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Multiple-Access Capability of Frequency-Hopped Spread-Spectum Revisited: An Exact Analysis of the Effect of Unequal Power Levels

by

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MULTIPLE-ACCESS CAPABILITY OF FREQUENCY-HOPPED SPREAD-SPECTRUM REVISITED: AN EXACT ANALYSIS OF THE EFFECT OF UNEQUAL POWER LEVELS

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ABSTRACT

In this paper we present a new method for the evaluation of the probability of error of uncoded frequency-hopped spread-spectrum multiple-access (FH/SSMA) communications. For systems with binary FSK modulation the method provides an accurate approximation and a tight upper bound to the bit error probability; for systems with M-ary FSK modulation it provides tight upper bounds to the symbol error probability. This method relies on the integration of the product of the characteristic function of the envelope of the branch of the BFSK demodulator, which carries the desired signal, and of the derivative of the characteristic function of the envelope of the other branch; it can achieve any desirable acccuracy and the computational effort required for its evaluation grows linearly with the number of interfering users. In the M-ary case tight upper bounds based on the union bound and the results of the binary case are derived.

The new method allows us to quantify accurately the effect of unequal power levels on the other-user interference for the first time. Comparison of the multiple-access capability of FH/SS systems as predicted by the bounds available in the literature and by the new method indicates that FH/SS systems without error-control can support (at a given error rate) considerably many more simultaneous users than previously thought when the relative received powers of the users are not significantly different. This trend is amplified further for systems with error-control. Our results indicate that the FH/SSMA systems also suffer from the near-far problem although less seriously than the direct-sequence SSMA systems.

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1. Introduction and Problem Statement

Previous research on frequency-hopped spread-spectrum multiple-access systems (FH/SSMA) has not provided exact results on the average probability of error, primarily because of the difficulty to evaluate accurately the conditional probability of error given that a number of interfering signals hit the desired signal. Due to the lack of accurate expressions for the error probability, the effect of unequal power levels of the interfering users--termed the near-far problem in the context of direct-sequence spread-spectrum (DS/SS) systems--has not been studied. In this paper we remedy this situation by deriving accurate approximations and tight upper bounds on the bit (and symbol) error probabilities of the FH/SSMA systems that take into consideration the effect of unequal power levels of the interfering signals. In the following we present a brief review of the existing results and identify the difficulties encountered in the evaluation of the error probabilities for the FH/SSMA systems.

In the performance of FH/SSMA and hybrid FH-DS/SSMA systems with M-ary FSK modulation and noncoherent demodulation, $\overline{P}_e(K)$, the probability of a symbol error given that K other users share the same channel with the user under consideration (desired signal), plays a seminal role. This probability is evaluated as follows. If $P_e(k)$ denotes the conditional probability of a symbol error when hits from k users occur, and P_h denotes the probability that any particular other user will hit a symbol of the desired signal, then $\overline{P}_e(K)$ is upper-bounded by

$$\overline{P}_e(K) \leq \sum_{k=0}^{K} \binom{K}{k} P_h^k (1-P_h)^{K-k} \; P_e(k). \tag{1}$$

In (1) $P_e(k)$ actually denotes the probability of error when k full hits occur. We say that a **full hit** from an interfering signal occurs when the signal is present in the same frequency bin (slot) for the entire duration of the particular M-ary symbol. Similarly a **partial hit** occurs when the interfering signal is present in the same frequency bin for part of the M-ary symbol's duration. The terms full and partial hits described above have been used in the relevant literature to mean exclusively hits that occur at the level of the frequency-hopping pattern. However, to evaluate the performance of FH/SSMA systems accurately, one has to distinguish between two levels of hits: those occuring at the frequency hopping level (on the frequency bins or slots used for frequency hopping) and those occuring at the level of the M-ary FSK tones (used for modulating the data); we name the former **FH hits** and the latter **tone hits**. The probability of an FH hit (full or partial) for an AWGN or a nonselective fading channel and random memoryless hopping patterns is given by [Geraniotis & Pursley, 1982] $P_h = \left(1+\frac{1}{N_s}\right)\frac{1}{q}$, where N_s is the number of M-ary FSK symbols per dwell time and q is the number of frequencies available for hopping. Therefore, (1) provides an upper bound since it is assumed that all the FH hits (occuring with probability P_h) are full FH hits.

In the existing literature hard bounds and approximations on $P_e(k)$ have been obtained. Specifically, in [Geraniotis & Pursley, 1982] the conditional probability of error given that k users cause full FH hits, was upper-bounded by 1 due to the difficulty of obtaining more accurate estimates of its value; that is, it was assumed that all (FH) hits result in symbol errors. Later in [Geraniotis, 1985] and [Geraniotis, 1986] a Gaussian-approximation technique was proposed for evaluating $P_e(k)$ of coherent and noncoherent hybrid FH-DS/SSMA systems. If the number of chips per bit N is set to 1, these results provide approximations to $P_e(k)$ for FH/SSMA systems. For coherent systems the accuracy of the Gaussian-approximation technique was checked via a more accurate characteristic-function technique. However, for noncoherent systems the accuracy of the Gaussian-approximation technique was never validated by more accurate results due to the lack of any.

It is advantageous to distinguish between full FH hits and partial FH hits (see [Geraniotis, 1985]). In this case we can write

$$\bar{P}_e(K) = \sum_{k_f=0}^{K} \sum_{k_p=0}^{K-k_f} \binom{K}{k_f} \binom{K-k_f}{k_p} P_f^{k_f} P_p^{k_p} (1-P_h)^{K-k_f-k_p} P_e(k_f, k_p) \tag{2}$$

where k_f and k_p denote the number of users causing full FH hits and partial FH hits, respectively, P_f and P_p denote the corresponding probabilities of full and partial FH hits, $P_h = P_f + P_p$, and $P_e(k_f, k_p)$ denotes the probability of symbol error conditioned on the occurence of k_f full hits and k_p partial hits. The probabilities of full and partial FH hits from a typical interfering user employing a random memoryless frequency-hopping pattern are given by [Geraniotis, 1985] the expressions $P_f = \left(1-\frac{1}{N_s}\right)\frac{1}{q}$ and $P_p = \frac{2}{N_s}\frac{1}{q}$, respectively. If $P_e(k_f, k_p)$ could be computed exactly, (2) would provide an expression for the exact probability of error of MFSK FH/SSMA systems given K interferers.

In [Geraniotis, 1985] and [Geraniotis, 1986] $P_e(k_f, k_p)$ has been evaluated via the aforementioned Gaussian-approximation technique. Again for noncoherent systems the accuracy of the results was not validated due to the lack of a more accurate computational technique.

In the following sections we derive *exact expressions* (arbitrarily accurate approximations) for $P_e(k)$ and $P_e(k_f, k_p)$ of binary FSK FH/SSMA systems, *tight upper bounds* for these quantities for M-ary FSK FH/SSMA systems, and *Gaussian approximations* to them for both binary and M-ary FH/SSMA systems. In particular, binary FSK

FH/SSMA systems are treated in Section 2 and M-ary FSK FH/SSMA systems are treated in Section 3. Numerical results are presented in Section 4 and conclusions are drawn in Section 5.

Both the exact expressions and the approximations that we develop in the following sections take into account the other-user interference in an accurate way. In particular, the (possibly different) received power levels of the interfering users (which depend in their transmitting powers and their distances from the receiver under consideration) enter in these expressions.

Before proceeding with the derivations of accurate expressions and approximations for $P_e(k)$ and $P_e(k_f, k_p)$ we need to provide an accurate characterization of the other-user interference. In particular, we distinguish between hits at the frequency-hopping pattern level (FH hits) and hits at the level of the frequency tones used by the M-ary FSK modulation scheme (tone hits) and derive the probabilities of occurence of these different events.

Under the assumption that all **FH hits** (hits at the frequency-hopping pattern level) during the reception of an M-ary FSK symbol are **full hits**, we can write the interference due to the k-th interfering signal which is present at the output of the matched filter of the in-phase component of the m-th branch ($m = 1, 2, \ldots, M$) of the noncoherent MFSK demodulator as [Geraniotis and Pursley, 1982]:

$$I_{c,m}^{(k)} = \sqrt{\frac{P_k}{2}}\, \delta(b_0^{(k)}, m)[R_\psi(\tau_k) + \hat{R}_\psi(\tau_k)] \cos\left(\theta_m^{(k)} + \alpha_k - \beta\right). \tag{3}$$

In (3) P_k is the power of the k-th signal, T_s the M-ary symbol duration, τ_k its delay $(mod T_s)$, $b_0^{(k)} = m' \in \{1, 2, \cdots, M\}$ the information symbol of the k-th user, $\theta_m^{(k)}$ the phase corresponding to the frequency tone $f_{m'}$ carrying the information symbol

m',

and α_k, β the hopping phase and dehopping phase (see [Geraniotis and Pursley, 1982]) for the k-th transmitter and the receiver under consideration, respectively. The function δ is defined as $\delta(u,v)=1$ for $u=v$ and 0 for $u\neq v$. The functions R_ψ and $\hat{R}_\psi$ denote the continuous partial autocorrelation functions of the shaping waveform $\psi(t)$ (time limited in $[0, T_s]$) and they are defined as

$$\hat{R}_\psi(\tau)=\int_\tau^{T_s}\psi(t)\psi(t-\tau)dt \quad \text{and} \quad R_\psi(\tau)=\hat{R}_\psi(T_s-\tau).$$

For a rectangular shaping waveform $R_\psi(\tau)=\tau$, $\hat{R}_\psi(\tau)=T_s-\tau$ and (3) becomes

$$I_{c,m}^{(k)}=\sqrt{\frac{P_k}{2}}\,T_s\,\delta(b_0^{(k)},m)\cos\left(\theta_{m'}^{(k)}+\alpha_k-\beta\right). \tag{4}$$

We assume that $b_0^{(k)}$ takes values in $\{1, 2, \cdots, M\}$ with equal probability, that the delay $\tau_k \ mod \ T_s$ is uniformly distributed in $[0, T_s]$, the phase angles are uniformly distributed in $[0, 2\pi]$, and that they are mutually independent random variables. Thus for the M-ary FSK system (3) results in a **full tone hit** with probability $\frac{1}{M}$ [the probability that the Kroencker δ in (3) equals 1] and in **no tone hit** with probability $1-\frac{1}{M}$. Finally, the interference present at the quadrature component of the m-th branch of the noncoherent MFSK demodulator can be obtained from (3) or (4) if we replace $\cos(\cdot)$ by $\sin(\cdot)$.

More accurately, since the k-th interfering signal causes both **full and partial FH hits** during the reception of a particular M-ary symbol we have

$$I_{c,m}^{(k)}=\sqrt{\frac{P_k}{2}}\,T_s\,\delta(b_{-1}^{(k)},m)R_\psi(\tau_k)\cos\left(\theta_{m'}^{(k)}+\alpha_k-\beta\right) \tag{5a}$$

or

$$I_{c,m}^{(k)} = \sqrt{\frac{P_k}{2}}\, T_s\, \delta(b_0^{(k)}, m)\hat{R}_\psi(\tau_k)\cos\left(\theta_m^{(k)} + \alpha_k - \beta\right) \tag{5b}$$

when a partial FH hit occurs, and

$$I_{c,m}^{(k)} = \sqrt{\frac{P_k}{2}}\, T_s \left[\delta\left(b_{-1}^{(k)}, m\right) R_\psi(\tau_k)\cos\left[\theta_m^{(k)} + \alpha_k - \beta\right) + \delta\left(b_0^{(k)}, m\right)\hat{R}_\psi(\tau_k)\cos\left(\theta_m^{(k)} + \alpha_k - \beta\right)\right] \tag{6}$$

when a full FH hit occurs. In (5a)-(5b) and (6) $b_{-1}^{(k)} = m'$ is the previous symbol and $b_0^{(k)} = m''$ is the current symbol of the k-th interfering user, whereas the rest of the quantities are as defined in the previous paragraph. Therefore, a **partial FH hit** causes a **partial tone hit** [to the particular (m-th) branch of the MFSK receiver] with probability $\frac{1}{M}$ and **no tone hits** with probability $1-\frac{1}{M}$. On the other hand, a **full FH hit** causes a **full tone hit** with probability $\frac{1}{M^2}$, a **partial tone hit** with probability $\frac{2(M-1)}{M^2}$ and **no tone hits** with probability $\frac{(M-1)^2}{M^2}$.

We close the description of the model for the other-user interference in FH/SSMA systems by mentioning that the cumulative interference at the output of the matched filters of each branch of the demodulator is additive and that all the random variables (data bits, phases, time delays) involved in the interference terms due to different users are mutually independent.

2. Error Probability for Binary FSK FH/SSMA Systems

The starting point for our derivation of exact expressions for $P_e(k)$ and $P_e(k_f, k_p)$ in this case is the work of [Lord, 1954] for circulary symmetric Gaussian random variables and the application of that to multiple-tone jamming of binary FSK systems by [Bird, 1985]. ~~Before using the results of these two papers for our purpose we need to characterize accurately the other-user interference in FH/SSMA systems.~~

Let R_1 and R_2 denote the outputs of the envelope detectors of the two branches ($m=1,2$) of the binary FSK system. Then the error probability becomes

$$P_e = \frac{1}{2}\left[Pr\{R_2 > R_1 \mid 1\} + Pr\{R_1 > R_2 \mid 2\}\right] = Pr\{R_2 > R_1 \mid 1\} \tag{7}$$

since it is assumed that the AWGN has the same spectral density in the channels of both branches and each one of the interfering users hits the two branches with equal probability, so that the conditional error probabilities (conditioned on messages 1 or 2 being transmitted) are equal. We can write

$$Pr\{R_1 < R_2 \mid 1\} = \int_0^\infty Pr\{R_1 < R_2 = r_2, 1\}\, p(r_2 \mid 1) dr_2 \tag{8}$$

where $p(r_2 \mid 1)$ is the *pdf* of R_2 given that message $m=1$ is transmitted and the conditional probability $Pr\{R_1 < R_2 \mid R_2 = r_2, 1\}$ is given (see [Lord, 1954]) because of circular symmetry by

$$Pr\{R_1 < R_2 \mid R_2 = r_2, 1\} = r_2 \int_0^\infty \Phi_1(u) J_1(r_2 u) du , \tag{9}$$

where $\Phi_1(u)$ is the characteristic function of R_1 and the Bessel function $J_1(x)$ is given by

$$J_1(x) = \frac{1}{\pi} \int_0^\pi \cos(x \sin\theta - \theta)\, d\theta. \tag{10}$$

Upon substitution from (9) into (8) we obtain

$$Pr\,\{R_1 < R_2 \mid 1\} = \int_0^\infty \Phi_1(u)\left[\int_0^\infty r_2 J_1(r_2 u)\, p(r_2 \mid 1) dr_2\right] du. \tag{11}$$

From the definition of the characteristic function of circularly symmetric random variables [Lord, 1954] we have that the characteristic function of R_2 is given by

$$\Phi_2(u) = \int_0^\infty J_0(r_2 u)\, p(r_2 \mid 1)\, dr_2, \tag{12}$$

where

$$J_0(x) = \frac{1}{\pi}\int_0^\pi \cos(x \sin\theta) d\theta. \tag{13}$$

Since $J_1(x) = -J_0'(x)$,

$$\frac{d\,\Phi_2(u)}{du} = -\int_0^\infty r_2 J_1(r_2 u)\, p(r_2 \mid 1) dr_2, \tag{14}$$

and (11) becomes

$$Pr\,(R_1 < R_2 \mid 1) = -\int_0^\infty \Phi_1(u)\,\frac{d\,\Phi_2(u)}{du} du. \tag{15}$$

The result of (15) is quite general and is applicable to all BFSK systems with possibly different interference present at the two branches.

In [Bird, 1985] it is shown that, if the outputs of the matched filters of the in-phase components of the two branches of the BFSK demodulator are

$$Z_{c,1} = s\cos\phi + \sum_{i=1}^{N_1} a_{1,i}\cos\gamma_{1,i} + n_{c,1} \tag{16a}$$

and

$$Z_{c,2} = \sum_{j=1}^{N_2} a_{2,j}\cos\gamma_{2,j} + n_{c,2}, \tag{16b}$$

then

$$\Phi_1(u) = \exp\left(-\frac{u^2\sigma_1^2}{2}\right) J_0(su) \prod_{i=1}^{N_1} J_0(a_{1,i}\, u) \tag{17a}$$

and

$$\Phi_2(u) = \exp\left(-\frac{u^2\sigma_2^2}{2}\right) \prod_{j=1}^{N_2} J_0(a_{2,j}\, u) \tag{17b}$$

In (16) and (17) s denotes the amplitude of the desired signal at the output of the correlator (assumed to be present at branch $m=1$), $a_{1,i}$ and $a_{2,j}$ the amplitudes of the i-th and j-th interfering signals present at branches 1 and 2, respectively, $\gamma_{1,i}$ and $\gamma_{2,j}$ their phases which are uniformly distributed in $[0,2\pi]$ and mutually independent, and $n_{c,1}$, $n_{c,2}$ the in-phase components of the AWGN. The outputs of the quadrature components $I_{s,1}$ and $I_{s,2}$ can be obtained from (16a)-(16b) by replacing all the $\cos(\cdot)$' s by $\sin(\cdot)$' s. The quantities σ_1^2 and σ_2^2 denote the variances of the AWGN present at the two branches. Finally, N_1 and N_2 denote the number of interfering signals present at the two branches.

Let us rewrite (17a) and (17b) in the form

$$\Phi_1(u) = \exp\left(-\frac{\sigma_1^2 u^2}{2}\right) J_0(su)\, \overline{\Phi}_1(u) \tag{18a}$$

$$\Phi_2(u) = \exp\left(-\frac{\sigma_2^2 u^2}{2}\right) \overline{\Phi}_2(u) \tag{18b}$$

where $\overline{\Phi}_1(u)$ and $\overline{\Phi}_2(u)$ denote the characteristic functions of the other-user interference terms and consist of products of $J_0(\cdot)$ functions. Consequently (15) becomes

$$Pr\,\{R_1 < R_2 \mid 1\} = \int_0^\infty \exp\left(-\frac{\sigma_1^2+\sigma_2^2}{2}u^2\right) J_0(su)\, \overline{\Phi}_1(u) \left[\sigma_2^2 u\, \overline{\Phi}_2(u) - \frac{d\,\overline{\Phi}_2(u)}{du}\right] du \tag{19}$$

2.1. Exact Expression for the Case of Full Hits

Let us assume now that there are n groups of interfering users with $\overline{K}_i$ users in the i-th group ($1 \leq i \leq n$) all having received power P_i. We will use the machinery presented above to evaluate the probability of error given that K_i of the total $\overline{K}_i$ interfering users of group i ($1 \leq i \leq n$) cause full hits. For simplicity we assume that a rectangular shaping waveform is employed so that we can use (4). Notice the similarity of (4) for $m = 1, 2$ to a typical term of (16a) and (16b).

We first derive the result for two groups of users and then extend it to the general case ($n > 2$). Suppose now that k_1 out of the K_1 interfering users hit branch 1 and $K_1 - k_1$ hit branch 2; similarly k_2 out of the K_2 interfering users hit branch 1 and $K_2 - k_2$ hit branch 2. The probability of this event is $\begin{pmatrix} K_1 \\ k_1 \end{pmatrix} 2^{-K_1} \cdot \begin{pmatrix} K_2 \\ k_2 \end{pmatrix} 2^{-K_2}$. The conditional characteristic functions $\overline{\Phi}_1(u \mid k_1, k_2)$ and $\overline{\Phi}_2(u \mid k_1, k_2)$ take the form

$$\overline{\Phi}_1 (u \mid k_1, k_2) = J_0(a_1 u)^{k_1} J_0(a_2 u)^{k_2} \tag{20a}$$

and

$$\overline{\Phi}_2 (u \mid k_1, k_2) = J_0(a_1 u)^{K_1 - k_1} J_0(a_2 u)^{K_2 - k_2}, \tag{20b}$$

where $a_i = \sqrt{\frac{P_i}{2}}\, T$, $i = 1, 2$. Then we can show that

$$\overline{\Phi}_{1,2} (u \mid k_1, k_2) = \overline{\Phi}_1 (u \mid k_1, k_2) \left[\sigma_2^2 u\, \overline{\Phi}_2 (u \mid k_1, k_2) - \frac{d\, \overline{\Phi}_2 (u \mid k_1, k_2)}{du} \right] \tag{21a}$$

$$= J_0(a_1 u)^{k_1} J_0(a_2 u)^{k_2} \Big[\sigma_2^2 u J_0(a_1 u)^{K_1 - k_1} J_0(a_2 u)^{K_2 - k_2}$$

$$+ (K_1 - k_1) a_1 J_1(a_1 u) J_0(a_1 u)^{K_1 - k_1 - 1} \cdot J_0(a_2 u)^{K_2 - k_2}$$

$$+ (K_2-k_2)a_2 J_1(a_2 u) J_0(a_2 u)^{K_2-k_2-1} \cdot J_0(a_1 u)^{K_1-k_1} \Big]$$

$$= J_0(a_1 u)^{K_1} J_0(a_2 u)^{K_2} \left[\sigma_2^2 u + (K_1-k_1)a_1 \frac{J_1(a_1 u)}{J_0(a_1 u)} + (K_2-k_2)a_2 \frac{J_1(a_2 u)}{J_0(a_2 u)}\right]. \tag{21b}$$

We now define

$$\overline{\Phi}_{1,2}(u) = \overline{\Phi}_1(u) \left[\sigma_2^2 u \,\overline{\Phi}_2(u) - \frac{d\,\overline{\Phi}_2(u)}{du}\right] \tag{22a}$$

$$= \sum_{k_1=0}^{K_1} \binom{K_1}{k_1} 2^{-K_1} \sum_{k_2=0}^{K_2} \binom{K_2}{k_2} 2^{-K_2} \,\overline{\Phi}_{1,2}(u \mid k_1, k_2)$$

and obtain that

$$\overline{\Phi}_{1,2}(u) = J_0(a_1 u)^{K_1} J_0(a_2 u)^{K_2} \left[\frac{K_1 a_1}{2} \frac{J_1(a_1 u)}{J_0(a_1 u)} + \frac{K_2 a_2}{2} \frac{J_1(a_2 u)}{J_0(a_1 u)} + \sigma_2^2 u\right]. \tag{23}$$

For n groups of users, with K_i users from group i causing full hits, (23) generalizes to

$$\overline{\Phi}_{1,2}(u) = \prod_{i=1}^{n} J_0(a_i u)^{K_i} \cdot \left[\sum_{i=1}^{n} \frac{K_i a_i}{2} \frac{J_1(a_i u)}{J_0(a_i u)} + \sigma_2^2 u\right]. \tag{24}$$

In (19) and (24) $\sigma_1^2 = \sigma_2^2 = \dfrac{N_0 T}{4}$ where N_0 is the spectral density of the AWGN and T is the duration of a data bit, and $s = \sqrt{\dfrac{P_0}{2}} T$, where P_0 is the power of the desired signal. Furthermore, if in (19) and (24) we replace u by $\dfrac{u}{\sqrt{\dfrac{P_0}{2}} T}$ we obtain

$$P_e(K_1, K_2, \ldots, K_n) = \int_0^{\infty} \exp\left(-\frac{u^2}{2E_b/N_0}\right) J_0(u) \prod_{i=1}^{n} J_0(\overline{a}_i u)^{K_i}$$

$$\left[\sum_{i=1}^{n} \frac{K_i \bar{a}_i}{2} \frac{J_1(\bar{a}_i u)}{J_0(\bar{a}_i u)} + \frac{u}{2E_b/N_0}\right] du ,$$

(25)

where $\bar{a}_i = \sqrt{\frac{P_i}{P_0}}$ for $i = 1,2, \ldots, n$, and $E_b = P_0 T$ is the energy per bit of the desired signal.

Equation (25) provides an exact expression for the conditional error probability given that K_i users cause full hits from the i-th group ($1 \leq i \leq n$) of users with received power P_i. To compute the infinite integral in (25) we need to truncate it and thus (25) actually provides an approximation whose accuracy improves at the expense of increasing computational effort. Nevertheless, the computational effort grows linearly with n the number of groups of users with identical recieved power; actually it grows linearly with $\sum_{i=1}^{n} K_i$.

If we relax the assumption that the shaping waveform is rectangular, we should use (3) instead of (4). This implies that the delay τ_k of each interfering user is now involved in all computations. The derivation of the new results is facilitated if we realize that a_i (and $\bar{a}_i$) is now replaced by $a_i\,[R_\psi(\tau) + \hat{R}_\psi(\tau)]/T$ and expectations with respect to $\tau \sim U\,[0,\,T\,]$ should be evaluated. Due to the independence assumptions these expectations can pass inside the products and the terms raised in the various powers. The final result is still provided by (25) if we replace $J_0(\bar{a}_i u)$ and $J_1(\bar{a}_i u)$ by

$$E_\tau \left\{ J_0 \left(\bar{a}_i \; \frac{R_\psi(\tau) + \hat{R}_\psi(\tau)}{T} u \right) \right\} \text{ and } E_\tau \left\{ J_1 \left(\bar{a}_i \; \frac{R_\psi(\tau) + \hat{R}_\psi(\tau)}{T} u \right) \right\}, \tag{26}$$

respectively.

Finally, to obtain the total probability of a bit error (BER) we must average the conditional error probability with respect to the distribution of full hits:

$$\overline{P}_e(\overline{K}_1, \overline{K}_2, \ldots, \overline{K}_n) = \sum_{K_1=0}^{\overline{K}_1} \sum_{K_2=0}^{\overline{K}_2} \cdots \sum_{K_n=0}^{\overline{K}_n} p(K_1, K_2, \ldots, K_n)\, P_e(K_1, K_2, \ldots, K_n), \tag{27}$$

where

$$p(K_1, K_2, \ldots, K_n) = \prod_{i=1}^{n} \left\{ \binom{\overline{K}_i}{K_i} P_h^{K_i} (1-P_h)^{\overline{K}_i - K_i} \right\}. \tag{28}$$

Equations (26), (27), and (28) can be combined to give the final result

$$\overline{P}_e(\overline{K}_1, \overline{K}_2, \ldots, \overline{K}_n) = \int_0^\infty \exp\left(-\frac{u^2}{2E_b/N_0}\right) J_0(u) \prod_{i=1}^{n} \left[1 - P_h + P_h J_0(\overline{a}_i u)\right]^{\overline{K}_i}$$
$$\left[\sum_{i=1}^{n} \frac{\overline{K}_i \overline{a}_i P_h J_1(\overline{a}_i u)}{2[1-P_h + P_h J_0(\overline{a}_i u)]} + \frac{u}{2E_b/N_0}\right] du. \tag{29}$$

Notice that, as was discussed above in connection with equation (25), the computational effort for the final expression about the error probability of FH/SSMA cited in (29) still grows linearly with $\sum_{i=1}^{n} \overline{K}_i$, where n is the number of groups of users with identical powers and $\overline{K}_i$ the number of users with the same power in the i-th group.

2.2 Exact Expression for the Case of Full and Partial Hits

Suppose now that of the $\overline{K}_i$ users, $K_{i,f}$ users cause full hits and $K_{i,p}$ users cause partial hits. We are interested in evaluating the overall probability of a bit error in this case. As in (27), we now write

$$\overline{P}_e(\overline{K}_1,\overline{K}_2, \ldots, \overline{K}_n) = \sum_{K_{1,f}=0}^{\overline{K}_1} \sum_{K_{1,p}=0}^{\overline{K}_1-K_{1,f}} \cdots \sum_{K_{n,f}=0}^{\overline{K}_n} \sum_{K_{n,p}=0}^{\overline{K}_n-K_{n,f}} p(K_{1,f}, K_{1,p}, \ldots, K_{n,f}, K_{n,p})$$

$$P_e(K_{1,f}, K_{1,p}, \ldots, K_{n,f}, K_{n,p}), \tag{30}$$

where

$$p(K_{1,f}, K_{1,p}, \ldots, K_{n,f}, K_{n,p}) = \prod_{i=1}^{n} \left\{ \binom{\overline{K}_i}{K_{i,f}} \binom{\overline{K}_i-K_{i,f}}{K_{i,p}} P_f^{K_{i,f}} P_p^{K_{i,p}} (1-P_h)^{\overline{K}_i-K_{i,f}-K_{i,p}} \right\} \tag{31}$$

and $P_e(K_{1,f}, K_{1,p}, \ldots, K_{n,f}, K_{n,p})$, the conditional probability of error given the number of users who cause full and partial hits, will be computed next.

Suppose that from the $K_{i,f}$ users of the i-th group causing full FH hits, $K_{i,f}'$ cause full tone hits to branch 1 (of the BFSK demodulator), $K_{i,f}''$ cause partial tone hits to both branches, and $K_{i,f}-K_{i,f}'-K_{i,f}''$ cause full tone hits to branch 2. Then since the probabilities of occurence of these events are 1/4, 1/2, and 1/4, respectively, the composite event occurs with probability

$$\binom{K_{i,f}}{K_{i,f}'} \binom{K_{i,f}-K_{i,f}'}{K_{i,f}''} \left(\frac{1}{4}\right)^{K_{i,f}'} \left(\frac{1}{2}\right)^{K_{i,f}''} \left(\frac{1}{4}\right)^{K_{i,f}-K_{i,f}'-K_{i,f}''} . \tag{32a}$$

Notice that when a partial FH hit occurs, there are two possible outcomes: (i) that the interference present at branch 1 is of the form (5a) for $m = 1$, while the interference present at branch 2 is of the form (5b) for $m = 2$, or (ii) the interference at branch 1 is as in (5b) and at branch 2 is as in (5a). These two cases occur with equal probability. Under the assumption that the shaping waveform is such that $\hat{R}_\psi(\tau) = R_\psi(T-\tau)$ (which is trivially satisfied for a rectangular shaping waveform) and since the delay τ_k for each interfering user is uniformly distributed in $[0, T]$, and so is $T - \tau_k$, the interference terms of (5a) and (5b) take the same values with equal probability (have identical statistics). Therefore, we do not distinguish between the aforementioned cases (i) and (ii), and consider case (i) only.

Also suppose that from the $K_{i,p}$ users of the i-th group who cause partial FH hits, $K_{i,p}'$ cause partial tone hits with interference of the form (5a) to branch 1 and the remaining $K_{i,p} - K_{i,p}'$ cause partial tone hits of the form (5a) to branch 2. This event occurs with probability

$$\begin{pmatrix} K_{i,p} \\ K_{i,p}' \end{pmatrix} 2^{-K_{i,p}}, \tag{32b}$$

since the individual cases have each probability 1/2. Indeed, we ought to have distinguished between the cases that the interference present at branch 1 (or branch 2) takes the form (5a) or (5b). However, since as discussed above, the interference terms in (5a) and (5b) take the same values with equal probability (have identical statistics), we consider (5a) only.

Under all the above assumptions the conditional characteristic functions of the interference present at branches 1 and 2 due to the i-th group of interfering users become

$$\overline{\Phi}_{1,i}(u \mid \underline{K}_{i,f}, \underline{K}_{i,p}, \underline{\tau}) = J_0(a_i u)^{K_{i,f}'} \cdot \prod_{j=1}^{K_{i,f}''} J_0\Big(a_i R_\psi(\tau_j) u\Big) \cdot \prod_{l=1}^{K_{i,p}'} J_0\Big(a_i R_\psi(\tau_l') u\Big) \tag{33a}$$

and

$$\overline{\Phi}_{2,i}(u \mid \underline{K}_{i,f}, \underline{K}_{i,p}, \underline{\tau}) = J_0(a_i u)^{K_{i,f} - K_{i,f}' - K_{i,f}''} \cdot \prod_{j=1}^{K_{i,f}''} J_0\Big(a_i \hat{R}_\psi(\tau_j) u\Big) \cdot \prod_{k=1}^{K_{i,p} - K_{i,p}'} J_0\Big(a_i R_\psi(\tau_k'') u\Big), \tag{33b}$$

where $\underline{K}_{i,f} = \left(K_{i,f}, K_{i,f}', K_{i,f}''\right)$, $\underline{K}_{i,p} = \left(K_{i,p}, K_{i,p}'\right)$, and $\underline{\tau}$ is the vector of the delays of all the interfering uses involved.

Next we compute $\overline{\Phi}_{1,2,i} = \overline{\Phi}_{1,i}\left[\sigma_2^2 u\, \overline{\Phi}_{2,i} - \frac{d\,\overline{\Phi}_{2,i}}{du}\right]$, where the argument of the $\overline{\Phi}$'s is $(\underline{K}_{i,f}, \underline{K}_{i,p}, \underline{\tau})$, and take the expectation with respect to all the mutually independent time delays involved in (33a) and (33b) to obtain after several manipulations

$$\overline{\Phi}_{1,2,i}(u \mid \underline{K}_{i,f}, \underline{K}_{i,p}) = J_0(a_i u)^{K_{i,f}-K_{i,f}''}\left[E_\tau\left\{J_0\left(a_i R_\psi(\tau)u\right)J_0\left(a_i \hat{R}_\psi(\tau)u\right)\right\}\right]^{K_{i,f}''}$$

$$\cdot\left[E_\tau\left\{J_0\left(a_i R_\psi(\tau)u\right)\right\}\right]^{K_{i,p}}$$

$$\cdot\left[\sigma_2^2 u + K_{i,f}''\, a_i \frac{E_\tau\left\{J_0\left(a_i R_\psi(\tau)u\right)J_1\left(a_i \hat{R}_\psi(\tau)u\right)\right\}}{E_\tau\left\{J_0\left(a_i R_\psi(\tau)u\right)J_0\left(a_i \hat{R}_\psi(\tau)u\right)\right\}}\right.$$

$$+ (K_{i,p} - K_{i,p}')a_i \frac{E_\tau\left\{J_1\left(a_i R_\psi(\tau)u\right)\right\}}{E_\tau\left\{J_0\left(a_i R_\psi(\tau)u\right)\right\}}$$

$$\left. + (K_{i,f} - K_{i,f}' - K_{i,f}'')a_i \frac{J_1(a_i u)}{J_0(a_i u)}\right]. \tag{34}$$

Subsequently, we average (34) with respect to the distribution of (32a)-(32b) to obtain

$$\overline{\Phi}_{1,2,i}(u) = \sum_{K_{i,f}'=0}^{K_{i,f}} \binom{K_{i,f}}{K_{i,f}'} \sum_{K_{i,f}''=0}^{K_{i,f}-K_{i,f}'} \binom{K_{i,f}-K_{i,f}'}{K_{i,f}''} \left(\frac{1}{4}\right)^{K_{i,f}'} \left(\frac{1}{2}\right)^{K_{i,f}''} \left(\frac{1}{4}\right)^{K_{i,f}-K_{i,f}'-K_{i,f}''}$$

$$\sum_{K'_{i,p}=0}^{K_{i,p}} \binom{K_{i,p}}{K'_{i,p}} \left(\frac{1}{2}\right)^{K'_{i,p}} \left(\frac{1}{2}\right)^{K_{i,p}-K'_{i,p}} \cdot \overline{\Phi}_{1,2,i}(u \mid \underline{K}_{i,f}, \underline{K}_{i,p})$$

$$= \left[\frac{1}{2}J_0(a_i u) + \frac{1}{2}E_\tau\left\{J_0\left(a_i R_\psi(\tau)u\right)J_0\left(a_i \hat{R}_\psi(\tau)u\right)\right\}\right]^{K_{i,f}} \left[E_\tau\left\{J_0\left(a_i R_\psi(\tau)u\right)\right\}\right]^{K_{i,p}}$$

$$\cdot\left\{\sigma_2^2 u + K_{i,f}\, a_i \frac{\frac{1}{4}J_1(a_i u) + \frac{1}{2}E_\tau\left\{J_0\left(a_i R_\psi(\tau)u\right)J_1\left(a_i \hat{R}_\psi(\tau)u\right)\right\}}{\frac{1}{2}J_0(a_i u) + \frac{1}{2}E_\tau\left\{J_0\left(a_i R_\psi(\tau)u\right)J_0\left(a_i \hat{R}_\psi(\tau)u\right)\right\}}\right.$$

$$\left. + \frac{K_{i,p}\, a_i}{2} \frac{E_\tau\left\{J_1\left(a_i R_\psi(\tau)u\right)\right\}}{E_\tau\left\{J_0\left(R_\psi(\tau)u\right)\right\}}\right\}. \qquad (35)$$

Equation (35) can be easily generalized to the case that n groups of users with $\left(K_{i,f}, K_{i,p}\right)$ users each (of received power P_i, $i=1,2,\ldots,n$) are interfering with the desired signal' s reception. The result is

$$P_e\left(K_{1,f}, K_{1,p}, \ldots, K_{n,f}, K_{n,p}\right) = \int_0^\infty \exp\left(-\frac{u^2}{2E_b/N_0}\right) J_0(u)\Phi_I(u)\,du, \qquad (36)$$

where

$$\Phi_I(u) = \prod_{i=1}^{n}\left[\frac{1}{2}J_0(\overline{a}_i u)+\frac{1}{2}E_\tau\left\{J_0\left(\overline{a}_i R_\psi(\tau)u\right)J_0\left(\overline{a}_i \hat{R}_\psi(\tau)u\right)\right\}\right]^{K_{i,f}} \left[E_\tau\left\{J_0\left(\overline{a}_i R_\psi(\tau)u\right)\right\}\right]^{K_{i,p}}$$

$$\cdot\left[\sum_{i=1}^{n}\left\{K_{i,f}\,\overline{a}_i \frac{\frac{1}{4}J_1(\overline{a}_i u)+\frac{1}{2}E_\tau\left\{J_0\left(\overline{a}_i R_\psi(\tau)u\right)J_1\left(\overline{a}_i \hat{R}_\psi(\tau)u\right)\right\}}{\frac{1}{2}J_0(\overline{a}_i u)+\frac{1}{2}E_\tau\left\{J_0\left(\overline{a}_i R_\psi(\tau)u\right)J_0\left(\overline{a}_i \hat{R}_\psi(\tau)u\right)\right\}}\right.\right.$$

$$+\quad \frac{K_{i,p}\,\overline{a}_i}{2}\frac{E_\tau\left\{J_1\left(\overline{a}_i\,R_\psi(\tau)u\right)\right\}}{E_\tau\left\{J_0\left(\overline{a}_i\,R_\psi(\tau)u\right)\right\}}\Bigg\}+\frac{u}{2E_b/N_0}\Bigg] \tag{37}$$

Finally, in order to obtain $\overline{P}_e\left(\overline{K}_1,\overline{K}_2,\ldots,\overline{K}_n\right)$ we combine (31), (32), and (37). The result is

$$\overline{P}_e\left(\overline{K}_1,\overline{K}_2,\ldots,\overline{K}_n\right)=\int_0^\infty \exp\left(-\frac{u^2}{2E_b/N_0}\right)J_0(u)\overline{\Phi}_I(u)\,du\,, \tag{38}$$

where

$$\overline{\Phi}_I(u)=\prod_{i=1}^{n}\Bigg\{1-P_h+P_f\left[\frac{1}{2}J_0(\overline{a}_i\,u)+\frac{1}{2}E_\tau\left\{J_0\left(\overline{a}_i\,R_\psi(\tau)u\right)J_0\left(\overline{a}_i\,\hat{R}_\psi(\tau)u\right)\right\}\right]$$
$$+P_p\left[E_\tau\left\{J_0\left(\overline{a}_i\,R_\psi(\tau)u\right)\right\}\right]\Bigg\}^{\overline{K}_i}$$

$$\Bigg[\sum_{i=1}^{n}\overline{K}_i\,\overline{a}_i\,\frac{P_f\left[\frac{1}{4}J_1(\overline{a}_i\,u)+\frac{1}{2}E_\tau\left\{J_0\left(\overline{a}_i\,R_\psi(\tau)u\right)J_1\left(\overline{a}_i\,\hat{R}_\psi(\tau)u\right)\right\}\right]+P_p\,\frac{1}{2}E_\tau\left\{J_1\left(\overline{a}_i\,R_\psi(\tau)u\right)\right\}}{1-P_h+P_f\left[\frac{1}{2}J_0(\overline{a}_i\,u)+\frac{1}{2}E_\tau\left\{J_0\left(\overline{a}_i\,R_\psi(\tau)u\right)J_0\left(\overline{a}_i\,\hat{R}_\psi(\tau)u\right)\right\}\right]+P_p\,E_\tau\left\{J_0\left(\overline{a}_i\,R_\psi(\tau)u\right)\right\}}$$
$$+\frac{u}{2E_b/N_0}\Bigg] \tag{39}$$

Notice that in comparing (39) to (29) we observe the required computational effort still grows linearly with $\sum_{i=1}^{n}\overline{K}_i$; however, (39) takes longer to compute due to the more complicated expressions involved.

2.3. Approximations

In order to derive convenient approximations to (26) for the case of full hits and to (38)-(39) for the case of full and partial hits we use the results of [Geraniotis, 1986] for noncoherent hybrid FH-DS/SSMA systems. In particular, we set the signature sequences employed to be equal to 1 for all time instants, and the number of chips per bit N to be equal to 1 in order that we obtain an equivalent FH/SSMA system from the hybrid system.

For binary FSK with noncoherent demodulation and full hits only the result is

$$P_e\left(K_1,K_2,\ldots,K_n\right)=\frac{1}{2}\exp\left\{-\frac{1}{4}\left[\left(\frac{2E_b}{N_0}\right)^{-1}+\left(\frac{m_\psi}{2}+\frac{m_\psi'}{4}\right)\sum_{i=1}^{n}K_i\,\overline{a}_i^{\,2}\right]^{-1}\right\}, \tag{40}$$

and $\overline{P}_e$ can be obtained from (27), (28), and (40). In (40) $m_\psi = \frac{1}{T^2}\int_0^T R_\psi^2(\tau)d\tau$ and $m_\psi' = \frac{1}{T^2}\int_0^T R_\psi(\tau)\hat{R}_\psi(\tau)d\tau$; they take the values 1/3 and 1/6, respectively, for a rectangular shaping waveform.

Notice that the computational effort required for evaluating $\overline{P}_e(\overline{K}_1,\overline{K}_2,\ldots,\overline{K}_n)$ from (40) via (27) and (28) grows linearly with $\prod_{i=1}^{n}\overline{K}_i$.

When both full and partial hits are accounted for, we have

$$P_e\left(K_{1,f},K_{1,p},\ldots,K_{n,f},K_{n,p}\right)=$$

$$=\frac{1}{2}\exp\left\{-\frac{1}{4}\left[\left(\frac{2E_b}{N_0}\right)^{-1}+\sum_{i=1}^{n}\left[\left(\frac{m_\psi}{2}+\frac{m_\psi'}{4}\right)K_{i,f}+\frac{m_\psi}{4}K_{i,p}\right]\overline{a}_i^{\,2}\right]^{-1}\right\} \tag{41}$$

Finally, $\overline{P}_e$ can be obtained from (30), (31), and (41).

Notice that the computational effort required for evaluating $\overline{P}_e(\overline{K}_1,\overline{K}_2,\ldots,\overline{K}_n)$ from (41) via (30) and (31) grows linearly with $\prod_{i=1}^{n}\overline{K}_i^{\,2}$.

3. Error Probability for M-ary FSK FH/SSMA Systems

Unfortunately the technique used in Sections 1.1 and 1.2 above cannot be extended to systems employing M-ary FSK modulation with noncoherent demodulation to provide accurate approximations. However, tight upper bounds to $P_e^{(M)}\left(K_1, K_2, \ldots, K_n\right)$, $P_e^{(M)}\left(K_{1,f}, K_{1,p}, \ldots, K_{n,f}, K_{n,p}\right)$, and $\bar{P}_e^{(M)}\left(\bar{K}_1, \bar{K}_2, \ldots, \bar{K}_n\right)$ for the M-ary case can be obtained from the corresponding exact expressions for the binary case using the union bound. In particular, we have

$$\bar{P}_e^{(M)}\left(\bar{K}_1, \bar{K}_2, \ldots, \bar{K}_n\right) \leq (1 - P_h)^{\sum_{i=1}^{n} \bar{K}_i} P_e^{(M)} + (M-1) \cdot \bar{P}_{e,M}^{\bar{0}}\left(\bar{K}_1, \bar{K}_2, \ldots, \bar{K}_n\right) \tag{42}$$

where $P_e^{(M)}$ denotes the error probability of a MFSK system in AWGN (no other-user interference) and $\bar{P}_{e,M}^{\bar{0}}\left(\bar{K}_1, \bar{K}_2, \ldots, \bar{K}_n\right)$ denotes the probability of deciding in favor of a any particular M-ary FSK symbol different than the one transmitted by the user under consideration, when there is at least one interfering signal present [i.e., the event "0": $(K_1=0, K_2=0, \ldots, K_n=0)$ of no users causing FH hits is excluded from the averaging with respect to the distribution of hits; that is why the superscript $\bar{0}$ (complement of the event "0") is used in (42)]; for the case of full hits this is given by a modified version of (29):

$$\bar{P}_{e,M}^{\bar{0}}(\bar{K}_1, \bar{K}_2, \ldots, \bar{K}_n) = \int_0^{\infty} \exp\left(-\frac{u^2}{2E_s/N_0}\right) J_0(u) \prod_{i=1}^{n} \left[1 - P_h + P_h\left(1 - \frac{2}{M} + \frac{2}{M} J_0(\bar{a}_i u)\right)\right]^{\bar{K}_i}$$
$$\left[\sum_{i=1}^{n} \frac{\bar{K}_i \, \bar{a}_i \, P_h \, J_1(\bar{a}_i u)}{M\left[1 - P_h + P_h\left(1 - \frac{2}{M} + \frac{2}{M} J_0(\bar{a}_i u)\right)\right]} + \frac{u}{2E_s/N_0}\right] du.$$

Table 1

Bit Error Probability of an Uncoded BFSK FH/SSMA System

($q = 100$, $N_b = 10$, $n = 2$, $\overline{K}_1 = 2$, $\overline{P}_1 = .5$, $\overline{K}_2 = 2$, $\overline{P}_2 = 1.$)

E_b/N_0	Exact			Gaussian Approx.			Upper Bound
	Full/Partial	Full Hits		Full/Partial	Full Hits		
8	2.88	3.01	(x 10^{-2})	2.71	2.78	(x 10^{-2})	6.37 x 10^{-2}
10	1.03	1.19	(x 10^{-2})	0.88	0.95	(x 10^{-2})	4.65 x 10^{-2}
12	0.69	0.79	(x 10^{-2})	0.51	0.58	(x 10^{-2})	4.34 x 10^{-2}
14	0.52	0.69	(x 10^{-2})	0.44	0.51	(x 10^{-2})	4.33 x 10^{-2}
16	0.45	0.63	(x 10^{-2})	0.41	0.48	(x 10^{-2})	4.33 x 10^{-2}

Table 2

Bit Error Probability of an Uncoded BFSK FH/SSMA System

($q = 100$, $N_b = 10$, $n = 2$, $\overline{K}_1 = 5$, $\overline{P}_1 = .5$, $\overline{K}_2 = 5$, $\overline{P}_2 = 1.$)

E_b/N_0	Exact			Gaussian Approx.			Upper Bound
	Full/Partial	Full Hits		Full/Partial	Full Hits		
8	3.98	4.29	(x 10^{-2})	3.56	3.72	(x 10^{-2})	1.24x10^{-1}
10	2.06	2.44	(x 10^{-2})	1.69	1.87	(x 10^{-2})	1.08 x 10^{-1}
12	1.52	1.93	(x 10^{-2})	1.23	1.41	(x 10^{-2})	1.05 x 10^{-1}
14	1.31	1.72	(x 10^{-2})	1.11	1.28	(x 10^{-2})	1.05 x 10^{-1}
16	1.16	1.58	(x 10^{-2})	1.04	1.21	(x 10^{-2})	1.05 x 10^{-1}

Table 3

Bit Error Probability of an Uncoded BFSK FH/SSMA System

($q = 100$, $N_b = 10$, $n = 2$, $\overline{K}_1 = 5$, $\overline{P}_1 = .25$, $\overline{K}_2 = 5$, $\overline{P}_2 = .5$)

E_b/N_0	Exact			Gaussian Approx.			Upper Bound
	Full/Partial	Full Hits		Full/Partial	Full Hits		
8	3.02	3.21	(x 10^{-2})	2.91	3.00	(x 10^{-2})	$1.24\text{x}10^{-1}$
10	1.01	1.19	(x 10^{-2})	0.99	1.08	(x 10^{-2})	1.08 x 10^{-1}
12	0.44	0.58	(x 10^{-2})	0.52	0.60	(x 10^{-2})	1.05 x 10^{-1}
14	0.23	0.33	(x 10^{-2})	0.40	0.47	(x 10^{-2})	1.05 x 10^{-1}
16	0.13	0.19	(x 10^{-2})	0.34	0.40	(x 10^{-2})	1.05 x 10^{-1}

Table 4

Bit Error Probability of an Uncoded BFSK FH/SSMA System

($q = 100$, $N_b = 10$, $n = 3$, $\overline{K}_1 = 5$, $\overline{P}_1 = .25$, $\overline{K}_2 = 5$, $\overline{P}_2 = .5$, $\overline{K}_3 = 5$, $\overline{P}_3 = 1.$)

E_b/N_0	Exact			Gaussian Approx.			Upper Bound
	Full/Partial	Full Hits		Full/Partial	Full Hits		
8	4.26	4.62	(x 10^{-2})	4.02	4.01	(x 10^{-2})	$1.71\text{x}10^{-1}$
10	2.25	2.67	(x 10^{-2})	1.98	2.09	(x 10^{-2})	1.56 x 10^{-1}
12	1.63	2.06	(x 10^{-2})	1.43	1.56	(x 10^{-2})	1.53 x 10^{-1}
14	1.37	1.80	(x 10^{-2})	1.25	1.39	(x 10^{-2})	1.53 x 10^{-1}
16	1.22	1.64	(x 10^{-2})	1.16	1.29	(x 10^{-2})	1.53 x 10^{-1}

Table 5

Bit Error Probability of an Uncoded BFSK FH/SSMA System

($q = 100$, $N_b = 10$, $n = 1$, $\bar{K}_1 = 4$, $\bar{P}_1 = 1.$)

E_b/N_0	Exact			Gaussian Approx.			Upper Bound
	Full/Partial	Full Hits		Full/Partial	Full Hits		
8	3.14	3.30	(x 10^{-2})	2.87	2.95	(x 10^{-2})	6.37x10^{-2}
10	1.34	1.54	(x 10^{-2})	1.06	1.15	(x 10^{-2})	4.65 x 10^{-2}
12	0.96	1.21	(x 10^{-2})	0.69	0.79	(x 10^{-2})	4.34 x 10^{-2}
14	0.89	1.17	(x 10^{-2})	0.64	0.73	(x 10^{-2})	4.33 x 10^{-2}
16	0.84	1.15	(x 10^{-2})	0.61	0.71	(x 10^{-2})	4.33 x 10^{-2}

Table 6

Bit Error Probability of an Uncoded BFSK FH/SSMA System

($q = 100$, $N_b = 10$, $n = 1$, $\bar{K}_1 = 4$, $\bar{P}_1 = 2.$)

E_b/N_0	Exact			Gaussian Approx.			Upper Bound
	Full/Partial	Full Hits		Full/Partial	Full Hits		
8	3.85	3.95	(x 10^{-2})	3.26	3.35	(x 10^{-2})	6.37x10^{-2}
10	2.17	2.32	(x 10^{-2})	1.49	1.59	(x 10^{-2})	4.65 x 10^{-2}
12	1.91	2.09	(x 10^{-2})	1.15	1.26	(x 10^{-2})	4.34 x 10^{-2}
14	1.90	2.09	(x 10^{-2})	1.11	1.23	(x 10^{-2})	4.33 x 10^{-2}
16	1.89	2.09	(x 10^{-2})	1.09	1.21	(x 10^{-2})	4.33 x 10^{-2}

Table 7

Symbol Error Probability of an Uncoded MFSK FH/SSMA System

(M =32, q = 100, N_s = 10, n = 2, $\bar{K}_1 = 2$, $\bar{P}_1 = .5$, $\bar{K}_2 = 2$, $\bar{P}_2 = 1.$)

E_b/N_0	E_s/N_0	Union Bound			Gaussian Approx.			Upper Bound
		F/P Hits	F Hits		F/P Hits	F Hits		
2.00	8.99	2.96	3.12	(x 10^{-1})	1.14	1.14	(x 10^{-1})	4.52 x 10^{-1}
3.00	9.99	1.07	1.22	(x 10^{-1})	0.52	0.52	(x 10^{-1})	3.43 x 10^{-1}
4.00	10.99	3.03	4.47	(x 10^{-2})	1.80	1.81	(x 10^{-2})	2.39 x 10^{-1}
5.00	11.99	0.69	2.05	(x 10^{-2})	0.45	0.46	(x 10^{-2})	1.52 x 10^{-1}
6.00	12.99	0.18	1.48	(x 10^{-2})	0.82	0.85	(x 10^{-3})	0.92 x 10^{-1}

Table 8

Symbol Error Probability of an Uncoded MFSK FH/SSMA System

(M =32, q = 100, N_s = 10, n = 2, $\bar{K}_1 = 5$, $\bar{P}_1 = .5$, $\bar{K}_2 = 5$, $\bar{P}_2 = 1.$)

E_b/N_0	E_s/N_0	Union Bound			Gaussian Approx.			Upper Bound
		F/P Hits	F Hits		F/P Hits	F Hits		
2.00	8.99	2.99	3.37	(x 10^{-1})	1.17	1.17	(x 10^{-1})	4.87x10^{-1}
3.00	9.99	1.10	1.47	(x 10^{-1})	0.54	0.54	(x 10^{-1})	3.85 x 10^{-1}
4.00	10.99	0.32	0.68	(x 10^{-1})	1.95	1.98	(x 10^{-2})	2.87 x 10^{-1}
5.00	11.99	0.88	4.30	(x 10^{-2})	0.54	0.55	(x 10^{-2})	2.06 x 10^{-1}
6.00	12.99	0.35	3.60	(x 10^{-2})	1.17	1.25	(x 10^{-3})	1.50 x 10^{-1}

Table 9

Symbol Error Probability of an Uncoded MFSK FH/SSMA System

(M =32, q = 100, N_s = 10, n = 2, $\overline{K}_1 = 5$, $\overline{P}_1 = .25$, $\overline{K}_2 = 5$, $\overline{P}_2 = .5$)

E_b/N_0	E_s/N_0	Union Bound			Gaussian Approx.			Upper Bound
		F/P Hits	F Hits		F/P Hits	F Hits		
2.00	8.99	2.96	3.14	(x 10^{-1})	1.15	1.15	(x 10^{-1})	4.87 x 10^{-1}
3.00	9.99	1.07	1.22	(x 10^{-1})	0.52	0.52	(x 10^{-1})	3.85 x 10^{-1}
4.00	10.99	2.00	4.20	(x 10^{-2})	1.81	1.83	(x 10^{-2})	2.87 x 10^{-1}
5.00	11.99	0.63	1.57	(x 10^{-2})	0.45	0.46	(x 10^{-2})	2.06 x 10^{-1}
6.00	12.99	1.12	8.08	(x 10^{-3})	0.77	0.79	(x 10^{-3})	1.50 x 10^{-1}

Table 10

Symbol Error Probability of an Uncoded MFSK FH/SSMA System

(M =32, q = 100, N_s = 10, n = 2, $\overline{K}_1 = 10$, $\overline{P}_1 = .5$, $\overline{K}_2 = 10$, $\overline{P}_2 = 1.$)

E_b/N_0	E_s/N_0	Union Bound			Gaussian Approx.			Upper Bound
		F/P Hits	F Hits		F/P Hits	F Hits		
2.00	8.99	3.04	3.80	(x 10^{-1})	1.21	1.22	(x 10^{-1})	5.41x10^{-1}
3.00	9.99	1.14	1.88	(x 10^{-1})	0.58	0.58	(x 10^{-1})	4.49 x 10^{-1}
4.00	10.99	0.36	1.08	(x 10^{-1})	0.22	0.23	(x 10^{-1})	3.62 x 10^{-1}
5.00	11.99	1.22	8.05	(x 10^{-2})	0.68	0.72	(x 10^{-2})	2.89 x 10^{-1}
6.00	12.99	0.65	7.14	(x 10^{-2})	0.18	0.20	(x 10^{-2})	2.39 x 10^{-1}

Table 11

Symbol Error Probability of an Uncoded MFSK FH/SSMA System

($M = 32$, $q = 100$, $N_b = 10$, $n = 1$, $\overline{K}_1 = 4$, $\overline{P}_1 = 1.$)

E_b/N_0	E_s/N_0	Union Bound			Gaussian Approx.			Upper Bound
		F/P Hits	F Hits		F/P Hits	F Hits		
8	14.99	0.16	2.27	(x 10^{-2})	0.43	0.51	(x 10^{-4})	4.72 x 10^{-2}
10	16.99	0.14	2.24	(x 10^{-2})	0.45	0.58	(x 10^{-5})	4.33 x 10^{-2}
12	18.99	0.13	2.22	(x 10^{-2})	0.91	0.13	(x 10^{-5})	4.33 x 10^{-2}
14	20.99	0.12	2.20	(x 10^{-2})	0.35	0.50	(x 10^{-6})	4.33 x 10^{-2}
16	22.99	0.11	2.19	(x 10^{-2})	0.19	0.28	(x 10^{-6})	4.33 x 10^{-2}

Table 12

Symbol Error Probability of an Uncoded MFSK FH/SSMA System

($M = 32$, $q = 100$, $N_b = 10$, $n = 1$, $\overline{K}_1 = 4$, $\overline{P}_1 = 2.$)

E_b/N_0	E_s/N_0	Union Bound			Gaussian Approx.			Upper Bound
		F/P Hits	F Hits		F/P Hits	F Hits		
8	14.99	0.50	4.21	(x 10^{-2})	0.45	0.54	(x 10^{-3})	4.72x10^{-2}
10	16.99	0.50	4.24	(x 10^{-2})	0.16	0.20	(x 10^{-3})	4.33 x 10^{-2}
12	18.99	0.50	4.25	(x 10^{-2})	0.71	0.89	(x 10^{-4})	4.33 x 10^{-2}
14	20.99	0.49	4.25	(x 10^{-2})	0.41	0.52	(x 10^{-4})	4.33 x 10^{-2}
16	22.99	0.49	4.25	(x 10^{-2})	0.28	0.37	(x 10^{-4})	4.33 x 10^{-2}

Table 13a

Maximum Number of Interfering Users Tolerated at a Codeword Error Probability

of P_E by a Reed-Solomon Coded BFSK FH/SS System

($q = 100$, $N_b = 10$, $E_b/N_0 = 16\ dB$; all interfering users have relative power $\overline{P}_1$)

P_E	RS Code	P_b	$\overline{P}_1 = .1$			$\overline{P}_1 = .5$			Bound
			Gauss	Exact		Gauss	Exact		
			F/P	F/P	F	F/P	F/P	F	
10^{-3}	(32 , 16) errors	.026	260	251	201	36	38	30	2
10^{-3}	(32 , 16) eras.	.058	436	398	326	77	72	61	5
10^{-3}	(32 , 8) errors	.039	222	207	168	39	34	28	3
10^{-3}	(32 , 8) eras.	.105	551	479	406	116	95	82	9
10^{-5}	(32 , 16) errors	.0146	183	185	145	20	23	18	1
10^{-5}	(32 , 16) eras.	.0402	342	320	259	54	53	44	3
10^{-5}	(32 , 8) errors	.0244	143	134	110	22	20	16	1
10^{-5}	(32 , 8) eras.	.0826	439	389	325	88	74	63	7

Table 13b

Maximum Number of Interfering Users Tolerated at a Codeword Error Probability of P_E by a Reed-Solomon Coded BFSK FH/SS System

($q = 100$, $N_b = 10$, $E_b/N_0 = 16\ dB$; all interfering users have relative power $\bar{P}_1$)

			$\bar{P}_1 = 1.$			$\bar{P}_1 = 2.$			
P_E	RS Code	P_b	Gauss	Exact		Gauss	Exact		Bound
			F/P	F/P	F	F/P	F/P	F	
10^{-3}	(32 , 16) errors	.026	16	11	8	9	5	4	2
10^{-3}	(32 , 16) eras.	.058	37	26	21	21	12	11	5
10^{-3}	(32 , 8) errors	.039	20	14	12	12	7	7	3
10^{-3}	(32 , 8) eras.	.105	63	44	38	39	24	23	9
10^{-5}	(32 , 16) errors	.0146	8	6	4	5	3	2	1
10^{-5}	(32 , 16) eras.	.0402	25	17	14	14	8	7	3
10^{-5}	(32 , 8) errors	.0244	11	8	7	7	4	4	1
10^{-5}	(32 , 8) eras.	.0826	47	34	29	30	18	17	7

Table 13c

Maximum Number of Interfering Users Tolerated at a Codeword Error Probability

of P_E by a Reed-Solomon Coded BFSK FH/SS System

($q = 100$, $N_b = 10$, $E_b/N_0 = 16\ dB$; all interfering users have relative power $\overline{P}_1$)

			$\overline{P}_1 = 4.$			$\overline{P}_1 = 10.$			
P_E	RS Code	P_b	Gauss	Exact		Gauss	Exact		Bound
			F/P	F/P	F	F/P	F/P	F	
10^{-3}	(32 , 16) errors	.026	6	4	4	5	4	4	2
10^{-3}	(32 , 16) eras.	.058	16	10	11	12	9	10	5
10^{-3}	(32 , 8) errors	.039	9	6	6	7	5	6	3
10^{-3}	(32 , 8) eras.	.105	29	19	21	24	17	20	9
10^{-5}	(32 , 16) errors	.0146	3	2	2	3	2	2	1
10^{-5}	(32 , 16) eras.	.0402	10	7	7	8	6	7	3
10^{-5}	(32 , 8) errors	.0244	5	3	3	4	3	3	1
10^{-5}	(32 , 8) eras.	.0826	22	14	15	18	13	15	7

Table 14a

Maximum Number of Interfering Users Tolerated at a Codeword Error Probability

of P_E by a Reed-Solomon Coded BFSK FH/SS System

($q = 100$, $N_b = 10$, $E_b/N_0 = 16\ dB$)

(half of the interfering users have relative power $\bar{P}_1$ and the other half have $\bar{P}_2$)

P_E	RS Code	P_b	$\bar{P}_1 = .5, \bar{P}_2 = 1$ Gauss	Exact		$\bar{P}_1 = 1, \bar{P}_2 = 2$ Gauss	Exact		Bound
			F	F/P	F	F	F/P	F	
10^{-3}	(32 , 16) errors	.026	19	18	15	10	7	6	2
10^{-3}	(32 , 16) eras.	.058	44	40	33	25	17	16	5
10^{-3}	(32 , 8) errors	.039	23	21	17	14	10	9	3
10^{-3}	(32 , 8) eras.	.105	73	62	54	44	31	29	9
10^{-5}	(32 , 16) errors	.0146	11	11	8	5	4	3	1
10^{-5}	(32 , 16) eras.	.0402	30	28	23	17	11	10	3
10^{-5}	(32 , 8) errors	.0244	13	12	10	8	5	5	1
10^{-5}	(32 , 8) eras.	.0826	55	47	41	33	24	21	7

Table 14b

Maximum Number of Interfering Users Tolerated at a Codeword Error Probability of P_E by a Reed-Solomon Coded BFSK FH/SS System

$$(q = 100, N_b = 10, E_b / N_0 = 16 \ dB)$$

(one third of the interfering users have relative power $\overline{P}_1$, one third have $\overline{P}_2$, and one third have $\overline{P}_3$)

			$\overline{P}_1$=.1, $\overline{P}_2$=.5, $\overline{P}_3$=1			$\overline{P}_1$=.5, $\overline{P}_2$=1, $\overline{P}_3$=2			
P_E	RS Code	P_b	Gauss	Exact		Gauss	Exact		Bound
			F	F/P	F	F	F/P	F	
10^{-3}	(32 , 16) errors	.026	29	26	22	14	11	8	2
10^{-3}	(32 , 16) eras.	.058	65	59	47	32	24	20	5
10^{-3}	(32 , 8) errors	.039	34	30	25	17	14	11	3
10^{-3}	(32 , 8) eras.	.105	104	89	77	55	41	38	9
10^{-5}	(32 , 16) errors	.0146	17	16	11	8	5	5	1
10^{-5}	(32 , 16) eras.	.0402	44	41	32	22	17	14	3
10^{-5}	(32 , 8) errors	.0244	20	17	14	10	8	7	1
10^{-5}	(32 , 8) eras.	.0826	79	68	59	41	32	29	7

Table 15

Maximum Number of Interfering Users Tolerated at a Codeword Error

Probability of P_E by a Reed-Solomon Coded MFSK FH/SS System

($M = 32$, $q = 100$, $N_s = 2$, $E_b/N_0 = 16\ dB$; all interfering users have relative power $\bar{P}_1$)

P_E	RS Code	P_s	$\bar{P}_1 = .5$ Union B. F/P	$\bar{P}_1 = .5$ Union B. F	$\bar{P}_1 = 1.$ Union B. F/P	$\bar{P}_1 = 1.$ Union B. F	$\bar{P}_1 = 2.$ Union B. F/P	$\bar{P}_1 = 2.$ Union B. F	Bound
10^{-3}	(32 , 16) errors	.13	1208	204	186	17	34	9	9
10^{-3}	(32 , 16) eras.	.29	1587	317	356	38	77	20	22
10^{-3}	(32 , 8) errors	.194	1428	212	210	25	51	13	14
10^{-3}	(32 , 8) eras.	.524	1897	400	487	69	138	36	49
10^{-5}	(32 , 16) errors	.0728	912	149	111	7	19	5	5
10^{-5}	(32 , 16) eras.	.201	1463	259	264	26	53	13	14
10^{-5}	(32 , 8) errors	.122	1169	157	139	15	32	8	8
10^{-5}	(32 , 8) eras.	.413	1736	344	400	54	108	28	35

Table 16

Maximum Number of Interfering Users Tolerated at a Codeword Error

Probability of P_E by a Reed-Solomon Coded MFSK FH/SS System

($M = 32$, $q = 100$, $N_s = 2$, $E_b/N_0 = 16$ dB)

(half of the interfering users have relative power $\bar{P}_1$ and the other half have $\bar{P}_2$)

(one third of the interfering users have relative power $\bar{P}_1$, one third have $\bar{P}_2$, and one third have $\bar{P}_3$)

			$\bar{P}_1$=.5 $\bar{P}_2$=1.		$\bar{P}_1$=1. $\bar{P}_2$=2.		$\bar{P}_1$=.5 $\bar{P}_2$=1. $\bar{P}_3$=2.		
P_E	RS Code	P_s	Union B.		Union B.		Union B.		Bound
			F/P	F	F/P	F	F/P	F	
10^{-3}	(32,16) err.	.13	303	35	59	11	89	17	9
10^{-3}	(32,16) eras.	.29	543	75	131	26	194	40	22
10^{-3}	(32,8) err.	.194	347	49	83	17	123	26	14
10^{-3}	(32,8) eras.	.524	739	129	221	48	320	71	49
10^{-5}	(32,16) err.	.0728	193	19	33	6	50	10	5
10^{-5}	(32,16) eras.	.201	419	53	92	18	137	28	14
10^{-5}	(32,8) err.	.122	239	31	53	11	79	17	8
10^{-5}	(32,8) eras.	.413	621	103	175	37	254	56	35

$$-\prod_{i=1}^{n}(1-P_h)^{\sum_{i=1}^{n}\overline{K}_i}\cdot\int_0^{\infty}\exp\left(-\frac{u^2}{2E_s/N_0}\right)J_0(u)\frac{u}{2E_s/N_0}du. \quad (43)$$

where E_s denotes the energy of each M-ary symbol. Equation (43) was obtained by following similar steps as for the derivation of (29); the main difference lies in that: (i) instead of $\prod_{i=1}^{n}\left\{\binom{K_i}{k_i}2^{-K_i}\right\}$, we used

$$\prod_{i=1}^{n}\left\{\binom{K_i}{k_i}\binom{K_i-k_i}{k_i'}\left(\frac{1}{M}\right)^{k_i}\left(\frac{1}{M}\right)^{k_i'}\left(1-\frac{2}{M}\right)^{K_i-k_i-k_i'}\right\} \quad (44)$$

where k_i denotes the number of the interfering users--out of the K_i uses causing full FH hits from a total of $\overline{K}_i$ in the i-th group--that cause full tone hits only to the branch of the MFSK demodulator carrying the desired signal and k_i' denotes the number of users--again out of the K_i ones--causing full tone hits to the particular branch of the MFSK demodulator that we compare to the branch carrying the desired signal; and (ii) in averaging (25) with respect to the distribution (44) we excluded the event $(K_1=0, K_2=0, \ldots, K_n=0)$; this resulted in subtracting the last term in (43). This last term can be put in the simpler form $-(1-P_h)^{\sum_{i=1}^{n}\overline{K}_i}\frac{1}{2}\exp\left(-\frac{E_s}{2N_0}\right)$ by using the fact (refer to [7, pp. 717, 6.63.4])

$$\int_0^{\infty}\exp(-u^2)J_0(au)u\,du = \frac{1}{2}\exp\left(-\frac{a^2}{4}\right) \quad (45)$$

for any $a > 0$. Finally, by combining (42) and (43) we obtain the result

$$\overline{P}_e^{(M)}\left(\overline{K}_1, \ldots, \overline{K}_n\right) \le (1-P_h)^{\sum_{i=1}^{n}\overline{K}_i} \cdot \left[P_e^{(M)} - \frac{M-1}{2}\exp\left(-\frac{E_s}{2N_0}\right)\right]$$

$$+(M-1)\int_0^{\infty}\exp\left(-\frac{u^2}{2E_s/N_0}\right)J_0(u)\prod_{i=1}^{n}\left[1-P_h+P_h\left(1-\frac{2}{M}+\frac{2}{M}J_0(\overline{a}_i u)\right)\right]^{\overline{K}_i}$$

$$\left[\sum_{i=1}^{n}\frac{\overline{K}_i\,\overline{a}_i\,P_h\,J_1(\overline{a}_i u)}{M\left[1-P_h+P_h\left(1-\frac{2}{M}+\frac{2}{M}J_0(\overline{a}_i u)\right)\right]}+\frac{u}{2E_s/N_0}\right]du.$$

(46)

We can also obtain an equation that provides $\overline{P}_{e,M}\left(\overline{K}_1,\overline{K}_2, \ldots, \overline{K}_n\right)$ for the case of full hits and partial hits. This is derived in a similar way as in (38)-(39). Instead of (32a) (binary case), we now use

$$\binom{K_{i,f}}{K_{i,f}'}\binom{K_{i,f}-K_{i,f}'}{K_{i,f}''}\binom{K_{i,f}-K_{i,f}'-K_{i,f}''}{K_{i,f}'''}\left(\frac{1}{M^2}\right)^{K_{i,f}'}\left(\frac{2}{M^2}\right)^{K_{i,f}''}\left(\frac{1}{M^2}\right)^{K_{i,f}'''}$$

$$\left(1-\frac{4}{M^2}\right)^{K_{i,f}-K_{i,f}'-K_{i,f}''-K_{i,f}'''} \tag{47a}$$

where $K_{i,f}$ is the number of users in the i-th group, out of the total $\overline{K}_i$ that cause full FH hits; out of them, $K_{i,f}'$ users cause full tone hits to the branch of the MFSK demodulation that carries the desired signal, $K_{i,f}'''$ users cause full tone hits to the other branch of the MFSK demodulator, that we compare the desired branch with, $K_{i,f}''$ users cause partial tone hits to both branches, and the remaining $K_{i,f} - K_{i,f}' - K_{i,f}'' - K_{i,f}'''$ users cause no tone hits to either of the two branches under consideration. Similarly, instead of (32b) we now use

$$\binom{K_{i,p}}{K_{i,p}'}\binom{K_{i,p}-K_{i,p}'}{K_{i,p}''}\left(\frac{1}{M}\right)^{K_{i,p}'}\left(\frac{1}{M}\right)^{K_{i,p}''}\left(1-\frac{2}{M}\right)^{K_{i,p}-K_{i,p}'-K_{i,p}''} \tag{47b}$$

where $K_{i,p}$ is the number of users in the i-th group, out of the total $\overline{K}_i$, that cause partial FH hits; out of them $K_{i,p}'$ users cause partial tone hits to the branch that carries the desired signal, $K_{i,p}''$ users cause partial tone hits to the branch compared with the desired one, and the remaining $K_{i,p} - K_{i,p}' - K_{i,p}''$ users cause no tone hits to either of the two branches under consideration.

The expression for $\overline{P}_e^{(M)}(\overline{K}_1, \overline{K}_2, \ldots, \overline{K}_n)$ is now given by

$$\overline{P}_e^{(M)}\left(\overline{K}_1, \overline{K}_2, \ldots, \overline{K}_n\right) = (1 - P_h)^{\sum_{i=1}^{n} \overline{K}_i} \cdot \left[P_e^{(M)} - \frac{M-1}{2}\exp\left(-\frac{E_s}{2N_0}\right)\right]$$

$$+ (M-1)\cdot\int_0^{\infty}\exp\left(-\frac{u^2}{2E_s/N_0}\right) J_0(u)\overline{\Phi}_I(u)\,du\,, \qquad (48)$$

where $\overline{\Phi}_I(u)$ takes the form

$$\overline{\Phi}_I(u) = \prod_{i=1}^{n} A(u; P_f, P_p, R_\psi, \hat{R}_\psi, M; \overline{a}_i)^{\overline{K}_i} \cdot$$

$$\left[\sum_{i=1}^{n} \overline{K}_i \, \overline{a}_i \frac{P_f \left[\frac{1}{M^2}J_1(\overline{a}_i u) + \frac{2}{M^2}E_\tau\left\{J_0\left(\overline{a}_i R_\psi(\tau)u\right)J_1\left(\overline{a}_i \hat{R}_\psi(\tau)u\right)\right\}\right] + P_p \frac{1}{M}E_\tau\left\{J_1\left(\overline{a}_i R_\psi(\tau)u\right)\right\}}{A(u; P_f, P_p, R_\psi, \hat{R}_\psi, M; \overline{a}_i)}\right.$$

$$\left. + \frac{u}{2E_s/N_0}\right] \qquad (49)$$

where

$$A(u; P_f, P_p, R_\psi, \hat{R}_\psi, M; \overline{a}_i) = 1 - P_h$$

$$+ P_f \left[1 - \frac{4}{M^2} + \frac{2}{M^2}J_0(\overline{a}_i u) + \frac{2}{M^2}E_\tau\left\{J_0\left(\overline{a}_i R_\psi(\tau)u\right)J_0\left(\overline{a}_i \hat{R}_\psi(\tau)u\right)\right\}\right]$$

$$+ P_p \left[1 - \frac{2}{M} + \frac{2}{M}E_\tau\left\{J_0\left(\overline{a}_i R_\psi(\tau)u\right)\right\}\right] \qquad (50)$$

We can also use the results of [Geraniotis, 1986] for hybrid FH-DS/SSMA systems with M-ary FSK modulation in the same way we did for binary FSK systems in Section

1.3 above. The result which corresponds to (40) is

$$P_e^{(M)}\left(K_1, K_2, \ldots, K_n\right) = \sum_{m=1}^{M-1} \binom{M-1}{m} \frac{(-1)^{m+1}}{m+1} \exp\left\{-\frac{m}{2(m+1)} \left[\left(\frac{2E_s}{N_0}\right)^{-1} + \left(\frac{m_\psi}{M} + \frac{m_\psi'}{M^2}\right) \sum_{i=1}^{n} K_i \, \overline{a}_i^{\,2}\right]^{-1}\right\}, \tag{51}$$

and the result which corresponds to (41) is

$$P_e^{(M)}\left(K_{1,f}, K_{1,p}, \ldots, K_{n,f}, K_{n,p}\right) =$$

$$= \sum_{m=1}^{M-1} \binom{M-1}{m} \frac{(-1)^{m+1}}{m+1} \exp\left\{-\frac{m}{2(m+1)} \left[\left(\frac{2E_s}{N_0}\right)^{-1} + \sum_{i=1}^{n} \left[\left(\frac{m_\psi}{M} + \frac{m_\psi'}{M^2}\right) K_{i,f} + \frac{m_\psi}{2M} K_{i,p}\right] \overline{a}_i^{\,2}\right]^{-1}\right\} \tag{52}$$

Then the error probabilities $\overline{P}_e^{(M)}\left(\overline{K}_1, \overline{K}_2, \ldots, \overline{K}_n\right)$ can be obtained from (27), (28), and (51) for the case of full hits and from (30), (31), and (52) for the case of full and partial hits.

The computational effort required to evaluate $P_e^{(M)}\left(\overline{K}_1, \overline{K}_2, \ldots, \overline{K}_n\right)$ grows linearly with $\sum_{i=1}^{n} \overline{K}_i$ when (46) or (48)-(50) is used, with $\prod_{i=1}^{n} \overline{K}_i$ when (51) is used, and with $\prod_{i=1}^{n} \overline{K}_i^{\,2}$ when (52) is used.

4. Numerical Results

(i) Uncoded Binary FSK Case

We start the presentation of numerical results with binary FSK FH/SSMA systems. Tables 1-6 illustrate the different expressions of the probability of error of these systems for different numbers of interfering users and relative power levels with respect to the received power of the user under consideration. In all tables q denotes the number of frequencies available for hopping, N_b the number of bits per dwell-time (hop), n the number of groups of interfering users with the same relative power, $\overline{K}_i$ the number of users (all with relative power $\overline{P}_i$) in the i-th group, and E_b/N_0 the bit signal-to-noise ratio.

In each table there are two columns of data under the headings "Exact" and "Gaussian Approximation"; they refer to the cases that full/partial hits and full hits only were taken into consideration in deriving the expression for the probability of error of the BFSK FH/SSMA system, which relies on the integration of the characteristic functions of the interference in the two branches of the BFSK demodulator, and in deriving the approximation based on the Central Limit Theorem. The expressions for full/partial hits are provided by equations (38)-(39) for the exact error probability, and by (30), (31), and (41) for the Gaussian approximation. The expressions for full hits only are provided by equations (29)--which serves as a tight upper bound for the error probability--and by (27), (28), and (40) for the Gaussian approximation. All these expressions take into consideration the different power levels of the interfering users. The infinite integrals involved in the "exact" expressions were truncated to finite integrals from 0 to 35 and a 700-point Simpson rule was employed; these values are sufficient for limiting the truncation and integration error to 10^{-5} or less for the range of E_b/N_0 of interest [i.e., 8 - 16 dB]. Finally, the last column of each table provides the hard upper bound on the error probability obtained in [1] for FH/SSMA systems in AWGN and also used in [6] for FH/SSMA systems in partial-band noise jamming; this upper bound is insensitive to the power levels of the interfering users; it assumes that hits from other users cause errors with probability 1. It takes the form

$$\overline{P}_e \leq P_e^{(M)} \cdot (1-P_h)^{\overline{K}} + \left[1-(1-P_h)^{\overline{K}}\right], \tag{53}$$

where $M = 2$ in the binary case, $P_e^{(M)}$ is the error probability for an MFSK FH/SS system in AWGN (and no other-user interference), P_h is the probability of a hit defined in Section 1, and $\overline{K} = \sum_{i=1}^{n} \overline{K}_i$ is the total number of interfering users.

Tables 1 to 4 show that, as the number of interfering users with relative power smaller than 1 increases, the difference between the hard upper bound and the exact expressions for the error probability becomes unacceptably large, finally reaching a difference of two orders of magnitude in the last entry of Table 3. However, the difference between the expressions based on the Gaussian approximation and the exact expressions remains relatively small over a wider range of values of the power levels and number of interfering users. By contrast, in Tables 5 and 6, where the relative powers of the interfering users are larger than or equal to 1, the difference between the hard upper bound and the exact expressions remains within narrower limits; the same holds true for the expressions based on the Gaussian approximation, which in this case provide slightly more optimistic results.

(ii) Uncoded M-ary FSK Case

In Tables 7 to 12 we illustrate the performance of M-ary FSK FH/SSMA systems. The notation is similar to that used in Tables 1 to 6 for binary FSK systems; the difference lies in M, which denotes the number of frequency tones used for the MFSK modulation, in $E_s/N_0 = log_2 M\ E_b/N_0$ being the symbol signal-to-noise ratio, and in N_s (which replaces N_b) as the number of M-ary symbols per dwell-time.

In each table we now present results about the probability of error in three groups of columns: The first group is under the heading "Union Bound" and has two columns: the first which is based on equations (48), (49), and (50) provides a tight upper bound on the error probability by taking into account both full and partial hits; the second which is based on equation (46) assumes that the interfering users cause only full hits thus providing a less tight upper bound on the exact probability of error. The second group, under the heading "Gaussian Approximation," also has two columns: the first refers to equations (30),(31), and (52) which are valid for the case of full and partial hits; the second refers to equations (27), (28), and (51), which are also valid for the case of full hits only. The expressions used for generating the results of these two columns take into consideration the different power levels of the interfering users. The infinite integrals involved in the aforementioned union bounds were truncated to finite integrals from 0 to 80 and a 1600-point Simpson rule was employed; these values are sufficient for limiting the truncation and integration error to 10^{-5} or less for the range of E_b/N_0 of interest (i.e., 2 - 16 dB)--recall that the effective signal-to-noise ratio is now $\log_2 M$ times larger. Finally, the third group contains only one column and provides the hard upper bound of (53) (see [1] and [6]), as applied to the M-ary FSK case ($M > 2$).

Tables 7 to 10, which are characterized by low bit signal-to-noise ratios, establish that, as the number of interfering users with relative power levels smaller than 1 increases, the difference between the upper bound of (53) and the tighter union bound widens to unacceptable levels, worse than those of the corresponding binary case. By contrast, the Gaussian approximation remains within restricted deviations from the union bound. These results parallel the corresponding results of Tables 1 to 4 for the binary case. By contrast, in Tables 11 and 12 where the bit signal-to-noise ratios take larger values and the interfering users have relative powers 1 or 2, the upper bound of (53), although sometimes 10 times larger than the tighter union bound for full/partial hits, is much closer to that than the Gaussian approximation which appears to be unacceptably optimistic.

(iii) Multiple-Access Capability of Coded FH/SS and The Near-Far Problem

The last four tables, Tables 13 to 16, provide the multiple-access capability of the BFSK and 32-ary FSK FH/SSMA systems, respectively, which employ Reed-Solomon error-control coding; that is the maximum number of interfering FH/SS signals with fixed relative power (with respect to the desired signal) that can be tolerated in the vicinity of the receiver at an error probability P_E. In all these tables the first column gives the desired value of P_E, the codeword error probability (or the packet error probability if one codeword per packet is used) and the second column provides the total number of symbols per codeword, the number of information symbols per codeword of the RS code, and the type of decoding: errors or erasures decoding.

In Tables 13a - 13c and 14a - 14b the third column provides P_b, the bit error probability of the uncoded system that corresponds to the particular P_E of the coded system--here we upperbounded the symbol error probability of the uncoded system by 5 times its bit error probability ($5P_b$) since 5 BFSK symbols are transmitted in each Reed-Solomon symbol. The remaining 7 columns of Table 13a provide the multiple-access capability as predicted by the Gaussian (full/partial hits) approximation, the exact error probability of (38)-(39) [full and partial hits] and of (29) [full hits only], and the upper bound of (53) for two distinct values of the relative powers: $\bar{P}_1 = .1$, and .5 (all the interfering users have the same relative power). This is repeated in Tables 13b for $\bar{P}_1 = 1.$ and 2. and in Table 13c for $\bar{P}_1 = 4$ and 10. As indicated by the results for $\bar{P}_1 = 1$ the upper bound is considerably pessimistic with respect to the exact value of the multiple-access capability--almost 5 times smaller for most cases, whereas the Gaussian approximation, although optimistic, is up to 30% larger than the exact value. For $\bar{P}_1 = 2$ the exact value is almost 3 times larger than the bound and less than 2 times smaller than the Gaussian approximation. Regarding the near-far problem, we observe that the multiple-access capability of the BFSK decreases considerably as the relative power of the interfering users increases from .1 to .5 and then to 1, 2, 4 and 10. Similarly, in Tables 14a and 14b we provide the multiple-access capability when the interfering users have two (in Table 14a) and three (in Table 14b) distinct relative power levels: specifically, results for $(\bar{P}_1,\bar{P}_2) = (.5,1.)$ and $(1.,2.)$ are presented in Table 14a and results for $(\bar{P}_1,\bar{P}_2,\bar{P}_3) = (.1,.5,1.)$ and $(.5,1.,2.)$ are presented in Table 14b. Again the near-far problem of FH/SSMA becomes evident from these results.

In Table 15 the third column gives P_s, the symbol error probability of the uncoded system that corresponds to the particular P_E of the coded system; the remaining 7 columns provide the multiple-access capability as determined by the union bound of (48), (49), and (50) [for full and partial hits] and (46) [for full hits only], and the upper bound of (53) for three different relative power levels .5, 1, and 2 (all interfering users have the same relative power level). These results indicate that the upper bound gives very pessimistic results, which can be 120 times smaller (for the parameters considered) than those obtained from the more accurate union bound. The Gaussian approximation was not included here because it gives overly optimistic results. Again the near-far problem manifests itself in that the multiple-access capability decreases considerably when the relative power of the interfering users increases from .5 to 1 and then to 2. In Table 16 we show the multiple-access capability for the situations that the interfering users have two or three distinct relative power levels, in particular for the cases that $(\bar{P}_1,\bar{P}_2) = (.5,1.)$ and $(1.,2.)$ and $(\bar{P}_1,\bar{P}_2,\bar{P}_3) = (.5,1.,2.)$. Again the near-far problem manifests itself in these cases.

5. Conclusions

In this paper we presented a method for the accurate evaluation of the probability of error in uncoded binary and M-ary FSK FH/SSMA systems. In the binary case we provided the exact expression (an arbitrarily accurate approximation) for the bit error probability of the FH/SSMA system based on the evaluation of the characteristic functions of the envelopes of the two branches of the BFSK demodulator; both full and partial hits were taken into consideration. Furthermore, a tight upper bound on the exact expression for the error probability was developed by considering full hits only. Both the exact expression and the tight upper bound based on full hits take into consideration the different power levels of the interfering users. We can improve their accuracy at will, while the required computational effort remains linear in the number of interfering signals. In addition to these results, the Central Limit Theorem (CLT) was used to provide two approximations to the error probability: one for the case of full hits only, and one for the case of full and partial hits; these approximations maintain the desirable feature of taking into consideration the power levels of the interfering users, while at the same time being easier to compute.

We established that the upper bound on the error probabilty of BFSK FH/SSMA systems developed in [1] and widely used in the literature becomes unacceptably loose, when compared to the exact results as the number of interfering users with relative powers smaller than or equal to 1 increases. By contrast, the approximations based on the CLT, termed Gaussian approximations, remain tight--relatively close to the exact results--for a wide range of numbers and power levels of interfering users. This implies that the multi-access capability of the FH/SSMA systems is in fact larger than originally believed based on the use of the aforementioned upper bound. We also showed that the multiple-access capability of Reed-Solomon coded FH/SSMA systems is considerably larger than it was thought previously.

Similar results were obtained for the M-ary FSK case. In this case, tight upper bounds on the exact error probability were derived based on the union upper bound and the exact results for full hits or full/partial hits of the binary case. In addition, the CLT was applied so as to provide easy to compute approximations. Both the union bound and the Gaussian approximation take into consideration the different power levels of the interfering users.

In the M-ary case we showed that the upper bound of [1] is unacceptably loose (even looser than in the binary case) for a wide range of relative power levels of the interfering users, whereas the Gaussian approximation maintains satisfactory accuracy for low bit signal-to-noise ratios; however, for higher bit signal-to-noise ratios the Gaussian approximation gives very optimistic results. Our results indicate that the multiple-access capability of M-ary FSK FH/SSMA systems is much larger than originally thought when the relative power levels of the interfering signals are this trend is amplified further when error-control coding is used, as shown for Reed-Solomon coded FH/SSMA systems.

Our results indicated that the multiple-access capability of binary and M-ary FSK FH/SSMA systems decreases considerably as the relative power of the interfering users increases. Therefore the near-far problem, which has been observed, quantified, and dealt with in direct-sequence (DS) SSMA systems, is also present in FH/SSMA systems, though it appears to be less serious, and should be further investigated.

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