ABSTRACT

Title of dissertation: COMBINATORICS OF K-THEORETIC JEU DE TAQUIN

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Thomas and Yong [17] introduced a theory of jeu de taquin which extended Schützenberger's [14] for Young tableaux. The extended theory computes structure constants for the K-theory of (type A) Grassmannians using combinatorial machinery similar to that for cohomology. This rule naturally generalizes to give a conjectural root-system uniform rule for any minuscule flag variety G/P.

In this dissertation, we see that the root-system uniform rule is well-defined for certain G/P other than the Grassmannian. This gives rise to combinatorially defined rings which are conjecturally isomorphic to $K(G/P)$. Although we do not prove that these rings are isomorphic to $K(G/P)$, we do produce a "Pieri rule" for computing the product of a general class with a generating class in the type B combinatorial case. We also investigate some symmetries which support the conjectural isomorphism. Moreover, our results combined with recent work of Buch and Ravikumar [3] imply that this conjecture is in fact true.

Lenart [9] gave a Pieri rule for the type A K-theory, demonstrating that the Pieri structure constants are binomial coefficients. In contrast, using techniques of

[10], we show that type B Pieri structure constants have no such simple closed forms.

COMBINATORICS OF K-THEORETIC JEU DE TAQUIN

by

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Dedication

To my parents, for supporting my seemingly unending education.

Acknowledgments

Thanks go to my advisor, Harry Tamvakis for suggesting the main problem in my thesis and for introducing me to tableau combinatorics. I would also like to acknowledge the helpful conversations and advice of my friends in the math department: Ben Lauser, Beth McLaughlin, Walter Ray-Dulany, and Kevin Wilson. Lastly, I could not have completed this thesis without Randy Baden's tireless moral support.

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Chapter 1

Introduction

1.1 Schubert calculus on $G(k, \mathbb{C}^n)$

Classically, one might ask for the number of some linear spaces of given dimensions which satisfy certain geometric conditions. In such an arrangement, for each of the linear spaces, the set of linear subspaces meeting it is a *Schubert variety.* Thus the answer to the question is to compute some number of intersections of Schubert varieties. Intersections of Schubert varieties can be identified as cup products in cohomology.

The (type A) Grassmannian $G(k, \mathbb{C}^n)$ is the set of k-dimensional subspaces of $\mathbb{C}^n$. Each Schubert variety Ω_λ within $G(k, \mathbb{C}^n)$ gives rise to a cohomology class $\sigma_\lambda = [\Omega_\lambda]$, making this projective variety precisely the place to compute intersections like those mentioned above. Schubert classes are in bijection with partitions, and indeed as an abelian group $H^*(G(k, \mathbb{C}^n), \mathbb{Z}) = \oplus_\lambda \mathbb{Z}\sigma_\lambda$. When expressed in the basis of Schubert classes, products in the ring give rise to *structure constants* $c^\nu_{\lambda,\mu}$ which compute intersections of appropriate Schubert varieties.

$$\sigma_\lambda \cup \sigma_\mu = \sum_\nu c^\nu_{\lambda,\mu} \sigma_\nu$$

As a ring, $H^*(G(k, \mathbb{C}^n))$ is generated by the *special* Schubert classes $\sigma_{(p)}$. It is therefore helpful to look for a formula for $c^\nu_{\lambda,(p)}$. Named for its 19^{th} century discoverer Mario Pieri, the Pieri rule computes exactly the product of a special class with an arbitrary class [11]. In $H^*(G(k,\mathbb{C}^n))$, $c^\nu_{\lambda,(p)}$ is 1 exactly when ν is the result of adding a size p horizontal strip to λ, and is 0 otherwise.

Using Young tableaux, Littlewood and Richardson formulated the first combinatorial enumeration of the structure constants $c^\nu_{\lambda,\mu}$. Known as the Littlewood-Richardson rule, $c^\nu_{\lambda,\mu}$ counts the number of Young tableaux on the diagram ν/λ satisfying an easily checked "word" condition.

1.2 Extending to maximal isotropic Grassmannians

Rather than parametrizing any k-dimensional subspace, we may parametrize subspaces of $\mathbb{C}^n$ which are isotropic with respect to a symplectic or orthogonal form. Computing intersections of Schubert varieties of the maximal isotropic Grassmannians, gives rise to theorems analogous to the Littlewood-Richardson rule. We call the set of n-planes in $2n$-space which are isotropic with respect to the standard symplectic form the Lagrangian Grassmannian, $LG(n,2n)$. The maximal orthogonal Grassmannians are denoted $OG(n,2n+1)$ and $OG(n+1,2n+2)$, depending on the parity of the dimension of the ambient space. $LG(n,2n)$ and $OG(n,2n+1)$ are examples of Hermitian symmetric spaces, and Schubert calculus can be extended to Hermitian symmetric spaces which are not Grassmannians.

Pieri rules in these contexts were given by Hiller and Boe in 1986 [6]. Pragacz [12] noted a connection to Schur's Q-functions and later Stembridge [15] provided a "Littlewood-Richardson"-type rule for structure constants of $H^*(LG(n, 2n))$ and the orthogonal cases.

1.3 K-theoretic and minuscule extensions

Geometric and combinatorial theorems in Schubert calculus can be generalized from singular cohomology to other theories, including equivariant or quantum cohomology. In this dissertation, we choose to view K-theoretic Schubert calculus.

As an abelian group $K(G(k, \mathbb{C}^n)) = \oplus_\lambda \mathbb{Z}[\mathcal{O}_\lambda]$ with a basis of classes of structure sheaves of the Schubert varieties Ω_λ, still corresponding to partitions λ fitting in the $k \times (n-k)$ partition. The structure constants for $K(G(k, \mathbb{C}^n))$ with respect to this basis are denoted $C^\nu_{\lambda,\mu}$, and are equal to $c^\nu_{\lambda,\mu}$ when $|\lambda| + |\mu| = |\nu|$.

In 2000, Lenart [9] proved a Pieri rule for $K(G(k, \mathbb{C}^n))$. The Pieri structure constants $C^\nu_{\lambda,(p)}$ are binomial coefficients $\binom{r(\nu/\lambda)-1}{|\nu/\lambda|-p}$ where $r(\nu/\lambda)$ is the number of rows in the skew shape ν/λ. Buch [2], in 2002, provided a Littlewood-Richardson rule, using new "set-valued" tableaux on ν/λ obeying a specific word condition. In this way, Buch recovered Lenart's Pieri rule. He also showed that polynomials coming from set-valued tableaux give rise to Grothendieck polynomials, first discussed in 1982 by Lascoux and Schützenberger [8].

All Grassmannians so far mentioned are examples of minuscule flag varieties. Thomas and Yong [16], in 2006, gave a root-system uniform rule for calculating in $H^*(G/P)$ for any minuscule flag variety G/P. This enabled them to see Littlewood-Richardson rules of cohomology in the light of Lie theory. It also gave a setting where tableau combinatorics and a process called *rectification* made computation of cohomological structure constants more uniform across the different Lie types.

Finally, in 2009, Thomas and Yong formulated their uniform rules in the setting of K-theory, using *increasing* and *superstandard* tableaux. In [17], they show how their framework gives rise to the Pieri rule in type A, provided by Lenart in 2000. Because the two rules agree on a generating set for $K(G(k, \mathbb{C}^n))$, the number of increasing tableaux on ν/λ which *K-rectify* to superstandard tableau S_μ equals the type A K-theoretic structure constant $C^{\nu}_{\lambda,\mu}$.

Structure constants of K-theory of other Lie types remained unproved. However, the root-system uniform language allowed Thomas and Yong to state a conjectural rule for calculating structure constants, providing that their K-rectification procedure is well-defined on a class of increasing tableaux.

1.4 Summary of new results

The primary result of this dissertation is to prove Thomas and Yong's conjectural rule in type B. To do this, our workhorse will be a tableau property called *tulginess* which is invariant under the K-theoretic sliding algorithm defined in [17].

After proving that type B K-rectification to superstandard tableaux is well-defined, we are able to create numbers which count the procedure. The integer $d^{\nu}_{\lambda,\mu}$ is the number tableaux on shape ν/λ which K-rectify to S_μ, the superstandard tableau of shape μ. Using techniques similar to Thomas and Yong, we show that the $d^{\nu}_{\lambda,\mu}$ are structure constants in a commutative ring. Unsurprisingly, the generating set for this ring corresponds to the special partitions (p), and we prove an explicit Pieri formula for computing $d^{\nu}_{\lambda,(p)}$. Unlike Lenart's type A result, the type B Pieri formula is not so simple as to be expressed as a binomial coefficient.

Without a *geometric* Pieri rule to compare with, the ring defined by the $\{d^{\nu}_{\lambda,\mu}\}$ as structure constants is only conjecturally isomorphic to $K(OG(n, 2n+1))$. If that isomorphism exists, we should expect to see certain symmetries in the structure constants. Indeed, we see an S_3 action on the coefficients: $d^{\nu}_{\lambda,\mu} = d^{\nu}_{\mu,\lambda} = d^{\lambda^\vee}_{\mu,\nu^\vee}$. Anders Buch and Vijay Ravikumar [3] have recently proved the required geometric Pieri rule, which agrees with our combinatorial one. This implies that the above two rings are isomorphic, and we therefore obtain a Littlewood-Richardson type rule for the K-theory of orthogonal Grassmannians.

Thomas and Yong have recently announced in [18] that they have arrived at proofs of Theorems 3.2.4 and 3.3.1 independently of work presented here.

In the final chapter, some attention is paid to minuscule cases other than $OG(n, 2n+1)$. We also discuss further generalizations of tableau combinatorics in relation to Schubert calculus.

Chapter 2

Preliminaries

In this chapter, we will review the definitions and results of Thomas and Yong [17] which are the motivation for this work. We will see that much of their work, while not explicitly stated by them, applies to all minuscule G/P.

2.1 Minuscule Schubert calculus

Let G be a complex connected reductive Lie group. Fix Borel and opposite Borel subgroups B and B_-, with maximal torus $T = B \cap B_-$ and Weyl group $W = N(T)/T$. Denote the root system Φ, the positive roots Φ^+, and a base of simple roots Δ. Every subset of Δ is associated to a parabolic subgroup P. The *generalized flag variety* G/P has *Schubert varieties*

$$X_w = \overline{B_- wP/P} \text{ for } wW_P \in W/W_P,$$

where W_P is the parabolic subgroup of W corresponding to P. Let $K(G/P)$ be the Grothendieck ring of algebraic vector bundles over G/P. $K(G/P)$ has a $\mathbb{Z}$-basis of Schubert structure sheaves $\{[\mathcal{O}_{X_w}]\}$.

Define Schubert structure constants $C_{u,v}^{w}(G/P) \in \mathbb{Z}$ by

$$\mathcal{O}_u \cdot \mathcal{O}_v = \sum_{wW_P \in W/W_P} C_{u,v}^{w}(G/P)\mathcal{O}_w.$$

Brion [1] has established that

$$(-1)^{l(w)-l(u)-l(v)} C_{u,v}^{w}(G/P) \in \{0, 1, 2, \ldots\},$$

where $l(w)$ is the *Coxeter length* of the minimal length coset representative of wW_P.

A simple root and a maximal parabolic subgroup P corresponding to the root are *minuscule* if the associated fundamental weight ω_P satisfies $\langle \omega_P, \alpha^\vee \rangle \leq 1$ for all $\alpha \in \Phi^+$ under the usual pairing between weights and coroots. The *minuscule flag varieties* G/P are classified into five infinite families and two exceptional cases, corresponding to the Lie type of G as seen in Figure 2.1.

Associated to each minuscule G/P is a planar poset $(\Lambda_{G/P}, \prec)$, obtained as a subposet of the positive roots $\Omega_{G^\vee}$ for the dual root system of G. Figure 2.2 gives two such examples of $\Lambda_{G/P} \subset \Omega_{G^\vee}$. The relation $\prec$ is the transitive closure of the covering relation $x \preceq y$, meaning $y - x \in \Delta$. Lower order ideals of $\Lambda_{G/P}$, called *shapes*, are in bijection with cosets wW_P indexing Schubert varieties. Under this bijection, if $wW_P \leftrightarrow \lambda$ then $l(w) = |\lambda|$, the number of roots in the shape λ.

Define a *skew shape* ν/λ to be the set theoretic difference of two shapes, $\nu \backslash \lambda$,

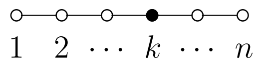
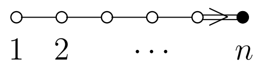
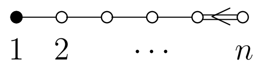
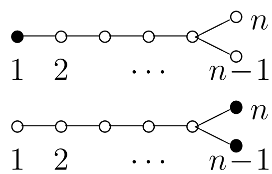
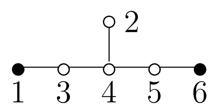
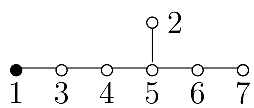
Root System	Dynkin Diagram	Nomenclature
A_n		Grassmannian $G(k, \mathbb{C}^n)$
B_n		Odd Orthogonal Grassmannian $OG(n, 2n+1)$
C_n, $n \geq 3$		Projective Space $\mathbb{P}^{2n-1}$
$D_n, n \geq 4$		Even dimensional quadric $\mathbb{Q}^{2n-2}$ Even Orthogonal Grassmannian $OG(n+1, 2n+2)$
E_6		Cayley Plane $\mathbb{OP}^2$
E_7		(unnamed) $G_\omega(\mathbb{O}^3, \mathbb{O}^6)$

Figure 2.1: Dynkin diagrams of all Lie types with miniscule simple roots shown.

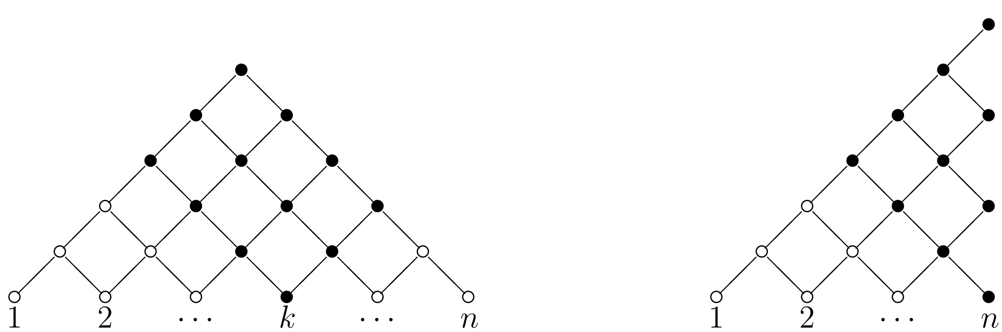


Figure 2.2: $\Lambda_{G/P} \subset \Omega_{G^\vee}$ for $G/P = G(k, \mathbb{C}^n)$ and $OG(n, 2n+1)$

when λ is a subideal of ν. An *increasing tableau* on skew shape ν/λ is an assignment

$$\texttt{label} : \nu/\lambda \to \{1, 2, \ldots, q\}$$

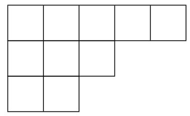
such that $\texttt{label}(x) < \texttt{label}(y)$ when $x \prec y$, and where each label $1, \ldots, q$ appears at least once. An inner corner of ν/λ is a maximal element $x \in \Lambda_{G/P}$ that is below some element in ν/λ.

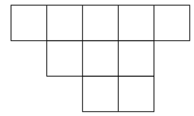
The geometric terms and diagrams are often cumbersome and unintuitive, so the next section will provide the combinatorial language we will use to talk about operations on the posets.

2.2 Combinatorial definitions

A *partition* λ of a positive integer n is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ of weakly decreasing positive integers λ_i whose sum is n. If $\lambda_1 > \lambda_2 > \cdots > \lambda_l$ then λ is called a *strict* partition of n. We write $l(\lambda) = l$ and say λ has *length* l. The unique length 0 partition is denoted $\emptyset$. The *size* of λ is the number n which it partitions and is denoted $|\lambda| = n$.

A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ can be represented by a *diagram*, consisting of l left-justified rows of λ_i boxes in row i. Strict partitions can also be represented by *shifted diagrams*, wherein row i is indented $i - 1$ boxes, rather than being left-justified. For example, consider the partition $(5, 3, 2)$. The diagram and shifted diagram for this partition are as below.





For partitions λ and ν, if $\nu_i \geq \lambda_i$ for each $i \leq l(\lambda)$, we say $\lambda \subseteq \nu$. Furthermore, we can create the skew diagram ν/λ consisting of the boxes of ν not in λ.

An *increasing tableau* is a filling of a (skew) diagram (shifted or not) with one number per box such that the labels strictly increase reading down any column or across any row. The *(shifted) (skew) shape* of a tableau is the partition which gives rise to the (shifted) (skew) diagram underlying that tableau. Here are two increasing tableaux on the shape $(5, 3, 2)$ and the shifted skew shape $(5, 3, 2)/(3)$, respectively.

1	2	4	5	6
3	4	5		
5	6			

			1	2
	1	3	4	
		5	6	

While not technically necessary, we also require that all positive numbers less than the maximal label be entries of the tableau. This convenience ensures that the alphabet from which labels are drawn is indexed merely by the largest integer entry of a tableau. Since most tableaux we mention will be increasing tableaux, we will often drop the word "increasing."

Note. Partitions whose diagrams fit inside a $k \times (n-k)$ rectangle correspond to lower order ideals of $\Lambda_{G(k,\mathbb{C}^n)}$, hence to type A_n Schubert varieties. Those which fit in an n-column staircase correspond to type B_n Schubert varieties. The relation

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 5 \\ \hline \multicolumn{1}{c|}{} & 6 & 7 & \multicolumn{2}{c}{} \\ \cline{2-3} \multicolumn{2}{c|}{} & 8 & \multicolumn{2}{c}{} \\ \cline{3-3} \end{array} \qquad\qquad \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 5 \\ \hline 6 & 7 & \multicolumn{3}{c}{} \\ \cline{1-2} 8 & \multicolumn{4}{c}{} \\ \cline{1-1} \end{array}$$

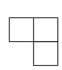
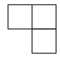
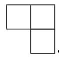
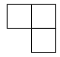
Figure 2.3: Superstandard tableaux on (5,2,1).

$x \prec y$ on $\Lambda_{G/P}$ corresponds to a box x being above and/or to the left of box y in the diagram corresponding to the entire ideal $\Lambda_{G/P}$. So we see that the increasing tableaux notions of both sections agree. Often we will use cardinal and intermediate directions when referring to diagrams. We will use the terms “diagram” and “shape” interchangeably. With this convention, $x \prec y$ iff x is weakly northeast of y in the diagram sense.

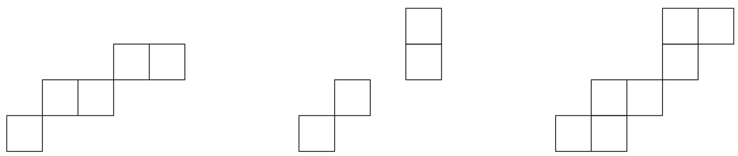
An inside corner of ν/λ is a maximally southeast box of λ. The set of increasing tableaux of shape ν/λ in $\Lambda_{G/P}$ is denoted $INC_{G/P}(\nu/\lambda)$, or just $INC(\nu/\lambda)$ if G/P is understood from context.

When no entry of a tableau T is repeated, we call T a (shifted) *standard* tableau. The (shifted) *superstandard* tableau S_μ is a the (shifted) standard filling of the (shifted) shape μ with $1, 2, \ldots, \mu_1$ in the first row, $\mu_1 + 1, \ldots, \mu_1 + \mu_2$ in the second row, etc. For example, $S_{(5,2,1)}$ shifted and not are as in Figure 2.3.

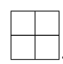
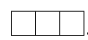
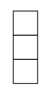
Skew shapes with no pair of boxes sharing a north-south border are called *horizontal strips*. *Vertical strips* are those sharing no east-west borders. A *border strip* is a skew shape ν/λ which can be written as a union of one vertical strip ρ/λ and one horizontal strip ν/ρ for some shape ρ.

Although $\nu/\lambda =$  is a union of a horizontal strip and a vertical strip, there is no ρ fulfilling the conditions of the definition of a border strip, so it is *not* a border strip. In fact, one can see that if  is contained in a skew shape, then the shape is not a border strip. Conversely, every shape which is not a border strip must have three boxes in the shape of . Thus an alternative definition of a border strip is any shape ν/λ not containing  as a subshape.

Example 2.2.1. The following skew shapes form a horizontal strip, vertical strip, and border strip respectively.



2.3 K-theoretic jeu de taquin

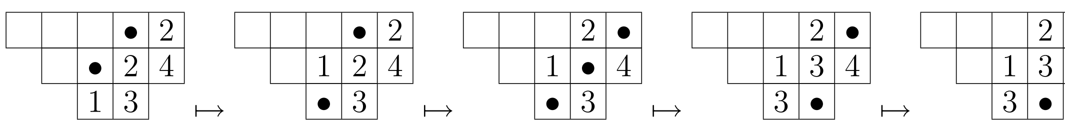
A *ribbon* is a skew shape which does not contain , , or  as a subshape. Each connected component is called a *short ribbon*. An *alternating ribbon* is a filling of a ribbon with two symbols where adjacent boxes are filled differently. We define $\texttt{switch}(R)$ to be the operation on an alternating ribbon R which switches the positions of each symbol. For example,

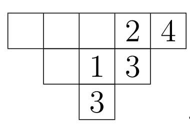
$$R = \begin{array}{ccc} & \circ & \bullet \\ \circ & \bullet & \\ \bullet & & \end{array} \qquad \texttt{switch}(R) = \begin{array}{ccc} & \bullet & \circ \\ \bullet & \circ & \\ \circ & & \end{array}$$

By definition, if an alternating short ribbon consists of a single box, $\texttt{switch}$ does nothing to the symbol in it.

We now define the process of K-theoretic *jeu de taquin*, an operation on increasing tableaux. Let $T \in INC_{G/P}(\nu/\lambda)$ have largest entry q. Choose $\{x_1, \ldots, x_s\}$, a subset of inside corners of ν/λ. Fill the boxes $\{x_i\}$ with the symbol "$\bullet$," called a *hole*. Form the alternating ribbon R_1, consisting of boxes with entry 1 or a hole. Perform `switch` on R_1. Then form R_2 consisting of boxes with entry 2 or a hole. Perform `switch` again. Proceed through each R_i until `switch` has operated on R_q. This process "moves holes past" the entries of T. Delete all boxes with holes. The resulting filling of numbers makes up the tableau $Kjdt_{\{x_i\}}(T)$.

Example 2.3.1. Here is an example of type B $Kjdt$. Let T and $\{x_i\}$ be as indicated as below. Each "$\mapsto$" indicates a `switch` within the $Kjdt_{\{x_i\}}(T)$ procedure.



So $Kjdt_{\{x_i\}}(T) =$  .

Outside corners of a shape are the boxes maximally northwest in $\Lambda_{G/P}$ which are southeast of the shape. A process of *reverse K-theoretic jeu de taquin*, $Krevjdt$, begins with a subset of outside corners to a tableau, and moves them through the tableau from right to left. This occurs by performing successive `switch` operations on R_q through R_1.

Since we begin with an increasing tableau and switch holes with each filling in numeric order, we see that $Kjdt_{\{x_i\}}(T)$ is an increasing tableau also. Given a $T \in INC_{G/P}(\nu/\lambda)$, we can iterate applications of $Kjdt$, choosing a sequence of inside corner subsets. When no (non-empty) inside corner subsets can be chosen, the result is called a *K-rectification* of T, denoted $Krect_R(T)$. The choice $R = (\{x_i^{(1)}\}, \ldots, \{x_j^{(t)}\})$ of holes is called the *rectification order.*

Example 2.3.2. In this type B example, the symbol "$\mapsto$" indicates an application of $Kjdt$ with the holes as indicated. Call the first tableau (minus the holes) T and the last U.

$$\begin{ytableau} \, & \, & \, & \bullet & 2 \\ \none & \, & \bullet & 2 & 4 \\ \none & \none & 1 & 3 \end{ytableau} \mapsto \begin{ytableau} \, & \, & \, & 2 & 4 \\ \none & \bullet & 1 & 3 \\ \none & \none & 3 \end{ytableau} \mapsto \begin{ytableau} \, & \, & \bullet & 2 & 4 \\ \none & 1 & 3 \end{ytableau} \mapsto \begin{ytableau} \, & \bullet & 2 & 4 \\ \none & 1 & 3 \end{ytableau} \mapsto \begin{ytableau} \bullet & 1 & 2 & 4 \\ \none & 3 \end{ytableau} \mapsto \begin{ytableau} 1 & 2 & 4 \\ \none & 3 \end{ytableau}$$

This shows $Krect_R(T) = U$.

A rectification order, R, can also be thought of as an increasing tableau itself. Let us suppose the sequence of hole choices is length q_R. We label the first set of inside corners with q_R, the second with $q_R - 1$, and so on. For future convenience, when displaying a rectification order as a tableau, the entries will be boldfaced. In the example above:

$$R = \begin{ytableau} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{5} \\ \none & \mathbf{4} & \mathbf{5} \end{ytableau}$$

Later, it will be helpful to consider modifying to partial K-rectification, called K-infusion. If $T \in INC(\nu/\lambda)$ and $S \in INC(\lambda/\rho)$, we can slide the boxes of S through the boxes of T as if S were a sequence of holes. This setup can be seen in Figure 2.4. The result is a pair of tableaux on shapes γ/ρ and ν/γ for some shape γ. The first will be denoted $Kinf_1(S, T)$, and the second $Kinf_2(S, T)$. Notice that if $R \in INC(\lambda)$ then $Krect_R(T) = Kinf_1(R, T)$.

Example 2.3.3. Let S be the bold skew tableau in $INC((4, 3, 1)/(3))$ and $T \in$

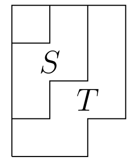
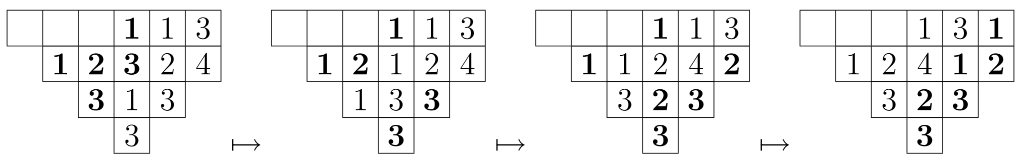
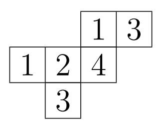
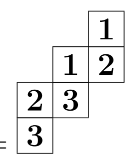


Figure 2.4: $T \in INC(\nu/\lambda)$ extending $S \in INC(\lambda/\rho)$

$INC((6,5,3,1)/(4,3,1))$ the other tableau "extending" S. Each "$\mapsto$" is an application of $Kjdt(T)$ using part of S as the choice of holes. We do not delete holes during K-infusion.



$Kinf_1(S,T) =$  $Kinf_2(S,T) =$ 

K-infusion is an involution. The proof given in [17] uses techniques we will not utilize in this dissertation, but does generalize to all minuscule G/P. Likewise, the lemma following the theorem will not be proved here, but will be used.

Theorem 2.3.1. ([17]) *Let* $S \in INC_{G/P}(\lambda/\rho)$ *and* $T \in INC_{G/P}(\nu/\lambda)$. *Then*

(1) $Kinf_1(Kinf_1(S,T), Kinf_2(S,T)) = S$ *and*

(2) $Kinf_2(Kinf_1(S,T), Kinf_2(S,T)) = T$.

As a special case of this theorem, we see that if $Kjdt_{\{x_i\}}(T) = U$ and $\{y_j\}$ are the positions of the holes before deletion after computing $Kjdt_{\{x_i\}}(T)$, then $Krevjdt_{\{y_j\}}(U) = T$.

Lemma 2.3.2. **([17])** *Let $T \in INC_{G/P}(\nu/\lambda)$, $R \in INC_{G/P}(\lambda)$, and $a \in \mathbb{N}$. If A is the subtableau of T consisting of entries $1, \ldots, a$, and $B = T \backslash A$ is the remaining tableau, then*

$$Krect_R(T) = Kinf_1(R, T) = Kinf_1(R, A) \cup Kinf_1(Kinf_2(R, A), B)).$$

Thomas and Yong [17] demonstrate that different rectification orders applied to the same starting tableau can yield different K-rectifications in Type A. There is, however, a class of tableaux for which K-rectification is well-defined without specifying rectification order. These are tableaux whose K-rectification is superstandard.

Theorem 2.3.3. **([17])** *Let $T \in INC_{G(k,\mathbb{C}^n)}(\nu/\lambda)$ such that $Krect_R(T) = S_\mu$ for some partition μ and rectification order R. Then $Krect_Q(T) = S_\mu$ for any rectification order Q.*

Let $c^{\nu}_{\lambda,\mu}$ be the number of (type A) increasing tableaux on ν/λ whose K-rectification is S_μ. These combinatorial numbers coincide with structure constants of type A K-theory, up to a sign. Let $\varepsilon^{\nu}_{\lambda,\mu} = (-1)^{|\nu|-|\lambda|-|\mu|}$.

Theorem 2.3.4. *For any λ, μ, and ν in $\Lambda_{G(k,\mathbb{C}^n)}$, $c^{\nu}_{\lambda,\mu} = \varepsilon^{\nu}_{\lambda,\mu} C^{\nu}_{\lambda,\mu}(G(k, \mathbb{C}^n))$*

This result led Thomas and Yong to conjecture that for every minuscule G/P, there would exist a class of tableaux on which K-rectification would be well defined. Moreover the number of tableaux K-rectifying to a tableau of that class should coincide with the K-theoretic structure constant, up to the predictable sign $\varepsilon^{\nu}_{\lambda,\mu}$.

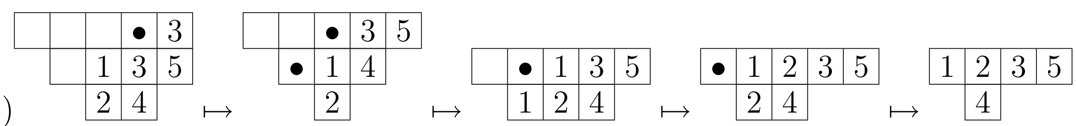
Conjecture 2.3.5. **([17])** (1) For any minuscule G/P, there is a class of tableau $\{C_\mu(G/P)\}$ indexed by shapes $\mu \subset \Lambda_{G/P}$ such that if $T \in INC_{G/P}(\nu/\lambda)$ and $Krect_R(T) = C_\mu$ for some $R \in INC_{G/P}(\lambda)$, then $Krect_Q(T) = C_\mu$ for all $Q \in INC_{G/P}(\lambda)$.

(2) The number of tableaux $T \in INC_{G/P}(\nu/\lambda)$ with $Krect(T) = C_\mu$ equals $\varepsilon^\nu_{\lambda,\mu} C^\nu_{\lambda,\mu}(G/P)$.

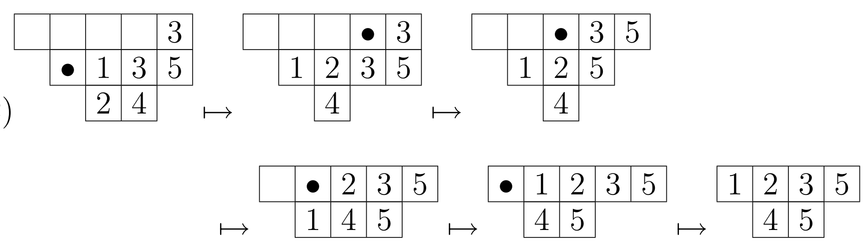
As in type A, different rectification orders applied to the same tableau can result in different K-rectifications in type B.

Example 2.3.4. Below are two K-rectifications of the same initial tableau, but with differing rectification orders. Note that in neither case is the K-rectification superstandard.

(1)



(2)



In type B, we will see that the first part of Conjecture 2.3.5, is true. The superstandard tableaux $\{S_\mu\}$ provide the class on which K-rectification is well-defined.

We let $d^\nu_{\lambda,\mu}$ count the number of type B tableaux on shape ν/λ whose K-rectification is S_μ. We will show that the $d^\nu_{\lambda,\mu}$, with appropriate signs, form structure constants of a ring, but it remains conjectural that this ring is isomorphic to $K(OG(n, 2n+1))$ as desired. Some progress toward that result will be presented, by counting certain $d^{\nu}_{\lambda,\mu}$ which conjecturally correspond to structure constants coming from the generators of $K(OG(n, 2n+1))$.

Chapter 3

Type B Structure

For this chapter, $G/P = OG(n, 2n+1)$, so the notation will often suppress the use of G/P subscripts.

3.1 Tulginess

Let $T \in INC(\nu/\lambda)$. Construct the column word of T, $col(T)$, to be the string formed by reading the the entries of T down each column from top to bottom starting at the right. Analogous to $col(T)$, we define $row(T)$ to be the row word obtained by keeping track of position while reading the entries of T from left to right starting from the bottom.

Example 3.1.1.

$$T = \begin{array}{|c|c|c|c|c|}\hline
\ & \ & \ & 1 & 4 \\ \hline
\multicolumn{1}{c|}{} & \ & \ & 2 & \multicolumn{1}{c}{} \\ \cline{2-4}
\multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 2 & 3 & \multicolumn{1}{c}{} \\ \cline{3-4}
\end{array}$$

$$col(T) = 41232 \qquad\qquad row(T) = 23214$$

Note that the first 2 of the row word corresponds to the second 2 of the column word, as they occupy the same box within the tableau, but the first 2 of $row(T)$ does not correspond with the first 2 of $col(T)$. The correspondence is a bijection of the two sequences and will be denoted σ. We will say that some k and $\sigma(k)$ are *partners* and frequently abuse notation, saying that some k in $row(T)$ is the partner of (the corresponding) k in $col(T)$.

We define the mix word of a tableau, $mix(T)$, as the concatenation of $col(T)$ with $row(T)$ along with the bijection σ relating partners. Referring again to the previous example, $mix(T) = (41232|23214, \sigma)$. The pipe symbol helps keep track of where the column word part ends and the row word begins. Although σ is part of the data of $mix(T)$, the notation will be suppressed when writing $mix(T)$.

For the next few paragraphs, the *expanse* of a tableau is the longest length among weakly increasing subsequence $v_1 \cdots v_q$ of $mix(T)$ such that no pair (v_i, v_j) are partners. This condition means that although each pair k and $\sigma(k)$ appear in $mix(T)$, the expanse counts at most one of them.

In the previous example, there are two such longest subsequences of $mix(T)$. They are the underlined portions of $mix(T)$: $4\underline{1}\underline{2}3\underline{2}|2\underline{3}21\underline{4}$ and $4\underline{1}\underline{2}32|\underline{2}\underline{3}21\underline{4}$. Thus the expanse of the tableau above is 5.

The expanse of a tableau is somewhat analogous to Thomas and Yong's $\texttt{LIS}(T)$ statistic [17]. While this statistic was invariant under type A $Kjdt$, the notion of expanse is not quite strong enough for type B. What will suffice for type B is the condition that the expanse of a tableau be as large as possible. This motivates the following definition and supplants the usage of "expanse" afterward.

Definition 3.1.1. A tableau $T \in INC(\nu/\lambda)$ is called *tulgey* if the expanse of T is $|\nu/\lambda|$.

The tableau of the previous example is thus tulgey. When a tableau is tulgey, an underlining of its mix word which exhibits the tulginess is called a *marking*.

Markings of the previous example's $mix(T)$ are denoted above. In every marking, for every k in $mix(T)$ either k or $\sigma(k)$ is underlined. It bears repeating that because longest weakly increasing subsequences of $mix(T)$ with at most one partner per pair underlined are not unique, there may be multiple markings for the mix word any tulgey T. Sometimes we use the phrase "marking of a tableau" as shorthand for "marking of the mix word of that tableau." Every tulgey tableau has a marking, and we do *not* use the word "marking" except when referring to tulgey tableaux.

Proposition 3.1.1. *If $T \in INC(\nu/\lambda)$ is a tulgey tableau then ν/λ is a border strip.*

Proof. Suppose ν/λ is not a border strip. Then $\begin{array}{|c|c|}\hline a & b \\ \hline \multicolumn{1}{c|}{} & c \\ \cline{2-2}\end{array}$ is contained in T somewhere for $a < b < c$. Moreover $bca|cab$ is a subsequence of $mix(T)$. If there is a marking with the first b underlined: $\underline{b}ca|cab$ (other underlines unknown) then neither a can be underlined in the marking, so T can not be tulgey. If $bca|ca\underline{b}$ is part of the marking of $mix(T)$ then neither c can be underlined, so again T can not be tulgey. Either way, we see that only tableaux which are border strips can be tulgey. $\square$

We extend the definitions above to tableaux "with holes" by inserting the hole symbol into the words where appropriate. It is also required that if a and b are *numbered* (non-hole) entries of a tableau with holes and $a \leq b$ then b is *not* weakly northwest of a. Tulginess is also extended, but is defined by ignoring holes in markings of a tableau. The following tableau is tulgey with holes:

$$\begin{array}{|c|c|c|c|c|}\hline & & & 1 & 4 \\ \hline \multicolumn{1}{c|}{} & & 2 & \bullet & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & \bullet & 3 & \multicolumn{1}{c}{} \\ \cline{3-4}\end{array}$$

This tableau with holes has a marking given by: $4\underline{1} \bullet 3\underline{2} \bullet | \bullet \underline{3}2 \bullet 1\underline{4}$. Since each numbered entry or its partner is underlined, this tableau with holes is tulgey (so the use of the word "marking" is justified).

It is convenient when choosing markings that they be in a certain standard form. For this we need the following proposition.

Proposition 3.1.2. *Let T be a tulgey tableau with holes. For any fixed k, a numbered label of T, there is a marking of $mix(T)$ such that all k are underlined on the same side of the pipe.*

Proof. Let k be a numbered label of the tableau T, tulgey with holes. For any marking M of T, define the statistic $k_c(M)$ to be the number of k underlined in $col(T)$ in M. Our goal is to build a marking M' for T wherein all k are underlined on the same side of the pipe in M'.

Pick a marking M which maximizes the statistic k_c. If $k_c(M)$ equals the number of k in $col(T)$, then M suffices to finish the proof. Let us assume neither this nor $k_c(M) = 0$. (If $k_c(M) = 0$, we are again done, as M provides a marking of T where all k are underlined in $row(T)$).

Now we may say there is at least one k underlined in $row(T)$ such that the result of switching that k's underlining to $\sigma(k)$, is not a valid marking. Let k_1 be the first such k in $row(T)$. Let k_2 be the first k underlined in $col(T)$. Then

$$M = \alpha \underline{k_2} \beta | \gamma \underline{k_1} \delta$$

with α, β, γ, and δ sequences of (underlined and not) numbers and holes. Note that nothing but k can be underlined in $\beta|\gamma$. Also, all numbers less than k are underlined in α and everything larger than k is underlined in δ. If $\sigma(k_1)$ is in β, then

$$M' = \alpha\underline{k_2}\beta_1\underline{k_1}\beta_2|\gamma k_1\delta \quad (\text{where } \beta_1 k_1 \beta_2 = \beta)$$

is a valid marking of $mix(T)$, but $k_c(M') = k_c(M)+1$, contradicting the maximality condition defining M. Thus $\sigma(k_1)$ is in α, making

$$M = \alpha_1 k_1 \alpha_2 \underline{k_2}\beta|\gamma\underline{k_1}\delta \quad (\text{with } \alpha_1 k_1 \alpha_2 = \alpha).$$

This means the box labeled with k_1 is strictly east of the box labeled with k_2. To be tulgey with holes, k_2's box may not be north of k_1's, so k_2 is in γ. This means we can shift the underline to $\sigma(k_2)$ to arrive at a new marking. Likewise, any k underlined in β is the label of a box strictly west of k_1's box so must be south of k_1's box, placing the partner of that k in γ. We can now create M' a marking of $mix(T)$ where all k are underlined in $row(T)$, and all other underlines are where they were in M. □

We may now assume without loss of generality that for a pair (T, k), where T is a tulgey tableau with holes and k is a numbered entry of T, all underlined k in the marking of $mix(T)$ occur entirely within $col(T)$ or within $row(T)$.

Theorem 3.1.3. *$Kjdt_{\{x_i\}}(T)$ is tulgey iff T is, regardless of initial holes $\{x_i\}$.*

The proof of this theorem follows immediately from the following two lemmas, as $Kjdt_{\{x_i\}}$ and $Krevjdt_{\{x_i\}}$ are merely repeated applications of switching.

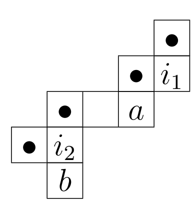
Set T_0 to be T along with holes at positions $\{x_j\}$ and T_i be the result of switching the $\bullet$'s with the i's of T_{i-1}.

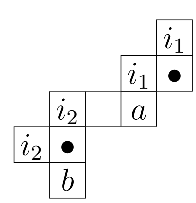
Lemma 3.1.4. *If T_{i-1} is tulgey with holes then T_i is also tulgey with holes.*

Proof. Let M be a marking of T_{i-1} with all i underlined on the same side of the pipe. We will create M' a marking for T_i. Doing so requires us to consider the two possibilities of where i is underlined in M.

Case 1. All i are underlined in $col(T_{i-1})$ in M. Let M_1 be the underlining of $mix(T_i)$ with all non i labels underlined where they were in M and all i's underlined in $col(T_i)$. If M_1 is a marking, $M' = M$, but if M_1 is not a marking, we will move one underline from a label in $col(T_i)$ to its partner to get the marking M'. This happens in a very predictable way as we shall see.

Because holes for i's shifting up in `switch` occur immediately before the i which will fill them, only i's moving left in `switch` can force M_1 not to be a marking. For all non-southwesternmost i, there is an i later in $col(T_{i-1})$ and $col(T_i)$, so specifically only the southwesternmost i shifting left can force M_1 not to be a marking. The following figure demonstrates this. The problem occurs only when b is underlined in $col(T_{i-1})$.





$\bullet\underline{i_1}\bullet a \bullet \underline{i_2}b\bullet|b\bullet i_1 \bullet a \bullet i_2\bullet \subseteq M$ $\qquad$ $\underline{i_1}\bullet\underline{i_1}a\underline{i_2}\bullet\underline{b}\underline{i_2}|bi_2\bullet i_2 a i_1 \bullet i_1 \subset M_1$

Notice how in M_1 there is the underlined sequence $\underline{bi_2}$ even though $i < b$.

Let us now consider the problem of the southwesternmost i shifting left locally. Let a, b, c, d, e, f, and g be either labels of T_{i-1} or outside the shape of T_{i-1}, in which case they are "empty" and can be discarded from considerations of M. Although c could be a hole, we have already dealt with any i shifting up, so we assume without loss of generality that $c \neq \bullet$.

$$\begin{array}{|c|c|c|} \hline a & b & c \\ \hline d & \bullet & i \\ \hline e & f & g \\ \hline \end{array} \subset T_{i-1} \qquad \text{and} \qquad \begin{array}{|c|c|c|} \hline a & b & c \\ \hline d & i & \bullet \\ \hline e & f & g \\ \hline \end{array} \subset T_i$$

$$c\underline{i}gb \bullet fade|efgd \bullet iabc \subset M$$

Immediately we see a, b, and d are empty, since there is no way to underline them in M. The underlining of c is irrelevant to our purposes as after `switch` its position relative to all i has not changed. We need only consider $\underline{i}g \bullet fe|efg \bullet i \subset M$.

If g is underlined in $row(T_{i-1}))$, then M_1 is a marking, so $M' = M_1$. However, if g is underlined in $col(T_{i-1})$, we see that e and f, being less than g can not be underlined in M, so are empty. This is the situation where g sits on the staircase. In this case, $\underline{i}\underline{g} \bullet |g \bullet i$ is the relevant part of M and $\bullet g\underline{i}|\underline{g}i\bullet$ is the corresponding part of M'. That is, M' is M_1 with the underlining of g moved to $\sigma(g)$.

Case 2. All i are underlined in $row(T_{i-1})$ in M. Let M_1 be the underlining consisting of non i labels underlined in $row(T_i)$ and all other numbers underlined in their same positions as in M. Again, if M_1 is a marking, let $M' = M_1$. Analogous to the last case, if M_1 is not a marking, only the easternmost i shifting up can potentially be a problem for creating M'. Isolating that i, we see the following.

$$\begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & \bullet & e \\ \hline f & i & g \\ \hline\end{array} \subset T_{i-1} \qquad\qquad \begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & i & e \\ \hline f & \bullet & g \\ \hline\end{array} \subset T_i$$

$$cegb \bullet iadf | f\underline{i}gd \bullet eabc \subset M$$

Since $i < e$ (if e is non-empty) we see g must be empty. Also, a must be empy lest we be unable to underline b in M. To be tulgey, d must be underlined in $col(T_{i-1})$, if it isn't empty. The important part of M is now $\bullet ia\underline{d}f | f\underline{i}d \bullet a$. What we have shown is that M_1 is indeed a marking, with no underlines needing to move to their partners to create M'. $\square$

Lemma 3.1.5. *If T_i is tulgey then T_{i-1} was tulgey too.*

Proof. This proof is almost identical to the last. Here, let M be the marking of T_i with all i underlined in the same half of $mix(T_i)$. We again create a marking M' of T_{i-1}, to show it too is tulgey. We must proceed by cases, just as before.

Case 1. All i are underlined in $col(T_i)$. Let M_1 be as in Case 1 of the previous proof. Only the northernmost i shifting right can present a problem for creating M' from M_1. Let a, b, c, d, e, f, and g be as in the last proof.

$$\begin{array}{|c|c|c|}\hline a & b & c \\ \hline i & \bullet & d \\ \hline e & f & g \\ \hline\end{array} \subset T_i \qquad\qquad cdgb \bullet fa\underline{i}e | efgi \bullet dabc \subset M$$

Other than a and f, all entries are in the same positions relative to all i in M and

M_1. We need only examine a and f in this local picture.

If non-empty, $a < i$, thus b would be non-empty too. We have $a < b < i$ but all three can not be underlined in M, so a must actually be empty. To be a tableau with holes, if f is non-empty, it must be underlined in $row(T_i)$. This means M_1 is a marking, so $M' = M_1$ is a marking for T_{i-1}.

Case 2. All i are underlined in $row(T_i)$. Let M_1 be as in Case 2 of the previous proof. This time, only the southernmost i shifting down can potentially present a problem for creating M'. This time we have the following piece of T_i and marking:

$$\begin{array}{|c|c|c|}\hline a & i & b \\ \hline c & \bullet & d \\ \hline e & f & g \\ \hline \end{array} \subset T_i \qquad bdgi \bullet face|efgc \bullet da\underline{i}b \subset M$$

For T_i to be tulgey, b and d must be empty, because they can not be underlined along with i in $row(T_i)$. This forces g to be empty too. Since $f > i$, it can not be underlined in M, so it is empty. The relevant piece of M is $i \bullet \underline{ace}|ec \bullet a\underline{i}$. After `switch`, the corresponding piece of M_1 is $\bullet i\underline{ace}|ec\underline{i}a\bullet$. M_1 is a marking, so $M' = M_1$ making T_{i-1} is tulgey with holes. $\square$

We now know that tulginess is an invariant of $Kjdt$, so $Krect_R(T)$ is tulgey iff T is. The shape of $Krect_R(T)$ is a shifted partition so box (1,1) has a label in $Krect_R(T)$ (unless $T = \emptyset$, but then we're not really saying much). Since tulgey tableaux must occupy border strips, the shape of $Krect_R(T)$ must be (p), the only type of border strip which is also a (non-skew) shape. Here, p is the number of distinct letters appearing as labels of T. Note that each label appears exactly once, by the "increasing" condition of $Krect_R(T)$. There are no more or fewer labels since no new labels are added during $Kdjt$, nor are any destroyed.

We have now categorized T such that $Krect_R(T) = S_{(p)}$. They are the tulgey tableaux with entries $1, 2, \ldots, p$. Moreover, we see that the K-rectification of any tulgey tableau is unique, regardless of rectification order! We state this formally as the following theorem.

Theorem 3.1.6. *Let $T \in INC(\nu/\lambda)$ with entries $1, \ldots, p$. (1) T is tulgey iff there is some $R \in INC(\lambda)$ such that $Krect_R(T) = S_{(p)}$. (2) If T is tulgey then $Krect_Q(T) = S_{(p)}$ for any $Q \in INC(\lambda)$.*

3.2 Uniqueness of K-rectification

We need some more machinery analogous to that found in [17] to prove that K-rectification to superstandard tableau is rectification order invariant.

Definition 3.2.1. For $S \in INC_{G(k,\mathbb{C}^n)}(\nu/\lambda)$, $LIS_A(S)$ is the length of a longest strictly increasing subsequence of $row(S)$. For $T \in INC_{OG(n,2n+1)}(\nu/\lambda)$, $LIS_B(T)$ is the length of a longest strictly increasing subsequence of $mix(T)$.

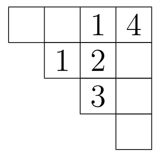
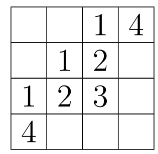
Theorem 3.2.1. ([17]) *$LIS_A(T) = LIS_A(Kjdt_{\{x_i\}})(T)$ for any set of inside corners $\{x_i\}$ and any $T \in INC_{G(k,\mathbb{C}^n)}(\nu/\lambda)$.*

Corollary. *$LIS_A(T) = LIS_A(Krect_R(T))$ for any $R \in INC_{G(k,\mathbb{C}^n)}(\lambda)$ and $T \in INC_{G(k,\mathbb{C}^n)}(\nu/\lambda)$. This number is also μ_1 where the shape of $Krect_R(T)$ is μ.*

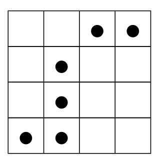
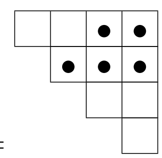
Definition 3.2.2. Let $T \in INC_{OG(n,2n+1)}(\nu/\lambda)$. By considering $\Lambda_{OG(n,2n+1)}$ as the upper triangular subposet of $\Lambda_{G(n,\mathbb{C}^{2n})}$, create a tableau $T^2 \in INC_{\Lambda_{G(n,\mathbb{C}^{2n})}}(\nu/\lambda)$ by

$T^2 = T \cup T^t$ where T^t is the transpose of T. This process can be done on shapes as well as tableau. Conversely, if A is a set of boxes in $\Lambda_{G(n,\mathbb{C}^{2n})}$, let $\sqrt{A}$ be the minimal set of boxes B in $\Lambda_{OG(n,2n+1)}$ such that $A \subseteq B^2$.

Example 3.2.1. Boxes filling out $\Lambda_{G/P}$ are shown for clarity.

$T =$  $T^2 =$ 

Let A be the set of boxes marked with a $\bullet$ in the first picture below, then $\sqrt{A}$ is the set of boxes marked with a $\bullet$ in the second.

$A =$  $\sqrt{A} =$ 

Now we have the tools to prove the following lemma, and thus the proposition following it as an immediate corollary.

Lemma 3.2.2. *If* $T \in INC_{OG(n,2n+1)}(\nu/\lambda)$ *then* $LIS_B(T) = LIS_A(T^2)$.

Proof. First see that T having no boxes on the staircase means T^2 misses the diagonal of the $n \times n$ boxes it sits inside. In this case $mix(T) = row(T^2)$, so we see immediately that $LIS_B(T) = LIS_A(T^2)$.

If T^2 intersects the diagonal in exactly one place filled with label k then $mix(T)$ is exactly $row(T^2)$ with the k duplicated. So again $LIS_B(T) = LIS_A(T^2)$.

Let I be a collection of boxes of T such that $LIS_B(T) = LIS_B(I) = |I|$. We can see that I^2 meets the diagonal in at most two places, by considering a, b labels of I with

$$I = \begin{array}{ccc} \boxed{a} & \boxed{} & \\ & \boxed{\cdots} & \boxed{} \\ & & \boxed{b} \end{array} \subset T$$

Since $a < b$, a is included in the $col(I)$ part of of the sequence enumerated by $LIS_B(I)$ and b is included in $row(I)$. If there were a third entry c on the diagonal to the southeast of b, it would be included in $row(I)$, but b would be included in $col(I)$. Thus there can be no third box in I on the diagonal.

We can also ensure that there is an I satisfying $LIS_B(T) = LIS_B(I) = |I|$ with at most one box of I on the diagonal. Suppose I such a set of boxes, but with I intersecting the diagonal at two positions (x, x) and (y, y), labeled with a and b respectively, with $a < b$. The entries of I in row y are in consecutive boxes, as if any were skipped, I could be made larger by including them, which of course is impossible. It is also interesting to see that a is the last entry counted by I from $col(T)$ and b is the first from $row(T)$.

If c is the label of a box in T directly above a box of I from row y, we see that c satisfies $a < c < b$, but c occurs in $col(T)$ before a and in $row(T)$ after b. Therefore it is impossible that c's box be in I.

Since row $y - 1$ is read immediately after row y in $row(T)$, we can create a new subset J of T formed by removing the boxes in row y of I and replacing them with the boxes immediately above, from row $y - 1$. In addition, we notice that we could increase the size of J by including the box in position $(y - 1, y - 1)$. Since $|J| = |I|$, which is maximal, that box must be in I already, and so being on the diagonal, must be filled with a in position (x, x), giving $x + 1 = y$.

To reiterate, we now have J satisfying $LIS_B(T) = LIS_B(J) = |J|$ with J containing only one box of the diagonal. This means $LIS_B(J) = LIS_A(J^2)$ and since $J^2 \subseteq T^2$, we have $LIS_B(T) \leq LIS_A(T^2)$.

To obtain the opposite inequality, let $I \subseteq T^2$ such that $LIS_A(I) = LIS_A(T^2) = |I|$. Because $\sqrt{I} \subseteq T$, we have $row(I) \subseteq mix(\sqrt{I}) \subseteq mix(T)$, and thus $LIS_A(T^2) = LIS_A(I) = LIS_B(\sqrt{I}) \leq LIS_B(T^2)$. $\square$

Proposition 3.2.3. *Let $T \in INC_{OG(n,2n+1)}(\nu/\lambda)$ and $R \in INC_{OG(n,2n+1)}(\lambda)$ with $Krect_R(T)$ having shape μ. Then $LIS_B(T) = \mu_1$. Moreover because $LIS_B(T)$ is independent of R, we see that $Krect_Q(T)$ has μ_1 boxes in its first row for any $Q \in INC_{OG(n,2n+1)}(\lambda)$.*

Proof. By considering R^2, a symmetric rectification order, we note that $Krect_{R^2}(T^2) = (Krect_R(T))^2$ since each step of type A $Kjdt$ using R^2 will mirror each step of type B $Kdjt$ using R. Also, note that for a tableau S on a non-skew shape ρ, we have $LIS_B(S) = LIS_A(S^2) = \rho_1$. Now we have

$$LIS_B(T) = LIS_A(T^2) = LIS_A(Krect_{R^2}(T^2)) = \mu_1$$

via the previous lemma and Theorem 3.2.1. $\square$

Theorem 3.2.4. *Let $T \in INC_{OG(n,2n+1)}(\nu/\lambda)$. If $Krect_R(T) = S_\mu$ for some rectification order R, then $Krect_Q(T) = S_\mu$ for any rectification order Q.*

Proof. Pick $R \in INC(\lambda)$ such that $Kinf_1(R, T) = S_\mu$. Let $Q \in INC(\lambda)$ and let U_1 be the subtableau of T consisting of all entries $1, \ldots, \mu_1$. Set $T_2 = T \backslash U_1$,

$R_2 = Kinf_2(R, T)$, and $Q_2 = Kinf_2(Q, T)$.

$$Krect_R(T) = Kinf_1(R, T) = Kinf_1(R, U_1) \cup Kinf_1(R_2, T_2)$$

$$Krect_Q(T) = Kinf_1(Q, T) = Kinf_1(Q, U_1) \cup Kinf_1(Q_2, T_2)$$

Because $Kinf_1(R, U_1) = S_{(\mu_1)}$, U_1 must be tulgey, which forces $Kinf_1(Q, U_1) = S_{(\mu_1)}$ as well.

$Krect_R(T)$ has μ_1 boxes in its first row, so by Proposition 3.2.3, $Krect_Q(T)$ has μ_1 boxes in its first row also. We now have that $Krect_R(T)$ and $Krect_Q(T)$ have the same first row. With no switches in $Kinf_1(Q_2, T_2)$ affecting the first row, it now suffices to prove $Kinf_1(R_2, T_2) = Kinf_1(Q_2, T_2)$.

We may think of $Kinf_1(R_2, T_2)$ as a K-rectification to a superstandard tableau of shape $(\mu_2, \ldots, \mu_{l(\mu)})$ with alphabet $\{\mu_1 + 1, \ldots, |\mu|\}$. Using T_2 in the place of T, R_2 instead of R, and Q_2 for Q, we can repeat the above argument to see that the first two rows of $Krect_R(T)$ and $Krect_Q(T)$ agree. After $l(\mu) - 2$ more iterations, we see that all rows must agree, thereby forcing $Krect_R(T) = Krect_Q(T)$. $\square$

Now knowing that K-rectifiction to superstandard tableaux is unique, we drop the subscript and simply write $Krect(T)$. Along with that, we can now count superstandard K-rectifications.

Definition 3.2.3. Define the symbol $d^{\nu}_{\lambda,\mu} = \#\{T \in INC(\nu/\lambda) | Krect(T) = S_\mu\}$. The number of tulgey tableau with entries $1, \ldots, p$ on shape ν/λ is $d^{\nu}_{\lambda,(p)}$.

3.3 Ring structure

We define $\mathcal{D} = \oplus_\lambda \mathbb{Z}\tau_\lambda$, with $\mathbb{Z}$-basis of formal symbols $\{\tau_\lambda\}$ indexed by strict partitions. Multiplication of basis elements in $\mathcal{D}$ is given by

$$\tau_\lambda * \tau_\mu = \sum_\nu d^\nu_{\lambda,\mu}\tau_\nu.$$

Thomas and Yong [17] explicitly provide a type A analogue of $\mathcal{D}$. We will follow their proof technique and give all details to show that $\mathcal{D}$ is a commutative ring.

Theorem 3.3.1. *$\mathcal{D}$ is a commutative ring, with identity $\tau_\emptyset$.*

Proof. Let α, β, and γ be strict partitions. Curiously, we will require a proof of commutativity within the proof of associativity, so let us show that first.

Let B be a tableau on γ/α such that $Krect(B) = S_\beta$, that is B is a tableau counted by $d^\gamma_{\alpha,\beta}$. We have $Kinf_1(S_\alpha, B) = S_\beta$. Create $A = Kinf_2(S_\alpha, B)$ a tableau on γ/β. We know the shape of A since it extends S_β. Note that $Kinf_1(S_\beta, A) = S_\alpha$ by involution, Theorem 2.3.1. This gives us a bijection

$$\{B \text{ on } \gamma/\alpha | Krect(B) = S_\beta\} \leftrightarrow \{A \text{ on } \gamma/\beta | Krect(A) = S_\alpha\}$$

This is precisely the statement $d^\gamma_{\alpha,\beta} = d^\gamma_{\beta,\alpha}$.

For multiplication in $\mathcal{D}$ to be associative, we must examine the following products.

$$(\tau_\alpha * \tau_\beta) * \tau_\gamma = \sum_\sigma d^\sigma_{\alpha,\beta}\tau_\sigma\tau_\gamma = \sum_\sigma d^\sigma_{\alpha,\beta}\sum_\nu d^\nu_{\sigma,\gamma}\tau_\nu = \sum_{\sigma,\nu} d^\sigma_{\alpha,\beta}d^\nu_{\sigma,\gamma}\tau_\nu$$

$$\tau_\alpha * (\tau_\beta * \tau_\gamma) = \sum_\rho d^\rho_{\beta,\gamma}\tau_\alpha\tau_\rho = \sum_\rho d^\rho_{\beta,\gamma}\sum_\nu d^\nu_{\alpha,\rho}\tau_\nu = \sum_{\rho,\nu} d^\rho_{\beta,\gamma}d^\nu_{\alpha,\rho}\tau_\nu$$

This means we must show equality $\sum_\sigma d^\sigma_{\alpha,\beta}d^\nu_{\sigma,\gamma} = \sum_\rho d^\rho_{\beta,\gamma}d^\nu_{\alpha,\rho}$ for any fixed ν. The first sum counts the number of pairs (B, C) such that B is a tableau on σ/α with $Krect(B) = S_\beta$ and C is a tableau on ν/σ with $Krect(C) = S_\gamma$. The second sum counts pairs of tableaux (F, D) where D is on ρ/β with $Krect(D) = S_\gamma$ and F is on ν/α with $Krect(F) = S_\rho$.

Given such a (B, C) pair, construct $A = Kinf_2(S_\alpha, B)$ on σ/β. This is the image of B under the commutativity bijection. Because $Krect(C) = S_\gamma$, we may pick whatever rectification order we choose. Our order will be to partially K-rectify C by infusion with A, then finish the order with a further K-infusion with S_β. Thus $Kinf_1(S_\beta, Kinf_1(A, C)) = S_\gamma$. Let $D = Kinf_1(A, C)$, on shape ρ/β for some ρ. Likewise, if $E = Kinf_2(A, C)$ then $Krect(E) = Krect(A) = S_\alpha$.

Tableaux like E are counted by $d^\nu_{\rho,\alpha}$. Under the commutativity bijection, E corresponds to F, a tableau on ν/α which K-rectifies to S_ρ.

Now we have a sequence of bijections mapping pairs.

$$(B, C) \leftrightarrow (A, C) \leftrightarrow (D, E) \leftrightarrow (F, D)$$

We see that F and D are counted by $d^\nu_{\alpha,\rho}$ and $d^\rho_{\beta,\gamma}$ as intended. Thus multiplication in $\mathcal{D}$ is associative.

To see that $\tau_\emptyset$ is the multiplicative identity, consider $\tau_\emptyset * \tau_\alpha = \sum_\nu d^\nu_{\emptyset,\alpha}\tau_\nu$. As $d^\nu_{\emptyset,\alpha}$ counts tableaux T on $\nu/\emptyset = \nu$ which K-rectify to S_α, T must already be K-rectified. This gives the fact that $d^\nu_{\emptyset,\alpha} = 1$ when $\alpha = \nu$ and is zero if $\alpha \neq \nu$. By commutativity, $\tau_\emptyset$ is also a right identity. □

We define $\mathcal{D}(n)$ to be the quotient $\mathcal{D}/\langle \tau_\lambda | \lambda_1 > n\rangle$. The only τ_λ that occur are those for which λ fits in the n column staircase. It is conjectured in [17], and given Theorem 3.2.4 proved in [3], that $\tau_\lambda \mapsto (-1)^{|\lambda|}[\mathcal{O}_\lambda]$ extends to a ring isomorphism from $\mathcal{D}(n)$ to $K(OG(n, 2n+1))$.

Because the lowest order piece of $K(X)$ is $H^*(X)$, cohomological structure constants must coincide with with the lowest graded structure constants of $K(OG(n, 2n+1))$. This means that the correspondence between combinatorics and geometry holds when $|\nu| = |\lambda| + |\mu|$ since in [16], it is proved that $d^\nu_{\lambda,\mu}$ do provide the structure constants for the cohomology ring $H^*(OG(n, 2n+1))$ for $|\nu| = |\lambda| + |\mu|$.

Proposition 3.3.2. *The ring $\mathcal{D}(n)$ is generated by the "special classes" $\tau_{(k)}$. That is, $\mathcal{D}(n) = \langle \tau_{(k)} | 1 \leq k \leq n\rangle$.*

Proof. Consider the order $\trianglelefteq$, on partitions. This is lexicographic order in the sense that $\lambda = (\lambda_1, \ldots, \lambda_l) \triangleleft (\mu_1, \ldots, \mu_k) = \mu$ iff for the smallest i where $\lambda_i \neq \mu_i$, then $\lambda_i < \mu_i$. When two partitions are of unequal length, the shorter may be padded with zeroes, if that is deemed convenient. The relation $\trianglelefteq$ provides a total order on the set of (non-skew) shapes.

Note that if $d^\nu_{\lambda,\mu} \neq 0$ then $\lambda \subseteq \nu$ so $\lambda \trianglelefteq \nu$. If $p < \lambda_{l(\lambda)}$ then $d^{(\lambda,p)}_{\lambda,(p)} = 1$, where

$(\lambda, p) = (\lambda_1, \ldots, \lambda_{l(\lambda)}, p)$. Combining these facts yields

$$\tau_{(\lambda_1)} * \tau_{(\lambda_2)} * \cdots * \tau_{(\lambda_{l(\lambda)})} = \tau_\lambda + \sum_{\lambda \lhd \mu} c_{\lambda,\mu} \tau_\mu$$

for some $c_{\lambda,\mu} \in \mathbb{Z}$. Because no shapes are larger than the entire staircase $(n, n-1, \ldots, 1)$ under this order,

$$\tau_{(n)} * \tau_{(n-1)} * \cdots * \tau_{(1)} = \tau_{(n,n-1,\ldots,1)} \in \langle \tau_{(k)} | 1 \leq k \leq n \rangle$$

Suppose $\tau_\mu \in \langle \tau_{(k)} | 1 \leq k \leq n \rangle$ for all $\mu \rhd \lambda$. Then $\tau_\lambda = \Pi_{1 \leq i \leq l(\lambda)} \tau_{(\lambda_i)} - \sum_{\lambda \lhd \mu} c_{\lambda,\mu} \tau_\mu$ is in $\langle \tau_{(k)} | 1 \leq k \leq n \rangle$ by induction, completing the proof. $\square$

The above proof is the standard argument to show that Pieri rules generate many rings in Schubert calculus.

In different language, Stembridge [15] defines a word condition on objects called marked shifted tableaux. In this context, cohomological structure constants $c^\nu_{\lambda,\mu}$ are counted by such tableaux on ν/λ of "content μ" satisfying the word condition. There is an explicit bijection between Stembridge's tableaux and standard tableaux on ν/λ K-rectifying to S_μ. This gives another proof that the $d^\nu_{\lambda,\mu} = C^\nu_{\lambda,\mu}(OG(n, 2n+1))$ when $|\nu| = |\lambda| + |\mu|$. Because a second proof is redundant and we have not introduced Stembridge's marked shifted tableaux, the technical details are omitted.

We are already familiar with the commutation symmetry $d^\nu_{\lambda,\mu} = d^\nu_{\mu,\lambda}$. In [17],

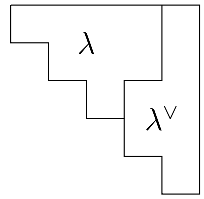


Figure 3.1: Strict partitions and their duals fill out the staircase.

there is an allusion to a $\mathbb{Z}/3\mathbb{Z}$ symmetry

$$C^{\nu}_{\lambda,\mu}(G/P) = C^{\lambda^\vee}_{\mu,\nu^\vee}(G/P) = C^{\mu^\vee}_{\nu^\vee,\lambda}(G/P)$$

for minuscule G/P. If $d^{\nu}_{\lambda,\mu} = \varepsilon^{\nu}_{\lambda,\mu} C^{\nu}_{\lambda,\mu}(OG(n, 2n+1))$ as expected, then $d^{\nu}_{\lambda,\mu}$ would share this symmetry. By including the commutation symmetry, we get an S_3-symmetry.

In the $OG(n, 2n+1)$ case, a strict partition λ uses some subset of $\{1, \ldots, n\}$. $\lambda^\vee$ is defined as the strict partition using the complementary subset. Pictorially $\lambda \cup \lambda^\vee = \Lambda_{OG(n,2n+1)}$ when $\lambda^\vee$ is reflected through the "$y = x$" line, as seen in Figure 3.1.

We are able to explicitly provide the bijection on tableaux to show $d^{\nu}_{\lambda,(p)} = d^{\lambda^\vee}_{\nu^\vee,(p)}$. After applying the commutation symmetry, this gives a special case of the expected $\mathbb{Z}/3\mathbb{Z}$ symmetry. We temporarily create an operation `flip` on tableaux which views an increasing tableau on ν/λ as a decreasing tableau on $\lambda^\vee/\nu^\vee$.

Proposition 3.3.3. *For λ, ν strict partitions and $p \in \mathbb{N}$, $d^{\nu}_{\lambda,(p)} = d^{\lambda^\vee}_{\nu^\vee,(p)}$.*

Proof. Let T be a tableau counted by $d^{\nu}_{\lambda,(p)}$, i.e. a tulgey tableau of shape ν/λ whose alphabet is $\{1, \ldots, p\}$. Thus `flip`(T) is a decreasing tableau on the shape $\lambda^\vee/\nu^\vee$.

An illustration of this process follows the proof. Because ν/λ is a border strip, the reverse of the column word of T is its row word, and vice-versa, so

$$mix(\texttt{flip}(T)) = rev(row(T))|rev(col(T)) = col(T)|row(T) = mix(T)$$

Create $\hat{T}$ by replacing entry x of $\texttt{flip}(T)$ with $p+1-x$. Now $\hat{T}$ is an increasing tableau. There are decreases in $mix(\hat{T})$ precisely when there are increases in $mix(T)$ and vice-versa. Thus, there is a marking of $mix(\hat{T})$ consisting of the underlines given by the complement of the underlines in a marking of $mix(T)$. Noticing that $\hat{\hat{T}} = T$ shows that this involution is the necessary bijection to tulgey tableaux on $\lambda^\vee/\nu^\vee$ with alphabet $\{1, \ldots, p\}$, counted by $d^{\lambda^\vee}_{\nu^\vee,(p)}$. $\square$

Example 3.3.1. Let T be the first tableau below. Then $\texttt{flip}(T)$ and $\hat{T}$ are the second and third, respectively. Extra empty boxes fill out $\Lambda_{OG(5,11)}$ for clarity. Markings for $mix(T)$ and $mix(\hat{T})$ have complementary positions underlined. Notice also that $mix(\texttt{flip}(T)) = mix(T)$.

$$\begin{ytableau} \ & \ & \ & \ & 1 \\ \none & \ & 1 & 2 & 3 \\ \none & \none & 2 & \ & \ \\ \none & \none & \none & \ & \ \\ \none & \none & \none & \none & \ \end{ytableau} \qquad \begin{ytableau} \ & \ & \ & 3 & 1 \\ \none & \ & \ & 2 & \ \\ \none & \none & 2 & 1 & \ \\ \none & \none & \none & \ & \ \\ \none & \none & \none & \none & \ \end{ytableau} \qquad \begin{ytableau} \ & \ & \ & 1 & 3 \\ \none & \ & \ & 2 & \ \\ \none & \none & 2 & 3 & \ \\ \none & \none & \none & \ & \ \\ \none & \none & \none & \none & \ \end{ytableau}$$

$$mix(T) = \underline{1}32\underline{1}2|\underline{2}1\underline{23}1 \qquad mix(\hat{T}) = 3\underline{12}3\underline{2}|2\underline{3}21\underline{3}$$

Chapter 4

Type B Pieri Rule

If $d^{\nu}_{\lambda,\mu} = \varepsilon^{\nu}_{\lambda,\mu} C^{\nu}_{\lambda,\mu}(OG(n, 2n+1))$ as conjectured, knowing $d^{\nu}_{\lambda,(p)}$ will allow us to compute any structure constants of $K(OG(n, 2n+1))$. Pieri structure constants, $d^{\nu}_{\lambda,(p)}$, are not so clean to count as their type A counterpart. This chapter will develop a few formulas which cover all possible cases to count $d^{\nu}_{\lambda,(p)}$ for any λ, ν, or p.

We will need the following lemma frequently in this chapter. A very similar statement was given in [17], as the tableaux described are those which K-rectify to $S_{(p)}$ in Type A.

Let $c(\nu/\lambda)$ be the number of columns of ν/λ and $r(\nu/\lambda)$ be the number of rows.

Lemma 4.0.4. *Let ν/λ be a horizontal strip and $p \geq 1$. The number of increasing tableaux on ν/λ with alphabet $1, \ldots, p$ such that $row(T)$ is weakly increasing is $\binom{r(\nu/\lambda)-1}{|\nu/\lambda|-p}$.*

Proof. Each row of ν/λ contains distinct and consecutive numbers as we are counting *increasing* tableaux. An entry can be repeated in a labeling if and only if it begins a row other than the first. There are $|\nu/\lambda| - p$ repeats. There are $r(\nu/\lambda) - 1$ new rows where the weak increases can occur. Choose $|\nu/\lambda| - p$ of the new rows in which to insert a weak increase. In all other new rows, a minimal strict increase is forced. $\square$

Corollary. *Let ν/λ be a vertical strip and $p \in \mathbb{N}$. Then the number of increasing tableaux on ν/λ with alphabet $1, \ldots, p$ such that col(T) is weakly increasing is $\binom{c(\nu/\lambda)-1}{|\nu/\lambda|-p}$.*

Proof. This is exactly the same as the previous proof with columns and rows transposed. □

4.1 A large component in the southwest

Two boxes of a shape are said to be *neighbors* if they share an edge. Set $x \sim y$ iff x and y are neighbors. A *component* of a shape is an equivalence class of the transitive closure of $\sim$. *Corners* of a shape are all boxes with the following two exceptions: (1) those boxes with neighbors both east and west of themselves, or (2) those boxes with neighbors both north and south of themselves. Figure 4.1 shows a shape consisting of two components, and six corners (marked with holes). Thus corners include all "turning points," *singletons* (single box components), and "extrema of components."

Let N denote the number of components of ν/λ, and N' be the number of components with more than one box. Also let $C(\mathcal{C})$ be the number of corners in a given component $\mathcal{C}$. $C(\nu/\lambda)$ is the sum of $C(\mathcal{C})$ for all components, and is the number of corners of the entire shape ν/λ. We suppress notation showing that N and N' are dependent on ν/λ.

Note that because we are considering $d^{\nu}_{\lambda,(p)}$, we will assume all tableaux, without holes, in this chapter are border strips. This also implies that within ν/λ, the

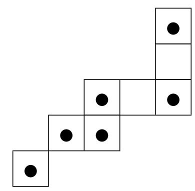
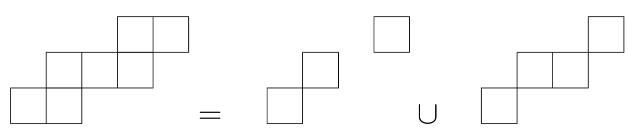


Figure 4.1: Corners of a skew shape.

southwesternmost box has at most one neighbor (also in ν/λ). For this section, we will assume that the southwesternmost component is *large*, meaning it has more than a single box. Thus, the southwesternmost box has a neighbor in ν/λ.

Definition 4.1.1. Let ν/λ be a border strip. A *separation* of ν/λ is a partition α such that $\lambda \subseteq \alpha \subseteq \nu$ with α/λ a vertical strip and ν/α a horizontal strip.

Example 4.1.1. Consider the border strip $\nu/\lambda = (7,5,2)/(5,2)$. It has two separations; one of which is given by $\alpha = (6,3,1)$. So $\nu/\lambda = \alpha/\lambda \cup \nu/\alpha$ a union of vertical and horizontal strips, respectively.



We say that two boxes of ν/λ are *separated* by a separation α if one is in α and the other is not. We say that a large component is separated by α if the southwesternmost box of that component is separated from its unique neighbor in ν/λ.

In this section, where the southwesternmost component of ν/λ is large, we make the extra assumption that all separations mentioned will *not* separate the southwesternmost component.

For each other large component $\mathcal{C}$, the only box of $\mathcal{C}$ which has the potential to be in one separation and not in another is the southwesternmost box of $\mathcal{C}$. If $\square\square \subseteq \nu/\lambda$ then the box on the right must be in ν/α. If $\begin{smallmatrix}\square\\ \square\end{smallmatrix} \subseteq \nu/\lambda$ the box above must lie in α/λ. We see immediately there are 2 to the power of the number of components of ν/λ minus 1 different separations of ν/λ, as singleton components can be in ν/α or α/λ.

We say that a tableau T on ν/λ respects a separation α, or is tulgey with respect to α when there is a marking M of $mix(T)$ such that

$$\alpha = \lambda \cup \{\text{boxes of } \nu \text{ containing entries underlined in } col(T) \text{ in } M\}.$$

Note that every tulgey tableau respects some α, i.e. the description above is a shape. To see this, let c be a box of α, constructed in such a way. Let a be the box immediately to its north (unless c is in the first row of the staircase) and b be the box immediately to the west of c (unless c is the westernmost box of its row). α is a shape provided that a is either in λ or in α; same with b. If $a \subseteq \lambda$, we're done. Since $a < c$, then the enty of T in box a must be underlined in M before c, so it is underlined in $col(T)$, so $a \subset \alpha$. The same line of reasoning works for b.

Proposition 4.1.1. *No tableau can be tulgey with respect to two distinct separations.*

Proof. Suppose there are different separations α and β with T tulgey with respect to both of them. This would require some southwesternmost box of some non-southwesternmost component to be in one separation and not in the other. T has some entry k in this box. Let q be the label of the southwesternmost box of T.

Case 1: The box to the east of q is in ν/λ. If T is tulgey with respect to α in which the box containing k is in α then $k \leq q$, but if T is tulgey with respect to β with the box containing k not in β then $k > q$.

Case 2: The box to the north of q is in ν/λ. In this case, the box with label k being in α implies $k < q$, and that box not being in β means $k \geq q$.

Both of these outcomes lead to contradictions. □

Notice that in the proof above we do use the fact that the the southwesternmost component of ν/λ is large. We also see that given a tulgey T (with a large southwesternmost component), there are precisely two markings of T, the one described by the creation of an α as above, and the marking due to a separation which does not respect our caveat that of keeping the southwesternmost component unseparated.

For the proof of the next proposition, we recall the Vandermonde identity:

$$\sum_k \binom{n}{a-k}\binom{m}{b+k} = \binom{n+m}{a+b}$$

A quick proof of this celebrated identity is given by asking, "What is the coefficient of x^{a+b} in $(1+x)^{n+m} = (1+x)^n(1+x)^m$?"

Proposition 4.1.2. *When the southwesternmost component of ν/λ is large,*

$$d^{\nu}_{\lambda,(p)} = \sum_{\alpha} \binom{c(\alpha/\lambda) + r(\nu/\alpha) - 1}{|\nu/\lambda| - p}$$

where the sum is over all separations α of ν/λ leaving the southwesternmost com-

ponent of ν/λ unseparated.

Proof. Let us count the number of tableaux which are tulgey with respect to a fixed α. By earlier considerations, because no tableau will be counted twice and each tulgey tableau is tulgey with respect to some separation, we simply vary α over all possible separations to arrive at $d^{\nu}_{\lambda,(p)}$.

Case 1: The southwesternmost box of ν/λ has its unique neighbor to its east. We index our count of tulgey T by the entry q in the southwesternmost box of T. The first part of the following sum comes from first filling α/λ with an alphabet of $1, \ldots, q-1$ and ν/α with $q, \ldots, p$. The second part of the sum fills α/λ with an alphabet $1, \ldots, q$. These are all the tulgey fillings of ν/λ with respect to α.

$$\sum_q \binom{c(\lambda/\alpha)-1}{|\alpha/\lambda|-(q-1)}\binom{r(\nu/\alpha)-1}{|\nu/\alpha|-(p-q+1)}+\binom{c(\lambda/\alpha)-1}{|\alpha/\lambda|-q}\binom{r(\nu/\alpha)-1}{|\nu/\alpha|-(p-q+1)}$$

$$=\binom{c(\alpha/\lambda)+r(\nu/\alpha)-2}{|\nu/\lambda|-p}+\binom{c(\alpha/\lambda)+r(\nu/\alpha)-2}{|\nu/\lambda|-p-1}=\binom{c(\alpha/\lambda)+r(\nu/\alpha)-1}{|\nu/\lambda|-p}$$

by the Vandermonde identity then the recursive definition of binomial coefficients.

Case 2: The southwesternmost box of ν/λ has its neighbor to the north. We perform the same steps as above, but notice that the alphabets filling α/λ must be $1, \ldots, q$, but here the alphabet for ν/α can include q or not.

$$\sum_q \binom{c(\lambda/\alpha)-1}{|\alpha/\lambda|-q}\binom{r(\nu/\alpha)-1}{|\nu/\alpha|-(p-q+1)}+\binom{c(\lambda/\alpha)-1}{|\alpha/\lambda|-q}\binom{r(\nu/\alpha)-1}{|\nu/\alpha|-(p-q)}$$

$$=\binom{c(\alpha/\lambda)+r(\nu/\alpha)-2}{|\nu/\lambda|-p}+\binom{c(\alpha/\lambda)+r(\nu/\alpha)-2}{|\nu/\lambda|-p-1}=\binom{c(\alpha/\lambda)+r(\nu/\alpha)-1}{|\nu/\lambda|-p}$$

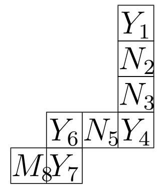


Figure 4.2: Sample for counting contribution to $s(\alpha)$.

again by Vandermonde and recursive identities. $\square$

Proposition 4.1.3. *When the southwesternmost component of ν/λ is large,*

$$d^{\nu}_{\lambda,(p)} = 2^{N-N'} \sum_{k=0}^{N'-1} \binom{N'-1}{k} \binom{C(\nu/\lambda)-k-2}{|\nu/\lambda|-p}.$$

Proof. For a separation α of ν/λ, let $s(\alpha) = c(\alpha/\lambda) + r(\nu/\alpha)$ for shorthand. Components of ν/λ contribute to $s(\alpha)$ independently, so we focus first on understanding how one component $\mathcal{C}$ of ν/α contributes to $s(\alpha)$.

If $\mathcal{C}$ consists of a single box, then it contributes exactly 1 to $s(\alpha)$, as the box is entirely in one column of α/λ or one row of ν/α.

To compute the contribution of a large component $\mathcal{C}$, let us begin in its northeasternmost box and trace through its boxes neighbor by neighbor, adjusting our count until we arrive at the southwesternmost box of $\mathcal{C}$. For purposes of clarity, we will work through the sample shape in Figure 4.2.

In Figure 4.2, a box is labeled Y if the box contributes to $s(\alpha)$, N if not, and M if it depends on α. The subscripts on the labels are in the order of the path through the component.

Notice that, yes, the first box starts us off in some column of α/λ, so our contribution count is currently 1. Moving to the second box, as it is in the same column of α/λ, it does not change the running total. The next box also does not contribute to $s(\alpha)$ for the same reason. The fourth box is no longer in α, so being in ν/α it contributes to $r(\nu/\alpha)$, increasing our running total for $s(\alpha)$. The fifth box is in the same row, so until the next corner, no new boxes will contribute in our running tally of $s(\alpha)$. The sixth and seventh boxes each contribute since they occupy new columns and rows not yet counted in $s(\alpha)$. See that the final box could contribute to $s(\alpha)$, but only if it is separated from its neighbor by α.

In general, any component works the same way a new box in the march through $\mathcal{C}$ contributes to $s(\alpha)$ if it is a corner, unless it is the southwesternmost box of $\mathcal{C}$ and α does not separate it from its neighbor. For each of the N' large components $\mathcal{C}_i$, let $c_i(\alpha)$ be $C(\mathcal{C}_i)$ if α separates $\mathcal{C}_i$ or $C(\mathcal{C}_i) - 1$ if α leaves $\mathcal{C}_i$ unseparated.

$$s(\alpha) = N - N' + c_1(\alpha) + \cdots + c_{N'}(\alpha)$$

If $k(\alpha)$ is the number of components $\mathcal{C}_i$ for which α does *not* separate $\mathcal{C}_i$ for $1 \leq i \leq n-1$, we have

$$s(\alpha) = N - N' + \sum_{i=1}^{N'} C(\mathcal{C}_i) - k(\alpha) - 1 = C(\nu/\lambda) - k(\alpha) - 1.$$

There are $2^{N-N'}\binom{N'-1}{k}$ separations α for which $k(\alpha) = k$, and each has $\binom{s(\alpha)-1}{|\nu/\lambda|-p} = \binom{C(\nu/\lambda)-k-2}{|\nu/\lambda|-p}$ tulgey tableaux with respect to it. Thus, by indexing by the number of components left unseparated, we have recovered the the expression for $d^{\nu}_{\lambda,(p)}$ as we desired. $\square$

4.2 All components are singletons

The next case we tackle is when ν/λ is composed of precisely N singleton components (and no large components). Jeu de taquin performed by choosing one box at a time can "shift" a box west or north until two boxes share a vertex. $Krevjdt$ can shift them east or south. Any N singleton shape can be shifted to any other. Tulginess is invariant under this kind of shifting, so the number of tulgey tableaux $d^{\nu}_{\lambda,(p)}$ depends only on p and the number of boxes, N. Let $T(N,p) = d^{\nu}_{\lambda(p)}$, in this instance.

When $p = 1$, there is only one filling of ν/λ, and it is tulgey, so $T(N,1) = 1$. When $p = 2$, any filling with at least one 1 and one 2 is tulgey, which makes $T(N,2) = 2^N - 2$. Set $T(N,p) = 0$ when $p > N$ or $p < 1$.

When $p \geq 2$, it should be clear that the first letter in $mix(T)$ is either a 1 or p. Consider the permutation in S_p which reverses the alphabet, i.e. $(1\ p)(2\ p-1)\cdots$. Acting on a tulgey tableau with this permutation gives a different tulgey tableau. This also switches those whose mix words begin with 1 and those that begin with p. The number of tulgey tableaux beginning with 1 is $T(N-1,p) + T(N-1,p-1)$. This can be seen by adding the number of those where 1 is repeated and those where it is not. These considerations together give the recurrence relation:

$$T(N,p) = 2T(N-1,p) + 2T(N-1,p-1)$$

Proposition 4.2.1. *$T(N,p) = 2^{p-1}\sum_{k=0}^{N-p} 2^k \binom{p+k-2}{p-2}$ for $p \geq 2$.*

Proof. Given base cases as above, it suffices to show that the alleged expression satisfies the recurrence relation above. Let's see what we get when we plug into the recurrence relation:

$$2^{p-1}\sum_{k=0}^{N-p} 2^k \binom{p+k-2}{p-2} \stackrel{?}{=} 2^p \sum_{k=0}^{N-p-1} 2^k \binom{p+k-2}{p-2} + 2^{p-1}\sum_{k=0}^{N-p} 2^k \binom{p+k-3}{p-3}$$

Combining two arithmetic steps in one line, the above is true iff the below is.

$$2^{N-p}\binom{N-2}{p-2} \stackrel{?}{=} 2^{N-p}\binom{N-3}{p-3} + \sum_{k=0}^{N-p-1} 2^k \left[\binom{p+k-2}{p-2} + \binom{p+k-3}{p-3}\right]$$

Substitute using the recursive binomial identity twice: $\binom{N-3}{p-3} = \binom{N-2}{p-2} - \binom{N-3}{p-2}$ and $\binom{p+k-3}{p-3} = \binom{p+k-2}{p-2} - \binom{p+k-3}{p-2}$. After moving a couple terms, we get:

$$2^{N-p}\binom{N-3}{p-2} \stackrel{?}{=} \sum_{k=0}^{N-p-1} 2^k \left[2\binom{p+k-2}{p-2} - \binom{p+k-3}{p-2}\right]$$

Moving the negative piece to the left side, and noticing that its $k = 0$ term does not contribute, we now have:

$$\sum_{k=1}^{N-p} 2^k \binom{p+k-3}{p-2} \stackrel{?}{=} 2\sum_{k=0}^{N-p-1} 2^k \binom{p+k-2}{p-2}$$

These two sides are indeed equal, and can be seen to be so by shifting the index k by one on either side. $\square$

4.3 Singletons in the southwest

We have only one case left to examine. It is a sort of combination of the two previous types. This is when there are exactly $M \geq 1$ consecutive singleton components in the southwest of ν/λ followed by a large component. There may be more components further to the northeast, singletons or large ones.

Let N and N' again be the number of components of ν/λ and the number of large components, respectively. Form a subshape $\bar{\nu}/\bar{\lambda}$ from the $N - M$ northeasternmost components. Thus $\bar{\nu}/\bar{\lambda}$ is a shape whose southwesternmost component is large.

To count the tulgey tableaux on ν/λ, pick s and t to index the minimal and maximal entries of the M singletons. There are $m = t - s + 1$ number of distinct entries in the M singletons. Next, choose a separation α of $\bar{\nu}/\bar{\lambda}$ as before, starting from the northeasternmost vertex of the northeasternmost box of $\bar{\nu}/\bar{\lambda}$, ending at the southwesternmost vertex of the southwesternmost box of $\bar{\nu}/\bar{\lambda}$.

Note. We make *no* stipulation that the separation must keep the last large component unseparated.

Given s, t, and separation α as above, there are four methods of filling in $\bar{\nu}/\bar{\lambda}$. The alphabet in α/λ can be $1, \ldots, s-1$ or $1, \ldots, s$ and the alphabet in ν/α can be $t, \ldots, p$ or $t+1, \ldots, p$. This *will* introduce overcounting $\mathcal{OV}$, which we will calculate later.

Adding those four alphabet choices and remembering to fill in the M singletons, we have:

$$d^{\nu}_{\lambda,(p)} = \sum_{1\leq s\leq t\leq p} T(M,m) \sum_{\alpha} \Big[\binom{c(\alpha/\lambda)-1}{|\alpha/\lambda|-s} \binom{r(\nu/\alpha)-1}{|\nu/\alpha/|-(p-t+1)}$$
$$+ \binom{c(\alpha/\lambda)-1}{|\alpha/\lambda|-s} \binom{r(\nu/\alpha)-1}{|\nu/\alpha/|-(p-t)} + \binom{c(\alpha/\lambda)-1}{|\alpha/\lambda|-(s-1)} \binom{r(\nu/\alpha)-1}{|\nu/\alpha/|-(p-t+1)}$$
$$+ \binom{c(\alpha/\lambda)-1}{|\alpha/\lambda|-(s-1)} \binom{r(\nu/\alpha)-1}{|\nu/\alpha/|-(p-t)} \Big] - \mathcal{OV}.$$

Replacing s with $t-m+1$, we can use the Vandermonde identity to eliminate t from our sum:

$$d^{\nu}_{\lambda,(p)} = \sum_{\alpha,1\leq m\leq p} T(M,m) \Big[\binom{c(\alpha/\lambda)+r(\nu/\alpha)-2}{|\bar{\nu}/\bar{\lambda}|-p+m-2}$$
$$+ 2\binom{c(\alpha/\lambda)+r(\nu/\alpha)-2}{|\bar{\nu}/\bar{\lambda}|-p+m-1} + \binom{c(\alpha/\lambda)+r(\nu/\alpha)-2}{|\bar{\nu}/\bar{\lambda}|-p+m} \Big] - \mathcal{OV}$$

Using the recursive binomial identity twice, we can clean this sum up even further:

$$d^{\nu}_{\lambda,(p)} = \sum_{\alpha,m} T(M,m) \binom{c(\alpha/\lambda)+r(\nu/\alpha)}{|\bar{\nu}/\bar{\lambda}|-p+m} - \mathcal{OV}$$

Just as previously, we see that $c(\alpha/\lambda)+r(\nu/\alpha)$ is the number of corners of $\bar{\nu}/\bar{\lambda}$ minus the number of large components left unseparated by α. This time, however, as α may or may not separate the southwesternmost box of $\bar{\nu}/\bar{\lambda}$ from its neighbor, we have

$$d^{\nu}_{\lambda,(p)} = 2^{N-N'-M} \sum_{m,k} T(M,m) \binom{N'}{k} \binom{C(\bar{\nu}/\bar{\lambda})-k}{|\bar{\nu}/\bar{\lambda}|-p+m} - \mathcal{OV}$$

by summing over the number of unseparated components k.

Let us now turn to the problem of overcounting. If $s < t$, given α, the four alphabet cases produce distinct classes of tableaux and no tableaux is counted twice in its class. Moreover, in this circumstance, since α can be recovered from a tableau, different separations can not yield the same tableau.

This means overcounting can only potentially occur when $s = t$, i.e. $m = 1$. In this case, there are two different separations for every tulgey tableau on ν/λ. They are the one which separates the southwesternmost box of the southwesternmost large component from its neighbor and the one that does not, provided the label in that box is also s.

This makes every single tulgey filling of $\bar{\nu}/\bar{\lambda}$ count precisely twice. Since $\bar{\nu}/\bar{\lambda}$ has a large lower left component, there are $2^{N-N'-M}\sum_k \binom{N'-1}{k}\binom{C(\bar{\nu}/\bar{\lambda})-k-2}{|\bar{\nu}/\bar{\lambda}|-p}$ such overcountings.

Sadly, we are not done overcounting. We have inadvertently counted some fillings which were not increasing tableaux. This occurred when α separated the southwesternmost box, filled with s, from its neighbor, filled with t. The problem is that because $s = t$, there is not an increase between these two adjacent boxes.

For each potential α separating the southwesternmost component, filled with s, there are $\binom{c(\alpha/\lambda)-1}{|\alpha/\lambda|-s}\binom{r(\nu/\alpha)-1}{|\nu/\alpha/|-(p-s+1)}$ such overcountings. The non-tableaux counted by the formula amount to:

$$\sum_{\alpha}\binom{c(\alpha/\lambda)+r(\nu/\alpha)-2}{|\bar{\nu}/\bar{\lambda}|-p-1} = 2^{N-N'-M}\sum_{k}\binom{N'-1}{k}\binom{C(\bar{\nu}/\bar{\lambda})-k-2}{|\bar{\nu}/\bar{\lambda}|-p-1}$$

Total overcounting amounts to:

$$\begin{aligned}\mathcal{OV} &= 2^{N-N'-M}\sum_{k=0}^{N'-1}\binom{N'-1}{k}\Big[\binom{C(\bar{\nu}/\bar{\lambda})-k-2}{|\bar{\nu}/\bar{\lambda}|-p}+\binom{C(\bar{\nu}/\bar{\lambda})-k-2}{|\bar{\nu}/\bar{\lambda}|-p-1}\Big]\\ &= 2^{N-N'-M}\sum_{k=0}^{N'-1}\binom{N'-1}{k}\binom{C(\bar{\nu}/\bar{\lambda})-k-1}{|\bar{\nu}/\bar{\lambda}|-p}\end{aligned}$$

This formula holds in the specialized case when there are no large components. Here $N=M$, $N'=0$, $\bar{\nu}/\bar{\lambda}=\emptyset$ making the sum include only terms $m=p$ and $k=0$. Each term of the overcounting is 0, so $d^{\nu}_{\lambda,(p)}=T(N,p)$ as needed.

In one expression, we have the following proposition.

Proposition 4.3.1. *When the southwesternmost component is a singleton,*

$$\begin{aligned}d^{\nu}_{\lambda,(p)} = 2^{N-N'-M}\Big[&\sum_{m,k}T(M,m)\binom{N'}{k}\binom{C(\bar{\nu}/\bar{\lambda})-k}{|\bar{\nu}/\bar{\lambda}|-p+m}\\ &-\sum_{k=0}^{N'-1}\binom{N'-1}{k}\binom{C(\bar{\nu}/\bar{\lambda})-k-1}{|\bar{\nu}/\bar{\lambda}|-p}\Big]\end{aligned}$$

Example 4.3.1. Here we compute the number of tableaux counted by $d^{\nu}_{\lambda,(p)}$ where $p=5$ and

$$\nu/\lambda=(14,13,12,10,8,7,3,1)/(13,12,11,9,7,4,2).$$

In this example $N=5$, $N'=2$, $M=2$, and

$$\bar{\nu}/\bar{\lambda}=(14,13,12,10,8,7)/(13,12,11,9,7)$$

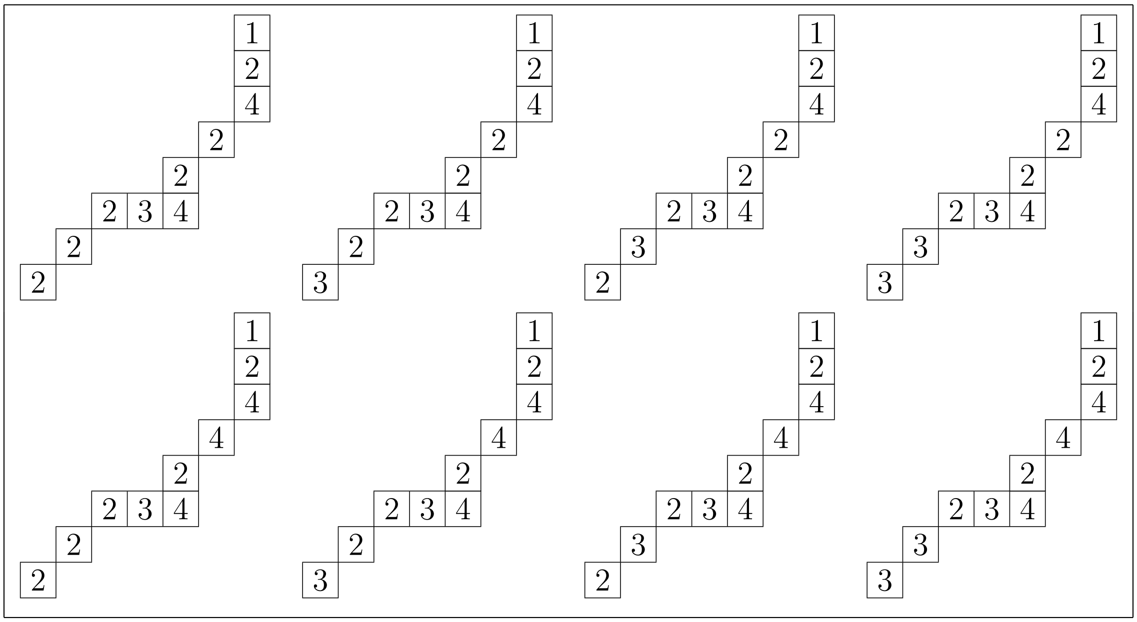


Figure 4.3: Tableaux enumerating $d^{\nu}_{\lambda,(p)}$.

which makes $C(\bar{\nu}/\bar{\lambda}) = 6$ and $|\bar{\nu}/\bar{\lambda}| = 8$.

$$
\begin{aligned}
d^{\nu}_{\lambda,(p)} &= 2^{5-2-2}\Big[\sum_{m,k} T(2,m)\binom{2}{k}\binom{6-k}{8-4+m} - \sum_{k=0}^{2-1}\binom{2-1}{k}\binom{6-k-1}{8-4}\Big] \\
&= 2\Big[\binom{2}{0}\binom{6}{5} + \binom{2}{1}\binom{5}{5} + 2\binom{2}{0}\binom{6}{6} - \binom{1}{0}\binom{5}{4} - \binom{1}{1}\binom{4}{4}\Big] \\
&= 2[6+2+2-5-1] = 20 - 12 = 8
\end{aligned}
$$

There are 2 tableaux double-counted and 10 things counted that were not tableaux. Figure 4.3 shows the 8 tableaux actually counted by $d^{\nu}_{\lambda,(p)}$, followed by Figure 4.4, showing the 10 “tableaux” which were counted that shouldn’t have been. The first tableau in each of the first two rows of Figure 4.3 is counted twice, which is why the overcounting is 10+2=12, not just 10.

Combining Propositions 4.1.3 and 4.3.1, we arrive at the type B Pieri Rule.

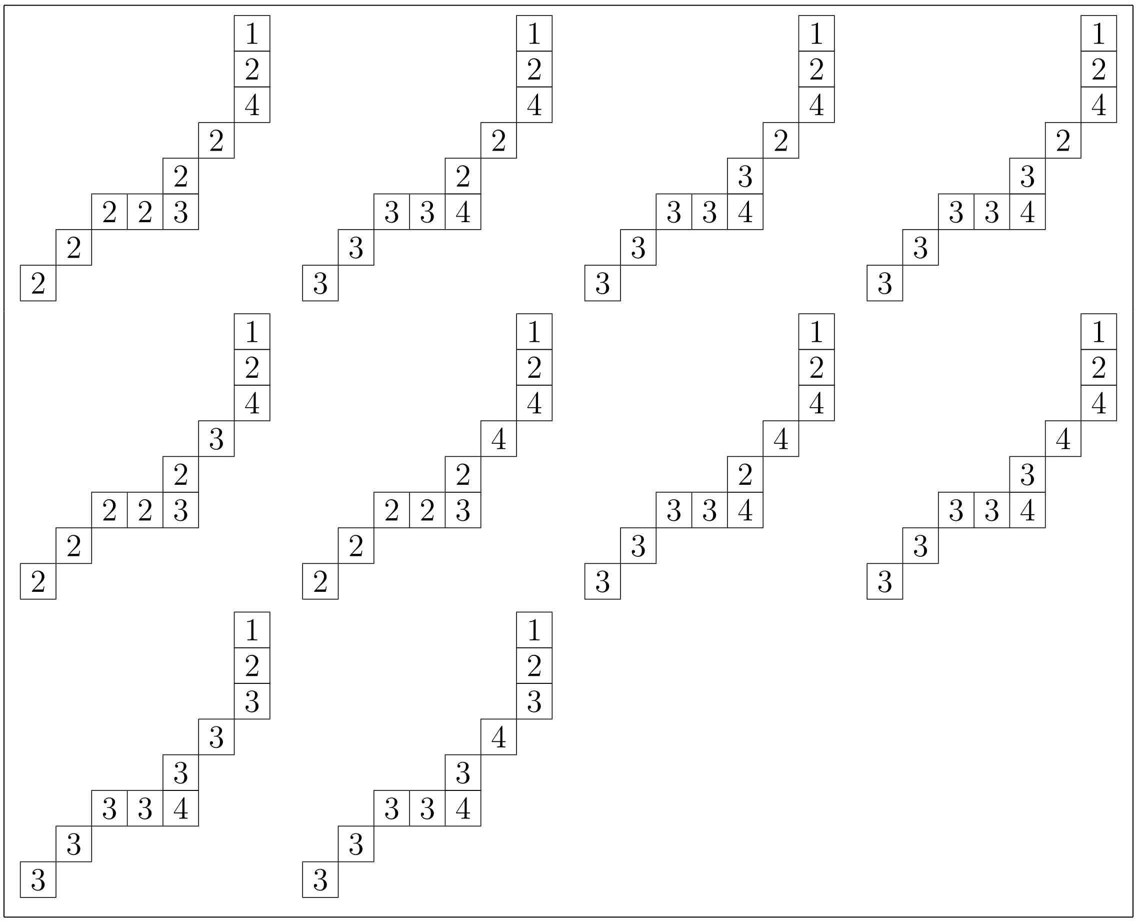


Figure 4.4: Non-tableaux contributing to overcounting of $d^{\nu}_{\lambda,(p)}$.

Theorem 4.3.2. *(Type B Pieri rule) (1) If the southwesternmost component of ν/λ is a singleton,*

$$d^{\nu}_{\lambda,(p)} = 2^{N-N'-M}\Big[\sum_{m,k} T(M,m)\binom{N'}{k}\binom{C(\bar{\nu}/\bar{\lambda})-k}{|\bar{\nu}/\bar{\lambda}|-p+m} - \sum_{k=0}^{N'-1}\binom{N'-1}{k}\binom{C(\bar{\nu}/\bar{\lambda})-k-1}{|\bar{\nu}/\bar{\lambda}|-p}\Big]$$

(2) If the southwesternmost component of ν/λ is large,

$$d^{\nu}_{\lambda,(p)} = 2^{N-N'}\sum_{k=0}^{N'-1}\binom{N'-1}{k}\binom{C(\nu/\lambda)-k-2}{|\nu/\lambda|-p}.$$

4.4 A Mathematica session

This short section takes us on a brief but interesting tangent. The techniques of the book $A=B$ by Petkovšek, Wilf, and Zeilberger [10], allow us to determine whether hypergeometric sums have simple closed forms. A hypergeometric term $f(k)$ is a function for which the ratio $\frac{f(k+1)}{f(k)}$ is a rational function in k. Hypergeometric terms may have multiple variables, provided the ratios corresponding to all variables are all rational functions. Expressible as a simple closed form roughly means that a sum can be written as a definite finite linear combination of hypergeometric terms. See [10] for the precise definition of "simple closed form."

Using $A{=}B$ packages, we examine the simplest of our Pieri rules:

$$T(N,p) = 2^{p-1} \sum_{k=0}^{N-p} 2^k \binom{p+k-2}{p-2}$$

If $T(N,p)$ has a simple closed form, then we may be able to express $d^{\nu}_{\lambda,(p)}$ in a simple closed form, as $T(N,p)$ is a specialization of a $d^{\nu}_{\lambda,(p)}$ calculation. However, if $T(N,p)$ is not "Gosper-summable," meaning that Gosper's algorithm does not yield a nice expression, we have no hope for $d^{\nu}_{\lambda,(p)}$ in general.

In the example below, `t[k,p]` is the summand of $T(N,p)$. Note that it is a hypergeometric term, so asking if it is Gosper-summable is exactly what we want. The `zb.m` Mathematica package (available through resources in [10]) will allow us to ask if a hypergeometric series is Gosper-summable. The command `Gosper` will only display an empty set when something is not Gosper-summable.

The other piece of information displayed is a recurrence relation held by `t[k,p]`. Below, `F` is `t` and $\Delta_k[f(k)] = f(k+1) - f(k)$. `R` is the "key" function used in the algorithms of $A{=}B$, shown here for the sake of those already familiar with the book.

Example 4.4.1. This is a Mathematica session showing that the Pieri rule given by $T(N,p)$ is not Gosper-summable.

```
In[1]:=<<zb.m

In[2]:=t[k,p]:=2^(p+k-1)Binomial[p+k-2,p-2]

In[3]:=Gosper[t[k,p],{k,0,N-p}]

Out[3]={}
```

```
In[4]:=Zb[t[k,p],k,p,1]
```

Out[4]= $\{2F[k,p]+F[k,1+p]==\Delta_k[F[k,p]R[k,p]]\}$

```
In[5]:=Show[R]
```

Out[5]= $\frac{2k}{-1+p}$

Chapter 5

Further Results and Conjectures

5.1 K-rectification in the other classical minuscule Lie types

Let us turn our attention to the classical Lie types other than A and B. Let $d^{\nu}_{\lambda,\mu}(G/P)$ be the number of increasing tableaux of shape ν/λ which K-rectify to S_μ within $\Lambda_{G/P}$. We shall see that in types C and D that $d^{\nu}_{\lambda,\mu}(G/P)$ is indeed well defined.

In the minuscule type C_n case, $G/P = \mathbb{P}^{2n-1}$ which gives $\Lambda_{G/P}$ the shape described by the partition $(1^{2n-1}) := (1, \ldots, 1)$ of length $2n-1$. On this very simple shape, all K-rectification is well defined as there is only one rectification order possible for any skew shape $(0^k, 1^{|\nu/\lambda|})$. Provided its alphabet is $1, \ldots, |\mu|$, the single possible tableau K-rectifies to S_μ. Thus $d^{\nu}_{\lambda,\mu}(\mathbb{P}^{2n-1})$ is 1 when $|\nu| = |\lambda| - |\mu|$ else it is 0.

There are two minuscule type D_n cases. Picking the first simple root to be the minuscule one makes $G/P = \mathbb{Q}^{2n-2}$, the complex quadric defined by the equation $z_1^2 + \cdots + z_{2n}^2 = 0$. The poset $\Lambda_{\mathbb{Q}^{2n-2}}$ is given in Figure 5.1. This poset is known as $d_n(1) = \Delta_{n-2,n-2}$ in [13], but we will return to this later.

Proposition 5.1.1. *Let $T \in INC_{\mathbb{Q}^{2n-2}}(\nu/\lambda)$. Then $Krect(T)$ is well-defined for any rectification order.*

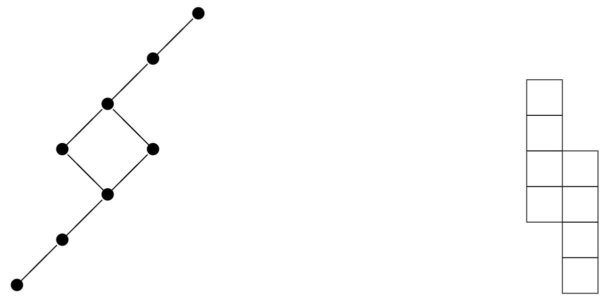


Figure 5.1: $\Lambda_{\mathbb{Q}^{2n-2}}$ as a poset and as shape $(1,\ldots,1,2,2,1,\ldots,1)$ when $n=5$.

Proof. Consider possible rectification orders of T, i.e. the set $INC(\lambda)$. Notice that if λ contains the unique instance of $\begin{smallmatrix} & \square \\ \square & \end{smallmatrix}$ then there are three tableaux in $INC(\lambda)$. If $\begin{smallmatrix} & \square \\ \square & \end{smallmatrix} \not\subset \lambda$ then there is a unique increasing tableau on λ, so there is only one potential rectification order. Thus $Krect(T)$ is well-defined if $\begin{smallmatrix} & \square \\ \square & \end{smallmatrix} \not\subset \lambda$.

Suppose $\begin{smallmatrix} & \square \\ \square & \end{smallmatrix} \subset \lambda$. It must then be the case that $|\nu/\lambda| \leq n-2$. K-rectification of T necessarily occupies a vertical strip, thus is $S_{(1^{|\nu/\lambda|})}$. $\square$

Let us count $d^{\nu}_{\lambda,\mu}(\mathbb{Q}^{2n-2})$. Let $\Lambda = \Lambda_{\mathbb{Q}^{2n-2}}$ for this section. Our analysis will proceed by exhausting all possible cases.

Case 1: $|\mu| \geq n$, i.e. $\begin{smallmatrix} & \square \\ \square & \end{smallmatrix} \subset \mu$. If $|\mu| = |\nu/\lambda|$, there's a unique sequence of $Krevjdt$ to get from S_μ to a tableau on ν/λ. This tableau must therefore K-rectify to S_μ. If $|\mu| \neq |\nu/\lambda|$ then there is no hope to get to $T \in INC(\nu/\lambda)$ via $Krevjdt$. Thus in Case 1, $d^{\nu}_{\lambda,\mu} = 1$ if $|\mu| = |\nu/\lambda|$ and 0 otherwise.

Case 2: μ is a vertical strip with $|\mu| \leq n-2$. Any tableau with an alphabet of $1,\ldots,|\mu|$ will K-rectify to S_μ. If $|\nu/\lambda| = |\mu|$ then $d^{\nu}_{\lambda,\mu} = 2$ when $\begin{smallmatrix} & \square \\ \square & \end{smallmatrix} \subset \nu/\lambda$ or 1 when $\begin{smallmatrix} & \square \\ \square & \end{smallmatrix} \not\subset \nu/\lambda$. If $|\nu/\lambda| = |\mu|+1$ then $d^{\nu}_{\lambda,\mu}$ is 1 iff $\begin{smallmatrix} & \square \\ \square & \end{smallmatrix} \subset \nu/\lambda$. In any other situation within Case 2, $d^{\nu}_{\lambda,\mu} = 0$.

Case 3: $|\mu| = n-1$. There are two possible shapes of μ: $(1^{n-2},2)$ and the vertical strip (1^{n-1}). Each of these has only one possible sequence of $Krevjdt$ to arrive at any potential ν/λ. That ν/λ must have $n-1$ boxes. Any tableau on $n-1$ distinctly filled boxes K-rectifies to one of the potential μ.

If $\ydiagram{1+1,1} \subset \nu/\lambda$ then there are precisely two standard tableaux on ν/λ. These can not K-rectify to the same tableau, so one must K-rectify to $(1^{n-2}, 2)$ and one to (1^{n-1}).So $d^{\nu}_{\lambda,\mu} = 1$ when $\ydiagram{1+1,1} \subset \nu/\lambda$ and $|\nu/\lambda| = n-1$.

If $\ydiagram{1+1,1} \not\subset \nu/\lambda$, then $\nu/\lambda = \Lambda/(1^{n-2}, 2)$ or $\Lambda/(1^{n-1})$. Each of these must K-rectify to one of the potential μ. The only thing to determine is which K-rectifies to which.

Subcase 1: $\nu/\lambda = \Lambda/(1^{n-2}, 2)$. Let T be the unique tableau on this shape. After any sequence of $Kjdt$, there is an odd entry in the position $(1, n-1)$. If $n-1$ is odd then during the $Kdjt$ applications within $Krect(T)$, the highest numbered box of T goes to this position and finishes there since the shape is now non-skew. Thus $Krect(T) = S_{(1^{n-1})}$. If $n-1$ is even, the highest numbered box ends in position $(2, n-2)$ instead, making $Krect(T) = S_{(1^{n-1},2)}$.

Subcase 2: $\nu/\lambda = \Lambda/(1^{n-1})$. Let U be the unique tableau on this shape. Opposite reasoning from the previous subcase gives us that when $n-1$ odd, $Krect(U) = S_{(1^{n-2},2)}$. When $n-1$ is even, $Krect(U) = S_{(1^{n-1})}$.

This finishes calculations for all structure constants $d^{\nu}_{\lambda,\mu}(\mathbb{Q}^{2n-2})$.

The type C minuscule case is trivial, as $\Lambda_{\mathbb{P}^{2n-1}}$ is a vertical strip. This makes $d^{\nu}_{\lambda,\mu}(\mathbb{P}^{2n-1}) = 1$ when $|\mu| = |\nu/\lambda|$ and 0 otherwise.

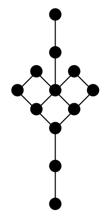


Figure 5.2: A d-complete poset sample.

5.2 Generalizations in various directions

Proctor [13] defined and examined the class of d-complete posets. These are posets where a particular *jeu de taquin* algorithm is well-defined. This allows us to ask if there is a class of tableaux $\{C_\mu\}$ to which rectification (iterated *jdt*) is also well-defined. For all minuscule G/P, $\Lambda_{G/P}$ is a d-complete poset, and standard tableaux provide the right targets for rectification [16].

Using techniques like Proctor's, it would be interesting to know if $Kjdt$ and $Krect$ are well-defined for all d-complete posets. For this, it would be necessary to create a new class of tableaux $\{C_\mu\}$. In this dissertation, we have seen that in $\Lambda_{G/P}$ the superstandard tableaux are a good choice. However, it is not even clear what a superstandard tableau on certain d-complete posets would be. Figure 5.2 is a special case of one of Proctor's d-complete classes.

If an analogue of K-rectification does hold on d-complete posets, we might be able to construct rings where the analogues of $d^{\nu}_{\lambda,\mu}$ provide structure constants.

Buch [2] showed that the type A K-theoretic structure constants count a wholly different sort of tableaux. These tableaux have sets as entries in each box, instead of individual numbers. The lowest degree (cohomological) structure constants specialize to the case where all sets are singletons, and cohomology is recovered. In addition, he defines a procedure of *jdt* as well.

It remains to be seen whether set valued tableaux can make the jump to other Lie types. A promising aspect of Buch's set-valued tableaux is that they give rise to *Grothendieck polynomials* which multiply like classes of $K(G(k, \mathbb{C}^n))$. To move to type B, as well as others, some connection between type B Grothendieck polynomials and (probably) shifted tableaux would likely first need to exist.

Type B Grothendieck polynomials themselves would be a treasure trove of combinatorics, but there are some hurdles to overcome before using them. As shown by Fomin and Kirillov [4], there are multiple "kinds" of type B analogues of Schubert polynomials. Each kind has its benefits, such as integral rather than rational coefficients, or being well-defined under cleaner divided difference operators. Unfortunately, unlike type A, all the desired properties can not be enjoyed by a single family. Still, using these constructions and the ideas of their work on Grothendieck polynomials and the Yang-Baxter equation [5], it may be possible to construct useful type B Grothendieck polynomials.

In yet another direction, it would be interesting to look at *equivariant*, *quantum*, or even quantum equivariant K-theory of minuscule G/P. It is an intriguing question to know how far increasing tableaux algorithms can be pushed to give meaningful answers to problems of more abstract cohomology theories. Quantum and equivariant cohomology is well studied, but in the type B case, little work has been done on the details when moving to K-theory.

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