cdots$	p
l_1	p	p-1	p-2	$\cdots$	0

For each set of (k_1, l_1), let

$$k_2 + l_2 = l_1 \tag{A.4}$$

Again, we have one equation in two variables. We may let k_2 to assume the values of 0 to l_1 and solve Eq. (A.4) for l_2. The results are as shown below:

Variables	Solutions				
k_2	0	1	2	$\cdots$	l_1
l_2	l_1	l_1-1	l_1-2	$\cdots$	0

The above procedure can be repeated until all sets of partitions are obtained. The procedure is very simple and can be easily implemented on a digital computer.

Example

To illustrate the principle, consider the following equation in three integers.

$$k_1 + k_2 + k_3 = 4 \tag{A.5}$$

Let

$$l_1 = k_2 + k_3 \tag{A.6}$$

then, Eq. (A.5) reduces to

$$k_1 + l_1 = 4 \tag{A.7}$$

Choosing k_1 from zero to 4, we obtain

Variables	Solutions				
k_1	0	1	2	3	4
l_1	4	3	2	1	0

Let

$$k_2 + k_3 = l_1 \tag{A.8}$$

Hence, by choosing k_2 from zero to l_1, we can solve Eq. (A.8) for k_3 for each pair of solutions obtained in the previous step.

For example, for $k_1 = 1$ and $l_1 = 3$, we obtain

k_2	0	1	2	3
k_3	3	2	1	0

Repeating the above process all sets of k_i , $i = 1, 2$, and 3 can be solved.

All the coefficients in Eq. (A.1) are equal to 1. In general the coefficients can assume any integers. The same procedure can be used to solve such a linear equation, but the solutions are valid only when they are non-negative integers. A more rigorus solution procedure for solving one linear equation in two unknowns can be found in (Gelfond, 1981).

A.2 N Linear Equations in m Integers

The following are n linear equations with m integers, k_i, $i = 1, \ldots, m$

$$\begin{array}{ccccccccccc} a_{1,1} & k_1 & + & a_{1,2} & k_2 & + & \cdots & + & a_{1,m} & k_m & = & p_1 \\ a_{2,1} & k_1 & + & a_{2,2} & k_2 & + & \cdots & + & a_{2,m} & k_m & = & p_2 \\ \vdots & & & & & & & & & & \vdots & \\ a_{n,1} & k_1 & + & a_{n,2} & k_2 & + & \cdots & + & a_{n,m} & k_m & = & p_n \end{array} \tag{A.9}$$

where the coefficients $a_{i,j}$ and p_j are all integers, and $m > n$.

Writing Eq. (A.9) in matrix form yields

$$G\,K = 0 \tag{A.10}$$

where

$$G = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} & -p_1 \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m} & -p_2 \\ \vdots & \vdots & & & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,m} & -p_n \end{bmatrix}$$

$$K = [k_1, k_2, \ldots, k_n, 1]^{\mathrm{T}}$$

Using the Gauss elimination method, G can be reduced to an upper triangular form:

$$G^* = \begin{bmatrix} g_{1,1} & g_{1,2} & \cdots & \cdots & \cdots & g_{1,m} & q_1 \\ 0 & g_{2,2} & \cdots & \cdots & \cdots & g_{2,m} & q_2 \\ 0 & 0 & \ddots & & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & g_{n,n} & \cdots & a_{n,m} & q_n \end{bmatrix} \tag{A.11}$$

Hence, Eq. (A.10) becomes

$$G^* \, K = 0 \tag{A.12}$$

Note that the last equation in Eq. (A.12) contains $(m - n + 1)$ unknowns. Specifically, we have

$$g_{n,n}k_n + g_{n,n+1}k_{n+1} + \cdots + g_{n,m}k_m = q_n \tag{A.13}$$

Equation (A.13) can be solved by the procedure outlined in Appendix A.1. Once $k_n, \ldots, k_m$ are solved. The remaining unknowns can be solved by backward substitution. Note that the coeeficients, q_{ij}, are all integers.

B Contracted Graphs with Six Loops

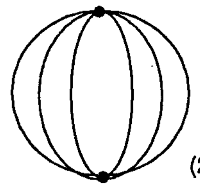
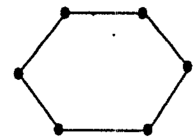
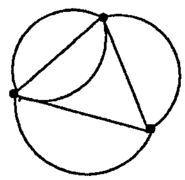
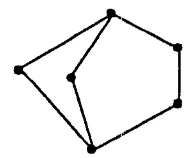
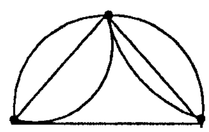
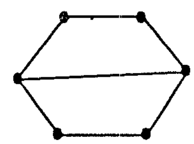
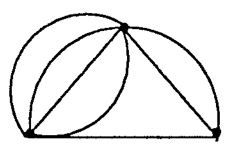
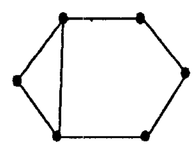
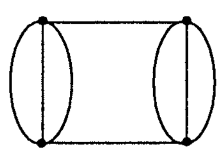
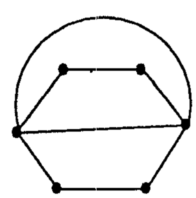
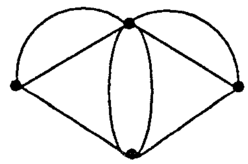
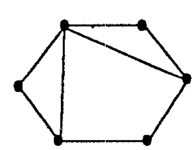
	Contracted graph	Dual of contracted graph
1	 (2,6)	 (6,6)
2	 (3,7)	 (6,7)
3	 (3,7)	 (6,7)
4	 (3,7)	 (6,7)
5	 (4,8)	 (6,8)
6	 (4,8)	 (6,8)

Table B: The contracted graphs with six loops

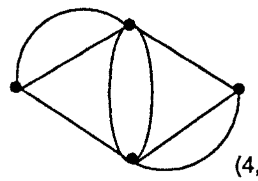
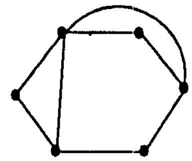
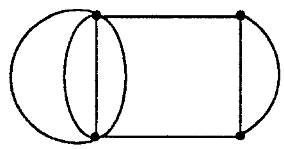
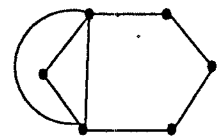
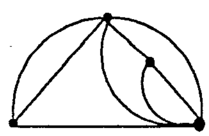
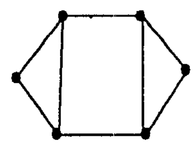
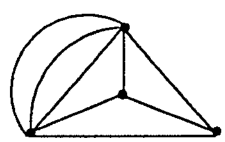
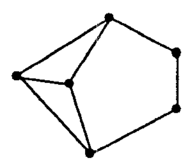
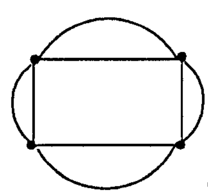
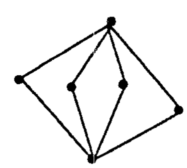
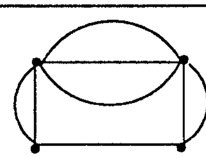
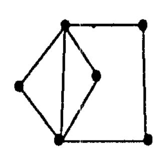
	Contracted graph	Dual of contracted graph
7	 (4,8)	 (6,8)
8	 (4,8)	 (6,8)
9	 (4,8)	 (6,8)
10	 (4,8)	 (6,8)
11	 (4,8)	 (6,8)
12	 (4,8)	 (6,8)

Table B: Continued

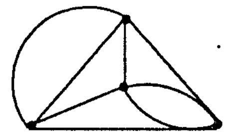
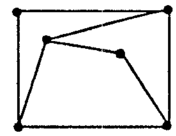
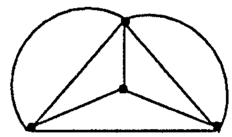
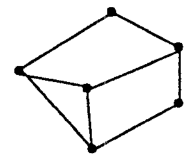
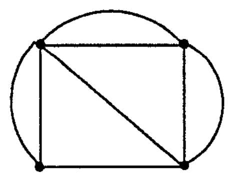
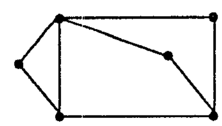
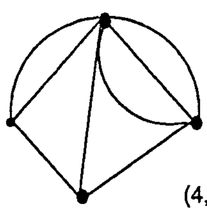
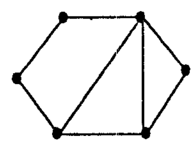
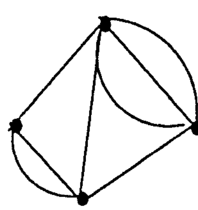
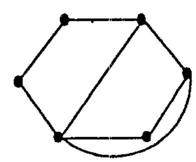
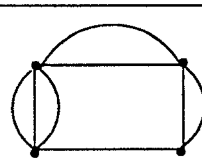
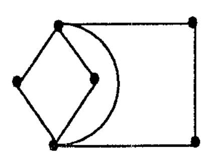
	Contracted graph	Dual of contracted graph
13	 (4,8)	 (6,8)
14	 (4,8)	 (6,8)
15	 (4,8)	 (6,8)
16	 (4,8)	 (6,8)
17	 (4,8)	 (6,8)
18	 (4,8)	 (6,8)

Table B: Continued

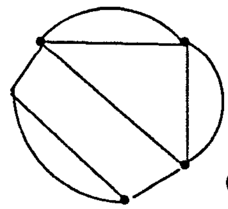
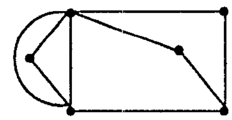
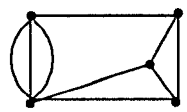
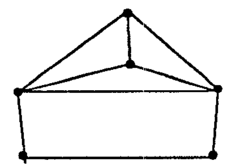
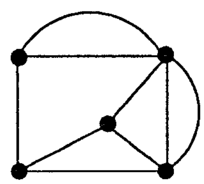
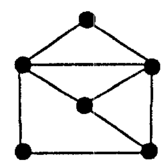
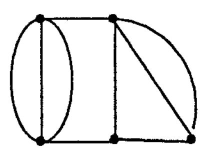
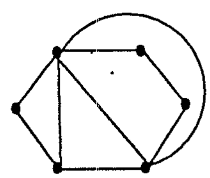
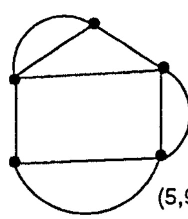
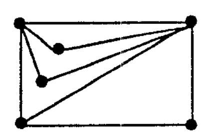
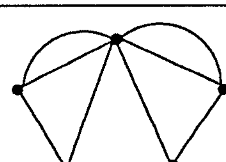
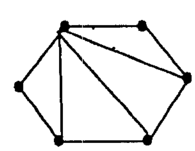
	Contracted graph	Dual of contracted graph
19	 (5,9)	 (6,9)
20	 (5,9)	 (6,9)
21	 (5,9)	 (6,9)
22	 (5,9)	 (6,9)
23	 (5,9)	 (6,9)
24	 (5,9)	 (6,9)

Table B: Continued

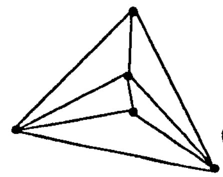
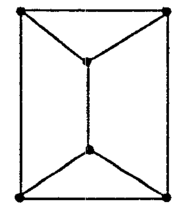
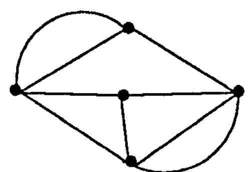
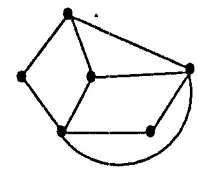
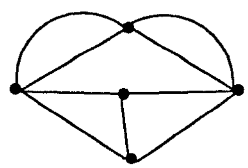
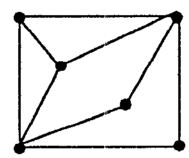
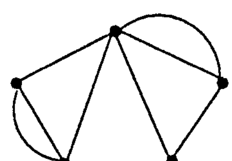
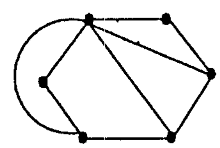
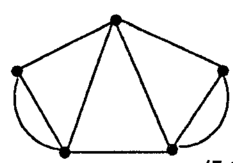
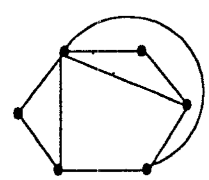
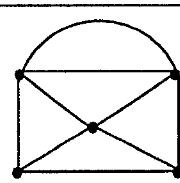
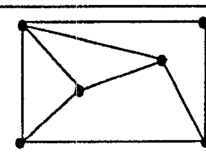
	Contracted graph	Dual of contracted graph
25	 (5,9)	 (6,9)
26	 (5,9)	 (6,9)
27	 (5,9)	 (6,9)
28	 (5,9)	 (6,9)
29	 (5,9)	 (6,9)
30	 (5,9)	 (6,9)

Table B: Continued

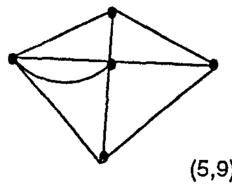
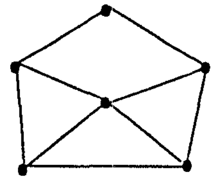
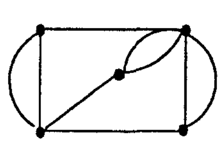
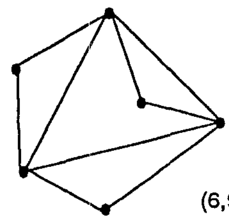
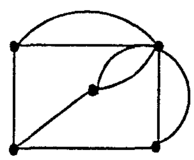
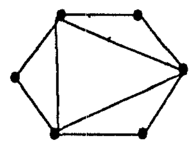
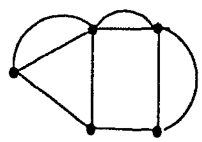
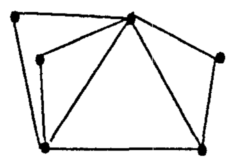
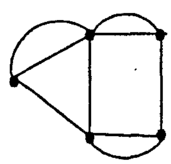
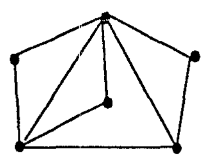
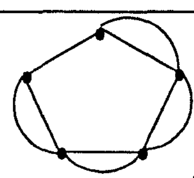
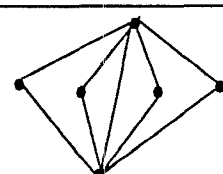
	Contracted graph	Dual of contracted graph
31	 (5,9)	 (6,9)
32	 (5,9)	 (6,9)
33	 (5,9)	 (6,9)
34	 (5,9)	 (6,9)
35	 (5,9)	 (6,9)
36	 (5,9)	 (6,9)

Table B: Continued

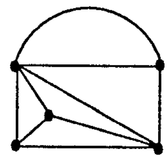
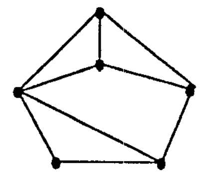
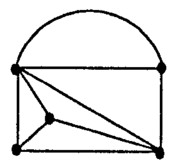
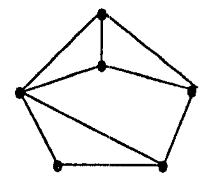
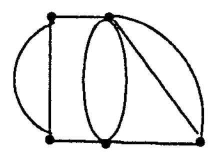
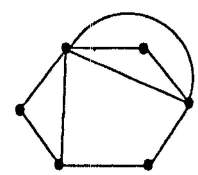
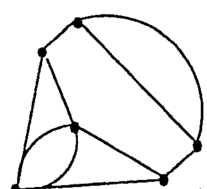
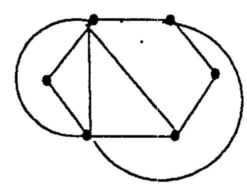
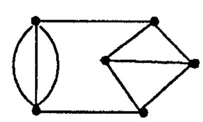
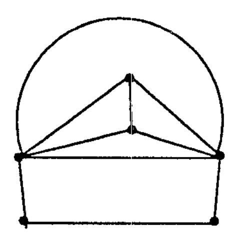
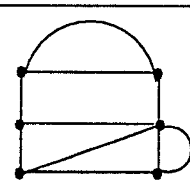
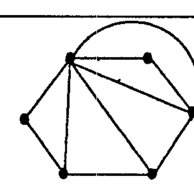
	Contracted graph	Dual of contracted graph
37	 (5,9)	 (6,9)
38	 (5,9)	 (6,9)
39	 (5,9)	 (6,9)
40	 (6,10)	 (6,10)
41	 (6,10)	 (6,10)
42	 (6,10)	 (6,10)

Table B: Continued

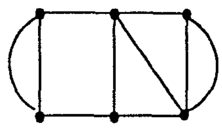
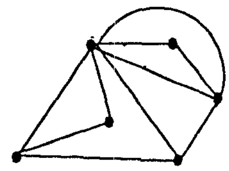
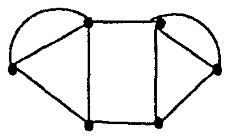
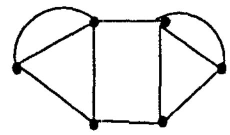
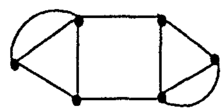
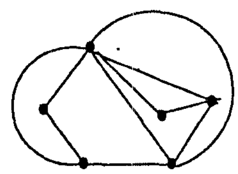
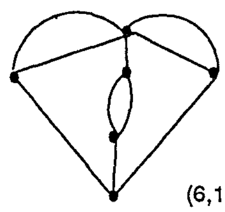
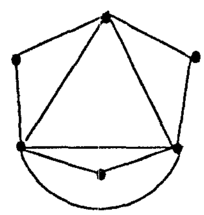
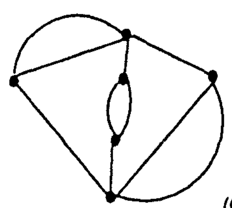
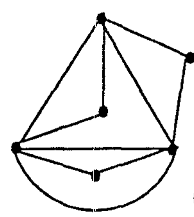
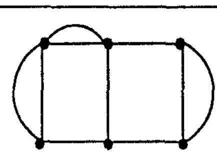
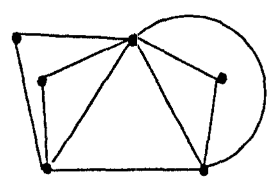
	Contracted graph	Dual of contracted graph
43	 (6,10)	 (6,10)
44	 (6,10)	 (6,10)
45	 (6,10)	 (6,10)
46	 (6,10)	 (6,10)
47	 (6,10)	 (6,10)
48	 (6,10)	 (6,10)

Table B: Continued

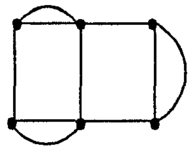
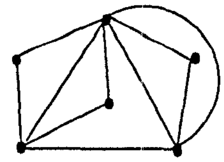
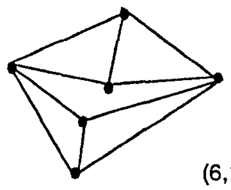
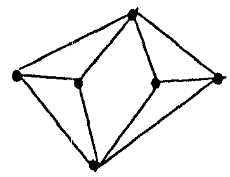
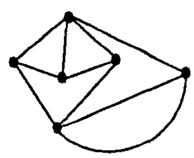
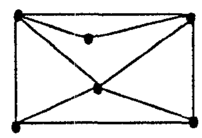
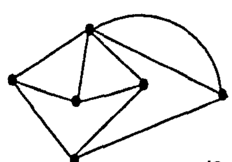
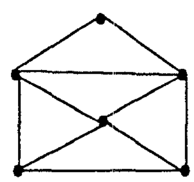
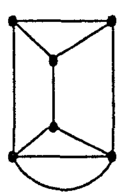
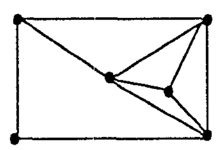
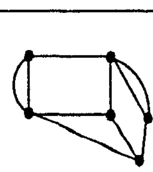
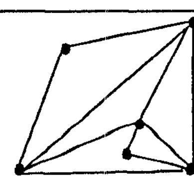
	Contracted graph	Dual of contracted graph
49	 (6,10)	 (6,10)
50	 (6,10)	 (6,9)
51	 (6,10)	 (6,10)
52	 (6,10)	 (6,10)
53	 (6,10)	 (6,10)
54	 (6,10)	 (6,10)

Table B: Continued

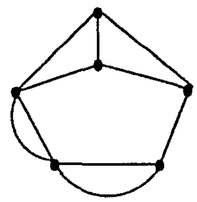
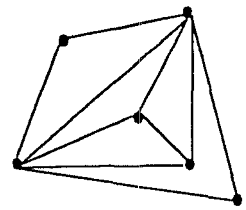
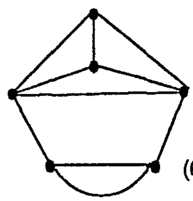
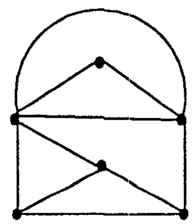
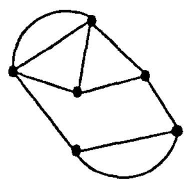
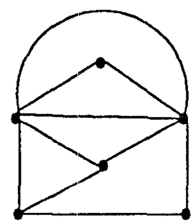
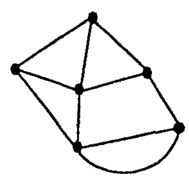
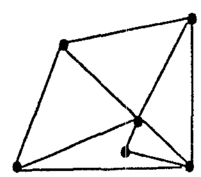
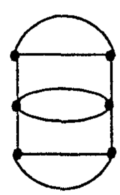
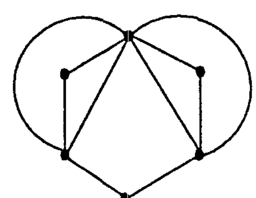
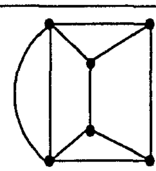
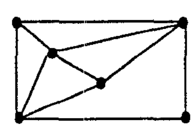
	Contracted graph	Dual of contracted graph
55	 (6,10)	 (6,10)
56	 (6,10)	 (6,10)
57	 (6,10)	 (6,10)
58	 (6,10)	 (6,10)
59	 (6,10)	 (6,10)
60	 (6,10)	 (6,10)

Table B: Continued

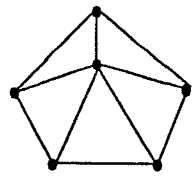
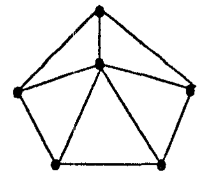
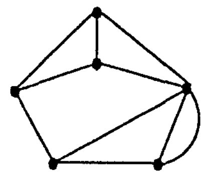
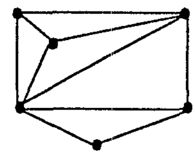
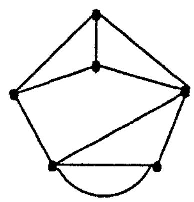
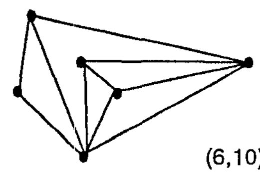
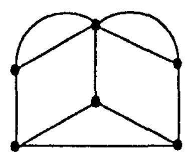
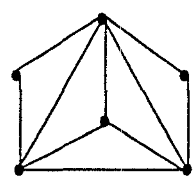
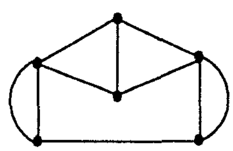
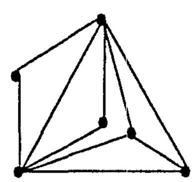
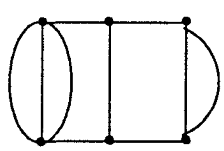
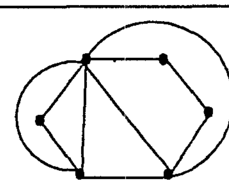
	Contracted graph	Dual of contracted graph
61	 (6,10)	 (6,10)
62	 (6,10)	 (6,10)
63	 (6,10)	 (6,10)
64	 (6,10)	 (6,10)
65	 (6,10)	 (6,10)
66	 (6,10)	 (6,10)

Table B: Continued

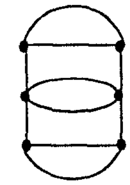
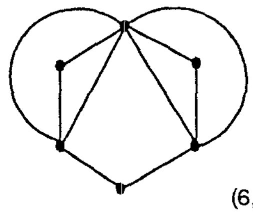
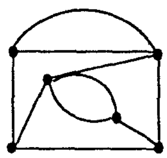
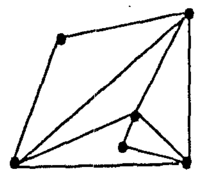
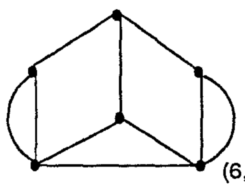
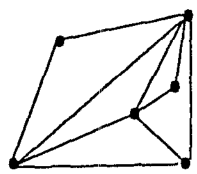
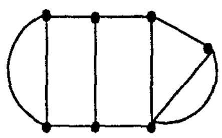
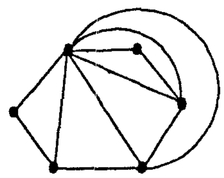
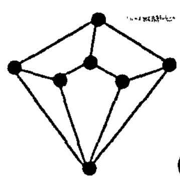
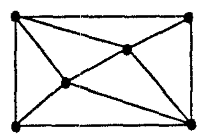
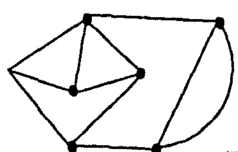
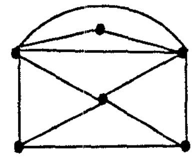
	Contracted graph	Dual of contracted graph
67	 (6,10)	 (6,10)
68	 (6,10)	 (6,10)
69	 (6,10)	 (6,10)
70	 (7,11)	 (6,11)
71	 (7,11)	 (6,11)
72	 (7,11)	 (6,11)

Table B: Continued

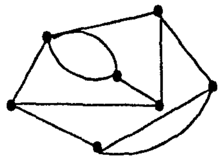
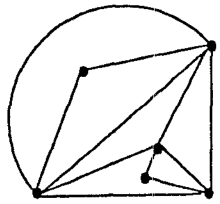
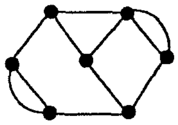
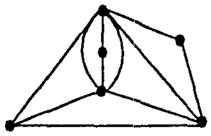
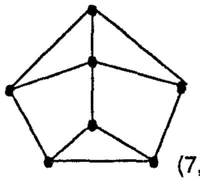
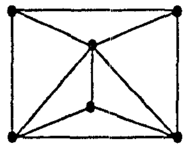
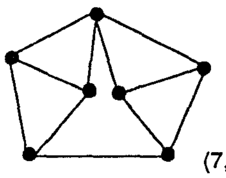
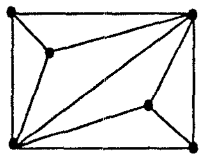
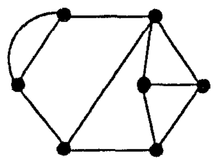
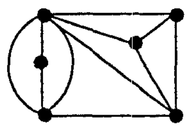
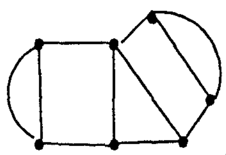
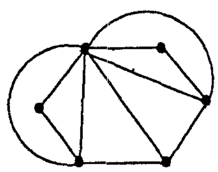
	Contracted graph	Dual of contracted graph
73	 (7,11)	 (6,11)
74	 (7,11)	 (6,11)
75	 (7,11)	 (6,11)
76	 (7,11)	 (6,11)
77	 (7,11)	 (6,11)
78	 (7,11)	 (6,11)

Table B: Continued

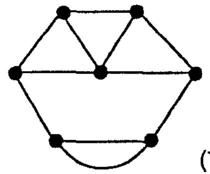
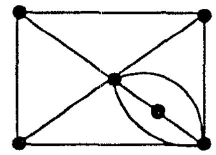
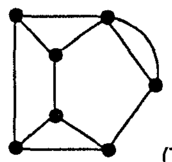
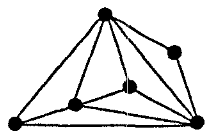
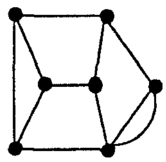
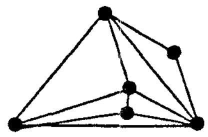
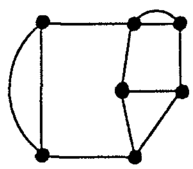
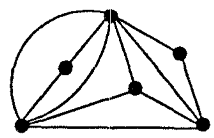
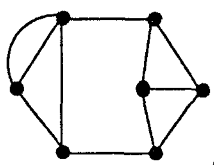
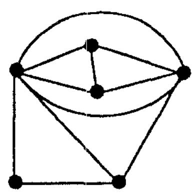
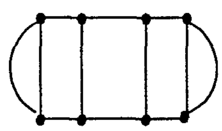
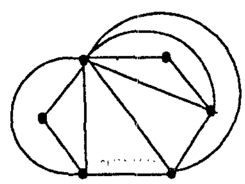
	Contracted graph	Dual of contracted graph
79	 (7,11)	 (6,11)
80	 (7,11)	 (6,11)
81	 (7,11)	 (6,11)
82	 (7,11)	 (6,11)
83	 (7,11)	 (6,11)
84	 (8,12)	 (6,12)

Table B: Continued

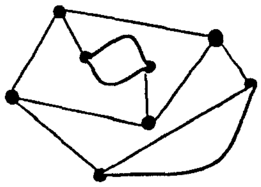
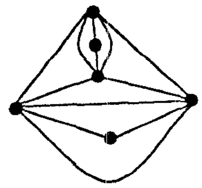
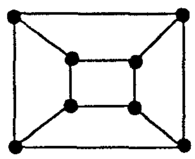
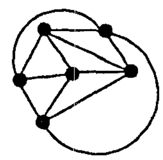
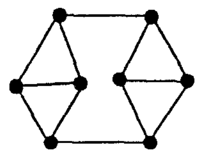
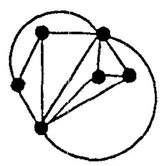
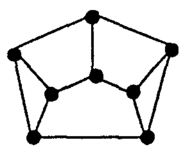
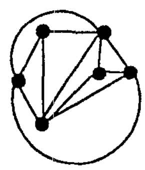
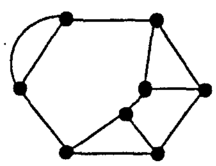
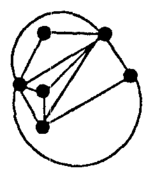
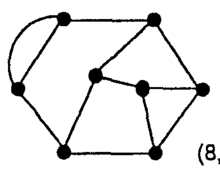
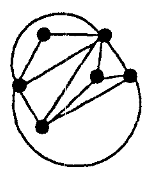
	Contracted graph	Dual of contracted graph
85	 (8,12)	 (6,12)
86	 (8,12)	 (6,12)
87	 (8,12)	 (6,12)
88	 (8,12)	 (6,12)
89	 (8,12)	 (6,12)
90	 (8,12)	 (6,12)

Table B: Continued

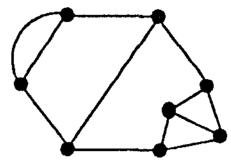
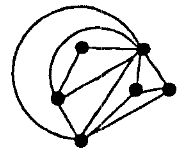
	Contracted graph	Dual of contracted graph
91	 (8,12)	 (6,12)

Table B: Continued

C Number of Graphs with Different Sets of Vertices and Joints

Number of vertices	Number of edges	Number of loops	Number of graphs
5	5	2	2
	6	3	2
	7	4	3
	8	5	2
	9	6	1
6	6	2	1
	7	3	3
	8	4	9
	9	5	13
	10	6	11
7	7	2	1
	8	3	4
	9	4	17
	10	5	26
	11	6	78
8	8	2	1
	9	3	6
	10	4	26
	11	5	140
	12	6	387

Table C

D Conventional Graphs with Seven Vertices

Let k_i denote the number of edges of a binary-vertex chain and k_{max} denote the maximum number of edges of a binary-vertex chain.

Table D.1 The (7,7), (7,8) and (7,9) conventional graphs.

k_i of the (7,9) conventional graphs shown in Table D.1 is not greater than three.

k_{max} of the (7,9) conventional graphs is equal to four.

Table D.2 The (7,10) conventional graphs ($k_i \leq 3$).

k_{max} of the (7,10) conventional graphs is equal to four.

Table D.3 The (7,11) conventional graphs ($k_i \leq 3$).

k_{max} of the (7,11) conventional graphs is equal to three.

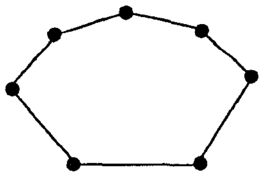
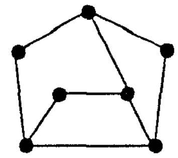
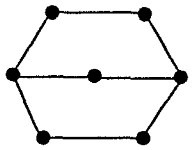
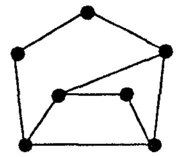
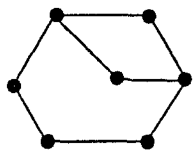
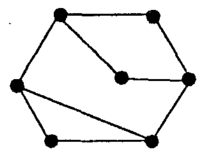
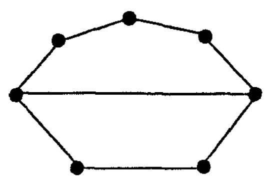
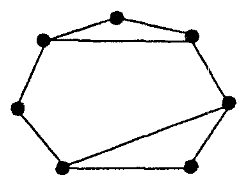
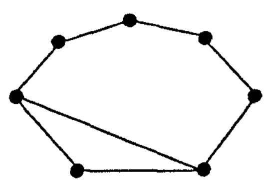
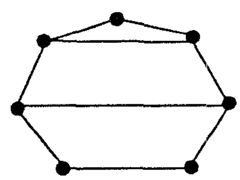
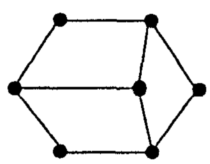
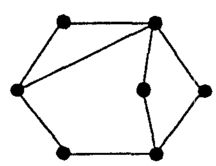
v	e	Conventional graph	v	e	Conventional graph
7	7	(1) 	7	9	(2) 
7	8	(1) 			(3) 
		(2) 			(4) 
		(3) 			(5) 
		(4) 			(6) 
7	9	‘(1) 			(7) 

Table D.1: The (7,7), (7,8) and (7,9) conventional graphs

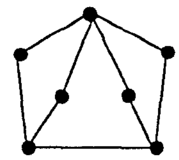
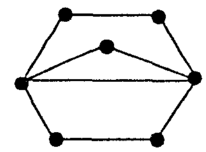
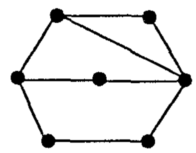
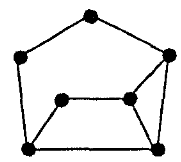
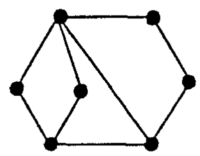
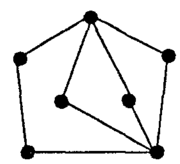
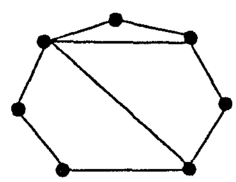
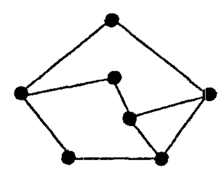
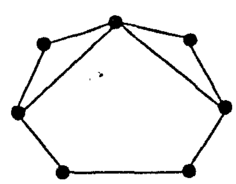
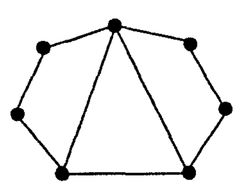
v	e	Conventional graph	v	e	Conventional graph
7	9	(8) 	7	9	(14) 
		(9) 			(15) 
		(10) 			(16) 
		(11) 			(17) 
		(12) 			
		(13) 			

Table D.1: Continued

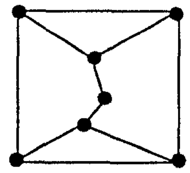
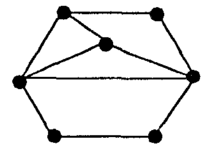
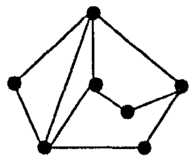
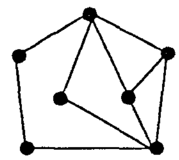
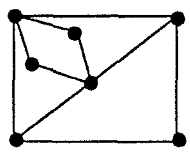
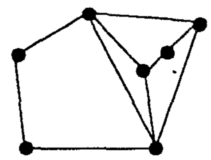
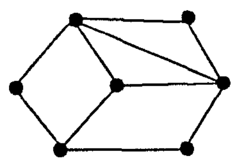
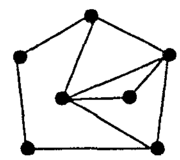
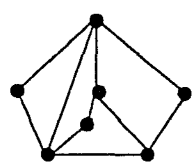
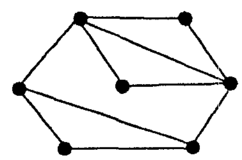
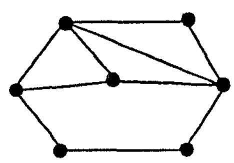
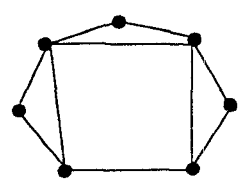
v	e	Conventional graph	v	e	Conventional graph
7	10	(1) 	7	10	(7) 
		(2) 			(8) 
		(3) 			(9) 
		(4) 			(10) 
		(5) 			(11) 
		(6) 			(12) 

Table D.2: The (7,10) conventional graphs

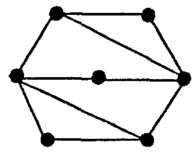
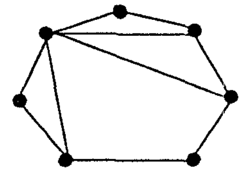
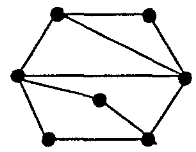
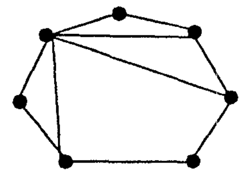
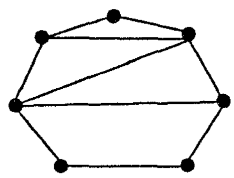
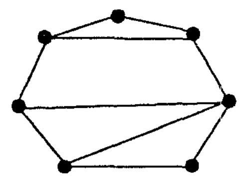
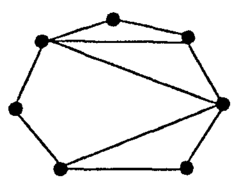
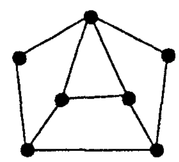
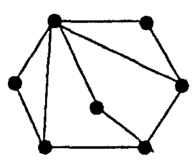
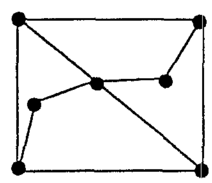
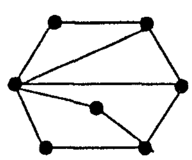
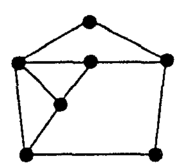
v	e	Conventional graph	v	e	Conventional graph
7	10	(13) 	7	10	(19) 
		(14) 			(20) 
		(15) 			(21) 
		(16) 			(22) 
		(17) 			(23) 
		(18) 			(24) 

Table D.2: Continued

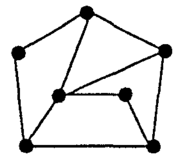
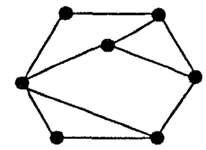
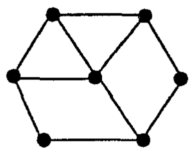
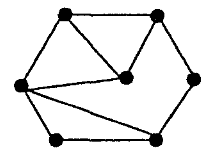
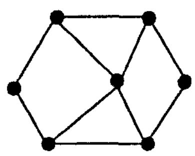
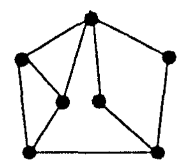
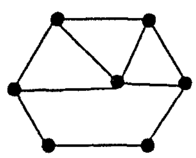
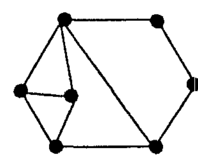
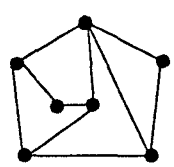
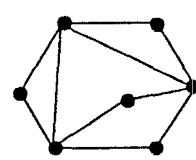
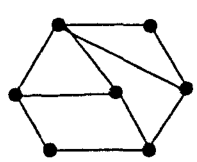
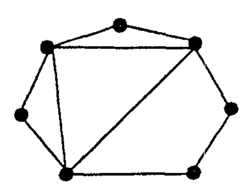
v	e	Conventional graph	v	e	Conventional graph
7	10	(25) 	7	10	(31) 
		(26) 			(32) 
		(27) 			(33) 
		(28) 			(34) 
		(29) 			(35) 
		(30) 			(36) 

Table D.2: Continued

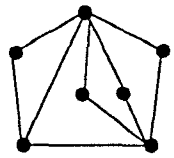
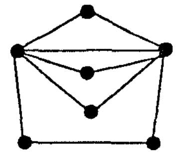
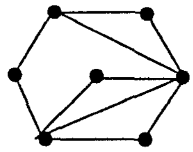
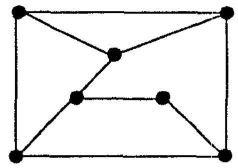
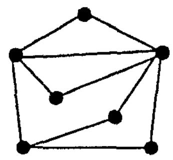
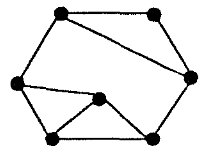
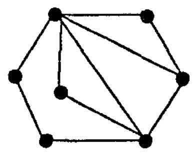
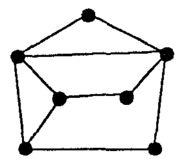
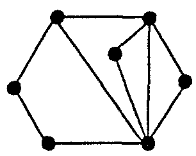
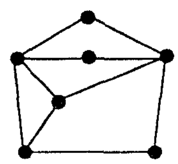
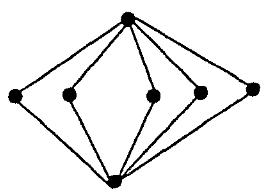
v	e	Conventional graph	v	e	Conventional graph
7	10	(37) 	7	10	(43) 
		(38) 			(44) 
		(39) 			(45) 
		(40) 			(46) 
		(41) 			(47) 
		(42) 			

Table D.2: Continued

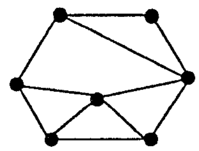
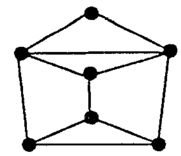
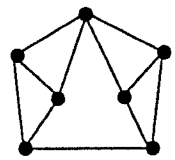
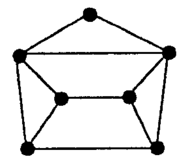
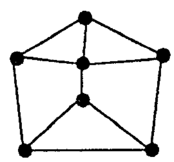
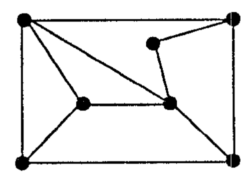
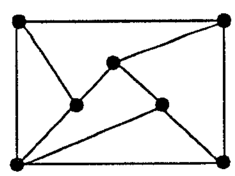
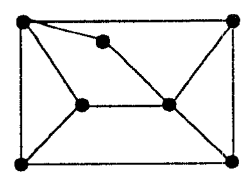
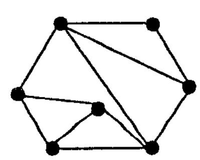
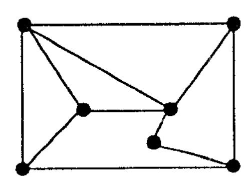
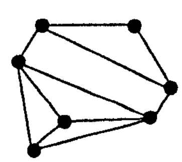
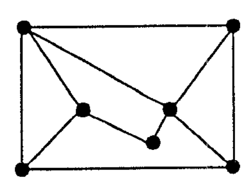
v	e	Conventional graph	v	e	Conventional graph
7	11	(1) 	7	11	(7) 
		(2) 			(8) 
		(3) 			(9) 
		(4) 			(10) 
		(5) 			(11) 
		(6) 			(12) 

Table D.3: The (7,11) conventional graphs

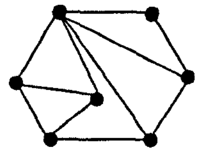
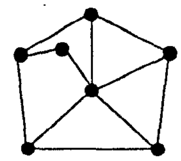
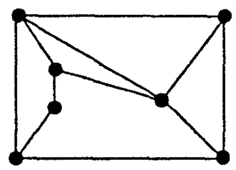
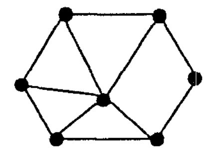
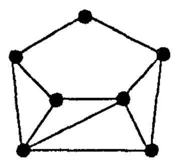
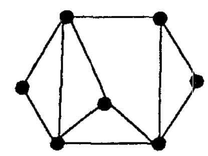
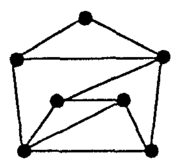
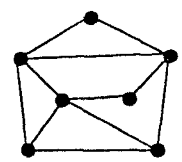
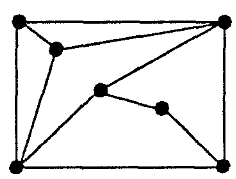
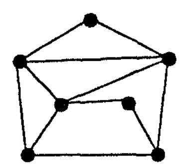
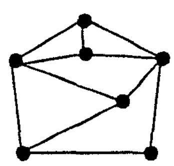
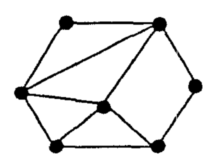
v	e	Conventional graph	v	e	Conventional graph
7	11	(13) 	7	11	(19) 
		(14) 			(20) 
		(15) 			(21) 
		(16) 			(22) 
		(17) 			(23) 
		(18) 			(24) 

Table D.3: Continued

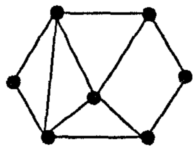
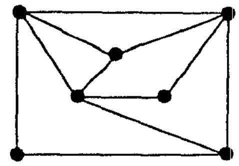
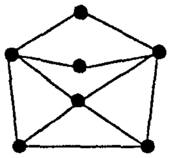
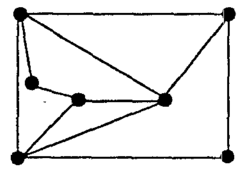
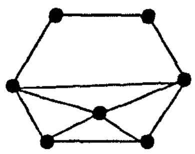
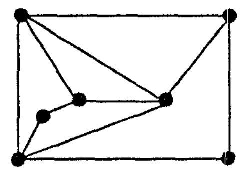
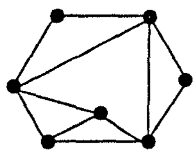
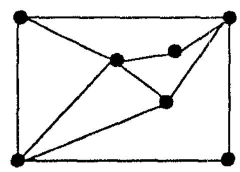
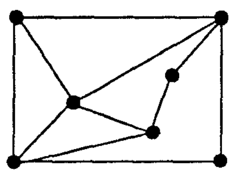
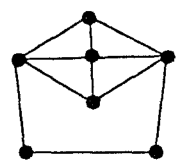
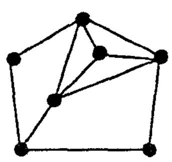
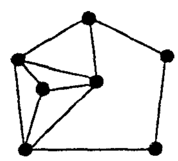
v	e	Conventional graph	v	e	Conventional graph
7	11	(25) 	7	11	(31) 
		(26) 			(32) 
		(27) 			(33) 
		(28) 			(34) 
		(29) 			(35) 
		(30) 			(36) 

Table D.3: Continued

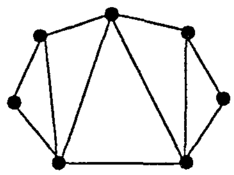
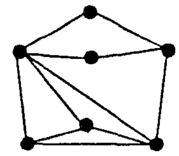
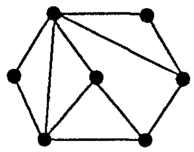
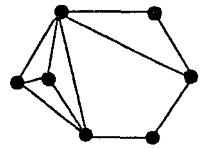
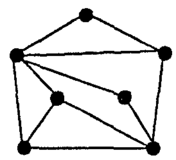
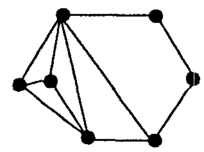
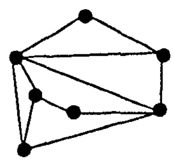
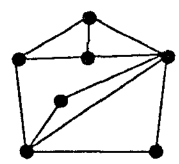
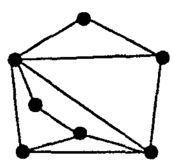
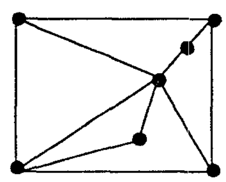
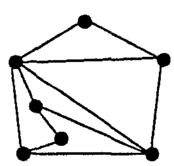
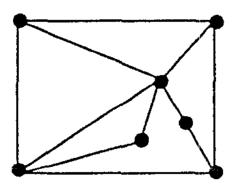
v	e	Conventional graph	v	e	Conventional graph
7	11	(37) 	7	11	(43) 
		(38) 			(44) 
		(39) 			(45) 
		(40) 			(46) 
		(41) 			(47) 
		(42) 			(48) 

Table D.3: Continued

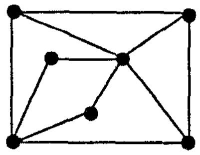
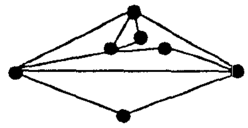
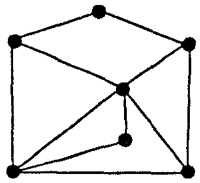
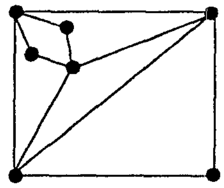
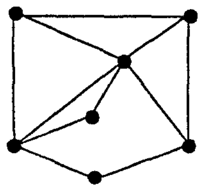
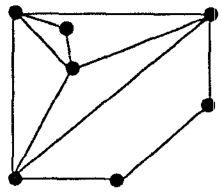
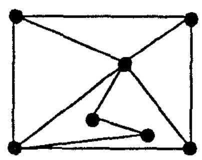
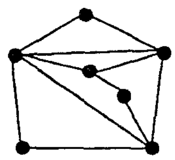
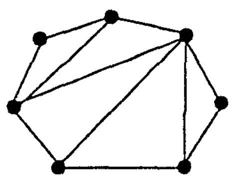
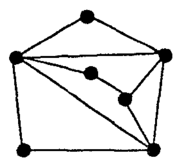
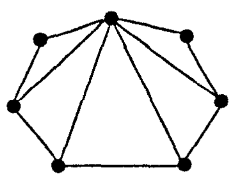
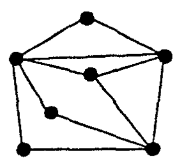
v	e	Conventional graph	v	e	Conventional graph
7	11	(49) 	7	11	(55) 
		(50) 			(56) 
		(51) 			(57) 
		(52) 			(58) 
		(53) 			(59) 
		(54) 			(60) 

Table D.3: Continued

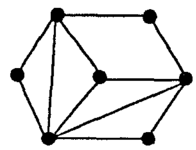
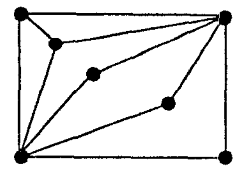
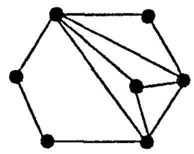
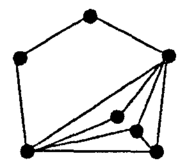
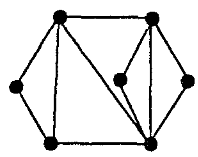
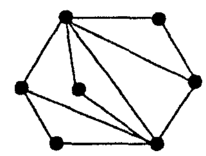
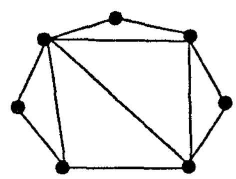
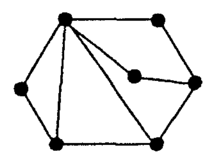
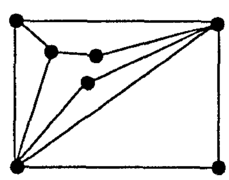
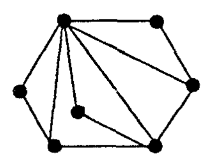
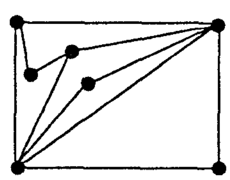
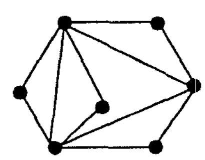
v	e	Conventional graph	v	e	Conventional graph
7	11	(61) 	7	11	(67) 
		(62) 			(68) 
		(63) 			(69) 
		(64) 			(70) 
		(65) 			(71) 
		(66) 			(72) 

Table D.3: Continued

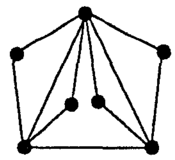
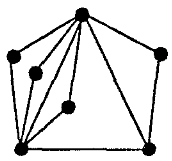
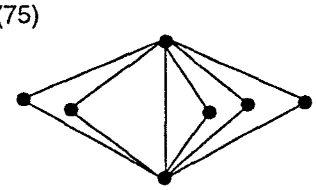
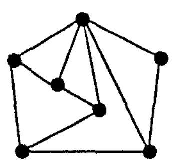
v	e	Conventional graph	v	e	Conventional graph
7	11	(73) 			
		(74) 			
		(75) 			
		(76) 			

Table D.3: Continued

E Conventional Graphs with Eight Vertices

Let k_i denote the number of edges of a binary-vertex chain and k_{max} denote the maximum number of edges of a binary-vertex chain.

Table E.1 The (8,8) and (8,9) conventional graphs.

k_{max} of the (8,9) conventional graphs is equal to six.

Table E.2 The (8,10) conventional graphs ($k_i \leq 3$ only).

k_{max} of the (8,10) conventional graphs is equal to five.

Table E.3 The (8,11) conventional graphs ($k_i \leq 3$ only).

k_{max} of the (8,11) conventional graphs is equal to five.

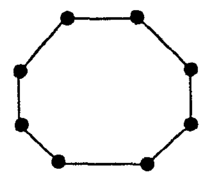
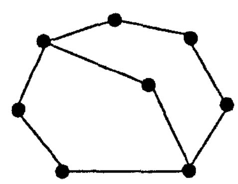
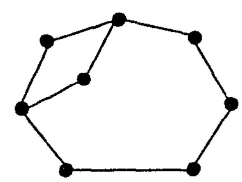
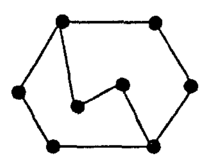
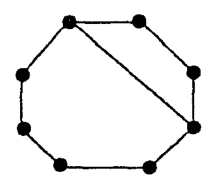
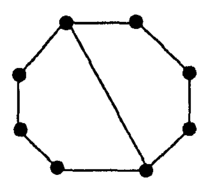
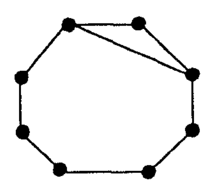
v	e	Conventional graph	
8	8	(1) 	
8	9	(1) 	(4) 
		(2) 	(5) 
		(3) 	(6) 

Table E.1: The (8,8) and (8,9) conventional graphs

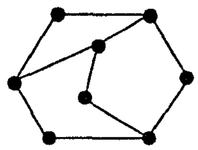
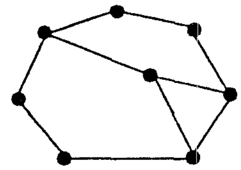
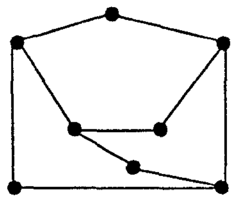
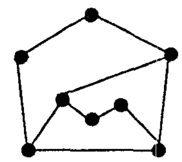
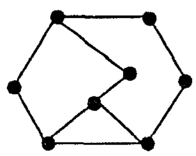
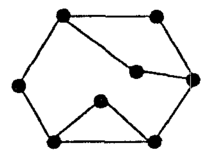
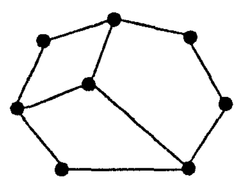
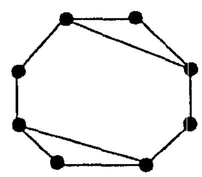
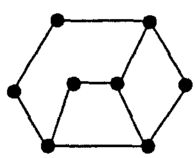
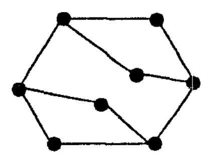
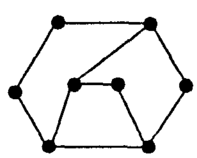
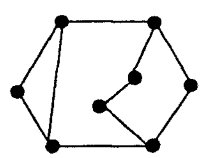
v	e	Conventional graph	v	e	Conventional graph
8	10	(1) 	8	10	(7) 
		(2) 			(8) 
		(3) 			(9) 
		(4) 			(10) 
		(5) 			(11) 
		(6) 			(12) 

Table E.2: The (8,10) conventional graphs

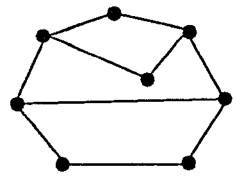
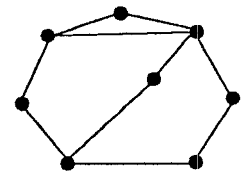
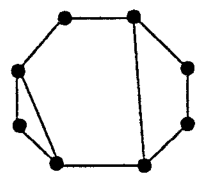
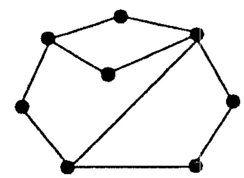
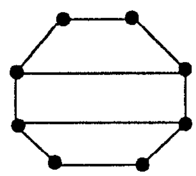
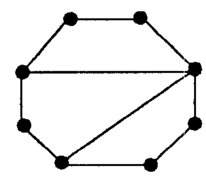
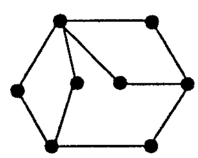
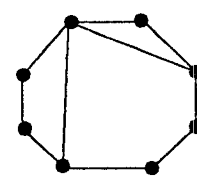
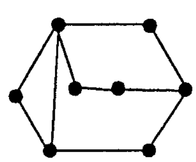
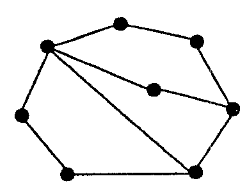
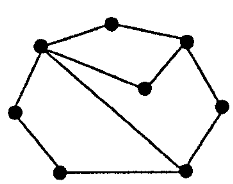
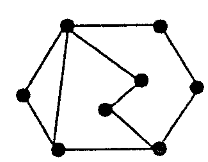
v	e	Conventional graph	v	e	Conventional graph
8	10	(13) 	8	10	(19) 
		(14) 			(20) 
		(15) 			(21) 
		(16) 			(22) 
		(17) 			(23) 
		(18) 			(24) 

Table E.2: Continued

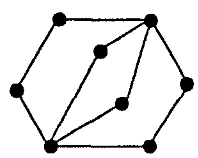
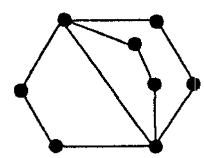
v	e	Conventional graph	v	e	Conventional graph
8	10	(25) 	8	10	(26) 

Table E.2: Continued

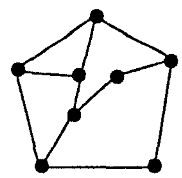
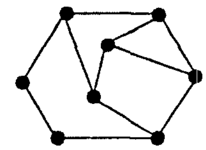
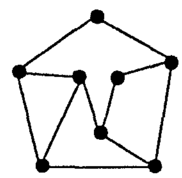
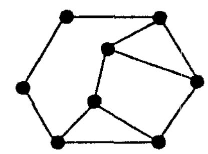
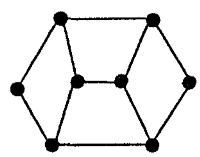
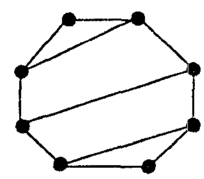
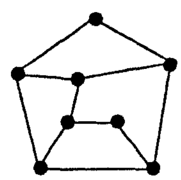
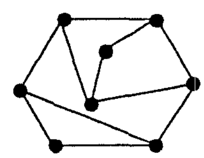
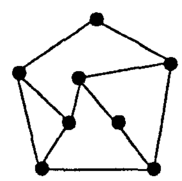
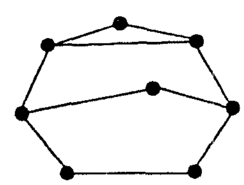
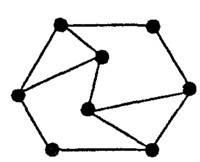
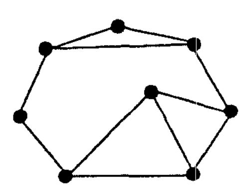
v	e	Conventional graph	v	e	Conventional graph
8	11	(1) 	8	11	(7) 
		(2) 			(8) 
		(3) 			(9) 
		(4) 			(10) 
		(5) 			(11) 
		(6) 			(12) 

Table E.3: The (8,11) conventional graphs

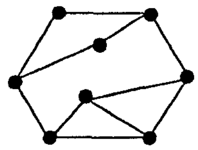
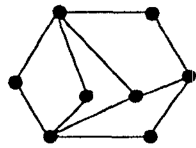
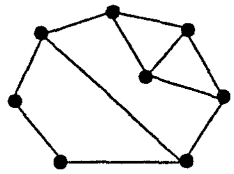
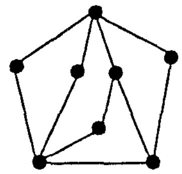
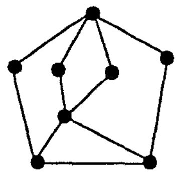
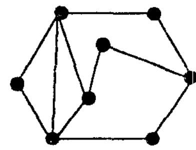
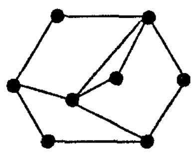
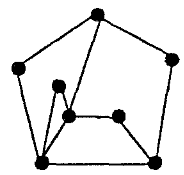
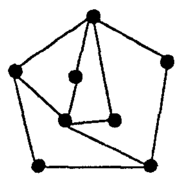
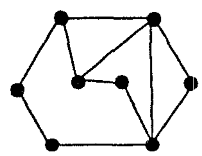
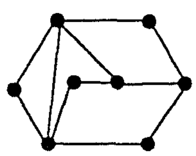
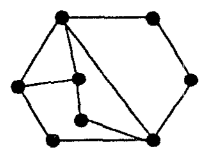
v	e	Conventional graph	v	e	Conventional graph
8	11	(13) 	8	11	(19) 
		(14) 			(20) 
		(15) 			(21) 
		(16) 			(22) 
		(17) 			(23) 
		(18) 			(24) 

Table E.3: Continued

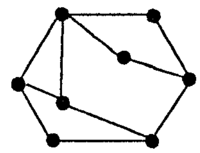
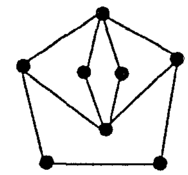
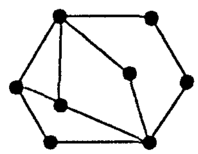
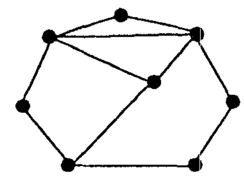
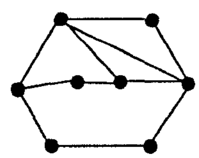
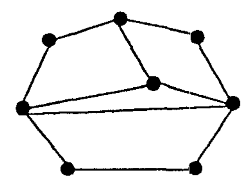
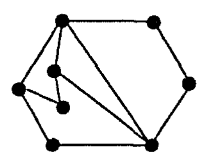
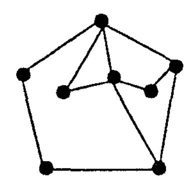
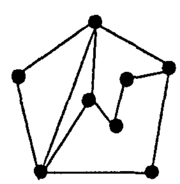
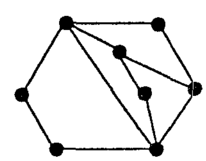
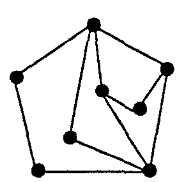
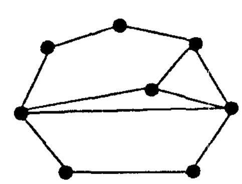
v	e	Conventional graph	v	e	Conventional graph
8	11	(25) 	8	11	(31) 
		(26) 			(32) 
		(27) 			(33) 
		(28) 			(34) 
		(29) 			(35) 
		(30) 			(36) 

Table E.3: Continued

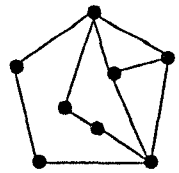
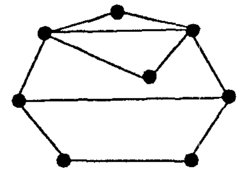
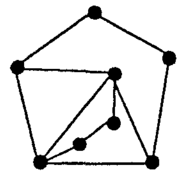
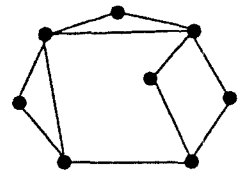
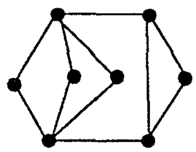
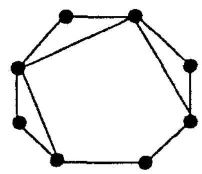
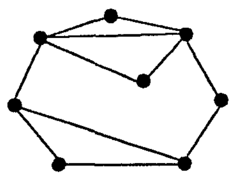
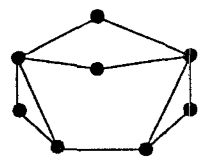
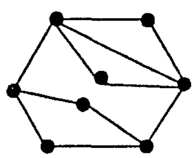
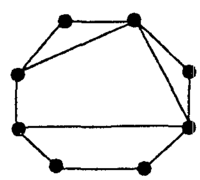
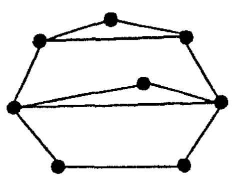
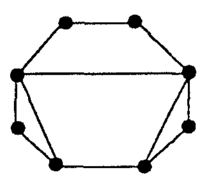
v	e	Conventional graph	v	e	Conventional graph
8	11	(37) 	8	11	(43) 
		(38) 			(44) 
		(39) 			(45) 
		(40) 			(46) 
		(41) 			(47) 
		(42) 			(48) 

Table E.3: Continued

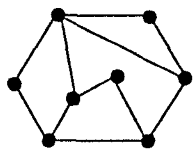
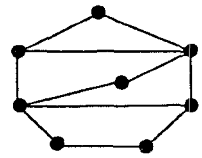
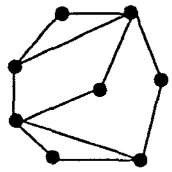
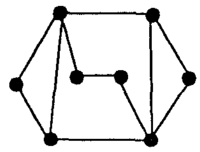
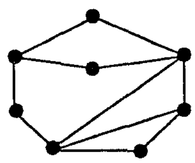
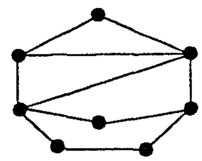
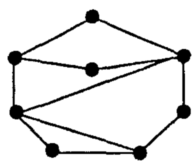
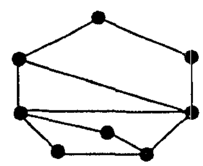
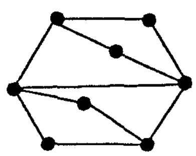
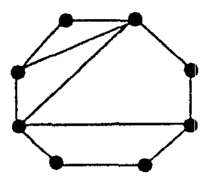
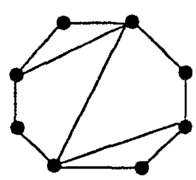
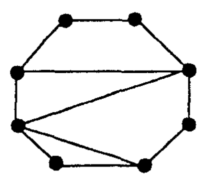
v	e	Conventional graph	v	e	Conventional graph
8	11	(49) 	8	11	(55) 
		(50) 			(56) 
		(51) 			(57) 
		(52) 			(58) 
		(53) 			(59) 
		(54) 			(60) 

Table E.3: Continued

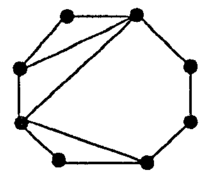
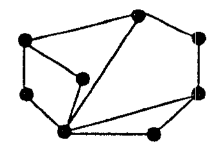
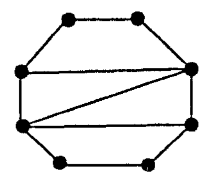
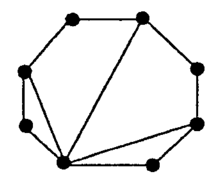
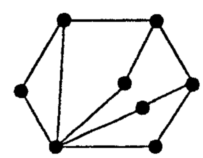
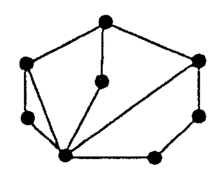
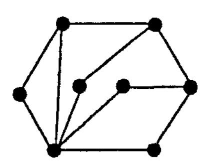
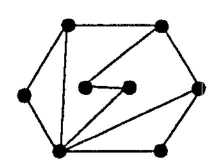
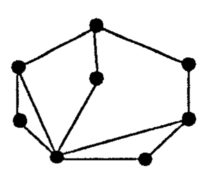
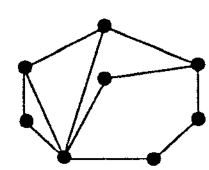
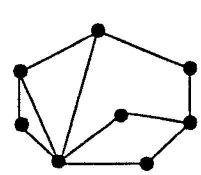
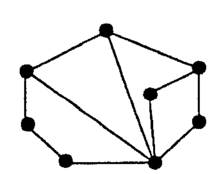
v	e	Conventional graph	v	e	Conventional graph
8	11	(61) 	8	11	(67) 
		(62) 			(68) 
		(63) 			(69) 
		(64) 			(70) 
		(65) 			(71) 
		(66) 			(72) 

Table E.3: Continued

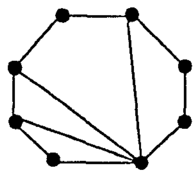
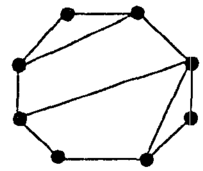
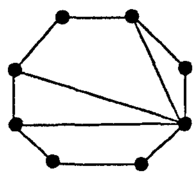
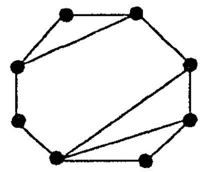
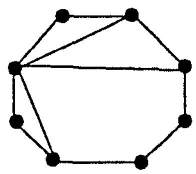
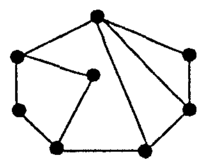
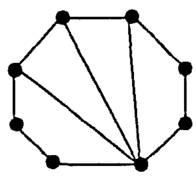
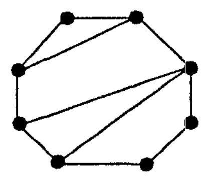
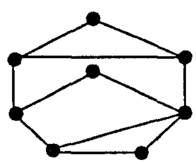
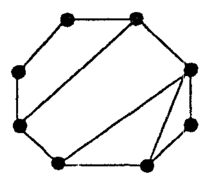
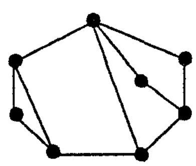
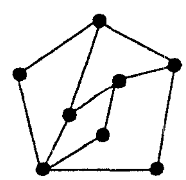
v	e	Conventional graph	v	e	Conventional graph
8	11	(73) 	8	11	(79) 
		(74) 			(80) 
		(75) 			(81) 
		(76) 			(82) 
		(77) 			(83) 
		(78) 			(84) 

Table E.3: Continued

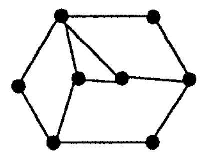
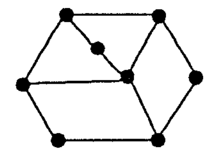
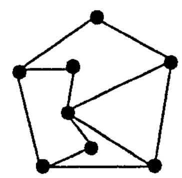
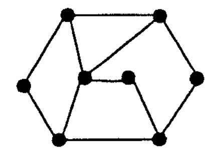
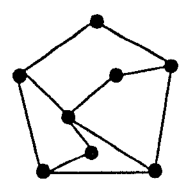
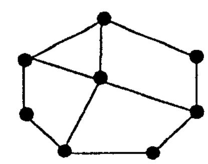
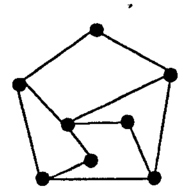
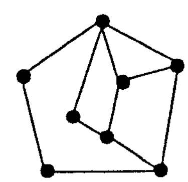
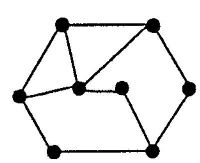
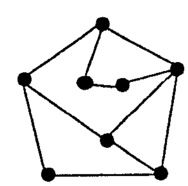
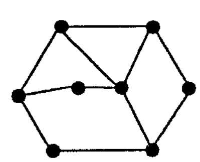
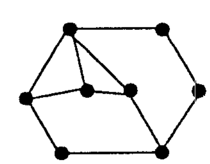
v	e	Conventional graph	v	e	Conventional graph
8	11	(85) 	8	11	(91) 
		(86) 			(92) 
		(87) 			(93) 
		(88) 			(94) 
		(89) 			(95) 
		(90) 			(96) 

Table E.3: Continued

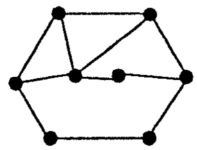
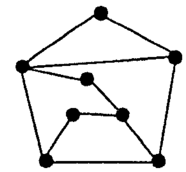
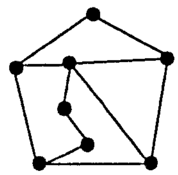
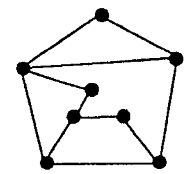
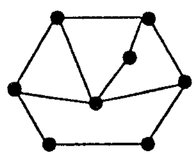
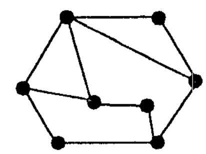
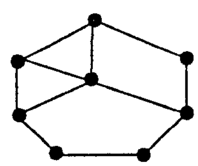
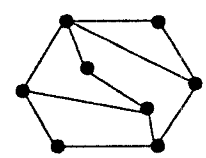
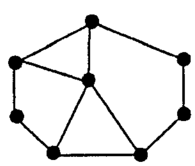
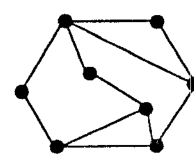
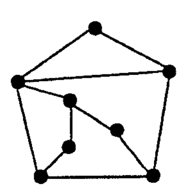
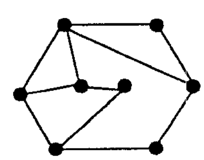
v	e	Conventional graph	v	e	Conventional graph
8	11	(97) 	8	11	(103) 
		(98) 			(104) 
		(99) 			(105) 
		(100) 			(106) 
		(101) 			(107) 
		(102) 			(108) 

Table E.3: Continued

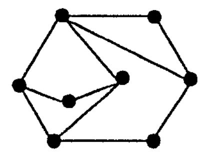
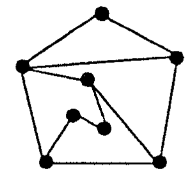
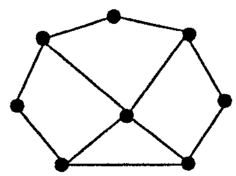
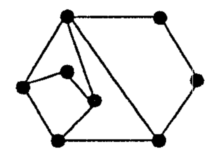
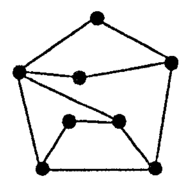
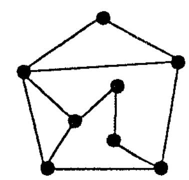
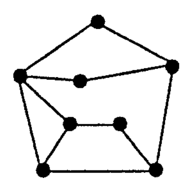
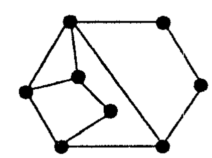
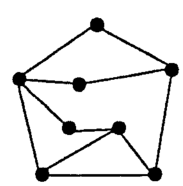
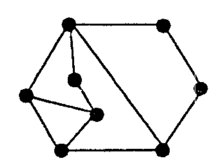
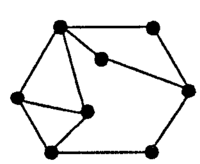
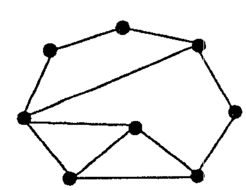
v	e	Conventional graph	v	e	Conventional graph
8	11	(109) 	8	11	(115) 
		(110) 			(116) 
		(111) 			(117) 
		(112) 			(118) 
		(113) 			(119) 
		(114) 			(120) 

Table E.3: Continued

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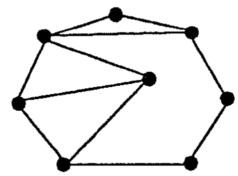
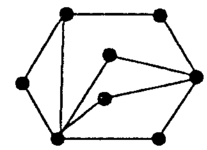
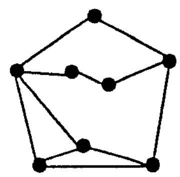
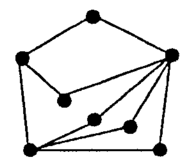
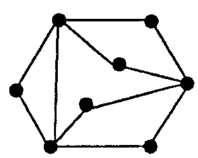
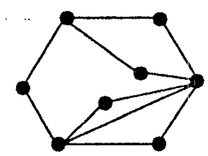
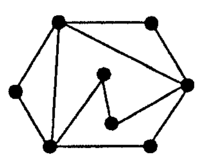
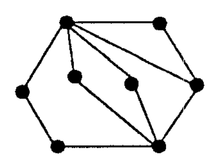
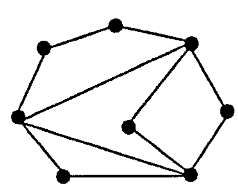
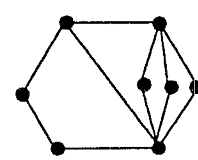
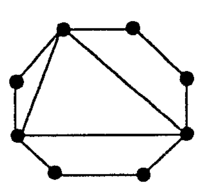
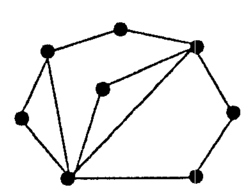
8	11	(121) 	8	11	(127) 
		(122) 			(128) 
		(123) 			(129) 
		(124) 			(130) 
		(125) 			(131) 
		(126) 			(132) 

Table E.3: Continued

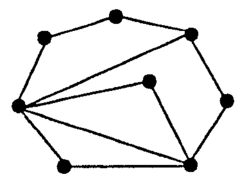
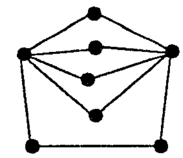
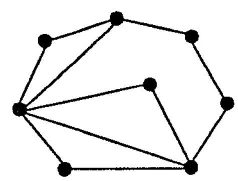
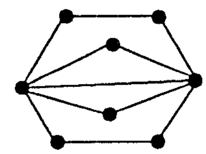
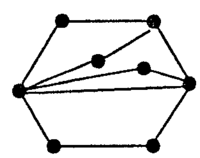
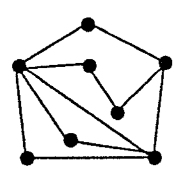
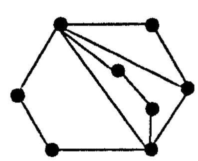
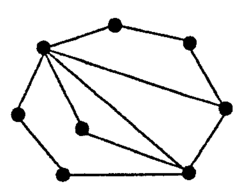
v	e	Conventional graph	v	e	Conventional graph
8	11	(133) 	8	11	(139) 
		(134) 			(140) 
		(135) 			
		(136) 			
		(137) 			
		(138) 			

Table E.3: Continued

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