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On the Crossing Minimization of Links in Computer Network Layout

by

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Abstract

We consider a layout problem of computer communication networks. It is formulated graph-theoretically as follows : Given a graph $G = (V, E)$, find an embedding of G in the plane with the least number of edge-crossings such that (i) the vertices in V are placed on the circumference of a circle C, and (ii) the edges in E are drawn only inside C. In this paper, we first present a linear time algorithm for embedding an outerplanar graph in the plane with no edge-crossings under constraints (i) and (ii). We then show that the problem is NP-hard for general graphs. This indicates that the computer network layout problem is, in general, very likely to be intractable.

1. Introduction

An important task of computer communication network management tools is to provide users with information on the global configuration of a network in an appropriate form. A natural and reasonable way of presentation is to draw the network on a graphics screen so that the users can easily grasp its entire logical structure. And, as such, an automatic network layout algorithm is provided in IBM's CNMgraf : Communications Network Management Graphics Facility [6]. Based on a classification of the links, this algorithm first partitions the set of the computing facilities into subsets, called sites. It then draws the sites as points located on the circumference of a "primary" circle and the links between different sites as straight lines inside the circle. Similarly, for each site, the algorithm places its computing facilities on a "regional" circle and draws the links between the different facilities of the site as straight lines inside the regional circle. The final drawing of the entire network is completed by placing the regional circles around the primary circle.

In this approach, all circles must be arranged in such a way as to make the resultant layout of the network suitable for the users' inspection. For this purpose, the algorithm used in the CNMgraf tries to find a drawing which results in as few crossings of links as possible for each of the circles. The problem of finding such a drawing may be formulated graph-theoretically as follows : Given a graph $G=(V, E)$, find an embedding of G in the plane with the least number of edge-crossings such that (i) the vertices in V are placed on the circumference of a circle C, and (ii) the edges in E are drawn inside C.

In this paper, we present complexity results of this problem. We first describe a linear time algorithm for embedding an outerplanar graph G in the particular form mentioned above in such a way that no edge-crossings occur. This algorithm is based on the characterization of outerplanar graphs obtained by Syslo and Iri [11]. We then show that the problem is NP-hard for general graphs, which implies that the network layout problem is, in general, very likely to be intractable. Note that in the case in which constraints (i) and (ii) are not imposed, the problem is an optimization version of the CROSSING NUMBER problem [3,4] which was recently shown to be NP-complete by Garey and Johnson [4]. However, their proof can not be extended for our problem.

In the next section, after defining some terms, we formally describe the decision problem version of our problem. In Section 3, we present a linear time algorithm for embedding an outerplanar graph with no edge-crossings under the above constraints. In Section 4, we introduce a modified version of the OPTIMAL LINEAR ARRANGEMENT problem which is known to be NP-complete [3,5]. Then, we show a polynomial transformation from this modified NP-complete problem to our decision problem. Section 5 concludes this paper with some comments.

2. Preliminaries

Let $G=(V, E)$ be an undirected graph with no self-loops and no multiple edges. Two distinct edges e and e' in E are said to be *adjacent* to each other in G if they share a vertex as their common endpoint. Let $\overline{G}$ be an embedding of G in the plane. We call $\overline{G}$ a *circle-confined drawing* associated with a circle C if it satisfies the following four constraints.

(i) Each vertex in V is drawn as a point located on the circumference of C.

(ii) Each edge in E is drawn as a continuous line inside C.

(iii) Any two adjacent edges touch or intersect with each other only at their common endpoint.

(iv) Any two non-adjacent edges do not touch each other, but they may intersect with each other at exactly one point.

Fig. 1 shows an example of a circle-confined drawing of a graph.

For a circle-confined drawing $\overline{G}$ of $G = (V, E)$ and two edges e and e' in E, we say that e and e' *cross* each other in $\overline{G}$ if they intersect in $\overline{G}$ but are not adjacent in G. Such an unordered pair of edges e and e' is called a *crossing* in $\overline{G}$. Note that two edges e and e' cross each other in $\overline{G}$ if and only if the endpoints of e and those of e' appear alternately on the circumference of the circle (if the remaining vertices are ignored). Therefore, the number of crossings in $\overline{G}$ depends on only the order of the appearances of the vertices on the circle.

For a subset V_1 of V, we say that V_1 is *consecutive* in $\overline{G}$ if all vertices in V_1 are placed successively on the circumference of the circle. For a subset E_1 of E, let $N_{\overline{G}}(E_1)$ denote the number of crossings of the edges of E_1 in $\overline{G}$. Furthermore, for two mutually disjoint subsets E_1 and E_2 of E, let $N_{\overline{G}}(E_1, E_2)$ denote the number of crossings in $\overline{G}$ between the edges of E_1 and those of E_2. Note that if $E_1 = \phi$, then $N_{\overline{G}}(E_1) = 0$. Similarly, if $E_1 = \phi$ or $E_2 = \phi$, then $N_{\overline{G}}(E_1, E_2) = 0$.

Using the terms and notations introduced above, the decision problem version of our problem is formally defined as follows :

Problem OPTIMAL CIRCLE-CONFINED DRAWING (OCCD)

Instance : Graph $G = (V, E)$, integer $B \geq 0$.

Question: Is there a circle-confined drawing $\overline{G}$ of G such that $N_{\overline{G}}(E) \leq B$? □

A *planar circle-confined drawing* is a circle-confined drawing with no edge-crossings. A graph is called *outerplanar* if it can be embedded in the plane so that all of the vertices lie on the same face and that no two edges cross each other. It is easy to see that a graph has a planar circle-confined drawing if and only if it is outerplanar. In Section 3, we present an efficient algorithm for finding a planar circle-confined drawing of an outerplanar graph. Then, we show in Section 4 that OCCD is in general NP-complete.

Before closing this section, we introduce four functions g_1, g_2, g_3, and g_4 which play important roles in the NP-completeness proof of OCCD.

Let $g_1(p)$ denote the number of crossings in a circle-confined drawing of graph $H_1 = K_p$, namely, a complete graph with $p (\geq 0)$ vertices. For a complete bipartite graph $H_2 = (V_1 \cup V_2, E)$ where $|V_1| = p \geq 0$, $|V_2| = q \geq 0$ and $E = \{ (v,w) \mid v \in V_1 \text{ and } w \in V_2 \}$, $g_2(p,q)$ is defined to be the number of crossings in a circle-confined drawing of H_2 in which V_1 and V_2 each are consecutive. Note that the value of $g_1(p)$ is uniquely determined for any non-negative integer p. Likewise, the value of $g_2(p,q)$ depends on the values of p and q only.

Next, consider the following graph:

$$H_3 = (V_1 \cup V_2, E_{11} \cup E_{12}),$$

where $|V_1| = p \geq 0$, $|V_2| = q \geq 0$,

$$E_{11} = \{ (v,w) \mid v,w \in V_1 \},$$

and $E_{12} = \{ (v,w) \mid v \in V_1 \text{ and } w \in V_2 \}$.

We define $g_3(p,q)$ to be $N_{\overline{H}_3}(E_{11}, E_{12})$, where $\overline{H}_3$ is a circle-confined drawing of H_3 in which V_1 and V_2 each are consecutive. Note that $g_3(p,q)$ is not always equal to $g_3(q,p)$, whereas $g_2(p,q) = g_2(q,p)$. For example, $g_3(2,3) = 0$ while $g_3(3,2) = 2$ as shown in Fig. 2.

Finally, let H_4 be a graph defined as follows:

$$H_4 = (V_1 \cup V_2 \cup V_3 \cup V_4, E_{12} \cup E_{23} \cup E_{34} \cup E_{41}),$$

where $|V_1| = p \geq 0$, $|V_2| = q \geq 0$, $|V_3| = r \geq 0$, $|V_4| = s \geq 0$,

$$E_{i(i+1)} = \{ (v,w) \mid v \in V_i \text{ and } w \in V_{i+1} \} \quad \text{for } i = 1,2,3,$$

and $E_{41} = \{ (v,w) \mid v \in V_4 \text{ and } w \in V_1 \}$.

We define $g_4(p,q,r,s)$ to be $N_{\overline{H}_4}(E_{12}, E_{23}) + N_{\overline{H}_4}(E_{34}, E_{41})$, where $\overline{H}_4$ is a circle-confined drawing of H_4 in which each of V_1, V_2, V_3 and V_4 is consecutive and they are placed on the circumference of the circle in this order in the clockwise direction. An example is given in Fig. 3, where the points enclosed by "□ " indicate the crossings between the edges in E_{12} and in E_{23} and those between the edges in E_{34} and in E_{41}.

The following four lemmas provide the expression for each of the above functions in terms of its parameters. The proofs are given in Appendix 1.

Lemma 1. $g_1(p) = \frac{1}{24} \cdot p \cdot (p-1) \cdot (p-2) \cdot (p-3)$ for $p \geq 0$. □

Lemma 2. $g_2(p, q) = \frac{1}{4} \cdot p \cdot q \cdot (p-1) \cdot (q-1)$ for $p, q \geq 0$. □

Lemma 3. $g_3(p, q) = \frac{1}{6} \cdot p \cdot q \cdot (p-1) \cdot (p-2)$ for $p, q \geq 0$. □

Lemma 4. $g_4(p, q, r, s) = \frac{1}{2} \cdot p \cdot r \cdot \{ q \cdot (q-1) + s \cdot (s-1) \}$ for $p, q, r, s \geq 0$. □

3. Circle-Confined Drawing of Outerplanar Graphs

Any outerplanar graph has a planar circle-confined drawing. We first show a straightforward method of finding such a drawing. Let $G = (V, E)$ be an outerplanar graph.

(1) Create a new vertex v^*.

(2) Set $E^* \leftarrow \{ (v^*, v) \mid v \in V \}$ and $G^* \leftarrow (V \cup \{v^*\}, E \cup E^*)$.

(3) Find a planar embedding $\overline{G}^*$ of G^*.

(4) Place the vertices in V on the circumference of a circle C in the order in which their incident edges in E^* appear, say in the counterclockwise direction, around v^* in $\overline{G}^*$. Draw the edges in E by straight line segments inside C.

An example is provided in Fig. 4.

Using existing linear time algorithms for finding a planar embedding of a planar graph, in particular, the one given by Chiba, et al. [2], the above procedure can be implemented to run in linear time. However, those algorithms are developed to deal with general planar graphs. And, as such, they are unnecessarily complicated for our purpose. In the remainder of this section, we will present another approach which is much simpler to implement.

We will first show an algorithm for finding a planar circle-confined drawing of a biconnected outerplanar graph. The following lemma due to Syslo [10] gives a useful property of biconnected outerplanar graphs.

Lemma 5 [10]. Any biconnected outerplanar graph with three or more vertices has a unique Hamiltonian cycle. □

Assume that the outerplanar graph $G = (V, E)$ is biconnected. If $|V| = 2$, it is trivial to obtain a planar circle-confined drawing of G. On the other hand, if $|V| \geq 3$, we can construct such a drawing in the following manner.

(1) Find a unique Hamiltonian cycle for G.

(2) Place the vertices in V on the circumference of a circle C in the same order as they appear on the Hamiltonian cycle.

(3) Draw the edges in E by straight line segments inside C.

In what follows, we assume that $|V| \geq 3$, and show how to construct the Hamiltonian cycle for G.

A simple path $P = [v_1, v_2, \ldots, v_j]$ in a graph is called a *branchless path* [1] if the degree of vertex v_i is equal to two for $i = 2, 3, \ldots, j-1$ and the degrees of v_1 and v_j are greater than two. It is known that any biconnected outerplanar graph with three or more vertices has at least two branchless paths if it is not a cycle [11].

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[1] Syslo and Iri [11] called such a path a maximal series of edges. We feel that the term "branchless path" is more appropriate to represent the situation.

Lemma 6 [11]. Suppose that an outerplanar graph has a branchless path $P = [v_1, v_2, \ldots, v_j]$. If the graph contains edge (v_1, v_j), remove all edges on P; otherwise replace P by a new single edge (v_1, v_j) (see Fig. 5). Then, the resultant graph is also outerplanar. □

The operation described in Lemma 6 is called the *reduction* of the branchless path P. Syslo and Iri [11] obtained the following characterization of outerplanar graphs.

Theorem 1 [11]. A biconnected graph with three or more vertices is outerplanar if and only if a cycle is obtained from the graph by zero or more reductions of branchless paths. □

Based on this theorem, they developed a linear time outerplanarity testing algorithm. We will show that such reductions are also useful to find a Hamiltonian cycle for G.

Let $P = [v_1, v_2, \ldots, v_j]$ be a branchless path in G. By definition, the degree of v_i is two for $i = 2, 3, \ldots, j-1$ and the degrees of v_1 and v_j are greater than two. Therefore, G contains at least one vertex different from the vertices on P. Let HC_G be the set of edges on the unique Hamiltonian cycle for G. Since P contains all edges incident upon vertices $v_2, v_3, \ldots, v_{j-1}$, HC_G must contain all edges on P. Let $G' = (V', E')$ be the graph obtained from G by performing the reduction of P. Then G' is outerplanar by Lemma 6, and has at least three vertices as mentioned above. Furthermore, it is easy to see that G' is biconnected. Therefore, by Lemma 5, G' has a unique Hamiltonian cycle. Let $HC_{G'}$ be the set of edges on such a cycle. Then, HC_G and $HC_{G'}$ have the following relationship.

Lemma 7. $HC_G = \{ (v_i, v_{i+1}) \mid i = 1, 2, \ldots, j-1 \} \cup (HC_{G'} - \{ (v_1, v_j) \})$.

Proof. The edges in the set $HC_G - \{ (v_i, v_{i+1}) \mid i = 1, 2, \ldots, j-1 \}$ constitute a Hamiltonian path for G' from v_1 to v_j. Therefore, from the uniqueness of the Hamiltonian cycle for G', we have $HC_{G'} = (HC_G - \{ (v_i, v_{i+1}) \mid i = 1, 2, \ldots, j-1 \}) \cup \{ (v_1, v_j) \}$. □

If G' itself is a cycle of $|V'|$ vertices, then $HC_{G'} = E'$, and hence HC_G is immediately determined by Lemma 7. On the other hand, if G' is not a cycle, it has at least one branchless path. The reduction of such a path will yield a smaller biconnected outerplanar graph. Repeating this process, the graph will eventually become a cycle as claimed in Theorem 1. During such repetitions, we can extract appropriate edges from each of the branchless paths in the following manner.

Algorithm HC

Input: A biconnected outerplanar graph $G = (V, E)$ with $|V| \geq 3$.

Output: The set HC_G of edges on the unique Hamiltonian cycle for G.

Method:

1. Set $HC_G \leftarrow \phi$ and $G' \leftarrow G$. Mark all edges in E "new".
2. **while** G' is not a cycle **do**
   a) Find a branchless path $P = [v_1, v_2, \ldots, v_j]$ in G'.
   b) Add all new edges on P to HC_G.
   c) Perform the reduction of P. Set G' to be the resultant graph, and mark the edge (v_1, v_j) "old".
3. Add all new edges of G' to HC_G. □

In their graph outerplanarity testing algorithm, Syslo and Iri [11] gave an efficient procedure for making a series of reductions of branchless paths. Using their method in Steps 2.a) and 2.c), Algorithm HC can be implemented to run in $O(|V|)$ time. Furthermore, one can easily determine, in $O(|V|)$ time, the Hamiltonian cycle for G from HC_G. Therefore, a planar circle-confined drawing of G can be constructed in $O(|V|)$ time.

Now let $G = (V, E)$ be any connected outerplanar graph. Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$, ..., $G_k = (V_k, E_k)$ be its biconnected components. Without any loss of generality, we assume that $(\cup_{j=1}^{i} V_j) \cap V_{i+1} \neq \phi$ for $i = 1,2,\ldots,k-1$. We denote by $G^{(i)}$ subgraph $(\cup_{j=1}^{i} V_j, \cup_{j=1}^{i} E_j)$ of G for $i = 1,2,\ldots,k$. Note that $G^{(k)} = G$. From the assumption, $G^{(i)}$ and G_{i+1} have one and only one vertex in common which is a cut vertex in G. Let c_i denote such a cut vertex.

Let i be an integer such that $1 \leq i \leq k-1$. Since any subgraph of an outerplanar graph is also outerplanar, $G^{(i)}$ and G_{i+1} each have a planar circle-confined drawing. Let $\overline{G^{(i)}}$ and $\overline{G_{i+1}}$ denote such drawings of $G^{(i)}$ and G_{i+1}, respectively. Then, we can construct a planar circle-confined drawing of $G^{(i+1)}$ by inserting the vertices in $V_{i+1} - \{c_i\}$ onto the circle associated with $\overline{G^{(i)}}$ in such a way that they are placed together between c_i and one of its neighboring vertices in $\overline{G^{(i)}}$ in the same order as in $\overline{G_{i+1}}$.

Since G is outerplanar, $|E| \leq 2 \cdot |V| - 3$, [7]. Using depth-first search [1], we can partition G into its biconnected components in $O(|V|)$ time. If necessary, renumber them in $O(|V|)$ time so that they satisfy the aforementioned assumption. Using Algorithm HC, we can find a planar circle-confined drawing of G_i in $O(|V_i|)$ time for $i = 1,2,\ldots,k$. Therefore, the construction of a planar circle-confined drawing of the entire graph G requires $O(|V|)$ time in total.

4. NP-completeness Proof of OPTIMAL CIRCLE-CONFINED DRAWING

As mentioned in Section 2, the number of crossings in a circle-confined drawing is uniquely determined by the ordering of the vertices on the circle. Therefore, by guessing such an ordering and then computing the number of crossings, OCCD is nondeterministically solvable in polynomial time. Thus, OCCD belongs to the class NP. In order to complete the NP-completeness proof, we have to show that a known NP-complete problem is polynomially transformable to OCCD.

We begin with the following NP-complete problem, called OPTIMAL LINEAR ARRANGEMENT (abbreviated to OLA) [3,5].

Problem OPTIMAL LINEAR ARRANGEMENT (OLA)

Instance : Graph $G = (V, E)$, integer $B > 0$.

Question: Is there a one-to-one function $f : V \rightarrow \{1,2,\ldots, | V |\}$ such that

$$\sum_{(u,v) \in E} | f(u) - f(v) | \leq B ? \square$$

With respect to the instance of OLA, we may assume that $| E | \leq B \leq | V | \cdot | E |$. Furthermore, if $B \geq | E |$, then it is clear that $\sum_{(u,v) \in E} | f(u) - f(v) | \leq B$ if and only if $\sum_{\substack{(u,v) \in E \\ f(u) < f(v)}} (f(v) - f(u) - 1) \leq B - | E |$. Therefore, the following decision problem, which we call MODIFIED OPTIMAL LINEAR ARRANGEMENT (abbreviated to MOLA), is also NP-complete.

Problem MODIFIED OPTIMAL LINEAR ARRANGEMENT (MOLA)

Instance: Graph $G = (V, E)$, integer B such that $0 \leq B \leq | V | \cdot (| E | - 1)$.

Question: Is there a one-to-one function $f : V \rightarrow \{1,2,\ldots, | V |\}$ such that

$$\sum_{\substack{(u,v) \in E \\ f(u) < f(v)}} (f(v) - f(u) - 1) \leq B ? \square$$

In the remainder of this section, we will show a polynomial transformation from MOLA to OCCD. Suppose that we are given a graph $G_0 = (V_0, E_0)$ with $| V_0 | = n$ and an integer B_0 such that $0 \leq B_0 \leq n \cdot (| E_0 | - 1)$. As an instance of OCCD, we construct another graph $G = (V, E)$ and an integer B in the following manner.

$G = (V, E)$,

where $V = V_0 \cup V_1 \cup V_2$,

with $V_1 = \{w_1, w_2, \ldots, w_{2n^4}\}$, and $V_2 = \{x_1, x_2, \ldots, x_{3n^{10}}\}$,

and $E = E_0 \cup E_{01} \cup E_{11} \cup E_{12} \cup E_{22}$,

with $E_{01} = \{ (v,w) \mid v \in V_0, w \in V_1\}$,

$E_{11} = \{ (v,w) \mid v,w \in V_1\}$,

$E_{12} = \{ (v,w) \mid v \in V_1, w \in V_2\}$,

and $E_{22} = \{ (v,w) \mid v,w \in V_2\}$,

and $B = g_1(3n^{10}+2n^4)+g_2(2n^4,n)+g_3(2n^4,n)+g_4(n,n^4,3n^{10},n^4)+n^4-1+2B_0 \cdot n^4$.

Since $B_0 \leq n \cdot (|E_0|-1)$ and $|E_0| \leq \frac{1}{2} \cdot n \cdot (n-1)$, we have $B_0 \leq \frac{1}{2} \cdot n \cdot (n+1)(n-2)$. Therefore, B is bounded by a polynomial function of n from Lemmas 1, 2, 3 and 4. Note that subgraphs $(V_1 \cup V_2, E_{11} \cup E_{12} \cup E_{22})$, (V_1, E_{11}) and (V_2, E_{22}) of G form complete graphs $K_{3n^{10}+2n^4}$, K_{2n^4} and $K_{3n^{10}}$, respectively. A rough sketch of graph G is shown in Fig. 6.

In order to complete the proof, we have to show the equivalence of the following two statements.

(I) There exists a one-to-one function $f_0 : V_0 \rightarrow \{1,2,\ldots,n\}$ such that

$$\sum_{\substack{(u,v) \in E_0 \\ f_0(u) < f_0(v)}} (f_0(v) - f_0(u) - 1) \leq B_0.$$

(II) There exists a circle-confined drawing $\overline{G}$ of G such that $N_{\overline{G}}(E) \leq B$.

Theorem 2. If Statement (I) holds for G_0 and B_0, then Statement (II) holds for G and B.

Proof. Let $f_0 : V_0 \rightarrow \{1,2,\ldots,n\}$ be a one-to-one function such that $\sum_{\substack{(u,v) \in E_0 \\ f_0(u) < f_0(v)}} (f_0(v) - f_0(u) - 1) \leq B_0$. Corresponding to f_0, we define a one-to-one function $f : V \rightarrow \{1,2,\ldots,|V|\}$ as follows.

$$f(v) = f_0(v) \qquad (v \in V_0)$$

$$f(w_i) = n + i \qquad (w_i \in V_1,\ i = 1,2,\ldots,n^4)$$

$$f(x_i) = n^4 + n + i \qquad (x_i \in V_2,\ i = 1,2,\ldots,3n^{10})$$

$$f(w_i) = 3n^{10} + n + i \quad (w_i \in V_1,\ i = n^4+1, n^4+2, \ldots, 2n^4).$$

Let $\overline{G}$ be a circle-confined drawing of G in which the vertices in V are placed on the circumference of the circle in ascending order of the values of $f(\cdot)$ in the clockwise direction (see Fig. 7). Then, it is clear that $N_{\overline{G}}(E_0, E_{11} \cup E_{12} \cup E_{22}) = 0$ and $N_{\overline{G}}(E_{01}, E_{22}) = 0$. Since $|E_0| \leq \frac{1}{2} \cdot n \cdot (n-1)$, $N_{\overline{G}}(E_0) \leq \frac{1}{2} \cdot |E_0| \cdot (|E_0| - 1) \leq n^4 - 1$. Furthermore, the following four equations hold from the definitions of g_1, g_2, g_3 and g_4.

$$N_{\overline{G}}(E_{11} \cup E_{12} \cup E_{22}) = g_1(3n^{10} + 2n^4).$$

$$N_{\overline{G}}(E_{01}) = g_2(2n^4, n).$$

$$N_{\overline{G}}(E_{01}, E_{11}) = g_3(2n^4, n).$$

$$N_{\overline{G}}(E_{01}, E_{12}) = g_4(n, n^4, 3n^{10}, n^4).$$

For each edge $e = (u,v) \in E_0$ such that $f(u) < f(v)$, $N_{\overline{G}}(\{e\}, E_{01}) = 2n^4 \cdot (f(v) - f(u) - 1)$. This implies that

$$\begin{aligned} N_{\overline{G}}(E_0, E_{01}) &= 2n^4 \cdot \sum_{\substack{(u,v) \in E_0 \\ f(u) < f(v)}} (f(v) - f(u) - 1) \\ &= 2n^4 \cdot \sum_{\substack{(u,v) \in E_0 \\ f_0(u) < f_0(v)}} (f_0(v) - f_0(u) - 1) \\ &\leq 2B_0 \cdot n^4. \end{aligned}$$

From the above equations, we have

$$\begin{aligned} N_{\overline{G}}(E) = {} & N_{\overline{G}}(E_0) + N_{\overline{G}}(E_{11} \cup E_{12} \cup E_{22}) + N_{\overline{G}}(E_{01}) + N_{\overline{G}}(E_{01}, E_{11}) + N_{\overline{G}}(E_{01}, E_{12}) \\ & + N_{\overline{G}}(E_0, E_{01}) \end{aligned}$$

$\leq (n^4-1)+g_1(3n^{10}+2n^4)+g_2(2n^4,n)+g_3(2n^4,n)+g_4(n,n^4,3n^{10},n^4)+2B_0 \cdot n^4$

$= B$. $\square$

A circle confined-drawing $\overline{G}$ of G is said to be *canonical* if it satisfies the following condition: For a partition of V_1 into two subsets $V_1^{(1)}$ and $V_1^{(2)}$ such that $|V_1^{(1)}| = |V_1^{(2)}| = n^4$, each of $V_0, V_1^{(1)}, V_2$ and $V_1^{(2)}$ is consecutive in $\overline{G}$ and they are placed on the circumference of the circle in this order in the clockwise direction (see Fig. 8).

For each vertex $v \in V_0$, let $E_{01}(v)$ be defined as $\{(v,w) \in E_{01} \mid w \in V_1\}$. To prove the converse of Theorem 2, we start with the following four lemmas. Since the proofs of Lemmas 9 and 10 are complicated, they are given in Appendix 2.

Lemma 8. For any circle-confined drawing $\overline{G}$ of G,

$N_{\overline{G}}(E_{11} \cup E_{12} \cup E_{22}) = g_1(3n^{10}+2n^4)$, and

$N_{\overline{G}}(E_{01}, E_{11}) = g_3(2n^4, n)$.

Proof. Since subgraph $(V_1 \cup V_2, E_{11} \cup E_{12} \cup E_{22})$ of G forms a complete graph $K_{3n^{10}+2n^4}$, the first equality clearly holds. Let v be a vertex in V_0, and let $\overline{G}_v$ denote the circle-confined drawing which is obtained from $\overline{G}$ by removing all vertices in $V_2 \cup (V_0 - \{v\})$ and the edges incident upon them. Then, V_1 is consecutive in $\overline{G}_v$. Since subgraph (V_1, E_{11}) forms K_{2n^4} and v is adjacent to every vertex in V_1, we have

$$N_{\overline{G}}(E_{01}(v), E_{11}) = N_{\overline{G}_v}(E_{01}(v), E_{11})$$

$$= g_3(2n^4, 1).$$

Applying the same argument to each of the other vertices in V_0, we have

$$N_{\overline{G}}(E_{01}, E_{11}) = \sum_{v \in V_0} N_{\overline{G}}(E_{01}(v), E_{11})$$

$$= n \cdot g_3(2n^4,1).$$

Since $n \cdot g_3(2n^4,1) = g_3(2n^4,n)$ from Lemma 3, we obtain

$$N_{\overline{G}}(E_{01}, E_{11}) = g_3(2n^4,n). \ \square$$

Lemma 9. For any circle-confined drawing $\overline{G}$ of G,

$$N_{\overline{G}}(E_{01}, E_{12}) \geq g_4(n, n^4, 3n^{10}, n^4). \ \square$$

Lemma 10. Let $\overline{G}$ be a circle-confined drawing of G. If V_2 is consecutive in $\overline{G}$ but $\overline{G}$ is not canonical, then

$$N_{\overline{G}}(E_{01}, E_{12}) \geq g_4(n, n^4, 3n^{10}, n^4) + 3n^{10}. \ \square$$

Lemma 11. Let $\overline{G}$ be a circle-confined drawing of G. If V_2 is not consecutive in $\overline{G}$, then

$$N_{\overline{G}}(E_{01}, E_{22}) > g_2(2n^4,n) + 2B_0 \cdot n^4 + n^4 - 1.$$

Proof. Suppose that V_2 is not consecutive in $\overline{G}$. Then, it is easy to see that there exist two edges $e = (v,w) \in E_{01}$ and $e' = (x,y) \in E_{22}$ which cross each other in $\overline{G}$. Starting from v, we visit the vertices on the circumference of the circle one by one in the clockwise direction. We denote by α the number of vertices in V_2 which are encountered before we arrive at w. Since exactly one of x and y is encountered by such a visit, $1 \leq \alpha \leq 3n^{10} - 1$. Therefore, we have

$$\begin{aligned} N_{\overline{G}}(E_{01}, E_{22}) &\geq N_{\overline{G}}(\{e\}, E_{22}) \\ &= \alpha \cdot (3n^{10} - \alpha) \\ &\geq 3n^{10} - 1 \\ &\geq n^{10} + n^7 + n^4 - 1 - (n^9 - n^7) - \frac{1}{2}(n^6 - n^5) \\ &= (n^{10} - n^9 - \frac{1}{2}n^6 + \frac{1}{2}n^5) + 2n^7 + n^4 - 1. \end{aligned}$$

By Lemma 2, $n^{10} - n^9 - \frac{1}{2}n^6 + \frac{1}{2}n^5 = g_2(2n^4,n)$. Furthermore, $B_0 \leq n \cdot (|E_0| - 1) < n^3$ by the assumption on B_0, and hence $2n^7 > 2B_0 \cdot n^4$. Thus, we obtain

$$N_{\overline{G}}(E_{01}, E_{22}) > g_2(2n^4, n) + 2B_0 \cdot n^4 + n^4 - 1. \square$$

The above four lemmas lead to the following lemma.

Lemma 12. If $\overline{G}$ is a circle-confined drawing of G such that $N_{\overline{G}}(E) \le B$, then $\overline{G}$ is canonical.

Proof. Assume that V_2 is not consecutive in $\overline{G}$. Then, $N_{\overline{G}}(E_{01}, E_{22}) > g_2(2n^4, n) + 2B_0 \cdot n^4 + n^4 - 1$ from Lemma 11. Furthermore, the following three equations hold by Lemmas 8 and 9.

$$N_{\overline{G}}(E_{11} \cup E_{12} \cup E_{22}) = g_1(3n^{10} + 2n^4).$$

$$N_{\overline{G}}(E_{01}, E_{11}) = g_3(2n^4, n).$$

$$N_{\overline{G}}(E_{01}, E_{12}) \ge g_4(n, n^4, 3n^{10}, n^4).$$

Therefore, we have

$$\begin{aligned} N_{\overline{G}}(E) &\ge N_{\overline{G}}(E_{01}, E_{22}) + N_{\overline{G}}(E_{11} \cup E_{12} \cup E_{22}) + N_{\overline{G}}(E_{01}, E_{11}) + N_{\overline{G}}(E_{01}, E_{12}) \\ &> g_2(2n^4, n) + 2B_0 \cdot n^4 + n^4 - 1 + g_1(3n^{10} + 2n^4) + g_3(2n^4, n) + g_4(n, n^4, 3n^{10}, n^4) \\ &= B. \end{aligned}$$

This contradicts the hypothesis that $N_{\overline{G}}(E) \le B$. Thus, if $N_{\overline{G}}(E) \le B$, then V_2 is consecutive in $\overline{G}$.

We now assume that $\overline{G}$ is not canonical. Since V_2 is consecutive in $\overline{G}$, $N_{\overline{G}}(E_{01}, E_{12}) \ge g_4(n, n^4, 3n^{10}, n^4) + 3n^{10}$ from Lemma 10. Furthermore, $3n^{10} > g_2(2n^4, n) + 2B_0 \cdot n^4 + n^4 - 1$ as is shown in the proof of Lemma 11. Thus, we have

$$N_{\overline{G}}(E_{01}, E_{12}) > g_4(n, n^4, 3n^{10}, n^4) + g_2(2n^4, n) + 2B_0 \cdot n^4 + n^4 - 1.$$

This and Lemma 8 yield

$$N_{\overline{G}}(E) \geq N_{\overline{G}}(E_{11} \cup E_{12} \cup E_{22}) + N_{\overline{G}}(E_{01}, E_{11}) + N_{\overline{G}}(E_{01}, E_{12})$$

$$> g_1(3n^{10}+2n^4) + g_3(2n^4, n) + g_4(n, n^4, 3n^{10}, n^4) + g_2(2n^4, n) + 2B_0 \cdot n^4 + n^4 - 1$$

$$= B.$$

This contradicts the hypothesis that $N_{\overline{G}}(E) \leq B$ again. Consequently, if $N_{\overline{G}}(E) \leq B$, then $\overline{G}$ is canonical. □

The following lemma is trivial from the definitions.

Lemma 13. For any canonical circle-confined drawing $\overline{G}$ of G,

$N_{\overline{G}}(E_{01}) = g_2(2n^4, n)$, and

$N_{\overline{G}}(E_{01}, E_{12}) = g_4(n, n^4, 3n^{10}, n^4)$. □

We are now ready to show the following theorem.

Theorem 3. If Statement (II) holds for G and B, then Statement (I) holds for G_0 and B_0.

Proof. Let $\overline{G}$ be a circle-confined drawing of G such that $N_{\overline{G}}(E) \leq B$. Then, $\overline{G}$ is canonical from Lemma 12, and hence V_0 is consecutive in $\overline{G}$. Therefore, one can define a one-to-one function $f : V \to \{1,2,\ldots,|V|\}$ such that (i) $\{f(v) \mid v \in V_0\} = \{1,2,\ldots,n\}$, and (ii) the vertices are placed on the circumference of the circle in ascending order of the values of $f(\cdot)$ in the clockwise direction.

Assume that $\sum_{\substack{(u,v) \in E_0 \\ f(u) < f(v)}} (f(v) - f(u) - 1) \geq B_0 + 1$. For each edge $e = (u, v) \in E_0$ such that $f(u) < f(v)$, we have $N_{\overline{G}}(\{e\}, E_{01}) = 2n^4 \cdot (f(v) - f(u) - 1)$. This implies that

$$N_{\overline{G}}(E_0, E_{01}) = 2n^4 \cdot \sum_{\substack{(u,v) \in E_0 \\ f(u) < f(v)}} (f(v) - f(u) - 1)$$

$$\geq 2n^4 \cdot (B_0 + 1)$$

$$> 2B_0 \cdot n^4 + n^4 - 1.$$

Therefore, from Lemmas 8 and 13, we have

$$N_{\overline{G}}(E) \geq N_{\overline{G}}(E_{11} \cup E_{12} \cup E_{22}) + N_{\overline{G}}(E_{01}) + N_{\overline{G}}(E_{01}, E_{11}) + N_{\overline{G}}(E_{01}, E_{12}) + N_{\overline{G}}(E_0, E_{01})$$

$$> g_1(3n^{10}+2n^4) + g_2(2n^4, n) + g_3(2n^4, n) + g_4(n, n^4, 3n^{10}, n^4) + 2B_0 \cdot n^4 + n^4 - 1$$

$$= B.$$

This contradicts the hypothesis that $N_{\overline{G}}(E) \leq B$. Therefore, we have $\sum_{\substack{(u,v) \in E_0 \\ f(u) < f(v)}} (f(v) - f(u) - 1) \leq B_0$. This completes the proof. □

As mentioned before, OCCD belongs to the class NP. Since G and B can be constructed in polynomial time, the following theorem is established from Theorems 2 and 3.

Theorem 4. MOLA is polynomially transformable to OCCD. Therefore, OCCD is NP-complete. □

5. Conclusion

In this paper, we considered a graph drawing problem which gives rise to a computer communication network layout problem. More specifically, the problem is to find an embedding of a graph with the minimum number of edge-crossings such that the vertices are placed on the circumference of a circle and that the edges are drawn inside the circle. We have first presented a linear time algorithm for embedding an outerplanar graph in such a way that no edge-crossings occur. We have then shown that the corresponding decision problem is in general NP-complete. If the graph is allowed to have multiple edges between some pairs of vertices, then the NP-completeness proof becomes much simpler. In fact, such a proof was given in the preliminary version of this paper [9].

It is important to develop an effective heuristic algorithm for the problem. As is usually the case with this type of problems, a reasonable approach is to find an initial solution and improve its quality to produce a good final solution. We have already obtained a good heuristic algorithm for the initial stage [8] and are currently engaged in the development of an algorithm for the second stage.

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References

[1] A. V. Aho, J. E. Hopcroft and J. D. Ullman, *The Design and Analysis of Computer Algorithms*, Addison-Wesley, Reading, Mass, 1974.

[2] N. Chiba, T. Nishizeki, S. Abe and T. Ozawa, "A linear algorithm for embedding planar graphs using PQ-trees", *J. of Computer and System Sciences*, vol. 30, pp. 54-76, 1985.

[3] M. R. Garey and D. S. Johnson, *Computers and Intractability - A Guide to the Theory of NP-Completeness*, W. H. Freeman and Co., San Francisco, Ca, 1979.

[4] M. R. Garey and D. S. Johnson, "CROSSING NUMBER is NP-complete", *SIAM J. on Algebraic and Discrete Methods*, vol. 4, pp. 312-316, 1983.

[5] M. R. Garey, D. S. Johnson and L. Stockmeyer, "Some simplified NP-complete graph problems", *Theoretical Computer Science*, vol. 1, pp. 237-267, 1976.

[6] R. S. Gilbert and W. B. Kleinoder, "CNMgraf - Graphic presentation services for network management", *Proc. 9th Data Communications Symposium*, Whistler, B.C., Canada, September 1985, pp. 199-206.

[7] F. Harary, *Graph Theory*, Addison-Wesley, Reading, Mass, 1974.

[8] P.-Y. F. Lin, S. Masuda and K. Nakajima, "Heuristic layout algorithms for computer networks", to appear in *Proc. IEEE Pacific Rim Conference on Communications, Computers and Signal Processing*, Victoria, B.C., Canada, June 1987.

[9] S. Masuda, T. Kashiwabara, K. Nakajima and T. Fujisawa, "On the NP-completeness of a

computer network layout problem", to appear in *Proc. 1987 IEEE International Symposium on Circuits and Systems,* Philadelphia, Pa, May 1987.

[10] M. M. Syslo, "Characterizations of outerplanar graphs", *Discrete Mathematics,* vol. 26, pp. 47-53, 1979.

[11] M. M. Syslo and M. Iri, "Efficient outerplanarity testing", *Ann. Soc. Math. Pol. Ser. IV: Fundamenta Informaticae,* vol. 2, pp. 261-275, 1979.

Appendix 1. Proofs of Lemmas 1, 2, 3 and 4

The proofs of Lemmas 2, 3, 1 and 4 are given here in this order.

Proof of Lemma 2. Let H_2 be a complete bipartite graph described as follows:

$$H_2 = (V_1 \cup V_2, E),$$

$$\text{where } V_1 = \{v_1, v_2, \ldots, v_p\},$$

$$V_2 = \{v_{p+1}, v_{p+2}, \ldots, v_{p+q}\},$$

$$\text{and } E = \{(v, w) \mid v \in V_1, w \in V_2\}.$$

Let $\overline{H}_2$ be a circle-confined drawing of H_2 in which the vertices are placed on the circumference of the circle in ascending order of their subscripts in the clockwise direction (see Fig. A-1). By the definition of g_2, $g_2(p,q) = N_{\overline{H}_2}(E)$. When $p = 0$ or $q = 0$, clearly $g_2(p,q) = 0$, and hence the lemma holds. Therefore, we only need to consider the case in which $p, q \geq 1$.

For $i = 1,2,\ldots,q$, let $e_i = (v_1, v_{p+i})$ and $E_i = \{(w, v_{p+i}) \mid w \in V_1 - \{v_1\}\}$. Furthermore, let $E^* = E_1 \cup E_2 \cup \cdots \cup E_q$, that is, $E^* = E - \{e_1, e_2, \ldots, e_q\}$. Then, $N_{\overline{H}_2}(E - E^*) = 0$, and hence $N_{\overline{H}_2}(E) = N_{\overline{H}_2}(E^*) + N_{\overline{H}_2}(E^*, E - E^*)$. Since subgraph $((V_1 - \{v_1\}) \cup V_2, E^*)$ of H_2 forms a complete bipartite graph, $N_{\overline{H}_2}(E^*) = g_2(p-1, q)$. Thus, we have

$$g_2(p,q) = g_2(p-1,q) + N_{\overline{H}_2}(E^*, E - E^*).$$

If $q \geq 2$, e_i crosses the edges in $E_{i+1} \cup E_{i+2} \cup \cdots \cup E_q \subset E^*$ for $i = 1,2,\ldots,q-1$, and e_q crosses no edges in E^*. Since $|E_i| = p - 1$ for $i = 1,2,\ldots,q$, we have

$$N_{\overline{H}_2}(E^*, E - E^*) = \Sigma_{i=1}^{q} N_{\overline{H}_2}(E^*, \{e_i\})$$

$$= \Sigma_{i=1}^{q-1}(q - i)\cdot(p - 1)$$

$$= \frac{1}{2}\cdot q\cdot(p - 1)\cdot(q - 1) \quad \text{for } q \geq 2.$$

If $q = 1$, $N_{\overline{H}_2}(E^*, E - E^*) = 0$, and hence the above equality still holds. Thus, we have

$$g_2(p, q) = g_2(p - 1, q) + \frac{1}{2}\cdot q\cdot(p - 1)\cdot(q - 1) \quad \text{for } p, q \geq 1.$$

Since $g_2(0,q)=0$, we obtain

$$g_2(p, q) = \Sigma_{i=1}^{p}\frac{1}{2}\cdot q\cdot(q - 1)\cdot(i - 1)$$

$$= \frac{1}{4}\cdot p\cdot q\cdot(p - 1)\cdot(q - 1). \ \square$$

Proof of Lemma 3. Let H_3 be a graph defined by:

$H_3 = (V_1 \cup V_2, E_{11} \cup E_{12})$,

where $V_1 = \{v_1, v_2, \ldots, v_p\}$,

$V_2 = \{v_{p+1}, v_{p+2}, \ldots, v_{p+q}\}$,

$E_{11} = \{ (v, w) \mid v, w \in V_1\}$,

and $E_{12} = \{ (v, w) \mid v \in V_1, w \in V_2\}$.

Let $\overline{H}_3$ be a circle-confined drawing of H_3 in which the vertices are placed on the circumference of the circle in ascending order of their subscripts in the clockwise direction. By the definition of g_3, $g_3(p, q) = N_{\overline{H}_3}(E_{11}, E_{12})$. The lemma trivially holds when $p = 0$ or $q = 0$. Therefore, we only need to consider the case in which p, $q \geq 1$.

For a vertex v in V_2, let $e_i = (v, v_i)$ for $i = 1,2,\ldots,p$. Note that e_i crosses the edges in $\{ (v_j, v_k) \mid 1 \leq j < i < k \leq p \} \subset E_{11}$. Let $E^* = \{ (v, w) \mid w \in V_1\} = \{e_1, e_2, \ldots, e_p\}$. Then, we have

$$N_{\overline{H}_3}(E_{11}, E^v) = \Sigma_{i=1}^{p} N_{\overline{H}_3}(E_{11}, \{e_i\})$$

$$= \Sigma_{i=1}^{p}(i-1)\cdot(p-i)$$

$$= \frac{1}{6}\cdot p\cdot(p-1)\cdot(p-2).$$

It is clear that $N_{\overline{H}_3}(E_{11}, E_{12} - E^v) = g_3(p, q-1)$. Furthermore, $N_{\overline{H}_3}(E_{11}, E_{12}) = N_{\overline{H}_3}(E_{11}, E^v) + N_{\overline{H}_3}(E_{11}, E_{12} - E^v)$. Thus, we have

$$g_3(p, q) = g_3(p, q-1) + \frac{1}{6}\cdot p\cdot(p-1)\cdot(p-2).$$

Since $g_3(p, 0) = 0$, we obtain

$$g_3(p, q) = \frac{1}{6}\cdot p\cdot q\cdot(p-1)\cdot(p-2). \ \square$$

Proof of Lemma 1. Let $H_1 = (V, E)$ be a complete graph with p vertices, and let $\overline{H}_1$ be a circle-confined drawing of H_1. By the definition of g_1, $g_1(p) = N_{\overline{H}_1}(E)$. The lemma clearly holds when $p = 0$. Suppose that $p \geq 1$. For a vertex v in V, let $E^v = \{(v, w) \in E \mid w \in V - \{v\}\}$. Then, $N_{\overline{H}_1}(E - E^v) = g_1(p-1)$ since subgraph $(V - \{v\}, E - E^v)$ of H_1 forms K_{p-1}. Furthermore, $N_{\overline{H}_1}(E - E^v, E^v) = g_3(p-1, 1) = \frac{1}{6}\cdot(p-1)\cdot(p-2)\cdot(p-3)$ from Lemma 3. Since $N_{\overline{H}_1}(E^v) = 0$, we have

$$g_1(p) = N_{\overline{H}_1}(E - E^v) + N_{\overline{H}_1}(E - E^v, E^v)$$

$$= g_1(p-1) + \frac{1}{6}\cdot(p-1)\cdot(p-2)\cdot(p-3).$$

Since $g_1(0) = 0$, we derive

$$g_1(p) = \frac{1}{6}\cdot\Sigma_{i=1}^{p}(i-1)\cdot(i-2)\cdot(i-3)$$

$$= \frac{1}{24}\cdot p\cdot(p-1)\cdot(p-2)\cdot(p-3). \ \square$$

Proof of Lemma 4. Let H_4 be a graph defined by:

$$H_4 = (V_1 \cup V_2 \cup V_3 \cup V_4, E_{12} \cup E_{23} \cup E_{34} \cup E_{41}),$$

where $|V_1| = p$, $|V_2| = q$, $|V_3| = r$, $|V_4| = s$,

$$E_{i(i+1)} = \{ (v,w) \mid v \in V_i, w \in V_{i+1} \} \quad \text{for } i=1,2,3,$$

and $E_{41} = \{ (v,w) \mid v \in V_4, w \in V_1 \}$.

Let $\overline{H}_4$ be a circle-confined drawing of H_4 in which each of V_1, V_2, V_3 and V_4 is consecutive and they are located on the circumference of the circle in this order in the clockwise direction. Then, from the definition of g_4, $g_4(p,q,r,s) = N_{\overline{H}_4}(E_{12}, E_{23}) + N_{\overline{H}_4}(E_{34}, E_{41})$. If $p = |V_1| = 0$, then $E_{12} = E_{41} = \phi$. On the other hand, if $r = |V_3| = 0$, then $E_{23} = E_{34} = \phi$. Thus, the lemma holds when $p = 0$ or $r = 0$. Therefore, we only need to consider the case in which $p, r \geq 1$.

It is clear that $N_{\overline{H}_4}(E_{12}, E_{23}) = N_{\overline{H}_4}(E_{12} \cup E_{23}) - N_{\overline{H}_4}(E_{12}) - N_{\overline{H}_4}(E_{23})$. Furthermore, subgraphs $((V_1 \cup V_3) \cup V_2, E_{12} \cup E_{23})$, $(V_1 \cup V_2, E_{12})$ and $(V_2 \cup V_3, E_{23})$ each are complete bipartite graphs, and hence we have

$$N_{\overline{H}_4}(E_{12} \cup E_{23}) = g_2(p+r, q),$$

$$N_{\overline{H}_4}(E_{12}) = g_2(p, q), \text{ and}$$

$$N_{\overline{H}_4}(E_{23}) = g_2(r, q).$$

Thus, from Lemma 2, we obtain

$$\begin{aligned} N_{\overline{H}_4}(E_{12}, E_{23}) &= g_2(p+r, q) - g_2(p, q) - g_2(r, q) \\ &= \frac{1}{4} \cdot q \cdot (q-1) \cdot \{ (p+r) \cdot (p+r-1) - p \cdot (p-1) - r \cdot (r-1) \}. \\ &= \frac{1}{2} \cdot p \cdot q \cdot r \cdot (q-1) \quad \text{for } q \geq 0. \end{aligned}$$

Similarly, we have

$$N_{\overline{H}_4}(E_{34}, E_{41}) = \frac{1}{2} \cdot p \cdot r \cdot s \cdot (s - 1) \quad \text{for } s \geq 0.$$

Therefore, we obtain

$$g_4(p, q, r, s) = N_{\overline{H}_4}(E_{12}, E_{23}) + N_{\overline{H}_4}(E_{34}, E_{41})$$

$$= \frac{1}{2} \cdot p \cdot r \cdot \{ q \cdot (q - 1) + s \cdot (s - 1) \} \quad \text{for } q, s \geq 0. \ \square$$

Appendix 2. Proofs of Lemmas 9 and 10

Let v be a vertex in V_0. Let G^* be the induced subgraph of G with respect to $V_1 \cup V_2 \cup \{v\}$, and let $\overline{G}^*$ be a circle-confined drawing of G^* in which V_2 is not consecutive. Then, V_2 is partitioned in $\overline{G}^*$ into t subsets $V_2(1), V_2(2), \ldots, V_2(t)$ for some integer $t \geq 2$ such that

(i) $V_2(1) \cup V_2(2) \cup \cdots \cup V_2(t) = V_2$ and $V_2(i) \cap V_2(j) = \phi$ for $1 \leq i \neq j \leq t$,

(ii) for $i = 1,2,\ldots,t$, $V_2(i)$ is not empty and is consecutive in $\overline{G}^*$, but $V_2(i) \cup \{w\}$ is not consecutive in $\overline{G}^*$ for any vertex $w \in V_2 - V_2(i)$, and

(iii) the vertices in $\{v\}, V_2(1), V_2(2), \ldots, V_2(t)$ are placed on the circumference of the circle in this order in the clockwise direction.

We call t the *partition number* of $\overline{G}^*$. For $i = 2,3,\ldots,t$, let $V_1(i)$ denote the set of vertices in V_1 such that $V_1(i)$ itself is consecutive in $\overline{G}^*$ and that $V_2(i-1) \cup V_1(i) \cup V_2(i)$ is also consecutive in $\overline{G}^*$. Note that the vertices in $\{v\}$, $V_2(1)$, $V_1(2)$, $V_2(2)$, $V_1(3)$, ..., $V_2(t-1)$, $V_1(t)$, $V_2(t)$ appear on the circle in this order in the clockwise direction. Let $V_1(1)$ (resp., $V_1(t+1)$) denote the set of vertices in $V_1 - \cup_{i=2}^{t} V_1(i)$ which are placed between v and the vertices in $V_2(1)$ (resp., $V_2(t)$). See Fig. A-2. Note that $V_1(1)$ and $V_1(t+1)$ may be empty while $V_1(2)$, $V_1(3)$, ..., $V_1(t)$ are not.

We define $\alpha(i)$, $\alpha(i,j)$, $\beta(i)$ and $\beta(i,j)$ as follows:

$$\alpha(i) = |V_1(i)| \quad \text{for } i = 1,2,\ldots,t+1.$$

$$\alpha(i,j) = \alpha(i)+\alpha(i+1)+\cdots+\alpha(j) \quad \text{for } 1 \leq i \leq j \leq t+1.$$

$$\beta(i) = |V_2(i)| \quad \text{for } i = 1,2,\ldots,t.$$

$$\beta(i,j) = \beta(i)+\beta(i+1)+\cdots+\beta(j) \quad \text{for } 1 \leq i \leq j \leq t.$$

Let $E_{01}^{(i)} = \{ (v,w) \mid w \in V_1(i) \}$ for $i = 1,2,\ldots,t+1$. Note that $E_{01}^{(1)} \cup E_{01}^{(2)} \cup \cdots \cup E_{01}^{(t+1)} = E_{01}(v)$. Let $E_{12}^{-}(i) = \{ (w,z) \mid w \in V_1(i),\ z \in V_2(1) \cup V_2(2) \cup \cdots \cup V_2(i-1) \}$ for $i = 2,3,\ldots,t+1$, and $E_{12}^{+}(i) = \{ (w,z) \mid w \in V_1(i),\ z \in V_2(i) \cup V_2(i+1) \cup \cdots \cup V_2(t) \}$ for $i = 1,2,\ldots,t$. See Fig. A-3. In particular, we define $E_{12}^{-}(1)$ and $E_{12}^{+}(t+1)$ to be empty.

Claim 1. For $i = 1,2,\ldots,t+1$, $N_{\bar{G}*}(E_{01}^{(i)}, E_{12}^{-}(i) \cup E_{12}^{+}(i)) = \frac{3}{2}\cdot\alpha(i)\cdot(\alpha(i)-1)\cdot n^{10}$.

Proof. For $i = 2,3,\ldots,t+1$, subgraphs $(V_1(i) \cup (\{v\} \cup V_2(1) \cup V_2(2) \cup \cdots \cup V_2(i-1)),\ E_{01}^{(i)} \cup E_{12}^{-}(i))$ and $(V_1(i) \cup (V_2(1) \cup V_2(2) \cup \cdots \cup V_2(i-1)),\ E_{12}^{-}(i))$ form complete bipartite graphs (see Fig. A-3 again). Since $|V_1(i)| = \alpha(i)$ and $|V_2(1) \cup V_2(2) \cup \cdots \cup V_2(i-1)| = \beta(1,i-1)$ by definition, we have

$$N_{\bar{G}*}(E_{01}^{(i)} \cup E_{12}^{-}(i)) = g_2(\alpha(i),\ \beta(1,i-1)+1), \text{ and}$$

$$N_{\bar{G}*}(E_{12}^{-}(i)) = g_2(\alpha(i),\ \beta(1,i-1)).$$

It is clear that $N_{\bar{G}*}(E_{01}^{(i)}) = 0$, and hence $N_{\bar{G}*}(E_{01}^{(i)} \cup E_{12}^{-}(i)) = N_{\bar{G}*}(E_{01}^{(i)}, E_{12}^{-}(i)) + N_{\bar{G}*}(E_{12}^{-}(i))$. This and Lemma 2 yield

$$\begin{aligned}
&N_{\bar{G}*}(E_{01}^{(i)}, E_{12}^{-}(i)) \\
&= N_{\bar{G}*}(E_{01}^{(i)} \cup E_{12}^{-}(i)) - N_{\bar{G}*}(E_{12}^{-}(i)) \\
&= g_2(\alpha(i),\ \beta(1,i-1)+1) - g_2(\alpha(i),\ \beta(1,i-1)) \\
&= \frac{1}{4}\cdot\alpha(i)\cdot(\alpha(i)-1)\cdot\{ (\beta(1,i-1)+1)\cdot\beta(1,i-1) - \beta(1,i-1)\cdot(\beta(1,i-1)-1) \} \\
&= \frac{1}{2}\cdot\alpha(i)\cdot(\alpha(i)-1)\cdot\beta(1,i-1) \quad \text{for } i = 2,3,\ldots,t+1.
\end{aligned}$$

In the case of $i=1$, $N_{\overline{G}^*}(E_{01}^{(1)}, E_{12}^-(1)) = 0$ since $E_{12}^-(1) = \phi$.

Similarly, we obtain

$$N_{\overline{G}^*}(E_{01}^{(i)}, E_{12}^+(i)) = \frac{1}{2}\cdot\alpha(i)\cdot(\alpha(i)-1)\cdot\beta(i,t) \quad \text{for } i=1,2,\ldots,t, \text{ and}$$

$$N_{\overline{G}^*}(E_{01}^{(t+1)}, E_{12}^+(t+1)) = 0.$$

From the above equalities, we have the following equations.

$$\begin{aligned}N_{\overline{G}^*}(E_{01}^{(1)}, E_{12}^-(1)\cup E_{12}^+(1)) &= N_{\overline{G}^*}(E_{01}^{(1)}, E_{12}^+(1))\\ &= \frac{1}{2}\cdot\alpha(1)\cdot(\alpha(1)-1)\cdot\beta(1,t).\end{aligned}$$

$$\begin{aligned}N_{\overline{G}^*}(E_{01}^{(i)}, E_{12}^-(i)\cup E_{12}^+(i)) &= N_{\overline{G}^*}(E_{01}^{(i)}, E_{12}^-(i)) + N_{\overline{G}^*}(E_{01}^{(i)}, E_{12}^+(i))\\ &= \frac{1}{2}\cdot\alpha(i)\cdot(\alpha(i)-1)\cdot(\beta(1,i-1)+\beta(i,t))\\ &= \frac{1}{2}\cdot\alpha(i)\cdot(\alpha(i)-1)\cdot\beta(1,t) \quad \text{for } i=2,3,\ldots,t.\end{aligned}$$

$$\begin{aligned}N_{\overline{G}^*}(E_{01}^{(t+1)}, E_{12}^-(t+1)\cup E_{12}^+(t+1)) &= N_{\overline{G}^*}(E_{01}^{(t+1)}, E_{12}^-(t+1))\\ &= \frac{1}{2}\cdot\alpha(t+1)\cdot(\alpha(t+1)-1)\cdot\beta(1,t).\end{aligned}$$

Since $\beta(1,t) = |V_2| = 3n^{10}$, the following equality holds for $i=1,2,\ldots,t+1$.

$$N_{\overline{G}^*}(E_{01}^{(i)}, E_{12}^-(i)\cup E_{12}^+(i)) = \frac{3}{2}\cdot\alpha(i)\cdot(\alpha(i)-1)\cdot n^{10}. \ \square$$

Claim 2. $N_{\overline{G}^*}(E_{01}^{(1)}, E_{12}) = \frac{3}{2}\cdot\alpha(1)\cdot(\alpha(1)-1)\cdot n^{10}.$

$$N_{\overline{G}^*}(E_{01}^{(t+1)}, E_{12}) = \frac{3}{2}\cdot\alpha(t+1)\cdot(\alpha(t+1)-1)\cdot n^{10}.$$

$$\begin{aligned}N_{\overline{G}^*}(E_{01}^{(i)}, E_{12}) = {}&\alpha(i)\cdot\{\,\alpha(1,i-1)\cdot\beta(i,t)+\alpha(i+1,t+1)\cdot\beta(1,i-1)\,\}\\ &+\frac{3}{2}\cdot\alpha(i)\cdot(\alpha(i)-1)\cdot n^{10} \quad \text{for } i=2,3,\ldots,t.\end{aligned}$$

Proof. It is clear that $N_{\overline{G}^*}(E_{01}^{(1)}, E_{12}) = N_{\overline{G}^*}(E_{01}^{(1)}, E_{12}^+(1))$ and that $N_{\overline{G}^*}(E_{01}^{(t+1)}, E_{12}) = N_{\overline{G}^*}(E_{01}^{(t+1)}, E_{12}^-(t+1))$. Therefore, the first two equalities are obtained from Claim 1.

For $i = 2,3,\ldots,t$, the edges in $E_{01}^{(i)}$ cross those in $\{(w,z) \mid w \in V_1(1) \cup V_1(2) \cup \cdots \cup V_1(i-1),\ z \in V_2(i) \cup V_2(i+1) \cup \cdots \cup V_2(t)\} \cup \{(w,z) \mid w \in V_1(i+1) \cup V_1(i+2) \cup \cdots \cup V_1(t+1),\ z \in V_2(1) \cup V_2(2) \cup \cdots \cup V_2(i-1)\} \subset E_{12} - (E_{12}^-(i) \cup E_{12}^+(i))$. Thus, from Claim 1, we derive

$$
\begin{aligned}
&N_{\overline{G}^*}(E_{01}^{(i)}, E_{12}) \\
&= \alpha(i)\cdot\{\ \alpha(1,i-1)\cdot\beta(i,t) + \alpha(i+1,t+1)\cdot\beta(1,i-1)\ \} + N_{\overline{G}^*}(E_{01}^{(i)}, E_{12}^-(i) \cup E_{12}^+(i)) \\
&= \alpha(i)\cdot\{\ \alpha(1,i-1)\cdot\beta(i,t) + \alpha(i+1,t+1)\cdot\beta(1,i-1)\ \} + \frac{3}{2}\cdot\alpha(i)\cdot(\alpha(i)-1)\cdot n^{10}. \ \square
\end{aligned}
$$

We now consider two embeddings $\tilde{G}^*$ and $\hat{G}^*$ of G^* defined as follows. $\tilde{G}^*$ (resp., $\hat{G}^*$) is the circle-confined drawing which is obtained from $\overline{G}^*$ by exchanging the position of $V_1(2)$ for that of $V_2(1)$ (resp., $V_2(2)$) in such a way that the orders of the vertices within $V_1(2)$ and $V_2(1)$ (resp., $V_2(2)$) are preserved (see Fig. A-4). Then, the following two claims can be established.

Claim 3. $N_{\tilde{G}^*}(E_{01}(v), E_{12}) = N_{\overline{G}^*}(E_{01}(v), E_{12}) + \alpha(2)\cdot\beta(1)\cdot(\ \alpha(1) - \alpha(3,t+1)\ )$.

Proof. From the definitions, $E_{01}(v) = E_{01}^{(1)} \cup E_{01}^{(2)} \cup \cdots \cup E_{01}^{(t+1)}$. Furthermore, it is clear that $N_{\tilde{G}^*}(E_{01}^{(i)}, E_{12}) = N_{\overline{G}^*}(E_{01}^{(i)}, E_{12})$ for $i = 3,4,\ldots,t+1$. Thus, $N_{\tilde{G}^*}(E_{01}(v), E_{12}) - N_{\overline{G}^*}(E_{01}(v), E_{12}) = N_{\tilde{G}^*}(E_{01}^{(1)} \cup E_{01}^{(2)}, E_{12}) - N_{\overline{G}^*}(E_{01}^{(1)} \cup E_{01}^{(2)}, E_{12})$. From Claim 2, we have

$$
\begin{aligned}
&N_{\overline{G}^*}(E_{01}^{(1)} \cup E_{01}^{(2)}, E_{12}) \\
&= N_{\overline{G}^*}(E_{01}^{(1)}, E_{12}) + N_{\overline{G}^*}(E_{01}^{(2)}, E_{12}) \\
&= \frac{3}{2}\cdot\alpha(1)\cdot(\alpha(1)-1)\cdot n^{10} + \frac{3}{2}\cdot\alpha(2)\cdot(\alpha(2)-1)\cdot n^{10} + \alpha(2)\cdot\{\ \alpha(1)\cdot\beta(2,t) + \alpha(3,t+1)\cdot\beta(1)\ \}.
\end{aligned}
$$

Furthermore, $V_1(1) \cup V_1(2)$ and $V_2(1) \cup V_2(2)$ each are consecutive in $\tilde{G}^*$. Thus, substituting $\alpha(1)+\alpha(2)$ for $\alpha(1)$ in the right hand side of the first equality in Claim 2, we obtain

$$N_{\check{G}^*}(E_{01}^{(1)} \cup E_{01}^{(2)}, E_{12}) = \frac{3}{2}\cdot(\ \alpha(1)+\alpha(2)\ )\cdot(\ \alpha(1)+\alpha(2)-1\ )\cdot n^{10}.$$

Therefore, we can obtain

$$N_{\check{G}^*}(E_{01}(v), E_{12}) - N_{\overline{G}^*}(E_{01}(v), E_{12})$$

$$= \frac{3}{2}\cdot\{\ (\alpha(1)+\alpha(2)\ )\cdot(\alpha(1)+\alpha(2)-1) - \alpha(1)\cdot(\alpha(1)-1) - \alpha(2)\cdot(\alpha(2)-1)\ \}\cdot n^{10}$$

$$- \alpha(2)\cdot\alpha(1)\cdot\beta(2,t) - \alpha(2)\cdot\alpha(3,t+1)\cdot\beta(1)$$

$$= 3\alpha(1)\cdot\alpha(2)\cdot n^{10} - \alpha(1)\cdot\alpha(2)\cdot(3n^{10} - \beta(1)\ ) - \alpha(2)\cdot\alpha(3,t+1)\cdot\beta(1)$$

$$= \alpha(2)\cdot\beta(1)\cdot(\ \alpha(1) - \alpha(3,t+1)\ ).\ \square$$

Claim 4. $N_{\hat{G}^*}(E_{01}(v), E_{12}) = N_{\overline{G}^*}(E_{01}(v), E_{12}) + \alpha(2)\cdot\beta(2)\cdot(\ \alpha(3,t+1) - \alpha(1)\ )$.

Proof. **Using an argument similar to the one made in the proof of Claim 3, we derive**

$$N_{\hat{G}^*}(E_{01}(v), E_{12}) - N_{\overline{G}^*}(E_{01}(v), E_{12})$$

$$= N_{\hat{G}^*}(E_{01}^{(2)} \cup E_{01}^{(3)}, E_{12}) - N_{\overline{G}^*}(E_{01}^{(2)} \cup E_{01}^{(3)}, E_{12})$$

$$= N_{\hat{G}^*}(E_{01}^{(2)} \cup E_{01}^{(3)}, E_{12}) - N_{\overline{G}^*}(E_{01}^{(2)}, E_{12}) - N_{\overline{G}^*}(E_{01}^{(3)}, E_{12})$$

$$= \frac{3}{2}\cdot(\alpha(2)+\alpha(3)\ )\cdot(\alpha(2)+\alpha(3)-1)\cdot n^{10} + (\alpha(2)+\alpha(3)\ )\cdot\{\ \alpha(1)\cdot\beta(3,t) + \alpha(4,t+1)\cdot\beta(1,2)\ \}$$

$$- \{\frac{3}{2}\cdot\alpha(2)\cdot(\alpha(2)-1)\cdot n^{10} + \alpha(2)\cdot(\ \alpha(1)\cdot\beta(2,t) + \alpha(3,t+1)\cdot\beta(1)\ )\ \}$$

$$- \{\frac{3}{2}\cdot\alpha(3)\cdot(\alpha(3)-1)\cdot n^{10} + \alpha(3)\cdot(\ \alpha(1,2)\cdot\beta(3,t) + \alpha(4,t+1)\cdot\beta(1,2)\ )\ \}$$

$$= \frac{3}{2}\cdot\{\ (\alpha(2)+\alpha(3)\ )\cdot(\alpha(2)+\alpha(3)-1) - \alpha(2)\cdot(\alpha(2)-1) - \alpha(3)\cdot(\alpha(3)-1)\ \}\cdot n^{10}$$

$$+ \alpha(3)\cdot\{\ \alpha(1)\cdot\beta(3,t) - \alpha(1,2)\cdot\beta(3,t) + \alpha(4,t+1)\cdot\beta(1,2) - \alpha(4,t+1)\cdot\beta(1,2)\ \}$$

$$+ \alpha(2)\cdot\{\ \alpha(1)\cdot\beta(3,t) - \alpha(1)\cdot\beta(2,t) + \alpha(4,t+1)\cdot\beta(1,2) - \alpha(3,t+1)\cdot\beta(1)\ \}$$

$$= 3\cdot\alpha(2)\cdot\alpha(3)\cdot n^{10} - \alpha(3)\cdot\alpha(2)\cdot\beta(3,t)$$

$$+ \alpha(2)\cdot\{\ \alpha(1)\cdot(\beta(3,t) - \beta(2,t)\ ) + \alpha(4,t+1)\cdot(\beta(1)+\beta(2)\ ) - (\alpha(3)+\alpha(4,t+1)\ )\cdot\beta(1)\ \}$$

$$= \alpha(2)\cdot\alpha(3)\cdot(3n^{10} - \beta(3,t)\ )$$

$$+ \alpha(2)\cdot\{\ -\alpha(1)\cdot\beta(2) + \alpha(4,t+1)\cdot\beta(2) - \alpha(3)\cdot\beta(1)\ \}$$

$= \alpha(2)\cdot\{\ \alpha(3)\cdot(\beta(1)+\beta(2)\ ) - \alpha(1)\cdot\beta(2)+\alpha(4,t+1)\cdot\beta(2) - \alpha(3)\cdot\beta(1)\ \}$

$= \alpha(2)\cdot\beta(2)\cdot(\ \alpha(3) - \alpha(1)+\alpha(4,t+1)\ )$

$= \alpha(2)\cdot\beta(2)\cdot(\ \alpha(3,t+1) - \alpha(1)\ )$. □

The following claim is an immediate consequence of Claims 3 and 4.

Claim 5. $N_{\overline{G}^*}(E_{01}(v), E_{12}) \geq Min\ \{\ N_{\check{G}^*}(E_{01}(v), E_{12}), N_{\hat{G}^*}(E_{01}(v), E_{12})\ \}$. □

The next lemma follows from Claim 5.

Lemma A-1. For any circle-confined drawing $\overline{G}^*$ of G^*, there exists a circle-confined drawing $\overline{\overline{G}}^*$ of G^* such that V_2 is consecutive in $\overline{\overline{G}}^*$ and that $N_{\overline{\overline{G}}^*}(E_{01}(v), E_{12}) \leq N_{\overline{G}^*}(E_{01}(v), E_{12})$.

Proof. If V_2 is consecutive in $\overline{G}^*$, then the lemma trivially holds. Otherwise, as can be seen from Claim 5, we can construct another circle-confined drawing of G^* whose partition number is smaller than that of $\overline{G}^*$ without increasing the number of crossings between the edges in $E_{01}(v)$ and those in E_{12}. Therefore, repeating such a construction as many times as needed would result in a desired circle-confined drawing of G^*. □

Lemma A-2. For any circle-confined drawing $\overline{G}^*$ of G^*,

$N_{\overline{G}^*}(E_{01}(v), E_{12}) \geq g_4(1,n^4,3n^{10},n^4)$. □

Proof. By Lemma A-1, there exists a circle-confined drawing $\overline{\overline{G}}^*$ of G^* such that V_2 is consecutive in $\overline{\overline{G}}^*$ and that $N_{\overline{\overline{G}}^*}(E_{01}(v), E_{12}) \leq N_{\overline{G}^*}(E_{01}(v), E_{12})$. Suppose, starting from v, we visit the vertices on the circumference of the circle associated with $\overline{\overline{G}}^*$ one by one in the clockwise or counterclockwise direction. Let α_v (resp., β_v) denote the number of the vertices in V_1 which are encountered before the vertices of V_2 when we go along the circle in the clockwise (resp., counterclockwise) direction. Note that $\alpha_v, \beta_v \geq 0$ and $\alpha_v+\beta_v = 2n^4$. By the definition of g_4, $N_{\overline{\overline{G}}^*}(E_{01}(v), E_{12}) = g_4(1, \alpha_v, 3n^{10}, \beta_v)$. Therefore, from Lemma 4, we have

$$N_{\overline{G}^*}(E_{01}(v), E_{12}) \geq N_{\overline{\overline{G}}^*}(E_{01}(v), E_{12})$$

$$= g_4(1, \alpha_v, 3n^{10}, \beta_v)$$

$$= \frac{3}{2} \cdot n^{10} \cdot (\alpha_v \cdot (\alpha_v - 1) + \beta_v \cdot (\beta_v - 1))$$

$$= \frac{3}{2} \cdot n^{10} \cdot (\alpha_v \cdot (\alpha_v - 1) + (2n^4 - \alpha_v) \cdot (2n^4 - \alpha_v - 1))$$

$$= 3n^{10} \cdot \{ (\alpha_v - n^4)^2 + n^8 - n^4 \}$$

$$= 3n^{10} \cdot (\alpha_v - n^4)^2 + 3n^{14} \cdot (n^4 - 1).$$

By Lemma 4 again, $3n^{14} \cdot (n^4 - 1) = g_4(1, n^4, 3n^{10}, n^4)$. Therefore, $N_{\overline{G}^*}(E_{01}(v), E_{12}) \geq g_4(1, n^4, 3n^{10}, n^4)$. □

We are now ready to show the proofs of Lemmas 9 and 10.

Proof of Lemma 9. From Lemma A-2, $N_{\overline{G}}(E_{01}, E_{12}) \geq n \cdot g_4(1, n^4, 3n^{10}, n^4)$. Therefore, $N_{\overline{G}}(E_{01}, E_{12}) \geq g_4(n, n^4, 3n^{10}, n^4)$ by Lemma 4. □

Proof of Lemma 10. Suppose that V_2 is consecutive in $\overline{G}$. For each vertex $v \in V_0$, let α_v be defined in the same manner as in the proof of Lemma A-2. Then, if $\overline{G}$ is not canonical, there exists a vertex v in V_0 such that $\alpha_v \neq n^4$. For such v, the argument used in the proof of Lemma A-2 immediately yields

$$N_{\overline{G}}(E_{01}(v), E_{12}) \geq g_4(1, n^4, 3n^{10}, n^4) + 3n^{10}.$$

For every other vertex $w \in V_0 - \{v\}$, $N_{\overline{G}}(E_{01}(w), E_{12}) \geq g_4(1, n^4, 3n^{10}, n^4)$ by Lemma A-2. Thus, from Lemma 4, we have

$$N_{\overline{G}}(E_{01}, E_{12}) \geq g_4(1, n^4, 3n^{10}, n^4) + 3n^{10} + (n - 1) \cdot g_4(1, n^4, 3n^{10}, n^4)$$

$$= g_4(n, n^4, 3n^{10}, n^4) + 3n^{10}. \; \square$$

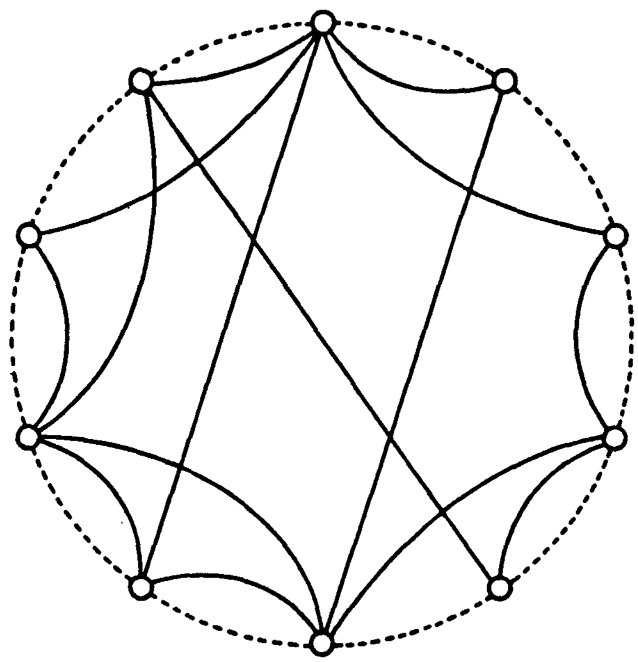

Fig. 1. An example of a circle-confined drawing of a graph.

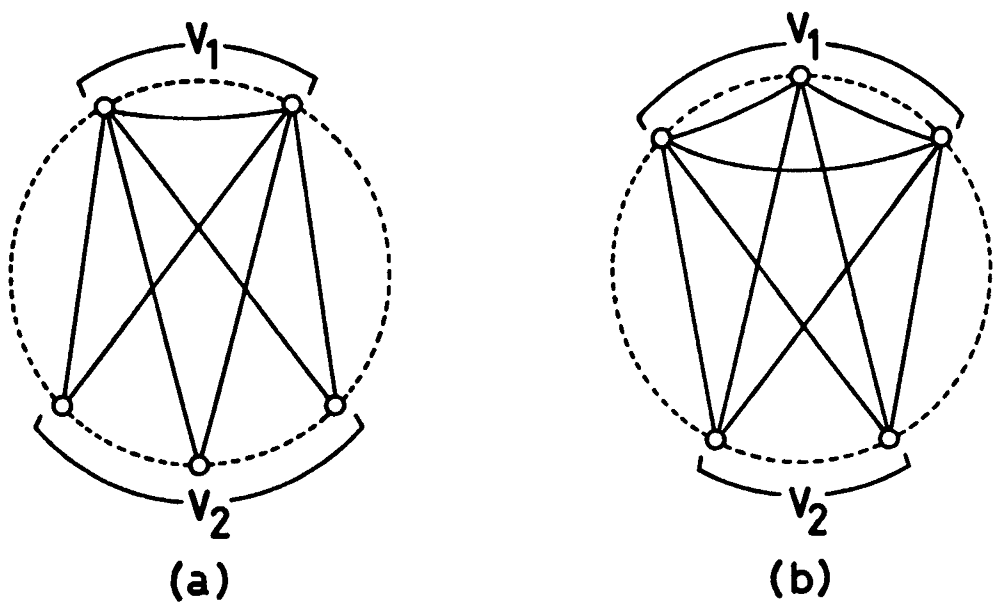

Fig. 2. $g_3(|V_1|,|V_2|)$. (a) $g_3(2,3) = 0$, (b) $g_3(3,2) = 2$.

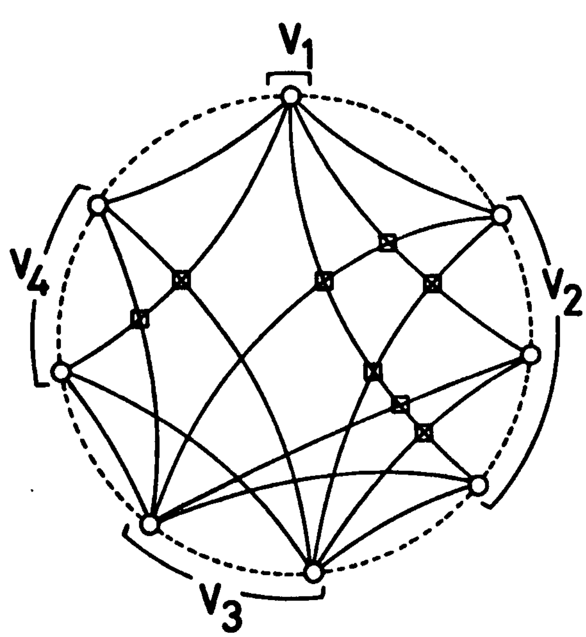

Fig. 3. $g_4(|V_1|,|V_2|,|V_3|,|V_4|) = g_4(1,3,2,2) = 8$.
□ 's indicate the points under consideration.

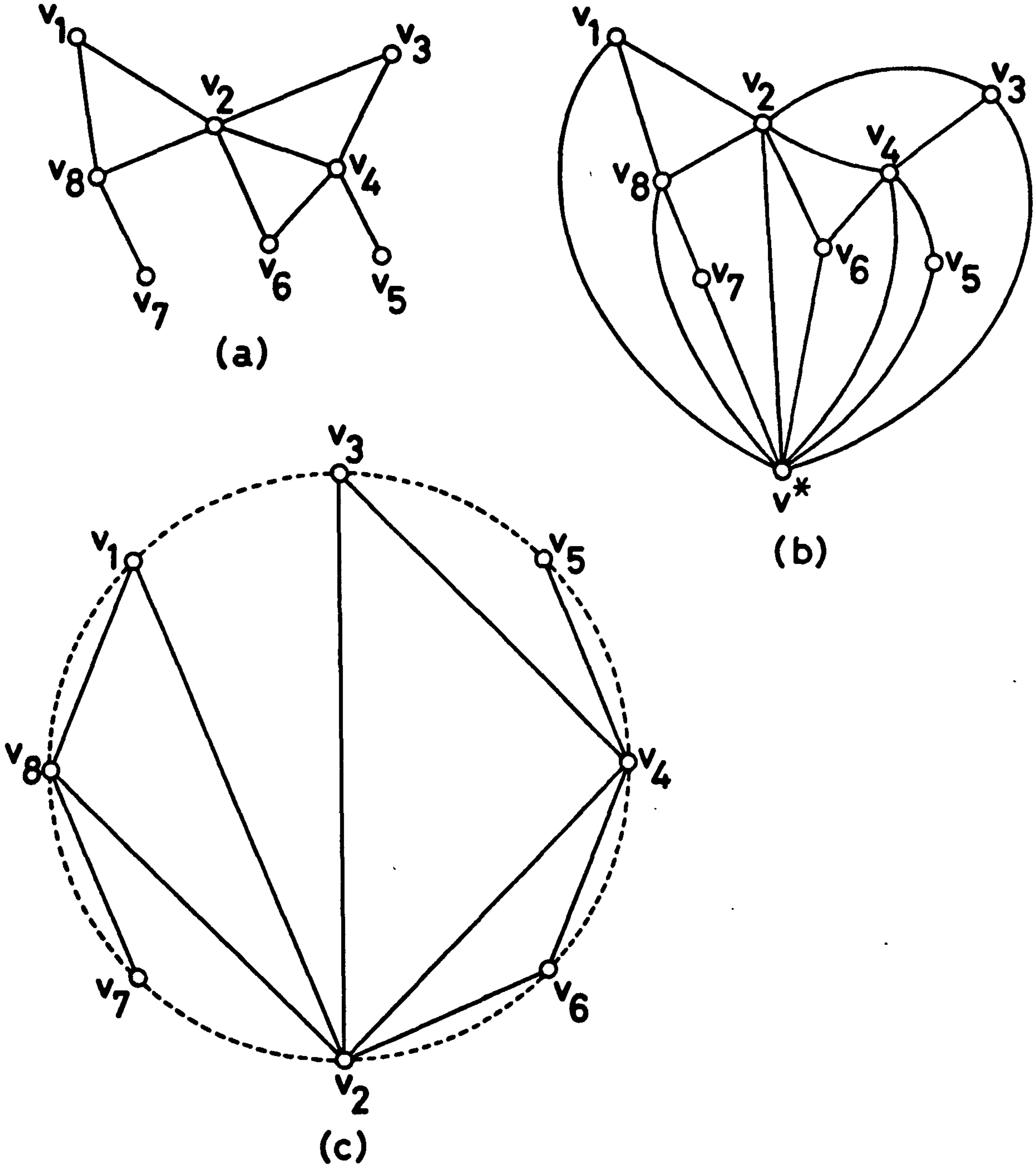

Fig. 4. A planar circle-confined drawing of an outerplanar graph.
(a) An outerplanar graph G.
(b) A planar embedding of G^*.
(c) A planar circle-confined drawing of G.

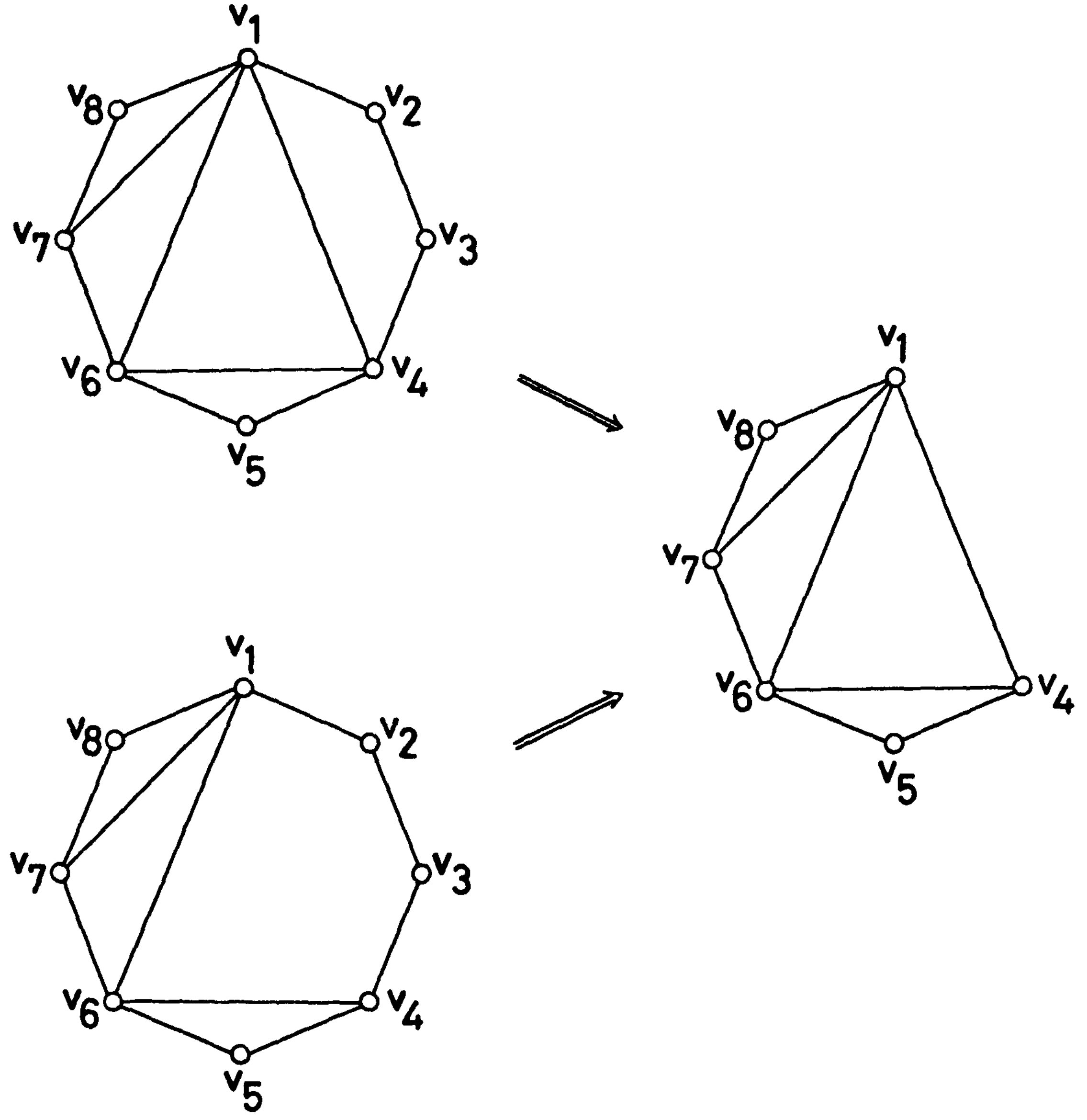

Fig. 5. Reduction of branchless path $[v_1, v_2, v_3, v_4]$.

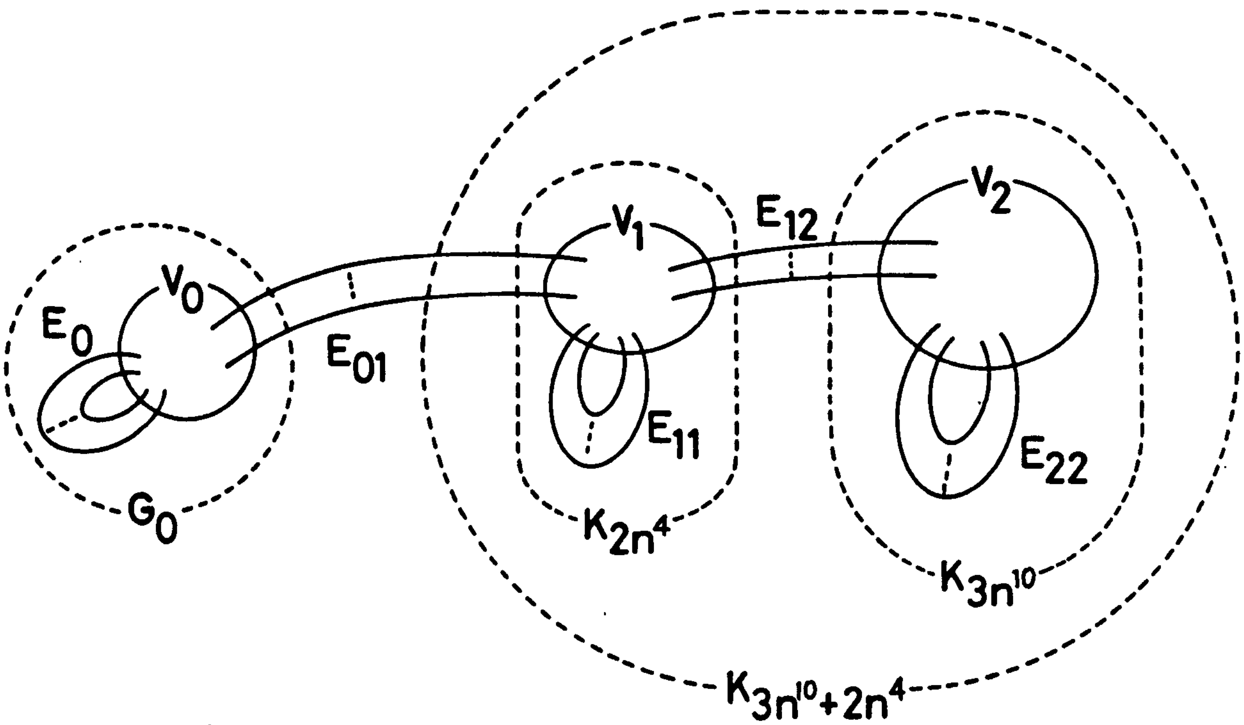

Fig. 6. A rough sketch of graph G.

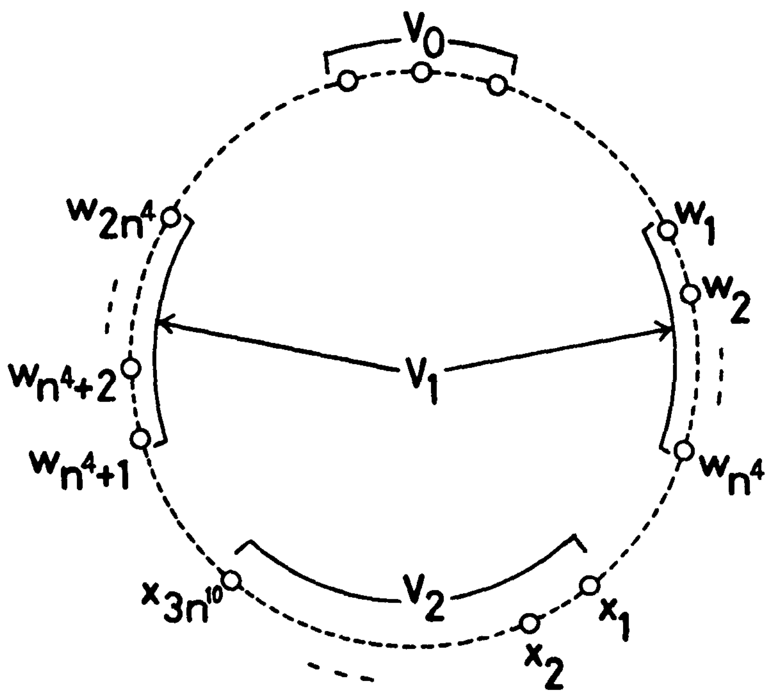

Fig. 7. An illustration for the proof of Theorem 2.

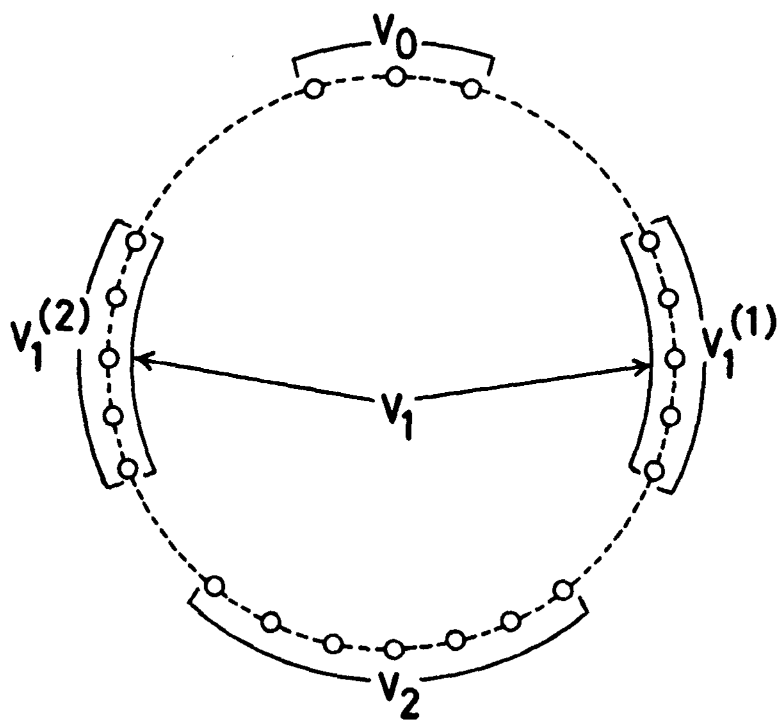

Fig. 8. Placement of the vertices in a canonical drawing of G.

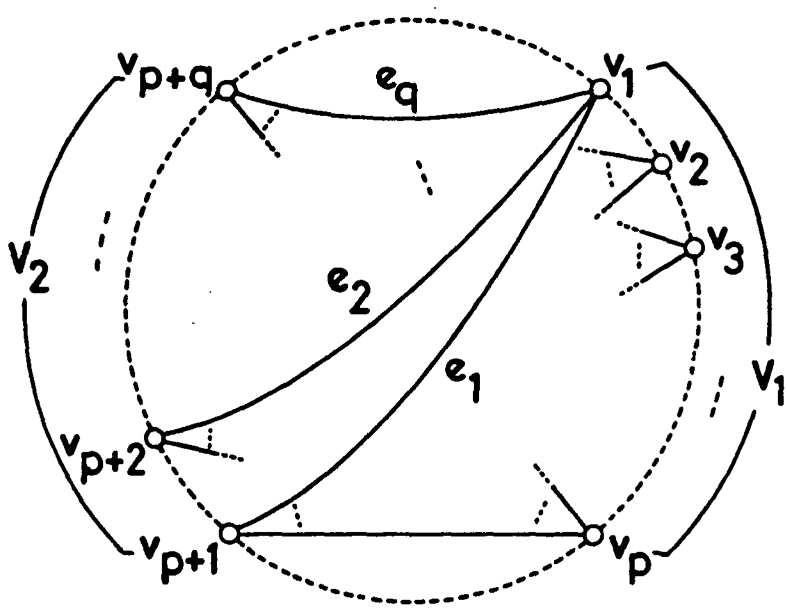

Fig. A-1. A circle-confined drawing of a complete bipartite graph H_2.

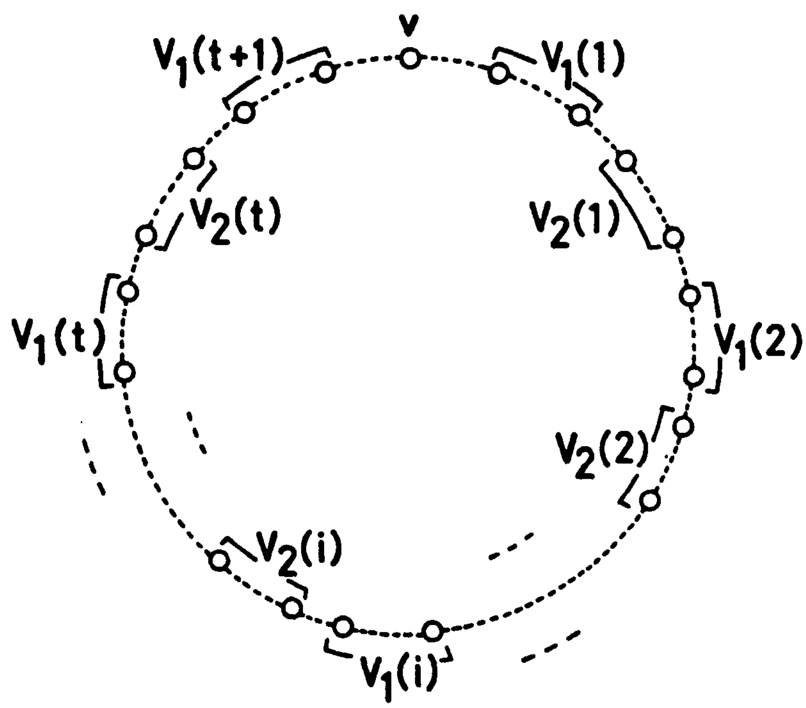

Fig. A-2. Placement of the vertices in $\overline{G}*$.

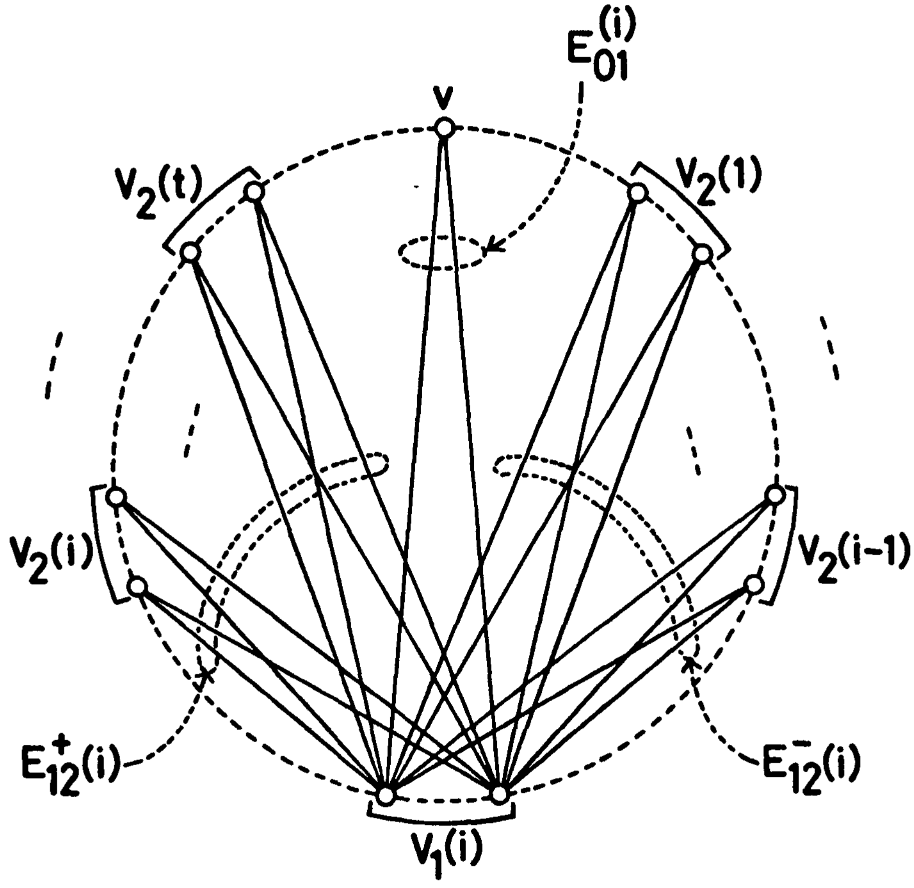

Fig. A-3. Sets of edges $E_2^{(i)}$, $E_4^-\ (i)$ and $E_4^+\ (i)$.

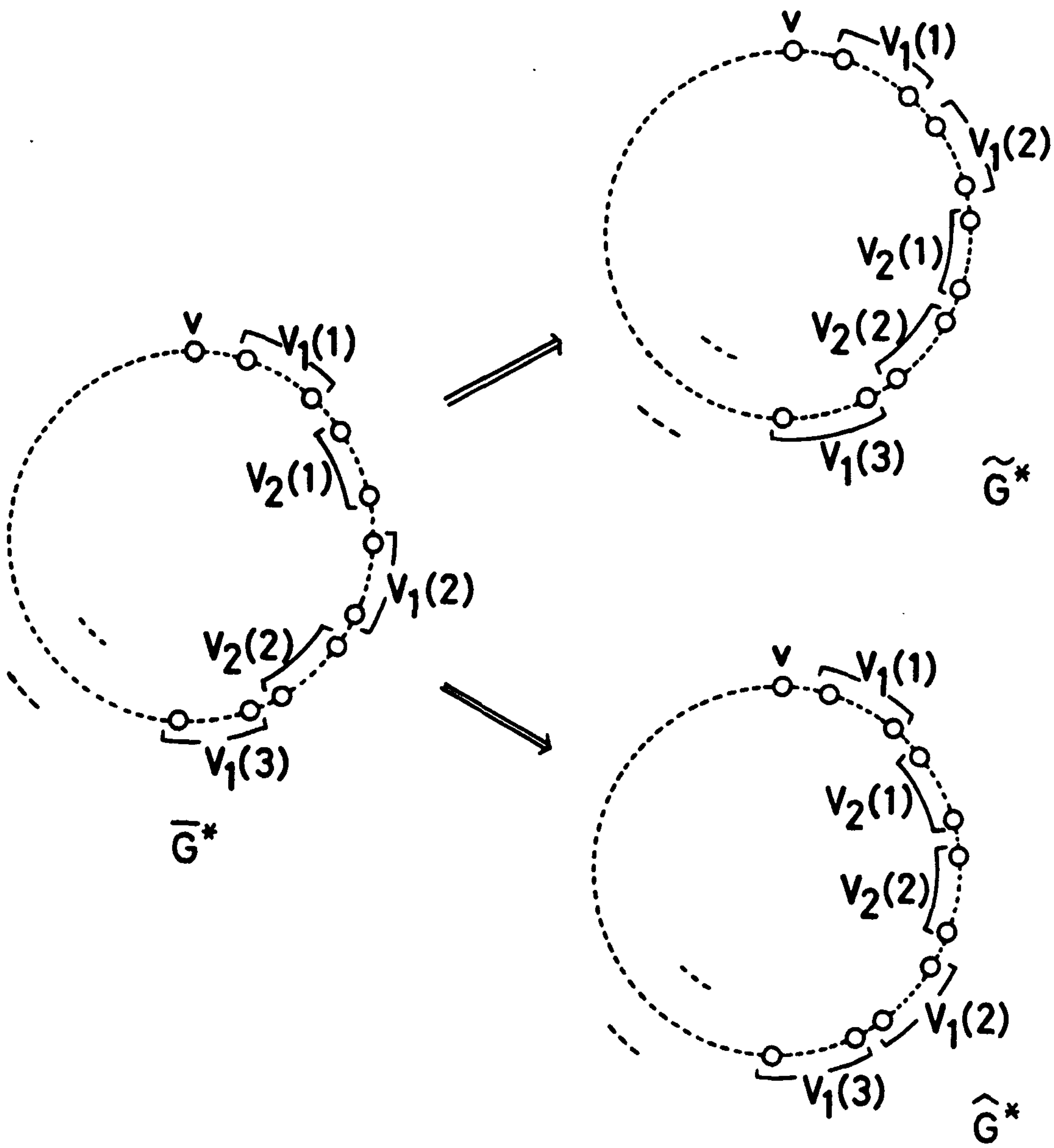

Fig. A-4. Two circle-confined drawings $\tilde{G}^*$ and $\hat{G}^*$.