ABSTRACT

Title of dissertation: DISTRIBUTED ESTIMATION AND
STABILITY OF EVOLUTIONARY GAME DYNAMICS
WITH APPLICATIONS TO STUDY OF ANIMAL MOTION

Shinkyu Park, Doctor of Philosophy, 2015

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In this dissertation, we consider three problems: in the first we investigate distributed state estimation of linear time-invariant (LTI) plants; in the second we study optimal remote state estimation of Markov processes; while in the third we examine stability of evolutionary game dynamics in large populations.

Problem 1: Consider that an autonomous LTI plant is given and that each member of a network of LTI observers accesses a portion of the output of the plant. The dissemination of information within the network is dictated by a pre-specified directed graph in which each vertex represents an observer. This work proposes a distributed estimation scheme that is a natural generalization of consensus in which each observer computes its own state estimate using only the portion of the output vector accessible to it and the state estimates of other observers that are available to it, according to the graph. Unlike straightforward high-order solutions in which each observer broadcasts its measurements throughout the network, the average size of the state of each observer in the proposed scheme does not exceed the order of the plant plus one. We determine necessary and suf-

ficient conditions for the existence of a parameter choice for which the proposed scheme attains asymptotic omniscience of the state of the plant at all observers. The conditions reduce to certain detectability requirements that imply that if omniscience is not possible under the proposed scheme then it is not viable under any other scheme – including higher order LTI, nonlinear, and time-varying ones – subject to the same graph. We apply the proposed scheme to distributed tracking of a group of water buffaloes.

Problem 2: Consider a two-block remote estimation framework in which a sensing unit accesses the full state of a Markov process and decides whether to transmit information about the state to a remotely located estimator given that each transmission incurs a communication cost. The estimator finds the best state estimate of the process using the information received from the sensing unit. The main purpose of this work is to design transmission policies and estimation rules that dictate decision making of the sensing unit and estimator, respectively, and that are optimal for a cost functional which combines the expectation of squared estimation error and communication costs. Our main results establish the existence of transmission policies and estimation rules that are jointly optimal, and propose an iterative procedure to find ones. Our convergence analysis shows that the sequence of sub-optimal solutions generated by the proposed procedure has a convergent subsequence, and the limit of any convergent subsequence is a person-by-person optimal solution.[1] We apply the proposed scheme to remote estimation of location of a water buffalo.

Problem 3: We investigate an energy conservation and dissipation (passivity) as-

[1] The definitions of joint optimality and person-by-person optimality are given in Definition 3.1.2 and Definition 3.1.3, respectively.

pect of evolutionary dynamics in evolutionary game theory. We define a notion of passivity for evolutionary dynamics, and describe conditions under which dynamics exhibit passivity. For dynamics that are defined on a finite-dimensional state space, we show that the conditions can be characterized in connection with state-space realizations of the dynamics. In addition, we establish stability of passive dynamics in terms of dissipation of stored energy defined by passivity, and present stability results in population games. We provide implications of stability for various passive dynamics both analytically and by means of numerical simulations.

DISTRIBUTED ESTIMATION AND STABILITY OF EVOLUTIONARY GAME DYNAMICS WITH APPLICATIONS TO STUDY OF ANIMAL MOTION

by

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To my family

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Chapter 1: Introduction

We consider the following three problems: **(i)** distributed state estimation of LTI plants (Chapter 2), **(ii)** remote state estimation of Markov processes (Chapter 3), and **(iii)** stability of evolutionary game dynamics (Chapter 4). The main objective and summary of main contributions to each problem are described in this chapter.

Main results of each chapter can be applied to study of animal motion: The estimation schemes that will be studied in Chapter 2 and Chapter 3 can be applied to tracking of animal groups. The data from tracking animal groups are then analyzed to identify and study collective animal motion. Based on results on stability of evolutionary game dynamics, which are presented in Chapter 4, we can find a reasoning over which certain collective motion emerges in animal groups.

1.1 Design of Distributed LTI Observers for State Omniscience

Consider the following linear time-invariant (LTI) plant in state-space form[1]:

$$\begin{aligned} x(k+1) &= Ax(k) \\ y(k) &= Cx(k) \end{aligned} \tag{1.1}$$

[1]In order to simplify the notation, without loss of generality, we omit noise terms in the state-space equation (1.1). See Section 2.2.1.1 for more discussions.

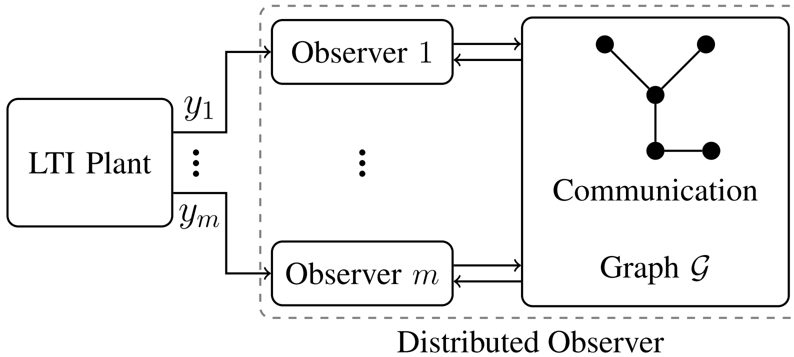


Figure 1.1: A framework for distributed state estimation.

where $x(k) \in \mathbb{R}^n$ and $y(k) = \begin{pmatrix} y_1^T(k) & \dots & y_m^T(k) \end{pmatrix}^T$, with $y_i(k) = C_i x(k) \in \mathbb{R}^{r_i}$, represent the state and output, respectively.

We consider the problem of designing a group of m observers so that each observer can asymptotically resolve the entire state x. Information exchange among observers is constrained by a pre-selected directed graph $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ with $\mathbb{V} = \{1, \cdots, m\}$, where each vertex in $\mathbb{V}$ represents an observer and the edges in $\mathbb{E} \subseteq \mathbb{V} \times \mathbb{V}$ determine the viability and direction of information transfer. We refer to a given $\mathcal{G}$ as the *communication graph* and we denote a group of m observers equipped with $\mathcal{G}$, with each observer accessing an element of $\{y_1, \dots, y_m\}$, as a *distributed observer* (see Figure 1.1 for an illustration).

The internal state of an observer consists of a state estimate $\hat{x}_i$ and an additional state z_i that is updated based on its portion y_i of the output of the plant and the state estimates of the other observers connected to it via the edges of $\mathcal{G}$. We later refer to z_i as the augmented state of observer i. A distributed observer is said to achieve *omniscience asymptotically* if $\lim_{k\to\infty} ||\hat{x}_i(k) - x(k)|| = 0$ holds for all i in $\mathbb{V}$, i.e., the state estimate at every observer converges to the state of the plant.

Our main goals are: **(i)** Given a plant (1.1) and a graph $\mathcal{G}$, we wish to determine necessary and sufficient conditions for the existence of a LTI distributed observer that achieves omniscience asymptotically. **(ii)** Provided it exists, we want to find an omniscience-achieving solution in which the dimension μ_i of z_i, i.e., $z_i \in \mathbb{R}^{\mu_i}$, satisfies the following *scalability condition*:

$$\sum_{i=1}^{m} \mu_i < m \tag{1.2}$$

The main technical challenges are: **(i)** Each observer accesses only a portion of the output of the plant. Hence, unless the pair (A, C_i) is detectable for all i in $\mathbb{V}$, state omniscience requires information exchange among observers. The exception being the trivial case in which the state of the plant can be resolved from the portion of the output available to every observer. **(ii)** Notice that (1.2) rules out simple LTI schemes in which observers share their measurements throughout the network.[2] **(iii)** The existence of an omniscience-achieving scheme that conforms with both $\mathcal{G}$ and (1.2) cannot be established by existing results on observer design.

1.1.1 Summary of the Main Contributions

In order to achieve the stated goals, this work focuses on the following two contributions: **(i)** We propose a parametrized class of LTI distributed observers within which information exchange conforms to a pre-specified directed communication graph $\mathcal{G}$. **(ii)** We find necessary and sufficient conditions for the existence of a parameter choice for the aforementioned class that is omniscience-achieving and satisfies the scalability constraint (1.2). We also outline a method to compute such a parameter choice, provided it exists.

[2] See Section 2.2.1.3 for more details.

In Section 2.4 we provide a detailed analysis that hinges on the fact that asymptotic omniscience for the proposed class of distributed observers can be cast as the stabilization of certain LTI systems via fully decentralized output feedback. Using this analogy, in Theorem 2.2.2 we show that an omniscience-achieving parameter choice satisfying (1.2) exists if and only if the state of the plant (1.1) is detectable from the combined output portions available to each source component[3] of $\mathcal{G}$. We also ascertain that if such a detectability condition holds then there exists an omniscience-achieving solution for which the resulting aggregate dimension of all additional internal (augmented) states satisfies:

$$\sum_{i=1}^{m} \mu_i \leq m - m_s \tag{1.3}$$

where m_s is the number of source components[4] of $\mathcal{G}$. It follows from our analysis that if there is no omniscience-achieving solution in the proposed class satisfying (1.2), then omniscience cannot be attained by any other scheme – including higher order LTI, nonlinear, and time-varying ones – subject to the same graph.

We apply the distributed estimation scheme to tracking of 4 water buffaloes using animal-borne wireless camera network. In Section 2.5, we present experimental results using a data set collected from the deployment of animal-borne wireless camera network in the Gorongosa National Park (Mozambique)

1.2 Optimal Remote State Estimation of Markov Processes

We study a two-block remote state estimation problem: Suppose that a *sensing unit* accesses the full state $\mathbf{x}_k$ of the process at time k, and decides whether to transmit

[3]The definition of the source component is given in Definition 2.2.1.

[4]The number of source components of $\mathcal{G}$ ranges from 1 to m.

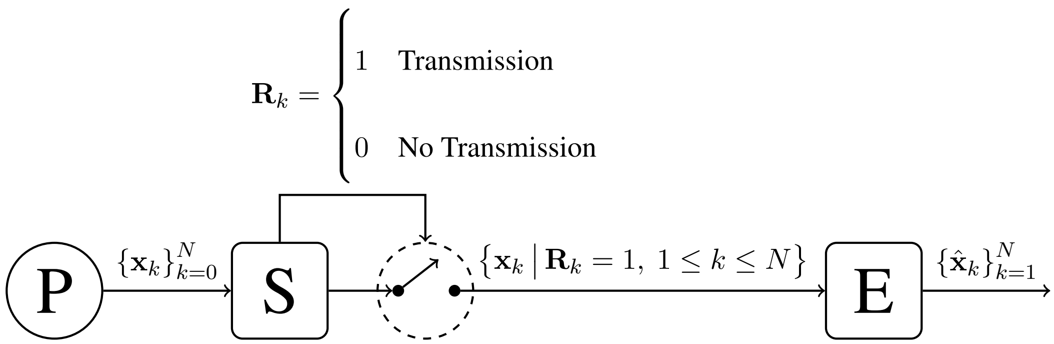


Figure 1.2: A framework for remote state estimation with a Markov process (P), a sensing unit (S), and an estimator (E).

$(\mathbf{R}_k = 1)$ the accessed information $\mathbf{x}_k$ to a *remotely located estimator* or not to transmit $(\mathbf{R}_k = 0)$, where each transmission incurs a positive communication cost c_k. The estimator computes a state estimate $\hat{\mathbf{x}}_k$ that is a causal function of information available to it. The diagram in Figure 1.2 depicts the framework adopted in this work.

Suppose that a transmission policy and an estimation rule at time k, denoted by $\boldsymbol{\mathcal{T}}_k$ and $\mathcal{E}_k$, respectively, dictate the random variable $\mathbf{R}_k$ and estimate $\hat{\mathbf{x}}_k$ as follows:

$$\mathbf{R}_k = \boldsymbol{\mathcal{T}}_k \left(\{\mathbf{x}_j\}_{j=0}^k, \{\mathbf{R}_j\}_{j=1}^{k-1} \right) \tag{1.4a}$$

$$\hat{\mathbf{x}}_k = \mathcal{E}_k \left(\left\{ \mathbf{x}_j \,\middle|\, \mathbf{R}_j = 1,\ 1 \leq j \leq k \right\}, \{\mathbf{R}_j\}_{j=1}^{k} \right) \tag{1.4b}$$

Based on (1.4a) and (1.4b), let us consider a cost functional given as follows:

$$\sum_{k=1}^{N} \mathbb{E} \left[d^2 \left(\mathbf{x}_k, \hat{\mathbf{x}}_k \right) + c_k \cdot \mathbf{R}_k \,\middle|\, \mathbf{x}_0 = x_0, \{\boldsymbol{\mathcal{T}}_k\}_{k=1}^N, \{\mathcal{E}_k\}_{k=1}^N \right] \tag{1.5}$$

subject to a Markov process $\{\mathbf{x}_k\}_{k=0}^N$ defined on a metric space $(\mathbb{X}, d)$. The total cost (1.5) consists of the expectation of squared estimation error $d^2(\mathbf{x}_k, \hat{\mathbf{x}}_k)$ and communication costs $c_k \cdot \mathbf{R}_k$.

Our goal is to find optimal transmission policies $\{\boldsymbol{\mathcal{T}}_k\}_{k=1}^N$ and estimation rules

$\{\mathcal{E}_k\}_{k=1}^N$ for (1.5). To achieve this, we formulate this as a two-player team decision problem, and find optimal decision functions for both players – sensing unit and estimator. To assess optimality of solutions obtained in this work, we adopt the notions of joint optimality and person-by-person optimality: A jointly optimal solution consists of the transmission policies $\{\mathcal{T}_k\}_{k=1}^N$ and estimation rules $\{\mathcal{E}_k\}_{k=1}^N$ that achieve the minimum of (1.5); while a person-by-person optimal solution consists of the transmission policies $\{\mathcal{T}_k\}_{k=1}^N$ and estimation rules $\{\mathcal{E}_k\}_{k=1}^N$ for which given $\{\mathcal{T}_k\}_{k=1}^N$, $\{\mathcal{E}_k\}_{k=1}^N$ minimizes (1.5), and vice versa.

1.2.1 Summary of the Main Contributions

Our main strategy, which is described in Section 3.1, is to divide the aforementioned problem into a set of N sub-problems, and sequentially solve each sub-problem. In Section 3.2, adapting the notions of joint optimality and person-by-person optimality to each sub-problem, we focus on the following contributions for each sub-problem: **(i)** We show that there exists a jointly optimal solution. As jointly optimal solutions are also person-by-person optimal, this result ensures that the set of person-by-person optimal solutions is non-empty. **(ii)** We propose an iterative procedure to compute a person-by-person optimal solution. The procedure, which is inspired from Lloyd's algorithm originally used to compute Centroidal Voronoi Tessellations [1–3], alternates between finding a best response (transmission policy) of the sensing unit to a decision function (estimation rule) of the estimator and vice versa, and it generates a sequence of sub-optimal solutions. Our analysis will show that the sequence has a convergent subsequence, and the limit of any convergent subsequence is a person-by-person optimal solution. In Section 3.3, we describe how to recover an optimal solution to the original problem from optimal solutions of the sub-problems. In Section 3.4, we consider two specific Markov process models – linear system models and self-propelled particle models, and verify that our main results are applicable to these models. Lastly, we apply the remote estimation scheme to tracking of a water buffalo using animal-borne wireless camera network. In Section 3.5, we present experimental results using a data set collected from the deployment of animal-borne wireless camera network in the Gorongosa National Park (Mozambique).

1.3 Evolutionary Game Dynamics and Passivity

Of central interest, in evolutionary game theory [4, 5], is the study of dynamics of strategically interacting players in large populations. This line of study involves an investigation of properties of behavioral rules adopted by players and asymptotes of trajectories induced by the rules in an effort to identify stable equilibria. In this work, we conduct the investigation by adapting the notion of passivity originated from dynamical system theory [6, 7].

Consider multiple populations of players engaged in a game in which each player chooses a strategy from a finite set of strategies, and repeatedly revises its strategy choice in response to given payoffs. Evolutionary dynamics describe such strategy revision processes and determine the time-evolution of the population state – the distribution of strategy choices across populations. Assuming that there are infinitely many players in each population, we express evolutionary dynamics with differential equations and regard these dynamics as dynamical systems. This point of view allows us to define passivity for evolutionary dynamics and to perform stability analysis based on passivity methods adopted from dynamical system theory literature.

The study of evolutionary dynamics and associated stability concepts has been one of active research areas in evolutionary game theory. Brown and von Neumann [8] studied *Brown-von Neumann-Nash (BNN) dynamics* to examine the existence of optimal strategies for a zero-sum two-player game. Taylor and Jonker [9] studied *replicator dynamics* and established a connection between *evolutionarily stable strategies* and stable equilibria of replicator dynamics. Later the result was strengthened by Zeeman [10] who also proposed a stability concept for games under replicator dynamics. Also Gilboa and Matshu [11] considered *cyclic stability* for games under dynamics exhibiting the best response choice.

In succeeding work, rather than working on specific forms of dynamics such as ones considered in [8–11], stability results were established for various classes of evolutionary dynamics. Swinkels [12] considered a class of *myopic adjustment dynamics*, and studied *strategic stability* of equilibria of these dynamics. Ritzberger and Weibull [13] considered a class of *sign-preserving selection dynamics*, and studied asymptotic stability of faces of the population state space. In particular, the authors discovered that a face spanned by a set of pure strategies is stable under sign-preserving selection dynamics if the face is closed under a *better-reply correspondence*.

In a recent development of evolutionary game theory, Hofbauer and Sandholm [14] studied *stable games* and established global stability results for a certain class of evolutionary dynamics, which includes *excess payoff/target (EPT) dynamics*, *pairwise compar-*

ison dynamics, and *perturbed best response (PBR) dynamics*. Fox and Shamma [15] later revealed that stable games and the aforementioned class of evolutionary dynamics exhibit passivity. Based on an input-output property of passive dynamical systems, the authors established $\mathbb{L}_2$-stability of evolutionary dynamics in a class of (generalized) stable games. In addition, applications of evolutionary game theory to study of animal group motion are found in [16, 17], where stable strategy choices in animal pursuit-evasion problems are examined.

Inspired on the passivity analysis in [15], we further investigate passivity in evolutionary game theory. **Our main goals are (i)** to define passivity for evolutionary dynamics that admit realizations in a finite-dimensional state space and present systematic methods to examine passivity of evolutionary dynamics of interest; and **(ii)** to establish stability of passive dynamics in population games.

1.3.1 Summary of the Main Contributions

1. We define three notions of passivity – (ordinary) passivity, strict passivity, and strict output passivity – and explain how passivity defines stored energy of evolutionary dynamics, which will be used to establish stability of the dynamics. We characterize passivity in terms of vector fields that define state-space realizations of evolutionary dynamics. Based on this characterization, we show that the EPT dynamics, pairwise comparison dynamics, and PBR dynamics are passive; while the replicator dynamics are not.

2. We investigate properties of passive evolutionary dynamics in relation to the *Nash*

stationarity (**NS**) condition and *positive correlation* (**PC**) condition[5]. We first show that for passive dynamics satisfying (**NS**), their equilibrium points coincide with the set of states that achieve the lowest level of stored energy of the dynamics. In addition, if the dynamics also satisfy (**PC**) then we show that these dynamics cannot be strictly output passive.

3. We show an equivalence between passivity of evolutionary dynamics and (a weak form of) stability of a closed-loop resulting from a feedback interconnection of evolutionary dynamics and a certain class of payoff operators. This result leads us to re-define passivity of evolutionary dynamics using a class of (generalized) population games. Furthermore, we study the effect of control costs on passivity where we establish a relation between convexity of control costs and passivity of evolutionary dynamics.

4. Based on the above contributions, we present stability results for passive evolutionary dynamics in population games. In particular, we consider a class of games that generalizes stable games [14], and show that in this class of games, stored energy of passive dynamics converges to its lowest level. We provide an interpretation of the convergence of stored energy for formerly established dynamics both analytically and by means of numerical simulations.

[5] See (**NS**) and (**PC**) in Section 4.2.3 for their respective definitions.

1.3.2 Stability Concept and Landscape Metaphor

A landscape metaphor from genetics suggests that each individual in populations would move up toward the peak of *fitness landscape*, and would reside unless external force is applied [18, 19]. This metaphor suggests a reasoning over which the state of populations evolves and a "stable equilibrium" emerges.

In this work, we adopt a concept of stability that is similar to the idea suggested by the landscape metaphor: Stability implies that along the trajectory of the population state, stored energy of evolutionary dynamics converges to its lowest level. The convergence to the lowest energy level would have distinct interpretations which are specific to individual evolutionary dynamics. In some cases, the convergence implies that the population state approaches equilibrium points of dynamics; and hence it establishes asymptotic stability of the equilibrium points. As a case in point, in Section 4.3, we will show that for the BNN dynamics and Smith dynamics, the convergence implies that the population state converges to Nash equilibria; and for the logit dynamics, it implies that the population state converges to the set of best-response strategy distributions.

Chapter 2: Design of Distributed LTI Observers for State Omniscience

2.1 Problem Formulation

2.1.1 Notation

- m is the number of observers that form the distributed observer.
- $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ is a graph[1] formed by a vertex set $\mathbb{V}$ and an edge set $\mathbb{E} \subseteq \mathbb{V} \times \mathbb{V}$.
- I_p is the p-dimensional identity matrix.
- $\mathbf{1}_p$ is the p-dimensional vector with all entries equal to one.
- $\otimes$ represents Kronecker product of matrices.
- For a set $\{K_1, \cdots, K_p\}$ of matrices, we define the following block diagonal matrix:

$$diag\,(K_1, \cdots, K_p) \overset{def}{=} \begin{pmatrix} K_1 & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & K_p \end{pmatrix}$$

- Given a set $\mathbb{V} = \{1, \cdots, |\mathbb{V}|\}$, $W = (w_{ij})_{i,j \in \mathbb{V}}$ is a matrix in $\mathbb{R}^{|\mathbb{V}| \times |\mathbb{V}|}$ whose i, j-th entry is w_{ij}.

[1] For notational convenience, we assume that every vertex of $\mathcal{G}$ has a self-loop, i.e., $(i, i) \in \mathbb{E}$ for all i in $\mathbb{V}$.

- For a set $\mathbb{J} = \{j_1, \cdots, j_p\} \subseteq \{1, \cdots, m\}$ and matrices B and C formed by concatenating conformal submatrices $\{B_i\}_{i=1}^m$ and $\{C_i\}_{i=1}^m$ as $B = \begin{pmatrix} B_1 & \dots & B_m \end{pmatrix}$ and $C = \begin{pmatrix} C_1^T & \dots & C_m^T \end{pmatrix}^T$, respectively, we define

$$B_{\mathbb{J}} \overset{def}{=} \begin{pmatrix} B_{j_1} & \dots & B_{j_p} \end{pmatrix} \text{ and } C_{\mathbb{J}} \overset{def}{=} \begin{pmatrix} C_{j_1}^T & \dots & C_{j_p}^T \end{pmatrix}^T$$

2.1.2 Problem Description

We consider that a LTI plant (1.1) and a directed communication graph $\mathcal{G}$ are given. Without loss of generality, we consider that the dynamic matrix A is nondegenerate (see Appendix A.2) and that the output matrices $\{C_i\}_{i=1}^m$ are nonzero. Each vertex i in $\mathbb{V}$ is associated with an observer that accesses $y_i(k) = C_i x(k)$. We adopt the convention that $\mathbb{E}$ includes edge (j, i) if information can be transmitted from observer j to observer i. The neighborhood of observer i, denoted as $\mathbb{N}_i$, is a subset of $\mathbb{V}$ that contains i and all other vertices with an outgoing edge towards i. Essentially, elements of $\mathbb{N}_i$ represent the observers that can transmit information to observer i.

In this work, we adopt the following parametrized class of distributed observers inspired on [20], where for each i in $\mathbb{V}$, observer i updates its internal state according to the following state-space equation:

$$\begin{aligned} \hat{x}_i(k+1) &= A \sum_{j \in \mathbb{N}_i} \mathbf{w}_{ij} \underbrace{\hat{x}_j(k)}_{\text{state estimate}} + \mathbf{K}_i \underbrace{\left(y_i(k) - C_i \hat{x}_i(k)\right)}_{\text{measurement innovation}} + \mathbf{P}_i \underbrace{z_i(k)}_{\text{augmented state}} \\ z_i(k+1) &= \mathbf{Q}_i \left(y_i(k) - C_i \hat{x}_i(k)\right) + \mathbf{S}_i z_i(k) \end{aligned} \tag{2.1}$$

where $\mathbf{w}_{ij} \in \mathbb{R}$, $\mathbf{K}_i \in \mathbb{R}^{n \times r_i}$, $\mathbf{P}_i \in \mathbb{R}^{n \times \mu_i}$, $\mathbf{Q}_i \in \mathbb{R}^{\mu_i \times r_i}$, $\mathbf{S}_i \in \mathbb{R}^{\mu_i \times \mu_i}$ are the design parameters and μ_i is the dimension of the augmented state z_i.[2] We refer to $\{\mathbf{K}_i, \mathbf{P}_i, \mathbf{Q}_i, \mathbf{S}_i\}_{i \in \mathbb{V}}$

[2]We use bold font to represent the parameters to be designed. The role of the augmented states in (2.1)

as gain matrices and $\mathbf{W} = (\mathbf{w}_{ij})_{i,j\in\mathbb{V}}$ as a weight matrix[3] that must satisfy $\sum_{j\in\mathbb{N}_i} \mathbf{w}_{ij} = 1$ for all i in $\mathbb{V}$. The update scheme (2.1) complies with $\mathcal{G}$ because the estimate $\hat{x}_i$ of observer i only depends on y_i and the estimates $\{\hat{x}_j\}_{j\in\mathbb{N}_i}$ of the observers in its neighborhood $\mathbb{N}_i$.

The following Definition of an omniscience-achieving parameter choice will be used throughout the chapter.

Definition 2.1.1 (**Omniscience-achieving Parameter Choice**). *Consider a LTI plant* (1.1) *with state x and a distributed observer whose state estimates $\{\hat{x}_i\}_{i\in\mathbb{V}}$ are computed according to* (2.1). *A parameter choice $\mathbf{W} = (\mathbf{w}_{ij})_{i,j\in\mathbb{V}}$ and $\{\mathbf{K}_i, \mathbf{P}_i, \mathbf{Q}_i, \mathbf{S}_i\}_{i\in\mathbb{V}}$ for* (2.1) *is referred to as omniscience-achieving if the resulting distributed observer achieves omniscience asymptotically, i.e., $\lim_{k\to\infty} ||\hat{x}_i(k) - x(k)|| = 0$ holds for all i in $\mathbb{V}$.*

The following is the main problem addressed in this work.

Problem 2.1.2. *Given a LTI plant* (1.1) *and a graph $\mathcal{G}$, determine necessary and sufficient conditions for the existence of an omniscience-achieving parameter choice for* (2.1) *that satisfies the scalability condition* (1.2).

2.1.3 Comparative Survey of Related Work

The work in [21, 22] introduced a computationally tractable distributed state estimation scheme for linear plants. The proposed method, so called *Distributed Kalman*

is explained in Section 2.2.1.2.

[3] We assume that $\mathbf{w}_{ij} = 0$ if $j \notin \mathbb{N}_i$ for all i in $\mathbb{V}$.

Filtering (DKF)[4], alternates between an estimation (Kalman filtering) step and a data-fusion step that can be viewed as consensus [24].

Results on the performance and stability of the DKF are presented in [25–28]. In particular, the authors of [25] showed non-convexity of performance optimization for a simple system model, e.g., a first-order LTI plant. In [27, 28], stability properties of the DKF are studied when multiple data-fusion steps are allowed between two consecutive estimation steps.

Subsequent work [29–36] investigates similar estimation schemes which have the structure of an estimation-data fusion alternation as in [21, 22]. In [31], the authors performed a stability analysis in terms of the plant model and underlying communication graph to obtain gain parameters for the estimation step; and in [35], these parameters are obtained via optimization of a quadratic estimation cost. Besides, the data-fusion step is realized using weighted averaging [29], diffusion strategies [30], gossip algorithms [32], and internal model average consensus [33].

Other notable approaches to distributed estimation are proposed in [37–41]. The authors of [37] introduced a design method for the DKF which is based on spatial decomposition of the plant and a distributed algorithm for matrix computation. In [38], a distributed estimation algorithm is proposed for plants that consist of *overlapping* subsystems. In addition, a moving horizon estimation scheme was used in [39], and a H_2/H_∞ optimization framework was adopted in [40, 41] for distributed state estimation.

Moreover, in [42, 43], the authors establish *necessary* conditions for achieving omniscience in distributed state estimation. These conditions specify observability/detectability

[4] An extensive review of the DKF schemes is found in [23].

requirements in terms of the plant model and underlying communication graph.

To achieve asymptotic omniscience, some of the existing schemes require **(i)** strong observability conditions [26, 30, 38], **(ii)** multiple data-fusion steps between two consecutive estimation steps [27,28], which imposes a two-time-scale structure, or **(iii)** imposition of algebraic constraints on the underlying graph [31, 33, 36], which are stronger than what is considered in our work.

In contrast to previous work, we propose a class of distributed observers that operate on a single time-scale, and we find necessary and sufficient conditions for the existence of an omniscience-achieving distributed observer in this class for which the scalability condition (1.2) holds. It will follow from our analysis that if asymptotic omniscience cannot be achieved under the proposed scheme then it is not possible under any other scheme – including higher order LTI, nonlinear, and time-varying ones – subject to the same communication graph.

The use of augmented states as in (2.1) was proposed in [20], where we also provided sufficient conditions for the existence of an omniscience-achieving parameter choice. In [44], we developed necessary and sufficient conditions for the existence of an omniscience-achieving parameter choice for the case where $\mathbf{W}$ is a pre-selected symmetric matrix. This work *extends* and *unifies* our prior results in the following way: we consider *directed communication graphs*, which allow asymmetric $\mathbf{W}$, and we investigate necessary and sufficient conditions for the existence of an omniscience-achieving parameter choice for (2.1). Unlike the methods proposed in [20] and [44], here we also consider the scalability constraint (1.2).

2.2 Main Result

In this section, we present our solution to Problem 2.1.2, and an example that illustrates it. We start by defining the source component of a graph.

Definition 2.2.1. *Given a directed graph $\mathcal{G} = (\mathbb{V}, \mathbb{E})$, a strongly connected component $\left(\mathbb{V}^C, \mathbb{E}^C\right)$ of $\mathcal{G}$ is said to be a source component*[5] *if there is no edge from $\mathbb{V} \setminus \mathbb{V}^C$ to $\mathbb{V}^C$. Also we define a set of source component representatives*[6] *as a subset $\mathbb{V}^R$ of $\mathbb{V}$ that contains exactly one element (representative) from each source component of $\mathcal{G}$.*

The following is our main Theorem.

Theorem 2.2.2. *Suppose that the plant is given as in* (1.1)*, that the communication graph $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ is pre-selected, and that the following hold:*

(i) *There are m_s source components*[7] *of $\mathcal{G}$ which are represented as $\{(\mathbb{V}_l, \mathbb{E}_l)\}_{l=1}^{m_s}$. Each source component $\mathcal{G}_l = (\mathbb{V}_l, \mathbb{E}_l)$ is associated with a subsystem of the plant described by the pair $(A, C_{\mathbb{V}_l})$.*

(ii) *Let $\mathbb{V}^R$ be a set of source component representatives. For each i in $\mathbb{V}^R$, we define ν_i to be the order (number of vertices) of the source component to which vertex i belongs.*

[5] We adopt the convention that if the graph $\mathcal{G}$ is strongly connected then $\mathcal{G}$ itself is a (unique) source component.

[6] The choice of $\mathbb{V}^R$ may not be unique.

[7] According to Definition 2.2.1, every graph has at least one source component, i.e., $m_s \geq 1$.

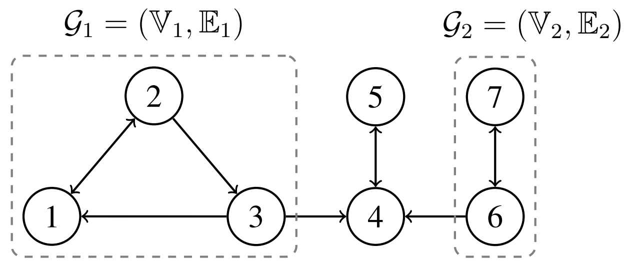


Figure 2.1: A communication graph $\mathcal{G}$ for Example 2.2.3.

There is an omniscience-achieving parameter choice for (2.1) *that satisfies (1.2) if and only if all the subsystems* $\{(A, C_{\mathbb{V}_l})\}_{l=1}^{m_s}$ *are detectable. In particular, if such a detectability condition holds then there is a parameter choice for which* μ_i *is given by*

$$\mu_i = \begin{cases} \nu_i - 1 & \textit{if } i \in \mathbb{V}^R \\ 0 & \textit{otherwise} \end{cases} \tag{2.2}$$

for all i *in* $\mathbb{V}$.

The proof is given in Section 2.4. When the conditions of the Theorem are verified, the method outlined in Appendix A.1.2 can be used to compute omniscience-achieving parameters for which (2.2) is satisfied. Notice that because $\sum_{i \in \mathbb{V}^R} \nu_i \leq m$ holds, we can conclude that μ_i given by (2.2) satisfies (1.2). In fact, since $\mathbb{V}^R$ has m_s elements, it also follows that (1.3) holds.

Example 2.2.3. *Consider the communication graph* $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ *depicted in Figure 2.1 and a LTI plant* (1.1) *with* $m = 7$. *From Definition 2.2.1, we conclude that* $\mathcal{G}_1$ *and* $\mathcal{G}_2$ *are the source components of* $\mathcal{G}$, *and we select* $\mathbb{V}^R = \{1, 6\}$. *From Theorem 2.2.2, we conclude that there exists an omniscience-achieving parameter choice for which* μ_i *is* *given by*

$$\mu_i = \begin{cases} 2 & \textit{if } i = 1 \\ 1 & \textit{if } i = 6 \\ 0 & \textit{otherwise} \end{cases}$$

if and only if $(A, C_{\mathbb{V}_1})$ *and* $(A, C_{\mathbb{V}_2})$ *are both detectable.* □

2.2.1 Additional Remarks on the Proposed Class of Distributed Observers

2.2.1.1 The Effect of Noise on the Estimation Performance

Although our formulation focuses on the noiseless case, the fact that the plant and the distributed observer are LTI guarantees graceful degradation with respect to noise in the communication links and/or measurements. In particular, if the noise amplitude is bounded by β then the limit $\max_{i \in \mathbb{V}} \lim_{k \to \infty} \|\hat{x}_i(k) - x(k)\|$ may be positive, but one can find an upper bound that scales linearly with β. Also, the effect of noise can be quantified using classical frequency-domain methods.

2.2.1.2 The Role of the Augmented States

As will be discussed in Section 2.4.3, asymptotic omniscience for the proposed class of distributed observers can be cast as the stabilization of certain LTI systems via fully decentralized output feedback. The augmented states in (2.1) are directly related with the internal dynamics of such a decentralized controller which gives us additional freedom in designing the way local state estimates and measurements are fused.

2.2.1.3 Complexity of the Proposed Scheme

We evaluate the complexity of the proposed scheme in terms of the dimensions of the augmented states required to achieve asymptotic omniscience.

For the sake of argument, we compare our method with a simple *relay-based centralized scheme* described as follows: Suppose that under the same configuration as in Figure 1.1, every observer would transmit its local measurement to its neighbors and, at the same time, would relay local measurements received from neighboring observers in which each transmission/relay incurs a unit time delay. Under this setting, the fixed-lag smoothing scheme [45] can be adopted at each observer to determine its update rule for state estimation. Similar to our scheme, the internal state of each observer in the centralized scheme consists of a state estimate and an augmented state to account for the time delay in transmission/relay. However, in what regards to achieving asymptotic omniscience, this centralized scheme would require augmented states whose dimensions would be much larger than our scheme. To see this, we note that in the centralized scheme, the dimension of the augmented state of each observer i is equal to the product of the order of the plant and the maximum length among the respective *shortest paths* from the other vertices to vertex i in the graph $\mathcal{G}$. In contrast, as stated in Theorem 2.2.2, in our scheme only one observer per source component needs an augmented state, whose dimension equals the order of the source component minus one. As a case in point, suppose that $\mathcal{G}$ is a directed ring, and let n and m be the orders of the plant and graph $\mathcal{G}$, respectively. Then, for the centralized scheme, the aggregate dimension of all augmented states could be as large as $n \cdot m \cdot (m-1)$; whereas, for the scheme we propose it is no larger than $m-1$.

2.3 Application to the Synchronization of Coupled Multi-agent Systems

Given a graph $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ with $\mathbb{V} = \{1, \cdots, m\}$ and a set $\mathbb{V}^I = \{1, \cdots, m_a\}$, consider a LTI multi-agent system whose state-space representation is given as follows:

$$\chi^{(i)}(k+1) = F_o\chi^{(i)}(k) + \sum_{j \in \mathbb{V}^I \setminus \{i\}} F_{ij}\left(\chi^{(j)}(k) - \chi^{(i)}(k)\right) + \sum_{j \in \mathbb{V}} G_{ij}u_j(k),\ i \in \mathbb{V}^I \tag{2.3a}$$

$$y_i(k) = H_i \begin{pmatrix} \chi^{(1)}(k) \\ \vdots \\ \chi^{(m_a)}(k) \end{pmatrix},\ i \in \mathbb{V} \tag{2.3b}$$

For each i in $\mathbb{V}^I$, $\chi^{(i)}(k)$ takes a value in $\mathbb{R}^n$ and represents the state of agent i. For each i in $\mathbb{V}$, $y_i(k)$ and $u_i(k)$ take values in $\mathbb{R}^{r_i}$ and $\mathbb{R}^{p_i}$, and represent the output and control input of the system (2.3) associated with vertex i of $\mathcal{G}$, respectively.

For each i, j in $\mathbb{V}^I$, the matrix F_{ij} in (2.3a) quantifies the coupling between the state $\chi^{(i)}$ of agent i and the state $\chi^{(j)}$ of agent j. For notational convenience, $\mathcal{G}^I = \left(\mathbb{V}^I, \mathbb{E}^I\right)$ represents the coupling among the states of agents in which, for each i in $\mathbb{V}^I$ and j in $\mathbb{V}^I \setminus \{i\}$, an edge (j, i) belongs $\mathbb{E}^I$ if and only if $F_{ij} \neq \mathbf{0}$ holds. We refer to $\mathcal{G}^I$ as the *interaction graph* of the multi-agent system (2.3). We remark that if all the agents are *synchronized* at time k_0 [8], i.e., $\chi^{(1)}(k_0) = \cdots = \chi^{(m_a)}(k_0)$, then they remain synchronized and the state of each agent i satisfies

$$\chi^{(i)}(k+1) = F_o\chi^{(i)}(k)$$

[8]In this case, we may assume that $u_i(k) = 0,\ \forall i \in \mathbb{V}$ and $k \geq k_0$, since there is no need to control synchronized agents.

for $k \geq k_0$. The agent model (2.3) is called *homogeneous* because the agents have the same internal dynamics as specified by the dynamic matrix F_o.

The goal is to design a set of controllers for which the agents of the system (2.3) are asymptotically synchronized, i.e., $\lim_{k\to\infty} \left\| \chi^{(i)}(k) - \chi^{(1)}(k) \right\| = 0$ holds for all i in $\mathbb{V}^I \setminus \{1\}$. In particular, we suppose that each controller i is represented by vertex i in $\mathbb{V}$ and has the following state-space representation:

$$\begin{aligned} \xi_i(k+1) &= \sum_{j\in\mathbb{N}_i} \mathbf{S}^c_{ij}\xi_j(k) + \mathbf{Q}^c_i y_i(k) \\ u_i(k) &= \sum_{j\in\mathbb{N}_i} \mathbf{P}^c_{ij}\xi_j(k) + \mathbf{K}^c_i y_i(k) \end{aligned} \tag{2.4}$$

where ξ_i is the internal state of controller i, and $\mathbb{N}_i$ is the neighborhood of controller i defined by $\mathcal{G}$, which represents the controllers that can transmit information to controller i. We refer to a set of controllers equipped with $\mathcal{G}$ as a *distributed controller*. The diagram in Figure 2.2 depicts the overall system considered here.

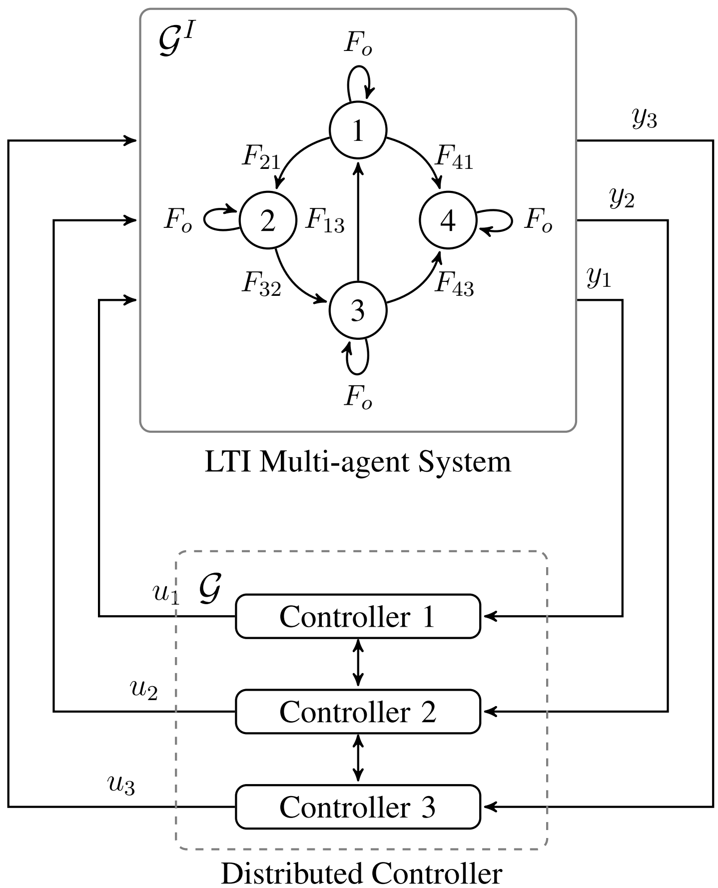


Figure 2.2: A diagram showing an example of an overall closed-loop system that consists of a LTI multi-agent system and distributed controller. See Example 2.3.3 for a numerical implementation of the closed-loop system.

We summarize the problem as follows.

Problem 2.3.1. *Given a graph* $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ *and a LTI multi-agent system as in* (2.3)*, we want to*

(i) *determine parameters* $\left\{\mathbf{K}_i^c, \mathbf{P}_{ij}^c, \mathbf{Q}_i^c, \mathbf{S}_{ij}^c\right\}_{i,j\in\mathbb{V}}$ *for* (2.4) *such that the interconnection of* (2.3) *and* (2.4) *results in the asymptotic synchronization of the system* (2.3)*, i.e.,*

$$\lim_{k\to\infty} \left\|\chi^{(i)}(k) - \chi^{(1)}(k)\right\| = 0$$

for all i *in* $\mathbb{V}^I \setminus \{1\}$*, and*

(ii) *show that the state of each agent converges to a solution of* $\chi_o(k+1) = F_o\chi_o(k)$ *for some initial value* $\chi_o(0) \in \mathbb{R}^n$*, i.e.,*

$$\lim_{k\to\infty} \left\|\chi^{(i)}(k) - \chi_o(k)\right\| = 0$$

for all i *in* $\mathbb{V}^I$*.*

The literature on the problem of designing distributed controllers for synchronization of multi-agent systems is vast (see, for instance, [46–49] and references therein). To mention a few, the work of [50] considered synchronization of linearly coupled nonlinear agents, and the authors of [51] formulate synchronization as mixed-integer nonlinear optimization. Also, there is recent work [52–59] that focused on studying synchronization problems with LTI multi-agent models. The framework in these articles assumes that the states of agents are completely decoupled, and each agent has an associated controller that accesses its full state and has full control of it.

Here, we consider a LTI multi-agent system in which **(i)** agents are interacting according to (2.3a), **(ii)** for each j in $\mathbb{V}$, the j-th control input $u_j(k)$ affects the state of the system according to $\{G_{ij}\}_{i\in\mathbb{V}^I}$, and **(iii)** for each i in $\mathbb{V}$, the i-th output vector $y_i(k)$ depends on the state of the system according to H_i. Due to **(i)-(iii)**, the formulation considered in Problem 2.3.1 may not be cast as one to which existing results for completely decoupled multi-agent models can be applied. More specifically, suppose that each agent has an associated controller that accesses its full state and has full control of it. To transform the agent model (2.3a) into a completely decoupled one, each controller needs to access the states of the agents on which the state of its associated agent depends, and generate control to cancel the coupling. However, this may not be possible since the interaction graph $\mathcal{G}^I$ and graph $\mathcal{G}$, whose edges determine the viability and direction of information transfer among controllers, may not be identical as depicted in Figure 2.2.

The proposed scheme can be applied to frequency synchronization in power grids [60–62], which ensures stable operation of grids and efficient power transfer from generators to loads.

Our solution to Problem 2.3.1 is given as follows.

Proposition 2.3.2. *Suppose that a graph $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ is pre-selected, that a LTI multi-agent system is given as in* (2.3)*, and that $\mathcal{G}$ and the matrices in* (2.5) *satisfy the following:*

(i) *The pair (A', B') is stabilizable, where A' and B' are defined in* (2.5b) *and* (2.5c)*, respectively.*

(ii) *The graph $\mathcal{G}$ and the pair (A, C) satisfy the detectability condition of Theorem 2.2.2, where A and C are defined in* (2.5a) *and* (2.5d)*, respectively.*

There exists a distributed controller (2.4) *that asymptotically synchronizes the system* (2.3)*, i.e.,*

$$\lim_{k \to \infty} \left\| \chi^{(i)}(k) - \chi^{(1)}(k) \right\| = 0$$

for all i in $\mathbb{V}^I \setminus \{1\}$. Furthermore, if all eigenvalues of F_o lie on or inside the unit circle in $\mathbb{C}$, then the state of each agent converges to a solution of

$$\chi_o(k+1) = F_o \chi_o(k)$$

for some initial value $\chi_o(0) \in \mathbb{R}^n$, i.e.,

$$\lim_{k \to \infty} \left\| \chi^{(i)}(k) - \chi_o(k) \right\| = 0$$

for all i in $\mathbb{V}^I$.

$$A = \left(\begin{array}{c|ccc} F_o & F_{12} & \cdots & F_{1m_a} \\ \hline \mathbf{0} & & A' & \end{array}\right) \tag{2.5a}$$

with $A' = \begin{pmatrix} A'_{1,1} & \cdots & A'_{1,m_a-1} \\ \vdots & \ddots & \vdots \\ A'_{m_a-1,1} & \cdots & A'_{m_a-1,m_a-1} \end{pmatrix}$

$$\text{and } A'_{i-1,j-1} = \begin{cases} F_{ij} - F_{1j} & \text{if } i \neq j \\ F_o - \sum_{l \in \mathbb{V}^I \setminus \{i\}} F_{il} - F_{1i} & \text{if } i = j \end{cases}, \tag{2.5b}$$

$$B' = \begin{pmatrix} B'_1 & \ldots & B'_m \end{pmatrix} \text{ with } B'_i = \begin{pmatrix} (G_{2i} - G_{1i})^T & \cdots & (G_{m_a i} - G_{1i})^T \end{pmatrix}^T, \tag{2.5c}$$

$$C = \begin{pmatrix} C_1^T & \ldots & C_m^T \end{pmatrix}^T \text{ with } C_i = H_i \left(\begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{1}_{m_a-1} & I_{m_a-1} \end{pmatrix} \otimes I_n \right) \tag{2.5d}$$

A constructive proof of Proposition 2.3.2 is given in Appendix A.3, where we use Theorem 2.2.2 to show the existence of $\left\{\mathbf{K}_i^c, \mathbf{P}_{ij}^c, \mathbf{Q}_i^c, \mathbf{S}_{ij}^c\right\}_{i,j\in\mathbb{V}}$ for (2.4) for which the interconnection of (2.3) and (2.4) results in the asymptotic synchronization of the system (2.3).

Example 2.3.3 (Numerical Example)**.** *Consider a multi-agent system* (2.3) *and the graph* $\mathcal{G}$ *depicted in Figure 2.2, where the matrices in* (2.3) *are numerically specified as follows:*

$$F_o = \begin{pmatrix} 0.9950 & 0.0998 \\ -0.0998 & 0.9950 \end{pmatrix}$$

$$F_{ij} = \begin{cases} -0.1 I_2 & \textit{if } (i,j) \in \{(1,3),(2,1),(3,2),(4,1),(4,3)\} \\ \mathbf{0} & \textit{otherwise} \end{cases}$$

$$\begin{pmatrix} G_{11} & G_{12} & G_{13} \end{pmatrix} = \left(\begin{array}{c|c|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{pmatrix} G_{21} & G_{22} & G_{23} \end{pmatrix} = \left(\begin{array}{c|c|cc} -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{pmatrix} G_{31} & G_{32} & G_{33} \end{pmatrix} = \left(\begin{array}{c|c|cc} 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{pmatrix} G_{41} & G_{42} & G_{43} \end{pmatrix} = \left(\begin{array}{c|c|cc} 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$(H_{11}\ H_{12}\ H_{13}\ H_{14}) = \left(\begin{array}{cc|cc|cc|cc} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$
$$(H_{21}\ H_{22}\ H_{23}\ H_{24}) = \left(\begin{array}{cc|cc|cc|cc} 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \end{array}\right)$$
$$(H_{31}\ H_{32}\ H_{33}\ H_{34}) = \left(\begin{array}{cc|cc|cc|cc} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{array}\right)$$

*The assumptions **(i)** and **(ii)** of Proposition 2.3.2 are satisfied; hence, the existence of a distributed controller* (2.4) *that synchronizes the system* (2.3) *is guaranteed. We compute a parameter choice for* (2.4) *according to the procedure described in Appendix A.3.1 and Appendix A.3.2. The state trajectories of the resulting closed-loop system are depicted in Figure 2.3.* □

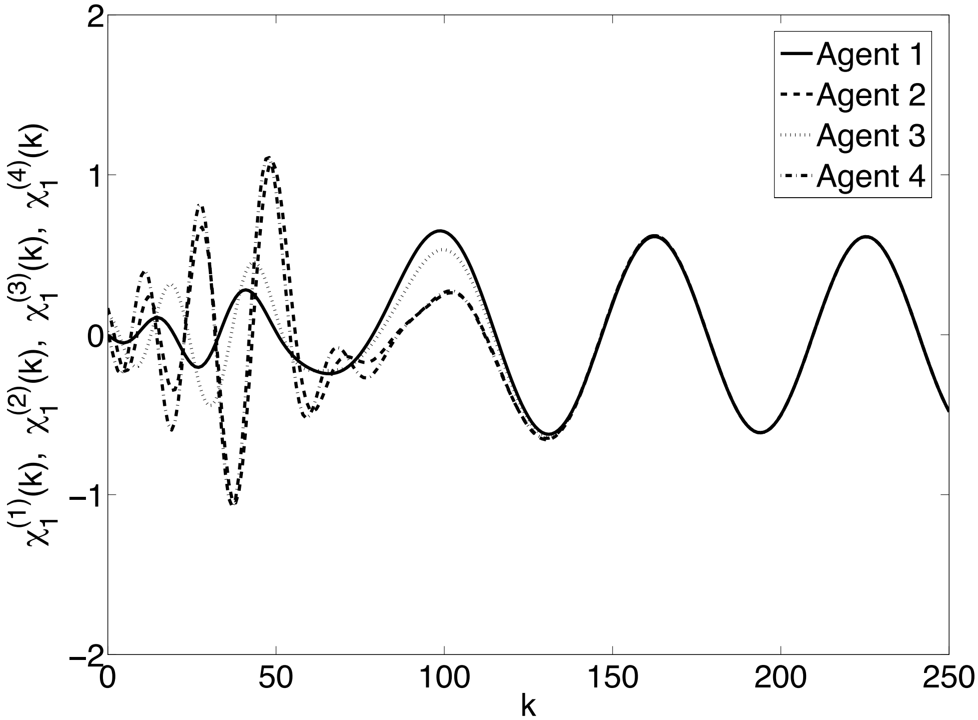


Figure 2.3: A simulation result of Example 2.3.3 which depicts the synchronization of the first components $\chi_1^{(1)}(k), \chi_1^{(2)}(k), \chi_1^{(3)}(k), \chi_1^{(4)}(k)$ of the states of agent $1, 2, 3, 4$.

Remark 2.3.4. *Since our results can be applied to any interaction graph $\mathcal{G}^I$, the assumption **(ii)** of Proposition 2.3.2 may be stronger than what would be needed for the cases in which the agents are completely decoupled (cf. Assumption 1 in [58]). As a case in point, consider a system configuration with the same number of agents and controllers and for which the agents are all decoupled, i.e., $\mathbb{V}^I = \mathbb{V}$ and $\mathbb{E}^I = \bigcup_{i \in \mathbb{V}^I}(i,i)$. In addition, assume that the input and output matrices of* (2.3)*, respectively, satisfy $G_{ij} = \mathbf{0}$ if $i \neq j$ for all i in $\mathbb{V}^I$ and j in $\mathbb{V}$, and $H_i = e_i^T \otimes H_i'$ for all i in $\mathbb{V}$ and for a matrix H_i' in $\mathbb{R}^{r_i \times n}$, where e_i is the i-th column of the m_a-dimensional identity matrix. Under this setting, **(ii)** of Proposition 2.3.2 requires the graph $\mathcal{G}$ to be strongly connected, while Assumption 1 in [58] only requires $\mathcal{G}$ to have a directed spanning tree.*

2.4 Proof of the Main Theorem

In this section, we provide a proof for Theorem 2.2.2. If the conditions of the Theorem hold then Appendix A.1.2 outlines a randomized method to obtain a choice of omniscience-achieving parameters.

The proof has two parts: It starts with Lemma 2.4.3, Lemma 2.4.4, and Theorem 2.4.5 that describe important spectral properties of a parametrized class of weight matrices $\mathbf{W}$. The second part, which consists of Proposition 2.4.8, Theorem 2.4.10, and Remark 2.4.11, determines conditions for a parameter choice $\mathbf{W}$ and $\{\mathbf{K}_i, \mathbf{P}_i, \mathbf{Q}_i, \mathbf{S}_i\}_{i \in \mathbb{V}}$ to be omniscience-achieving. The structure of the proof is outlined in the diagram of Figure 2.4.

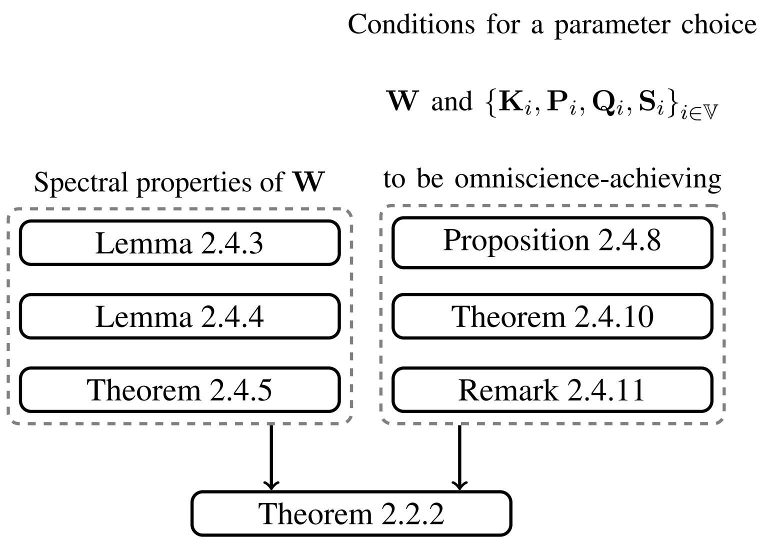


Figure 2.4: A precedence diagram for the proof of Theorem 2.2.2

2.4.1 Key Results on Weighted Laplacian Matrices

Definition 2.4.1. *Consider a graph $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ with $\mathbb{V} = \{1, \cdots, |\mathbb{V}|\}$. A matrix $L = (l_{ij})_{i,j\in\mathbb{V}}$ is said to be a Weighted Laplacian Matrix (WLM) of $\mathcal{G}$ if the following three conditions hold:*

(i) *If $(j, i) \notin \mathbb{E}$ then $l_{ij} = 0$ for all i in $\mathbb{V}$ and j in $\mathbb{V} \setminus \{i\}$.*

(ii) *If $(j, i) \in \mathbb{E}$ then $l_{ij} < 0$ for all i in $\mathbb{V}$ and j in $\mathbb{V} \setminus \{i\}$.*

(iii) *It holds that $\sum_{j=1}^{|\mathbb{V}|} l_{ij} = 0$ for all i in $\mathbb{V}$.*

For notational convenience, we define the set of WLMs of $\mathcal{G}$ as follows:

$$\mathbb{L}(\mathcal{G}) \stackrel{def}{=} \left\{ L \in \mathbb{R}^{|\mathbb{V}|\times|\mathbb{V}|} \,\middle|\, L \text{ is a WLM of } \mathcal{G} \right\}$$

Definition 2.4.2 (UEPP). *Given square matrices A and B, $A \otimes B$ is said to have the so called Unique Eigenvalue Product Property (UEPP) if every nonzero eigenvalue λ of*

$A \otimes B$ can be uniquely expressed[9] *as a product $\lambda = \lambda_A \cdot \lambda_B$, where λ_A and λ_B are eigenvalues of A and B, respectively.*

Lemma 2.4.3. *Let $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ with $\mathbb{V} = \{1, \cdots, |\mathbb{V}|\}$ be a directed graph, and A be a matrix in $\mathbb{R}^{n \times n}$. Suppose that a matrix $\mathbf{W}$ in $\mathbb{R}^{|\mathbb{V}| \times |\mathbb{V}|}$ is defined as $\mathbf{W} = I_{|\mathbb{V}|} - \alpha L$, where α is a positive real number and $L = (l_{ij})_{i,j \in \mathbb{V}}$ is a WLM of $\mathcal{G}$. Given L and α' satisfying $0 < \alpha' \leq \left(\max_{1 \leq i \leq |\mathbb{V}|} l_{ii}\right)^{-1}$* [10]*, for almost every α in $(0, \alpha')$, $\mathbf{W}$ is a stochastic matrix and $\mathbf{W} \otimes A$ satisfies the UEPP.*

Lemma 2.4.4. *Let a matrix $\mathbf{W}$ in $\mathbb{R}^{m \times m}$ and a matrix A in $\mathbb{R}^{n \times n}$ be given. If all eigenvalues of $\mathbf{W}$ are simple*[11] *and $\mathbf{W} \otimes A$ satisfies the UEPP, then each eigenvector q of $\mathbf{W} \otimes A$ associated with a nonzero eigenvalue λ can be written as a Kronecker product $q = v \otimes p$, where v and p are, respectively, eigenvectors of $\mathbf{W}$ and A (associated with the eigenvalues $\lambda_{\mathbf{W}}$ and λ_A for which $\lambda = \lambda_{\mathbf{W}} \cdot \lambda_A$ holds).*

The proofs of Lemmas 2.4.3 and 2.4.4 are given in Appendix A.4.

Theorem 2.4.5. *Let $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ be a strongly connected graph. For almost every element L of the set $\mathbb{L}(\mathcal{G})$, the following hold:*

(i) *Every right and left eigenvectors of L have no zero entries.*

(ii) *Every eigenvalue of L is simple.*

[9] For an eigenvalue λ of $A \otimes B$, let λ_A, λ'_A be the eigenvalues of A and λ_B, λ'_B be the eigenvalues of B for which $\lambda = \lambda_A \cdot \lambda_B = \lambda'_A \cdot \lambda'_B$ holds. The eigenvalue λ is said to be uniquely expressed as a product $\lambda = \lambda_A \cdot \lambda_B$ if it holds that $\lambda_A = \lambda'_A$ and $\lambda_B = \lambda'_B$.

[10] If $l_{ii} = 0$ for all i in $\mathbb{V}$, then we consider that $\left(\max_{1 \leq i \leq |\mathbb{V}|} l_{ii}\right)^{-1} = \infty$.

[11] An eigenvalue of a matrix is simple if both the geometric and algebraic multiplicities of the eigenvalue are equal to 1.

Since Theorem 2.4.5 hinges on structured linear system theory, in Appendix A.5 we provide a review of key concepts followed by a proof.

2.4.2 A Brief Introduction to Stabilization via Decentralized Control

We review certain classical results in decentralized control that will be used in the proof of Theorem 2.2.2. Of special relevance are the fundamental work of [63–66] that investigates the notion of fixed modes[12] for LTI plants, and the work of [68] that studies the effect of decentralized output feedback on LTI plants. To introduce these results, we consider the following state-space representation for a LTI plant:

$$\begin{aligned} \widetilde{x}(k+1) &= \widetilde{A}\widetilde{x}(k) + \sum_{i=1}^{|\mathbb{V}|} \widetilde{B}_i \widetilde{u}_i(k) \\ \widetilde{y}_i(k) &= \widetilde{C}_i \widetilde{x}(k) \end{aligned} \tag{2.6}$$

for each i in $\mathbb{V} = \{1, \cdots, |\mathbb{V}|\}$, where $\widetilde{x}(k) \in \mathbb{R}^{\widetilde{n}}$, $\widetilde{u}_i(k) \in \mathbb{R}^{\widetilde{p}_i}$, and $\widetilde{y}_i(k) \in \mathbb{R}^{\widetilde{r}_i}$ are the state, i-th control input, and i-th output, respectively.

Definition 2.4.6. *[63, 64] A given $\lambda \in \mathbb{C}$ is a fixed mode of* (2.6) *if it is an eigenvalue of $\widetilde{A} + \sum_{i=1}^{|\mathbb{V}|} \widetilde{B}_i K_i \widetilde{C}_i$ for all K_i in $\mathbb{R}^{\widetilde{p}_i \times \widetilde{r}_i}$.*

Remark 2.4.7. *The fixed mode is an eigenvalue of the plant* (2.6) *which is invariant under the decentralized output feedback $\widetilde{u}_i(k) = K_i \widetilde{y}_i(k)$ for all i in $\mathbb{V}$, where K_i is a matrix in $\mathbb{R}^{\widetilde{p}_i \times \widetilde{r}_i}$. In addition, if the plant* (2.6) *has an unstable fixed mode then it cannot be stabilized by any decentralized controller that is causal and LTI (see [63, 64] for the details).*

[12] The notion of fixed modes is analogous to the concept of uncontrollable or unobservable modes adopted in centralized control problems [67].

The fixed modes can be characterized by an algebraic rank test described in the following Proposition.

Proposition 2.4.8. *[65, 66] Consider that a LTI plant is given as in* (2.6)*. Let*

$$\widetilde{B} = \begin{pmatrix} \widetilde{B}_1 & \dots & \widetilde{B}_{|\mathbb{V}|} \end{pmatrix} \textit{ and } \widetilde{C} = \begin{pmatrix} \widetilde{C}_1^T & \dots & \widetilde{C}_{|\mathbb{V}|}^T \end{pmatrix}^T$$

A given $\lambda \in \mathbb{C}$ is a fixed mode of the plant if and only if there exists a subset $\mathbb{J} \subseteq \mathbb{V}$ for which

$$rank \begin{pmatrix} \widetilde{A} - \lambda \cdot I_{\widetilde{n}} & \widetilde{B}_{\mathbb{J}} \\ \widetilde{C}_{\mathbb{J}^c} & \mathbf{0} \end{pmatrix} < \widetilde{n} \tag{2.7}$$

holds, where $\widetilde{n}$ is the dimension of $\widetilde{A}$, and $\mathbb{J}^c = \mathbb{V} \setminus \mathbb{J}$.

Definition 2.4.9. *Let $\left(\mathbb{V}, \mathbb{E}^P\right)$ be a graph of a LTI plant* (2.6) *in which the edge set $\mathbb{E}^P$ satisfies $(j,i) \in \mathbb{E}^P$ if and only if $\widetilde{C}_i \left(z \cdot I_{\widetilde{n}} - \widetilde{A}\right)^{-1} \widetilde{B}_j$ is nonzero. The plant* (2.6) *is said to be strongly connected if its graph $\left(\mathbb{V}, \mathbb{E}^P\right)$ is strongly connected.*

In the following Theorem, based on Theorem 4 of [68], we specify the effect of decentralized output feedback of the following form on a strongly connected LTI plant.

$$\begin{aligned} z_1(k+1) &= S_1 z_1(k) + Q_1 \widetilde{y}_1(k) \\ \widetilde{u}_1(k) &= P_1 z_1(k) + K_1 \widetilde{y}_1(k) \end{aligned} \tag{2.8a}$$

$$\widetilde{u}_i(k) = K_i \widetilde{y}_i(k), \; i \in \mathbb{V} \setminus \{1\} \tag{2.8b}$$

where $z_1(k)$ takes a value in $\mathbb{R}^{\mu_1}$ for a nonnegative integer μ_1.

Theorem 2.4.10. *Consider a LTI plant given as in* (2.6) *and decentralized output feedback* (2.8)*. Suppose that the plant is strongly connected and has no unstable fixed mode.*

Then, for almost every choice of $\{K_i\}_{i\in\mathbb{V}\setminus\{1\}}$*, there exists a choice of* $K_1,\ P_1,\ Q_1,\ S_1$ *for which the closed-loop system obtained from the interconnection of* (2.6) *and* (2.8) *described by*

$$\begin{pmatrix} \widetilde{x}(k+1) \\ z_1(k+1) \end{pmatrix} = \begin{pmatrix} \widetilde{A} + \sum_{i=1}^{|\mathbb{V}|} \widetilde{B}_i K_i \widetilde{C}_i & \widetilde{B}_1 P_1 \\ Q_1 \widetilde{C}_1 & S_1 \end{pmatrix} \begin{pmatrix} \widetilde{x}(k) \\ z_1(k) \end{pmatrix} \tag{2.9}$$

is stable.

Remark 2.4.11. *The system* (2.9) *also can be viewed as the closed-loop system obtained by applying a (centralized) controller described by* (2.8a) *to a LTI system described by the triple*

$$\left(\widetilde{A} + \sum_{i=2}^{|\mathbb{V}|} \widetilde{B}_i K_i \widetilde{C}_i,\ \widetilde{B}_1,\ \widetilde{C}_1 \right) \tag{2.10}$$

We can find a parameter choice $K_1,\ P_1,\ Q_1,\ S_1$ *for which* (2.9) *is stable using results on finding stabilizing (centralized) controllers for LTI systems. In particular, by adopting the result of [69], we can find a stabilizing controller* (2.8a) *of order* μ_1 *equal to the controllability index of* (2.10) *minus one.*

2.4.3 Additional Preliminary Results

Let m_s be the number of source components of $\mathcal{G}$ in which we denote each source component as $\mathcal{G}_l = (\mathbb{V}_l, \mathbb{E}_l)$, and $\mathbb{V}^R$ be the set of source component representatives (see Definition 2.2.1). Notice that the source components of $\mathcal{G}$ impose the following structure on $\mathbf{W}$:

$$\mathbf{W} = \begin{pmatrix} \mathbf{W}_1 & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \cdots & \mathbf{W}_{m_s} & \mathbf{0} \\ \mathbf{W}_{m_s+1,1} & \cdots & \mathbf{W}_{m_s+1,m_s} & \mathbf{W}_{m_s+1,m_s+1} \end{pmatrix} \tag{2.11}$$

For each l in $\{1, \cdots, m_s\}$, the sparsity pattern of $\mathbf{W}_l \in \mathbb{R}^{|\mathbb{V}_l| \times |\mathbb{V}_l|}$ must be consistent[13] with $\mathcal{G}_l$ so that under a suitable choice of $\{\mathbf{W}_{m_s+1,l}\}_{l=1}^{m_s+1}$, the sparsity pattern of $\mathbf{W}$ given above is consistent with $\mathcal{G}$.

For notational convenience, we consider that $\mathbb{V}_l = \{1, \cdots, |\mathbb{V}_l|\}$ and $\mathbb{V}_l \cap \mathbb{V}^R = \{1\}$. To analyze the asymptotic omniscience of the proposed estimation scheme, under the parameter choice of $\mathbf{W}$ and $\{\mathbf{K}_i, \mathbf{P}_i, \mathbf{Q}_i, \mathbf{S}_i\}_{i \in \mathbb{V}}$ described in Appendix A.1.2, we derive the state-space representation for the dynamics for the estimation error of (2.1) associated with $\mathcal{G}_l$ as follows:

$$\begin{pmatrix} \widetilde{x}(k+1) \\ z_1(k+1) \end{pmatrix} = \begin{pmatrix} \mathbf{W}_l \otimes A - \sum_{i=1}^{|\mathbb{V}_l|} \overline{B}_i \mathbf{K}_i \overline{C}_i & -\overline{B}_1 \mathbf{P}_1 \\ \mathbf{Q}_1 \overline{C}_1 & \mathbf{S}_1 \end{pmatrix} \begin{pmatrix} \widetilde{x}(k) \\ z_1(k) \end{pmatrix} \tag{2.12}$$

where $\widetilde{x} = \begin{pmatrix} \widetilde{x}_1^T & \dots & \widetilde{x}_{|\mathbb{V}_l|}^T \end{pmatrix}^T$ with $\widetilde{x}_i = x - \hat{x}_i$, and $\mathbf{W}_l$ is a submatrix of $\mathbf{W}$ associated with $\mathcal{G}_l$ as in (2.11). For each i in $\mathbb{V}_l$, $\overline{B}_i = e_i \otimes I_n$ and $\overline{C}_i = e_i^T \otimes C_i$ where e_i is the i-th column of the $|\mathbb{V}_l|$-dimensional identity matrix. Notice that (2.12) can be viewed as the state-space representation of the closed-loop system obtained by applying decentralized output feedback, parametrized by $\mathbf{K}_1, \mathbf{P}_1, \mathbf{Q}_1, \mathbf{S}_1, \{\mathbf{K}_i\}_{i \in \mathbb{V}_l \setminus \{1\}}$, to a LTI system described

[13] The sparsity pattern of a matrix $\mathbf{W} = (\mathbf{w}_{ij})_{i,j \in \mathbb{V}}$ is consistent with a graph $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ if $\mathbf{w}_{ij} = 0$ for $(j, i) \notin \mathbb{E}$.

by the triple

$$\left(\mathbf{W}_l \otimes A,\ -\begin{pmatrix}\overline{B}_1 & \cdots & \overline{B}_{|\mathbb{V}_l|}\end{pmatrix},\ \begin{pmatrix}\overline{C}_1\\ \vdots \\ \overline{C}_{|\mathbb{V}_l|}\end{pmatrix}\right) \tag{2.13}$$

Hence, we can write (2.12) as in (2.9) by selecting $\widetilde{A} = \mathbf{W}_l \otimes A$, $P_1 = \mathbf{P}_1$, $Q_1 = \mathbf{Q}_1$, $S_1 = \mathbf{S}_1$ and $\widetilde{B}_i = -\overline{B}_i$, $\widetilde{C}_i = \overline{C}_i$, $K_i = \mathbf{K}_i$ for all i in $\mathbb{V}_l$. This idea, in conjunction with Theorem 2.4.10, allows us to connect the stability of the estimation error dynamics (2.12) with strong connectivity of (2.13) and the absence of unstable fixed modes in (2.13).

The following Lemma states certain spectral properties of $\mathbf{W}$ determined by Procedure 3 in Appendix A.1.2.1. The proof of Theorem 2.2.2 is then followed.

Lemma 2.4.12. *The submatrices* $\{\mathbf{W}_l\}_{l=1}^{m_s}$ *and* $\mathbf{W}_{m_s+1,m_s+1}$ *of* $\mathbf{W}$ *in* (2.11) *satisfy the following with probability one:*

(P1) *For each l in $\{1, \cdots, m_s\}$, every right and left eigenvectors of $\mathbf{W}_l$ have no zero entries.*

(P2) *For each l in $\{1, \cdots, m_s\}$, every eigenvector q of $\mathbf{W}_l \otimes A$ associated with an unstable eigenvalue λ can be written as a Kronecker product $q = v \otimes p$, where v and p are, respectively, eigenvectors of $\mathbf{W}_l$ and A (associated with the eigenvalue $\lambda_{\mathbf{W}_l}$ and the unstable eigenvalue λ_A for which $\lambda = \lambda_{\mathbf{W}_l} \cdot \lambda_A$ holds).*

(P3) *Every eigenvalue of $\mathbf{W}_{m_s+1,m_s+1} \otimes A$ is zero.*

Proof. Notice that for each l in $\{1, \cdots, m_s\}$, in Procedure 3 (Line 3-10), we have set $\mathbf{W}_l = I_{|\mathbb{V}_l|} - \alpha L$ where α is chosen according to a uniform distribution defined on $(0, 1)$, and $L = (l_{ij})_{i,j \in \mathbb{V}_l}$ is a WLM of $\mathcal{G}_l$, each of its nonzero off-diagonal entries l_{ij} is chosen according to a uniform distribution defined on $\left(-\frac{1}{|\mathbb{N}_i|-1}, 0\right)$ independent of choices of other entries. By Theorem 2.4.5, L satisfies **(i)** of Theorem 2.4.5 which ensures that **(P1)** holds with probability one.

In addition, according to Lemma 2.4.3 and Theorem 2.4.5, this choice of α and L ensures that $\mathbf{W}_l$ is a stochastic matrix and has all simple eigenvalues, and $\mathbf{W}_l \otimes A$ satisfies the UEPP (see Definition 2.4.2) with probability one. Since $\mathbf{W}_l$ is stochastic, its eigenvalues lie on or inside the unit circle in $\mathbb{C}$; hence, an unstable eigenvalue λ of $\mathbf{W}_l \otimes A$ can be written as $\lambda = \lambda_{\mathbf{W}_l} \cdot \lambda_A$ where $\lambda_{\mathbf{W}_l}$ is an eigenvalue of $\mathbf{W}_l$ and λ_A is an unstable eigenvalue of A. Therefore, invoking Lemma 2.4.4, we conclude that **(P2)** holds with probability one.

Lastly, the way entries of $\mathbf{W}_{m_s+1,m_s+1}$ are chosen by Procedure 3 (Line 11-14) ensures that all eigenvalues of $\mathbf{W}_{m_s+1,m_s+1}$ are zero and **(P3)** holds. $\square$

2.4.4 Proof of Theorem 2.2.2

First of all notice that if the subsystem $(A, C_{\mathbb{V}_l})$ of the plant (1.1) is not detectable, then for any choice of $\mathbf{W}_l$, the system (2.13) has an unstable fixed mode. By Remark 2.4.7, there is no parameter choice for which the estimation error dynamics (2.12) is stable; hence, no omniscience-achieving parameter choice exists for (2.1). This proves the necessity of Theorem 2.2.2.

Let $\mathbb{V}_{m_s+1} = \mathbb{V} \setminus \bigcup_{l=1}^{m_s} \mathbb{V}_l$. Consider that $\{\mathbf{W}_{m_s+1,l}\}_{l=1}^{m_s+1}$ and $\{\mathbf{K}_i\}_{i \in \mathbb{V}_{m_s+1}}$, $\{\mu_i\}_{i \in \mathbb{V}_{m_s+1}}$ are determined by Procedure 3 (Line 11-14) and Procedure 4 (Line 9-11) in Appendix A.1.2, respectively. Notice that by **(P3)** of Lemma 2.4.12 and due to the choice of $\{\mathbf{K}_i\}_{i \in \mathbb{V}_{m_s+1}}$, $\{\mu_i\}_{i \in \mathbb{V}_{m_s+1}}$, to prove the sufficiency of Theorem 2.2.2, we only need to show that for each l in $\{1, \cdots, m_s\}$, under the detectability condition of Theorem 2.2.2, there exists a choice of $\mathbf{W}_l$ and $\{\mathbf{K}_i\}_{i \in \mathbb{V}_l}$, $\mathbf{P}_1, \mathbf{Q}_1, \mathbf{S}_1$ for which the estimation error dynamics (2.12) is stable.[14]

Suppose that the choice of $\mathbf{W}_l$ and $\{\mathbf{K}_i\}_{i \in \mathbb{V}_l \setminus \{1\}}$, determined by Procedure 3 (Line 3-10) and Procedure 4 (Line 4-7), respectively, ensures that, with probability one, the LTI system (2.13) is **(i)** strongly connected and has **(ii)** no unstable fixed mode, and **(iii)** the controllability index of the LTI system described by the triple

$$\left(\mathbf{W}_l \otimes A - \sum_{i=2}^{|\mathbb{V}_l|} \overline{B}_i \mathbf{K}_i \overline{C}_i, \ -\overline{B}_1, \ \overline{C}_1 \right) \tag{2.14}$$

is equal to $|\mathbb{V}_l|$. By Theorem 2.4.10 and Remark 2.4.11, there exist matrices $\mathbf{K}_1, \mathbf{P}_1, \mathbf{Q}_1, \mathbf{S}_1$ with $\mu_1 = |\mathbb{V}_l| - 1$ that, in conjunction with the chosen $\mathbf{W}_l$ and $\{\mathbf{K}_i\}_{i \in \mathbb{V}_l \setminus \{1\}}$, ensure that the estimation error dynamics (2.12) is stable, where these matrices can be determined by Procedure 4 (Line 8). Hence, we conclude that the detectability condition is sufficient for the existence of an omniscience-achieving parameter choice, and for the parameter choice determined by Procedure 3 and Procedure 4 to be omniscience-achieving with probability one. It remains to prove the arguments **(i)**-**(iii)**.

Proof of (i): Suppose that the transfer function matrix given by

$$\overline{C}_i \left(z \cdot I_{|\mathbb{V}_l| \cdot n} - \mathbf{W}_l \otimes A \right)^{-1} \overline{B}_j \tag{2.15}$$

is zero for some i, j in $\mathbb{V}_l$, or equivalently $\overline{C}_i \left(\mathbf{W}_l \otimes A \right)^k \overline{B}_j = \mathbf{0}$ holds for all nonnegative

[14] From the overall estimation error dynamics for (2.1), it can be verified that if (2.12) is stable for every l in $\{1, \cdots, m_s\}$, then it holds that $\lim_{k \to \infty} \|\hat{x}_i(k) - x(k)\| = 0$ for all $i \in \mathbb{V}_{m_s+1}$.

integer k. This yields that

$$\overline{C}_i \left(\mathbf{W}_l \otimes A\right)^k \overline{B}_j = \left(e_i^T \mathbf{W}_l^k e_j\right) C_i A^k = \mathbf{0} \tag{2.16}$$

where we use the fact that $\overline{B}_j = e_j \otimes I_n$ and $\overline{C}_i = e_i^T \otimes C_i$. Since $\mathcal{G}_l$ is strongly connected, due to the choice of $\mathbf{W}_l$ by Procedure 3 (Line 3-10), we can see that $e_i^T \mathbf{W}_l^{k_0} e_j \neq 0$ for a positive integer k_0, and hence $C_i A^{k_0} = \mathbf{0}$ holds. However, this contradicts the fact that A is nondegenerate and C_i is nonzero (see Section 2.1.2). Therefore, the transfer function matrix (2.15) is nonzero for all i, j in $\mathbb{V}_l$ which, by definition, implies that the system (2.13) is strongly connected with probability one.

Proof of (ii): Let us define $\overline{B} = \begin{pmatrix} \overline{B}_1 & \dots & \overline{B}_{|\mathbb{V}_l|} \end{pmatrix}$ and $\overline{C} = \begin{pmatrix} \overline{C}_1^T & \dots & \overline{C}_{|\mathbb{V}_l|}^T \end{pmatrix}^T$. According to Proposition 2.4.8, we need to show that the following inequality holds for every unstable eigenvalue λ of $\mathbf{W}_l \otimes A$:

$$rank \begin{pmatrix} \mathbf{W}_l \otimes A - \lambda \cdot I_{|\mathbb{V}_l| \cdot n} & \overline{B}_{\mathbb{J}} \\ \overline{C}_{\mathbb{J}^c} & \mathbf{0} \end{pmatrix} \geq |\mathbb{V}_l| \cdot n \tag{2.17}$$

where $\mathbb{J}$ is an arbitrary subset of $\mathbb{V}_l$, and $\mathbb{J}^c = \mathbb{V}_l \setminus \mathbb{J}$ is its complement.

Suppose that $\mathbb{J}$ is not empty then by **(P1)**, **(P2)** of Lemma 2.4.12 and by the definition of $\overline{B}$, it holds that

$$\begin{aligned} & rank \begin{pmatrix} \mathbf{W}_l \otimes A - \lambda \cdot I_{|\mathbb{V}_l| \cdot n} & \overline{B}_{\mathbb{J}} \\ \overline{C}_{\mathbb{J}^c} & \mathbf{0} \end{pmatrix} \\ & \geq rank \begin{pmatrix} \mathbf{W}_l \otimes A - \lambda \cdot I_{|\mathbb{V}_l| \cdot n} & \overline{B}_{\mathbb{J}} \end{pmatrix} = |\mathbb{V}_l| \cdot n \end{aligned} \tag{2.18}$$

Otherwise, since $\mathbb{J}^c = \mathbb{V}_l$, by **(P1)**, **(P2)** of Lemma 2.4.12, by the definition of $\overline{C}$, and by the detectability of the subsystem $(A, C_{\mathbb{V}_l})$, it holds that

$$\begin{aligned} &rank \begin{pmatrix} \mathbf{W}_l \otimes A - \lambda \cdot I_{|\mathbb{V}_l| \cdot n} & \overline{B}_{\mathbb{J}} \\ \overline{C}_{\mathbb{J}^c} & \mathbf{0} \end{pmatrix} \\ &= rank \begin{pmatrix} \mathbf{W}_l \otimes A - \lambda \cdot I_{|\mathbb{V}_l| \cdot n} \\ \overline{C} \end{pmatrix} = |\mathbb{V}_l| \cdot n \end{aligned} \tag{2.19}$$

Therefore, from (2.18) and (2.19), we can observe that the inequality in (2.17) holds for every unstable eigenvalue λ of $\mathbf{W}_l \otimes A$, and by Proposition 2.4.8 the system (2.13) has no unstable fixed mode with probability one.

Proof of (iii): To verify this, we consider a matrix given by

$$\begin{pmatrix} \overline{B}_1 & (\mathbf{W}_l \otimes A)\, \overline{B}_1 & \cdots & (\mathbf{W}_l \otimes A)^{|\mathbb{V}_l|-1}\, \overline{B}_1 \end{pmatrix} \tag{2.20}$$

Note that (2.20) can be rewritten as

$$\left(\begin{pmatrix} e_1 & \mathbf{W}_l e_1 & \cdots & \mathbf{W}_l^{|\mathbb{V}_l|-1} e_1 \end{pmatrix} \otimes I_n \right) \cdot diag\left(I_n,\ A,\ \cdots,\ A^{|\mathbb{V}_l|-1} \right) \tag{2.21}$$

where we use the fact that $\overline{B}_1 = e_1 \otimes I_n$.

By the nondegeneracy of A, the rank of the matrix in (2.20) equals

$$rank \begin{pmatrix} e_1 & \mathbf{W}_l e_1 & \cdots & \mathbf{W}_l^{|\mathbb{V}_l|-1} e_1 \end{pmatrix} \cdot n$$

and by **(P1)** of Lemma 2.4.12, we can see that the matrix in (2.20) has rank $|\mathbb{V}_l| \cdot n$. Hence, the following matrix has *generic* rank $|\mathbb{V}_l| \cdot n$, i.e., for almost every choice of $\{\mathbf{K}_i\}_{i \in \mathbb{V}_l \setminus \{1\}}$, the matrix has rank $|\mathbb{V}_l| \cdot n$.

$$\begin{pmatrix} \overline{B}_1 & \left(\mathbf{W}_l \otimes A - \sum_{i=2}^{|\mathbb{V}_l|} \overline{B}_i \mathbf{K}_i \overline{C}_i \right) \overline{B}_1 & \cdots & \left(\mathbf{W}_l \otimes A - \sum_{i=2}^{|\mathbb{V}_l|} \overline{B}_i \mathbf{K}_i \overline{C}_i \right)^{|\mathbb{V}_l|-1} \overline{B}_1 \end{pmatrix} \tag{2.22}$$

Therefore, due to the choice of $\{\mathbf{K}_i\}_{i \in \mathbb{V}_l \setminus \{1\}}$ by Procedure 4 (Line 4-7), the controllability index of (2.14) is equal to $|\mathbb{V}_l|$ with probability one. $\square$

2.5 Application to Tracking of Animal Groups and Experimental Results

In this section, we apply the proposed distributed estimation scheme to tracking of animal groups, and show preliminary experimental results using a data set collected from the deployment of *animal-borne wireless camera network* in the Gorongosa National Park (Mozambique) in August 2015.[15] The main purpose of the development and deployment of the system was to collect biologically meaningful measurements and videos using GPS, IMU, and Camera all integrated in a single tracking device, where the proposed estimation scheme can be used to determine how to fuse sensor measurements and location estimates of tracking devices within the network so that each tracking device in the network learns locations of all other devices connected to the same communication network (see Figure 2.5). The sensor measurements and videos are used to study animal group motion. During the deployment, 15 tracking devices were installed on waterbucks and water buffaloes. Figure 2.6 and Figure 2.7 show GPS tracks of 4 water buffaloes (Buffalo 1, 2, 3, 4).

[15]The development and deployment of animal-borne wireless camera network were performed under a research grant NSF ECCS 1135726.

Disclaimer: The author of this dissertation was NOT involved in the deployment in the Gorongosa National Park.

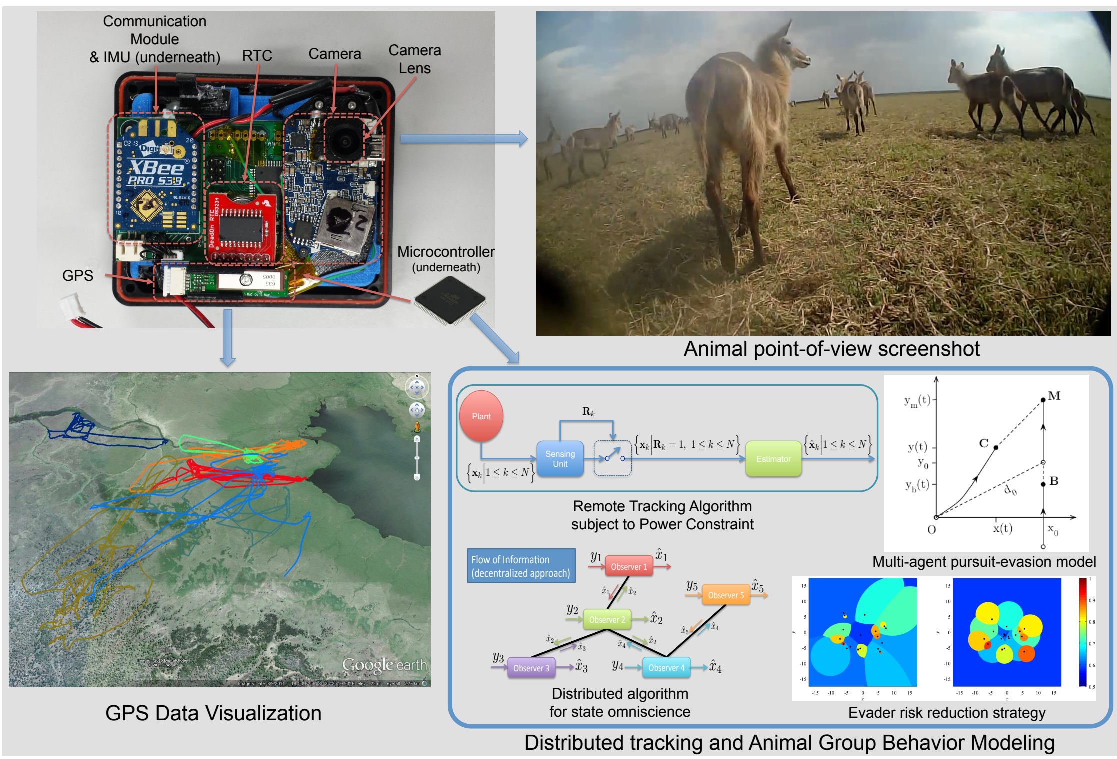


Figure 2.5: System overview of the tracking device in the animal-borne wireless camera network

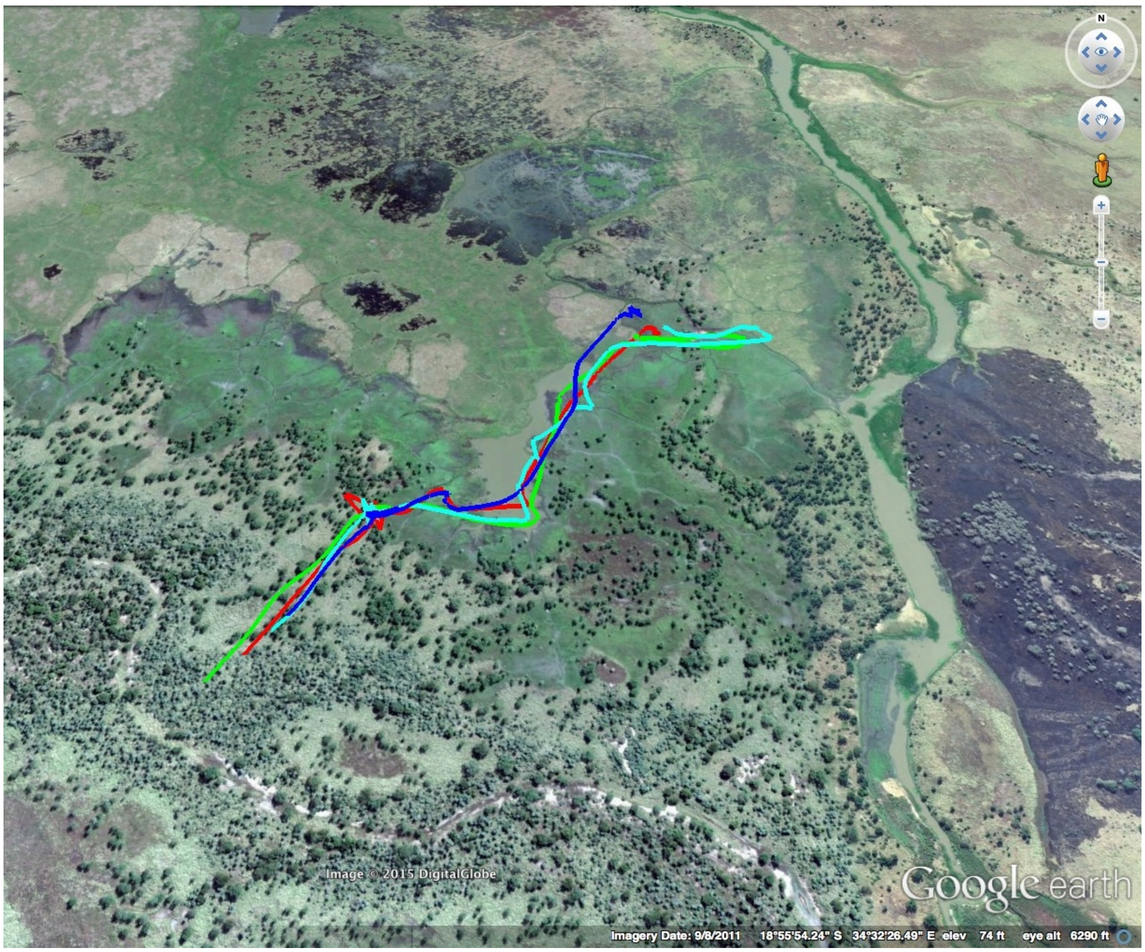


Figure 2.6: A screenshot of GPS tracks of water buffaloes in the Google earth

(Timespan: 2015-08-06T00:00:00Z $\sim$ 2015-08-06T04:00:00Z)

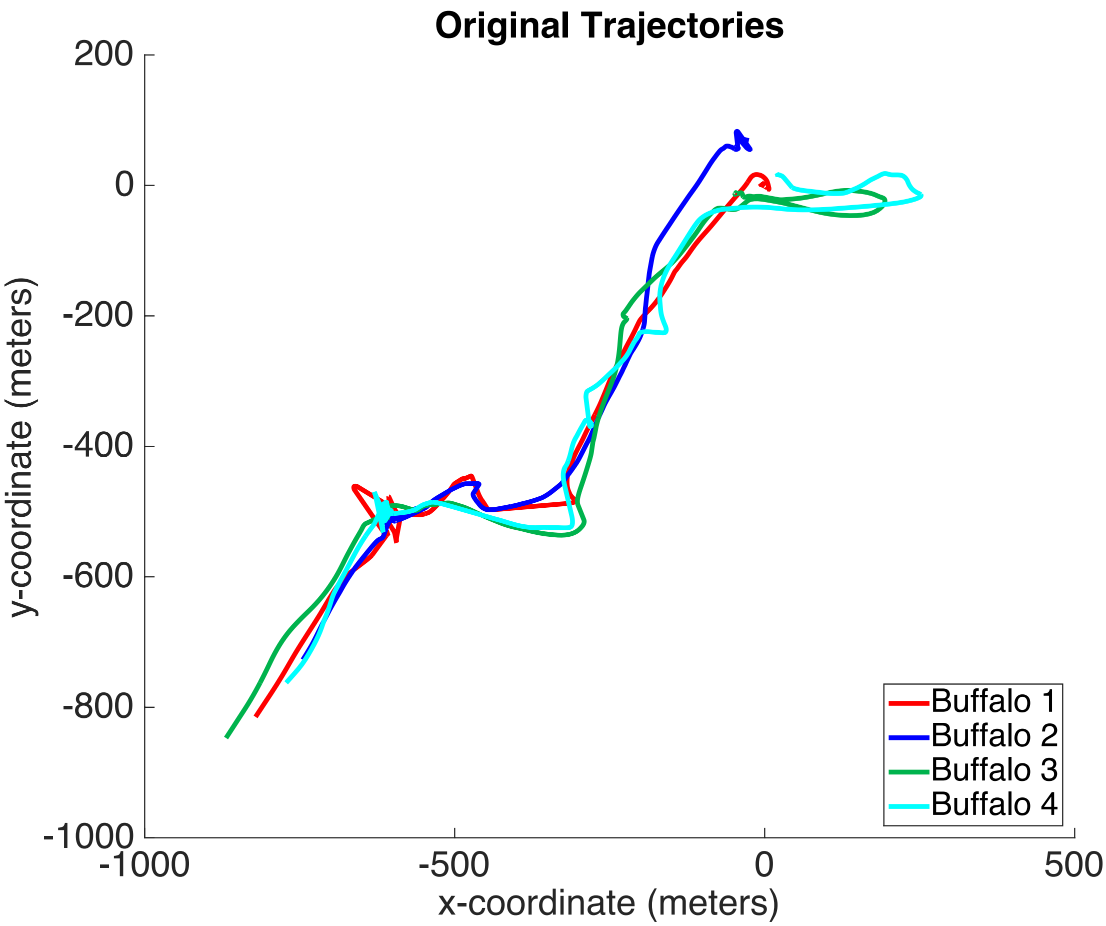


Figure 2.7: Original trajectories of water buffaloes in a local North East Up (NEU) coordinate system (The origin of the coordinate system: Latitude = -18.9279877268328, Longitude = 34.5457567343343)

To represent animal group motion, we adopt a continuous-time LTI model described by the following state-space equation: For each i in $\{1, \cdots, 4\}$,

$$\dot{p}_x^{(i)}(t) = v_x^{(i)}(t) \tag{2.23a}$$

$$\dot{p}_y^{(i)}(t) = v_y^{(i)}(t) \tag{2.23b}$$

$$\dot{v}_x^{(i)}(t) = -\sum_{j=1}^{4} a_{ij}\left(v_x^{(i)}(t) - v_x^{(j)}(t)\right) \tag{2.23c}$$

$$\dot{v}_y^{(i)}(t) = -\sum_{j=1}^{4} a_{ij}\left(v_y^{(i)}(t) - v_y^{(j)}(t)\right) \tag{2.23d}$$

where $\begin{pmatrix} p_x^{(i)}(t) & p_y^{(i)}(t) \end{pmatrix}^T$ and $\begin{pmatrix} v_x^{(i)}(t) & v_y^{(i)}(t) \end{pmatrix}^T$ denote the location and velocity of Buffalo i, respectively. We have assumed that $a_{ij} = a_{ji}$ for all i, j in $\{1, \cdots, 4\}$, and that a node (observer) is associated with each water buffalo and the location and velocity measurements of each buffalo are available to its associated node every 10 seconds. By discretizing (2.23), we obtain a discrete-time LTI model described as follows:

$$x(k+1) = Ax(k) \tag{2.24a}$$

$$y_i(k) = C_i x(k) \tag{2.24b}$$

for i in $\{1, \cdots, 4\}$, where

$$x(k) = \begin{pmatrix} p_x^{(1)}(k) & p_y^{(1)}(k) & v_x^{(1)}(k) & v_y^{(1)}(k) & \cdots & p_x^{(4)}(k) & p_y^{(4)}(k) & v_x^{(4)}(k) & v_y^{(4)}(k) \end{pmatrix}^T$$
$$y_i(k) = \begin{pmatrix} p_x^{(i)}(k) & p_y^{(i)}(k) & v_x^{(i)}(k) & v_y^{(i)}(k) \end{pmatrix}^T$$

and the system matrices A and C_i are determined as follows:

$$A = \exp\left(10 A_c\right)$$

$$C_i = e_i^T \otimes I_4$$

where

$$A_C = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\sum_{j\neq 1} a_{1j} & 0 & a_{12} & 0 & a_{13} & 0 & a_{14} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & a_{21} & 0 & -\sum_{j\neq 2} a_{2j} & 0 & a_{23} & 0 & a_{24} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & a_{31} & 0 & a_{32} & 0 & -\sum_{j\neq 3} a_{3j} & 0 & a_{34} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & a_{41} & 0 & a_{42} & 0 & a_{43} & 0 & -\sum_{j\neq 4} a_{4j} \end{pmatrix} \otimes I_2$$

and e_i is the i-column of I_4. Note that, according to C_i, each node has access to the location and velocity of its associated buffalo.

In practice, the model (2.24) may include a noise term $w(k)$ which is due to the un-modeled dynamics of animal motion:

$$x(k+1) = Ax(k) + w(k)$$
$$y_i(k) = C_i x(k)$$

To minimize the noise $w(k)$, we have chosen the entries a_{ij} of A_C that minimize the cost given by

$$\sum_{k=0}^{N-1} \|w(k)\|^2$$

where N is the number of available location and velocity measurements, and the resulting

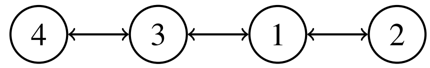


Figure 2.8: A communication graph $\mathcal{G}$ for designing a distributed observer.

choice of a_{ij} is given by

$$a_{12} = a_{21} = 0.002$$

$$a_{13} = a_{31} = 0.002$$

$$a_{14} = a_{41} = 0$$

$$a_{23} = a_{32} = 0.001$$

$$a_{24} = a_{42} = 0$$

$$a_{34} = a_{43} = 0.003$$

To design a distributed observer, we assume that the communication graph $\mathcal{G}$ is pre-selected as depicted in Figure 2.8. We find the omniscience-achieving parameter based on Procedure 3 and Procedure 4 described in Appendix A.1.2. Figure 2.9 shows the estimate $\begin{pmatrix} \hat{p}_x^{(i)}(k) & \hat{p}_y^{(i)}(k) \end{pmatrix}^T$ of the original trajectory of each buffalo, depicted in Figure 2.7, by every node; and Figure 2.10 shows normalized estimation error computed by

$$\frac{1}{\text{total traveled distance of Buffalo } i} \cdot \left\| \begin{pmatrix} p_x^{(i)}(k) \\ p_y^{(i)}(k) \end{pmatrix} - \begin{pmatrix} \hat{p}_x^{(i)}(k) \\ \hat{p}_y^{(i)}(k) \end{pmatrix} \right\|$$

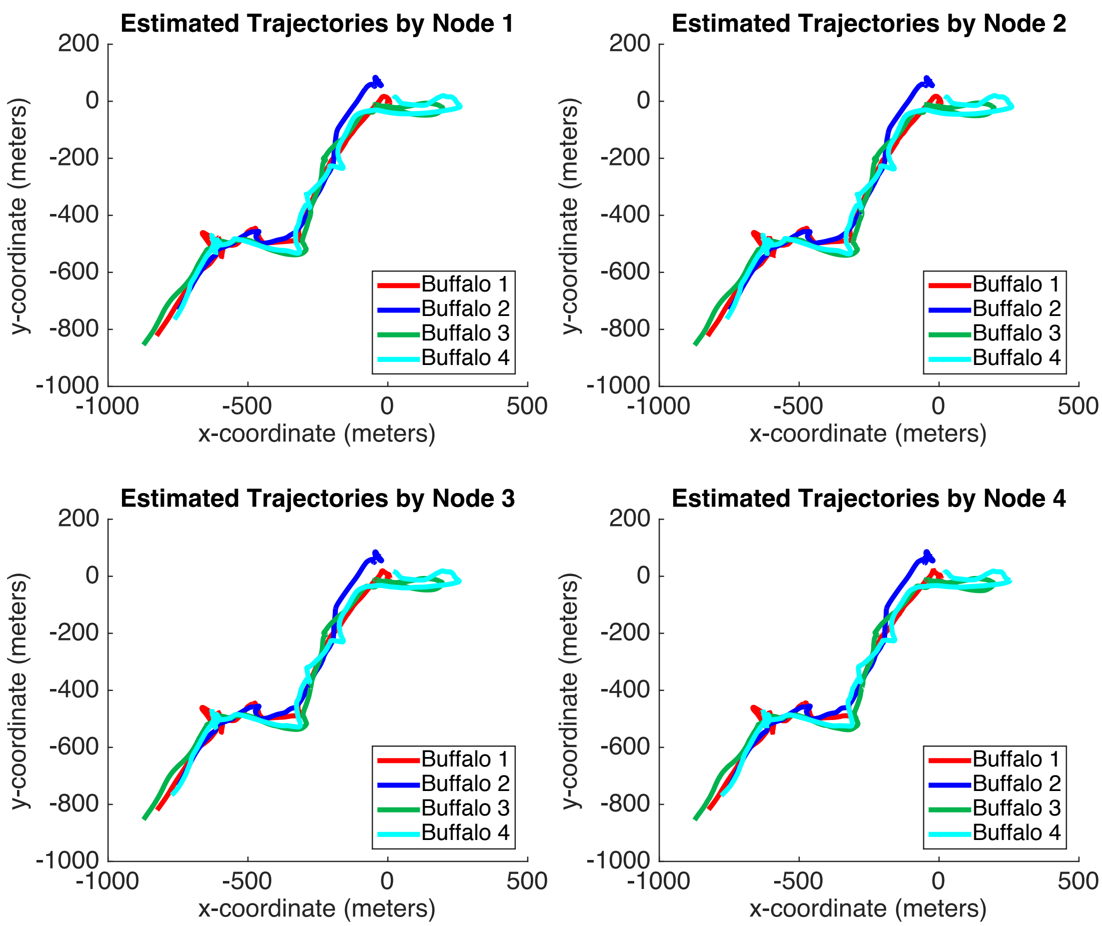


Figure 2.9: Estimated trajectories of water buffaloes using the proposed distributed estimation scheme in a local NEU coordinate system

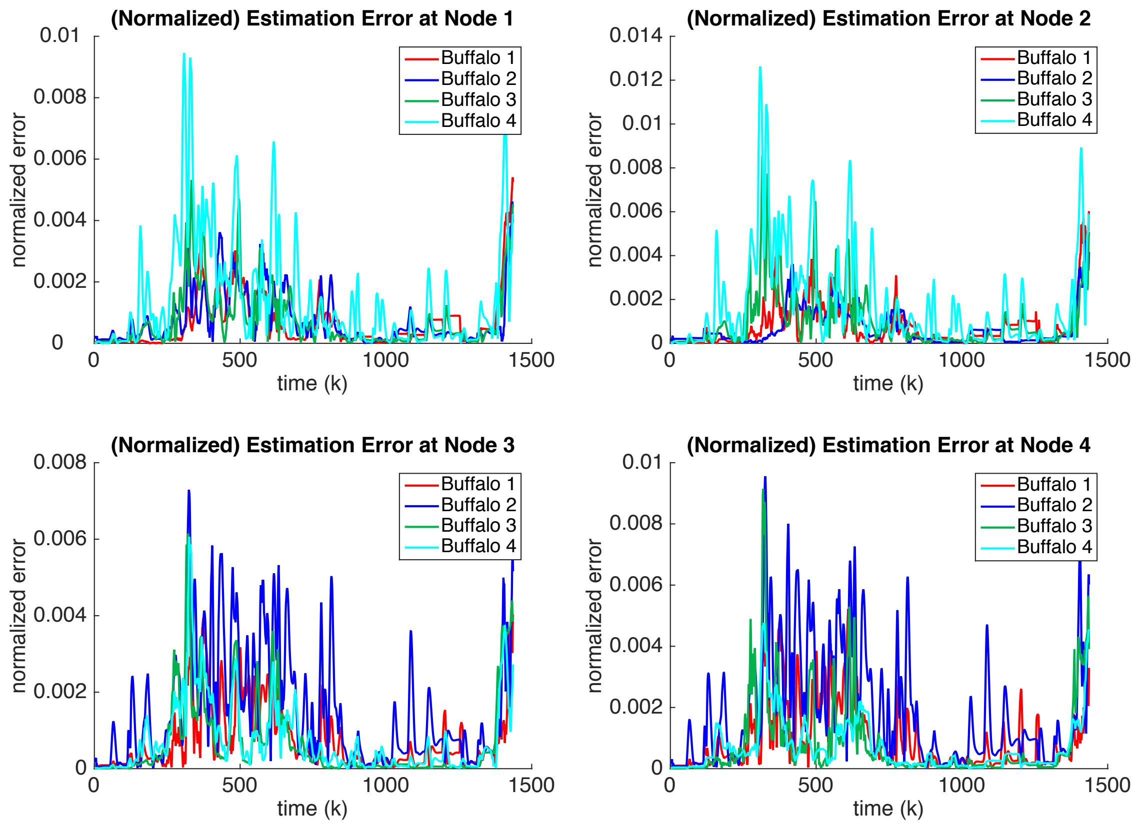


Figure 2.10: Normalized estimation error at every node (Total traveled distance: Buffalo 1 = 1694 m, Buffalo 2 = 1505.8 m, Buffalo 3 = 1981.8 m, Buffalo 4 = 2129 m)

2.6 Summary and Future Work

We described a parametrized class of LTI distributed observers for state estimation of a LTI plant, where the information exchange among the members of a distributed observer is constrained by a pre-selected communication graph. We developed necessary and sufficient conditions for the existence of a parameter choice for a distributed observer that ensures asymptotic omniscience and satisfies the scalability constraint (1.2). These conditions can be described by the detectability of the subsystems of the plant that are associated with the source components of the graph.

As a future direction, we suggest performance analysis of the proposed scheme and parameter optimization to minimize estimation error in the presence of noise in measurement and communication link. Also it is important to consider distributed state estimation over time-varying communication graphs.

Chapter 3: Optimal Remote State Estimation of Markov Processes

3.1 Problem Formulation

3.1.1 Notation and Terminologies

- For elements $a_1, \cdots, a_N$ of $\mathbb{X}$, we define $a_{1:N} \stackrel{def}{=} (a_1, \cdots, a_N)$.

- For functions $\mathcal{A}_1, \cdots, \mathcal{A}_N$, we define $\mathcal{A}_{1:N} \stackrel{def}{=} (\mathcal{A}_1, \cdots, \mathcal{A}_N)$.

- When the random variable $\mathbf{R}_k$ is dictated by a policy $\boldsymbol{\mathcal{T}}_k$, we use same superscript for $\mathbf{R}_k$ and $\boldsymbol{\mathcal{T}}_k$, e.g., $\mathbf{R}_k^*$ and $\boldsymbol{\mathcal{T}}_k^*$, or $\mathbf{R}_k^{(i)}$ and $\boldsymbol{\mathcal{T}}_k^{(i)}$.

- We define

  $$\boldsymbol{\tau}_k \stackrel{def}{=} \begin{cases} \max\left\{ j \in \{1, \cdots, k-1\} \,\middle|\, \mathbf{R}_j = 1 \right\} & \text{if } \mathbf{R}_j = 1 \text{ for some } j \in \{1, \cdots, k-1\} \\ 0 & \text{otherwise} \end{cases}$$

  The value of $\boldsymbol{\tau}_k$ indicates the most recent time when a transmission has occurred from the sensing unit to the estimator. We refer to $\boldsymbol{\tau}_k$ as the *last transmission time* before time k.

3.1.2 Problem Description

In this section, we describe the problem formulation considered throughout the work in which we seek transmission policies $\mathcal{T}_{1:N}$ and (state) estimation rules $\mathcal{E}_{1:N}$ that dictate decision making of the sensing unit and estimator, respectively, and that are optimal for the cost functional (1.5). Throughout the work, without loss of optimality, we consider that transmission policies and estimation rules have the following structures[1]: The transmission policy at time k depends only on the last transmission time $\boldsymbol{\tau}_k$, the information $\mathbf{x}_{\boldsymbol{\tau}_k}$ transmitted to the estimator at time $\boldsymbol{\tau}_k$, and the current state $\mathbf{x}_k$ of the process. In particular, we adopt a class of *randomized* transmission policies.[2] The estimation rule at time k depends only on the last transmission time $\boldsymbol{\tau}_k$ and the information $\mathbf{x}_{\boldsymbol{\tau}_k}$ received from the sensing unit at time $\boldsymbol{\tau}_k$. Given a transmission policy $\mathcal{T}_k$ and an estimation rule $\mathcal{E}_k$, the decision variables $\mathbf{R}_k$ and $\hat{\mathbf{x}}_k$ are dictated by $\mathcal{T}_k$ and $\mathcal{E}_k$, respectively, as follows:

$$\mathbf{R}_k = \mathcal{T}_k\left(\boldsymbol{\tau}_k, \mathbf{x}_{\boldsymbol{\tau}_k}, \mathbf{x}_k\right) \tag{3.1a}$$

$$\hat{\mathbf{x}}_k = \begin{cases} \mathcal{E}_k\left(\boldsymbol{\tau}_k, \mathbf{x}_{\boldsymbol{\tau}_k}\right) & \text{if } \mathbf{R}_k = 0 \\ \mathbf{x}_k & \text{otherwise} \end{cases} \tag{3.1b}$$

[1]We do not lose optimality of resulting solutions from the imposition of these structures. This can be verified by similar arguments as in Lemma 1 and Lemma 3 of [70].

[2]See Appendix B.2 for a detailed description of randomized policies.

We formally state our main problem as follows.

Problem 3.1.1. *Given a Markov process* $\{\mathbf{x}_k\}_{k=0}^{N}$*, find optimal transmission policies* $\boldsymbol{\mathcal{T}}_{1:N}$ *and estimation rules* $\mathcal{E}_{1:N}$ *for the cost functional given by*

$$\mathcal{J}\left(x_0, \boldsymbol{\mathcal{T}}_{1:N}, \mathcal{E}_{1:N}\right) = \sum_{k=1}^{N} \mathbb{E}\left[d^2\left(\mathbf{x}_k, \hat{\mathbf{x}}_k\right) + c_k \cdot \mathbf{R}_k \,\middle|\, \mathbf{x}_0 = x_0, \boldsymbol{\mathcal{T}}_{1:N}, \mathcal{E}_{1:N}\right] \tag{3.2}$$

subject to (3.1).[3]

We consider two notions of optimality for solutions of Problem 3.1.1 described as follows.

Definition 3.1.2. *Transmission policies* $\boldsymbol{\mathcal{T}}^*_{1:N}$ *and estimation rules* $\mathcal{E}^*_{1:N}$ *are said to be jointly optimal for* (3.2) *if they achieve the global minimum for every* x_0 *in* $\mathbb{X}$*.*

Definition 3.1.3. *Transmission policies* $\boldsymbol{\mathcal{T}}^*_{1:N}$ *and estimation rules* $\mathcal{E}^*_{1:N}$ *are said to be person-by-person optimal for* (3.2) *if the following relations hold for every* x_0 *in* $\mathbb{X}$*:*

$$\begin{aligned}\mathcal{J}\left(x_0, \boldsymbol{\mathcal{T}}^*_{1:N}, \mathcal{E}^*_{1:N}\right) &= \min_{\boldsymbol{\mathcal{T}}_{1:N}} \mathcal{J}\left(x_0, \boldsymbol{\mathcal{T}}_{1:N}, \mathcal{E}^*_{1:N}\right) \\ &= \min_{\mathcal{E}_{1:N}} \mathcal{J}\left(x_0, \boldsymbol{\mathcal{T}}^*_{1:N}, \mathcal{E}_{1:N}\right) \end{aligned} \tag{3.3}$$

Equation (3.3) *implies that given decision functions* $\boldsymbol{\mathcal{T}}^*_{1:N}$ *of one player (sensing unit),* $\mathcal{E}^*_{1:N}$ *are the best decision functions of the other player (estimator), and vice versa.*

We maintain the following assumptions throughout the work.

[3] The initial condition $\mathbf{x}_0 = x_0$ and the process model is common knowledge to both the sensing unit and estimator.

Assumption 3.1.4. $(\mathbb{X}, d)$ *is a complete, separable, proper metric space.*[4]

Assumption 3.1.5. *Let* $p_k : \mathbb{X} \times \mathfrak{B} \to \mathbb{R}$ *be a transition probability of the process, where* $\mathfrak{B}$ *is the Borel* σ*-algebra generated by the metric topology associated with* $(\mathbb{X}, d)$*. We assume that the following are true:*

1. *For every non-empty open set* $\mathbb{O}$ *in* $\mathfrak{B}$*, the function* $x \mapsto p_k(x, \mathbb{O})$ *is positive for all* x *in* $\mathbb{X}$*.*

2. *For each* $\mathbb{A}$ *in* $\mathfrak{B}$*, the function* $x \mapsto p_k(x, \mathbb{A})$ *is continuous.*

Assumption 3.1.6. *For each* j *in* $\{k-1, \cdots, N\}$*,* k *in* $\{1, \cdots, N\}$*, and* x_{k-1} *in* $\mathbb{X}$*, we assume that there is a transformation* $M_j(k-1, x_{k-1}, \cdot) : \mathbb{X} \to \mathbb{X}$ *for which*

1. *It holds that* $M_{k-1}(k-1, x_{k-1}, x_{k-1}) = 0$*.*

2. *The function* $x_j \mapsto M_j(k-1, x_{k-1}, x_j)$ *is continuous and has a continuous inverse. We denote the inverse by* $M_j^{-1}(k-1, x_{k-1}, \cdot)$*.*

3. *For every* x_{j-1} *in* $\mathbb{X}$ *and* $\mathbb{A}$ *in* $\mathfrak{B}$*, the transition probability* p_j *satisfies*

$$p_j\left(x'_{j-1}, \mathbb{A}'\right) = p_j\left(x_{j-1}, \mathbb{A}\right)$$

where

$$x'_{j-1} = M_{j-1}(k-1, x_{k-1}, x_{j-1})$$
$$\mathbb{A}' = M_j\left(k-1, x_{k-1}, \mathbb{A}\right)$$

[4]For notional convenience, we suppose that $0 \in \mathbb{X}$.

4. *The metric d is invariant under M_j, i.e., it holds that*

$$d\left(x_j, \hat{x}_j\right) = d\left(M_j(k-1, x_{k-1}, x_j), M_j(k-1, x_{k-1}, \hat{x}_j)\right)$$

for all $x_j, \hat{x}_j$ in $\mathbb{X}$.

To find a solution to Problem 3.1.1, we divide the problem into a set of N sub-problems, and sequentially solve each sub-problem. We proceed by describing the so-called *Two-Player Optimal Stopping Problem* from time k, and show how each sub-problem can be related to the optimal stopping problem.

Problem 3.1.7. *Suppose that a Markov process $\{\mathbf{x}_j\}_{j=k-1}^{N}$ with a transition probability $p_j : \mathbb{X} \times \mathfrak{B} \to \mathbb{R}$ and positive constants $\left\{c'_j\right\}_{j=k}^{N}$ are given. Find optimal policies $\boldsymbol{\mathcal{T}}_{k:N}^{<k-1>}$ and rules $\mathcal{E}_{k:N}^{<k-1>}$ for the cost functional given by*

$$\mathbb{E}\left[\sum_{j=k}^{\mathbf{K}} d^2\left(\mathbf{x}_j, \hat{\mathbf{x}}_j\right) + c'_{\mathbf{K}} \cdot \mathbf{R}_{\mathbf{K}} \,\middle|\, \mathbf{x}_{k-1} = x_{k-1}, \boldsymbol{\mathcal{T}}_{k:N}^{<k-1>}, \mathcal{E}_{k:N}^{<k-1>}\right] \tag{3.4}$$

where

$$\mathbf{K} = \begin{cases} \min\left\{j \in \{k, \cdots, N\} \,\middle|\, \mathbf{R}_j = 1\right\} & \text{if } \mathbf{R}_j = 1 \text{ for some } j \in \{k, \cdots, N\} \\ N & otherwise \end{cases}$$

The policy $\boldsymbol{\mathcal{T}}_j^{<k-1>} : \mathbb{X} \times \mathbb{X} \to \{0, 1\}$ and the rule $\mathcal{E}_j^{<k-1>} : \mathbb{X} \to \mathbb{X}$, respectively, dictate $\mathbf{R}_j$ and $\hat{\mathbf{x}}_j$ as follows:

$$\mathbf{R}_j = \boldsymbol{\mathcal{T}}_j^{<k-1>}\left(x_{k-1}, \mathbf{x}_j\right) \tag{3.5a}$$

$$\hat{\mathbf{x}}_j = \begin{cases} \mathcal{E}_j^{<k-1>}\left(x_{k-1}\right) & \text{if } \mathbf{R}_j = 0 \\ \mathbf{x}_j & otherwise \end{cases} \tag{3.5b}$$

We adopt two notions of optimality for Problem 3.1.7 as follows.

Definition 3.1.8. *Policies $\boldsymbol{\mathcal{T}}^{*<k-1>}_{k:N}$ and rules $\mathcal{E}^{*<k-1>}_{k:N}$ are said to be jointly optimal for* (3.4) *if they achieve the global minimum for every x_{k-1} in $\mathbb{X}$.*

Definition 3.1.9. *Policies $\boldsymbol{\mathcal{T}}^{*<k-1>}_{k:N}$ and rules $\mathcal{E}^{*<k-1>}_{k:N}$ are said to be person-by-person optimal for* (3.4) *if the following relations hold for every x_{k-1} in $\mathbb{X}$:*

$$\begin{aligned}
&\mathbb{E}\left[\sum_{j=k}^{\mathbf{K}} d^2\left(\mathbf{x}_j, \hat{\mathbf{x}}_j\right) + c'_{\mathbf{K}} \cdot \mathbf{R}_{\mathbf{K}} \,\middle|\, \mathbf{x}_{k-1} = x_{k-1}, \boldsymbol{\mathcal{T}}^{*<k-1>}_{k:N}, \mathcal{E}^{*<k-1>}_{k:N}\right] \\
&= \min_{\boldsymbol{\mathcal{T}}^{<k-1>}_{k:N}} \mathbb{E}\left[\sum_{j=k}^{\mathbf{K}} d^2\left(\mathbf{x}_j, \hat{\mathbf{x}}_j\right) + c'_{\mathbf{K}} \cdot \mathbf{R}_{\mathbf{K}} \,\middle|\, \mathbf{x}_{k-1} = x_{k-1}, \boldsymbol{\mathcal{T}}^{<k-1>}_{k:N}, \mathcal{E}^{*<k-1>}_{k:N}\right] \\
&= \min_{\mathcal{E}^{<k-1>}_{k:N}} \mathbb{E}\left[\sum_{j=k}^{\mathbf{K}} d^2\left(\mathbf{x}_j, \hat{\mathbf{x}}_j\right) + c'_{\mathbf{K}} \cdot \mathbf{R}_{\mathbf{K}} \,\middle|\, \mathbf{x}_{k-1} = x_{k-1}, \boldsymbol{\mathcal{T}}^{*<k-1>}_{k:N}, \mathcal{E}^{<k-1>}_{k:N}\right]
\end{aligned} \tag{3.6}$$

Problem 3.1.7 can be viewed as a team decision problem [71] in which two players are involved and the main objective is to find optimal decision functions $\boldsymbol{\mathcal{T}}^{<k-1>}_{k:N}$ and $\mathcal{E}^{<k-1>}_{k:N}$ for the players. Note that the total expected cost (3.4) consists of a *running cost* $d^2\left(\mathbf{x}_j, \hat{\mathbf{x}}_j\right)$ and a *stopping cost* c'_j.

Remark 3.1.10. *In Section 3.2, we show that using the transformation described in Assumption 3.1.6, the value of* (3.4) *evaluated at an optimal solution does not depend on the initial condition $\mathbf{x}_{k-1} = x_{k-1}$ (see Remark 3.2.2).*

Remark 3.1.11. *For any polices $\boldsymbol{\mathcal{T}}^{<k-1>}_{\boldsymbol{k:N}}$ and rules $\mathcal{E}^{<k-1>}_{k:N}$, if $c'_j \leq c''_j$ holds for all j in*

$\{k, \cdots, N\}$*, then we have the following inequality:*

$$\mathbb{E}\left[\sum_{j=k}^{\mathbf{K}} d^2\left(\mathbf{x}_j, \hat{\mathbf{x}}_j\right) + c'_{\mathbf{K}} \cdot \mathbf{R}_{\mathbf{K}} \middle| \mathbf{x}_{k-1} = x_{k-1}, \boldsymbol{\mathcal{T}}_{k:N}^{<k-1>}, \mathcal{E}_{k:N}^{<k-1>}\right]$$
$$\leq \mathbb{E}\left[\sum_{j=k}^{\mathbf{K}} d^2\left(\mathbf{x}_j, \hat{\mathbf{x}}_j\right) + c''_{\mathbf{K}} \cdot \mathbf{R}_{\mathbf{K}} \middle| \mathbf{x}_{k-1} = x_{k-1}, \boldsymbol{\mathcal{T}}_{k:N}^{<k-1>}, \mathcal{E}_{k:N}^{<k-1>}\right] \tag{3.7}$$

Associated with Problem 3.1.7, we describe the k-th sub-problem of Problem 3.1.1 as follows.

Sub-problem k**:** *Given* $\left\{\boldsymbol{\mathcal{T}}_{j:N}^{<j-1>}\right\}_{j=k+1}^{N}$ *and* $\left\{\mathcal{E}_{j:N}^{<j-1>}\right\}_{j=k+1}^{N}$*, let us define constants* $\left\{c'_j\right\}_{j=k}^{N}$ *as follows:*

$$c'_j = c_j + \mathbb{E}\left[\sum_{l=j+1}^{\mathbf{K}_j} d^2\left(\mathbf{x}_l, \hat{\mathbf{x}}_l\right) + c'_{\mathbf{K}_j} \cdot \mathbf{R}_{\mathbf{K}_j} \middle| \mathbf{x}_j = 0, \boldsymbol{\mathcal{T}}_{j+1:N}^{<j>}, \mathcal{E}_{j+1:N}^{<j>}\right] \tag{3.8}$$

with $c'_N = c_N$*, where* c_j *is the communication cost at time* j *given as in* (3.2)*, and*

$$\mathbf{K}_j = \begin{cases} \min\left\{l \in \{j+1, \cdots, N\} \middle| \mathbf{R}_l = 1\right\} & \textit{if } \mathbf{R}_l = 1 \textit{ for some } l \in \{j+1, \cdots, N\} \\ N & \textit{otherwise} \end{cases}$$

With the stopping costs $\{c'_j\}_{j=k}^{N}$ *determined by* (3.8)*, find a solution* $\boldsymbol{\mathcal{T}}_{k:N}^{<k-1>}$ *and* $\mathcal{E}_{k:N}^{<k-1>}$ *to Problem 3.1.7.*

Note that Sub-problem k assumes that $\left\{\boldsymbol{\mathcal{T}}_{j:N}^{<j-1>}\right\}_{j=k+1}^{N}$ and $\left\{\mathcal{E}_{j:N}^{<j-1>}\right\}_{j=k+1}^{N}$ are given parameters.

Our main strategy in solving Problem 3.1.1 can be described as follows: We solve the Sub-problems backward in time starting from $k = N$, where for each Sub-problem k, we provide solutions $\left\{\boldsymbol{\mathcal{T}}_{j:N}^{<j-1>}\right\}_{j=k+1}^{N}$ and $\left\{\mathcal{E}_{j:N}^{<j-1>}\right\}_{j=k+1}^{N}$ for preceding Sub-problems

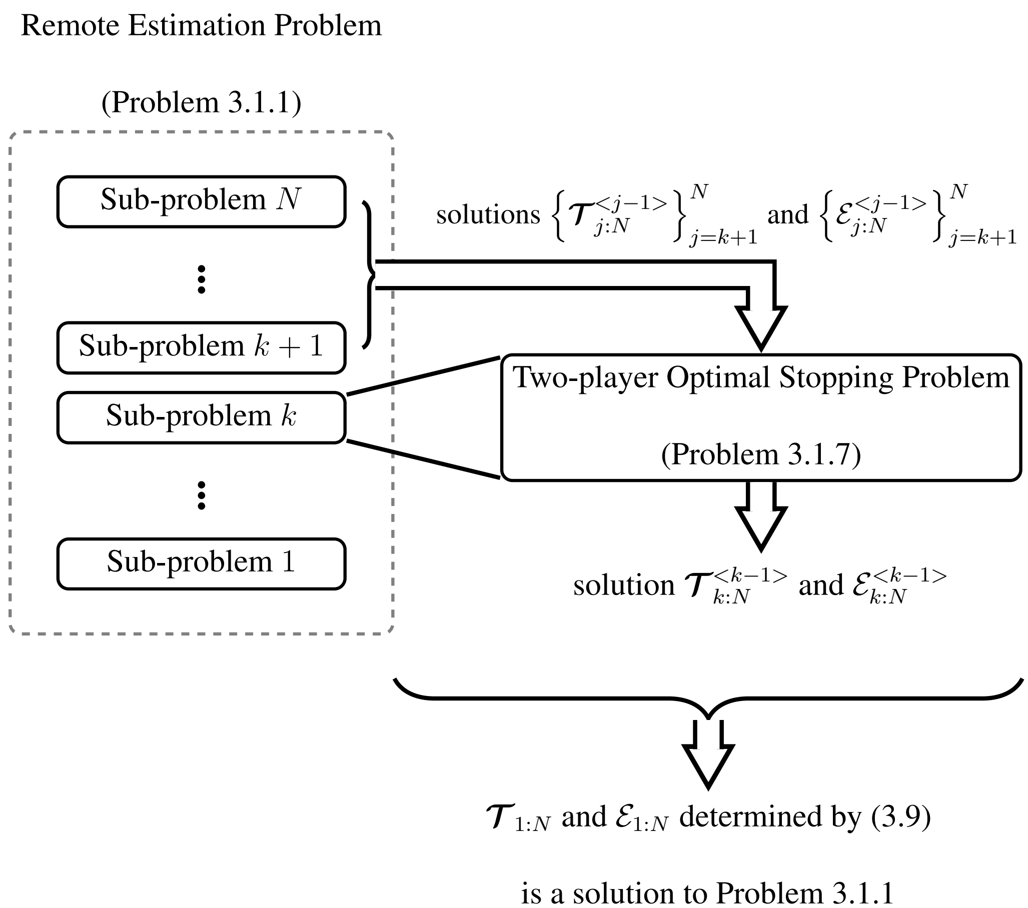


Figure 3.1: The problem solving strategy for Problem 3.1.1

to construct the stopping costs $\left\{c'_j\right\}_{j=k}^{N}$. Once solutions to all the Sub-problems are found, we determine transmission policies $\mathcal{T}_{1:N}$ and estimation rules $\mathcal{E}_{1:N}$ for Problem 3.1.1 in the following way:

$$\mathcal{T}_j(k-1, x_{k-1}, x_j) = \mathcal{T}_j^{<k-1>}(x_{k-1}, x_j) \tag{3.9a}$$

$$\mathcal{E}_j(k-1, x_{k-1}) = \mathcal{E}_j^{<k-1>}(x_{k-1}) \tag{3.9b}$$

for each j in $\{k, \cdots, N\}$ and k in $\{1, \cdots, N\}$.

In Section 3.2, we solve Sub-problem k. In particular, we show that there exists a jointly optimal solution and describe an iterative procedure for finding a person-by-person optimal solution. In Section 3.3, based on the results of Section 3.2, we verify that the transmission policies and estimation rules determined by (3.9) are a solution to Problem 3.1.1. The diagram in Figure 3.1 depicts the aforementioned problem solving strategy.

3.1.3 Comparative Survey of Related Work

The effect of communication costs in remote state estimation problems was studied in [70, 72–78]. Finite time-horizon problem formulations are consider in [70, 74, 76, 78]. The authors of [74] found a jointly optimal solution for first-order linear processes driven by Gaussian noise where it is shown that the transmission policy for jointly optimal solutions is of threshold-type. An iterative procedure for finding a transmission policy and estimation rule for first-order linear processes is proposed in [70]. The authors performed a convergence analysis on the proposed procedure for first-order linear processes driven by Gaussian noise, which essentially provides an alternative proof of the main result of [74]. The work of [76] considered a problem setting where the sensing unit has energy harvesting capability. The authors showed that the transmission policy for jointly optimal solutions is of threshold-type for a certain class of multi-dimensional linear processes. Preliminary results of our work were presented in [78] for linear processes.

Infinite time-horizon formulations are considered in [72, 73, 75, 77]. The authors of [72] studied the structure of optimal transmission policies for linear processes driven by Gaussian noise, and proposed a procedure to compute an optimal policy based on a value iteration algorithm. In [73], an algorithm for finding a sub-optimal solution was proposed.

For linear processes driven by Gaussian noise, the authors showed that the cost incurred by the proposed algorithm is within a constant factor of the optimum. While the question of whether the transmission policy for jointly optimal solutions is of threshold-type for multi-dimensional linear processes remains unanswered, the authors of [75] analyzed the performance of threshold-type transmission policies for such processes. A computationally efficient method for finding a sub-optimal transmission policy based on polynomial approximation is proposed in [77].

Other interesting remote estimation schemes are reported in [79–92]. The authors of [79] studied the structure of optimal transmission policies and estimation rules for the case where a finite number of transmissions is allowed to the sensing unit. The authors of [80] considered a problem setting where the sensing unit operates with two different sensing qualities, and found an optimal *time-periodic* transmission policies for linear processes driven by Gaussian noise. A remote estimation problem for continuous dynamical systems are studied in [85] where performance of various types of transmission policies is investigated. Results of [85] indicate that for remote estimation under a communication rate constraint, the transmission policy for jointly optimal solutions may not be of threshold-type. A framework in which the sensing unit observes noisy outputs of the process and transmits observed noisy outputs to the estimator is adopted in [83, 84, 87]. On the other hand, a framework in which the sensing unit accesses noisy observations of the state of the process and transmits its best state estimate to the estimator is adopted in [81, 82, 89]. The authors of [92] adopted a certain class of stochastic transmission policies which ensures that linear estimation rules are optimal. The authors of [86] proposed an approximate state estimation scheme based on a sum of Gaussians approach. Remote estimation over shared communication networks is considered in [90]. A problem of scheduling transmission power level for remote estimation was recently studied in [91].

The problem formulation considered in this work is technically different from previous ones found in literature in following ways:

1. We adopt random process models that may neither be linear nor have unimodal or symmetric probability distributions.

2. We consider classes of transmission policies and estimation rules on which no structural assumption is imposed under which the optimality of resulting solutions is lost.

3. We investigate optimization of the given performance criteria over both transmission policies and estimation rules.

3.2 Two-Player Optimal Stopping Problem

In this section, we find a solution to Sub-problem k where the constants $\{c'_j\}_{j=k}^N$ are determined by (3.8) using solutions $\left\{\boldsymbol{\mathcal{T}}_{j:N}^{<j-1>}\right\}_{j=k+1}^N$ and $\left\{\mathcal{E}_{j:N}^{<j-1>}\right\}_{j=k+1}^N$ to preceding sub-problems – Sub-problem $k+1$ to Sub-problem N. We consider two notions of optimality – joint optimality and person-by-person optimality. Our main results state the existence of a jointly optimal solution (Section 3.2.2) and describe an iterative procedure to find a person-by-person optimal solution (Section 3.2.3).

We proceed by re-writing (3.4) into a suitable form using the following Definition.

Definition 3.2.1. *Define a (random) function* $\boldsymbol{\mathcal{P}}_j : \mathbb{X} \to \{0,1\}$ *and a variable* $\hat{x}_j \in \mathbb{X}$ *for* *each j in $\{k, \cdots, N\}$ as follows:*[5]

$$\boldsymbol{\mathcal{P}}_j (x_j) = \boldsymbol{\mathcal{T}}_j^{<k-1>} (0, x_j) \tag{3.10a}$$

$$\hat{x}_j = \mathcal{E}_j^{<k-1>} (0) \tag{3.10b}$$

We refer to $\boldsymbol{\mathcal{P}}_j$ and $\hat{x}_j$ as the (randomized) policy[6] *and estimate at time j (for the initial condition $\mathbf{x}_{k-1} = 0$), respectively.*

Given that $\mathbf{x}_{k-1} = 0$, we can re-write (3.4) as follows:

$$\mathbb{E}_{\mathbf{x}_k} \left[J_k \left(\mathbf{x}_k, \boldsymbol{\mathcal{P}}_{k:N}, \hat{x}_{k:N} \right) \right] \tag{3.11}$$

where for each j in $\{k, \cdots, N\}$,

$$\begin{aligned} & J_j \left(\mathbf{x}_j, \boldsymbol{\mathcal{P}}_{j:N}, \hat{x}_{j:N} \right) \\ & = \left(d^2 \left(\mathbf{x}_j, \hat{x}_j \right) + \mathbb{E}_{\mathbf{x}_{j+1}} \left[J_{j+1} \left(\mathbf{x}_{j+1}, \boldsymbol{\mathcal{P}}_{j+1:N}, \hat{x}_{j+1:N} \right) \,\middle|\, \mathbf{x}_j \right] \right) \cdot \left(1 - \mathbf{R}_j \right) + c'_j \cdot \mathbf{R}_j \end{aligned} \tag{3.12}$$

with $J_{N+1} = 0$, and $\boldsymbol{\mathcal{P}}_j$ dictates $\mathbf{R}_j$ as follows:

$$\mathbf{R}_j = \boldsymbol{\mathcal{P}}_j \left(\mathbf{x}_j \right) \tag{3.13}$$

Note that J_j satisfies

$$\begin{aligned} & \mathbb{E}_{\mathbf{x}_j} \left[J_j \left(\mathbf{x}_j, \boldsymbol{\mathcal{P}}_{j:N}, \hat{x}_{j:N} \right) \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_{j-1} = 0 \right] \\ & = \Bigg(\mathbb{E}_{\mathbf{x}_j} \left[d^2 \left(\mathbf{x}_j, \hat{x}_j \right) \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 0 \right] \\ & \qquad + \mathbb{E}_{\mathbf{x}_{j+1}} \left[J_{j+1} \left(\mathbf{x}_{j+1}, \boldsymbol{\mathcal{P}}_{j+1:N}, \hat{x}_{j+1:N} \right) \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 0 \right] \Bigg) \\ & \quad \cdot \mathbb{P} \left(\mathbf{R}_j = 0 \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_{j-1} = 0 \right) + c'_j \cdot \mathbb{P} \left(\mathbf{R}_j = 1 \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_{j-1} = 0 \right) \end{aligned} \tag{3.14}$$

[5] As k is fixed in Sub-problem k, throughout the section, we drop the dependence of policies and estimates on k.

[6] See Appendix B.2 for a detailed description of randomized policies.

for all j in $\{k, \cdots, N\}$.

Remark 3.2.2. *Let $\boldsymbol{\mathcal{P}}^*_{k:N}$ and $\hat{x}^*_{k:N}$ be optimal policies and estimates for* (3.11)*, respectively. From* (3.14) *and our main results, we can see that*

$$\mathbb{E}_{\mathbf{x}_k}\left[J_k\left(\mathbf{x}_k, \boldsymbol{\mathcal{P}}^*_{k:N}, \hat{x}^*_{k:N}\right)\right] = \mathbb{E}_{\mathbf{x}_k}\left[J^*_k\left(\mathbf{x}_k, \hat{x}^*_{k:N}\right)\right] \tag{3.15}$$

holds where

$$J^*_j\left(x_j, \hat{x}^*_{j:N}\right) = \min\left\{d^2\left(x_j, \hat{x}^*_j\right) + \mathbb{E}_{\mathbf{x}_{j+1}}\left[J^*_{j+1}\left(\mathbf{x}_{j+1}, \hat{x}^*_{j+1:N}\right) \,\middle|\, \mathbf{x}_j = x_j\right], c'_j\right\} \tag{3.16}$$

with $J^*_{N+1} = 0$.

For any other initial condition $\mathbf{x}_{k-1} = x_{k-1}$, define estimates

$$\hat{x}'^*_j = M^{-1}_j\left(k-1, x_{k-1}, \hat{x}^*_j\right) \tag{3.17}$$

for each j in $\{k, \cdots, N\}$, where M_j is the transformation described in Assumption 3.1.6. Notice that from (3.16) *and by the definition of the transformation M_j, we can observe that*

$$\mathbb{E}_{\mathbf{x}_k}\left[J^*_k\left(\mathbf{x}_k, \hat{x}^*_{k:N}\right) \,\middle|\, \mathbf{x}_{k-1} = 0\right] = \mathbb{E}_{\mathbf{x}_k}\left[J^*_k\left(\mathbf{x}_k, \hat{x}'^*_{k:N}\right) \,\middle|\, \mathbf{x}_{k-1} = x_{k-1}\right]$$

This implies that the value of (3.4) *evaluated at an optimal solution does not depend on the initial condition, and by finding an optimal solution to* (3.11)*, we can derive a solution to Sub-problem k using the following relation:*

$$\boldsymbol{\mathcal{T}}^{<k-1>}_j\left(x_{k-1}, x_j\right) = \boldsymbol{\mathcal{P}}^*_j\left(M_j\left(k-1, x_{k-1}, x_j\right)\right) \tag{3.18a}$$

$$\mathcal{E}^{<k-1>}_j\left(x_{k-1}\right) = M^{-1}_j\left(k-1, x_{k-1}, \hat{x}^*_j\right) \tag{3.18b}$$

■

Based on Remark 3.2.2, to solve Sub-problem k, we will find optimal policies and estimates for the initial condition $\mathbf{x}_{k-1} = 0$, and derive a solution to Sub-problem k using (3.18).

3.2.1 Definitions and Preliminary Results

We restate Definition 3.1.8 and Definition 3.1.9 as follows.

Definition 3.2.3. *Policies $\boldsymbol{\mathcal{P}}^*_{k:N}$ and estimates $\hat{x}^*_{k:N}$ are said to be a jointly optimal solution for* (3.11) *if they achieve the global minimum.*

Definition 3.2.4. *Policies $\boldsymbol{\mathcal{P}}^*_{k:N}$ and estimates $\hat{x}^*_{k:N}$ are said to be a person-by-person optimal solution for* (3.11) *if the following relations hold:*

$$\begin{aligned}\mathbb{E}_{\mathbf{x}_k}\left[J_k\left(\mathbf{x}_k, \boldsymbol{\mathcal{P}}^*_{k:N}, \hat{x}^*_{k:N}\right)\right] &= \min_{\boldsymbol{\mathcal{P}}_{k:N}} \mathbb{E}_{\mathbf{x}_k}\left[J_k\left(\mathbf{x}_k, \boldsymbol{\mathcal{P}}_{k:N}, \hat{x}^*_{k:N}\right)\right] \\ &= \min_{\hat{x}_{k:N}} \mathbb{E}_{\mathbf{x}_k}\left[J_k\left(\mathbf{x}_k, \boldsymbol{\mathcal{P}}^*_{k:N}, \hat{x}_{k:N}\right)\right] \end{aligned} \tag{3.19}$$

Equation (3.19) *implies that given decision functions $\boldsymbol{\mathcal{P}}^*_{k:N}$ of one player (sensing unit), $\hat{x}^*_{k:N}$ are optimal decision variables of the other player (estimator), and vice versa.*

To find an optimal solution for (3.11), we define *best response mappings* $\mathfrak{P}$ and $\mathfrak{X}$ as follows.

Definition 3.2.5. *Given estimates $\hat{x}_{k:N}$, define $\mathfrak{P}\left(\hat{x}_{k:N}\right)$ as a collection of policies $\boldsymbol{\mathcal{P}}_{k:N}$ for which it holds that*

$$\mathbb{E}_{\mathbf{x}_k}\left[J_k\left(\mathbf{x}_k, \boldsymbol{\mathcal{P}}_{k:N}, \hat{x}_{k:N}\right)\right] = \min_{\boldsymbol{\mathcal{P}}'_{k:N}} \mathbb{E}_{\mathbf{x}_k}\left[J_k\left(\mathbf{x}_k, \boldsymbol{\mathcal{P}}'_{k:N}, \hat{x}_{k:N}\right)\right] \tag{3.20}$$

Definition 3.2.6. *Given policies $\boldsymbol{\mathcal{P}}_{k:N}$, define $\mathfrak{X}\left(\boldsymbol{\mathcal{P}}_{k:N}\right)$ as a collection of estimates $\hat{x}_{k:N}$ for which it holds that*

$$\mathbb{E}_{\mathbf{x}_k}\left[J_k\left(\mathbf{x}_k, \boldsymbol{\mathcal{P}}_{k:N}, \hat{x}_{k:N}\right)\right] = \min_{\hat{x}'_{k:N}} \mathbb{E}_{\mathbf{x}_k}\left[J_k\left(\mathbf{x}_k, \boldsymbol{\mathcal{P}}_{k:N}, \hat{x}'_{k:N}\right)\right] \tag{3.21}$$

Definition 3.2.7. *Policies $\boldsymbol{\mathcal{P}}_{k:N}$ are said to be degenerate if there exists $j_0 \in \{k, \cdots, N\}$ for which it holds that*

$$\mathbb{P}\left(\mathbf{R}_{j_0} = 0 \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_{j_0-1} = 0\right) = 0 \tag{3.22}$$

Remark 3.2.8. *Let $\boldsymbol{\mathcal{P}}_{k:N}$ be degenerate policies for which* (3.22) *holds. Then, from* (3.14)*, we can derive that*

$$\mathbb{E}_{\mathbf{x}_{j_0}}\left[J_{j_0}\left(\mathbf{x}_{j_0}, \boldsymbol{\mathcal{P}}_{j_0:N}, \hat{x}_{j_0:N}\right) \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_{j_0-1} = 0\right] = c'_{j_0} \tag{3.23}$$

Proposition 3.2.9. *Consider that policies $\boldsymbol{\mathcal{P}}_{k:N}$ and estimates $\hat{x}_{k:N}$ are given. Suppose that the policies are non-degenerate. Then $\boldsymbol{\mathcal{P}}_{k:N}$ belong to $\mathfrak{P}\left(\hat{x}_{k:N}\right)$ if and only if*

$$\begin{aligned} &\mathbb{E}_{\mathbf{x}_j}\left[J_j\left(\mathbf{x}_j, \boldsymbol{\mathcal{P}}_{j:N}, \hat{x}_{j:N}\right) \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_{j-1} = 0\right] \\ &= \mathbb{E}_{\mathbf{x}_j}\left[J_j^*\left(\mathbf{x}_j, \hat{x}_{j:N}\right) \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_{j-1} = 0\right] \end{aligned} \tag{3.24}$$

holds for all j in $\{k, \cdots, N\}$, where

$$J_j^*\left(x_j, \hat{x}_{j:N}\right) = \min\left\{d^2\left(x_j, \hat{x}_j\right) + \mathbb{E}_{\mathbf{x}_{j+1}}\left[J_{j+1}^*\left(\mathbf{x}_{j+1}, \hat{x}_{j+1:N}\right) \,\middle|\, \mathbf{x}_j = x_j\right], c'_j\right\} \tag{3.25}$$

with $J_{N+1}^* = 0$.

The proof follows from (3.14), Definition 3.2.5, and the fact that

$$\min_{\boldsymbol{\mathcal{P}}'_{k:N}} \mathbb{E}_{\mathbf{x}_k}\left[J_k\left(\mathbf{x}_k, \boldsymbol{\mathcal{P}}'_{k:N}, \hat{x}_{k:N}\right)\right] = \mathbb{E}_{\mathbf{x}_k}\left[J_k^*\left(\mathbf{x}_k, \hat{x}_{k:N}\right)\right]$$

We omit the detail for brevity.

Corollary 3.2.10. *Given estimates $\hat{x}_{k:N}$, consider (deterministic) policies $\mathcal{P}_{k:N}$ defined by*

$$\mathcal{P}_j(\mathbf{x}_j) = \begin{cases} 0 & \textit{if } \mathbf{x}_j \in \mathbb{D}_j \\ \\ 1 & \textit{otherwise} \end{cases} \tag{3.26}$$

for each j in $\{k, \cdots, N\}$, where $\mathbb{D}_j$ is a measurable set for which $\underline{\mathbb{D}}_j \subseteq \mathbb{D}_j \subseteq \overline{\mathbb{D}}_j$ holds with

$$\overline{\mathbb{D}}_j = \left\{ x_j \in \mathbb{X} \,\middle|\, d^2\left(x_j, \hat{x}_j\right) + \mathbb{E}_{\mathbf{x}_{j+1}}\left[J^*_{j+1}\left(\mathbf{x}_{j+1}, \hat{x}_{j+1:N}\right) \,\middle|\, \mathbf{x}_j = x_j \right] \leq c'_j \right\} \tag{3.27a}$$

$$\underline{\mathbb{D}}_j = \left\{ x_j \in \mathbb{X} \,\middle|\, d^2\left(x_j, \hat{x}_j\right) + \mathbb{E}_{\mathbf{x}_{j+1}}\left[J^*_{j+1}\left(\mathbf{x}_{j+1}, \hat{x}_{j+1:N}\right) \,\middle|\, \mathbf{x}_j = x_j \right] < c'_j \right\} \tag{3.27b}$$

Then it holds that $\mathcal{P}_{k:N} \in \mathfrak{P}\left(\hat{x}_{k:N}\right)$.

Remark 3.2.11. *Given estimates $\hat{x}_{k:N}$, let $\boldsymbol{\mathcal{P}}_{k:N}$ be non-degenerate policies for which $\boldsymbol{\mathcal{P}}_{k:N} \in \mathfrak{P}\left(\hat{x}_{k:N}\right)$ holds. Then Proposition 3.2.9 implies that*

1. $\mathbb{P}\left(\mathbf{x}_j \in \overline{\mathbb{D}}_j \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 0\right) = 1$
2. $\mathbb{P}\left(\mathbf{x}_j \in \underline{\mathbb{D}}_j \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 1\right) = 0$

where $\overline{\mathbb{D}}_j$ and $\underline{\mathbb{D}}_j$ are given in (3.27).

Proposition 3.2.12. *Consider that policies $\boldsymbol{\mathcal{P}}_{k:N}$ and estimates $\hat{x}_{k:N}$ are given. Suppose that the policies are non-degenerate. Then $\hat{x}_{k:N}$ belong to $\mathfrak{X}\left(\boldsymbol{\mathcal{P}}_{k:N}\right)$ if and only if*

$$\mathbb{E}_{\mathbf{x}_j}\left[d^2\left(\mathbf{x}_j, \hat{x}_j\right) \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 0 \right] = \min_{\hat{x}'_j \in \mathbb{X}} \mathbb{E}_{\mathbf{x}_j}\left[d^2\left(\mathbf{x}_j, \hat{x}'_j\right) \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 0 \right] \tag{3.28}$$

holds for all j in $\{k, \cdots, N\}$.

The proof follows from (3.14) and Definition 3.2.6. We omit the detail for brevity.

Corollary 3.2.13. *Given non-degenerate policies $\boldsymbol{\mathcal{P}}_{k:N}$, consider estimates $\hat{x}_{k:N}$ defined by*

$$\hat{x}_j \in \arg\min_{\hat{x}_j \in \mathbb{X}} \mathbb{E}_{\mathbf{x}_j} \left[d^2\left(\mathbf{x}_j, \hat{x}_j\right) \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 0 \right] \tag{3.29}$$

for each j in $\{k, \cdots, N\}$. Then it holds that $\hat{x}_{k:N} \in \mathfrak{X}\left(\boldsymbol{\mathcal{P}}_{k:N}\right)$.

Proposition 3.2.14. *Consider functions $\{\mathcal{G}_j\}_{j=k}^{N}$ defined by*

$$\mathcal{G}_j\left(x_{j-1}, \hat{x}_{j:N}\right) \stackrel{def}{=} \mathbb{E}_{\mathbf{x}_j} \left[J_j^*\left(\mathbf{x}_j, \hat{x}_{j:N}\right) \,\middle|\, \mathbf{x}_{j-1} = x_{j-1} \right] \tag{3.30}$$

where J_j^* is given in (3.25)*. $\{\mathcal{G}_j\}_{j=k}^{N}$ are all continuous functions.*[7]

The proof is given in Appendix B.4. The following is a consequence of Proposition 3.2.14.

Corollary 3.2.15. *Given estimates $\hat{x}_{k:N}$, the sets $\overline{\mathbb{D}}_j$ and $\underline{\mathbb{D}}_j$ defined in* (3.27) *are closed and open, respectively, for all j in $\{k, \cdots, N\}$.*

3.2.2 Existence of a Jointly Optimal Solution

Let us define

$$\mathcal{G}\left(\hat{x}_{k:N}\right) \stackrel{def}{=} \mathbb{E}_{\mathbf{x}_k}\left[J_k^*\left(\mathbf{x}_k, \hat{x}_{k:N}\right)\right] \tag{3.31}$$

where J_k^* is given in (3.25). Note that $\mathcal{G}\left(\hat{x}_{k:N}\right) = \min_{\boldsymbol{\mathcal{P}}'_{k:N}} \mathbb{E}_{\mathbf{x}_k}\left[J_k\left(\mathbf{x}_k, \boldsymbol{\mathcal{P}}'_{k:N}, \hat{x}_{k:N}\right)\right]$.

[7] Note that $\mathcal{G}_j$ is a function defined on $\mathbb{X}^{N-j+2}$. See Appendix B.1 for some remarks on the continuity of functions on a product space.

Proposition 3.2.16. *Let $\hat{x}^*_{k:N}$ be the estimates that achieve the global minimum of* (3.31). *The policies $\boldsymbol{\mathcal{P}}^*_{k:N}$ satisfying*

$$\boldsymbol{\mathcal{P}}^*_{k:N} \in \mathfrak{P}\left(\hat{x}^*_{k:N}\right)$$

are not degenerate in the sense of Definition 3.2.7.

The proof is given in Appendix B.5.

Theorem 3.2.17. *There exist estimates $\hat{x}^*_{k:N}$ that achieve the global minimum of* (3.31). *Furthermore, in conjunction with these estimates $\hat{x}^*_{k:N}$, the policies $\boldsymbol{\mathcal{P}}^*_{k:N}$ satisfying*

$$\boldsymbol{\mathcal{P}}^*_{k:N} \in \mathfrak{P}\left(\hat{x}^*_{k:N}\right)$$

are a jointly optimal solution for (3.11).

To prove Theorem 3.2.17, we need the following Lemma.

Lemma 3.2.18. *There exists a compact set $\mathbb{K} \subset \mathbb{X}^{N-k+1}$ for which*

$$\min_{\hat{x}_{k:N} \in \mathbb{K}} \mathcal{G}\left(\hat{x}_{k:N}\right) \leq \mathcal{G}\left(\hat{x}'_{k:N}\right)$$

holds for all $\hat{x}'_{k:N}$ in $\mathbb{X}^{N-k+1}$.

The proof is given in Appendix B.5.

Proof of Theorem 3.2.17: Recall the definitions of $\mathcal{G}_k$ and $\mathcal{G}$ given in (3.30) and (3.31), respectively. According to Proposition 3.2.14 and by the fact that $\mathcal{G}\left(\hat{x}_{k:N}\right) = \mathcal{G}_k\left(0, \hat{x}_{k:N}\right)$, we can see that $\mathcal{G}$ is a continuous function. Due to Lemma 3.2.18, in what regards to finding a global minimizer of $\mathcal{G}$, we may assume that the domain of $\mathcal{G}$ is compact. Hence, by the continuity of $\mathcal{G}$ and compactness of its domain, there exist estimates $\hat{x}^*_{k:N}$ that achieve the global minimum of $\mathcal{G}$. Let us choose policies $\boldsymbol{\mathcal{P}}^*_{k:N}$ satisfying

$\boldsymbol{\mathcal{P}}_{k:N}^{*} \in \mathfrak{P}\left(\hat{x}_{k:N}^{*}\right)$ using, for instance, Corollary 3.2.10. Since $\hat{x}_{k:N}^{*}$ is a global minimizer of $\mathcal{G}$ and $\boldsymbol{\mathcal{P}}_{k:N}^{*} \in \mathfrak{P}\left(\hat{x}_{k:N}^{*}\right)$ holds, we conclude that the solution $\boldsymbol{\mathcal{P}}_{k:N}^{*}$ and $x_{k:N}^{*}$ is jointly optimal for (3.11). $\square$

3.2.3 Iterative Procedure for Finding a Person-by-Person Optimal Solution

As numerically illustrated in [70], the function $\mathcal{G}$ in (3.31) may be non-convex, and finding a jointly optimal solution to (3.11) would be computationally intractable. Instead, in this section, we seek a person-by-person optimal solution to (3.11). An iterative procedure for finding such a solution is described in Procedure 1, where η is a pre-selected non-negative constant that determines a stopping criterion of the procedure.

Procedure 1: Finding a Person-by-Person Optimal Solution

```
input  : η ≥ 0, x̂_{k:N}^{(0)}
output: P_{k:N}^{(i+1)}, x̂_{k:N}^{(i)}
begin
    j ← N
    while j ≥ k do
        Choose P_j^{(1)} using (3.26)
        j ← j − 1
    i ← 0
    repeat
        i ← i + 1
        j ← k
        while j ≤ N do
            Choose x̂_j^{(i)} using (3.29)
            j ← j + 1
        j ← N
        while j ≥ k do
            Choose P_j^{(i+1)} using (3.26)
            j ← j − 1
    until |G(x̂_{k:N}^{(i)}) − G(x̂_{k:N}^{(i−1)})| ≤ η
```

In what follows, we analyze convergence properties of the sequence of solutions computed by Line $8-16$ of Procedure 1. We first define convergence of policies and estimates. To proceed, for each $\mathbb{A}$ in $\mathfrak{B}$, let us define

$$\mu_{j|j}^{(i)}(\mathbb{A}) = \mathbb{P}\left(\mathbf{x}_j \in \mathbb{A} \,\middle|\, \mathbf{R}_k^{(i)} = 0, \cdots, \mathbf{R}_j^{(i)} = 0\right)$$
$$\mu_{j|j}(\mathbb{A}) = \mathbb{P}\left(\mathbf{x}_j \in \mathbb{A} \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 0\right)$$

Definition 3.2.19. *Let $\left\{\boldsymbol{\mathcal{P}}_{k:N}^{(i)}\right\}_{i\in\mathbb{N}}$ be a sequence of policies. The sequence is said to converge to $\boldsymbol{\mathcal{P}}_{k:N}$ if it holds that*[8]

$$\mu_{j|j}^{(i)} \xrightarrow{w} \mu_{j|j} \tag{3.32a}$$

and

$$\begin{aligned}&\mathbb{P}\left(\mathbf{R}_j^{(i)} = 0 \,\middle|\, \mathbf{R}_k^{(i)} = 0, \cdots, \mathbf{R}_{j-1}^{(i)} = 0\right)\\ &\xrightarrow{i\to\infty} \mathbb{P}\left(\mathbf{R}_j = 0 \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_{j-1} = 0\right)\end{aligned} \tag{3.32b}$$

for all j in $\{k, \cdots, N\}$. In addition, two sets of policies $\boldsymbol{\mathcal{P}}_{k:N}$ and $\boldsymbol{\mathcal{P}}'_{k:N}$ are said to be equal if it holds that[9]

$$\mu_{j|j} = \mu'_{j|j} \tag{3.33a}$$

and

$$\begin{aligned}&\mathbb{P}\left(\mathbf{R}_j = 0 \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_{j-1} = 0\right)\\ &= \mathbb{P}\left(\mathbf{R}'_j = 0 \,\middle|\, \mathbf{R}'_k = 0, \cdots, \mathbf{R}'_{j-1} = 0\right)\end{aligned} \tag{3.33b}$$

for all j in $\{k, \cdots, N\}$.

[8] See Definition B.3.6 for the weak convergence of probability measures.

[9] $\mu_{j|j} = \mu'_{j|j}$ implies that $\mu_{j|j}(\mathbb{A}) = \mu'_{j|j}(\mathbb{A})$ holds for all $\mathbb{A}$ in $\mathfrak{B}$.

Remark 3.2.20 (Uniqueness of the Limit of Policies)**.** *Let* $\left\{\boldsymbol{\mathcal{P}}^{(i)}_{k:N}\right\}_{i\in\mathbb{N}}$ *be a sequence of policies that converge to* $\boldsymbol{\mathcal{P}}_{k:N}$ *and* $\boldsymbol{\mathcal{P}}'_{k:N}$*. Then the two sets of the policies* $\boldsymbol{\mathcal{P}}_{k:N}$ *and* $\boldsymbol{\mathcal{P}}'_{k:N}$ *are equal. To see this, using the definition of the weak convergence of probability measures, we can derive that*

$$\int_{\mathbb{X}} g \,\mathrm{d}\mu_{j|j} = \int_{\mathbb{X}} g \,\mathrm{d}\mu'_{j|j} \tag{3.34}$$

for every bounded, continuous function $g : \mathbb{X} \to \mathbb{R}$*. Then, by applying Lemma 9.3.2 in [93], we can see that* (3.33) *holds for all* j *in* $\{k, \cdots, N\}$*.*

Definition 3.2.21. *Let* $\left\{\hat{x}^{(i)}_{k:N}\right\}_{i\in\mathbb{N}}$ *be a sequence of estimates. The sequence is said to converge to* $\hat{x}_{k:N}$ *if it holds that*

$$\lim_{i\to\infty} d\left(\hat{x}^{(i)}_j, \hat{x}_j\right) = 0 \tag{3.35}$$

for all j *in* $\{k, \cdots, N\}$*. In addition, two sets of estimates* $\hat{x}_{k:N}$ *and* $\hat{x}'_{k:N}$ *are said to be equal if it holds that*

$$d\left(\hat{x}_j, \hat{x}'_j\right) = 0 \tag{3.36}$$

for all j *in* $\{k, \cdots, N\}$*.*

In the following Theorem, we examine convergence properties of sequences $\left\{\boldsymbol{\mathcal{P}}^{(i)}_{k:N}\right\}_{i\in\mathbb{N}}$ and $\left\{\hat{x}^{(i)}_{k:N}\right\}_{i\in\mathbb{N}}$ that satisfy

$$\boldsymbol{\mathcal{P}}^{(i)}_{k:N} \in \mathfrak{P}\left(\hat{x}^{(i-1)}_{k:N}\right) \tag{3.37a}$$

$$\hat{x}^{(i)}_{k:N} \in \mathfrak{X}\left(\boldsymbol{\mathcal{P}}^{(i)}_{k:N}\right) \tag{3.37b}$$

To proceed, we make the following assumption.

Assumption 3.2.22. *Consider sequences* $\left\{\boldsymbol{\mathcal{P}}_{k:N}^{(i)}\right\}_{i\in\mathbb{N}}$ *and* $\left\{\hat{x}_{k:N}^{(i)}\right\}_{i\in\mathbb{N}}$ *that satisfy* (3.37). *Suppose that the subsequences* $\left\{\boldsymbol{\mathcal{P}}_{k:N}^{(i_l)}\right\}_{l\in\mathbb{N}}$, $\left\{\hat{x}_{k:N}^{(i_l)}\right\}_{l\in\mathbb{N}}$, *and* $\left\{\hat{x}_{k:N}^{(i_l-1)}\right\}_{l\in\mathbb{N}}$ *converge to* $\boldsymbol{\mathcal{P}}_{k:N}$, $\hat{x}_{k:N}$, *and* $\hat{x}'_{k:N}$, *respectively. We assume that* $\hat{x}_{k:N}\in\mathfrak{X}\left(\boldsymbol{\mathcal{P}}_{k:N}\right)$.

Theorem 3.2.23. *Let* $\left\{\boldsymbol{\mathcal{P}}_{k:N}^{(i)}\right\}_{i\in\mathbb{N}}$ *and* $\left\{\hat{x}_{k:N}^{(i)}\right\}_{i\in\mathbb{N}}$ *be a sequence of solutions satisfying* (3.37). *Suppose that Assumption 3.2.22 holds and the policies* $\left\{\boldsymbol{\mathcal{P}}_{k:N}^{(i)}\right\}_{i\in\mathbb{N}}$ *are strictly non-degenerate, i.e., there exists a positive constant* ϵ *for which*

$$\mathbb{P}\left(\mathbf{R}_j^{(i)}=0\,\middle|\,\mathbf{R}_k^{(i)}=0,\cdots,\mathbf{R}_{j-1}^{(i)}=0\right)>\epsilon \tag{3.38}$$

holds for all i *in* $\mathbb{N}$ *and* j *in* $\{k,\cdots,N\}$. *Then, the sequence of the solutions* $\left\{\boldsymbol{\mathcal{P}}_{k:N}^{(i)}\right\}_{i\in\mathbb{N}}$ *and* $\left\{\hat{x}_{k:N}^{(i)}\right\}_{i\in\mathbb{N}}$ *has a convergent subsequence, and the limit of any convergence subsequence is a person-by-person optimal solution.*

To prove Theorem 3.2.23, we need the following Lemma.

Lemma 3.2.24. *Consider sequences* $\left\{\boldsymbol{\mathcal{P}}_{k:N}^{(i)}\right\}_{i\in\mathbb{N}}$ *and* $\left\{\hat{x}_{k:N}^{(i)}\right\}_{i\in\mathbb{N}}$ *satisfying* (3.37). *Suppose that* $\left\{\boldsymbol{\mathcal{P}}_{k:N}^{(i)}\right\}_{i\in\mathbb{N}}$ *are strictly non-degenerate and* $\left\{\hat{x}_{k:N}^{(i-1)}\right\}_{i\in\mathbb{N}}$ *converges to* $\hat{x}'_{k:N}$. *Then, the sequence* $\left\{\boldsymbol{\mathcal{P}}_{k:N}^{(i)}\right\}_{i\in\mathbb{N}}$ *has a convergent subsequence, and the limit* $\boldsymbol{\mathcal{P}}_{k:N}$ *of any convergence subsequence satisfies* $\boldsymbol{\mathcal{P}}_{k:N}\in\mathfrak{P}\left(\hat{x}'_{k:N}\right)$.

The proof is given in Appendix B.6.

Proof of Theorem 3.2.23: We first note that according to Lemma B.5.2, the sequence of the estimates $\left\{\hat{x}_{k:N}^{(i)}\right\}_{i\in\mathbb{N}}$ is contained in a compact set. By the compactness, there exists an infinite subset $\mathbb{I}$ of $\mathbb{N}$ for which the subsequences $\left\{\hat{x}_{k:N}^{(i)}\right\}_{i\in\mathbb{I}}$ and $\left\{\hat{x}_{k:N}^{(i-1)}\right\}_{i\in\mathbb{I}}$ are convergent. Let $\hat{x}_{k:N}$ and $\hat{x}'_{k:N}$ be respective limits of $\left\{\hat{x}_{k:N}^{(i)}\right\}_{i\in\mathbb{I}}$ and $\left\{\hat{x}_{k:N}^{(i-1)}\right\}_{i\in\mathbb{I}}$. Also, according to Lemma 3.2.24, there is an infinite subset $\mathbb{I}'$ of $\mathbb{I}$ for which the subsequence $\left\{\boldsymbol{\mathcal{P}}_{k:N}^{(i)}\right\}_{i\in\mathbb{I}'}$ is convergent. Let $\boldsymbol{\mathcal{P}}_{k:N}$ be the limit of $\left\{\boldsymbol{\mathcal{P}}_{k:N}^{(i)}\right\}_{i\in\mathbb{I}'}$.

To complete the proof, it remains to show that $\boldsymbol{\mathcal{P}}_{k:N}$ and $\hat{x}_{k:N}$ constitute a person-by-person optimal solution, i.e., it holds that

$$\boldsymbol{\mathcal{P}}_{k:N} \in \mathfrak{P}\left(\hat{x}_{k:N}\right) \tag{3.39a}$$

$$\hat{x}_{k:N} \in \mathfrak{X}\left(\boldsymbol{\mathcal{P}}_{k:N}\right) \tag{3.39b}$$

Equation (3.39b) is ensured by Assumption 3.2.22, and it remains to show that (3.39a) is true.

To see this, by contradiction, suppose that $\boldsymbol{\mathcal{P}}_{k:N} \notin \mathfrak{P}\left(\hat{x}_{k:N}\right)$ holds. Note that by Lemma 3.2.24, $\boldsymbol{\mathcal{P}}_{k:N}$ belong to $\mathfrak{P}\left(\hat{x}'_{k:N}\right)$. Then we can see that the following relations hold for policies $\boldsymbol{\mathcal{P}}'_{k:N}$ satisfying $\boldsymbol{\mathcal{P}}'_{k:N} \in \mathfrak{P}\left(\hat{x}_{k:N}\right)$:

$$\begin{aligned}\mathcal{G}\left(\hat{x}_{k:N}\right) &= \mathbb{E}_{\mathbf{x}_k}\left[J_k\left(\mathbf{x}_k, \boldsymbol{\mathcal{P}}'_{k:N}, \hat{x}_{k:N}\right)\right] \\ &\overset{(1)}{<} \mathbb{E}_{\mathbf{x}_k}\left[J_k\left(\mathbf{x}_k, \boldsymbol{\mathcal{P}}_{k:N}, \hat{x}_{k:N}\right)\right] \\ &\overset{(2)}{\leq} \mathbb{E}_{\mathbf{x}_k}\left[J_k\left(\mathbf{x}_k, \boldsymbol{\mathcal{P}}_{k:N}, \hat{x}'_{k:N}\right)\right] = \mathcal{G}\left(\hat{x}'_{k:N}\right)\end{aligned} \tag{3.40}$$

(1) follows from the hypothesis that $\boldsymbol{\mathcal{P}}_{k:N} \notin \mathfrak{P}\left(\hat{x}_{k:N}\right)$; and (2) is due to (3.39b). On the other hand, since $\mathcal{G}$ is decreasing along the sequence $\left\{\hat{x}_{k:N}^{(i)}\right\}_{i\in\mathbb{N}}$, i.e.,

$$\mathcal{G}\left(\hat{x}_{k:N}^{(i)}\right) \geq \mathcal{G}\left(\hat{x}_{k:N}^{(i+1)}\right)$$

holds for all i in $\mathbb{N}$, it holds that $\lim_{i\to\infty}\mathcal{G}\left(\hat{x}_{k:N}^{(i)}\right) = \alpha$ for some real number α. In conjunction with the continuity of $\mathcal{G}$ (see Proposition 3.2.14), this implies that

$$\mathcal{G}\left(\hat{x}_{k:N}\right) = \mathcal{G}\left(\hat{x}'_{k:N}\right) = \alpha$$

which contradicts (3.40). Therefore we conclude that $\boldsymbol{\mathcal{P}}_{k:N} \in \mathfrak{P}\left(\hat{x}_{k:N}\right)$. $\square$

3.3 Optimal Remote State Estimation

Based on the results of Section 3.2, we find a solution to Problem 3.1.1. To proceed, consider a procedure described in Procedure 2.

Procedure 2: Finding solutions to Sub-problems

---

output: $\left\{\boldsymbol{\mathcal{T}}^{*<k-1>}_{k:N}\right\}_{k=1}^{N}$ and $\left\{\mathcal{E}^{*<k-1>}_{k:N}\right\}_{k=1}^{N}$

begin

- $k \leftarrow N$
- **while** $k \geq 1$ **do**
  - **Step** 1**:** Compute constants $\left\{c'^{*}_{j}\right\}_{j=k}^{N}$ according to (3.8) for the solutions $\left\{\boldsymbol{\mathcal{T}}^{*<j-1>}_{j:N}\right\}_{j=k+1}^{N}$ and $\left\{\mathcal{E}^{*<j-1>}_{j:N}\right\}_{j=k+1}^{N}$ to Sub-problem $k+1$ to Sub-problem N.
  - **Step** 2**:** Find $\boldsymbol{\mathcal{T}}^{*<k-1>}_{k:N}$ and $\mathcal{E}^{*<k-1>}_{k:N}$ that is an optimal solution of Sub-problem k with the constants $\left\{c'^{*}_{j}\right\}_{j=k}^{N}$ obtain in **Step** 1.
  - $k \leftarrow k-1$

Based on solutions obtained via Procedure 2, we can state the following Theorem.

Theorem 3.3.1. *Let* $\left\{\boldsymbol{\mathcal{T}}^{*<k-1>}_{k:N}\right\}_{k=1}^{N}$ *and* $\left\{\mathcal{E}^{*<k-1>}_{k:N}\right\}_{k=1}^{N}$ *be solutions to Sub-problems obtained via Procedure 2. For each* k *in* $\{1, \cdots, N\}$*, if* $\boldsymbol{\mathcal{T}}^{*<k-1>}_{k:N}$ *and* $\mathcal{E}^{*<k-1>}_{k:N}$ *are a jointly optimal (person-by-person optimal) solution of Sub-problem* k*, then the transmission policies* $\boldsymbol{\mathcal{T}}^{*}_{1:N}$ *and the estimation rules* $\mathcal{E}^{*}_{1:N}$ *determined by*

$$\boldsymbol{\mathcal{T}}^{*}_{j}(k-1, x_{k-1}, x_{j}) = \boldsymbol{\mathcal{T}}^{*<k-1>}_{j}(x_{k-1}, x_{j}) \tag{3.41a}$$

$$\mathcal{E}^{*}_{j}(k-1, x_{k-1}) = \mathcal{E}^{*<k-1>}_{j}(x_{k-1}) \tag{3.41b}$$

for each j in $\{k, \cdots, N\}$ and k in $\{1, \cdots, N\}$ are jointly optimal (person-by-person optimal) for (3.2).

The proof is given in Appendix B.7

3.4 Application to Specific System Models

In this section, we apply our main results to linear system models and self-propelled particle models.

3.4.1 Linear System Models

Consider

$$\mathbf{x}_{k+1} = A_k \mathbf{x}_k + \mathbf{w}_k \tag{3.42}$$

where $\mathbf{w}_k$ is a random variable with a Gaussian distribution in $\mathbb{R}^n$. We define the metric d using the Euclidean norm $\|\cdot\|_2$ as $d\left(x_k, x'_k\right) = \|x_k - x'_k\|_2$. We note that the metric is invariant under the transformation defined by

$$M_j\left(k-1, x_{k-1}, x_j\right) = x_j - \prod_{l=k-1}^{j-1} A_l x_{k-1}$$

for j in $\{k-1, \cdots, N\}$ and k in $\{1, \cdots, N\}$, where we adopt the convention that $\prod_{l=k-1}^{j-1} A_l = I_n$ if $j = k-1$. It can be verified that Assumption 3.1.4 - Assumption 3.1.6 hold.

With the Euclidean norm, it is straight-forward to see that for given polices $\boldsymbol{\mathcal{P}}_{k:N}$, for each j in $\{k, \cdots, N\}$,

$$\hat{x}_j \in \underset{\hat{x}_j \in \mathbb{R}^n}{\arg\min}\, \mathbb{E}_{\mathbf{x}_j}\left[\|\mathbf{x}_j - \hat{x}_j\|^2 \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 0\right]$$

and

$$\hat{x}_j = \mathbb{E}_{\mathbf{x}_j}\left[\mathbf{x}_j \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 0\right] \tag{3.43}$$

are equivalent, provided that $\mathbb{E}_{\mathbf{x}_j}\left[\|\mathbf{x}_j - \hat{x}_j\|^2 \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 0\right]$ is well-defined for all $\hat{x}_j$ in $\mathbb{R}^n$. In the following Proposition, we show that the statement in Assumption 3.2.22 is valid for the linear system models.

Proposition 3.4.1. *Consider sequences $\left\{\boldsymbol{\mathcal{P}}_{k:N}^{(i)}\right\}_{i\in\mathbb{N}}$ and $\left\{\hat{x}_{k:N}^{(i)}\right\}_{i\in\mathbb{N}}$ that satisfy*

$$\boldsymbol{\mathcal{P}}_{k:N}^{(i)} \in \mathfrak{P}\left(\hat{x}_{k:N}^{(i-1)}\right) \tag{3.44a}$$

$$\hat{x}_{k:N}^{(i)} \in \mathfrak{X}\left(\boldsymbol{\mathcal{P}}_{k:N}^{(i)}\right) \tag{3.44b}$$

Suppose that the subsequences $\left\{\boldsymbol{\mathcal{P}}_{k:N}^{(i_l)}\right\}_{l\in\mathbb{N}}$, $\left\{\hat{x}_{k:N}^{(i_l)}\right\}_{l\in\mathbb{N}}$, and $\left\{\hat{x}_{k:N}^{(i_l-1)}\right\}_{l\in\mathbb{N}}$ converge to $\boldsymbol{\mathcal{P}}_{k:N}$, $\hat{x}_{k:N}$, and $\hat{x}'_{k:N}$, respectively. Then it holds that $\hat{x}_{k:N} \in \mathfrak{X}\left(\boldsymbol{\mathcal{P}}_{k:N}\right)$.

Proof. Let $\mu_{j|j}^{(i)}$ and $\mu_{j|j}$ be probability measures defined as

$$\mu_{j|j}^{(i)}\left(\mathbb{A}\right) = \mathbb{P}\left(\mathbf{x}_j \in \mathbb{A} \,\middle|\, \mathbf{R}_k^{(i)} = 0, \cdots, \mathbf{R}_j^{(i)} = 0\right) \tag{3.45}$$

$$\mu_{j|j}\left(\mathbb{A}\right) = \mathbb{P}\left(\mathbf{x}_j \in \mathbb{A} \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 0\right) \tag{3.46}$$

where $\mathbb{A}$ belongs to $\mathfrak{B}$, and the random variables $\mathbf{R}_j^{(i)}$ and $\mathbf{R}_j$ are dictated by $\boldsymbol{\mathcal{P}}_j^{(i)}$ and $\boldsymbol{\mathcal{P}}_j$, respectively. Since $\left\{\boldsymbol{\mathcal{P}}_{k:N}^{(i_l)}\right\}_{l\in\mathbb{N}}$ converges to $\boldsymbol{\mathcal{P}}_{k:N}$, it holds that $\mu_{j|j}^{(i_l)} \xrightarrow{w} \mu_{j|j}$ for all j in $\{k, \cdots, N\}$. Since $(\mathbb{R}^n, \|\cdot\|_2)$ is a complete, separable metric space, by the weak convergence of $\left\{\mu_{j|j}^{(i_l)}\right\}_{l\in\mathbb{N}}$ and the Skorokhod representation theorem [94], there exist a sequence of random variables $\left\{\mathbf{y}_j^{(i_l)}\right\}_{l\in\mathbb{N}}$ and a random variable $\mathbf{y}_j$ all defined on a common probability space $(\Omega, \mathfrak{F}, \nu)$ in which the following three facts are true:

(F1) $\mu_{j|j}^{(i_l)}$ is the probability measure of $\mathbf{y}_j^{(i_l)}$, i.e., $\nu\left(\left\{\omega \in \Omega \,\middle|\, \mathbf{y}_j^{(i_l)}(\omega) \in \mathbb{A}\right\}\right) = \mu_{j|j}^{(i_l)}(\mathbb{A})$ for each $\mathbb{A}$ in $\mathfrak{B}$.

(F2) $\mu_{j|j}$ is the probability measure of $\mathbf{y}_j$, i.e., $\nu\left(\left\{\omega \in \Omega \,\middle|\, \mathbf{y}_j(\omega) \in \mathbb{A}\right\}\right) = \mu_{j|j}(\mathbb{A})$ for each $\mathbb{A}$ in $\mathfrak{B}$.

(F3) $\left\{\mathbf{y}_j^{(i_l)}\right\}_{l\in\mathbb{N}}$ converges to $\mathbf{y}_j$ almost surely.

Hence, by Proposition 3.2.12 and (3.43), we can derive that

$$\begin{aligned}
\hat{x}_j^{(i_l)} &= \mathbb{E}_{\mathbf{x}_j}\left[\mathbf{x}_j \,\middle|\, \mathbf{R}_k^{(i_l)} = 0, \cdots, \mathbf{R}_j^{(i_l)} = 0\right] \\
&= \int_{\mathbb{R}^n} x \,\mathrm{d}\mu_{j|j}^{(i_l)} \\
&= \int_{\Omega} \mathbf{y}_j^{(i_l)}(\omega)\,\mathrm{d}\nu
\end{aligned} \tag{3.47}$$

Since $\left\{\hat{x}_j^{(i_l-1)}\right\}_{l\in\mathbb{N}}$ is a convergent sequence, it is bounded. By Remark 3.2.11, (3.44a), and Lemma B.3.9, there is a compact set $\mathbb{K}_j$ for which $\mu_{j|j}^{(i_l)}(\mathbb{K}_j) = 1$ for all l in $\mathbb{N}$. Hence, there is a positive real α for which it holds that

$$\int_{\left\{\omega\in\Omega \,\middle|\, \left\|\mathbf{y}_j^{(i_l)}(\omega)\right\|>\alpha\right\}} \mathbf{y}_j^{(i_l)}(\omega)\,\mathrm{d}\nu = \int_{\left\{x\in\mathbb{X} \,\middle|\, \|x\|>\alpha\right\}} x\,\mathrm{d}\mu_{j|j}^{(i_l)} = 0 \tag{3.48}$$

for all l in $\mathbb{N}$. In conjunction with **(F3)**, by an application of Theorem 10.3.6 in [93], we have that

$$\lim_{l\to\infty}\int_{\Omega} \mathbf{y}_j^{(i_l)}(\omega)\,\mathrm{d}\nu = \int_{\Omega} \mathbf{y}_j(\omega)\,\mathrm{d}\nu \tag{3.49}$$

Therefore, from (3.47) and (3.49), we can see that

$$\begin{aligned}
\hat{x}_j = \lim_{l\to\infty} \hat{x}_j^{(i_l)} &= \lim_{l\to\infty} \int_\Omega \mathbf{y}_j^{(i_l)}(\omega)\,\mathrm{d}\nu \\
&= \int_\Omega \mathbf{y}_j(\omega)\,\mathrm{d}\nu \\
&= \int_{\mathbb{X}} x\,\mathrm{d}\mu_{j|j} \\
&= \mathbb{E}\left[\mathbf{x}_j \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 0\right] \qquad (3.50)
\end{aligned}$$

Since this holds for every j in $\{k, \cdots, N\}$, by Proposition 3.2.12, we conclude that $\hat{x}_{k:N} \in \mathfrak{X}\left(\boldsymbol{\mathcal{P}}_{k:N}\right)$. □

3.4.2 Self-Propelled Particle Models

Consider

$$\begin{pmatrix} \mathbf{p}_{1,k+1} \\ \mathbf{p}_{2,k+1} \\ \boldsymbol{\theta}_{k+1} \end{pmatrix} = \begin{pmatrix} \mathbf{p}_{1,k} + \mathbf{r}_k \cdot \cos\left(\boldsymbol{\theta}_k + \boldsymbol{\phi}_k\right) \\ \mathbf{p}_{2,k} + \mathbf{r}_k \cdot \sin\left(\boldsymbol{\theta}_k + \boldsymbol{\phi}_k\right) \\ \boldsymbol{\theta}_k + \boldsymbol{\phi}_k \end{pmatrix} \qquad (3.51)$$

where $\mathbf{p}_{1,k}$, $\mathbf{p}_{2,k}$ take values in $\mathbb{R}$, and $\boldsymbol{\theta}_k$ takes a value in $[-\pi, \pi)$. $\mathbf{r}_k$ and $\boldsymbol{\phi}_k$ are random variables with a Weibull distribution and Wrapped Cauchy distribution, respectively. We define the metric d using the Frobenius norm $\|\cdot\|_F$ as follows:

$$d\left(\begin{pmatrix} p_{1,k} \\ p_{2,k} \\ \theta_k \end{pmatrix}, \begin{pmatrix} p'_{1,k} \\ p'_{2,k} \\ \theta'_k \end{pmatrix}\right) = \left\| \begin{pmatrix} \cos\theta_k & -\sin\theta_k & p_{1,k} \\ \sin\theta_k & \cos\theta_k & p_{2,k} \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} \cos\theta'_k & -\sin\theta'_k & p'_{1,k} \\ \sin\theta'_k & \cos\theta'_k & p'_{2,k} \\ 0 & 0 & 1 \end{pmatrix} \right\|_F$$

where the metric is invariant under the transformation given by

$$M_j\left(k-1, \begin{pmatrix} p_{1,k-1} \\ p_{2,k-1} \\ \theta_{k-1} \end{pmatrix}, \begin{pmatrix} p_{1,j} \\ p_{2,j} \\ \theta_j \end{pmatrix}\right) = \begin{pmatrix} \cos\theta_{k-1} & \sin\theta_{k-1} & 0 \\ -\sin\theta_{k-1} & \cos\theta_{k-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} p_{1,j} - p_{1,k-1} \\ p_{2,j} - p_{2,k-1} \\ \theta_j - \theta_{k-1} \end{pmatrix}$$

It can be verified that Assumption 3.1.4 - Assumption 3.1.6 hold.

By defining $\mathbf{x}_j = \begin{pmatrix} \mathbf{p}_{1,j} & \mathbf{p}_{2,j} & \boldsymbol{\theta}_j \end{pmatrix}^T$ and $\hat{x}_j = \begin{pmatrix} \hat{p}_{1,j} & \hat{p}_{2,j} & \hat{\theta}_j \end{pmatrix}^T$, we can derive

$$\begin{aligned} &\mathbb{E}\left[d^2\left(\mathbf{x}_j, \hat{x}_j\right) \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 0\right] \\ &= \mathbb{E}\left[\left(\mathbf{p}_{1,j} - \hat{p}_{1,j}\right)^2 + \left(\mathbf{p}_{2,j} - \hat{p}_{2,j}\right)^2 + 4 \cdot \left(1 - \cos\left(\boldsymbol{\theta}_j - \hat{\theta}_j\right)\right) \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 0\right] \end{aligned} \tag{3.52}$$

provided that the conditional expectation is well-defined for all $\hat{x}_j$ in $\mathbb{R} \times \mathbb{R} \times [-\pi, \pi)$. The first and second order conditions of optimality for (3.52) yield that a minimizer $\hat{x}_j$ satisfies

$$\hat{p}_{1,j} = \mathbb{E}\left[\mathbf{p}_{1,j} \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 0\right] \tag{3.53a}$$

$$\hat{p}_{2,j} = \mathbb{E}\left[\mathbf{p}_{2,j} \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 0\right] \tag{3.53b}$$

$$\sin\hat{\theta}_j = \frac{\mathbb{E}\left[\sin\boldsymbol{\theta}_j \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 0\right]}{\mathbb{E}^2\left[\sin\boldsymbol{\theta}_j \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 0\right] + \mathbb{E}^2\left[\cos\boldsymbol{\theta}_j \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 0\right]} \tag{3.53c}$$

$$\cos\hat{\theta}_j = \frac{\mathbb{E}\left[\cos\boldsymbol{\theta}_j \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 0\right]}{\mathbb{E}^2\left[\sin\boldsymbol{\theta}_j \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 0\right] + \mathbb{E}^2\left[\cos\boldsymbol{\theta}_j \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 0\right]} \tag{3.53d}$$

provided that at least one of

$$\mathbb{E}\left[\sin\boldsymbol{\theta}_j \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 0\right] \tag{3.54a}$$

$$\mathbb{E}\left[\cos\boldsymbol{\theta}_j \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 0\right] \tag{3.54b}$$

is non-zero. In this case, there exists a unique $\hat{\theta}_j$ that satisfies (3.53c) and (3.53d). If both (3.54a) and (3.54b) are zero, then the value of (3.52) does not depend on $\hat{\theta}_j$. In the following Proposition, we show that the statement in Assumption 3.2.22 is valid for the self-propelled particle models.

Proposition 3.4.2. *Consider sequences* $\left\{\boldsymbol{\mathcal{P}}_{k:N}^{(i)}\right\}_{i\in\mathbb{N}}$ *and* $\left\{\hat{x}_{k:N}^{(i)}\right\}_{i\in\mathbb{N}}$ *that satisfy*

$$\boldsymbol{\mathcal{P}}_{k:N}^{(i)} \in \mathfrak{P}\left(\hat{x}_{k:N}^{(i-1)}\right) \tag{3.55a}$$

$$\hat{x}_{k:N}^{(i)} \in \mathfrak{X}\left(\boldsymbol{\mathcal{P}}_{k:N}^{(i)}\right) \tag{3.55b}$$

Suppose that the subsequences $\left\{\boldsymbol{\mathcal{P}}_{k:N}^{(i_l)}\right\}_{l\in\mathbb{N}}$, $\left\{\hat{x}_{k:N}^{(i_l)}\right\}_{l\in\mathbb{N}}$, *and* $\left\{\hat{x}_{k:N}^{(i_l-1)}\right\}_{l\in\mathbb{N}}$ *converge to* $\boldsymbol{\mathcal{P}}_{k:N}$, $\hat{x}_{k:N}$, *and* $\hat{x}'_{k:N}$, *respectively. Then it holds that* $\hat{x}_{k:N} \in \mathfrak{X}\left(\boldsymbol{\mathcal{P}}_{k:N}\right)$.

Proof. By a similar argument as in the proof of Proposition 3.4.1, we can show that

$$\lim_{l\to\infty} \mathbb{E}\left[\mathbf{p}_{1,j} \,\middle|\, \mathbf{R}_k^{(i_l)} = 0, \cdots, \mathbf{R}_j^{(i_l)} = 0\right] = \mathbb{E}\left[\mathbf{p}_{1,j} \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 0\right] \tag{3.56a}$$

$$\lim_{l\to\infty} \mathbb{E}\left[\mathbf{p}_{2,j} \,\middle|\, \mathbf{R}_k^{(i_l)} = 0, \cdots, \mathbf{R}_j^{(i_l)} = 0\right] = \mathbb{E}\left[\mathbf{p}_{2,j} \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 0\right] \tag{3.56b}$$

$$\lim_{l\to\infty} \mathbb{E}\left[\sin\boldsymbol{\theta}_j \,\middle|\, \mathbf{R}_k^{(i_l)} = 0, \cdots, \mathbf{R}_j^{(i_l)} = 0\right] = \mathbb{E}\left[\sin\boldsymbol{\theta}_j \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 0\right] \tag{3.56c}$$

$$\lim_{l\to\infty} \mathbb{E}\left[\cos\boldsymbol{\theta}_j \,\middle|\, \mathbf{R}_k^{(i_l)} = 0, \cdots, \mathbf{R}_j^{(i_l)} = 0\right] = \mathbb{E}\left[\cos\boldsymbol{\theta}_j \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 0\right] \tag{3.56d}$$

Suppose that at least one of

$$\mathbb{E}\left[\sin\boldsymbol{\theta}_j \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 0\right] \tag{3.57}$$

$$\mathbb{E}\left[\cos\boldsymbol{\theta}_j \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 0\right] \tag{3.58}$$

is non-zero. Note that by Proposition 3.2.12 and (3.53), $\hat{x}_j^{(i_l)} = \begin{pmatrix} \hat{p}_{1,j}^{(i_l)} & \hat{p}_{2,j}^{(i_l)} & \hat{\theta}_j^{(i_l)} \end{pmatrix}^T$ and

$\hat{x}_j = \begin{pmatrix} \hat{p}_{1,j} & \hat{p}_{2,j} & \hat{\theta}_j \end{pmatrix}^T$ satisfy

$$
\begin{aligned}
p_{1,j}^{(i_l)} &= \mathbb{E}\left[\mathbf{p}_{1,j} \,\middle|\, \mathbf{R}_k^{(i_l)} = 0, \cdots, \mathbf{R}_j^{(i_l)} = 0\right] \\
p_{2,j}^{(i_l)} &= \mathbb{E}\left[\mathbf{p}_{2,j} \,\middle|\, \mathbf{R}_k^{(i_l)} = 0, \cdots, \mathbf{R}_j^{(i_l)} = 0\right] \\
\sin \hat{\theta}_j^{(i_l)} &= \frac{\mathbb{E}\left[\sin \boldsymbol{\theta}_j \,\middle|\, \mathbf{R}_k^{(i_l)} = 0, \cdots, \mathbf{R}_j^{(i_l)} = 0\right]}{\mathbb{E}^2\left[\sin \boldsymbol{\theta}_j \,\middle|\, \mathbf{R}_k^{(i_l)} = 0, \cdots, \mathbf{R}_j^{(i_l)} = 0\right] + \mathbb{E}^2\left[\cos \boldsymbol{\theta}_j \,\middle|\, \mathbf{R}_k^{(i_l)} = 0, \cdots, \mathbf{R}_j^{(i_l)} = 0\right]} \\
\cos \hat{\theta}_j^{(i_l)} &= \frac{\mathbb{E}\left[\cos \boldsymbol{\theta}_j \,\middle|\, \mathbf{R}_k^{(i_l)} = 0, \cdots, \mathbf{R}_j^{(i_l)} = 0\right]}{\mathbb{E}^2\left[\sin \boldsymbol{\theta}_j \,\middle|\, \mathbf{R}_k^{(i_l)} = 0, \cdots, \mathbf{R}_j^{(i_l)} = 0\right] + \mathbb{E}^2\left[\cos \boldsymbol{\theta}_j \,\middle|\, \mathbf{R}_k^{(i_l)} = 0, \cdots, \mathbf{R}_j^{(i_l)} = 0\right]}
\end{aligned}
$$

and

$$\lim_{l \to \infty} d\left(\hat{x}_j^{(i_l)}, \hat{x}_j\right) = 0 \tag{3.59}$$

For each j in $\{k, \cdots, N\}$, let us define $\hat{x}_j^* = \begin{pmatrix} \hat{p}_{1,j}^* & \hat{p}_{2,j}^* & \hat{\theta}_j^* \end{pmatrix}$ as

$$
\begin{aligned}
p_{1,j}^* &= \mathbb{E}\left[\mathbf{p}_{1,j} \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 0\right] \\
p_{2,j}^* &= \mathbb{E}\left[\mathbf{p}_{2,j} \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 0\right] \\
\sin \hat{\theta}_j^* &= \frac{\mathbb{E}\left[\sin \boldsymbol{\theta}_j \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 0\right]}{\mathbb{E}^2\left[\sin \boldsymbol{\theta}_j \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 0\right] + \mathbb{E}^2\left[\cos \boldsymbol{\theta}_j \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 0\right]} \\
\cos \hat{\theta}_j^* &= \frac{\mathbb{E}\left[\cos \boldsymbol{\theta}_j \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 0\right]}{\mathbb{E}^2\left[\sin \boldsymbol{\theta}_j \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 0\right] + \mathbb{E}^2\left[\cos \boldsymbol{\theta}_j \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 0\right]}
\end{aligned}
$$

Note that by Proposition 3.2.12 and (3.53), it holds that $\hat{x}_{k:N}^* \in \mathfrak{X}\left(\boldsymbol{\mathcal{P}}_{k:N}\right)$. From (3.56) and (3.59), we can observe that

$$d\left(\hat{x}_j, \hat{x}_j^*\right) \leq d\left(\hat{x}_j^{(i_l)}, \hat{x}_j^*\right) + d\left(\hat{x}_j^{(i_l)}, \hat{x}_j\right) \overset{l \to \infty}{\longrightarrow} 0$$

Therefore, we conclude that $\hat{x}_{k:N} \in \mathfrak{X}\left(\boldsymbol{\mathcal{P}}_{k:N}\right)$.

If both (3.57) and (3.58) are zero, then the value of (3.52) does not depend on $\hat{\theta}_j$, and by Proposition 3.2.12, we can show that $\hat{x}_{k:N} \in \mathfrak{X}\left(\boldsymbol{\mathcal{P}}_{k:N}\right)$ if it holds that

$$p_{1,j} = \mathbb{E}\left[\mathbf{p}_{1,j} \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 0\right]$$
$$p_{2,j} = \mathbb{E}\left[\mathbf{p}_{2,j} \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 0\right]$$

This can be verified by similar arguments given above and (3.56). □

3.5 Application to Animal Tracking and Experimental Results

In this section, we apply the proposed remote estimation scheme to animal tracking, and show preliminary experimental results using a data set collected from the deployment of *animal-borne wireless camera network* in the Gorongosa National Park (Mozambique) in August 2015.[10] The main purpose of the development and deployment of the system was to collect biologically meaningful measurements and videos using GPS, IMU, and Camera all integrated in a single tracking device, where the proposed estimation scheme can be used to determine when to share sensor measurements between tracking devices and how to determine the best location estimates of nearby devices (see Figure 2.5). The sensor measurements and videos are used to study animal group motion. During the deployment, 15 tracking devices were installed on waterbucks and water buffaloes. Figure 3.2 shows the GPS track of a water buffalo, and Figure 3.3 depicts the x-coordinate $(p_{x,k})$,

[10]The development and deployment of animal-borne wireless camera network were performed under a research grant NSF ECCS 1135726.

Disclaimer: The author of this dissertation was NOT involved in the deployment in the Gorongosa National Park.

y-coordinate $(p_{y,k})$, and heading angle θ_k of a portion of the GPS track (contained in the red rectangle in Figure 3.2) in a local North East Up (NEU) coordinate system.

Figure 3.2: A screenshot of the GPS track of a water buffalo in the Google earth

(Timespan: 2015-08-06T00:00:00Z ∼ 2015-08-06T06:00:00Z)

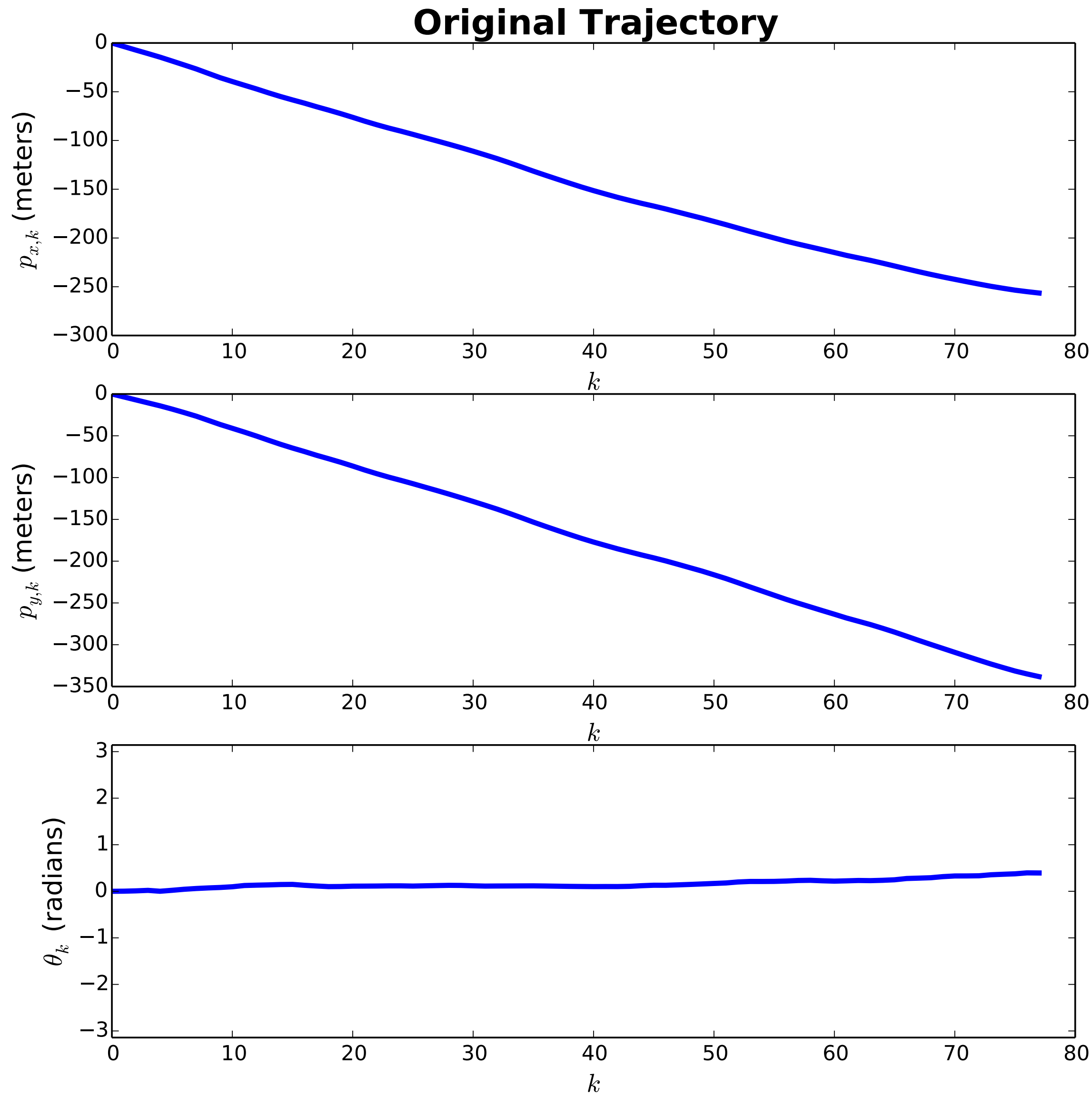


Figure 3.3: The x-coordinate $(p_{x,k})$, y-coordinate $(p_{y,k})$, and heading angle θ_k of a portion of the GPS track (contained in the red rectangle in Figure 3.2) in a local NEU coordinate system. (Timespan: 2015-08-06T05:40:00Z $\sim$ 2015-08-06T05:53:00Z / The origin of the coordinate system : Latitude = -18.9401136372457, Longitude = 34.5337888580266)

We model the movement of the water buffalo using the self-propelled particle model described in Section 3.4.2:

$$\begin{pmatrix} \mathbf{p}_{x,k+1} \\ \mathbf{p}_{y,k+1} \\ \boldsymbol{\theta}_{k+1} \end{pmatrix} = \begin{pmatrix} \mathbf{p}_{x,k} + \mathbf{r}_k \cdot \cos\left(\boldsymbol{\theta}_k + \boldsymbol{\phi}_k\right) \\ \mathbf{p}_{y,k} + \mathbf{r}_k \cdot \sin\left(\boldsymbol{\theta}_k + \boldsymbol{\phi}_k\right) \\ \boldsymbol{\theta}_k + \boldsymbol{\phi}_k \end{pmatrix} \tag{3.60}$$

We assume that the sensing unit makes a decision on transmission of information to the estimator every 10 seconds.

The Weibull random variable $\mathbf{r}_k$ and Wrapped Cauchy random variable $\boldsymbol{\phi}_k$ for the model have respective probability density functions given as follows:

$$f_{\mathbf{r}_k}(r) = \frac{a}{b}\left(\frac{r}{b}\right)^{a-1} e^{-(r/b)^a}, \text{ for } r \geq 0 \tag{3.61a}$$

$$f_{\boldsymbol{\phi}_k}(\phi) = \frac{1}{2\pi} \cdot \frac{\sinh\gamma}{\cosh\gamma - \cos(\phi - \mu)} \tag{3.61b}$$

Using the collected GPS data, we compute the maximum likelihood estimates (MLE) of the parameters for (3.61):

$$(a, b) = (10.3214, 5.9553)$$

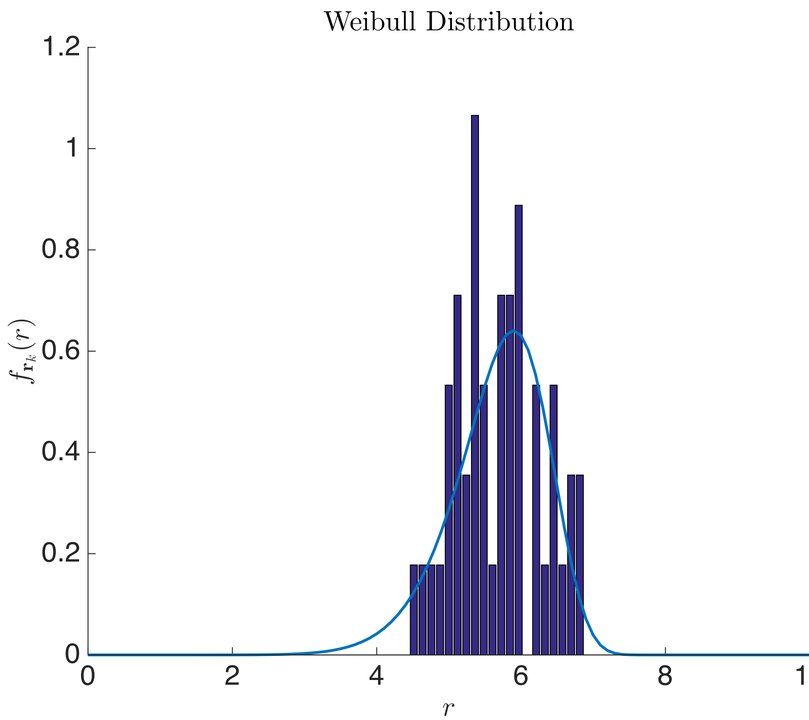
$$(\mu, \gamma) = (0.004, 0.001)$$

Under these parameter choices, the probability density functions are depicted in Figure 3.4.

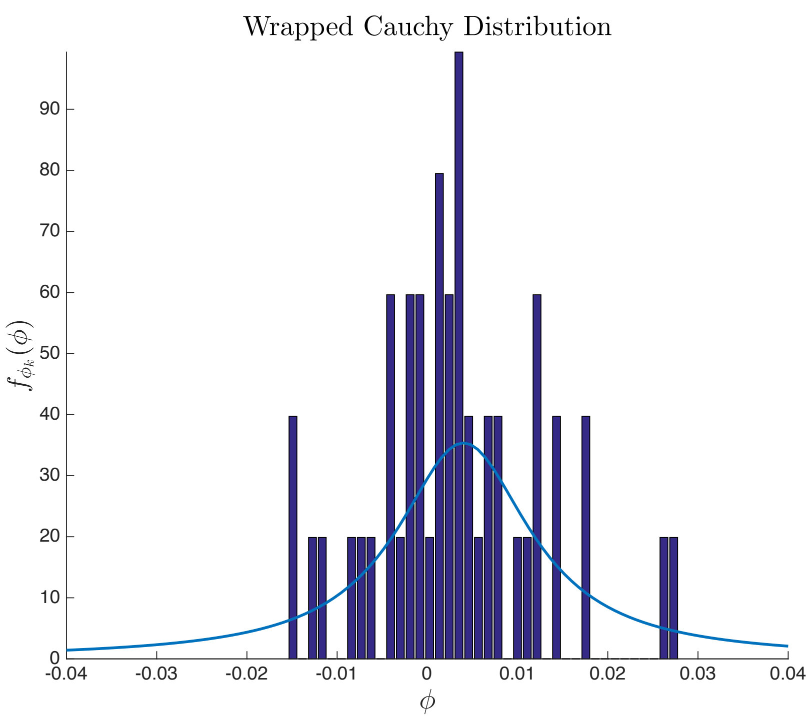
Transmission policies and estimation rules are determined based on Procedure 1 and Procedure 2 for the communication costs $c_k = 5$ for all k in $\{1, \cdots, N\}$ with $N = 78$. Figure 3.5 shows the estimate $\begin{pmatrix} \hat{p}_{x,k} & \hat{p}_{y,k} & \hat{\theta}_k \end{pmatrix}^T$ of the original trajectory of the buffalo, depicted in Figure 3.3; and Figure 3.6 shows the estimation error computed by

$$d\left(x_k, \hat{x}_k\right) = \left\| \begin{pmatrix} \cos\theta_k & -\sin\theta_k & p_{x,k} \\ \sin\theta_k & \cos\theta_k & p_{y,k} \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} \cos\hat{\theta}_k & -\sin\hat{\theta}_k & \hat{p}_{x,k} \\ \sin\hat{\theta}_k & \cos\hat{\theta}_k & \hat{p}_{y,k} \\ 0 & 0 & 1 \end{pmatrix} \right\|_F$$

Note that $d(x_k, \hat{x}_k) = 0$ at time k in Figure 3.6 implies that the sensing unit transmitted information on the full state x_k of the process to the estimator, and the state estimate $\hat{x}_k$ at the estimator was set to the state of the process, i.e., $\hat{x}_k = x_k$.



(a) The probability density function of $\mathbf{r}_k$



(b) The probability density function of $\boldsymbol{\phi}_k$

Figure 3.4: Comparisons between the probability density functions of $\mathbf{r}_k$ and $\boldsymbol{\phi}_k$ under the computed parameter choices and the GPS data.

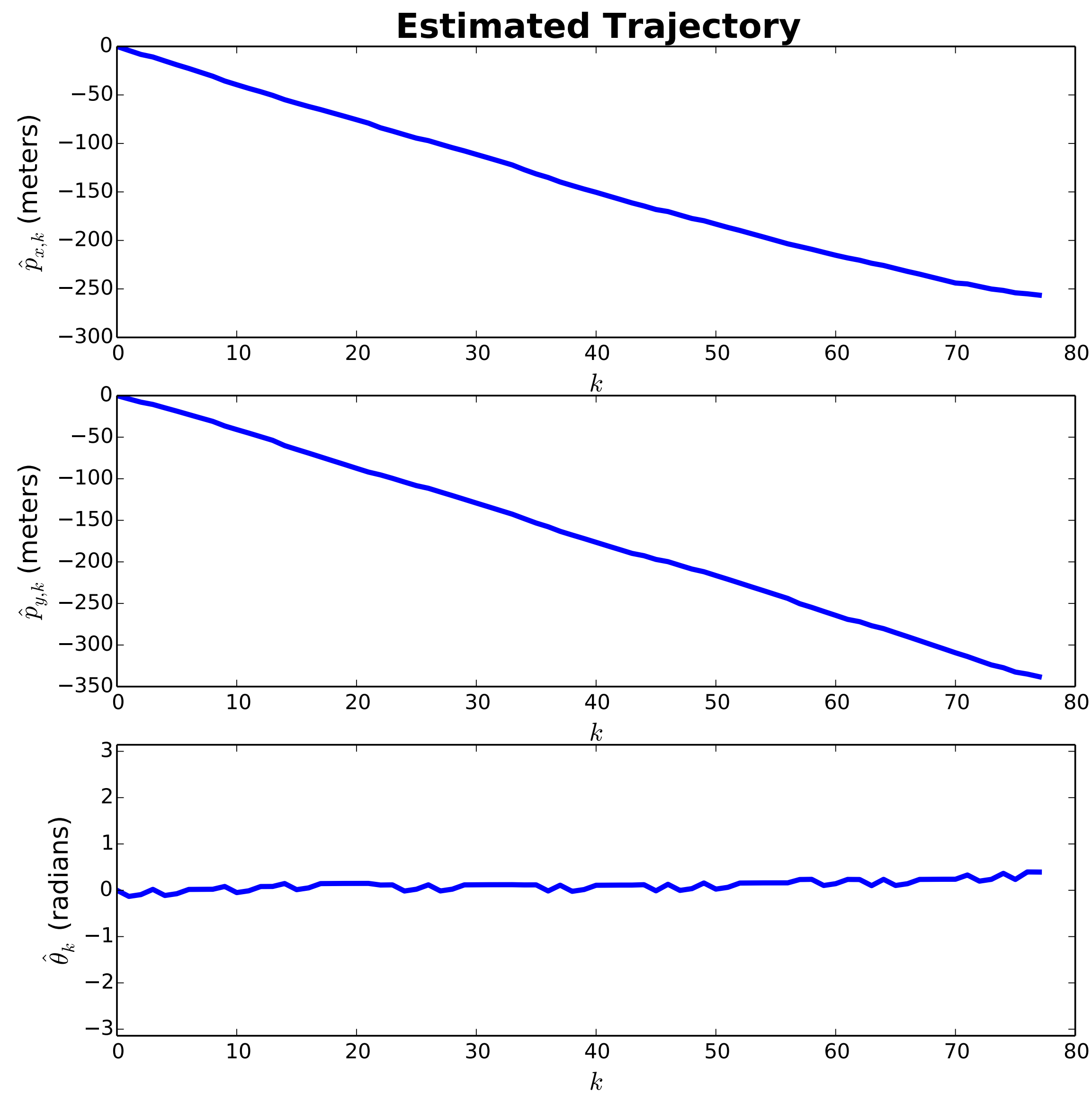


Figure 3.5: Estimated trajectory of the water buffalo by the proposed remote estimation scheme

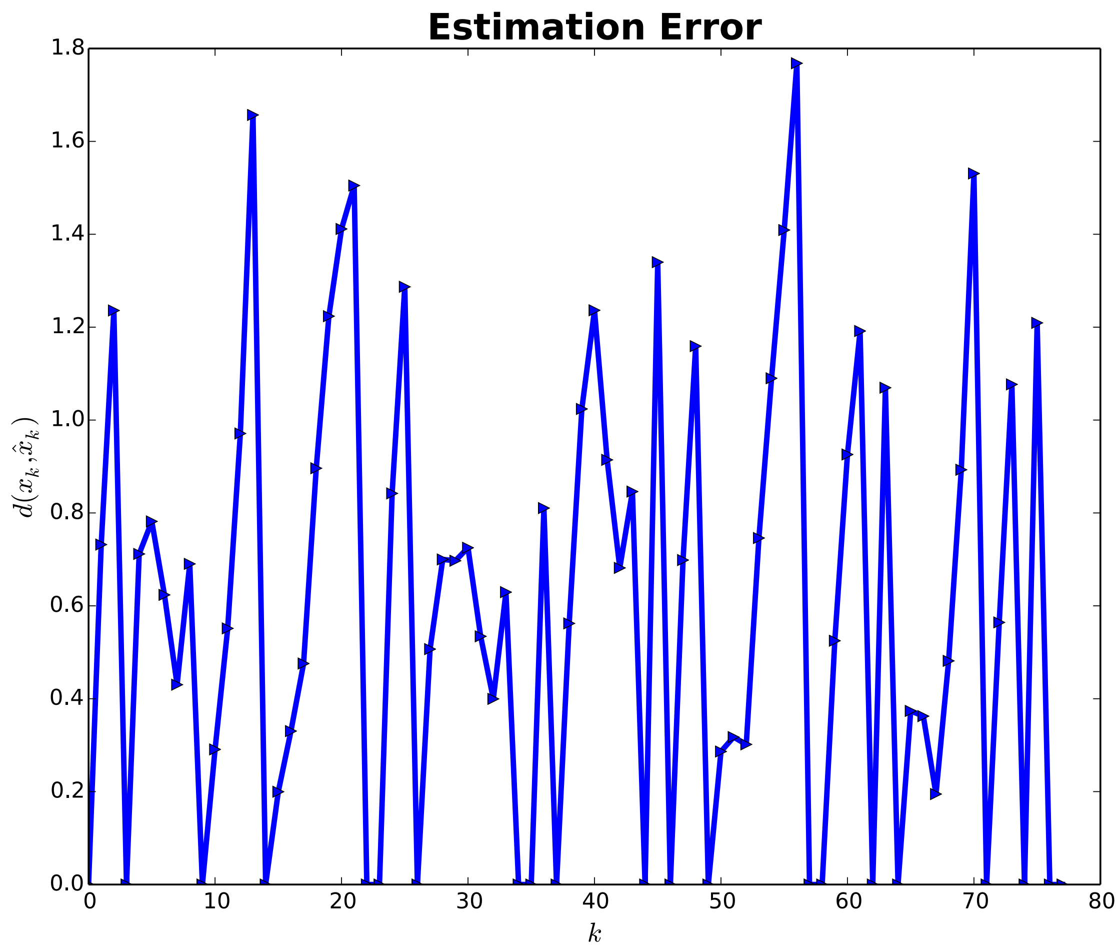


Figure 3.6: Estimation error of the remote estimation scheme

3.6 Summary and Future Work

We have studied the remote state estimation problem formulated in Section 3.1. To find a solution to the problem, we re-write the problem into N sub-problems and sequentially solve each Sub-problem k. We show that optimal solutions to all the sub-problems constitute an optimal solution to the original problem. Based on this idea, our main results show the existence of a jointly optimal solution, and describe an iterative procedure for finding a person-by-person optimal solution. In addition we have applied the proposed scheme to the experimental data obtained from the real-world deployment.

As future work, we will find the convergence rate of the proposed procedure described in Procedure 1, and if it exists, search for a new algorithm that achieves a faster convergence rate. Also we are interested in extending the presented results to large scale dynamical systems which may consist of multiple sensing units and estimators.

Chapter 4: Evolutionary Game Dynamics and Passivity

4.1 Background

4.1.1 Notation

- For a vector a in $\mathbb{R}^n$, its i-th entry is denoted by a_i, and we define

$$[a_i]_+ \stackrel{def}{=} \max\{a_i, 0\}$$
$$[a]_+ \stackrel{def}{=} \begin{pmatrix} [a_1]_+ & \cdots & [a_n]_+ \end{pmatrix}^T$$

- We denote the gradient and Hessian of a real-valued function $x \mapsto f(x)$ with respect to x by $\nabla_x f$ and $\nabla_x^2 f$, respectively, provided they exist.

- We denote the interior and the boundary of a set $\mathbb{A}$ by $\mathrm{int}(\mathbb{A})$ and $\mathrm{bd}(\mathbb{A})$, respectively.

- $\mathbb{R}_+^n$ $\left(\mathbb{R}_-^n\right)$ is the set of n-dimensional element-wise non-negative (non-positive) vectors. For $n = 1$, we omit superscript n and adopt $\mathbb{R}_+$ $(\mathbb{R}_-)$.

- $\mathbf{1}$ is the vector with all entries 1, I is the identity matrix, and e_i is the i-th column of I.

- $\|\cdot\|$ is the Euclidean norm.

4.1.2 Population Games and Evolutionary Dynamics

Consider a population of players engaged in a game where each player selects a (pure) strategy from the set of available strategies represented by $\{1, \cdots, n\}$.[1] Suppose that the population consists of a continuum of players. Population states, which describe the distributions of strategy choices by players, constitute a simplex

$$\mathbb{X} \overset{def}{=} \left\{ x \in \mathbb{R}^n_+ \,\middle|\, \sum_{i=1}^n x_i = 1 \right\}$$

We denote the tangent space of $\mathbb{X}$ as $T\mathbb{X} = \left\{ z \in \mathbb{R}^n \,\middle|\, \sum_{i=1}^n z_i = 0 \right\}$. A payoff vector $p \in \mathbb{R}^n$ is assigned to each population state x: p_i represents a payoff given to the i-strategists, the players choosing strategy i. Based on this notation, we describe population games and evolutionary dynamics in Section 4.1.2.1 and Section 4.1.2.2, respectively.

4.1.2.1 Population Games

We identify population games with payoff operators defined as follows:

$$p(\cdot) = \mathfrak{G}x(\cdot) \tag{4.1}$$

$\mathfrak{G} : \mathcal{X} \to \mathcal{P}$ is a causal operator where $\mathcal{X}$ is the set of all differentiable $\mathbb{X}$-valued time-dependent functions $x(\cdot) : \mathbb{R}_+ \to \mathbb{X}$, and $\mathcal{P}$ is the set of all differentiable $\mathbb{R}^n$-valued time-dependent functions $p(\cdot) : \mathbb{R}_+ \to \mathbb{R}^n$. Equation (4.1) suggests that a payoff trajectory $p(\cdot)$ is a causal function of a population state trajectory $x(\cdot)$; hence, under (4.1), the payoff $p(t)$

[1] Population games, in general, account multiple populations of players, and the strategy sets are allowed to be distinct across populations. However, for simple and clear presentation, we restrict our attention to single-population games. Results for the single-population cases can be extended to multi-population cases.

at each time t may depend on the entire history of a population state. We note that this formalism of population games generalizes the conventional ones presented, for instance, in [4, 95].

The following are a few examples of payoff operators (4.1).

Example 4.1.1. *Let F be a C^1 mapping from $\mathbb{X}$ to $\mathbb{R}^n$.*

Time-Delayed Payoff:	$p(t) = F(x(t-d))$
Contrarian Effect Payoff [15]:	$p(t) = F(x(t)) - \Lambda(x(t) - x(t-d))$
Cumulative Payoff:	$\dot{p}(t) = F(x(t))$
Anticipatory Payoff [15]:	$\dot{q}(t) = \lambda(F(x(t)) - q(t))$
	$p(t) = F(x(t)) + k(F(x(t)) - q(t))$

where Λ is a matrix in $\mathbb{R}^{n\times n}$, d is a positive constant, and k, λ are real numbers. □

We adopt the notion of Nash equilibrium in the following way.[2]

Definition 4.1.2 (Nash Equilibrium)**.** *Let $x \in \mathbb{X}$ be a population state and $p \in \mathbb{R}^n$ be a payoff vector assigned to it. The population state x is a Nash equilibrium if every strategy in use receives the maximum payoff, i.e., if x_i is positive then $p_i = \max_{j\in\{1,\cdots,n\}} p_j$ holds.*

4.1.2.2 Evolutionary Dynamics

Evolutionary dynamics describe how the population state evolves over time in response to a payoff trajectory. Throughout the chapter, we consider dynamics that can be

[2]When a game is described by a payoff function $p = F(x)$, Definition 4.1.2 coincides with the conventional definition of Nash equilibrium.

represented by a differential equation[3] given by

$$\dot{x} = V(p, x),\ x(0) = x_0 \in \mathbb{X} \tag{4.2}$$

where $p(t)$, $x(t)$, and $\dot{x}(t)$ take values in $\mathbb{R}^n$, $\mathbb{X}$, and $T\mathbb{X}$, respectively. We assume that the vector field $V : \mathbb{R}^n \times \mathbb{X} \to T\mathbb{X}$ is *well-defined* in a sense that for each initial value x_0 in $\mathbb{X}$ and payoff trajectory $p(\cdot)$ in $\mathcal{P}$, there exists a unique solution $x(\cdot)$ to (4.2) that belongs to $\mathcal{X}$.

We define the set of equilibrium points of (4.2) as

$$\mathbb{S} \overset{def}{=} \left\{(p, x) \in \mathbb{R}^n \times \mathbb{X} \,\middle|\, V(p, x) = \mathbf{0}\right\}$$

and for each x in $\mathbb{X}$, its projection on $\mathbb{R}^n \times \{x\}$ as $\mathbb{S}_x \overset{def}{=} \left\{p \in \mathbb{R}^n \,\middle|\, (p, x) \in \mathbb{S}\right\}$. We assume that for each x in $\mathbb{X}$, the set $\mathbb{S}_x$ is path-connected, i.e., for every p_0, p_1 in $\mathbb{S}_x$, there exists a piece-wise smooth path from p_0 to p_1. As a case in point, consider a set $\mathbb{S}_{NE}$ that consists of (p, x) for which given the payoff p, the population state x is a Nash equilibrium. It can be verified that if (p_0, x) and (p_1, x) both belong to $\mathbb{S}_{NE}$ then so does $(\lambda \cdot p_0 + (1 - \lambda) \cdot p_1, x)$ for all λ in $[0, 1]$. Hence $\mathbb{S}_{NE}$ satisfies the path-connectedness assumption.

The following are a few examples of evolutionary dynamics that are found in literature.

Example 4.1.3. *The replicator dynamics [9], BNN dynamics [8], Smith dynamics [96], and logit dynamics [97] are representative instances of evolutionary dynamics. The state-*

[3] In the literature of evolutionary game theory, it is a convention to represent evolutionary dynamics as $\dot{x} = V(F(x), x)$ to make the dependence on a payoff function $F : \mathbb{X} \to \mathbb{R}^n$ explicit. While, in this work, we remove the dependence to study evolutionary dynamics under a generalized class of games (4.1).

space representations of these dynamics are given as follows:

$$\textbf{\textit{Replicator:}} \qquad \dot{x}_i = x_i \hat{p}_i \tag{4.3}$$

$$\textbf{\textit{BNN:}} \qquad \dot{x}_i = [\hat{p}_i]_+ - x_i \sum_{j=1}^{n} [\hat{p}_j]_+ \tag{4.4}$$

$$\textbf{\textit{Smith:}} \qquad \dot{x}_i = \sum_{j=1}^{n} x_j [p_i - p_j]_+ - x_i \sum_{j=1}^{n} [p_j - p_i]_+ \tag{4.5}$$

$$\textbf{\textit{Logit:}} \qquad \dot{x}_i = \frac{\exp(\eta^{-1} \cdot p_i)}{\sum_{j=1}^{n} \exp(\eta^{-1} \cdot p_j)} - x_i \tag{4.6}$$

for each i in $\{1, \cdots, n\}$, where $\hat{p}$ is the excess payoff vector defined as $\hat{p} = p - p^T x \cdot \mathbf{1}$, and η is a positive real number. □

4.2 Passivity of Evolutionary Dynamics

We define a notion of passivity for evolutionary dynamics (4.2), and characterize passivity in terms of the vector field V in (4.2). Based on the characterization, we examine passivity of previously established dynamics, and investigate properties of passive dynamics.

4.2.1 Definition of Passivity for Evolutionary Dynamics

To define passivity for (4.2), let us consider the following inequality for a C^1 function $S_{ED} : \mathbb{R}^n \times \mathbb{X} \to \mathbb{R}_+$ and a constant η:

$$S_{ED}(p(t), x(t)) \leq S_{ED}(p(t_0), x(t_0)) + \int_{t_0}^{t} \left[\dot{p}^T(\tau)\dot{x}(\tau) - \eta \cdot \dot{x}^T(\tau)\dot{x}(\tau) \right] \, \mathrm{d}\tau \tag{4.7}$$

for $t \geq t_0 \geq 0$, where $x(\cdot) \in \mathcal{X}$ is the trajectory of the population state determined by (4.2) in response to a payoff trajectory $p(\cdot) \in \mathcal{P}$. In terms of (4.7), we state the definition of passivity as follows.

Definition 4.2.1. *Consider an evolutionary dynamic given as in* (4.2).

1. *The dynamic is said to be passive if there exists a C^1 function $S_{ED} : \mathbb{R}^n \times \mathbb{X} \to \mathbb{R}_+$ for which* (4.7) *holds with $\eta = 0$ for $t \geq t_0 \geq 0$ and every payoff trajectory $p(\cdot)$ in $\mathcal{P}$.*

2. *The dynamic is said to be strictly passive if there exists a C^1 function $S_{ED} : \mathbb{R}^n \times \mathbb{X} \to \mathbb{R}_+$ for which* (4.7) *holds with $\eta = 0$ for $t \geq t_0 \geq 0$ and every payoff trajectory $p(\cdot)$ in $\mathcal{P}$, and if $\nabla_x^T S_{ED}(p,x)V(p,x) = 0$ implies $V(p,x) = 0$.*

3. *The dynamic is said to be strictly output passive if there exists a C^1 function $S_{ED} : \mathbb{R}^n \times \mathbb{X} \to \mathbb{R}_+$ for which* (4.7) *holds with $\eta > 0$ for $t \geq t_0 \geq 0$ and every payoff trajectory $p(\cdot)$ in $\mathcal{P}$.*

We refer to S_{ED} as a *storage function* and (4.7) as the *passivity inequality*. Since S_{ED} is a non-negative function, without loss of generality, we assume that

$$\inf_{(p,x) \in \mathbb{R}^n \times \mathbb{X}} S_{ED}(p,x) = 0$$

It follows from Definition 4.2.1 that strict output passivity entails strict passivity and strict passivity entails passivity.

Remark 4.2.2. *The definition of passivity for evolutionary dynamics is closely related with the notion of dissipativity from dynamical system theory [98]. To see this, let us rewrite* (4.2) *in the following form:*

$$\begin{pmatrix} \dot{p} \\ \dot{x} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ V(p,x) \end{pmatrix} + \begin{pmatrix} u \\ \mathbf{0} \end{pmatrix} \tag{4.8a}$$

$$y = V(p,x) \tag{4.8b}$$

Note that (4.8) *is a state-space representation of a control-affine nonlinear system with the input* u*, state* (p, x)*, and output* y*. By the traditional notion of dissipativity [98], the system* (4.8) *is dissipative with respect to the supply rate* $s(u, y) = u^T y - \eta \cdot y^T y$ *if there exists a* C^1 *function* $S_{ED} : \mathbb{R}^n \times \mathbb{X} \to \mathbb{R}_+$ *for which*

$$S_{ED}(p(t), x(t)) \leq S_{ED}(p(t_0), x(t_0)) + \int_{t_0}^{t} \left[u^T(\tau) y(\tau) - \eta \cdot y^T(\tau) y(\tau) \right] \mathrm{d}\tau \qquad (4.9)$$

holds for $t \geq t_0 \geq 0$ *and every real-valued function* $u : \mathbb{R}_+ \to \mathbb{R}^n$*. Then, by the equivalence between* (4.2) *and* (4.8)*, we can verify that the passivity inequality* (4.7) *is satisfied for* (4.2) *if and only if the inequality* (4.9) *is satisfied for* (4.8)*.*

4.2.2 Characterization of Passivity of Evolutionary Dynamics

Let us consider the following two conditions:

$$\nabla_p S_{ED}(p, x) = V(p, x) \qquad \textbf{(P1)}$$

$$\nabla_x^T S_{ED}(p, x) V(p, x) \leq -\eta \cdot V^T(p, x) V(p, x) \qquad \textbf{(P2)}$$

where $S_{ED} : \mathbb{R}^n \times \mathbb{X} \to \mathbb{R}_+$ is a C^1 function, $V : \mathbb{R}^n \times \mathbb{X} \to T\mathbb{X}$ is the vector field given in (4.2), and η is a real number. In the following Theorem, we show that (**P1**) and (**P2**) are passivity requirements for evolutionary dynamics. This result not only provides an alternative definition of passivity but also is useful in studying properties of passive evolutionary dynamics.

Theorem 4.2.3. *Consider an evolutionary dynamic given as in* (4.2)*. The following statements are true:*

(S1) *The dynamic is passive if and only if there exists a C^1 function $S_{ED} : \mathbb{R}^n \times \mathbb{X} \to \mathbb{R}_+$ for which the conditions* (**P1**) *and* (**P2**) *hold with $\eta = 0$.*

(S2) *The dynamic is strictly passive if and only if there exists a C^1 function $S_{ED} : \mathbb{R}^n \times \mathbb{X} \to \mathbb{R}_+$ for which the conditions* (**P1**) *and* (**P2**) *hold with $\eta = 0$, and the equality in* (**P2**) *holds only if $V(p, x) = \mathbf{0}$ holds.*

(S3) *The dynamic is strictly output passive if and only if there exists a C^1 function $S_{ED} : \mathbb{R}^n \times \mathbb{X} \to \mathbb{R}_+$ for which the conditions* (**P1**) *and* (**P2**) *hold with $\eta > 0$.*

Proof. To prove the Theorem, as noted in Remark 4.2.2, recall that (4.2) can be rewritten as in (4.8) and that passivity of evolutionary dynamics (4.2) is equivalent to dissipativity of control-affine nonlinear systems (4.8) with the supply rate $s(u, y) = u^T y - \eta \cdot y^T y$. Using dissipativity characterization Theorem (see, for instance, Theorem 1 in [99]), we can see that there exists a C^1 function S_{ED} for which the conditions (**P1**) and (**P2**) hold if and only if S_{ED} satisfies the passivity inequality (4.7) for $t \geq t_0 \geq 0$ and every payoff trajectory $p(\cdot)$ in $\mathcal{P}$. The rest of the proof follows from Definition 4.2.1 and the equivalence between passivity for (4.2) and dissipativity for (4.8). □

Implications of the conditions (**P1**) and (**P2**) are as follows: For a fixed $x \in \mathbb{X}$, consider an integral $\int_P V(\mathbf{p}, x) \bullet \mathrm{d}\mathbf{p}$ of the vector field V along a piece-wise smooth path P from p_0 to p_1 in the direction of $\mathbf{p}$. (**P1**) is equivalent to the fact that the value of the integral does not depend on the choice of the path.[4] Dissatisfaction of (**P1**) could lead to a limit cycle and non-convergence to an equilibrium. (see Example 6.1 in [14]).

[4] A game-theoretic interpretation of (**P1**) is given in [100].

Next, suppose that the payoff vector $p(t)$ is constant, i.e., $p(t) = p_0$ for all t in $\mathbb{R}_+$. According to (**P2**), the population state $x(t)$ evolves along a trajectory for which the stored energy quantified by $S_{ED}(p(t), x(t))$ decreases. In particular, if the dynamic (4.2) is strictly passive then we can establish asymptotic stability of the set $\mathbb{S}$ using LaSalle's theorem [101]. In Section 4.3, based on this observation, we establish stability of passive dynamics in a class of population games.

As an application of Theorem 4.2.3, we evaluate passivity of evolutionary dynamics found in literature. We start with the replicator dynamics.

Proposition 4.2.4. *The replicator dynamics* (4.3) *are not passive.*

The proof is given in Appendix C.1.

In what follows, we examine passivity of the EPT dynamics [102], (impartial) pairwise comparison dynamics [103], and PBR dynamics [97].

EPT Dynamics:

$$\dot{x}_i = \varrho_i(\hat{p}) - x_i \cdot \mathbf{1}^T \varrho(\hat{p}) \tag{4.10}$$

where the excess payoff vector is defined as $\hat{p} = p - p^T x \cdot \mathbf{1}$. The function $\varrho = \begin{pmatrix} \varrho_1 & \cdots & \varrho_n \end{pmatrix}^T$ is called the *revision protocol* in which each entry is defined as $\varrho_i : \mathbb{R}^n \to \mathbb{R}_+$ and satisfies following two conditions – Integrability (**I**) and Acuteness (**A**):

$$\nabla_{\hat{p}} \gamma(\hat{p}) = \varrho(\hat{p}) \text{ for a } C^1 \text{ function } \gamma : \mathbb{R}^n \to \mathbb{R} \tag{\textbf{I}}$$

$$\hat{p}^T \varrho(\hat{p}) > 0 \text{ if } \hat{p} \in \mathbb{R}^n \setminus \mathbb{R}^n_- \tag{\textbf{A}}$$

Proposition 4.2.5. *The EPT dynamics* (4.10) *are strictly passive with a storage function*[5]

$$S_{EPT}(p, x) = \gamma(\hat{p})$$

The proof is given in Appendix C.2.

Pairwise Comparison Dynamics:

$$\dot{x}_i = \sum_{j=1}^{n} x_j \varrho_i(p_i - p_j) - x_i \sum_{j=1}^{n} \varrho_j(p_j - p_i) \tag{4.11}$$

The function $\varrho = \begin{pmatrix} \varrho_1 & \dots & \varrho_n \end{pmatrix}^T$ is called the *revision protocol* in which each entry is defined as $\varrho_i : \mathbb{R} \to \mathbb{R}_+$ and satisfies the following condition – Sign Preservation (**SP**):[6]

$$\operatorname{sgn}\left(\varrho_i(p_i - p_j)\right) = \operatorname{sgn}\left([p_i - p_j]_+\right) \tag{SP}$$

Proposition 4.2.6. *[15] The pairwise comparison dynamics* (4.11) *are strictly passive with a storage function* $S_{PC}(p, x) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i \int_0^{p_j - p_i} \varrho_j(s)\,\mathrm{d}s$.

PBR Dynamics:

$$\dot{x} = C(p) - x \tag{4.12}$$

where $C : \mathbb{R}^n \to \mathbb{X}$ is defined as $C(p) = \arg\max_{y \in \mathrm{int}(\mathbb{X})} \left(p^T y - v(y)\right)$, where $v : \mathrm{int}(\mathbb{X}) \to \mathbb{R}$ is a strictly convex C^2 function that satisfies $z^T \nabla_x^2 v(x) z > 0$ for all x in $\mathbb{X}$ and z in $T\mathbb{X} \setminus \{\mathbf{0}\}$, and $\|\nabla_x v(x)\| \to \infty$ as $x \to \mathrm{bd}(\mathbb{X})$. We refer to such C as a *choice function*, and to such v as an *admissible (deterministic) perturbation*.

[5] In the proof of Proposition 4.2.5, we show that there is a non-negative potential function γ that satisfies both (**I**) and (**A**).

[6] $\operatorname{sgn} : \mathbb{R} \to \{-1, 0, 1\}$ is defined as $\operatorname{sgn}(a) \overset{def}{=} \begin{cases} 1 & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ -1 & \text{if } a < 0 \end{cases}$.

Proposition 4.2.7. *The PBR dynamics* (4.12) *are strictly passive with a storage function*

$$S_{PBR}(p, x) = \max_{y \in \mathrm{int}(\mathbb{X})} \big(p^T y - v(y)\big) - \big(p^T x - v(x)\big) \tag{4.13}$$

Suppose that v is strongly convex satisfying

$$z^T \nabla_x^2 v(x) z \geq \eta' \cdot z^T z$$

for all x in $\mathbb{X}$ and z in $T\mathbb{X}$, where $\eta' > 0$. Then the dynamics are strictly output passive and satisfy the passivity inequality (4.7) *for $\eta = \eta'$.*

The proof is given in Appendix C.3.

4.2.3 Properties of Passive Evolutionary Dynamics

4.2.3.1 Payoff Monotonicity and Passivity

Using the characterization of passivity given in Theorem 4.2.3, we study properties of passive evolutionary dynamics in connection with the following two conditions[7] – *Nash Stationarity* **(NS)** and *Positive Correlation* **(PC)**:

$V(p, x) = \mathbf{0}$ if and only if

given the payoff p, the population state x is a Nash equilibrium (**NS**)

$p^T V(p, x) \geq 0$ holds for all $(p, x) \in \mathbb{R}^n \times \mathbb{X}$ (**PC**)

Consider evolutionary dynamics (4.2) in a game in which the payoff is constant, i.e., $p(t) = p_0$ for all t in $\mathbb{R}_+$. The conditions (**NS**) and (**PC**) have the following implications:

[7]These conditions are previously considered in literature to establish stability of evolutionary dynamics [14, 104].

(**PC**) implies that the population state trajectory $x(\cdot)$ determined by (4.2) evolves along which the average payoff $p_0^T x(t)$ is increasing, i.e., $\frac{\mathrm{d}}{\mathrm{d}t} p_0^T x(t) \geq 0$; and (**NS**) implies that the population state does not change if and only if the maximum average payoff is attained, i.e., $\dot{x}(t) = 0$ if and only if $p_0^T x(t) = \max_{x \in \mathbb{X}} p_0^T x$. We will refer to these phenomena as *payoff monotonicity*, and to the dynamics satisfying both (**NS**) and (**PC**) as *payoff monotonic*.

Proposition 4.2.8. *Consider passive evolutionary dynamics* (4.2) *with a storage function* $S_{ED} : \mathbb{R}^n \times \mathbb{X} \to \mathbb{R}_+$. *A global minimizer of* S_{ED} *is an equilibrium point of* (4.2). *In addition, if the dynamics satisfy* (**NS**) *then every equilibrium point of* (4.2) *is a global minimizer of* S_{ED}.

The proof is given in Appendix C.4.

Remark 4.2.9. *According to Proposition 4.2.8, given that* $\min_{(p,x) \in \mathbb{R}^n \times \mathbb{X}} S_{ED}(p, x) = 0$, *the inverse image* $S_{ED}^{-1}(0)$ *is a subset of the set* $\mathbb{S}$ *of equilibrium points of* (4.2), *and it is identical to* $\mathbb{S}$ *if the dynamics satisfy* (**NS**).

Based on Definition 4.2.1, we note that strict output passivity is a stronger notion, and leads to stronger stability than does ordinary passivity in a sense that strictly output passive dynamics are stable in a larger class of population games. Hence, in what regards to achieving stability, it is desired to adopt strictly output passive dynamics. However, in the following Proposition, we show that the evolutionary dynamics exhibiting the payoff monotonicity cannot be strictly output passive. Therefore, the payoff monotonicity and strict output passivity cannot be attained simultaneously.

Proposition 4.2.10. *For $n \geq 3$, no payoff monotonic evolutionary dynamics are strictly output passive.*

The proof is given in Appendix C.5.

The following is a direct consequence of Proposition 4.2.10.

Corollary 4.2.11. *The EPT dynamics* (4.10) *and the pairwise comparison dynamics* (4.11) *are not strictly output passive.*

As it can be verified that both dynamics are payoff monotonic, the proof directly follows from Proposition 4.2.10.

4.2.3.2 Equivalence to Closed-loop Stability

In Definition 4.2.1, we defined passivity as an input-output property of evolutionary dynamics: Satisfaction of the passivity inequality (4.7) for each payoff trajectory (input) $p(\cdot)$ in $\mathcal{P}$ and the population state trajectory (output) $x(\cdot)$ determined by (4.2) in response to $p(\cdot)$. In evolutionary game theory, the main interest lies in examining the time-evolution of the population state induced in specific games. Hence, it is more natural to define passivity of evolutionary dynamics in connection with games of interest. To achieve this, let us consider population games identified by cumulative payoff functions of the following form:

$$\dot{p}(t) = F(x(t)) \tag{4.14}$$

where F admits a C^1 potential function, i.e., there is a C^1 function $f : \mathbb{X} \to \mathbb{R}$ that satisfies $\nabla_x f = \Phi F$ where $\Phi = I - \frac{1}{n}\mathbf{1}\mathbf{1}^T$. Then, with a C^1 function defined by

$$S_G(x) = \max_{x \in \mathbb{X}} f(x) - f(x) \tag{4.15}$$

we can derive that

$$\begin{aligned} S_G(x(t)) - S_G(x(t_0)) &= -\int_{t_0}^{t} \frac{\mathrm{d}}{\mathrm{d}\tau} f(x(\tau))\,\mathrm{d}\tau \\ &= -\int_{t_0}^{t} \dot{p}^T(\tau)\dot{x}(\tau)\,\mathrm{d}\tau \end{aligned} \tag{4.16}$$

In the following Proposition, we show that passivity can be defined as (a weak form of) stability of closed-loops formed by cumulative payoff functions (4.14) and evolutionary dynamics. This result implies that passivity of evolutionary dynamics is equivalent to satisfaction of the passivity inequality (4.7) in the class of population games identified by (4.14).

Proposition 4.2.12. *Consider the following closed loop formed by a cumulative payoff function* (4.14) *and an evolutionary dynamic* (4.2)*:*

$$\dot{p} = F(x) \tag{4.17a}$$

$$\dot{x} = V(p, x) \tag{4.17b}$$

The dynamic (4.2) *is passive if and only if for each cumulative payoff function* (4.14)*, the closed-loop* (4.17) *has a Lyapunov function $E : \mathbb{R}^n \times \mathbb{X} \to \mathbb{R}_+$ of the form $E = S_G + E_{ED}$, where $S_G : \mathbb{X} \to \mathbb{R}_+$ is given in* (4.15) *and $E_{ED} : \mathbb{R}^n \times \mathbb{X} \to \mathbb{R}_+$ is a fixed C^1 function.*

The proof is given in Appendix C.6.

The following is a direct consequence of Proposition 4.2.12, where the proof is given in Corollary C.7.

Corollary 4.2.13. *An evolutionary dynamic* (4.2) *is passive if and only if there exists a* C^1 *function* $S_{ED} : \mathbb{R}^n \times \mathbb{X} \to \mathbb{R}_+$ *for which the passivity inequality* (4.7) *holds for* $\eta = 0$ *in the class of population games identified by cumulative payoff functions* (4.14)*, i.e., there exists a* C^1 *function* $S_{ED} : \mathbb{R}^n \times \mathbb{X} \to \mathbb{R}_+$ *for which*

$$S_{ED}(p(t), x(t)) \leq S_{ED}(p(t_0), x(t_0)) + \int_{t_0}^{t} F^T(x(\tau))\dot{x}(\tau)\, \mathrm{d}\tau$$

holds for every function $F : \mathbb{X} \to \mathbb{R}^n$ *that admits a* C^1 *potential function.*

4.2.3.3 Effect of Control Costs on Passivity

Consider the *total payoff function* $u : \mathbb{R}^n \times \mathbb{X} \to \mathbb{R}$ given by

$$u(p, x) = p^T x - v(x) \tag{4.18}$$

where a C^2 function $v : \mathbb{X} \to \mathbb{R}$ is referred to as a *control cost* [105] or a *deterministic perturbation* [106]. A control cost is said to be admissible if it is strictly convex satisfying $z^T \nabla_x^2 v(x) z > 0$ for all x in $\mathbb{X}$ and z in $T\mathbb{X} \setminus \{\mathbf{0}\}$, and $\|\nabla_x v(x)\| \to \infty$ as $x \to \mathrm{bd}\,(\mathbb{X})$. Notice that when there is no control cost, i.e., $v = 0$, the total payoff coincides with the average payoff. The idea of imposing control costs on the total payoff appeared in game theory and economics to study the effect of random perturbations [97, 106] or disutility [105] on choice models, to model human choice behavior [107], and to analyze the effect of social norms in economic problems [108]

We consider evolutionary dynamics that depend on u, and investigate the effect of control costs on passivity of the dynamics. We refer to the dynamics as *unperturbed* if $v = 0$; otherwise they are called *perturbed*. To proceed, let us consider the state-space representation of evolutionary dynamics in terms of revision protocols [14]: For each i in $\{1, \cdots, n\}$,

$$\dot{x}_i = \sum_{j=1}^{n} x_j \varrho_{ji}(p, x) - x_i \sum_{j=1}^{n} \varrho_{ij}(p, x) \tag{4.19}$$

where $\varrho_{ji} : \mathbb{R}^n \times \mathbb{X} \to \mathbb{R}_+$ is called the revision protocol and denotes the rate at which j-strategists switch to strategy i given a payoff p and population state x.

For instance, the revision protocols can be realized as follows:

$$\varrho_{ji}(p, x) = \left[\nabla_x^T u(p, x)\, (e_i - x)\right]_+ \tag{4.20a}$$

$$\varrho_{ji}(p, x) = \left[\nabla_x^T u(p, x)\, (e_i - e_j)\right]_+ \tag{4.20b}$$

The protocol (4.20a) depends on the (instantaneous) increase of $u(p, x)$ when the population state changes in the direction of $e_i - x$; and the protocol (4.20b) depends on the (instantaneous) increase of $u(p, x)$ when the population state changes in the direction of $e_i - e_j$. Based on the revision protocols (4.20a) and (4.20b), we can derive the following evolutionary dynamics: For each i in $\{1, \cdots, n\}$,

$$\dot{x}_i = \left[\nabla_x^T u(p, x)\, (e_i - x)\right]_+ - x_i \sum_{j=1}^{n} \left[\nabla_x^T u(p, x)\, (e_j - x)\right]_+ \tag{4.21a}$$

$$\dot{x}_i = \sum_{j=1}^{n} x_j \left[\nabla_x^T u(p, x)\, (e_i - e_j)\right]_+ - x_i \sum_{j=1}^{n} \left[\nabla_x^T u(p, x)\, (e_j - e_i)\right]_+ \tag{4.21b}$$

Note that when no control cost is imposed, i.e., $v = 0$, (4.21a) and (4.21b) become the state-space representations of the BNN dynamics (4.4) and Smith dynamics (4.5), respectively. According to Proposition 4.2.5 and Proposition 4.2.6, the unperturbed dynamics of (4.21a) and (4.21b) are strictly passive.

Suppose that the control costs are strongly convex. Then, it can be verified that the resulting perturbed dynamics of (4.21a) and (4.21b) are strictly output passive. In what follows, we formalize this idea and show how convexity of control costs affects passivity of evolutionary dynamics. For our purpose, we consider evolutionary dynamics (4.19) whose revision protocols depend only on the gradient $\nabla_x u(p, x)$ of the total payoff function (4.18) and the population state x, i.e., ϱ_{ji} is a function of $\nabla_x u(p, x)$ and x as in (4.20).

Proposition 4.2.14. *Consider evolutionary dynamics* (4.19) *whose revision protocols* ϱ_{ji} *depend only on the gradient* $\nabla_x u(p, x)$ *of the total payoff function* (4.18) *and the population state* x*. Suppose that the unperturbed dynamics of* (4.19) *are passive. Then the following are true:*

1. *If the control cost is admissible then the resulting perturbed dynamics are strictly passive.*
2. *If the control cost is admissible and strongly convex then the resulting perturbed dynamics are strictly output passive.*

The proof is given in Proposition C.8.

4.3 Stability of Passive Evolutionary Dynamics

In this section, we establish stability of passive evolutionary dynamics in population games in terms of dissipation of stored energy of the dynamics. To achieve this, we regard evolutionary dynamics and payoff operators as dynamical systems, and we investigate population state and payoff trajectories induced by a closed-loop formed by these dynamical systems (see Figure 4.1 for an illustration).

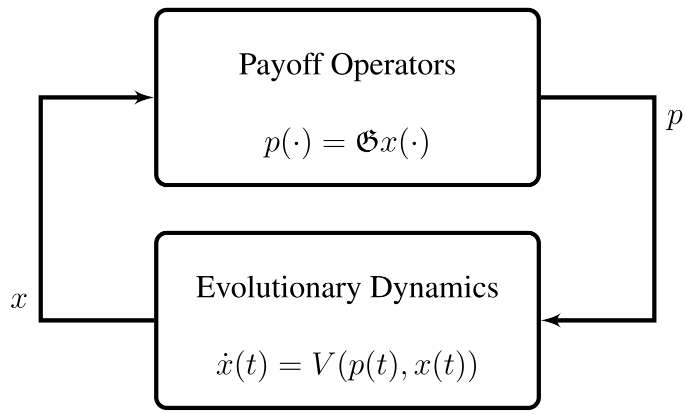


Figure 4.1: A closed-loop obtained by a feedback interconnection of payoff operators (4.1) and evolutionary dynamics (4.2).

Consider population games identified by (4.1) that satisfy the following inequality for some positive α: [8]

$$\int_0^t \left[\dot{p}^T(\tau)\dot{x}(\tau) - \nu \cdot \dot{x}^T(\tau)\dot{x}(\tau)\right] \, \mathrm{d}\tau \leq \alpha \tag{4.22}$$

for every population state trajectory $x(\cdot)$ in $\mathcal{X}$ and t in $\mathbb{R}_+$, where ν is non-negative real number and $p(\cdot) \in \mathcal{P}$ is a payoff trajectory determined by (4.1) in response to $x(\cdot)$. We represent the closed loop formed by (4.1) and (4.2) as follows:

$$p(\cdot) = \mathfrak{G}x(\cdot) \tag{4.23a}$$

$$\dot{x}(t) = V(p(t), x(t)) \tag{4.23b}$$

In the following Theorem, we state stability results for the closed-loop described by (4.23) in which

(CL1) (4.22) holds for $\nu = 0$, and (4.23b) is strictly passive.

[8]In [15], population games (4.1) satisfying (4.22) for $\nu = 0$ are called δ-*anti-passive*.

(CL2) (4.22) holds for $\nu > 0$, and (4.23b) is strictly output passive for a constant η satisfying $\eta > \nu$.

Theorem 4.3.1. *Consider* (4.23) *in which* ***(CL1)*** *or* ***(CL2)*** *holds. Let* $S_{ED} : \mathbb{R}^n \times \mathbb{X} \to \mathbb{R}_+$ *be a storage function of* (4.23b)*. Suppose that the following assumptions hold: For any sequence* $\left\{\left(p^{(l)}, x^{(l)}\right)\right\}_{l \in \mathbb{N}}$ *in* $\mathbb{R}^n \times \mathbb{X}$*,*

(A1) $\left\|V\left(p^{(l)}, x^{(l)}\right)\right\| \overset{l\to\infty}{\longrightarrow} \infty$ *implies* $S_{ED}\left(p^{(l)}, x^{(l)}\right) \overset{l\to\infty}{\longrightarrow} \infty$

(A2) $\nabla_x^T S_{ED}\left(p^{(l)}, x^{(l)}\right) V\left(p^{(l)}, x^{(l)}\right) \overset{l\to\infty}{\longrightarrow} 0$ *implies* $S_{ED}\left(p^{(l)}, x^{(l)}\right) \overset{l\to\infty}{\longrightarrow} 0$

If the time-derivative $\dot{p}$ *of the payoff is bounded, i.e, there is a positive real* M *for which* $\|\dot{p}(t)\| < M$ *holds for all* t *in* $\mathbb{R}_+$*, then it holds that* $\lim_{t\to\infty} S_{ED}\left(p(t), x(t)\right) = 0$*.*

The proof is given in Appendix C.9.

We note that the class of cumulative payoff functions presented in Section 4.2.3.2 satisfy (4.22). In the following Proposition, we present another class of payoff operators that satisfy (4.22).

Proposition 4.3.2. *For* C^1 *mappings* F_1 *and* F_2*, consider a (time-delayed) payoff function given by*

$$p(t) = F_1(x(t)) + F_2(x(t-d)) \tag{4.24}$$

where d *is a positive real.*[9] *Suppose that*

$$z^T DF_1(x) z \leq \nu_1 \cdot z^T z \tag{4.25a}$$

$$z^T DF_2^T(x) DF_2(x) z \leq \nu_2^2 \cdot z^T z \tag{4.25b}$$

[9]Here we assume that $x(t)$ for t in $[-d, 0)$ satisfies $\int_{-d}^{0} \|\dot{x}(\tau)\|^2 \, \mathrm{d}\tau < \beta$ for some positive real number β

hold for all x in $\mathbb{X}$ and z in $T\mathbb{X}$, where ν_1 is a real and ν_2 is a non-negative real. Then, the payoff function (4.24) *satisfies* (4.22) *with $\nu = \nu_1 + \nu_2$.*

The proof is given in Appendix C.10.

Based on Theorem 4.3.1, in what follows, we present stability results with the EPT dynamics (4.10), pairwise comparison dynamics (4.11), and PBR dynamics (4.12). As the EPT dynamics and pairwise comparison dynamics are at most strictly passive, we only consider **(CL1)** for these dynamics.

Proposition 4.3.3. *Consider* (4.23) *with the EPT dynamics* (4.10) *in which* ***(CL1)*** *holds. Let $S_{EPT} : \mathbb{R}^n \times \mathbb{X} \to \mathbb{R}_+$ be a storage function of* (4.10)*. Suppose that the revision protocol $\varrho : \mathbb{R}^n \to \mathbb{R}_+^n$ of* (4.10) *satisfies the following conditions: For any sequence $\left\{\hat{p}^{(l)}\right\}_{l \in \mathbb{N}}$ in $\mathbb{R}^n$,*

(C1) $\left\|\varrho\left(\hat{p}^{(l)}\right)\right\| \overset{l\to\infty}{\longrightarrow} \infty$ *implies* $S\left(\hat{p}^{(l)}\right) \overset{l\to\infty}{\longrightarrow} \infty$.

(C2) $\left\|\varrho\left(\hat{p}^{(l)}\right)\right\| \overset{l\to\infty}{\longrightarrow} 0$ *implies* $\left(\hat{p}^{(l)}\right)^T \varrho\left(\hat{p}^{(l)}\right) \overset{l\to\infty}{\longrightarrow} 0$*, and*

$\left(\hat{p}^{(l)}\right)^T \varrho\left(\hat{p}^{(l)}\right) \overset{l\to\infty}{\longrightarrow} 0$ *implies* $S\left(\hat{p}^{(l)}\right) \overset{l\to\infty}{\longrightarrow} 0$.

where $S(\hat{p}) = \int_P \varrho(\hat{\mathbf{p}}) \bullet \mathrm{d}\hat{\mathbf{p}}$ is an integral of ϱ along a piece-wise smooth path P from $\mathbf{0}$ to $\hat{p}$ in the direction of $\hat{\mathbf{p}}$.[10] *If the time-derivative $\dot{p}$ of the payoff is bounded, then it holds that $\lim_{t\to\infty} S_{EPT}\left(p(t), x(t)\right) = 0$.*

The proof is given in Appendix C.11.

Example 4.3.4. *The BNN dynamics* (4.4) *are the EPT dynamics with a revision protocol given by $\varrho\left(\hat{p}\right) = \left[\hat{p}\right]_+$ and a storage function given by $S_{EPT}\left(p, x\right) = \frac{1}{2}\left\|\left[\hat{p}\right]_+\right\|^2$. Notice*

[10] Note that by the condition (**I**) of the EPT dynamics, $\int_P \varrho(\mathbf{p}) \bullet \mathrm{d}\mathbf{p}$ does not depend on the choice of P.

that

$$S(\hat{p}) = \int_0^1 s \cdot \left\| [\hat{p}]_+ \right\|^2 \, \mathrm{d}s = \frac{1}{2} \left\| [\hat{p}]_+ \right\|^2 = \frac{1}{2} \left\| \varrho\left(\hat{p}\right) \right\|^2 \tag{4.26}$$

and

$$\hat{p}^T \varrho\left(\hat{p}\right) = \left\| \varrho\left(\hat{p}\right) \right\|^2 \tag{4.27}$$

It follows from (4.26) *and* (4.27) *that* ***(C1)*** *and* ***(C2)*** *of Proposition 4.3.3 hold. We also note that* $\lim_{t \to \infty} S_{EPT}(p(t), x(t)) = 0$ *implies* $\lim_{t\to\infty} \left\| [\hat{p}(t)]_+ \right\| = 0$*. Notice that given a payoff* p*, a population state* x *is a Nash equilibrium if and only if the excess payoff* $\hat{p} = p - p^T x \cdot \mathbf{1}$ *satisfies* $\left\| [\hat{p}]_+ \right\| = 0$*. Hence the convergence of* S_{EPT} *implies that the population state trajectory converges to a set of Nash equilibria.*

Proposition 4.3.5. *Consider* (4.23) *with the pairwise comparison dynamics* (4.11) *in which* ***(CL1)*** *holds. Let* $S_{PC} : \mathbb{R}^n \times \mathbb{X} \to \mathbb{R}_+$ *be a storage function of* (4.11)*. Suppose that the revision protocol* $\varrho = \begin{pmatrix} \varrho_1 & \cdots & \varrho_n \end{pmatrix}^T$ *of* (4.11) *satisfies the following conditions for each* i, j *in* $\{1, \cdots, n\}$*: For any sequence* $\left\{ \left(p^{(l)}, x^{(l)}\right) \right\}_{l \in \mathbb{N}}$ *in* $\mathbb{R}^n \times \mathbb{X}$*,*

(C1) $x_i^{(l)} \varrho_j \left(p_j^{(l)} - p_i^{(l)} \right) \overset{l\to\infty}{\longrightarrow} \infty$ *implies* $S_j\left(p_j^{(l)} - p_i^{(l)}, x_i^{(l)} \right) \overset{l\to\infty}{\longrightarrow} \infty$

(C2) $\varrho_j \left(p_j^{(l)} - p_i^{(l)} \right) \overset{l\to\infty}{\longrightarrow} 0$ *implies* $S_j\left(p_j^{(l)} - p_i^{(l)}, x_i^{(l)} \right) \overset{l\to\infty}{\longrightarrow} 0.$

where $S_j(p_j - p_i, x_i) = x_i \int_0^{p_j - p_i} \varrho_j(s) \, \mathrm{d}s$*. If the time-derivative* $\dot{p}$ *of the payoff is bounded, then it holds that* $\lim_{t\to\infty} S_{PC}\left(p(t), x(t)\right) = 0$*.*

The proof is given in Appendix C.12.

Example 4.3.6. *The Smith dynamics* (4.5) *are the pairwise comparison dynamics* (4.11) *with a revision protocol given by* $\varrho_j\left(p_j - p_i\right) = \left[p_j - p_i\right]_+$ *and a storage function given* *by* $S_{PC}(p,x) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} x_i[p_j - p_i]_+^2$. *We can derive the following:*

$$
\begin{aligned}
S_j(p_j - p_i, x_i) &= x_i \int_0^{p_j - p_i} [s]_+ \,\mathrm{d}s && \text{(4.28a)}\\
&= \frac{1}{2} x_i [p_j - p_i]_+^2 && \text{(4.28b)}\\
&= \frac{1}{2} x_i \varrho_j^2 (p_j - p_i) && \text{(4.28c)}\\
&\geq \frac{1}{2}\left[x_i \varrho_j (p_j - p_i)\right]^2 && \text{(4.28d)}
\end{aligned}
$$

It follows from (4.28c) *and* (4.28d) *that **(C1)** and **(C2)** of Proposition 4.3.5 hold. By the fact that* $\sum_{i=1}^{n} x_i \left[p_j - p_i\right]_+ \geq [\hat{p}_j]_+$, *we note that* $\lim_{t\to\infty} S_{PC}(p(t), x(t)) = 0$ *implies* $\lim_{t\to\infty} \left\| [\hat{p}(t)]_+ \right\| = 0$. *Hence, the convergence of* S_{PC} *implies that the population state trajectory converges to a set of Nash equilibria.*

Proposition 4.3.7. *Consider* (4.23) *with the PBR dynamics* (4.12) *in which **(CL1)** or **(CL2)** holds. Let* $S_{PBR} : \mathbb{R}^n \times \mathbb{X} \to \mathbb{R}_+$ *be a storage function of* (4.12)*. If the time-derivative* $\dot{p}$ *of the payoff is bounded, then it holds that* $\lim_{t\to\infty} S_{PBR}(p(t), x(t)) = 0$.

The proof is given in Appendix C.13.

Remark 4.3.8. *Due to* (C.6b)*, by letting* $y = C(p)$*, it holds that* $\left[p - \nabla_x v(C(p))\right]^T z = 0$ *for all* z *in* $T\mathbb{X}$*, where* C *is the choice function of the PBR dynamics* (4.12)*. Then, we can see that the storage function* (4.13) *satisfies*

$$
S_{PBR}(p,x) = \nabla_x^T v(y)\,(y - x) - (v(y) - v(x))
$$

By strict convexity of v *and by the fact that* $y \in \mathbb{X}$*, we can see that*

$$
\lim_{t\to\infty} S_{PBR}(p(t), x(t)) = 0 \textit{ implies } \lim_{t\to\infty} \|C(p(t)) - x(t)\| = 0
$$

i.e., the population state trajectory converges to the best response choice $C(p)$.

4.4 Numerical Examples

To illustrate the main results, we provide numerical examples and simulations. We consider two different types of examples. In the first example, we consider the replicator dynamics and BNN dynamics in population games identified by a cumulative payoff function. In the second example, we consider the BNN dynamics and logit dynamics in the Hypnodisk game [104], which is used in the proof of Proposition 4.2.10. Simulation results will show that the population state trajectories induced by the BNN dynamics converge to a limit cycle; while the trajectories induced by the logit dynamics converge to an equilibrium point. In addition, we examine the case in which the BNN dynamics are perturbed by the control cost $v(x) = \eta \cdot \sum_{i=1}^{3} x_i \ln x_i$ as in (4.21a). Simulation results will show that the perturbed BNN dynamics have strong stability properties than the unperturbed ones.

4.4.1 Replicator dynamics and BNN dynamics under a cumulative payoff function

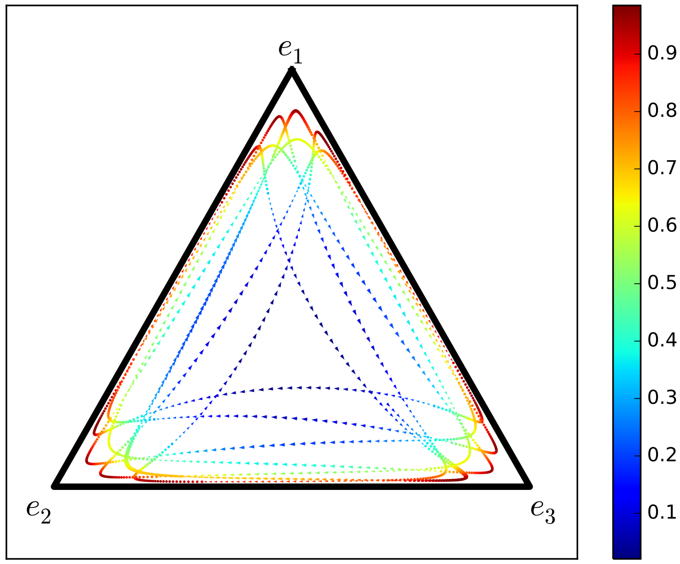
Consider a cumulative payoff function given by

$$\begin{pmatrix} \dot{p}_1(t) \\ \dot{p}_2(t) \\ \dot{p}_3(t) \end{pmatrix} = \begin{pmatrix} -x_1(t) + \frac{1}{3} \\ -x_2(t) + \frac{1}{3} \\ -x_3(t) + \frac{1}{3} \end{pmatrix} \tag{4.29}$$

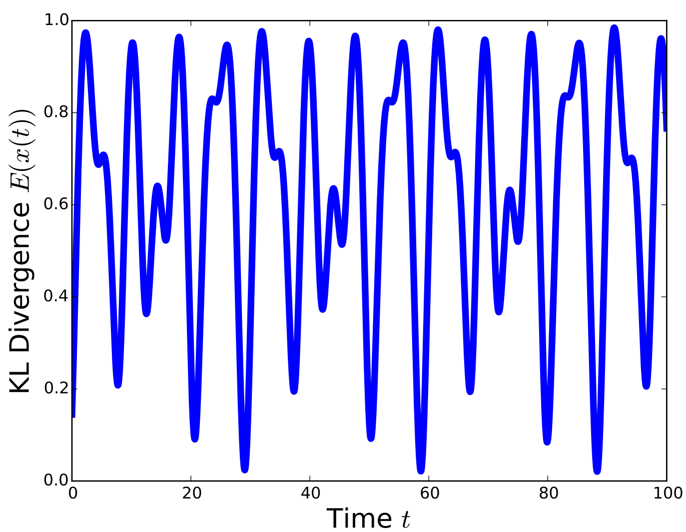
We note that under (4.29), both the replicator dynamics (4.3) and BNN dynamics (4.4) are stationary at each element of the set given by

$$\left\{ (p, x) \in \mathbb{R}^3 \times \mathbb{X} \;\middle|\; p_1 = p_2 = p_3 \text{ and } x_1 = x_2 = x_3 = \frac{1}{3} \right\} \tag{4.30}$$

The population state trajectories induced by these dynamics under (4.29) are illustrated in Figure 4.2 and Figure 4.3, respectively. From the illustrations, we can observe that for the replicator dynamics, the Kullback-Leibler (KL) divergence between the population state $x(t)$ and the equilibrium $\frac{1}{3} \cdot \mathbf{1}$ does not converge to zero; while the storage function of the BNN dynamics converges to zero, which implies that the population state trajectory converges to the equilibrium $\frac{1}{3} \cdot \mathbf{1}$.

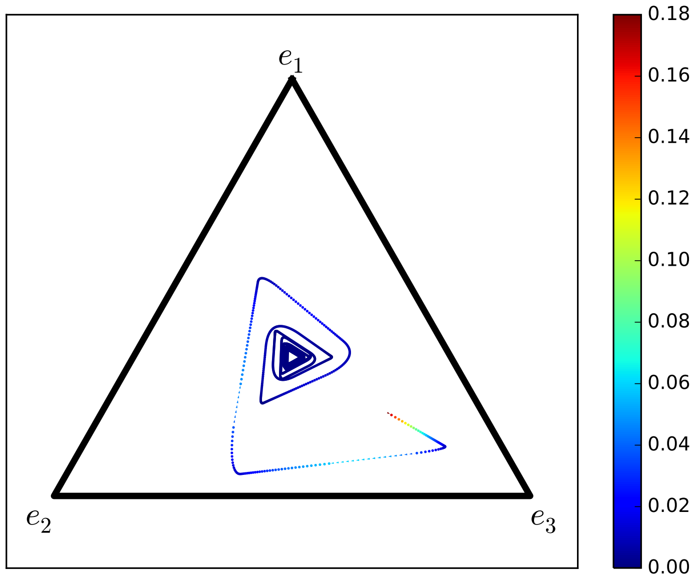


(a) Population state trajectory on simplex $\mathbb{X}$

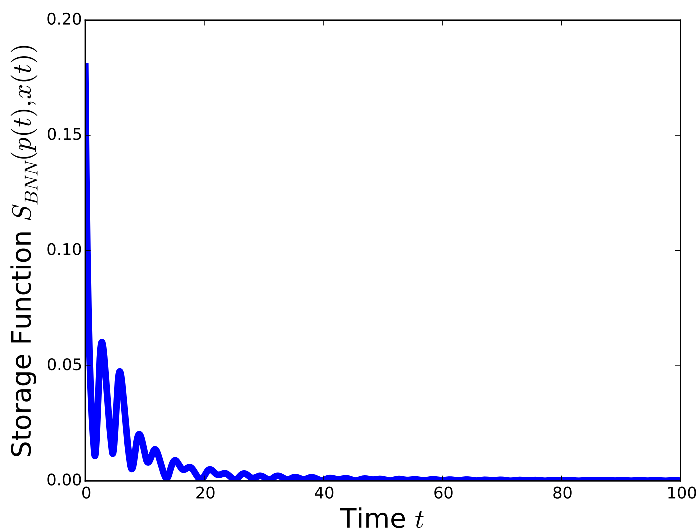


(b) Time-evolution of the KL divergence $E(x(t))$ along the trajectory starting from $(p_0, x_0) = (-1, 0, 1, 0.2, 0.2, 0.6)$

Figure 4.2: Simulation results for the replicator dynamics under a cumulative payoff given by (4.29). $E(x) = \sum_{i=1}^{3} x_i^* \ln \frac{x_i^*}{x}$ where $x^* = \frac{1}{3} \cdot \mathbf{1}$



(a) Population state trajectory on simplex $\mathbb{X}$



(b) Time-evolution of storage function $S_{BNN}(p(t), x(t))$ along the trajectory starting from $(p_0, x_0) = (-1, 0, 1, 0.2, 0.2, 0.6)$

Figure 4.3: Simulation results for the BNN dynamics under a cumulative payoff given by (4.29). $S_{BNN}(p, x) = \frac{1}{2}\sum_{i=1}^{3} [\hat{p}_i]_+^2$

4.4.2 BNN dynamics and logit dynamics in the Hypnodisk game

Consider the Hypnodisk game whose payoff function is given by

$$\begin{aligned}
\begin{pmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{pmatrix} &= H \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} \\
&= \cos\left(\theta(x_1(t), x_2(t), x_3(t))\right) \begin{pmatrix} x_1(t) - \frac{1}{3} \\ x_2(t) - \frac{1}{3} \\ x_3(t) - \frac{1}{3} \end{pmatrix} \\
&\quad + \frac{\sqrt{3}}{3} \sin\left(\theta(x_1(t), x_2(t), x_3(t))\right) \begin{pmatrix} x_2(t) - x_3(t) \\ x_3(t) - x_1(t) \\ x_1(t) - x_2(t) \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}
\end{aligned} \tag{4.31}$$

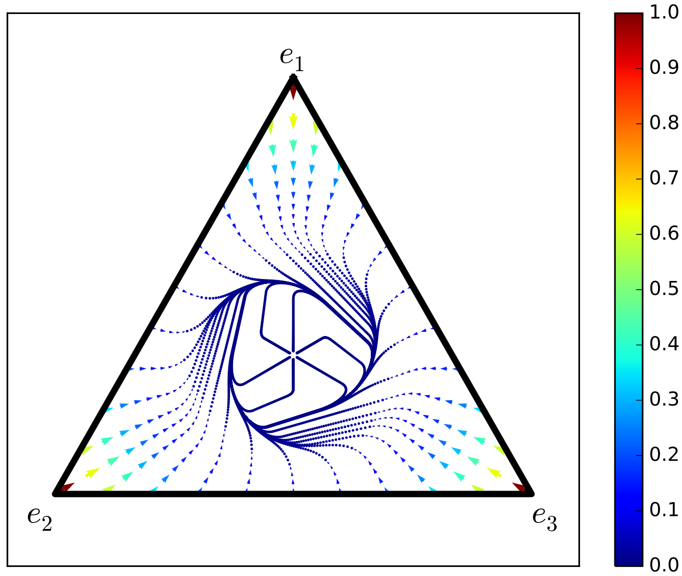
where θ is a smooth function as described in the proof of Proposition 4.2.10.

We note that under (4.31), both the BNN dynamics (4.4) and logit dynamics (4.6) have a unique equilibrium point at $(p, x) = \left(\frac{1}{3} \cdot \mathbf{1}, \frac{1}{3} \cdot \mathbf{1}\right)$. The population state trajectories induced by these dynamics under (4.31) are illustrated in Figure 4.4 and Figure 4.5, respectively. From the illustrations, we can observe that the logit dynamics have a stronger stability property than do the BNN dynamics as the population state trajectory of the former converges to the equilibrium $x = \frac{1}{3} \cdot \mathbf{1}$; while that of the latter dynamics forms a limit cycle.

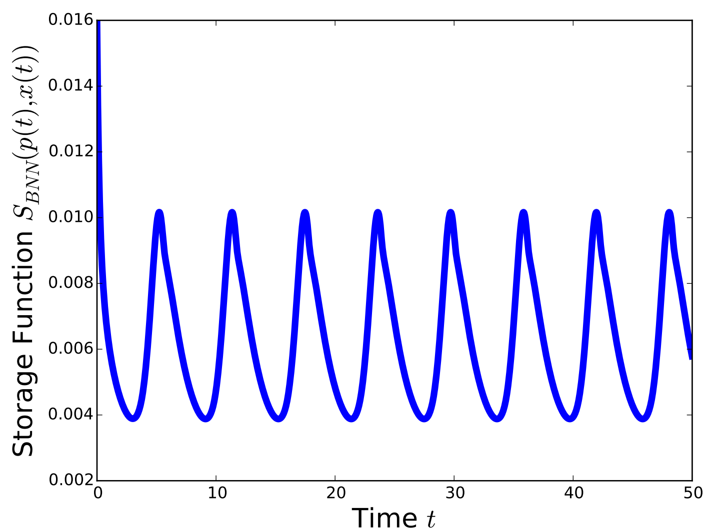
Now consider that a control cost given by $v(x) = \eta \cdot \sum_{i=1}^{3} x_i \ln x_i$ is imposed on the BNN dynamics as in (4.21a). The resulting perturbed dynamics are represented by

$$\dot{x}_i = \left[\tilde{p}_i - \tilde{p}^T x\right]_+ - x_i \sum_{j=1}^{3} \left[\tilde{p}_j - \tilde{p}^T x\right]_+ \tag{4.32}$$

where $\tilde{p} = p - \nabla_x v$. Since v is a strongly convex function, according to Proposition 4.2.14, we can see that (4.32) is strictly output passive. The population state trajectory induced by (4.32) under the payoff function (4.31) is depicted in Figure 4.6, which shows that the trajectory converges to equilibrium $x = \frac{1}{3} \cdot \mathbf{1}$.

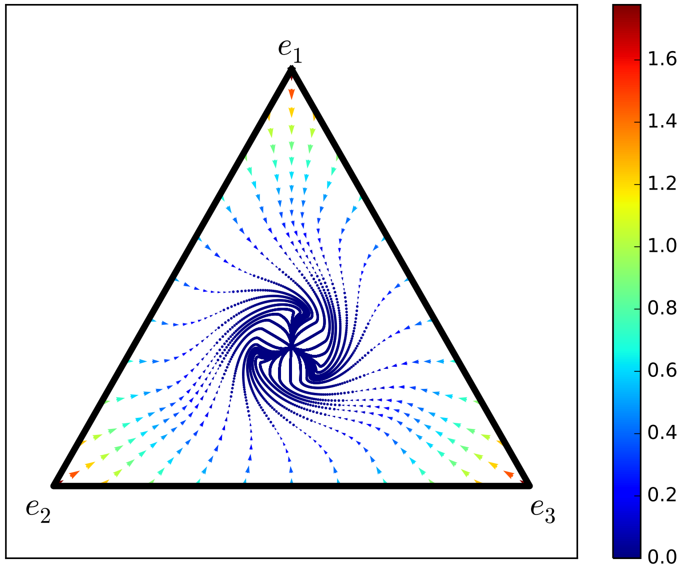


(a) Population state trajectory on simplex $\mathbb{X}$

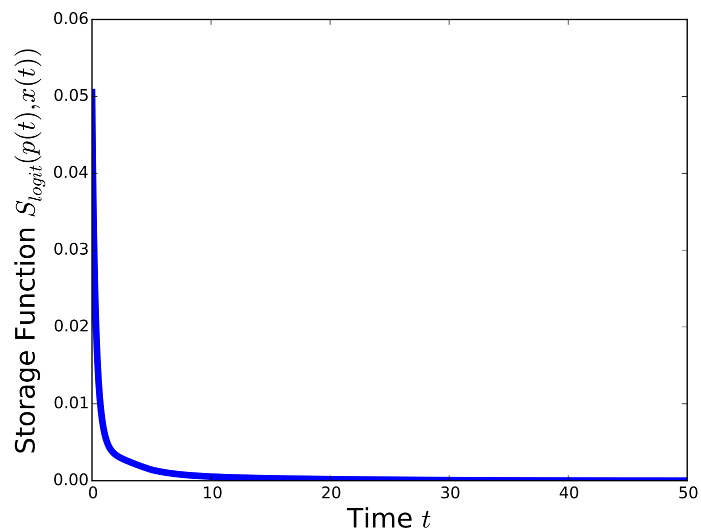


(b) Time evolution of storage function $S_{BNN}(p(t), x(t))$ along the trajectory starting from $p_0 = H(x_0)$, $x_0 = (0.5203, 0.3394, 0.1403)$

Figure 4.4: Simulation results for the BNN dynamics in the Hypnodisk game (4.31). $S_{BNN}(p, x) = \frac{1}{2}\sum_{i=1}^{3} [\hat{p}_i]_+$

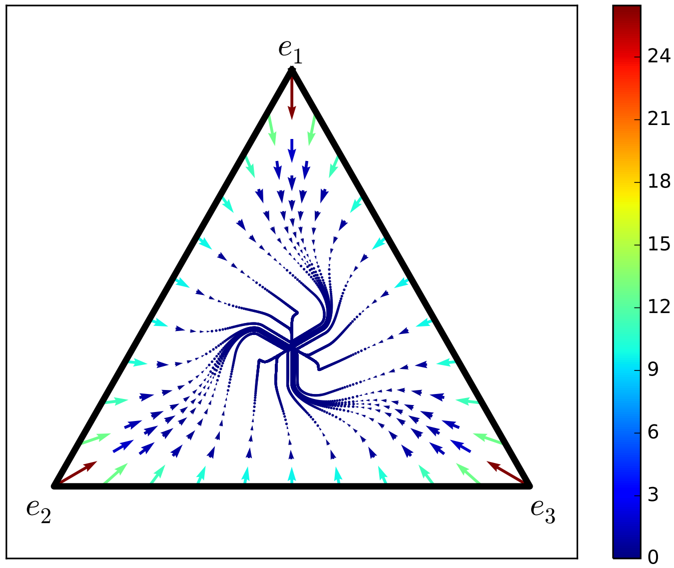


(a) Population state trajectory on simplex $\mathbb{X}$

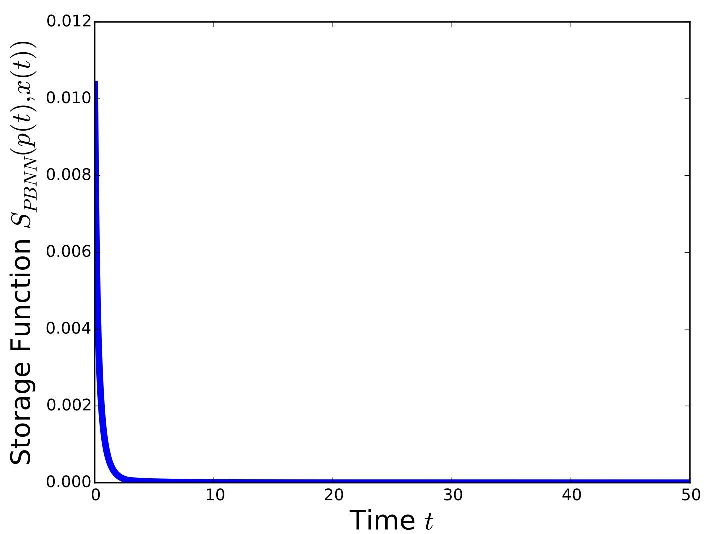


(b) Time evolution of storage function $S_{logit}(p(t), x(t))$ along the trajectory starting from $p_0 = H(x_0)$, $x_0 = (0.4677, 0.3709, 0.1614)$

Figure 4.5: Simulation results for the logit dynamics ($\eta' = 0.36$) in the Hypnodisk game (4.31). $S_{logit}(p, x) = \max_{y \in \mathrm{int}(\mathbb{X})} \left[p^T y - \eta' \cdot \sum_{i=1}^{3} y_i \ln y_i\right] - p^T x + \eta' \cdot \sum_{i=1}^{3} x_i \ln x_i$



(a) Population state trajectory on simplex $\mathbb{X}$



(b) Time evolution of storage function $S_{PBNN}(p(t), x(t))$ along the trajectory starting from $p_0 = H(x_0)$, $x_0 = (0.4269, 0.3876, 0.1855)$

Figure 4.6: Simulation results for the perturbed BNN dynamics $(\eta' = 0.36)$ in the Hypnodisk game (4.31). $S_{PBNN}(p, x) = \frac{1}{2} \sum_{i=1}^{3} \left[\tilde{p}_i - \tilde{p}^T x\right]_+$

4.5 Summary and Future Work

We have exploited the notion of passivity in evolutionary game theory. We have defined passivity for evolutionary dynamics and characterized it in terms of vector fields that define the state-space realizations of the dynamics. Based on the characterization, we have studied certain properties of passive dynamics and established stability in a generalized class of population games. Numerical simulations are provided to illustrate the stability results.

To benefit from the presented passivity methods, as a future direction, we suggest to investigate the following design problems: How to design passive evolutionary dynamics whose storage function achieves its minimum at desired states; while maintaining their stability in population games of interest.

Appendix A: Auxiliary Results for Chapter 2

A.1 Computational Considerations

We proceed to outlining how to find the source components of a directed graph and how to compute an omniscience-achieving parameter choice for (2.1) that satisfies (1.2), provided that the conditions of Theorem 2.2.2 hold.

A.1.1 Finding Source Components

In Chapter 22.5 of [109], the Strongly-Connected Components (SCC) algorithm is described for finding all strongly connected components of a directed graph. For each strongly connected component given by the SCC algorithm, we can check whether there are no incoming edges from outside of it, in which case it is a source component. Since both the SCC algorithm and subsequent checks have linear-time complexity, the overall procedure for finding source components is a linear-time algorithm.

A.1.2 Computing an Omniscience-achieving Parameter Choice

We proceed to describing randomized procedures to compute a parameter choice for (2.1). Under the detectability condition of Theorem 2.2.2, it follows from our analysis in Section 2.4.3 that the parameter choice obtained from the following randomized procedures satisfies (1.2) and is omniscience-achieving with probability one.

A.1.2.1 Computation of $\mathbf{W} = (\mathbf{w}_{ij})_{i,j\in\mathbb{V}}$

Let $\{\mathcal{G}_l\}_{l=1}^{m_s}$ with $\mathcal{G}_l = (\mathbb{V}_l, \mathbb{E}_l)$ be the source components of $\mathcal{G}$, and let us define $\mathbb{V}_{m_s+1} = \mathbb{V} \setminus \bigcup_{l=1}^{m_s} \mathbb{V}_l$. We first find a spanning subgraph $\mathcal{G}'$ of $\mathcal{G}$ for which $\{\mathcal{G}_l\}_{l=1}^{m_s}$ are the source components of $\mathcal{G}'$, and the subgraph of $\mathcal{G}'$ induced[1] by $\mathbb{V}_{m_s+1}$ has no cycle. Then we select a weight matrix $\mathbf{W}$ whose sparsity pattern is consistent with $\mathcal{G}'$ and, hence, with $\mathcal{G}$.

In order to obtain $\mathcal{G}'$, we perform multiple rounds of the depth-first search on $\mathcal{G}$ where each round of the search starts from a (unvisited) vertex in $\mathbb{V}_{m_s+1}$ that is a neighbor of a source component. We continue this search until every vertex in $\mathbb{V}_{m_s+1}$ is visited exactly once. The overall search operation gives a disjoint collection of directed trees. Next, we eliminate certain edges of $\mathcal{G}$ to obtain a new graph $\mathcal{G}'$ such that $\mathcal{G}'$ is same as $\mathcal{G}$ except that the subgraph of $\mathcal{G}'$ induced by $\mathbb{V}_{m_s+1}$ is the union of the trees obtained from the aforementioned search operation. By our construction of $\mathcal{G}'$, it can be verified that $\{\mathcal{G}_l\}_{l=1}^{m_s}$ are the source components of $\mathcal{G}'$, and the subgraph of $\mathcal{G}'$ induced by $\mathbb{V}_{m_s+1}$ has no cycle.

For each i in $\mathbb{V}$, let $\mathbb{N}_i'$ be the neighborhood of vertex i defined by $\mathcal{G}'$. We choose the submatrices of $\mathbf{W}$ in (2.11) as described by the following randomized procedure (Procedure 3). Here, we use $q \sim U\,(a, b)$ to denote a randomization in which q is the realization of a random variable uniformly distributed in the interval (a, b). We assume that the ran-

[1] $(\mathbb{V}', \mathbb{E}')$ is a subgraph of $(\mathbb{V}, \mathbb{E})$ induced by $\mathbb{V}' \subseteq \mathbb{V}$ if $\mathbb{E}'$ contains every edge (i, j) in $\mathbb{E}$ with $i, j \in \mathbb{V}'$.

domizations presented in the procedure are drawn from independent random variables.

Procedure 3: Computation of $\mathbf{W} = (\mathbf{w}_{ij})_{i,j \in \mathbb{V}}$

input : $\mathcal{G}' = (\mathbb{V}, \mathbb{E}')$

output: $\mathbf{W} = (\mathbf{w}_{ij})_{i,j \in \mathbb{V}}$

begin

  for $l \in \{1, \cdots, m_s\}$ **do**

    `/* computation of` $\mathbf{W}_l$ `in` (2.11) `*/`

    $\alpha \sim U(0, 1)$

    for $i \in \mathbb{V}_l$ **do**

      for $j \in \mathbb{V}_l \setminus \{i\}$ **do**

        if $j \in \mathbb{N}_i' \setminus \{i\}$ **then**

          $q \sim U\left(-\frac{1}{|\mathbb{N}_i'|-1}, 0\right)$

          $\mathbf{w}_{ij} \leftarrow -\alpha \cdot q$

        else $\mathbf{w}_{ij} \leftarrow 0$

      $\mathbf{w}_{ii} \leftarrow 1 - \sum_{j \in \mathbb{N}_i' \setminus \{i\}} \mathbf{w}_{ij}$

  `/* computation of` $\{\mathbf{W}_{m_s+1,l}\}_{l=1}^{m_s+1}$ `in` (2.11) `*/`

  for $i \in \mathbb{V}_{m_s+1}$ **do**

    for $j \in \mathbb{V}$ **do**

      if $j \in \mathbb{N}_i' \setminus \{i\}$ **then** $\mathbf{w}_{ij} \leftarrow \frac{1}{|\mathbb{N}_i'|-1}$

      else $\mathbf{w}_{ij} \leftarrow 0$

It follows from Lemmas 2.4.3 - 2.4.4 and Theorem 2.4.5 that Procedure 3 will select a weight matrix $\mathbf{W}$ randomly from a parametrized set for which almost all parameters lead to a suitable choice, with the possible exception of a subset of measure zero. Our particular choice for the distributions governing the randomizations in Procedure 3 is not important, and any other choice that assigns null probability to a subset of measure zero would work.

A.1.2.2 Computation of $\{\mathbf{K}_i, \mathbf{P}_i, \mathbf{Q}_i, \mathbf{S}_i\}_{i\in\mathbb{V}}$

In what follows, we describe a randomized method (Procedure 4) to choose gain matrices $\{\mathbf{K}_i, \mathbf{P}_i, \mathbf{Q}_i, \mathbf{S}_i\}_{i\in\mathbb{V}}$ that, in conjunction with $\mathbf{W}$ obtained from Procedure 3, are omniscience-achieving with probability one, provided that the conditions of Theorem 2.2.2 are satisfied. In fact, it follows from Theorem 2.4.10 that if the conditions of Theorem 2.2.2 are satisfied then the procedure will be selecting from a set in which almost all choices are omniscience-achieving.

Given a positive real c, we use $\mathbf{K} \sim U_{n\cdot r}\left((-c,c)^{n\cdot r}\right)$ to denote a randomization leading to a matrix $\mathbf{K}$ in $\mathbb{R}^{n\times r}$ whose entries are the realizations of $n \cdot r$ independent random variables uniformly distributed in the interval $(-c, c)$. For each i in $\mathbb{V}_l$, let $\overline{B}_i = e_i \otimes I_n$ and $\overline{C}_i = e_i^T \otimes C_i$, where e_i is the i-th column of the $|\mathbb{V}_l|$-dimensional identity matrix. We choose $\{\mathbf{K}_i, \mathbf{P}_i, \mathbf{Q}_i, \mathbf{S}_i\}_{i\in\mathbb{V}}$ as described below, where repeated randomizations are drawn from independent random variables.

Procedure 4: Computation of $\{\mathbf{K}_i, \mathbf{P}_i, \mathbf{Q}_i, \mathbf{S}_i\}_{i\in\mathbb{V}}$

input : $\mathcal{G}' = (\mathbb{V}, \mathbb{E}')$, $\mathbf{W}$ given as in (2.11), and (A, C) given as in (1.1)

output: $\{\mathbf{K}_i, \mathbf{P}_i, \mathbf{Q}_i, \mathbf{S}_i\}_{i\in\mathbb{V}}$

begin

  select $\mathbb{V}^R$ as in Definition 2.2.1

  for $l \in \{1, \cdots, m_s\}$ **do**

    `/* computation of` $\{\mathbf{K}_i, \mathbf{P}_i, \mathbf{Q}_i, \mathbf{S}_i\}_{i\in\mathbb{V}_l}$ `*/`

    $i_l \in \mathbb{V}_l \cap \mathbb{V}^R$ `// a singleton`

    for $i \in \mathbb{V}_l \setminus \{i_l\}$ **do**

      $\mathbf{K}_i \sim U_{n\cdot r_i}\left((-c, c)^{n\cdot r_i}\right)$

      $\mu_i = 0$

    compute $\mathbf{K}_{i_l}, \mathbf{P}_{i_l}, \mathbf{Q}_{i_l}, \mathbf{S}_{i_l}$ with $\mu_{i_l} = |\mathbb{V}_l| - 1$ for which

$$\begin{pmatrix} \mathbf{W}_l \otimes A - \sum_{i\in\mathbb{V}_l} \overline{B}_i \mathbf{K}_i \overline{C}_i & -\overline{B}_{i_l}\mathbf{P}_{i_l} \\ \mathbf{Q}_{i_l}\overline{C}_{i_l} & \mathbf{S}_{i_l} \end{pmatrix}$$

    is stable, provided they exist.

  `/* computation of` $\{\mathbf{K}_i, \mathbf{P}_i, \mathbf{Q}_i, \mathbf{S}_i\}_{i\in\mathbb{V}_{m_s+1}}$ `*/`

  for $i \in \mathbb{V}_{m_s+1}$ **do**

    $\mathbf{K}_i \leftarrow \mathbf{0}$

    $\mu_i = 0$

A.2 Nondegeneracy of the Dynamic Matrix A

Here, we justify the nondegeneracy assumption on the dynamic matrix A in (1.1). Suppose that the dynamic matrix A is degenerate. Let M be the real matrix for which $J = M^{-1}AM$ is a real block diagonal matrix in the following form:

$$J = diag\,(J_1, \cdots, J_p) \tag{A.1}$$

where for each i in $\{1, \cdots, p\}$, the submatrix J_i is the i-th real Jordan block. In particular, suppose that $J_{p_0+1}, \cdots, J_p$ are all the Jordan blocks associated with the zero eigenvalue. Notice that there exists a positive integer k_0 for which $J_i^k = \mathbf{0}$ for all $k \geq k_0$ and i in $\{p_0 + 1, \cdots, p\}$.

By applying a similarity transform to the plant (1.1) with M, we obtain

$$\begin{aligned} \begin{pmatrix} x_a(k+1) \\ x_b(k+1) \end{pmatrix} &= \begin{pmatrix} A_a & \mathbf{0} \\ \mathbf{0} & A_b \end{pmatrix} \begin{pmatrix} x_a(k) \\ x_b(k) \end{pmatrix} \\ y(k) &= \begin{pmatrix} C_a & C_b \end{pmatrix} \begin{pmatrix} x_a(k) \\ x_b(k) \end{pmatrix} \end{aligned} \tag{A.2}$$

where $\begin{pmatrix} x_a(k) \\ x_b(k) \end{pmatrix} = M^{-1}x(k)$, $A_a = diag\,(J_1, \cdots, J_{p_0})$, $A_b = diag\,(J_{p_0+1}, \cdots, J_p)$, and $\begin{pmatrix} C_a & C_b \end{pmatrix} = CM$. Since the block diagonal elements of A_b are the Jordan blocks associated with the zero eigenvalue, it holds that $x_b(k) = 0$ for all $k \geq k_0$, and from (A.2) we can derive the following state-space equation:

$$\begin{aligned} x_a(k+1) &= A_a x_a(k) \\ y(k) &= C_a x_a(k) \end{aligned} \tag{A.3}$$

for $k \geq k_0$, where A_a is a nondegenerate matrix. For this reason, in what regards to achieving asymptotic omniscience, we may design a distributed observer (2.1) for (A.3) to asymptotically resolve the state $x_a(k)$, from which the state $x(k)$ of (1.1) can be obtained using the relation $x(k) = M \begin{pmatrix} x_a(k) \\ 0 \end{pmatrix}$ for $k \geq k_0$.

A.3 Preliminary Concepts and Proof of Proposition 2.3.2

Let us define

$$x(k) = \begin{pmatrix} \chi^{(1)}(k) \\ x'(k) \end{pmatrix} \text{ where } x'(k) = \begin{pmatrix} \chi^{(2)}(k) - \chi^{(1)}(k) \\ \vdots \\ \chi^{(m_a)}(k) - \chi^{(1)}(k) \end{pmatrix}$$

We can re-write (2.3) as follows:

$$\begin{aligned} x(k+1) &= Ax(k) + \sum_{i \in \mathbb{V}} B_i u_i(k) \\ y_i(k) &= C_i x(k) \end{aligned} \tag{A.4}$$

for each i in $\mathbb{V}$, where

$$\begin{aligned} A &= \left(\begin{array}{c|ccc} F_o & F_{12} & \cdots & F_{1m_a} \\ \hline \mathbf{0} & & A' & \end{array}\right) \\ B_i &= \begin{pmatrix} G_{1i}^T & (B_i')^T \end{pmatrix}^T \\ C_i &= H_i \left(\begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{1}_{m_a-1} & I_{m_a-1} \end{pmatrix} \otimes I_n \right) \end{aligned} \tag{A.5}$$

In (A.5), F_o, $\{F_{1j}\}_{j=2}^{m_a}$, G_{1i}, H_i are from (2.3), and A' and B_i' are defined in (2.5b) and (2.5c), respectively.

To achieve asymptotic synchronization of the system (2.3), we need to find a distributed controller for which the partial state $x'(k)$ of (A.4) converges to zero as $k \to \infty$. To find such distributed controller, we adopt the following procedure (also see Figure A.1 for an illustration):

1. Using the method described in Section 2.2, we first design a distributed observer (2.1) for the multi-agent system (A.4) subject to the pre-selected graph $\mathcal{G}$.

2. Then, using results on the synthesis of decentralized control systems, we find fully decentralized controllers for the multi-agent/distributed observer system obtained in Step 1.

3. Finally, we recover a distributed controller from the distributed observer and the fully decentralized controllers obtained in Step 1 and Step 2, respectively.

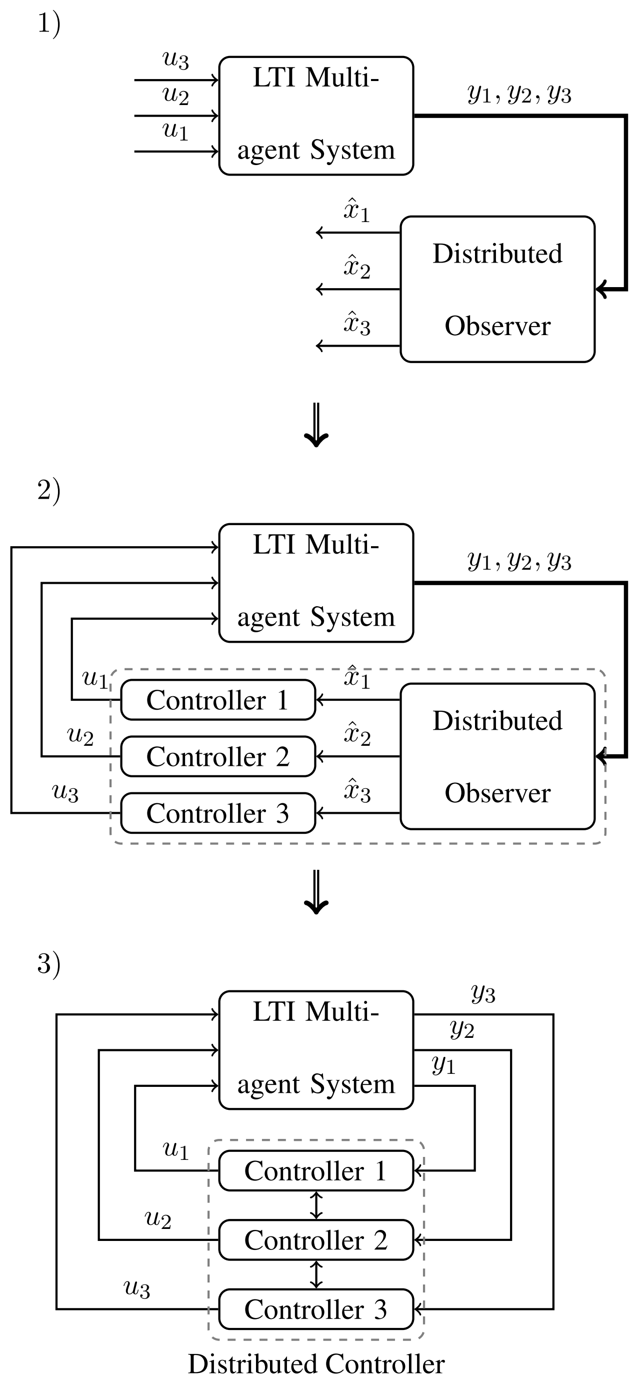


Figure A.1: Diagrams depicting a design procedure for finding a distributed controller.

Based on the aforementioned design procedure, we note that the resulting distributed controller has the following state-space representation:

$$\begin{pmatrix} \hat{x}_i(k+1) \\ z_i(k+1) \\ w_i(k+1) \end{pmatrix} = \begin{pmatrix} A\sum_{j\in\mathbb{N}_i} \mathbf{w}_{ij}\hat{x}_j(k) + \mathbf{K}_i\left(y_i(k) - C_i\hat{x}_i(k)\right) + \mathbf{P}_i z_i(k) \\ \mathbf{Q}_i\left(y_i(k) - C_i\hat{x}_i(k)\right) + \mathbf{S}_i z_i(k) \\ \mathbf{S}_i^d w_i(k) + \mathbf{Q}_i^d \hat{x}_i'(k) \end{pmatrix} \tag{A.6a}$$

$$u_i(k) = \mathbf{P}_i^d w_i(k) + \mathbf{K}_i^d \hat{x}_i'(k) \tag{A.6b}$$

for each i in $\mathbb{V}$, where $\hat{x}_i'(k) = \begin{pmatrix} \mathbf{0} & I_{(m_a-1)\cdot n} \end{pmatrix} \hat{x}_i(k)$, and A and C_i are defined in (A.5). It can be verified that (A.6) is a special case of (2.4). Hence, it remains to consider a parameter choice for (A.6) such that the resulting distributed controller synchronizes the system (2.3). In what follows, we describe particular choices of

$$\mathbf{W} = \left(\mathbf{w}_{ij}\right)_{i,j\in\mathbb{V}}, \{\mathbf{K}_i, \mathbf{P}_i, \mathbf{Q}_i, \mathbf{S}_i\}_{i\in\mathbb{V}} \text{ and } \left\{\mathbf{K}_i^d, \mathbf{P}_i^d, \mathbf{Q}_i^d, \mathbf{S}_i^d\right\}_{i\in\mathbb{V}}$$

for (A.6) in Appendix A.3.1 and Appendix A.3.2, respectively. A proof of Proposition 2.3.2 is then followed.

A.3.1 A Choice of $\mathbf{W} = \left(\mathbf{w}_{ij}\right)_{i,j\in\mathbb{V}}$, $\{\mathbf{K}_i, \mathbf{P}_i, \mathbf{Q}_i, \mathbf{S}_i\}_{i\in\mathbb{V}}$

We design a distributed observer (2.1) for the system (A.4) subject to the given graph $\mathcal{G}$. The estimation error $\widetilde{x}_i = x - \hat{x}_i$ of (2.1) evolves according to the following state-space equation:

$$\begin{pmatrix} \widetilde{x}(k+1) \\ z(k+1) \end{pmatrix} = \begin{pmatrix} \mathbf{W}\otimes A - \overline{\mathbf{K}}\,\overline{C} & -\overline{\mathbf{P}} \\ \overline{\mathbf{Q}}\,\overline{C} & \overline{\mathbf{S}} \end{pmatrix} \begin{pmatrix} \widetilde{x}(k) \\ z(k) \end{pmatrix} + \begin{pmatrix} \mathbf{1}_m \otimes \left(\sum_{i\in\mathbb{V}} B_i u_i(k)\right) \\ \mathbf{0} \end{pmatrix} \tag{A.7}$$

where

$$
\begin{aligned}
\widetilde{x} &= \begin{pmatrix} \widetilde{x}_1^T & \dots & \widetilde{x}_m^T \end{pmatrix}^T, \qquad z = \begin{pmatrix} z_1^T & \dots & z_m^T \end{pmatrix}^T, \\
\overline{C} &= \begin{pmatrix} \overline{C}_1^T & \dots & \overline{C}_m^T \end{pmatrix}^T \quad \text{with } \overline{C}_i = e_i^T \otimes C_i, \\
\mathbf{W} &= (\mathbf{w}_{ij})_{i,j\in\mathbb{V}}, \\
\overline{\mathbf{K}} &= diag\left(\mathbf{K}_1, \cdots, \mathbf{K}_m\right), \qquad \overline{\mathbf{P}} = diag\left(\mathbf{P}_1, \cdots, \mathbf{P}_m\right), \\
\overline{\mathbf{Q}} &= diag\left(\mathbf{Q}_1, \cdots, \mathbf{Q}_m\right), \qquad \overline{\mathbf{S}} = diag\left(\mathbf{S}_1, \cdots, \mathbf{S}_m\right)
\end{aligned}
\tag{A.8}
$$

In (A.8), e_i is the i-th column of the m-dimensional identity matrix.

By writing (A.4) and (A.7) altogether and by omitting $\chi^{(1)}$ from x in (A.4), we can derive the following equation:[2]

$$
\begin{pmatrix} x'(k+1) \\ \widetilde{x}(k+1) \\ z(k+1) \end{pmatrix} = \begin{pmatrix} A' & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}\otimes A - \overline{\mathbf{K}}\,\overline{C} & -\overline{\mathbf{P}} \\ \mathbf{0} & \overline{\mathbf{Q}}\,\overline{C} & \overline{\mathbf{S}} \end{pmatrix} \begin{pmatrix} x'(k) \\ \widetilde{x}(k) \\ z(k) \end{pmatrix} + \begin{pmatrix} \sum_{i\in\mathbb{V}} B_i' u_i(k) \\ \mathbf{1}_m \otimes \left(\sum_{i\in\mathbb{V}} B_i u_i(k)\right) \\ \mathbf{0} \end{pmatrix}
\tag{A.9a}
$$

$$
\hat{x}_i'(k) = E_i \begin{pmatrix} x'(k) \\ \widetilde{x}(k) \\ z(k) \end{pmatrix}
\tag{A.9b}
$$

for each i in $\mathbb{V}$, where $E_i = \begin{pmatrix} I_{(m_a-1)\cdot n} & -e_i^T \otimes \begin{pmatrix} \mathbf{0} & I_{(m_a-1)\cdot n} \end{pmatrix} & \mathbf{0} \end{pmatrix}$, e_i is the i-th column of the m-dimensional identity matrix, and A, A', B_i, B_i' and $\overline{C}_i$ are defined in

[2]This is valid since (A.9) does not depend on $\chi^{(1)}$.

(A.5) and (A.8), respectively. In (A.9b), we note that

$$E_i \begin{pmatrix} x'(k) \\ \widetilde{x}(k) \\ z(k) \end{pmatrix} = x'(k) - \begin{pmatrix} \mathbf{0} & I_{(m_a-1)\cdot n} \end{pmatrix} \widetilde{x}_i(k) = \begin{pmatrix} \mathbf{0} & I_{(m_a-1)\cdot n} \end{pmatrix} \hat{x}_i(k)$$

Essentially, the state-space equation (A.9) describes a LTI system with state $\begin{pmatrix} x'^T & \widetilde{x}^T & z^T \end{pmatrix}^T$, output vector $\hat{x}'_i$, and inputs $\{u_i\}_{i\in\mathbb{V}}$. If there is no input, i.e., $u_i = 0$ for all i in $\mathbb{V}$, and the matrix given by

$$\begin{pmatrix} \mathbf{W} \otimes A - \overline{\mathbf{K}}\,\overline{C} & -\overline{\mathbf{P}} \\ \overline{\mathbf{Q}}\,\overline{C} & \overline{\mathbf{S}} \end{pmatrix} \tag{A.10}$$

is stable, then we can see that the output $\hat{x}'_i(k)$ converges to $x'(k)$ as $k \to \infty$.

The following Lemma states the stabilizability and detectability of (A.9).

Lemma A.3.1. *Let a graph $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ and a LTI system* (A.4) *be given. Suppose that assumptions **(i)** and **(ii)** of Proposition 2.3.2 hold. We can find $\mathbf{W}$, $\overline{\mathbf{K}}$, $\overline{\mathbf{P}}$, $\overline{\mathbf{Q}}$, $\overline{\mathbf{S}}$ in* (A.9) *for which the resulting system* (A.9) *is both stabilizable and detectable for all i in $\mathbb{V}$.*

Proof. Notice that because of **(ii)** of Proposition 2.3.2, by Theorem 2.2.2 and the procedures in Appendix A.1.2, we can find $\mathbf{W}$, $\overline{\mathbf{K}}$, $\overline{\mathbf{P}}$, $\overline{\mathbf{Q}}$, $\overline{\mathbf{S}}$ for which the matrix (A.10) is stable. Under this choice of $\mathbf{W}$, $\overline{\mathbf{K}}$, $\overline{\mathbf{P}}$, $\overline{\mathbf{Q}}$, $\overline{\mathbf{S}}$, we show the stabilizability and detectability of (A.9). The stabilizability directly follows from **(i)** of Proposition 2.3.2. The detectability can be verified by the fact that if $u_i = 0$ for all i in $\mathbb{V}$, then it holds that $\lim_{k\to\infty} \|\hat{x}'_i(k) - x'(k)\| = 0$ for all i in $\mathbb{V}$ and $\lim_{k\to\infty} \left\| \begin{pmatrix} \widetilde{x}(k) \\ z(k) \end{pmatrix} \right\| = 0$. $\square$

A.3.2 A Choice of $\left\{\mathbf{K}_i^d, \mathbf{P}_i^d, \mathbf{Q}_i^d, \mathbf{S}_i^d\right\}_{i \in \mathbb{V}}$

Consider a set of fully decentralized controllers whose state-space representation is given as follows:

$$\begin{aligned} w_i(k+1) &= \mathbf{S}_i^d w_i(k) + \mathbf{Q}_i^d \hat{x}_i'(k) \\ u_i(k) &= \mathbf{P}_i^d w_i(k) + \mathbf{K}_i^d \hat{x}_i'(k) \end{aligned} \tag{A.11}$$

for each i in $\mathbb{V}$.

Consider the closed-loop system obtained by interconnecting (A.9) and (A.11) where the parameters $\mathbf{W}, \overline{\mathbf{K}}, \overline{\mathbf{P}}, \overline{\mathbf{Q}}, \overline{\mathbf{S}}$ of (A.9) are chosen as described in Lemma A.3.1. By the stabilizability and detectability of (A.9) for all i in $\mathbb{V}$, using the results on the synthesis of decentralized control systems [63,64], we can find a parameter choice $\left\{\mathbf{K}_i^d, \mathbf{P}_i^d, \mathbf{Q}_i^d, \mathbf{S}_i^d\right\}_{i \in \mathbb{V}}$ for (A.11) that ensures the stability of the closed-loop system.

A.3.3 Proof of Proposition 2.3.2

Suppose that $\mathbf{W}, \overline{\mathbf{K}}, \overline{\mathbf{P}}, \overline{\mathbf{Q}}, \overline{\mathbf{S}}$ and $\left\{\mathbf{K}_i^d, \mathbf{P}_i^d, \mathbf{Q}_i^d, \mathbf{S}_i^d\right\}_{i \in \mathbb{V}}$ are respective parameter choices made by the procedures described in Appendix A.3.1 and Appendix A.3.2. By Lemma A.3.1 and the discussion in Appendix A.3.2, such parameter choices ensure the stability of the closed-loop system resulting from an interconnection of (A.9) and (A.11). We note that, under the same parameter choice, the stability of this closed-loop ensures the synchronization of the multi-agent system (2.3) via the distributed controller described by (A.6), which is a special case of (2.4). Therefore, with the aforementioned parameter choices, we conclude that the distributed controller (A.6) (hence (2.4)) synchronizes the multi-agent system (2.3). This proves the first statement.

Next, we prove the second statement of Proposition 2.3.2. We proceed by writing the state-space equation for agent 1 using (2.3a) and (A.6b) as follows:

$$\chi^{(1)}(k+1) = F_o \chi^{(1)}(k) + \sum_{j=2}^{m_a} F_{1j}\left(\chi^{(j)}(k) - \chi^{(1)}(k)\right) + \sum_{j=1}^{m} G_{1j}\left(\mathbf{P}_j^d w_j(k) + \mathbf{K}_j^d \hat{x}'_j(k)\right) \tag{A.12}$$

Without loss of generality, suppose that $F_o = \begin{pmatrix} F_{o,U} & \mathbf{0} \\ \mathbf{0} & F_{o,S} \end{pmatrix}$ where $F_{o,U}$ and $F_{o,S}$ are unstable and stable parts of F_o, respectively. Accordingly, we obtain a partition $\begin{pmatrix} \chi_U^{(1)} \\ \chi_S^{(1)} \end{pmatrix}$ of $\chi^{(1)}$ and a partition $\begin{pmatrix} G_U \\ G_S \end{pmatrix}$ of $\left(F_{12} \ \cdots \ F_{1m_a} \ G_{11}\mathbf{P}_1^d \ \cdots \ G_{1m}\mathbf{P}_m^d \ G_{11}\mathbf{K}_1^d \ \cdots \ G_{1m}\mathbf{K}_m^d\right)$. For notational convenience, let

$$x' = \begin{pmatrix} \chi^{(2)} - \chi^{(1)} \\ \vdots \\ \chi^{(m_a)} - \chi^{(1)} \end{pmatrix}, \ w = \begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix}, \ \hat{x}' = \begin{pmatrix} \hat{x}'_1 \\ \vdots \\ \hat{x}'_m \end{pmatrix}$$

Since $x'(k)$, $w(k)$, $\hat{x}'(k)$ converge to zero exponentially as $k \to \infty$, it holds that $\lim_{k\to\infty} \left\|\chi_S^{(1)}(k)\right\| = 0$. Now, we consider the unstable dynamics of (A.12), which can be represented by the following state-space equation:

$$\chi_U^{(1)}(k+1) = F_{o,U}\chi_U^{(1)}(k) + G_U \begin{pmatrix} x'(k) \\ w(k) \\ \hat{x}'(k) \end{pmatrix} \tag{A.13}$$

Since eigenvalues of $F_{o,U}$ lie on or outside the unit circle in $\mathbb{C}$, we can verify that a solution to (A.13) satisfies

$$(F_{o,U})^{-k}\,\chi_U^{(1)}(k) = \chi_U^{(1)}(0) + \sum_{l=0}^{k-1} (F_{o,U})^{-l-1}\, G_U \begin{pmatrix} x'(l) \\ w(l) \\ \hat{x}'(l) \end{pmatrix} \tag{A.14}$$

where the right hand side of (A.14) converges exponentially as $k \to \infty$. Let

$$\chi_{o,U} = \chi_U^{(1)}(0) + \sum_{l=0}^{\infty} (F_{o,U})^{-l-1}\, G_U \begin{pmatrix} x'(l) \\ w(l) \\ \hat{x}'(l) \end{pmatrix}$$

be the limit point of (A.14).

To complete the proof, let us consider the following state-space equation:

$$\chi_o(k+1) = F_o \chi_o(k),\ \chi_o(0) = \begin{pmatrix} \chi_{o,U} \\ \chi_{o,S} \end{pmatrix}$$

for any vector $\chi_{o,S}$ of a proper dimension. Since $F_{o,U}$ has the unit spectral radius by the assumption of the second statement, due to the exponential convergence rate of (A.14), we can see that $\lim_{k\to\infty} \left\|\chi^{(1)}(k) - \chi_o(k)\right\| = 0$ holds.

Using the fact that $\lim_{k\to\infty} \left\|\chi^{(i)}(k) - \chi^{(1)}(k)\right\| = 0$ holds for all i in $\mathbb{V}^I \setminus \{1\}$, we conclude that $\lim_{k\to\infty} \left\|\chi^{(i)}(k) - \chi_o(k)\right\| = 0$ holds for all i in $\mathbb{V}^I$. This proves the Proposition.

A.4 Proofs of Lemmas 2.4.3 and 2.4.4

Proof of Lemma 2.4.3: Since the matrix L is a WLM of the graph $\mathcal{G}$ and the positive real number α' satisfies $\alpha' \leq \left(\max_{1\leq i\leq|\mathbb{V}|} l_{ii}\right)^{-1}$, for every α in $(0, \alpha')$, the matrix $\mathbf{W}$ is stochastic. Hence, it remains to show that for almost every α in $(0, \alpha')$, $\mathbf{W} \otimes A$ satisfies the UEPP.

Let $\{\upsilon_1, \cdots, \upsilon_s\}$ and $\{\lambda_1, \cdots, \lambda_t\}$ be the sets of distinct eigenvalues of A and L, respectively. Under the choice $\mathbf{W} = I_{|\mathbb{V}|} - \alpha L$, we can observe that if $\mathbf{W} \otimes A$ does not satisfy the UEPP, then its nonzero eigenvalue can be expressed as a product

$$(1 - \alpha\lambda)\upsilon = (1 - \alpha\lambda')\upsilon'$$

for distinct λ, λ' in $\{\lambda_1, \cdots, \lambda_t\}$ and for distinct υ, υ' in $\{\upsilon_1, \cdots, \upsilon_s\}$. Since the sets of distinct eigenvalues of A and L are both finite, we conclude that the set of the values of α for which the UEPP does not hold is finite. Hence, for almost every α in $(0, \alpha')$, $\mathbf{W} \otimes A$ satisfies the UEPP. $\square$

Proof of Lemma 2.4.4: By the UEPP of $\mathbf{W} \otimes A$, for each nonzero eigenvalue λ of $\mathbf{W} \otimes A$, we can find the unique pair of eigenvalues $\lambda_{\mathbf{W}}$ and λ_A of $\mathbf{W}$ and A, respectively, for which $\lambda = \lambda_{\mathbf{W}} \cdot \lambda_A$ holds. Since $\mathbf{W}$ has all simple eigenvalues and $\mathbf{W} \otimes A$ satisfies the UEPP, we can show that there is a unique eigenvector (unique up to a scale factor), say v, associated with $\lambda_{\mathbf{W}}$, and the geometric multiplicities of λ and λ_A are equal[3]. Hence, we can see that an eigenvector q of $\mathbf{W} \otimes A$ associated with λ can be written as $q = v \otimes p$ where p is an eigenvector of A associated with λ_A. This proves the Lemma. $\square$

A.5 Preliminary Results and Proof of Theorem 2.4.5

In this section, we provide a proof of Theorem 2.4.5. The proof hinges on some results from structured linear system theory [110, 111]. To this end, we briefly review the

[3] A proof of this argument is along similar lines as that of Lemma 3.8 in [20]

structural controllability and observability of structured linear systems in Appendix A.5.1 and provide the detailed proof of Theorem 2.4.5 in Appendix A.5.3.

A.5.1 Structural Controllability and Observability

Consider a graph $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ with $\mathbb{V} = \{1, \cdots, |\mathbb{V}|\}$ and an associated structured linear system whose state-space representation is given as follows:

$$\begin{aligned} x(k+1) &= [A]x(k) + [b_i]u(k) \\ y(k) &= [c_j]^T x(k) \end{aligned} \tag{A.15}$$

where $[A] \in \mathbb{R}^{|\mathbb{V}|\times|\mathbb{V}|}$ is a structure matrix, and $[b_i] \in \mathbb{R}^{|\mathbb{V}|}$ and $[c_j] \in \mathbb{R}^{|\mathbb{V}|}$ are structure vectors. Depending on respective sparse structures, entries of structure matrices and vectors are either zero or indeterminate. In particular, we suppose that $[A]$ is consistent[4] with the graph $\mathcal{G}$, and all entries of $[b_i]$ and $[c_j]$ are zero except the i-th entry and j-th entry, respectively. Under this setting, there are $(|\mathbb{E}| + 2)$ indeterminate entries of $[A]$, $[b_i]$, and $[c_j]$, and if we allow each indeterminate entry to take a value in $\mathbb{R}$, then a choice of these entries can be represented by a vector in $\mathbb{R}^{|\mathbb{E}|+2}$. In other words, the vectors in $\mathbb{R}^{|\mathbb{E}|+2}$ specify all *numerical realizations* of (A.15).

The following Definition describes the structural controllability and observability of structured linear systems (A.15).

Definition A.5.1. *Let a graph* $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ *and an associated structured linear system as in* (A.15) *be given. Let* p *be a vector in* $\mathbb{R}^{|\mathbb{E}|+2}$ *that specifies a numerical realization*

[4]A structure matrix $[A]$ is consistent with a graph $\mathcal{G}$ if the (i, j)-th entry of $[A]$ is indeterminate if $(j, i) \in \mathbb{E}$, and the entry is zero otherwise.

of (A.15). *The pair* $([A], [b_i])$ *is said to be structurally controllable if for almost all* p *in* $\mathbb{R}^{|\mathbb{E}|+2}$*, the resulting numerical realizations of* $([A], [b_i])$ *are controllable. The structural observability is similarly defined for the pair* $\left([A], [c_j]^T\right)$.

We can characterize the structural controllability and observability for the system (A.15) in terms of its associated graph as in the following Proposition.

Proposition A.5.2. *Let a strongly connected graph* $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ *and an associated structured linear system as in* (A.15) *be given. Then, for each* i, j *in* $\mathbb{V}$*, the pair* $([A], [b_i])$ *is structurally controllable and the pair* $\left([A], [c_j]^T\right)$ *is structurally observable.*

Proof. The proof directly follows from relevant results from the structured linear system literature (see, for instance, Theorem 1 in [110]). The detail is omitted for brevity. $\square$

A.5.2 A Key Lemma

The following Lemma is used in the proof of Theorem 2.4.5.

Lemma A.5.3. *Given a strongly connected graph* $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ *with* $\mathbb{V} = \{1, \cdots, |\mathbb{V}|\}$*, for any fixed vertex* r *in* $\mathbb{V}$*, the following are true:*

(i) *There exists a WLM* L_1 *of* $\mathcal{G}$ *for which the pair* $\left(L_1, e_r^T\right)$ *is observable.*

(ii) *There exists a WLM* L_2 *of* $\mathcal{G}$ *for which the pair* (L_2, e_r) *is controllable.*

(iii) *There exists a WLM* L_3 *of* $\mathcal{G}$ *for which all eigenvalues of* L_3 *are simple.*

where e_r *is the* r*-th column of the* $|\mathbb{V}|$*-dimensional identity matrix.*

Proof. We provide a two-part proof: In the first part, we prove **(i)** and **(ii)** using Proposition A.5.2; and then we provide a constructive proof of **(iii)**.

Proof of (i) and (ii): Consider a structured linear system that is associated with $\mathcal{G}$ as in (A.15). By Proposition A.5.2, we can find numerical realizations $\left(A_1, c_r^T\right)$ and (A_2, b_r) that are, respectively, observable and controllable. In particular, we may choose A_1 and A_2 to be irreducible and (element-wise) nonnegative.

We compute L_1 from A_1 by applying a special similarity transform used in [112]. This procedure is described as follows: By the Perron-Frobenius Theorem, we can find a right eigenvector $\tilde{v}$ (of A_1) with all positive entries, which corresponds to the Perron-Frobenius eigenvalue $\tilde{\lambda}$. Let M be a diagonal matrix whose diagonal elements are the entries of $\tilde{v}$. Then, by applying a similarity transform to $\left(A_1, c_r^T\right)$ with M, we obtain $\left(M^{-1}A_1M, c_r^TM\right)$. Since the observability is preserved under any similarity transform, the new pair $\left(M^{-1}A_1M, c_r^TM\right)$ is also observable. Note that $M^{-1}A_1M$ and A_1 have the same sparsity pattern, and so do c_r^T and c_r^TM.

Let us define

$$L_1 = I_{|\mathbb{V}|} - \frac{1}{\tilde{\lambda}}M^{-1}A_1M \tag{A.16}$$

Notice that L_1 is a WLM of $\mathcal{G}$, and that eigenvectors of L_1 are same as those of $M^{-1}A_1M$. Since $\left(M^{-1}A_1M, c_r^TM\right)$ is observable, by the PBH rank test, we can see that $\left(L_1, e_r^T\right)$ is observable.

By a similar argument, we can explicitly find a WLM L_2 of $\mathcal{G}$ for which (L_2, e_r) is a controllable pair. This completes the first part of the proof.

Proof of (iii): For a WLM L of $\mathcal{G}$, we represent L as follows:

$$L = \begin{pmatrix} l_1 & \cdots & l_{|\mathbb{V}|} \end{pmatrix}^T \tag{A.17}$$

where l_i^T is the i-th row of L. By re-scaling each row of L, we construct a WLM L_3 of $\mathcal{G}$ whose eigenvalues are all simple.

First of all, it is not difficult to show that for a positive real number α_1, the following matrix has all simple eigenvalues except at the origin.

$$\begin{pmatrix} \alpha_1 l_1 & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix}^T \in \mathbb{R}^{|\mathbb{V}|\times|\mathbb{V}|}, \tag{A.18}$$

where $\mathbf{0}$ is the $|\mathbb{V}|$-dimensional zero vector. Suppose that for some positive real numbers $\alpha_1, \cdots, \alpha_k$, the following matrix has all simple eigenvalues except at the origin.

$$\begin{pmatrix} \alpha_1 l_1 & \cdots & \alpha_k l_k & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix}^T \in \mathbb{R}^{|\mathbb{V}|\times|\mathbb{V}|} \tag{A.19}$$

Recall that eigenvalues of a matrix depend continuously on entries of the matrix. Since (A.19) has all simple eigenvalues except at the origin, for a sufficiently small positive real number α_{k+1}, the following matrix has all simple eigenvalues except at the origin.

$$\begin{pmatrix} \alpha_1 l_1 & \cdots & \alpha_k l_k & \alpha_{k+1} l_{k+1} & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix}^T \in \mathbb{R}^{|\mathbb{V}|\times|\mathbb{V}|} \tag{A.20}$$

By induction, we obtain

$$L_3 = \begin{pmatrix} \alpha_1 l_1 & \cdots & \alpha_{|\mathbb{V}|} l_{|\mathbb{V}|} \end{pmatrix}^T \tag{A.21}$$

that is a WLM of $\mathcal{G}$ and has all simple eigenvalues except at the origin for the selected positive real numbers $\alpha_1, \cdots, \alpha_{|\mathbb{V}|}$. Since $\mathcal{G}$ is a strongly connected graph, the eigenvalue of L_3 at the origin is also simple [113]. This completes the second part of the proof. $\square$

A.5.3 Proof of Theorem 2.4.5

To begin with, for the graph $\mathcal{G} = (\mathbb{V}, \mathbb{E})$, we define sets

$$\mathbb{L}^c_{1,r}(\mathcal{G}) \overset{def}{=} \left\{ L \in \mathbb{L}(\mathcal{G}) \,\middle|\, \left(L, e_r^T\right) \text{ is not observable} \right\}$$

$$\mathbb{L}^c_{2,r}(\mathcal{G}) \overset{def}{=} \left\{ L \in \mathbb{L}(\mathcal{G}) \,\middle|\, (L, e_r) \text{ is not controllable} \right\}$$

$$\mathbb{L}^c_{1}(\mathcal{G}) \overset{def}{=} \left\{ L \in \mathbb{L}(\mathcal{G}) \,\middle|\, \text{A right eigenvector of } L \text{ has a zero entry} \right\}$$

$$\mathbb{L}^c_{2}(\mathcal{G}) \overset{def}{=} \left\{ L \in \mathbb{L}(\mathcal{G}) \,\middle|\, \text{A left eigenvector of } L \text{ has a zero entry} \right\}$$

$$\mathbb{L}^c_{3}(\mathcal{G}) \overset{def}{=} \left\{ L \in \mathbb{L}(\mathcal{G}) \,\middle|\, \text{An eigenvalue of } L \text{ is not simple} \right\}$$

and a natural bijective mapping

$$\pi : \mathbb{L}(\mathcal{G}) \to \mathbb{R}^{|\mathbb{E}|-|\mathbb{V}|}_{<0},$$

where e_r is the r-th column of the $|\mathbb{V}|$-dimensional identity matrix, and $\mathbb{R}^{|\mathbb{E}|-|\mathbb{V}|}_{<0}$ is the set of the $(|\mathbb{E}| - |\mathbb{V}|)$-dimensional vectors whose entries are all negative. To prove Theorem 2.4.5, it is sufficient to show that the sets $\pi\left(\mathbb{L}^c_1(\mathcal{G})\right)$, $\pi\left(\mathbb{L}^c_2(\mathcal{G})\right)$, $\pi\left(\mathbb{L}^c_3(\mathcal{G})\right)$ have the Lebesgue measure zero in $\mathbb{R}^{|\mathbb{E}|-|\mathbb{V}|}_{<0}$.

In [110], the observability is shown to be a *generic property* of structured linear systems. In words, unless every numerical realization of a given structured linear system is not observable, *almost* every numerical realization is observable. By an application of this principle, we can show that unless $\mathbb{L}^c_{1,r}(\mathcal{G}) = \mathbb{L}(\mathcal{G})$ holds, $\pi\left(\mathbb{L}^c_{1,r}(\mathcal{G})\right)$ has the Lebesgue measure zero in $\mathbb{R}^{|\mathbb{E}|-|\mathbb{V}|}_{<0}$. By a similar argument for the controllability of structured linear systems, we conclude that unless $\mathbb{L}^c_{2,r}(\mathcal{G}) = \mathbb{L}(\mathcal{G})$ holds, $\pi\left(\mathbb{L}^c_{2,r}(\mathcal{G})\right)$ has the Lebesgue measure zero in $\mathbb{R}^{|\mathbb{E}|-|\mathbb{V}|}_{<0}$.

Since $\mathcal{G}$ is a strongly connected graph, by Lemma A.5.3, we can show that for any r in $\mathbb{V}$, $\mathbb{L}_{1,r}^c(\mathcal{G})$ and $\mathbb{L}_{2,r}^c(\mathcal{G})$ are proper subsets of $\mathbb{L}(\mathcal{G})$; hence, $\pi\left(\mathbb{L}_{1,r}^c(\mathcal{G})\right)$ and $\pi\left(\mathbb{L}_{2,r}^c(\mathcal{G})\right)$ have the Lebesgue measure zero in $\mathbb{R}_{<0}^{|\mathbb{E}|-|\mathbb{V}|}$. Since it holds that $\mathbb{L}_1^c(\mathcal{G}) = \bigcup_{r\in\mathbb{V}} \mathbb{L}_{1,r}^c(\mathcal{G})$ and $\mathbb{L}_2^c(\mathcal{G}) = \bigcup_{r\in\mathbb{V}} \mathbb{L}_{2,r}^c(\mathcal{G})$, we conclude that $\pi\left(\mathbb{L}_1^c(\mathcal{G})\right)$ and $\pi\left(\mathbb{L}_2^c(\mathcal{G})\right)$ have the Lebesgue measure zero in $\mathbb{R}_{<0}^{|\mathbb{E}|-|\mathbb{V}|}$.

Next, to prove that $\pi\left(\mathbb{L}_3^c(\mathcal{G})\right)$ has the Lebesgue measure zero, we adopt the following argument from algebra (see, for instance, Chapter 14.6 of [114]). For a matrix L in $\mathbb{R}^{m\times m}$, all solutions to a polynomial equation

$$\Delta(\lambda) \overset{def}{=} det(L - \lambda I) = a_m\lambda^m + \cdots + a_1\lambda + a_0 = 0 \tag{A.22}$$

are distinct if the discriminant

$$D(\Delta) \overset{def}{=} a_m^{2m-1} \prod_{1\le i<j\le m} (\lambda_i - \lambda_j)^2$$

is nonzero, where λ_i and λ_j are solutions to (A.22). This particular discriminant can be written as the resultant $R\left(\Delta, \frac{\mathrm{d}}{\mathrm{d}\lambda}\Delta\right)$ of $\Delta(\lambda)$ and $\frac{\mathrm{d}}{\mathrm{d}\lambda}\Delta(\lambda)$ as follows:

$$D(\Delta) = (-1)^{\frac{m(m-1)}{2}} \cdot R\left(\Delta, \frac{\mathrm{d}}{\mathrm{d}\lambda}\Delta\right) \tag{A.23}$$

The resultant $R(f,g)$ of two polynomials f and g is a polynomial function of coefficients of f and g. Hence, by (A.23), we can see that the discriminant $D(\Delta)$ can be written as a polynomial function of coefficients of $\Delta(\lambda)$ and $\frac{\mathrm{d}}{\mathrm{d}\lambda}\Delta(\lambda)$. Since coefficients of $\Delta(\lambda)$ and $\frac{\mathrm{d}}{\mathrm{d}\lambda}\Delta(\lambda)$ are polynomial functions of entries of L, the discriminant $D(\Delta)$ is a polynomial function of entries of L.

Also, notice that for a polynomial function $\widetilde{D}$ defined on $\mathbb{R}^{\widetilde{m}}$, solutions

$$\left\{q \in \mathbb{R}^{\widetilde{m}} \,\middle|\, \widetilde{D}\left(q\right) = 0\right\}$$

to the polynomial equation $\widetilde{D}\,(q) = 0$ form either the entire space $\mathbb{R}^{\widetilde{m}}$ or a proper algebraic variety in $\mathbb{R}^{\widetilde{m}}$, which has the Lebesgue measure zero [115].

For a strongly connected graph $\mathcal{G}$, we have seen from Lemma A.5.3 that there exists $L_3 \in \mathbb{L}(\mathcal{G})$ whose eigenvalues are all simple. Therefore, $\mathbb{L}_3^c(\mathcal{G})$ is a proper subset of $\mathbb{L}(\mathcal{G})$, and by the aforementioned principles, $\pi\left(\mathbb{L}_3^c(\mathcal{G})\right)$ has the Lebesgue measure zero in $\mathbb{R}_{<0}^{|\mathbb{E}|-|\mathbb{V}|}$. $\square$

Appendix B: Auxiliary Results for Chapter 3

B.1 On Product Metric Space

Given a metric space $(\mathbb{X}, d)$, we define a metric $\overline{d}$ on the product space $\mathbb{X}^k$ as follows: For $x_{1:k} = (x_1, \ \cdots \ , \ x_k)$ and $y_{1:k} = (y_1, \ \cdots \ , \ y_k)$ in $\mathbb{X}^k$,

$$\overline{d}(x_{1:k}, y_{1:k}) = \left[d^2\left(x_1, y_1\right) + \cdots + d^2\left(x_k, y_k\right)\right]^{1/2} \tag{B.1}$$

Note that $\left(\mathbb{X}^k, \overline{d}\right)$ is a (product) metric space.

For $(\mathbb{X}, d)$ and $\left(\mathbb{X}^k, \overline{d}\right)$, the following are true:

(F1) For each i in $\{1, \cdots, k\}$, let $\mathbb{K}_i$ be a compact subset of $\mathbb{X}$. Then $\mathbb{K}_1 \times \cdots \times \mathbb{K}_k$ is a compact subset of $\mathbb{X}^k$.

(F2) For a sequence $\left\{x_{1:k}^{(i)}\right\}_{i \in \mathbb{N}}$ in $\mathbb{X}^k$, $\left\{x_{1:k}^{(i)}\right\}_{i \in \mathbb{N}}$ converges to $x_{1:k}$ in $\mathbb{X}^k$, i.e.,

$$\lim_{i \to \infty} \overline{d}\left(x_{1:k}^{(i)}, x_{1:k}\right) = 0$$

if and only if $\left\{x_j^{(i)}\right\}_{i \in \mathbb{N}}$ converges to x_j for all j in $\{1, \cdots, k\}$, i.e.,

$$\lim_{i \to \infty} d\left(x_j^{(i)}, x_j\right) = 0$$

for all j in $\{1, \cdots, k\}$.

Consider a function $\mathcal{G} : \mathbb{X}^k \to \mathbb{R}$. By **(F2)**, $\mathcal{G}$ is continuous at $x_{1:k} \in \mathbb{X}^k$ if for any sequence $\left\{x_{1:k}^{(i)}\right\}_{i\in\mathbb{N}}$ for which $\lim_{i\to\infty} d\left(x_j^{(i)}, x_j\right) = 0$ holds for all j in $\{1, \cdots, k\}$, it holds that $\lim_{i\to\infty} \left|\mathcal{G}\left(x_{1:k}^{(i)}\right) - \mathcal{G}\left(x_{1:k}\right)\right| = 0$.

B.2 On Randomized Policies

In our problem formulation described in Section 3.1, the randomized transmission policy $\boldsymbol{\mathcal{T}}_k : \mathbb{N} \times \mathbb{X} \times \mathbb{X} \to \{0, 1\}$ dictates the random variable $\mathbf{R}_k$ as in (3.1). Then given respective realizations τ_k, x_{τ_k}, and x_k of the last transmission time $\boldsymbol{\tau}_k$, state $\mathbf{x}_{\boldsymbol{\tau}_k}$ of the underlying process at time $\boldsymbol{\tau}_k$, and state $\mathbf{x}_k$ of the process at time k, it holds that

$$\mathbf{R}_k = \begin{cases} 0 & \text{with probability } \mathbb{P}\left(\boldsymbol{\mathcal{T}}_k\left(\boldsymbol{\tau}_k, \boldsymbol{x}_{\boldsymbol{\tau}_k}, \mathbf{x}_k\right) = 0 \,\middle|\, \boldsymbol{\tau}_k = \tau_k, \boldsymbol{x}_{\boldsymbol{\tau}_k} = x_{\tau_k}, \mathbf{x}_k = x_k\right) \\ 1 & \text{with probability } \mathbb{P}\left(\boldsymbol{\mathcal{T}}_k\left(\boldsymbol{\tau}_k, \boldsymbol{x}_{\boldsymbol{\tau}_k}, \mathbf{x}_k\right) = 1 \,\middle|\, \boldsymbol{\tau}_k = \tau_k, \boldsymbol{x}_{\boldsymbol{\tau}_k} = x_{\tau_k}, \mathbf{x}_k = x_k\right) \end{cases}$$

In Section 3.2, the randomized policy $\boldsymbol{\mathcal{P}}_j : \mathbb{X} \to \{0, 1\}$ dictates the random variable $\mathbf{R}_j$ as in (3.13). Then given a realization x_j of the state $\mathbf{x}_j$ of the underlying process at time j, it holds that

$$\mathbf{R}_j = \begin{cases} 0 & \text{with probability } \mathbb{P}\left(\boldsymbol{\mathcal{P}}_j\left(\mathbf{x}_j\right) = 0 \,\middle|\, \mathbf{x}_j = x_j\right) \\ 1 & \text{with probability } \mathbb{P}\left(\boldsymbol{\mathcal{P}}_j\left(\mathbf{x}_j\right) = 1 \,\middle|\, \mathbf{x}_j = x_j\right) \end{cases}$$

Throughout the work, we restrict our attention to the policies in which

$$\mathbb{P}\left(\boldsymbol{\mathcal{P}}_j\left(\mathbf{x}_j\right) = 0 \,\middle|\, \mathbf{x}_j = x_j\right)$$

is a measurable function of x_j on the measurable space $(\mathbb{X}, \mathfrak{B})$. As a case in point, consider a (deterministic) policy defined by

$$\boldsymbol{\mathcal{P}}_j(\mathbf{x}_j) = \begin{cases} 0 & \text{if } \mathbf{x}_j \in \mathbb{D}_j \\ \\ 1 & \text{otherwise} \end{cases}$$

where $\mathbb{D}_j \in \mathfrak{B}$. It can be verified that

$$\mathbb{P}\left(\boldsymbol{\mathcal{P}}_j\left(\mathbf{x}_j\right) = 0 \,\middle|\, \mathbf{x}_j = x_j\right) = \begin{cases} 0 & \text{if } x_j \in \mathbb{D}_j \\ \\ 1 & \text{otherwise} \end{cases}$$

is a measurable function of x_j.

B.3 Preliminary Concepts and Results

We first review some of key definitions and results from probability theory [93,116]. Let $(\mathbb{X}, d)$ be the metric space defined in Section 3.1, and let $\mathcal{T}$ and $\mathfrak{B}$ be a topology and a Borel σ-algebra derived from the metric, respectively. Recall that $(\mathbb{X}, d)$ is assumed to be complete, separable, and proper (see Assumption 3.1.4).

Definition B.3.1. *Let μ be a probability measure on $(\mathbb{X}, \mathfrak{B})$. The probability measure is said to be tight if for each $\epsilon > 0$, there exists a compact subset $\mathbb{K}$ of $(\mathbb{X}, \mathcal{T})$ for which $\mu\left(\mathbb{K}\right) > 1 - \epsilon$ holds.*

The following is adopted from Theorem 7.1.4 in [93].

Lemma B.3.2. *Any probability measure μ on $(\mathbb{X}, \mathfrak{B})$ is tight.*

Definition B.3.3. *A probability measure μ defined on $(\mathbb{X}, \mathfrak{B})$ is said to be closed regular if for every $\mathbb{A}$ in $\mathfrak{B}$, it holds that*

$$\mu(\mathbb{A}) = \sup\left\{\mu(\mathbb{F}) \,\middle|\, \mathbb{F} \in \mathfrak{B} \textit{ closed}, \mathbb{F} \subset \mathbb{A}\right\} \tag{B.2}$$

From Theorem 7.1.3 in [93], we can state the following Lemma.

Lemma B.3.4. *Any probability measure μ on $(\mathbb{X}, \mathfrak{B})$ is closed regular.*

Remark B.3.5. *For a probability measure μ,*

$$\mu(\mathbb{A}) = 1 - \mu(\mathbb{X} \setminus \mathbb{A}) \tag{B.3}$$

where $\mathbb{A}$ in $\mathfrak{B}$. If μ is closed regular then for every $\delta > 0$, there exists a closed set $\mathbb{F}$ for which $\mathbb{F} \subset \mathbb{X} \setminus \mathbb{A}$ and $\mu(\mathbb{X} \setminus \mathbb{A}) < \mu(\mathbb{F}) + \delta$. Let us define an open set $\mathbb{O} = \mathbb{X} \setminus \mathbb{F}$. We can see that $\mathbb{O}$ satisfies $\mathbb{O} \supset \mathbb{A}$ and $\mu(\mathbb{O}) < \mu(\mathbb{A}) + \delta$. Hence we conclude that

$$\mu(\mathbb{A}) = \inf\left\{\mu(\mathbb{O}) \;\middle|\; \mathbb{O} \in \mathfrak{B} \text{ open}, \mathbb{O} \supset \mathbb{A}\right\} \tag{B.4}$$

Definition B.3.6 (Convergence of Probability Measures)**.** *Let $\left\{\mu^{(i)}\right\}_{i\in\mathbb{N}}$ and μ be a sequence of probability measures and a probability measure defined on $(\mathbb{X}, \mathfrak{B})$, respectively, and $\mathcal{C}_b(\mathbb{X})$ be the set of all bounded, continuous, real-valued functions on $\mathbb{X}$. The sequence is said to weakly converge to μ if it holds that*

$$\lim_{i\to\infty} \int g \,\mathrm{d}\mu^{(i)} = \int g \,\mathrm{d}\mu$$

for every g in $\mathcal{C}_b(\mathbb{X})$. We denote the weak convergence as $\mu^{(i)} \xrightarrow{w} \mu$.

Definition B.3.7. *Let $\left\{\mu^{(i)}\right\}_{i\in\mathbb{N}}$ be a sequence of probability measures defined on $(\mathbb{X}, \mathfrak{B})$. The probability measures are said to be uniformly tight if for each $\epsilon > 0$, there exists a compact subset $\mathbb{K}$ of $(\mathbb{X}, \mathcal{T})$ for which $\mu^{(i)}(\mathbb{K}) > 1 - \epsilon$ holds for all i in $\mathbb{N}$.*

A subset $\mathbb{A}$ of $\mathbb{X}$ is said to be a *μ-continuity set* if its boundary set has the zero measure with respect to μ, i.e., $\mu(\mathrm{bd}(\mathbb{A})) = 0$. The following is the *portmanteau theorem* (See Theorem 11.1.1 in [93]).

Theorem B.3.8. *For a sequence* $\left\{\mu^{(i)}\right\}_{i\in\mathbb{N}}$ *of probability measures and a probability measure* μ *on* $(\mathbb{X}, \mathfrak{B})$*, the following are equivalent:*

1. $\mu^{(i)} \xrightarrow{w} \mu$
2. $\limsup_{i\to\infty} \mu^{(i)}(\mathbb{F}) \leq \mu(\mathbb{F})$ *for any closed subset* $\mathbb{F}$ *of* $\mathbb{X}$
3. $\liminf_{i\to\infty} \mu^{(i)}(\mathbb{O}) \geq \mu(\mathbb{O})$ *for any open subset* $\mathbb{O}$ *of* $\mathbb{X}$
4. $\lim_{i\to\infty} \mu^{(i)}(\mathbb{A}) = \mu(\mathbb{A})$ *for all* μ*-continuity subset* $\mathbb{A}$ *of* $\mathbb{X}$*.*

Based on Remark 3.2.11, we can state the following Lemma.

Lemma B.3.9. *Given estimates* $\hat{x}_{k:N}$*, let* $\boldsymbol{\mathcal{P}}_{k:N}$ *be non-degenerate policies satisfying* $\boldsymbol{\mathcal{P}}_{k:N} \in \mathfrak{P}(\hat{x}_{k:N})$*. Consider compact sets* $\{\mathbb{K}_j\}_{j=k}^{N}$ *given by*[1]

$$\mathbb{K}_j = \left\{ x \in \mathbb{X} \,\middle|\, d^2(x, \hat{x}_j) \leq c_j' \right\} \tag{B.5}$$

Then under the policies $\boldsymbol{\mathcal{P}}_{k:N}$ *it holds that*

$$\mathbb{P}\left(\mathbf{x}_j \in \mathbb{K}_j \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 0 \right) = 1 \tag{B.6}$$

for all j *in* $\{k, \cdots, N\}$*.*

The proof follows from the fact that for each j in $\{k, \cdots, N\}$, $\mathbb{K}_j$ contains the set $\overline{\mathbb{D}}_j$ defined in (3.27a) and Remark 3.2.11-1.

Remark B.3.10. *Given policies* $\boldsymbol{\mathcal{P}}_{k:N}$*, for each* j *in* $\{k, \cdots, N\}$*, let us define*

$$\mu_{j|j}(\mathbb{A}) = \mathbb{P}\left(\mathbf{x}_j \in \mathbb{A} \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 0 \right) \tag{B.7a}$$

$$\mu_{j|j-1}(\mathbb{A}) = \mathbb{P}\left(\mathbf{x}_j \in \mathbb{A} \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_{j-1} = 0 \right) \tag{B.7b}$$

[1] Due to the properness assumption on the metric space $(\mathbb{X}, d)$, every closed ball is a compact set.

where $\mathbb{A}$ belongs to $\mathfrak{B}$. Then, the probability measure of the process $\{\mathbf{x}_j\}_{j=k}^{N}$ evolves as follows:

1. *Policy update:*

$$\mu_{j|j}(\mathbb{A}) = \frac{\int_{\mathbb{A}} \mathbb{P}\left(\mathbf{R}_j = 0 \,\middle|\, \mathbf{x}_j = x, \mathbf{R}_k = 0, \cdots, \mathbf{R}_{j-1} = 0\right) \mathrm{d}\mu_{j|j-1}}{\mathbb{P}\left(\mathbf{R}_j = 0 \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_{j-1} = 0\right)}$$

provided

$$\begin{aligned}&\mathbb{P}\left(\mathbf{R}_j = 0 \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_{j-1} = 0\right) \\ &= \int_{\mathbb{X}} \mathbb{P}\left(\mathbf{R}_j = 0 \,\middle|\, \mathbf{x}_j = x, \mathbf{R}_k = 0, \cdots, \mathbf{R}_{j-1} = 0\right) \mathrm{d}\mu_{j|j-1} > 0\end{aligned}$$

where $\mathbb{P}\left(\mathbf{R}_j = 0 \,\middle|\, \mathbf{x}_j = x, \mathbf{R}_k = 0, \cdots, \mathbf{R}_{j-1} = 0\right)$ is the probability specified by the given policy $\boldsymbol{\mathcal{P}}_j$, i.e.,

$$\mathbb{P}\left(\mathbf{R}_j = 0 \,\middle|\, \mathbf{x}_j = x, \mathbf{R}_k = 0, \cdots, \mathbf{R}_{j-1} = 0\right) = \mathbb{P}\left(\boldsymbol{\mathcal{P}}_j(\mathbf{x}_j) = 0 \,\middle|\, \mathbf{x}_j = x\right)$$

2. *Process update:*

$$\mu_{j|j-1}(\mathbb{A}) = \int_{\mathbb{X}} p_j(x, \mathbb{A}) \, \mathrm{d}\mu_{j-1|j-1}$$

where p_j is the transition probability of the process $\{\mathbf{x}_j\}_{j=k}^{N}$.

B.4 Proof of Proposition 3.2.14

To start with, we note that for each j in $\{k, \cdots, N\}$, $\mathcal{G}_j$ can be written as follows:

$$\mathcal{G}_j(x_{j-1}, \hat{x}_{j:N}) = \mathbb{E}_{\mathbf{x}_j}\left[\min\left\{d^2(\mathbf{x}_j, \hat{x}_j) + \mathcal{G}_{j+1}(\mathbf{x}_j, \hat{x}_{j+1:N}), c_j'\right\} \,\middle|\, \mathbf{x}_{j-1} = x_{j-1}\right]$$

with $\mathcal{G}_{N+1} = 0$.

We prove the Proposition by induction. First, notice that since $\mathcal{G}_{N+1} = 0$, it is a continuous function. Suppose that $\mathcal{G}_{j+1}$ is a continuous function. To show that $\mathcal{G}_j$ is continuous, we rewrite $\mathcal{G}_j$ as follows:

$$\mathcal{G}_j\left(x_{j-1}, \hat{x}_{j:N}\right) = \mathbb{E}_{\mathbf{x}_j}\left[g\left(\mathbf{x}_j, \hat{x}_{j:N}\right) \,\middle|\, \mathbf{x}_{j-1} = x_{j-1}\right] \tag{B.8}$$

where $g(x, \hat{x}_{j:N}) = \min\left\{d^2\left(x, \hat{x}_j\right) + \mathcal{G}_{j+1}\left(x, \hat{x}_{j+1:N}\right), c'_j\right\}$. Note that g is a continuous function.

To verify the continuity of $\mathcal{G}_j$, let $\{x_{j-1}^{(i)}\}_{i\in\mathbb{N}}$ and $\{\hat{x}_{j:N}^{(i)}\}_{i\in\mathbb{N}}$ be sequences that converge to x_{j-1} and $\hat{x}_{j:N}$, respectively. For each set $\mathbb{A}$ in $\mathfrak{B}$, let us define

$$\mu_j^{(i)}\left(\mathbb{A}\right) = \mathbb{P}\left(\mathbf{x}_j \in \mathbb{A} \,\middle|\, \mathbf{x}_{j-1} = x_{j-1}^{(i)}\right) \tag{B.9}$$

$$\mu_j\left(\mathbb{A}\right) = \mathbb{P}\left(\mathbf{x}_j \in \mathbb{A} \,\middle|\, \mathbf{x}_{j-1} = x_{j-1}\right) \tag{B.10}$$

By Assumption 3.1.5-2 and Theorem B.3.8, $\left\{\mu_j^{(i)}\right\}_{i\in\mathbb{N}}$ weakly converges to μ_j. Since $(\mathbb{X}, d)$ is a complete, separable metric space (see Assumption 3.1.4), by the Skorokhod representation theorem [94], there exist a sequence of random variables $\left\{\mathbf{y}_j^{(i)}\right\}_{i\in\mathbb{N}}$ and a random variable $\mathbf{y}_j$ all defined on a common probability space $(\Omega, \mathfrak{F}, \nu)$ in which the following three facts are true:

(F1) $\mu_j^{(i)}$ is the probability measure of $\mathbf{y}_j^{(i)}$, i.e., $\nu\left(\left\{\omega \in \Omega \,\middle|\, \mathbf{y}_j^{(i)}(\omega) \in \mathbb{A}\right\}\right) = \mu_j^{(i)}\left(\mathbb{A}\right)$ for each $\mathbb{A}$ in $\mathfrak{B}$.

(F2) μ_j is the probability measure of $\mathbf{y}_j$, i.e., $\nu\left(\left\{\omega \in \Omega \,\middle|\, \mathbf{y}_j(\omega) \in \mathbb{A}\right\}\right) = \mu_j\left(\mathbb{A}\right)$ for each $\mathbb{A}$ in $\mathfrak{B}$.

(F3) $\left\{\mathbf{y}_j^{(i)}\right\}_{i\in\mathbb{N}}$ converges to $\mathbf{y}_j$ almost surely.

From **(F1)** and **(F2)**, we can derive

$$
\begin{aligned}
&\mathcal{G}_j\left(x_{j-1}^{(i)}, \hat{x}_{j:N}^{(i)}\right)-\mathcal{G}_j\left(x_{j-1}, \hat{x}_{j:N}\right) \\
&=\mathbb{E}_{\mathbf{x}_j}\left[g\left(\mathbf{x}_j, \hat{x}_{j:N}^{(i)}\right) \,\middle|\, \mathbf{x}_{j-1}=x_{j-1}^{(i)}\right]-\mathbb{E}_{\mathbf{x}_j}\left[g\left(\mathbf{x}_j, \hat{x}_{j:N}\right) \,\middle|\, \mathbf{x}_{j-1}=x_{j-1}\right] \\
&=\int_{\Omega} g\left(\mathbf{y}_j^{(i)}(\omega), \hat{x}_{j:N}^{(i)}\right) \mathrm{d}\nu-\int_{\Omega} g\left(\mathbf{y}_j(\omega), \hat{x}_{j:N}\right) \mathrm{d}\nu
\end{aligned} \tag{B.11}
$$

Notice that by the fact that c_j' is a fixed constant, the sequence $\left\{g\left(\mathbf{y}_j^{(i)}(\cdot), \hat{x}_{j:N}^{(i)}\right)\right\}_{i\in\mathbb{N}}$ is uniformly bounded. i.e., for every $\omega \in \Omega$, it holds that $g\left(\mathbf{y}_j^{(i)}(\omega), \hat{x}_{j:N}^{(i)}\right) \leq c_j'$ for all i in $\mathbb{N}$.[2] Also, by the continuity of g and **(F3)**, it holds that

$$
\lim_{i\to\infty} g\left(\mathbf{y}_j^{(i)}(\omega), \hat{x}_{j:N}^{(i)}\right)=g\left(\mathbf{y}_j(\omega), \hat{x}_{j:N}\right)
$$

for almost every ω in Ω. Using the bounded convergence theorem (see Theorem 16.5 in [116]), we have that

$$
\begin{aligned}
&\lim_{i\to\infty}\left|\mathcal{G}_j\left(x_{j-1}^{(i)}, \hat{x}_{j:N}^{(i)}\right)-\mathcal{G}_j\left(x_{j-1}, \hat{x}_{j:N}\right)\right| \\
&=\lim_{i\to\infty}\left|\int_{\Omega} g\left(\mathbf{y}_j^{(i)}(\omega), \hat{x}_{j:N}^{(i)}\right) \mathrm{d}\nu-\int_{\Omega} g\left(\mathbf{y}_j(\omega), \hat{x}_{j:N}\right) \mathrm{d}\nu\right|=0
\end{aligned}
$$

Finally, by induction, we conclude that the functions $\{\mathcal{G}_j\}_{j=k}^{N}$ are all continuous. □

B.5 Proofs of Proposition 3.2.16 and Lemma 3.2.18

Lemma B.5.1. *For each j_0 in $\{k, \cdots, N\}$, there exist estimates $\hat{x}_{j_0:N}$ for which the set given by*

$$
\underline{\mathbb{D}}_{j_0}=\left\{x_{j_0} \in \mathbb{X} \,\middle|\, d^2\left(x_{j_0}, \hat{x}_{j_0}\right)+\mathbb{E}_{\mathbf{x}_{j_0+1}}\left[J_{j_0+1}^*\left(\mathbf{x}_{j_0+1}, \hat{x}_{j_0+1:N}\right) \,\middle|\, \mathbf{x}_{j_0}=x_{j_0}\right]<c_{j_0}'\right\} \tag{B.12}
$$

is non-empty, where $J_{j_0+1}^*$ is defined in (3.25).

[2] Note that g is a non-negative function.

Proof. Recall how $\{c'_j\}_{j=k}^N$ are determined by (3.8) with the solutions $\{\boldsymbol{\mathcal{T}}_{j:N}^{<j-1>}\}_{j=k+1}^N$ and $\{\mathcal{E}_{j:N}^{<j-1>}\}_{j=k+1}^N$ to the preceding sub-problems – Sub-problem $k+1$ to Sub-problem N. Let us fix x_{j_0} in $\mathbb{X}$. Under the choice of $\hat{x}_{j_0} = x_{j_0}$ and $\hat{x}_l = \mathcal{E}_l^{<j_0>}(x_{j_0})$ for each l in $\{j_0+1, \cdots, N\}$, by a similar argument as in Remark 3.2.2, we can see that

$$\mathbb{E}_{\mathbf{x}_{j_0+1}}\left[J^*_{j_0+1}\left(\mathbf{x}_{j_0+1}, \hat{x}_{j+1:N}\right) \,\middle|\, \mathbf{x}_{j_0} = x_{j_0}\right] = \mathbb{E}_{\mathbf{x}_{j_0+1}}\left[J^*_{j_0+1}\left(\mathbf{x}_{j_0+1}, \hat{x}'_{j_0+1:N}\right) \,\middle|\, \mathbf{x}_{j_0} = 0\right]$$

where $\hat{x}'_l = \mathcal{E}_l^{<j_0>}(0)$ for each l in $\{j_0+1, \cdots, N\}$. Hence, at $x_{j_0} = \hat{x}_{j_0}$, it holds that

$$\begin{aligned}
&d^2\left(x_{j_0}, \hat{x}_{j_0}\right) + \mathbb{E}_{\mathbf{x}_{j_0+1}}\left[J^*_{j_0+1}\left(\mathbf{x}_{j_0+1}, \hat{x}_{j_0+1:N}\right) \,\middle|\, \mathbf{x}_{j_0} = x_{j_0}\right] \\
&= \mathbb{E}_{\mathbf{x}_{j_0+1}}\left[J^*_{j_0+1}\left(\mathbf{x}_{j_0+1}, \hat{x}_{j_0+1:N}\right) \,\middle|\, \mathbf{x}_{j_0} = x_{j_0}\right] \\
&< c_{j_0} + \mathbb{E}_{\mathbf{x}_{j_0+1}}\left[J^*_{j_0+1}\left(\mathbf{x}_{j_0+1}, \hat{x}_{j_0+1:N}\right) \,\middle|\, \mathbf{x}_{j_0} = x_{j_0}\right] \\
&= c_{j_0} + \mathbb{E}_{\mathbf{x}_{j_0+1}}\left[J^*_{j_0+1}\left(\mathbf{x}_{j_0+1}, \hat{x}'_{j_0+1:N}\right) \,\middle|\, \mathbf{x}_{j_0} = 0\right] = c'_{j_0}
\end{aligned}$$

This proves the Lemma. □

Proof of Proposition 3.2.16: By contradiction, suppose that $\hat{x}^*_{k:N}$ is a global minimizer of (3.31) in which the policies $\boldsymbol{\mathcal{P}}^*_{k:N}$ satisfying $\boldsymbol{\mathcal{P}}^*_{k:N} \in \mathfrak{P}\left(\hat{x}^*_{k:N}\right)$ are degenerate. Let $j_0 \in \{k, \cdots, N\}$ be the smallest integer for which

$$\mathbb{P}\left(\mathbf{R}^*_{j_0} = 0 \,\middle|\, \mathbf{R}^*_k = 0, \cdots, \mathbf{R}^*_{j_0-1} = 0\right) = 0 \tag{B.13}$$

holds. Since j_0 is the smallest such integer, by Remark B.3.10, the probability measure $\mu_{j_0|j_0-1}$ of $\mathbf{x}_{j_0}$ is well-defined.

Using Lemma B.5.1, let us choose $\hat{x}^o_{j_0:N} \in \mathbb{X}^{N-j_0+1}$ for which the set given by

$$\underline{\mathbb{D}}_{j_0} = \left\{x_{j_0} \in \mathbb{X} \,\middle|\, d^2\left(x_{j_0}, \hat{x}^o_{j_0}\right) + \mathbb{E}_{\mathbf{x}_{j_0+1}}\left[J^*_{j_0+1}\left(\mathbf{x}_{j_0+1}, \hat{x}^o_{j_0+1:N}\right) \,\middle|\, \mathbf{x}_{j_0} = x_{j_0}\right] < c'_{j_0}\right\} \tag{B.14}$$

is non-empty. Note that according to Corollary 3.2.15, the set $\underline{\mathbb{D}}_{j_0}$ is open; hence, from Assumption 3.1.5-1 and Remark B.3.10-2, we have that

$$\mathbb{P}\left(\mathbf{x}_{j_0} \in \underline{\mathbb{D}}_{j_0} \,\middle|\, \mathbf{R}_k^* = 0, \cdots, \mathbf{R}_{j_0-1}^* = 0\right) > 0 \tag{B.15}$$

Consider functions $\mathcal{P}_{j_0:N}^o$ defined as

$$\mathcal{P}_j^o(x_j) = \begin{cases} 0 & \text{if } x_j \in \underline{\mathbb{D}}_j \\ 1 & \text{otherwise} \end{cases}$$

where

$$\underline{\mathbb{D}}_j = \left\{ x_j \in \mathbb{X} \,\middle|\, d^2\left(x_j, \hat{x}_j^o\right) + \mathbb{E}_{\mathbf{x}_{j+1}}\left[J_{j+1}^*\left(\mathbf{x}_{j+1}, \hat{x}_{j+1:N}^o\right) \,\middle|\, \mathbf{x}_j = x_j \right] < c_j' \right\}$$

Let us select new policies $\boldsymbol{\mathcal{P}}_{k:N}'$ and estimates $\hat{x}_{k:N}'$ as follows:

$$\boldsymbol{\mathcal{P}}_j' = \begin{cases} \boldsymbol{\mathcal{P}}_j^* & \text{for } j \in \{k, \cdots, j_0 - 1\} \\ \mathcal{P}_j^o & \text{for } j \in \{j_0, \cdots, N\} \end{cases}$$

$$\hat{x}_j' = \begin{cases} \hat{x}_j^* & \text{for } j \in \{k, \cdots, j_0 - 1\} \\ \hat{x}_j^o & \text{for } j \in \{j_0, \cdots, N\} \end{cases}$$

By definition, under the new policies $\boldsymbol{\mathcal{P}}_{k:N}'$, it holds that

$$\mathbb{P}\left(\mathbf{x}_{j_0} \in \underline{\mathbb{D}}_{j_0} \,\middle|\, \mathbf{R}_k' = 0, \cdots, \mathbf{R}_{j_0}' = 1\right) = 0 \tag{B.16}$$

which implies that

$$\begin{aligned} &\mathbb{P}\left(\mathbf{x}_{j_0} \in \underline{\mathbb{D}}_{j_0} \,\middle|\, \mathbf{R}_k' = 0, \cdots, \mathbf{R}_{j_0-1}' = 0\right) \\ &= \mathbb{P}\left(\mathbf{x}_{j_0} \in \underline{\mathbb{D}}_{j_0} \,\middle|\, \mathbf{R}_k' = 0, \cdots, \mathbf{R}_{j_0}' = 0\right) \cdot \mathbb{P}\left(\mathbf{R}_{j_0}' = 0 \,\middle|\, \mathbf{R}_k' = 0, \cdots, \mathbf{R}_{j_0-1}' = 0\right) \end{aligned} \tag{B.17}$$

By the fact that $\boldsymbol{\mathcal{P}}'_j = \boldsymbol{\mathcal{P}}^*_j$ for j in $\{k, \cdots, j_0 - 1\}$, from (B.15) and (B.17), we can see that

$$\mathbb{P}\left(\mathbf{R}'_{j_0} = 0 \,\middle|\, \mathbf{R}'_k = 0, \cdots, \mathbf{R}'_{j_0-1} = 0\right) > 0 \tag{B.18}$$

Due to (B.13), from Remark 3.2.8, we can see that

$$\mathbb{E}_{\mathbf{x}_{j_0}}\left[J_{j_0}\left(\mathbf{x}_{j_0}, \boldsymbol{\mathcal{P}}^*_{j_0:N}, \hat{x}^*_{j_0:N}\right) \,\middle|\, \mathbf{R}^*_k = 0, \cdots, \mathbf{R}^*_{j_0-1} = 0\right] = c'_{j_0}$$

While, by the way new policies and estimates are defined for j in $\{j_0, \cdots, N\}$, using (3.14) and (B.18), we can see that

$$\begin{aligned}
&\mathbb{E}_{\mathbf{x}_{j_0}}\left[J_{j_0}\left(\mathbf{x}_{j_0}, \boldsymbol{\mathcal{P}}'_{j_0:N}, \hat{x}'_{j_0:N}\right) \,\middle|\, \mathbf{R}'_k = 0, \cdots, \mathbf{R}'_{j_0-1} = 0\right] \\
&= \mathbb{E}_{\mathbf{x}_{j_0}}\left[J^*_{j_0}\left(\mathbf{x}_{j_0}, \hat{x}'_{j_0:N}\right) \,\middle|\, \mathbf{R}'_k = 0, \cdots, \mathbf{R}'_{j_0-1} = 0\right] < c'_{j_0}
\end{aligned}$$

These relations imply that

$$\begin{aligned}
&\mathbb{E}_{\mathbf{x}_{j_0}}\left[J_{j_0}\left(\mathbf{x}_{j_0}, \boldsymbol{\mathcal{P}}'_{j_0:N}, \hat{x}'_{j_0:N}\right) \,\middle|\, \mathbf{R}'_k = 0, \cdots, \mathbf{R}'_{j_0-1} = 0\right] \\
&< \mathbb{E}_{\mathbf{x}_{j_0}}\left[J_{j_0}\left(\mathbf{x}_{j_0}, \boldsymbol{\mathcal{P}}^*_{j_0:N}, \hat{x}^*_{j_0:N}\right) \,\middle|\, \mathbf{R}^*_k = 0, \cdots, \mathbf{R}^*_{j_0-1} = 0\right]
\end{aligned}$$

Using the facts that $\boldsymbol{\mathcal{P}}'_j = \boldsymbol{\mathcal{P}}^*_j$ and $\hat{x}'_j = \hat{x}^*_j$ for j in $\{k, \cdots, j_0 - 1\}$, and j_0 is the smallest integer for which (B.13) holds, from (3.14), we can infer that

$$\mathcal{G}\left(\hat{x}'_{k:N}\right) \leq \mathbb{E}_{\mathbf{x}_k}\left[J_k\left(\mathbf{x}_k, \boldsymbol{\mathcal{P}}'_{k:N}, \hat{x}'_{k:N}\right)\right] < \mathbb{E}_{\mathbf{x}_k}\left[J_k\left(\mathbf{x}_k, \boldsymbol{\mathcal{P}}^*_{k:N}, \hat{x}^*_{k:N}\right)\right] = \mathcal{G}\left(\hat{x}^*_{k:N}\right)$$

which violates the optimality of the global minimizer. Therefore, the global minimizer has to be non-degenerate. $\square$

Lemma B.5.2. *Consider policies* $\left\{\boldsymbol{\mathcal{P}}^{(i)}_{k:N}\right\}_{i\in\mathbb{N}}$ *and estimates* $\left\{\hat{x}^{(i)}_{k:N}\right\}_{i\in\mathbb{N}}$ *that satisfy*

$$\boldsymbol{\mathcal{P}}^{(i)}_{k:N} \in \mathfrak{P}\left(\hat{x}^{(i-1)}_{k:N}\right)$$

Suppose that the policies are strictly non-degenerate, i.e., there exists $\epsilon > 0$ for which

$$\mathbb{P}\left(\mathbf{R}_j^{(i)} = 0 \,\middle|\, \mathbf{R}_k^{(i)} = 0, \cdots, \mathbf{R}_{j-1}^{(i)} = 0\right) \geq \epsilon \tag{B.19}$$

holds for all i in $\mathbb{N}$ and j in $\{k, \cdots, j_0\}$. Then the sequence $\left\{\hat{x}_j^{(i)}\right\}_{i \in \mathbb{N}}$ is bounded for all j in $\{k, \cdots, j_0\}$.

Proof. By contradiction, suppose that there exists j' in $\{k, \cdots, j_0\}$ such that for a subsequence $\left\{\hat{x}_{j'}^{(i_l-1)}\right\}_{l \in \mathbb{N}}$ of $\left\{\hat{x}_{j'}^{(i)}\right\}_{i \in \mathbb{N}}$, it holds that

$$d\left(0, \hat{x}_{j'}^{(i_l-1)}\right) \overset{l \to \infty}{\longrightarrow} \infty \tag{B.20}$$

For each l in $\mathbb{N}$, let us choose a compact set $\mathbb{K}_{j'}^{(i_l)} = \left\{x \in \mathbb{X} \,\middle|\, d^2\left(x, \hat{x}_{j'}^{(i_l-1)}\right) \leq c'_j\right\}$. Then, according to Lemma B.3.9, under the policies $\left\{\boldsymbol{\mathcal{P}}_{k:N}^{(i_l)}\right\}_{l \in \mathbb{N}}$, it holds that

$$\mathbb{P}\left(\mathbf{x}_{j'} \in \mathbb{K}_{j'}^{(i_l)} \,\middle|\, \mathbf{R}_k^{(i_l)} = 0, \cdots, \mathbf{R}_{j'}^{(i_l)} = 0\right) = 1 \tag{B.21}$$

for all l in $\mathbb{N}$. Using (B.19), we can derive the following:

$$\begin{aligned}
&\mathbb{P}\left(\mathbf{R}_{j'}^{(i_l)} = 0 \,\middle|\, \mathbf{R}_k^{(i_l)} = 0, \cdots, \mathbf{R}_{j'-1}^{(i_l)} = 0\right) \\
&= \frac{\mathbb{P}\left(\mathbf{R}_{j'}^{(i_l)} = 0 \,\middle|\, \mathbf{x}_{j'} \in \mathbb{K}_{j'}^{(i_l)}, \mathbf{R}_k^{(i_l)} = 0, \cdots, \mathbf{R}_{j'-1}^{(i_l)} = 0\right)}{\mathbb{P}\left(\mathbf{x}_{j'} \in \mathbb{K}_{j'}^{(i_l)} \,\middle|\, \mathbf{R}_k^{(i_l)} = 0, \cdots, \mathbf{R}_{j'}^{(i_l)} = 0\right)} \\
&\qquad \cdot \mathbb{P}\left(\mathbf{x}_{j'} \in \mathbb{K}_{j'}^{(i_l)} \,\middle|\, \mathbf{R}_k^{(i_l)} = 0, \cdots, \mathbf{R}_{j'-1}^{(i_l)} = 0\right) \\
&\leq \mathbb{P}\left(\mathbf{x}_{j'} \in \mathbb{K}_{j'}^{(i_l)} \,\middle|\, \mathbf{R}_k^{(i_l)} = 0, \cdots, \mathbf{R}_{j'-1}^{(i_l)} = 0\right) \\
&= \frac{\mathbb{P}\left(\mathbf{x}_{j'} \in \mathbb{K}_{j'}^{(i_l)}, \mathbf{R}_k^{(i_l)} = 0, \cdots, \mathbf{R}_{j'-1}^{(i_l)} = 0\right)}{\prod_{j=k}^{j'-1} \mathbb{P}\left(\mathbf{R}_j^{(i_l)} = 0 \,\middle|\, \mathbf{R}_k^{(i_l)} = 0, \cdots, \mathbf{R}_{j-1}^{(i_l)} = 0\right)} \\
&\leq \epsilon^{k-j'} \cdot \mathbb{P}\left(\mathbf{x}_{j'} \in \mathbb{K}_{j'}^{(i_l)}\right)
\end{aligned} \tag{B.22}$$

holds for all l in $\mathbb{N}$. Hence, by Lemma B.3.2 and (B.20), we can see that

$$\lim_{l \to \infty} \mathbb{P}\left(\mathbf{x}_{j'} \in \mathbb{K}_{j'}^{(i_l)}\right) = 0 \tag{B.23}$$

In conjunction with (B.22), we conclude that

$$\lim_{l\to\infty} \mathbb{P}\left(\mathbf{R}_{j'}^{(i_l)} = 0 \,\middle|\, \mathbf{R}_{k}^{(i_l)} = 0, \cdots, \mathbf{R}_{j'-1}^{(i_l)} = 0\right) = 0 \tag{B.24}$$

This contradicts the fact that (B.19) holds for all j in $\{k, \cdots, j_0\}$. $\square$

Proof of Lemma 3.2.18: For a positive real r, let us define

$$\mathbb{K}_r \stackrel{def}{=} \left\{\hat{x}_{k:N} \in \mathbb{X}^{N-k+1} \,\middle|\, d\left(0, \hat{x}_j\right) \leq r \text{ for all } j \text{ in } \{k, \cdots, N\}\right\} \tag{B.25}$$

To prove the Lemma, it is sufficient to show that there exists $r > 0$ for which with $\mathbb{K} = \mathbb{K}_r$, the statement of the Lemma is true. By contradiction, suppose that there exists a sequence $\left\{\hat{x}_{k:N}^{(i)}\right\}_{i\in\mathbb{N}} \subset \mathbb{X}^{N-k+1}$ that satisfies the following hypotheses:

(H1) For each element $\hat{x}_{k:N}^{(i)}$ of the sequence, it holds that $\hat{x}_{k:N}^{(i)} \notin \mathbb{K}_i$.

(H2) For every $\hat{x}_{k:N}$ in $\mathbb{K}_i$, it holds that $\mathcal{G}\left(\hat{x}_{k:N}\right) > \mathcal{G}\left(\hat{x}_{k:N}^{(i)}\right)$.

We constructively show that the hypothesis **(H2)** is violated for sufficiently large i in $\mathbb{N}$. To proceed, let us select policies $\left\{\boldsymbol{\mathcal{P}}_{k:N}^{(i)}\right\}_{i\in\mathbb{N}}$ that satisfy $\boldsymbol{\mathcal{P}}_{k:N}^{(i)} \in \mathfrak{P}\left(\hat{x}_{k:N}^{(i-1)}\right)$. Let $j_0 \in \{k, \cdots, N\}$ be the smallest integer such that there is a subsequence $\left\{\boldsymbol{\mathcal{P}}_{k:j_0}^{(i_l)}\right\}_{l\in\mathbb{N}}$ of $\left\{\boldsymbol{\mathcal{P}}_{k:j_0}^{(i)}\right\}_{i\in\mathbb{N}}$ satisfying[3]

$$\lim_{l\to\infty} \mathbb{P}\left(\mathbf{R}_{j_0}^{(i_l)} = 0 \,\middle|\, \mathbf{R}_{k}^{(i_l)} = 0, \cdots, \mathbf{R}_{j_0-1}^{(i_l)} = 0\right) = 0 \tag{B.26}$$

Note that according to (3.14), (B.26) implies that

$$\lim_{l\to\infty} \mathbb{E}_{\mathbf{x}_{j_0}}\left[J_{j_0}\left(\mathbf{x}_{j_0}, \boldsymbol{\mathcal{P}}_{j_0:N}^{(i_l)}, \hat{x}_{j_0:N}^{(i_l-1)}\right) \,\middle|\, \mathbf{R}_{k}^{(i_l)} = 0, \cdots, \mathbf{R}_{j_0-1}^{(i_l)} = 0\right] = c'_{j_0} \tag{B.27}$$

[3]Such j_0 always exists; otherwise according to Lemma B.5.2, the sequence $\left\{\hat{x}_{k:N}^{(i)}\right\}_{i\in\mathbb{N}}$ is bounded, which violates **(H1)**.

Also, according to Lemma B.5.2, the sequence $\left\{\hat{x}_j^{(i)}\right\}_{i\in\mathbb{N}}$ is bounded for all j in $\{k, \cdots, j_0-1\}$.

Using Lemma B.5.1, let us choose $\hat{x}^o_{j_0:N} \in \mathbb{X}^{N-j_0+1}$ for which the set given by

$$\underline{\mathbb{D}}_{j_0} = \left\{x_{j_0} \in \mathbb{X} \,\middle|\, d^2\left(x_{j_0}, \hat{x}^o_{j_0}\right) + \mathbb{E}_{\mathbf{x}_{j_0+1}}\left[J^*_{j_0+1}\left(\mathbf{x}_{j_0+1}, \hat{x}^o_{j_0+1:N}\right) \,\middle|\, \mathbf{x}_{j_0} = x_{j_0}\right] < c'_{j_0}\right\} \tag{B.28}$$

is non-empty, where $J^*_{j_0+1}$ is defined in (3.25). Note that by Proposition 3.2.14

$$d^2\left(x_{j_0}, \hat{x}^o_{j_0}\right) + \mathbb{E}_{\mathbf{x}_{j_0+1}}\left[J^*_{j_0+1}\left(\mathbf{x}_{j_0+1}, \hat{x}^o_{j_0+1:N}\right) \,\middle|\, \mathbf{x}_{j_0} = x_{j_0}\right]$$

is a continuous function of x_{j_0}. Hence, for some $\epsilon > 0$, the set defined by

$$\mathbb{B} = \left\{x_{j_0} \in \mathbb{X} \,\middle|\, d^2\left(x_{j_0}, \hat{x}^o_{j_0}\right) + \mathbb{E}_{\mathbf{x}_{j_0+1}}\left[J^*_{j_0+1}\left(\mathbf{x}_{j_0+1}, \hat{x}^o_{j_0+1:N}\right) \,\middle|\, \mathbf{x}_{j_0} = x_{j_0}\right] < c'_{j_0} - \epsilon\right\} \tag{B.29}$$

is non-empty and open.

Consider functions $\mathcal{P}^o_{j_0:N}$ defined as

$$\mathcal{P}^o_j(x_j) = \begin{cases} 0 & \text{if } x_j \in \underline{\mathbb{D}}_j \\ 1 & \text{otherwise} \end{cases}$$

where

$$\underline{\mathbb{D}}_j = \left\{x_j \in \mathbb{X} \,\middle|\, d^2\left(x_j, \hat{x}^o_j\right) + \mathbb{E}_{\mathbf{x}_{j+1}}\left[J^*_{j+1}\left(\mathbf{x}_{j+1}, \hat{x}^o_{j+1:N}\right) \,\middle|\, \mathbf{x}_j = x_j\right] < c'_j\right\}$$

Let us select sequences of policies $\left\{\boldsymbol{\mathcal{P}}'^{(i)}_{k:N}\right\}_{i\in\mathbb{N}}$ and estimates $\left\{\hat{x}'^{(i)}_{k:N}\right\}_{i\in\mathbb{N}}$ as follows:

$$\boldsymbol{\mathcal{P}}'^{(i)}_j = \begin{cases} \boldsymbol{\mathcal{P}}^{(i)}_j & \text{for } j \in \{k, \cdots, j_0-1\} \\ \mathcal{P}^o_j & \text{for } j \in \{j_0, \cdots, N\} \end{cases}$$

$$\hat{x}'^{(i)}_j = \begin{cases} \hat{x}^{(i)}_j & \text{for } j \in \{k, \cdots, j_0-1\} \\ \hat{x}^o_j & \text{for } j \in \{j_0, \cdots, N\} \end{cases}$$

We argue that for sufficiently large i, it holds that $\hat{x}'^{(i-1)}_{k:N} \in \mathbb{K}_{i-1}$ and $\mathcal{G}\left(\hat{x}'^{(i-1)}_{k:N}\right) < \mathcal{G}\left(\hat{x}^{(i-1)}_{k:N}\right)$. This contradicts the hypothesis **(H2)**, and completes the proof of the Lemma. In what follows, we show that this argument is valid.

Since the sequence $\left\{\hat{x}^{(i-1)}_{j_0-1}\right\}_{i\in\mathbb{N}}$ is bounded, by Lemma B.3.9, and by the fact that $\boldsymbol{\mathcal{P}}'^{(i)}_j = \boldsymbol{\mathcal{P}}^{(i)}_j$ for i in $\mathbb{N}$ and j in $\{k,\cdots,j_0-1\}$, there exists a compact set $\mathbb{K}_{j_0-1}$ for which

$$\begin{aligned}&\mathbb{P}\left(\mathbf{x}_{j_0-1} \in \mathbb{K}_{j_0-1} \,\middle|\, \mathbf{R}'^{(i)}_k = 0,\cdots,\mathbf{R}'^{(i)}_{j_0-1} = 0\right)\\ &= \mathbb{P}\left(\mathbf{x}_{j_0-1} \in \mathbb{K}_{j_0-1} \,\middle|\, \mathbf{R}^{(i)}_k = 0,\cdots,\mathbf{R}^{(i)}_{j_0-1} = 0\right) = 1\end{aligned}$$

holds for all i in $\mathbb{N}$. Hence, due to Assumption 3.1.5 and the compactness of $\mathbb{K}_{j_0-1}$, for some $\delta_{j_0} > 0$, it holds that

$$\begin{aligned}&\mathbb{P}\left(\mathbf{x}_{j_0} \in \mathbb{B} \,\middle|\, \mathbf{R}'^{(i)}_k = 0,\cdots,\mathbf{R}'^{(i)}_{j_0-1} = 0\right)\\ &= \int_{\mathbb{K}_{j_0-1}} p_{j_0}(x,\mathbb{B})\, \mathrm{d}\mu'^{(i)}_{j_0-1|j_0-1} \geq \delta_{j_0}\cdot \mu'^{(i)}_{j_0-1|j_0-1}(\mathbb{K}_{j_0-1}) = \delta_{j_0}\end{aligned} \tag{B.30}$$

for all i in $\mathbb{N}$, where the set $\mathbb{B}$ is given in (B.29) and the probability measure $\mu'^{(i)}_{j_0-1|j_0-1}$ is defined as

$$\mu'^{(i)}_{j_0-1|j_0-1}(\mathbb{A}) = \mathbb{P}\left(\mathbf{x}_{j_0-1} \in \mathbb{A} \,\middle|\, \mathbf{R}'^{(i)}_k = 0,\cdots,\mathbf{R}'^{(i)}_{j_0-1} = 0\right)$$

for each $\mathbb{A}$ in $\mathfrak{B}$.

Since the sequence $\left\{\hat{x}^{(i-1)}_j\right\}_{i\in\mathbb{N}}$ is bounded for all j in $\{k,\cdots,j_0-1\}$, for sufficiently large i, we can see that $\hat{x}'^{(i-1)}_{k:N} \in \mathbb{K}_{i-1}$. In addition, by the way the policies $\left\{\boldsymbol{\mathcal{P}}'^{(i)}_{k:N}\right\}_{i\in\mathbb{N}}$ and estimates $\left\{\hat{x}'^{(i)}_{k:N}\right\}_{i\in\mathbb{N}}$ are defined, using (3.14), (B.29) and (B.30), we can derive the following relations:

$$\mathbb{E}_{\mathbf{x}_{j_0}}\left[J_{j_0}\left(\mathbf{x}_{j_0}, \boldsymbol{\mathcal{P}}'^{(i)}_{j_0:N}, \hat{x}'^{(i-1)}_{j_0:N}\right) \,\middle|\, \mathbf{R}'^{(i)}_k = 0,\cdots,\mathbf{R}'^{(i)}_{j_0-1} = 0\right]$$

$$
\begin{aligned}
&= \Bigg(\mathbb{E}_{\mathbf{x}_{j_0}}\left[d^2\left(\mathbf{x}_{j_0},\hat{x}^o_{j_0}\right)\,\Big|\,\mathbf{R}'^{(i)}_k=0,\cdots,\mathbf{R}'^{(i)}_{j_0}=0\right] \\
&\qquad + \mathbb{E}_{\mathbf{x}_{j_0+1}}\left[J^*_{j_0+1}\left(\mathbf{x}_{j_0+1},\hat{x}^o_{j_0+1:N}\right)\,\Big|\,\mathbf{R}'^{(i)}_k=0,\cdots,\mathbf{R}'^{(i)}_{j_0}=0\right]\Bigg) \\
&\quad \cdot \mathbb{P}\left(\mathbf{R}'^{(i)}_{j_0}=0\,\Big|\,\mathbf{R}'^{(i)}_k=0,\cdots,\mathbf{R}'^{(i)}_{j_0-1}=0\right) \\
&\quad + c'_{j_0}\cdot\left(1-\mathbb{P}\left(\mathbf{R}'^{(i)}_{j_0}=0\,\Big|\,\mathbf{R}'^{(i)}_k=0,\cdots,\mathbf{R}'^{(i)}_{j_0-1}=0\right)\right) \\
&\overset{(1)}{\le} \Bigg(\mathbb{E}_{\mathbf{x}_{j_0}}\left[d^2\left(\mathbf{x}_{j_0},\hat{x}^o_{j_0}\right)\,\Big|\,\mathbf{R}'^{(i)}_k=0,\cdots,\mathbf{R}'^{(i)}_{j_0-1}=0,\mathbf{x}_{j_0}\in\mathbb{B}\right] \\
&\qquad + \mathbb{E}_{\mathbf{x}_{j_0+1}}\left[J^*_{j_0+1}\left(\mathbf{x}_{j_0+1},\hat{x}^o_{j_0+1:N}\right)\,\Big|\,\mathbf{R}'^{(i)}_k=0,\cdots,\mathbf{R}'^{(i)}_{j_0-1}=0,\mathbf{x}_{j_0}\in\mathbb{B}\right]\Bigg) \\
&\quad \cdot \mathbb{P}\left(\mathbf{x}_{j_0}\in\mathbb{B}\,\Big|\,\mathbf{R}'^{(i)}_k=0,\cdots,\mathbf{R}'^{(i)}_{j_0-1}=0\right) \\
&\quad + c'_{j_0}\cdot\left(1-\mathbb{P}\left(\mathbf{x}_{j_0}\in\mathbb{B}\,\Big|\,\mathbf{R}'^{(i)}_k=0,\cdots,\mathbf{R}'^{(i)}_{j_0-1}=0\right)\right) \\
&\overset{(2)}{<} \left(c'_{j_0}-\epsilon\right)\cdot\mathbb{P}\left(\mathbf{x}_{j_0}\in\mathbb{B}\,\Big|\,\mathbf{R}'^{(i)}_k=0,\cdots,\mathbf{R}'^{(i)}_{j_0-1}=0\right) \\
&\quad + c'_{j_0}\cdot\left(1-\mathbb{P}\left(\mathbf{x}_{j_0}\in\mathbb{B}\,\Big|\,\mathbf{R}'^{(i)}_k=0,\cdots,\mathbf{R}'^{(i)}_{j_0-1}=0\right)\right) \\
&\overset{(3)}{\le} c'_{j_0}-\epsilon\cdot\delta_{j_0}
\end{aligned}
\tag{B.31}
$$

holds for all i in $\mathbb{N}$. To obtain (1), we use the fact that

$$d^2\left(\mathbf{x}_{j_0},\hat{x}^o_{j_0}\right)+\mathbb{E}_{\mathbf{x}_{j_0+1}}\left[J^*_{j_0+1}\left(\mathbf{x}_{j_0+1},\hat{x}^o_{j_0+1:N}\right)\,\Big|\,\mathbf{x}_{j_0}\right]<c'_{j_0}$$

if $\mathbf{R}'^{(i)}_{j_0}=0$ (or equivalently $\mathbf{x}_{j_0}\in\underline{\mathbb{D}}_{j_0}$), and $\mathbb{B}$ is a subset of $\underline{\mathbb{D}}_{j_0}$; whereas (2) and (3) follow from (B.29) and (B.30), respectively. By a similar argument as in the proof of Proposition 3.2.16, from (B.27) and (B.31), we can observe that for sufficiently large i, there exists $\hat{x}^{(i-1)}_{k:N}$ in $\mathbb{K}_{i-1}$ for which it holds that

$$\mathcal{G}\left(\hat{x}'^{(i-1)}_{k:N}\right)\le\mathbb{E}_{\mathbf{x}_k}\left[J_k\left(\mathbf{x}_k,\boldsymbol{\mathcal{P}}'^{(i)}_{k:N},\hat{x}'^{(i-1)}_{k:N}\right)\right]<\mathbb{E}_{\mathbf{x}_k}\left[J_k\left(\mathbf{x}_k,\boldsymbol{\mathcal{P}}^{(i)}_{k:N},\hat{x}^{(i-1)}_{k:N}\right)\right]=\mathcal{G}\left(\hat{x}^{(i-1)}_{k:N}\right)$$

$\square$

B.6 Proof of Lemma 3.2.24

Lemma B.6.1. *Let* $\left\{\hat{x}_{k:N}^{(i)}\right\}_{i\in\mathbb{N}}$ *be a sequence of estimates that converges to* $\hat{x}_{k:N}$*. The following hold for all* j *in* $\{k,\cdots,N\}$*:*

$$\overline{\mathbb{D}}_j \supset \bigcap_{i\in\mathbb{N}}\bigcup_{l\geq i}\overline{\mathbb{D}}_j^{(l)} \tag{B.32a}$$

$$\underline{\mathbb{D}}_j \subset \bigcup_{i\in\mathbb{N}}\bigcap_{l\geq i}\underline{\mathbb{D}}_j^{(l)} \tag{B.32b}$$

where

$$\overline{\mathbb{D}}_j = \left\{x_j \in \mathbb{X} \,\middle|\, d^2\left(x_j, \hat{x}_j\right) + \mathbb{E}_{\mathbf{x}_{j+1}}\left[J_{j+1}^*\left(\mathbf{x}_{j+1}, \hat{x}_{j+1:N}\right) \,\middle|\, \mathbf{x}_j = x_j\right] \leq c_j'\right\}$$
$$\underline{\mathbb{D}}_j = \left\{x_j \in \mathbb{X} \,\middle|\, d^2\left(x_j, \hat{x}_j\right) + \mathbb{E}_{\mathbf{x}_{j+1}}\left[J_{j+1}^*\left(\mathbf{x}_{j+1}, \hat{x}_{j+1:N}\right) \,\middle|\, \mathbf{x}_j = x_j\right] < c_j'\right\}$$

and

$$\overline{\mathbb{D}}_j^{(i)} = \left\{x_j \in \mathbb{X} \,\middle|\, d^2\left(x_j, \hat{x}_j^{(i)}\right) + \mathbb{E}_{\mathbf{x}_{j+1}}\left[J_{j+1}^*\left(\mathbf{x}_{j+1}, \hat{x}_{j+1:N}^{(i)}\right) \,\middle|\, \mathbf{x}_j = x_j\right] \leq c_j'\right\}$$
$$\underline{\mathbb{D}}_j^{(i)} = \left\{x_j \in \mathbb{X} \,\middle|\, d^2\left(x_j, \hat{x}_j^{(i)}\right) + \mathbb{E}_{\mathbf{x}_{j+1}}\left[J_{j+1}^*\left(\mathbf{x}_{j+1}, \hat{x}_{j+1:N}^{(i)}\right) \,\middle|\, \mathbf{x}_j = x_j\right] < c_j'\right\}$$

where J_{j+1}^* *is defined in* (3.25).

Proof. Let x_j be an element of $\bigcap_{i\in\mathbb{N}}\bigcup_{l\geq i}\overline{\mathbb{D}}_j^{(l)}$. By definition, there exists an infinite index set $\{i_l\}_{l\in\mathbb{N}}$ for which $x_j \in \overline{\mathbb{D}}_j^{(i_l)}$ holds for all l in $\mathbb{N}$. Hence, we can see that

$$d^2\left(x_j, \hat{x}_j^{(i_l)}\right) + \mathbb{E}_{\mathbf{x}_{j+1}}\left[J_{j+1}^*\left(\mathbf{x}_{j+1}, \hat{x}_{j+1:N}^{(i_l)}\right) \,\middle|\, \mathbf{x}_j = x_j\right] \leq c_j' \tag{B.33}$$

holds for all l in $\mathbb{N}$. Using Proposition 3.2.14 and by the fact that $\left\{\hat{x}_{k:N}^{(i)}\right\}_{i\in\mathbb{N}}$ converges to $\hat{x}_{k:N}$, we can derive

$$d^2\left(x_j, \hat{x}_j\right) + \mathbb{E}_{\mathbf{x}_{j+1}}\left[J_{j+1}^*\left(\mathbf{x}_{j+1}, \hat{x}_{j+1:N}\right) \,\middle|\, \mathbf{x}_j = x_j\right] \leq c_j' \tag{B.34}$$

which shows that $x_j \in \overline{\mathbb{D}}_j$. This proves (B.32a).

To show that (B.32b) is true, we consider

$$\underline{\mathbb{D}}_j^c \supset \bigcap_{i\in\mathbb{N}} \bigcup_{l\geq i} \left(\underline{\mathbb{D}}_j^{(l)}\right)^c \tag{B.35}$$

As the rest of the proof is similar to the above arguments, we omit the detail for brevity.

□

Proof of Lemma 3.2.24: Under the policies $\left\{\boldsymbol{\mathcal{P}}_{k:N}^{(i)}\right\}_{i\in\mathbb{N}}$, let us define

$$\mu_{j|j-1}^{(i)}(\mathbb{A}) = \mathbb{P}\left(\mathbf{x}_j \in \mathbb{A} \,\middle|\, \mathbf{R}_k^{(i)} = 0, \cdots, \mathbf{R}_{j-1}^{(i)} = 0\right) \tag{B.36a}$$

$$\mu_{j|j}^{(i)}(\mathbb{A}) = \mathbb{P}\left(\mathbf{x}_j \in \mathbb{A} \,\middle|\, \mathbf{R}_k^{(i)} = 0, \cdots, \mathbf{R}_j^{(i)} = 0\right) \tag{B.36b}$$

for each i in $\mathbb{N}$ and j in $\{k, \cdots, N\}$, where $\mathbb{A}$ belongs to $\mathfrak{B}$. Since $\left\{\hat{x}_{k:N}^{(i-1)}\right\}_{i\in\mathbb{N}}$ is a convergent sequence, according to Lemma B.3.9, there exist compact subsets $\{\mathbb{K}_j\}_{j=k}^N$ for which

$$\mu_{j|j}^{(i)}(\mathbb{K}_j) = 1 \tag{B.37}$$

holds for all i in $\mathbb{N}$ and j in $\{k, \cdots, N\}$. Hence, the probability measures $\left\{\mu_{j|j}^{(i)}\right\}_{i\in\mathbb{N}}$ are uniformly tight in the sense of Definition B.3.7 for all j in $\{k, \cdots, N\}$. According to Theorem 11.5.4 in [93], for each j in $\{k, \cdots, N\}$, there exists a subsequence $\left\{\mu_{j|j}^{(i_l)}\right\}_{l\in\mathbb{N}}$ of $\left\{\mu_{j|j}^{(i)}\right\}_{i\in\mathbb{N}}$ that weakly converges to a probability measure $\mu_{j|j}$. In addition, since $\mathbb{P}\left(\mathbf{R}_j^{(i)} = 0 \,\middle|\, \mathbf{R}_k^{(i)} = 0, \cdots, \mathbf{R}_{j-1}^{(i)}\right)$ takes a value in a compact set $[\epsilon, 1]$, for an infinite index set $\{i_l\}_{l\in\mathbb{N}}$, it holds that

$$\lim_{l\to\infty} \mathbb{P}\left(\mathbf{R}_j^{(i_l)} = 0 \,\middle|\, \mathbf{R}_k^{(i_l)} = 0, \cdots, \mathbf{R}_{j-1}^{(i_l)} = 0\right) = q_j$$

where q_j takes a value in $[\epsilon, 1]$. For these reasons, without loss of generality, we prove the Lemma under the following assumptions: For each j in $\{k, \cdots, N\}$,

(F1) There exists a probability measure $\mu_{j|j}$ such that $\mu_{j|j}^{(i)} \xrightarrow{w} \mu_{j|j}$ holds.

(F2) $\lim_{i\to\infty} \mathbb{P}\left(\mathbf{R}_j^{(i)} = 0 \,\middle|\, \mathbf{R}_k^{(i)} = 0, \cdots, \mathbf{R}_{j-1}^{(i)} = 0\right) = q_j$ whose value belongs to $[\epsilon, 1]$.

We proceed by showing that there exist policies $\boldsymbol{\mathcal{P}}_{k:N}$ for which

(A1) For every $\mathbb{A}$ in $\mathfrak{B}$, it holds that

$$\mu_{j|j}\left(\mathbb{A}\right) = \mathbb{P}\left(\mathbf{x}_j \in \mathbb{A} \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 0\right) \tag{B.38a}$$

and

$$q_j = \mathbb{P}\left(\mathbf{R}_j = 0 \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_{j-1} = 0\right) \tag{B.38b}$$

where $\mathbf{R}_j$ is dictated by $\boldsymbol{\mathcal{P}}_j$ for each j in $\{k, \cdots, N\}$.

(A2) $\boldsymbol{\mathcal{P}}_{k:N}$ belongs to $\mathfrak{P}\left(\hat{x}'_{k:N}\right)$, where $\hat{x}'_{k:N}$ is the limit of $\left\{\hat{x}_{k:N}^{(i-1)}\right\}_{i\in\mathbb{N}}$.

For our purpose, we define

$$\overline{\mathbb{D}}_j = \left\{x_j \in \mathbb{X} \,\middle|\, d^2\left(x_j, \hat{x}'_j\right) + \mathbb{E}_{\mathbf{x}_{j+1}}\left[J_{j+1}^*\left(\mathbf{x}_{j+1}, \hat{x}'_{j+1:N}\right) \,\middle|\, \mathbf{x}_j = x_j\right] \leq c'_j\right\} \tag{B.39a}$$

$$\underline{\mathbb{D}}_j = \left\{x_j \in \mathbb{X} \,\middle|\, d^2\left(x_j, \hat{x}'_j\right) + \mathbb{E}_{\mathbf{x}_{j+1}}\left[J_{j+1}^*\left(\mathbf{x}_{j+1}, \hat{x}'_{j+1:N}\right) \,\middle|\, \mathbf{x}_j = x_j\right] < c'_j\right\} \tag{B.39b}$$

and

$$\overline{\mathbb{D}}_j^{(i)} = \left\{x_j \in \mathbb{X} \,\middle|\, d^2\left(x_j, \hat{x}_j^{(i-1)}\right) + \mathbb{E}_{\mathbf{x}_{j+1}}\left[J_{j+1}^*\left(\mathbf{x}_{j+1}, \hat{x}_{j+1:N}^{(i-1)}\right) \,\middle|\, \mathbf{x}_j = x_j\right] \leq c'_j\right\} \tag{B.40a}$$

$$\underline{\mathbb{D}}_j^{(i)} = \left\{x_j \in \mathbb{X} \,\middle|\, d^2\left(x_j, \hat{x}_j^{(i-1)}\right) + \mathbb{E}_{\mathbf{x}_{j+1}}\left[J_{j+1}^*\left(\mathbf{x}_{j+1}, \hat{x}_{j+1:N}^{(i-1)}\right) \,\middle|\, \mathbf{x}_j = x_j\right] < c'_j\right\} \tag{B.40b}$$

for each i in $\mathbb{N}$ and j in $\{k, \cdots, N\}$, where J_{j+1}^* is defined in (3.25). Note that according to Corollary 3.2.15, the sets (B.39a) and (B.40a) are closed, and the sets (B.39b) and (B.40b) are open.

We first make the following two claims to show that **(A1)** is true.

Claim 1: For each $\mathbb{A}$ in $\mathfrak{B}$, let us define

$$\mu_{j|j-1}(\mathbb{A}) \stackrel{def}{=} \int_{\mathbb{X}} p_j(x, \mathbb{A}) \, \mathrm{d}\mu_{j-1|j-1} \tag{B.41}$$

where p_j is the transition probability of the process $\{\mathbf{x}_j\}_{j=k}^N$. Then, $\mu_{j|j-1}$ is a probability measure on $(\mathbb{X}, \mathfrak{B})$, and the following holds:

$$\lim_{i \to \infty} \mu_{j|j-1}^{(i)}(\mathbb{A}) = \mu_{j|j-1}(\mathbb{A})$$

for all $\mathbb{A}$ in $\mathfrak{B}$.

To prove the claim, based on Remark B.3.10-2, we note that

$$\mu_{j|j-1}^{(i)}(\mathbb{A}) = \int_{\mathbb{X}} p_j(x, \mathbb{A}) \, \mathrm{d}\mu_{j-1|j-1}^{(i)} \tag{B.42}$$

holds for each $\mathbb{A}$ in $\mathfrak{B}$, and by definition, for each x in $\mathbb{X}$, $\mathbb{A} \mapsto p_j(x, \mathbb{A})$ is a probability measure on $(\mathbb{X}, \mathfrak{B})$. In conjunction with Assumption 3.1.5-2, we can see that $x \mapsto p_j(x, \mathbb{A})$ is a bounded, continuous function. Hence, using **(F1)**, we have that

$$\begin{aligned} \lim_{i \to \infty} \mu_{j|j-1}^{(i)}(\mathbb{A}) &= \lim_{i \to \infty} \int_{\mathbb{X}} p_j(x, \mathbb{A}) \, \mathrm{d}\mu_{j-1|j-1}^{(i)} \\ &= \int_{\mathbb{X}} p_j(x, \mathbb{A}) \, \mathrm{d}\mu_{j-1|j-1} = \mu_{j|j-1}(\mathbb{A}) \end{aligned} \tag{B.43}$$

Lastly, the claim that $\mu_{j|j-1}$ is a probability measure on $(\mathbb{X}, \mathfrak{B})$ follows from the fact that $\mathbb{A} \mapsto p_j(x, \mathbb{A})$ is a probability measure on $(\mathbb{X}, \mathfrak{B})$. $\square$

Claim 2: There exists a measurable function $f_j : \mathbb{X} \to [0, 1]$ for which

$$\mu_{j|j}(\mathbb{A}) = \frac{\int_{\mathbb{A}} f_j \, \mathrm{d}\mu_{j|j-1}}{q_j} \tag{B.44}$$

holds for all $\mathbb{A}$ in $\mathfrak{B}$, where $\mu_{j|j-1}$ is defined in (B.41).

Based on Lemma B.3.8 and Remark B.3.10-1, for any open set $\mathbb{O}$, we can see that the following relations hold:

$$\begin{aligned}\mu_{j|j}(\mathbb{O}) \leq \liminf_{i\to\infty} \mu^{(i)}_{j|j}(\mathbb{O}) &\overset{(1)}{\leq} \lim_{i\to\infty} \frac{\mu^{(i)}_{j|j-1}(\mathbb{O})}{\mathbb{P}\left(\mathbf{R}^{(i)}_j = 0 \,\middle|\, \mathbf{R}^{(i)}_k = 0, \cdots, \mathbf{R}^{(i)}_{j-1} = 0\right)} \\ &\overset{(2)}{=} \frac{\mu_{j|j-1}(\mathbb{O})}{q_j}\end{aligned} \tag{B.45}$$

where (1) follows from Remark B.3.10-1, and (2) follows from Claim 1 and **(F2)**. We argue that

$$\mu_{j|j}(\mathbb{A}) \leq \frac{\mu_{j|j-1}(\mathbb{A})}{q_j} \tag{B.46}$$

holds for any set $\mathbb{A}$ in $\mathfrak{B}$. To justify the argument, by contradiction, suppose that for a set $\mathbb{A}$ in $\mathfrak{B}$ it holds that

$$\mu_{j|j}(\mathbb{A}) > \frac{\mu_{j|j-1}(\mathbb{A})}{q_j} \tag{B.47}$$

By the closed regularity theorem (see Theorem 7.1.3 in [93]) and Remark B.3.5, we can choose an open set $\mathbb{O}$ containing $\mathbb{A}$ for which

$$\begin{aligned}\mu_{j|j}(\mathbb{O}) \geq \mu_{j|j}(\mathbb{A}) > &\frac{\mu_{j|j-1}(\mathbb{O})}{q_j} \\ \geq &\frac{\mu_{j|j-1}(\mathbb{A})}{q_j}\end{aligned} \tag{B.48}$$

holds. This contradicts (B.45).

Notice that (B.46) implies that $\mu_{j|j}$ is absolutely continuous with respect to $\mu_{j|j-1}$. According to the Radon-Nikodym theorem, there is a measurable function $f_j : \mathbb{X} \to \mathbb{R}_+$ for which

$$\mu_{j|j}(\mathbb{A}) = \frac{\int_{\mathbb{A}} f_j \, \mathrm{d}\mu_{j|j-1}}{q_j} \tag{B.49}$$

holds for all $\mathbb{A}$ in $\mathfrak{B}$. In addition, it can be verified that $f_j(x) \leq 1$ for almost every x in $\mathbb{X}$; otherwise (B.46) would be violated. □

Proof of (A1): Define policies $\boldsymbol{\mathcal{P}}_{k:N}$ as follows:

$$\mathbb{P}\left(\boldsymbol{\mathcal{P}}_j(\mathbf{x}_j) = 0 \,\middle|\, \mathbf{x}_j = x\right) = f_j(x) \tag{B.50}$$

Then we can verify that under the policies $\boldsymbol{\mathcal{P}}_{k:N}$

$$\mathbb{P}\left(\mathbf{R}_j = 0 \,\middle|\, \mathbf{x}_j = x, \mathbf{R}_k = 0, \cdots, \mathbf{R}_{j-1} = 0\right) = f_j(x)$$

and from (B.49), we have that

$$\mu_{j|j}(\mathbb{A}) = \frac{\int_{\mathbb{A}} \mathbb{P}\left(\mathbf{R}_j = 0 \,\middle|\, \mathbf{x}_j = x, \mathbf{R}_k = 0, \cdots, \mathbf{R}_{j-1} = 0\right) \, \mathrm{d}\mu_{j|j-1}}{q_j} \tag{B.51}$$

Since it holds that $\mu_{k|k-1}(\mathbb{A}) = \mathbb{P}(\mathbf{x}_k \in \mathbb{A})$, from (B.51), we can see that

$$q_k = \mathbb{P}(\mathbf{R}_k = 0)$$

Hence, in conjunction with Remark B.3.10-1, we can see that

$$\mu_{k|k}(\mathbb{A}) = \mathbb{P}\left(\mathbf{x}_k \in \mathbb{A} \,\middle|\, \mathbf{R}_k = 0\right)$$

Then, using (B.41) and Remark B.3.10-2, we can observe that

$$\mu_{k+1|k}(\mathbb{A}) = \mathbb{P}\left(\mathbf{x}_{k+1} \in \mathbb{A} \,\middle|\, \mathbf{R}_k = 0\right)$$

By repeating this verification for each j in $\{k, \cdots, N\}$, we conclude that

$$\mu_{j|j}(\mathbb{A}) = \mathbb{P}\left(\mathbf{x}_j \in \mathbb{A} \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 0\right)$$

and

$$q_j = \mathbb{P}\left(\mathbf{R}_j = 0 \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_{j-1} = 0\right)$$

hold for all j in $\{k, \cdots, N\}$. $\square$

Henceforth, we make two additional claims under the policies $\boldsymbol{\mathcal{P}}_{k:N}$ determined as in (B.50) to show that **(A2)** is valid.

Claim 3: For any Borel measurable subset $\mathbb{A}$ contained in $\overline{\mathbb{D}}_j^c$, it holds that

$$\mathbb{P}\left(\mathbf{x}_j \in \mathbb{A} \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 0\right) = 0$$

To prove the claim, let $\mathbb{O}$ be an open set contained in $\overline{\mathbb{D}}_j^c$. By Remark 3.2.11-1 and Remark B.3.10-1, we can derive the following:

$$\mu_{j|j}^{(i)}(\mathbb{O}) = \mu_{j|j}^{(i)}\left(\mathbb{O} \cap \overline{\mathbb{D}}_j^{(i)}\right) \leq \frac{\mu_{j|j-1}^{(i)}\left(\mathbb{O} \cap \overline{\mathbb{D}}_j^{(i)}\right)}{\mathbb{P}\left(\mathbf{R}_j^{(i)} = 0 \,\middle|\, \mathbf{R}_k^{(i)} = 0, \cdots, \mathbf{R}_{j-1}^{(i)} = 0\right)} \tag{B.53}$$

By applying Lemma B.3.8, we obtain

$$\begin{aligned}
\mu_{j|j}(\mathbb{O}) &\leq \liminf_{i\to\infty} \mu_{j|j}^{(i)}(\mathbb{O}) \\
&\leq \liminf_{i\to\infty} \frac{\mu_{j|j-1}^{(i)}\left(\mathbb{O} \cap \overline{\mathbb{D}}_j^{(i)}\right)}{\mathbb{P}\left(\mathbf{R}_j^{(i)} = 0 \,\middle|\, \mathbf{R}_k^{(i)} = 0, \cdots, \mathbf{R}_{j-1}^{(i)} = 0\right)} \\
&\leq \liminf_{i\to\infty} \frac{\mu_{j|j-1}^{(i)}\left(\mathbb{O} \cap \left(\bigcup_{l\geq i} \overline{\mathbb{D}}_j^{(l)}\right)\right)}{\mathbb{P}\left(\mathbf{R}_j^{(i)} = 0 \,\middle|\, \mathbf{R}_k^{(i)} = 0, \cdots, \mathbf{R}_{j-1}^{(i)} = 0\right)} \\
&\overset{(1)}{\leq} \frac{\mu_{j|j-1}\left(\mathbb{O} \cap \left(\bigcup_{l\geq i_0} \overline{\mathbb{D}}_j^{(l)}\right)\right)}{\mathbb{P}\left(\mathbf{R}_j = 0 \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_{j-1} = 0\right)}
\end{aligned} \tag{B.54}$$

holds for all i_0 in $\mathbb{N}$, where (1) follows from Claim 1, **(A1)**, and the fact that $\left\{\bigcup_{l\geq i}\overline{\mathbb{D}}_j^{(l)}\right\}_{i\in\mathbb{N}}$ is a decreasing sequence of measurable sets. Hence, from Lemma B.6.1, we have that

$$\mu_{j|j}\left(\mathbb{O}\right) \leq \frac{\mu_{j|j-1}\left(\mathbb{O}\cap\left(\bigcap_{i\in\mathbb{N}}\bigcup_{l\geq i}\overline{\mathbb{D}}_j^{(l)}\right)\right)}{\mathbb{P}\left(\mathbf{R}_j=0\,\middle|\,\mathbf{R}_k=0,\cdots,\mathbf{R}_{j-1}=0\right)}=0 \tag{B.55}$$

Since $\overline{\mathbb{D}}_j^c$ is an open set, by selecting $\mathbb{O}=\overline{\mathbb{D}}_j^c$, we conclude that

$$\mu_{j|j}\left(\mathbb{A}\right)\leq\mu_{j|j}\left(\overline{\mathbb{D}}_j^c\right)=0 \tag{B.56}$$

holds for every Borel measurable subset $\mathbb{A}$ of $\overline{\mathbb{D}}_j^c$. □

Claim 4: Suppose that $\mathbb{P}\left(\mathbf{R}_j=1\,\middle|\,\mathbf{R}_k=0,\cdots,\mathbf{R}_{j-1}=0\right)>0$. Then, for any Borel measurable subset $\mathbb{A}$ contained in $\underline{\mathbb{D}}_j$, it holds that

$$\mathbb{P}\left(\mathbf{x}_j\in\mathbb{A}\,\middle|\,\mathbf{R}_k=0,\cdots,\mathbf{R}_j=1\right)=0$$

To prove the claim, let $\mathbb{F}$ be a closed set contained in $\underline{\mathbb{D}}_j$. Notice that by Remark 3.2.11-2

$$\begin{aligned}&\mathbb{P}\left(\mathbf{x}_j\in\mathbb{F}\cap\underline{\mathbb{D}}_j^{(i)}\,\middle|\,\mathbf{R}_k^{(i)}=0,\cdots,\mathbf{R}_j^{(i)}=0\right)\\&=\frac{\mathbb{P}\left(\mathbf{x}_j\in\mathbb{F}\cap\underline{\mathbb{D}}_j^{(i)}\,\middle|\,\mathbf{R}_k^{(i)}=0,\cdots,\mathbf{R}_{j-1}^{(i)}=0\right)}{\mathbb{P}\left(\mathbf{R}_j^{(i)}=0\,\middle|\,\mathbf{R}_k^{(i)}=0,\cdots,\mathbf{R}_{j-1}^{(i)}=0\right)}\end{aligned} \tag{B.57}$$

Using (B.57) and Theorem B.3.8, we can derive the following:

$$\begin{aligned}
\mu_{j|j}(\mathbb{F}) &\geq \limsup_{i\to\infty} \mu_{j|j}^{(i)}(\mathbb{F}) \\
&\geq \limsup_{i\to\infty} \mu_{j|j}^{(i)}\left(\mathbb{F}\cap \underline{\mathbb{D}}_j^{(i)}\right) \\
&= \limsup_{i\to\infty} \frac{\mu_{j|j-1}^{(i)}\left(\mathbb{F}\cap \underline{\mathbb{D}}_j^{(i)}\right)}{\mathbb{P}\left(\mathbf{R}_j^{(i)}=0 \,\middle|\, \mathbf{R}_k^{(i)}=0,\cdots,\mathbf{R}_{j-1}^{(i)}=0\right)} \\
&\geq \limsup_{i\to\infty} \frac{\mu_{j|j-1}^{(i)}\left(\mathbb{F}\cap \left(\bigcap_{l\geq i}\underline{\mathbb{D}}_j^{(l)}\right)\right)}{\mathbb{P}\left(\mathbf{R}_j^{(i)}=0 \,\middle|\, \mathbf{R}_k^{(i)}=0,\cdots,\mathbf{R}_{j-1}^{(i)}=0\right)} \\
&\overset{(1)}{\geq} \frac{\mu_{j|j-1}\left(\mathbb{F}\cap \left(\bigcap_{l\geq i_0}\underline{\mathbb{D}}_j^{(l)}\right)\right)}{\mathbb{P}\left(\mathbf{R}_j=0 \,\middle|\, \mathbf{R}_k=0,\cdots,\mathbf{R}_{j-1}=0\right)}
\end{aligned} \tag{B.58}$$

holds for all i_0 in $\mathbb{N}$, where (1) follows from Claim 1, **(A1)**, and the fact that $\left\{\bigcap_{l\geq i}\underline{\mathbb{D}}_j^{(l)}\right\}_{i\in\mathbb{N}}$ is an increasing sequence of measurable sets. Hence, from Lemma B.6.1, we have that

$$\begin{aligned}
\mu_{j|j}(\mathbb{F}) &\geq \frac{\mu_{j|j-1}\left(\mathbb{F}\cap \left(\bigcup_{i\in\mathbb{N}}\bigcap_{l\geq i}\underline{\mathbb{D}}_j^{(l)}\right)\right)}{\mathbb{P}\left(\mathbf{R}_j=0 \,\middle|\, \mathbf{R}_k=0,\cdots,\mathbf{R}_{j-1}=0\right)} \\
&= \frac{\mu_{j|j-1}(\mathbb{F})}{\mathbb{P}\left(\mathbf{R}_j=0 \,\middle|\, \mathbf{R}_k=0,\cdots,\mathbf{R}_{j-1}=0\right)}
\end{aligned} \tag{B.59}$$

Using this relation, we can see that

$$\begin{aligned}
&\mu_{j|j-1}(\mathbb{F}) \\
&= \mathbb{P}\left(\mathbf{x}_j\in\mathbb{F} \,\middle|\, \mathbf{R}_k=0,\cdots,\mathbf{R}_{j-1}=0\right) \\
&= \mathbb{P}\left(\mathbf{x}_j\in\mathbb{F} \,\middle|\, \mathbf{R}_k=0,\cdots,\mathbf{R}_j=0\right)\cdot\mathbb{P}\left(\mathbf{R}_j=0 \,\middle|\, \mathbf{R}_k=0,\cdots,\mathbf{R}_{j-1}=0\right) \\
&\quad + \mathbb{P}\left(\mathbf{x}_j\in\mathbb{F} \,\middle|\, \mathbf{R}_k=0,\cdots,\mathbf{R}_j=1\right)\cdot\mathbb{P}\left(\mathbf{R}_j=1 \,\middle|\, \mathbf{R}_k=0,\cdots,\mathbf{R}_{j-1}=0\right) \\
&\geq \mu_{j|j-1}(\mathbb{F}) \\
&\quad + \mathbb{P}\left(\mathbf{x}_j\in\mathbb{F} \,\middle|\, \mathbf{R}_k=0,\cdots,\mathbf{R}_j=1\right)\cdot\mathbb{P}\left(\mathbf{R}_j=1 \,\middle|\, \mathbf{R}_k=0,\cdots,\mathbf{R}_{j-1}=0\right)
\end{aligned} \tag{B.60}$$

Using the fact that $\mathbb{P}\left(\mathbf{R}_j = 1 \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_{j-1} = 0\right) > 0$, we can derive that

$$\mathbb{P}\left(\mathbf{x}_j \in \mathbb{F} \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 1\right) = 0 \tag{B.61}$$

From the closed regularity theorem (see Theorem 7.1.3 in [93]), for any Borel measurable set $\mathbb{A}$ contained in $\underline{\mathbb{D}}_j$, we obtain

$$\mathbb{P}\left(\mathbf{x}_j \in \mathbb{A} \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 1\right) = 0 \tag{B.62}$$

□

Proof of (A2): Recall the definitions of

$$\mathbb{E}_{\mathbf{x}_j}\left[J_j\left(\mathbf{x}_j, \boldsymbol{\mathcal{P}}_{j:N}, \hat{x}'_{j:N}\right) \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_{j-1} = 0\right]$$

and J_j^* given in (3.14) and (3.25), respectively.

For $j = N$, using Claim 3 and Claim 4, we can derive the following:

$$\begin{aligned} &\mathbb{E}_{\mathbf{x}_N}\left[d^2\left(\mathbf{x}_N, \hat{x}'_N\right) \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_N = 0\right] \\ &= \mathbb{E}_{\mathbf{x}_N}\left[\min\left\{d^2\left(\mathbf{x}_N, \hat{x}'_N\right), c'_N\right\} \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_N = 0\right] \end{aligned} \tag{B.63a}$$

and

$$c'_N = \mathbb{E}_{\mathbf{x}_N}\left[\min\left\{d^2\left(\mathbf{x}_N, \hat{x}'_N\right), c'_N\right\} \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_N = 1\right] \tag{B.63b}$$

provided that $\mathbb{P}\left(\mathbf{R}_N = 1 \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_{N-1} = 0\right) > 0$. From (3.14), (3.25), and (B.63), we can derive that

$$\begin{aligned} &\mathbb{E}_{\mathbf{x}_N}\left[J_N\left(\mathbf{x}_N, \boldsymbol{\mathcal{P}}_N, \hat{x}'_N\right) \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_{N-1} = 0\right] \\ &= \mathbb{E}_{\mathbf{x}_N}\left[J_N^*\left(\mathbf{x}_N, \hat{x}'_N\right) \,\middle|\, \mathbf{R}_k = 0, \cdots, \mathbf{R}_{N-1} = 0\right] \end{aligned} \tag{B.64}$$

Suppose that

$$\mathbb{E}_{\mathbf{x}_{j+1}}\left[J_{j+1}\left(\mathbf{x}_{j+1}, \boldsymbol{\mathcal{P}}_{j+1:N}, \hat{x}'_{j+1:N}\right)\,\middle|\,\mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 0\right]$$
$$= \mathbb{E}_{\mathbf{x}_{j+1}}\left[J^*_{j+1}\left(\mathbf{x}_{j+1}, \hat{x}'_{j+1:N}\right)\,\middle|\,\mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 0\right] \tag{B.65}$$

holds. Then, using Claim 3 and Claim 4, we can derive the following:

$$\mathbb{E}_{\mathbf{x}_j}\left[d^2\left(\mathbf{x}_j, \hat{x}'_j\right) + \mathbb{E}_{\mathbf{x}_{j+1}}\left[J_{j+1}\left(\mathbf{x}_{j+1}, \boldsymbol{\mathcal{P}}_{j+1:N}, \hat{x}'_{j+1:N}\right)\,\middle|\,\mathbf{x}_j\right]\,\middle|\,\mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 0\right]$$
$$= \mathbb{E}_{\mathbf{x}_j}\left[\min\left\{d^2\left(\mathbf{x}_j, \hat{x}'_j\right) + \mathbb{E}_{\mathbf{x}_{j+1}}\left[J^*_{j+1}\left(\mathbf{x}_{j+1}, \hat{x}'_{j+1:N}\right)\,\middle|\,\mathbf{x}_j\right], c'_j\right\}\,\middle|\,\mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 0\right]$$
$$= \mathbb{E}_{\mathbf{x}_j}\left[J^*_j\left(\mathbf{x}_j, \hat{x}'_{j:N}\right)\,\middle|\,\mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 0\right] \tag{B.66a}$$

and

$$c'_j = \mathbb{E}_{\mathbf{x}_j}\left[\min\left\{d^2\left(\mathbf{x}_j, \hat{x}'_j\right) + \mathbb{E}_{\mathbf{x}_{j+1}}\left[J^*_{j+1}\left(\mathbf{x}_{j+1}, \hat{x}'_{j+1:N}\right)\,\middle|\,\mathbf{x}_j\right], c'_j\right\}\,\middle|\,\mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 1\right]$$
$$= \mathbb{E}_{\mathbf{x}_j}\left[J^*_j\left(\mathbf{x}_j, \hat{x}'_{j:N}\right)\,\middle|\,\mathbf{R}_k = 0, \cdots, \mathbf{R}_j = 1\right] \tag{B.66b}$$

provided that $\mathbb{P}\left(\mathbf{R}_j = 1\,\middle|\,\mathbf{R}_k = 0, \cdots, \mathbf{R}_{j-1} = 0\right) > 0$. From (3.14), (3.25), and (B.66), we can derive that

$$\mathbb{E}_{\mathbf{x}_j}\left[J_j\left(\mathbf{x}_j, \boldsymbol{\mathcal{P}}_{j:N}, \hat{x}'_{j:N}\right)\,\middle|\,\mathbf{R}_k = 0, \cdots, \mathbf{R}_{j-1} = 0\right]$$
$$= \mathbb{E}_{\mathbf{x}_j}\left[J^*_j\left(\mathbf{x}_j, \hat{x}'_{j:N}\right)\,\middle|\,\mathbf{R}_k = 0, \cdots, \mathbf{R}_{j-1} = 0\right] \tag{B.67}$$

By induction, we conclude that (B.67) holds for all j in $\{k, \cdots, N\}$. By Definition 3.2.5 and the fact that

$$\min_{\boldsymbol{\mathcal{P}}'_{k:N}} \mathbb{E}_{\mathbf{x}_k}\left[J_k\left(\mathbf{x}_k, \boldsymbol{\mathcal{P}}'_{k:N}, \hat{x}'_{k:N}\right)\right] = \mathbb{E}_{\mathbf{x}_k}\left[J^*_k\left(\mathbf{x}_k, \hat{x}'_{k:N}\right)\right]$$

it holds that $\boldsymbol{\mathcal{P}}_{k:N} \in \mathfrak{P}\left(\hat{x}'_{k:N}\right)$. $\square$

B.7 Proof of Theorem 3.3.1

Notice that under (3.41), the cost-to-go of (3.2) from time k can be written as follows:

$$\begin{aligned}
&\mathcal{J}_k\left(x_{k-1}, \boldsymbol{\mathcal{T}}_{k:N}, \mathcal{E}_{k:N}\right) \\
&= \sum_{j=k}^{N} \mathbb{E}\left[d^2\left(\mathbf{x}_j, \hat{\mathbf{x}}_j\right) + c_j \cdot \mathbf{R}_j \,\middle|\, \mathbf{x}_{k-1} = x_{k-1}, \mathbf{R}_{k-1} = 1, \boldsymbol{\mathcal{T}}_{k:N}, \mathcal{E}_{k:N}\right] \\
&= \mathbb{E}\Bigg[\sum_{j=k}^{\mathbf{K}} d^2\left(\mathbf{x}_j, \hat{\mathbf{x}}_j\right) \\
&\qquad + \left(c_{\mathbf{K}} + \mathcal{J}_{\mathbf{K}+1}\left(\mathbf{x}_{\mathbf{K}}, \boldsymbol{\mathcal{T}}_{\mathbf{K}+1:N}, \mathcal{E}_{\mathbf{K}+1:N}\right)\right) \cdot \mathbf{R}_{\mathbf{K}} \,\Bigg|\, \mathbf{x}_{k-1} = x_{k-1}, \mathbf{R}_{k-1} = 1, \boldsymbol{\mathcal{T}}_{k:N}, \mathcal{E}_{k:N}\Bigg] \\
&= \mathbb{E}\Bigg[\sum_{j=k}^{\mathbf{K}} d^2\left(\mathbf{x}_j, \hat{\mathbf{x}}_j\right) \\
&\qquad + \left(c_{\mathbf{K}} + \mathcal{J}_{\mathbf{K}+1}\left(\mathbf{x}_{\mathbf{K}}, \boldsymbol{\mathcal{T}}_{\mathbf{K}+1:N}, \mathcal{E}_{\mathbf{K}+1:N}\right)\right) \cdot \mathbf{R}_{\mathbf{K}} \,\Bigg|\, \mathbf{x}_{k-1} = x_{k-1}, \boldsymbol{\mathcal{T}}_{k:N}^{<k-1>}, \mathcal{E}_{k:N}^{<k-1>}\Bigg]
\end{aligned} \tag{B.68}$$

with $\mathcal{J}_{N+1} = 0$, where

$$\mathbf{K} = \begin{cases} \min\left\{j \in \{k, \cdots, N\} \,\middle|\, \mathbf{R}_j = 1\right\} & \text{if } \mathbf{R}_j = 1 \text{ for some } j \in \{k, \cdots, N\} \\ N & \text{otherwise} \end{cases}$$

First we consider the case where for each k in $\{1, \cdots, N\}$, ${\boldsymbol{\mathcal{T}}^*}_{k:N}^{<k-1>}$ and ${\mathcal{E}^*}_{k:N}^{<k-1>}$ are a jointly optimal solution of Sub-problem k. We will show that for any transmission polices $\boldsymbol{\mathcal{T}}_{1:N}$ and estimation rules $\mathcal{E}_{1:N}$,

$$\mathcal{J}_k\left(x_{k-1}, \boldsymbol{\mathcal{T}}^*_{k:N}, \mathcal{E}^*_{k:N}\right) \leq \mathcal{J}_k\left(x_{k-1}, \boldsymbol{\mathcal{T}}_{k:N}, \mathcal{E}_{k:N}\right) \tag{B.69}$$

holds for all x_{k-1} in $\mathbb{X}$ and all k in $\{1, \cdots, N\}$.

For $k = N$, note that (B.68) can be written as

$$\begin{aligned}&\mathcal{J}_N\left(x_{N-1}, \boldsymbol{\mathcal{T}}_N, \mathcal{E}_N\right)\\&= \mathbb{E}\left[d^2\left(\mathbf{x}_N, \hat{\mathbf{x}}_N\right) + c_N \cdot \mathbf{R}_N \,\middle|\, \mathbf{x}_{N-1} = x_{N-1}, \boldsymbol{\mathcal{T}}_N^{<N-1>}, \mathcal{E}_N^{<N-1>}\right]\end{aligned}$$

By joint optimality of the solution $\boldsymbol{\mathcal{T}}^{*<N-1>}_N$ and $\mathcal{E}^{*<N-1>}_N$, we can see that

$$\mathcal{J}_N\left(x_{N-1}, \boldsymbol{\mathcal{T}}_N^*, \mathcal{E}_N^*\right) \leq \mathcal{J}_N\left(x_{N-1}, \boldsymbol{\mathcal{T}}_N, \mathcal{E}_N\right)$$

holds for all x_{N-1} in $\mathbb{X}$.

Suppose that

$$\mathcal{J}_{j+1}\left(x_j, \boldsymbol{\mathcal{T}}_{j+1:N}^*, \mathcal{E}_{j+1:N}^*\right) \leq \mathcal{J}_{j+1}\left(x_j, \boldsymbol{\mathcal{T}}_{j+1:N}, \mathcal{E}_{j+1:N}\right) \tag{B.70}$$

holds for all x_j in $\mathbb{X}$ and all j in $\{k, \cdots, N-1\}$. Let $\{c_j'^*\}_{j=k}^N$ be the stopping costs determined by (3.8) with the jointly optimal solutions $\left\{\boldsymbol{\mathcal{T}}^{*<j-1>}_{j:N}\right\}_{j=k+1}^N$ and $\left\{\mathcal{E}^{*<j-1>}_{j:N}\right\}_{j=k+1}^N$, and let $\{c_j'\}_{j=k}^N$ be constants determined by

$$c_j' = c_j + \inf_{x_j \in \mathbb{X}} \mathcal{J}_{j+1}\left(x_j, \boldsymbol{\mathcal{T}}_{j+1:N}, \mathcal{E}_{j+1:N}\right) \tag{B.71}$$

From Remark 3.1.10 and (3.8), we note that the stopping costs $\left\{c_j'^*\right\}_{j=k}^N$ satisfy

$$c_j'^* = c_j + \mathcal{J}_{j+1}\left(x_j, \boldsymbol{\mathcal{T}}_{k+1:N}^*, \mathcal{E}_{k+1:N}^*\right)$$

which does not depend on x_j. Then, from (B.68), we can derive the following relations for each x_{k-1} in $\mathbb{X}$:

$$\begin{aligned}
&\mathcal{J}_k\left(x_{k-1}, \boldsymbol{\mathcal{T}}^*_{k:N}, \mathcal{E}^*_{k:N}\right) \\
&= \mathbb{E}\left[\sum_{j=k}^{\mathbf{K}} d^2\left(\mathbf{x}_j, \hat{\mathbf{x}}_j\right) + c'^*_{\mathbf{K}} \cdot \mathbf{R}_{\mathbf{K}} \,\middle|\, \mathbf{x}_{k-1} = x_{k-1}, \boldsymbol{\mathcal{T}}^{*<k-1>}_{k:N}, \mathcal{E}^{*<k-1>}_{k:N}\right] \\
&\overset{(1)}{\leq} \mathbb{E}\left[\sum_{j=k}^{\mathbf{K}} d^2\left(\mathbf{x}_j, \hat{\mathbf{x}}_j\right) + c'^*_{\mathbf{K}} \cdot \mathbf{R}_{\mathbf{K}} \,\middle|\, \mathbf{x}_{k-1} = x_{k-1}, \boldsymbol{\mathcal{T}}^{<k-1>}_{k:N}, \mathcal{E}^{<k-1>}_{k:N}\right] \\
&\overset{(2)}{\leq} \mathbb{E}\left[\sum_{j=k}^{\mathbf{K}} d^2\left(\mathbf{x}_j, \hat{\mathbf{x}}_j\right) + c'_{\mathbf{K}} \cdot \mathbf{R}_{\mathbf{K}} \,\middle|\, \mathbf{x}_{k-1} = x_{k-1}, \boldsymbol{\mathcal{T}}^{<k-1>}_{k:N}, \mathcal{E}^{<k-1>}_{k:N}\right] \\
&\overset{(3)}{\leq} \mathcal{J}_k\left(x_{k-1}, \boldsymbol{\mathcal{T}}_{k:N}, \mathcal{E}_{k:N}\right)
\end{aligned} \tag{B.72}$$

where (1) follows from the fact that $\boldsymbol{\mathcal{T}}^{*<k-1>}_{k:N}$ and $\mathcal{E}^{*<k-1>}_{k:N}$ are a jointly optimal solution of Sub-problem k, (2) follows from Remark 3.1.11 and the fact that $c'^*_j \leq c'_j$ holds for all j in $\{k, \cdots, N\}$, and (3) follows from (B.68) and (B.71).

By induction, we can see that (B.69) holds for all x_{k-1} in $\mathbb{X}$ and all k in $\{1, \cdots, N\}$. Hence, we conclude that the solution $\boldsymbol{\mathcal{T}}^*_{1:N}$ and $\mathcal{E}^*_{1:N}$ determined by (3.41) is jointly optimal for (3.2).

To prove the statement for person-by-person optimality, we note that for every k in $\{1, \cdots, N\}$, with $\boldsymbol{\mathcal{T}}^{*<k-1>}_{k:N}$ $\left(\mathcal{E}^{*<k-1>}_{k:N}\right)$ fixed, $\mathcal{E}^{*<k-1>}_{k:N}$ $\left(\boldsymbol{\mathcal{T}}^{*<k-1>}_{k:N}\right)$ is a global minimizer of (3.4) for Sub-problem k. By a similar argument as for the joint optimality case, we can observe that with $\boldsymbol{\mathcal{T}}^*_{k:N}$ fixed, $\mathcal{E}^*_{k:N}$ determined by (3.41) minimizes (B.68), and vice versa for all x_{k-1} in $\mathbb{X}$ and all k in $\{1, \cdots, N\}$. This proves that $\boldsymbol{\mathcal{T}}^*_{1:N}$ and $\mathcal{E}^*_{1:N}$ are person-by-person optimal for (3.2). $\square$

Appendix C: Auxiliary Results for Chapter 4

C.1 Proof of Proposition 4.2.4

We proceed by showing that for any C_1 function $S_{RD} : \mathbb{R}^n \times \mathbb{X} \to \mathbb{R}_+$ satisfying (**P1**), the condition (**P2**) does not hold. Then, from Theorem 4.2.3, we conclude that the replicator dynamics (4.3) are not passive.

Let us re-write (4.3) in the following form:

$$\dot{x}_i = \sum_{j=1}^{n} x_i x_j \left(p_i - p_j\right) \tag{C.1}$$

Note that any C^1 function S_{RD} satisfying (**P1**) should be of the form

$$S_{RD}(p, x) = \frac{1}{4} \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j (p_i - p_j)^2 + S(x)$$

where S is a C^1 function. By taking a partial derivative of S_{RD} with respect to x, we obtain

$$\nabla_x^T S_{RD}(p, x) V(p, x) = \left[\begin{pmatrix} \frac{1}{2} \sum_{j=1}^{n} x_j (p_1 - p_j)^2 \\ \vdots \\ \frac{1}{2} \sum_{j=1}^{n} x_j (p_n - p_j)^2 \end{pmatrix} + \nabla_x S(x) \right]^T \begin{pmatrix} \sum_{j=1}^{n} x_1 x_j (p_1 - p_j) \\ \vdots \\ \sum_{j=1}^{n} x_n x_j (p_n - p_j) \end{pmatrix}$$

Let us choose $x_j = 0$ for all $j \geq 3$. Then, we obtain

$$\nabla_x^T S_{RD}(p,x) V(p,x) = \begin{pmatrix} \frac{1}{2}x_2(p_1-p_2)^2 + \frac{\partial S}{\partial x_1}(x) \\ \\ \frac{1}{2}x_1(p_1-p_2)^2 + \frac{\partial S}{\partial x_2}(x) \end{pmatrix}^T \begin{pmatrix} x_1 x_2 (p_1-p_2) \\ \\ -x_1 x_2 (p_1 - p_2) \end{pmatrix}$$
$$= -\frac{1}{2}x_1 x_2 (p_1-p_2)\left[(x_1-x_2)(p_1-p_2)^2 + 2\left(\frac{\partial S}{\partial x_2}(x) - \frac{\partial S}{\partial x_1}(x)\right)\right]$$

Note that for fixed x_1, x_2 (except for the points at which $\nabla_x^T S_{RD}(p,x)V(p,x) = 0$ holds for all p in $\mathbb{R}^n$), there exists $p \in \mathbb{R}^n$ for which $\nabla_x^T S_{RD}(p,x)V(p,x) > 0$ holds. Therefore, the function S_{RD} does not satisfy (**P2**). Since we made an arbitrary choice of S_{RD}, we conclude that no C^1 function satisfies (**P1**) and (**P2**) for the replicator dynamics. $\square$

C.2 Proof of Proposition 4.2.5

We first note that the condition (**A**) implies so-called the **Strict Positive Correlation** (**SPC**) [102] given by

$$V(p,x) \neq 0 \text{ implies } p^T V(p,x) > 0 \tag{SPC}$$

Let $\gamma : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function for which (**I**) holds. It can be verified that γ satisfies

$$\nabla_p \gamma(\hat{p}) = V(p,x) \tag{C.2a}$$

$$\nabla_x^T \gamma(\hat{p}) V(p,x) = -\left(\mathbf{1}^T \varrho(\hat{p})\right)\left(p^T V(p,x)\right) \tag{C.2b}$$

Let us select a candidate storage function as $S_{EPT}(p,x) = \gamma(\hat{p})$ where $\hat{p} = p - \mathbf{1} \cdot p^T x$. Due to (C.2a), the function S_{EPT} satisfies (**P1**). In conjunction with the fact that $\varrho(\hat{p}) = \mathbf{0}$ implies $V(p,x) = \mathbf{0}$, due to (**SPC**) and (C.2b), we can see that (**P2**) holds for $\eta = 0$

and the equality in (**P2**) holds only if $V(p,x) = \mathbf{0}$. According to the statement **(S2)** in Theorem 4.2.3, to complete the proof, it remains to show that γ is non-negative.

We argue that the following inequality holds for every $\hat{p}$ in $\mathbb{R}^n$:

$$\gamma(\hat{p}) \geq \gamma(\mathbf{0}) \tag{C.3}$$

Then, without loss of generality by setting $\gamma(\mathbf{0}) = 0$, we conclude that $S_{EPT}(p,x) = \gamma(\hat{p})$ is non-negative. In what follows, we show that (C.3) is valid.

We first claim that (C.3) holds for all (p,x) in $\mathbb{S}$, where $\mathbb{S}$ is the set of equilibrium points of (4.10). Note that for each x in $\mathbb{X}$, due to (**SPC**), it holds that $V(\mathbf{0},x) = \mathbf{0}$, i.e., $\mathbf{0} \in \mathbb{S}_x$. By the fact that $\nabla_p \gamma(\hat{p}) = V(p,x)$, for fixed x, the following equality holds for all p in $\mathbb{R}^n$:

$$\gamma\left(\hat{p}\right) = \gamma\left(\mathbf{0}\right) + \int_0^1 \dot{\mathbf{p}}^T(s) V(\mathbf{p}(s), x)\,\mathrm{d}s \tag{C.4}$$

where $\mathbf{p} : [0,1] \to \mathbb{R}$ is a parameterization of a piece-wise smooth path from $\mathbf{0}$ to p. According the path-connectedness assumption (see Section 4.1.2.2), for each p in $\mathbb{S}_x$, there is a path from $\mathbf{0}$ to p in which the entire path is contained in $\mathbb{S}_x$, i.e., $V(\mathbf{p}(s),x) = \mathbf{0}$ holds for all s in $[0,1]$; hence

$$\gamma\left(\hat{p}\right) - \gamma\left(\mathbf{0}\right) = \int_0^1 \dot{\mathbf{p}}^T(s) V(\mathbf{p}(s), x)\,\mathrm{d}s = \mathbf{0} \tag{C.5}$$

holds for every p in $\mathbb{S}_x$. Since (C.5) holds for every (p,x) in $\mathbb{S}$, this proves the claim.

To see that (C.3) also holds for (p,x) in $(\mathbb{R}^n \times \mathbb{X}) \backslash \mathbb{S}$, by contradiction, let us assume that there is $(p',x') \notin \mathbb{S}$ for which $S_{EPT}(p',x') = \gamma(\hat{p}') < \gamma(\mathbf{0})$. Let $x(\cdot)$ be a population state trajectory induced by (4.10) for the initial condition $x(0) = x'$ and constant payoff $p(t) = p'$ for all t in $\mathbb{R}_+$. By (**SPC**) and (C.2b), the value of $S_{EPT}(p', x(t))$ is strictly decreasing unless $V(p', x(t)) = \mathbf{0}$. By the hypothesis that $S_{EPT}(p', x') < \gamma(\mathbf{0})$ and by (C.5), for every (p, x) in $\mathbb{S}$, it holds that

$$S_{EPT}(p', x') < S_{EPT}(p, x)$$

and the trajectory $(p', x(\cdot))$ never converges to $\mathbb{S}$. On the other hand, by LaSalle's Theorem [101], since $p(t)$ is constant and the population state $x(t)$ is contained in a compact set, $x(t)$ converges to an invariant subset of $\left\{x \in \mathbb{X} \,\middle|\, \nabla_x^T S_{EPT}(p', x) V(p', x) = 0\right\}$. By (**SPC**) and (C.2b), the invariant set is contained in $\mathbb{S}$. This contradicts the fact that the trajectory $(p', x(\cdot))$ does not converge to $\mathbb{S}$; hence $\gamma(\hat{p}) \geq \gamma(\mathbf{0})$ holds for all (p, x) in $(\mathbb{R}^n \times \mathbb{X}) \setminus \mathbb{S}$. □

C.3 Proof of Proposition 4.2.7

The analysis used in Theorem 2.1 of [106] suggests that the following hold:

$$z^T \nabla_p \left[\max_{y \in \mathrm{int}(\mathbb{X})} (p^T y - v(y)) \right] = z^T C(p) \tag{C.6a}$$

$$z^T \nabla_x v(x) = z^T p \text{ if and only if } C(p) = x \tag{C.6b}$$

for all p in $\mathbb{R}^n$, x in $\mathbb{X}$, and z in $T\mathbb{X}$. Using (C.6), we can show that

$$\nabla_p S_{PBR}(p, x) = C(p) - x = V(p, x) \tag{C.7}$$

and

$$\begin{aligned} \nabla_x^T S_{PBR}(p, x) V(p, x) &= -\left(p - \nabla_x v(x)\right)^T V(p, x) \\ &= -\left(\nabla_y v(y) - \nabla_x v(x)\right)^T (y - x) \end{aligned} \tag{C.8}$$

where $y = C(p)$. By the fact that v is strictly convex, it holds that $\nabla_x^T S_{PBR}(p,x)V(p,x) \leq 0$ where the equality holds only if $V(p,x) = \mathbf{0}$. According to Theorem 4.2.3, we conclude that the PBR dynamics are strictly passive.

Furthermore, if the perturbation v is strongly convex, i.e.,

$$\left(\nabla_y v(y) - \nabla_x v(x)\right)^T (y - x) \geq \eta' \cdot \|y - x\|^2$$

holds for all x, y in $\mathbb{X}$, then from (C.8), we can derive that

$$\nabla_x^T S_{PBR}(p,x)V(p,x) \leq -\eta' \cdot \|V(p,x)\|^2 \tag{C.9}$$

Hence, by Theorem 4.2.3, we conclude that the PBR dynamics are strictly output passive and satisfies the passivity inequality (4.7) for $\eta = \eta'$. □

C.4 Proof of Proposition 4.2.8

The first part of the statement directly follows from the condition (**P1**) and the fact that at a global minimizer (p^*, x^*) of S_{ED}, it holds that $\nabla_p S_{ED}(p^*, x^*) = 0$.

Now suppose that the dynamics satisfy (**NS**). To prove the second statement, it is sufficient to show that at each equilibrium point (p, x) of (4.2), it holds that $S_{ED}(p,x) = 0$. To this end, let us consider an *anti-coordination game* whose payoff function is given by $F_{x_o}(x) = -(x - x_o)$ for a fixed $x_o \in \mathbb{X}$. Notice that x_o is a unique Nash equilibrium of the game. In what follows, we show that $S_{ED}(p_o, x_o) = 0$ holds for any choice of x_o from $\mathbb{X}$ and p_o from $\mathbb{S}_{x_o}$.

Let (p^*, x^*) be a global minimizer of S_{ED}, i.e., $S_{ED}(p^*, x^*) = 0$. By the first part of the statement and (**NS**), we have that $V(s \cdot p^*, x^*) = \mathbf{0}$ for all s in $\mathbb{R}_+$; hence it holds that

$$S_{ED}(\mathbf{0}, x^*) = S_{ED}(p^*, x^*) - \int_0^1 (p^*)^T V(s \cdot p^*, x^*) \, \mathrm{d}s = 0$$

By the continuity of S_{ED}, for each $\epsilon > 0$ there exists $\delta > 0$ for which it holds that $S_{ED}(\delta \cdot F_{x_o}(x^*), x^*) < \epsilon$.

According to the passivity conditions (**P1**) and (**P2**) for $\eta = 0$, the following relation holds for every positive constant δ:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} S_{ED}(\delta \cdot F_{x_o}(x(t)), x(t)) \\ &\leq \delta \cdot V^T(\delta \cdot F_{x_o}(x(t)), x(t)) DF_{x_0}(x(t)) \dot{x}(t) \\ &= -\delta \cdot \| V(\delta \cdot F_{x_o}(x(t)), x(t)) \|^2 \end{aligned} \tag{C.10}$$

where the trajectory $x(\cdot)$ starts from x^*. By an application of LaSalle's theorem [101] and by (**NS**), we can verify that $(\delta \cdot F_{x_o}(x(t)), x(t))$ converges to $(\mathbf{0}, x_o)$ as $t \to \infty$. In addition, due to (C.10), we have that

$$S_{ED}(\mathbf{0}, x_o) \leq S_{ED}(\delta \cdot F_{x_0}(x^*), x^*) < \epsilon$$

Since this holds for every $\epsilon > 0$, we conclude that $S_{ED}(\mathbf{0}, x_o) = 0$. By the fact that $V(s \cdot p_o, x_o) = \mathbf{0}$ for all s in $\mathbb{R}_+$ if p_o belongs to $\mathbb{S}_{x_o}$, we can see that

$$S_{ED}(p_o, x_o) = S_{ED}(\mathbf{0}, x_o) + \int_0^1 p_o^T V(s \cdot p_o, x_o) \, \mathrm{d}s = 0 \tag{C.11}$$

holds for every p_o in $\mathbb{S}_{x_o}$.

Since the choice of x_o from $\mathbb{X}$ in constructing the game was arbitrary, we conclude that (C.11) holds for every (p_o, x_o) in $\mathbb{S}$. This proves the Proposition. $\square$

C.5 Proof of Proposition 4.2.10

We first construct a game F based on the *Hypnodisk game* [104], which is described by the following payoff function:

$$\begin{pmatrix} H_1(x_1,x_2,x_3) \\ H_2(x_1,x_2,x_3) \\ H_3(x_1,x_2,x_3) \end{pmatrix} = \cos\left(\theta(x_1,x_2,x_3)\right) \begin{pmatrix} x_1 - \frac{1}{3} \\ x_2 - \frac{1}{3} \\ x_3 - \frac{1}{3} \end{pmatrix} + \frac{\sqrt{3}}{3} \sin\left(\theta(x_1,x_2,x_3)\right) \begin{pmatrix} x_2 - x_3 \\ x_3 - x_1 \\ x_1 - x_2 \end{pmatrix}$$
$$+ \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

where

$$\theta(x_1,x_2,x_3) = \pi \cdot \left[1 - b\left(\sum_{i=1}^{3} \left| x_i - \frac{1}{3} \right|^2 \right) \right]$$

and b is a bump function that satisfies

1. $b(a) = 1$ if $a \leq R_I^2$
2. $b(a) = 0$ if $a \geq R_O^2$
3. $b(a)$ is decreasing if $R_I^2 < a < R_O^2$

with $0 < R_I < R_O < \frac{1}{\sqrt{6}}$. Consider a payoff function F' given by

$$F_i'(x) = \begin{cases} H_i(x_1, x_2, x_3 + \cdots + x_n) & \text{if } i \in \{1,2\} \\ H_3(x_1, x_2, x_3 + \cdots + x_n) & \text{otherwise} \end{cases} \tag{C.12}$$

Note that the set of Nash equilibria for F' is given by

$$\left\{ x \in \mathbb{X} \,\middle|\, x_1 = x_2 = x_3 + \cdots + x_n = \frac{1}{3} \right\} \tag{C.13}$$

Since θ is a smooth function, F' is continuously differentiable, and its differential map DF' is bounded, i.e., for some $\delta > 0$, it holds that $z^T DF'(x) z < \delta \cdot z^T z$ for all x in $\mathbb{X}$ and z in $T\mathbb{X}$. Finally, for a given constant $\nu > 0$, we define a new payoff function by $F_\nu = \frac{\nu}{\delta} \cdot F'$.

Using the payoff function F_ν, we prove the statement of the Proposition. By contradiction, suppose that there is a payoff monotonic dynamic, i.e., the dynamic satisfies both (**NS**) and (**PC**), that is strictly output passive. By definition, the dynamic satisfies the passivity inequality (4.7) for $\eta > 0$. Consider a population game described by F_ν in which $\nu < \eta$ holds. Then the time derivative of the storage function $S_{ED}(F_\nu(x(t)), x(t))$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} S_{ED}(F_\nu(x(t)), x(t)) \leq -(\eta - \nu) \cdot \|V(F_\nu(x(t)), x(t))\|$$

By an application of LaSalle's theorem [101] and by (**NS**), we can verify that $x(t)$ converges to the set of Nash equilibria given as in (C.13).

On the other hand, when $x(t)$ is contained in the set

$$\left\{ x \in \mathbb{X} \;\middle|\; \left(x_1 - \frac{1}{3}\right)^2 + \left(x_2 - \frac{1}{3}\right)^2 + \left(x_3 + \cdots + x_n - \frac{1}{3}\right)^2 \leq R_I^2 \right\}$$

by the condition (**PC**), it holds that

$$\begin{aligned} & F_\nu^T(x(t)) V(F_\nu(x(t)), x(t)) \\ & = \frac{\nu}{2\delta} \cdot \frac{\mathrm{d}}{\mathrm{d}t} \left[\left(x_1(t) - \frac{1}{3}\right)^2 + \left(x_2(t) - \frac{1}{3}\right)^2 + \left(x_3(t) + \cdots + x_n(t) - \frac{1}{3}\right)^2 \right] \geq 0 \end{aligned} \tag{C.14}$$

Hence, the population state $x(t)$ never converges to the set of Nash equilibria. This is a contradiction. Therefore, the dynamic cannot be strictly output passive. $\square$

C.6 Proof of Proposition 4.2.12

The sufficiency directly follows from the choice of $E_{ED} = S_{ED}$ and the inequalities in (4.7) and (4.16), where S_{ED} is a storage function of the passive evolutionary dynamic. To prove the necessity, we consider a set of population games represented by cumulative payoff functions $\dot{p} = Ax$, where A is a symmetric matrix in $\mathbb{R}^{n\times n}$. It can be verified that each game satisfies (4.16) for a C^1 function

$$S_G(x) = \frac{1}{2}\max_{x\in\mathbb{X}} x^T A x - \frac{1}{2} x^T A x$$

Let us select a candidate Lyapunov function $E = S_G + E_{ED}$ for a C^1 function $E_{ED} : \mathbb{R}^n \times \mathbb{X} \to \mathbb{R}_+$. The time derivative of E leads to the following:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E(p,x) &= \nabla_x^T S_G(x)\dot{x} + \nabla_p^T E_{ED}(p,x)\dot{p} + \nabla_x^T E_{ED}(p,x)\dot{x} \\ &= -V^T(p,x)Ax + \nabla_p^T E_{ED}(p,x)Ax + \nabla_x^T E_{ED}(p,x)V(p,x) \\ &= \left(\nabla_p E_{ED}(p,x) - V(p,x)\right)^T Ax + \nabla_x^T E_{ED}(p,x)V(p,x) \leq 0 \qquad \text{(C.15)}\end{aligned}$$

Since x belongs to $\mathbb{X}$ and A could be any symmetric matrix in $\mathbb{R}^{n\times n}$, the inequality in (C.15) holds for every choice of A if and only if $\nabla_p E_{ED}(p,x) = V(p,x)$ and $\nabla_x^T E_{ED}(p,x)V(p,x) \leq 0$ hold. By Theorem 4.2.3, we conclude that the dynamic is passive with a storage function $S_{ED} = E_{ED}$. □

C.7 Proof of Corollary 4.2.13

The sufficiency directly follows from the definition of passivity. To prove the necessity, by contradiction, suppose that the dynamic is not passive. Then, by Proposition 4.2.12, for any choice of S_{ED}, we can construct a game identified by a cumulative payoff function given as in (4.14) for which

$$S_G\left(x(t)\right) + S_{ED}\left(p(t), x(t)\right) > S_G\left(x(t_0)\right) + S_{ED}\left(p(t_0), x(t_0)\right) \tag{C.16}$$

holds, where S_G is given in (4.15). According to (4.16), this yields that

$$S_{ED}(p(t), x(t)) > S_{ED}(p(t_0), x(t_0)) + \int_{t_0}^{t} F^T(x(\tau))\dot{x}(\tau)\,\mathrm{d}\tau \tag{C.17}$$

which contradicts the fact that the dynamic satisfies the passivity inequality (4.7) for every cumulative payoff function (4.14). □

C.8 Proof of Proposition 4.2.14

As the revision protocol depends only on the the gradient $\nabla_x u(p, x) = p - \nabla_x v(x)$ and the population state x, we represent the revision protocol and vector field of the dynamic as $\varrho_{ji}\left(\nabla_x u(p, x), x\right)$ and $V\left(\nabla_x u(p, x), x\right)$, respectively. Since unperturbed dynamics are passive, by Theorem 4.2.3, we can find a storage function $S_{ED} : \mathbb{R}^n \times \mathbb{X} \to \mathbb{R}_+$ for which the conditions (**P1**) and (**P2**) hold for $\eta = 0$. In what follows, we show that the resulting perturbed dynamics are strictly passive with a storage function

$$\widetilde{S}_{ED}(p, x) \overset{def}{=} S_{ED}\left(p - \nabla_x v(x), x\right)$$

We first compute the gradient of $\widetilde{S}_{ED}$ with respect to p and x as follows:

$$\nabla_p \widetilde{S}_{ED}\left(p, x\right) = \nabla_{p'} S_{ED}\left(p', x\right)\Big|_{p'=p-\nabla_x v(x)} = V\left(p - \nabla_x v(x), x\right) \tag{C.18}$$

and

$$\begin{aligned}
&\nabla_x \widetilde{S}_{ED}(p,x) \\
&= \nabla_x^T \left(p - \nabla_x v(x)\right) \; \nabla_{p'} S_{ED}(p',x)\Big|_{p'=p-\nabla_x v(x)} + \nabla_x S_{ED}(p',x)\Big|_{p'=p-\nabla_x v(x)} \\
&= -\left(\nabla_x^2 v(x)\right)^T V\left(p - \nabla_x v(x), x\right) + \nabla_x S_{ED}(p',x)\Big|_{p'=p-\nabla_x v(x)}
\end{aligned} \tag{C.19}$$

Using (C.19), we can derive the following:

$$\begin{aligned}
&\nabla_x^T \widetilde{S}_{ED}(p,x) V\left(p - \nabla_x v(x), x\right) \\
&= -V^T\left(p - \nabla_x v(x), x\right) \nabla_x^2 v(x) V\left(p - \nabla_x v(x), x\right) + \nabla_x^T S_{ED}(p',x) V(p',x)\Big|_{p'=p-\nabla_x v(x)} \\
&\overset{(1)}{\leq} -V^T\left(p - \nabla_x v(x), x\right) \nabla_x^2 v(x) V\left(p - \nabla_x v(x), x\right)
\end{aligned} \tag{C.20}$$

where (1) is due to passivity of the unperturbed dynamics. Since v satisfies $z^T \nabla_x^2 v(x) z > 0$ for all x in $\mathbb{X}$ and nonzero z in $T\mathbb{X}$,

$$\nabla_x^T \widetilde{S}_{ED}(p,x) V\left(p - \nabla_x v(x), x\right) = 0$$

holds only if $V\left(p - \nabla_x v(x), x\right) = \mathbf{0}$. Using Theorem 4.2.3, we can see that the perturbed dynamics are strictly passive.

Now suppose that v is strongly convex satisfying

$$z^T \nabla_x^2 v(x) z \geq \eta \cdot z^T z \tag{C.21}$$

for all x in $\mathbb{X}$ and z in $T\mathbb{X}$, where η' is a positive constant. From (C.20), we can see that

$$\begin{aligned}
&\nabla_x^T \widetilde{S}_{ED}(p,x) V\left(p - \nabla_x v(x), x\right) \\
&\leq -\eta \cdot V^T\left(p - \nabla_x v(x), x\right) V\left(p - \nabla_x v(x), x\right)
\end{aligned}$$

holds. Hence, from Theorem 4.2.3, we conclude that the perturbed dynamics are strongly output passive. $\square$

C.9 Proof of Theorem 4.3.1

First of all, we note that for any sequence $\left\{\left(p^{(l)}, x^{(l)}\right)\right\}_{l\in\mathbb{N}}$ in $\mathbb{R}^n \times \mathbb{X}$, if

$$\nabla_x^T S_{ED}\left(p^{(l)}, x^{(l)})V(p^{(l)}, x^{(l)}\right) + \nu \cdot V^T\left(p^{(l)}, x^{(l)}\right) V\left(p^{(l)}, x^{(l)}\right) \stackrel{l\to\infty}{\longrightarrow} 0$$

then, in both **(CL1)** and **(CL2)**, we can see that $\nabla_x^T S_{PBR}(p^{(l)}, x^{(l)})V(p^{(l)}, x^{(l)}) \stackrel{l\to\infty}{\longrightarrow} 0$. Hence, in conjunction with **(A2)**, we have that the following relation: For any sequence $\left\{\left(p^{(l)}, x^{(l)}\right)\right\}_{l\in\mathbb{N}}$ in $\mathbb{R}^n \times \mathbb{X}$,

(A2') $\nabla_x^T S_{ED}\left(p^{(l)}, x^{(l)}\right) V\left(p^{(l)}, x^{(l)}\right) + \nu \cdot V^T\left(p^{(l)}, x^{(l)}\right) V\left(p^{(l)}, x^{(l)}\right) \stackrel{l\to\infty}{\longrightarrow} 0$

implies $S_{ED}\left(p^{(l)}, x^{(l)}\right) \stackrel{l\to\infty}{\longrightarrow} 0$

To prove the Theorem, for each $\epsilon > 0$ let us define a set given by

$$\mathbb{O}_\epsilon \stackrel{def}{=} \left\{t > 0 \,\middle|\, S_{ED}(p(t), x(t)) > \frac{\epsilon}{2}\right\}$$

Since $S_{ED}(p(t), x(t)) > \frac{\epsilon}{2}$ holds for all t in $\mathbb{O}_\epsilon$, by the contrapositive of **(A2')**, there exists $\delta_1 > 0$ for which

$$\nabla_x^T S_{ED}(p(t), x(t))V(p(t), x(t)) + \nu \cdot V^T(p(t), x(t))V(p(t), x(t)) \leq -\delta_1$$

holds[1] for all t in $\mathbb{O}_\epsilon$. Note that using (4.22), we can derive the following relations:

$$\begin{aligned}
&S_{ED}(p(t), x(t)) - S_{ED}(p(0), x(0)) - \alpha \\
&\leq \int_0^t \frac{\mathrm{d}}{\mathrm{d}\tau} S_{ED}(p(\tau), x(\tau))\,\mathrm{d}\tau + \int_0^t \left[-\dot{p}^T(\tau)\dot{x}(\tau) + \nu \cdot \dot{x}^T(\tau)\dot{x}(\tau)\right]\,\mathrm{d}\tau \\
&= \int_0^t \left[\nabla_x^T S_{ED}(p(\tau), x(\tau)) V(p(\tau), x(\tau)) + \nu \cdot V^T(p(\tau), x(\tau)) V(p(\tau), x(\tau))\right]\,\mathrm{d}\tau
\end{aligned} \tag{C.22}$$

Since S_{ED} is a non-negative function, we can infer that (C.22) is lower-bounded by $-S_{ED}(p(0), x(0)) - \alpha$ for all $t \geq 0$, which yields that

$$\begin{aligned}
&- S_{ED}(p(0), x(0)) - \alpha \\
&\leq \int_0^\infty \left[\nabla_x^T S_{ED}(p(\tau), x(\tau)) V(p(\tau), x(\tau)) + \nu \cdot V^T(p(\tau), x(\tau)) V(p(\tau), x(\tau))\right]\,\mathrm{d}\tau \\
&\leq \int_{\mathbb{O}_\epsilon} \left[\nabla_x^T S_{ED}(p(\tau), x(\tau)) V(p(\tau), x(\tau)) + \nu \cdot V^T(p(\tau), x(\tau)) V(p(\tau), x(\tau))\right]\,\mathrm{d}\tau \\
&\leq -\delta_1 \cdot \mathcal{L}\left(\mathbb{O}_\epsilon\right)
\end{aligned} \tag{C.23}$$

where $\mathcal{L}\left(\mathbb{O}_\epsilon\right)$ is the Lebesgue measure of $\mathbb{O}_\epsilon$. Hence, we have that $\mathcal{L}\left(\mathbb{O}_\epsilon\right) \leq \frac{S_{ED}(p(0),x(0))+\alpha}{\delta_1}$.

Since $\mathbb{O}_\epsilon$ is an open set, we can represent $\mathbb{O}_\epsilon$ as an union of disjoint open intervals, i.e., $\mathbb{O}_\epsilon = \bigcup_{i\in\mathbb{N}} \mathbb{I}_i$ where $\{\mathbb{I}_i\}_{i\in\mathbb{N}}$ is a set of disjoint open intervals. Notice that by our construction of $\mathbb{O}_\epsilon$, by letting $\mathbb{I}_i = (a_i, b_i)$, we have that $S_{ED}(p(a_i), x(a_i)) \leq \frac{\epsilon}{2}$ and $S_{ED}(p(t), x(t)) > \frac{\epsilon}{2}$ for $t \in \mathbb{I}_i$. Since $\mathbb{O}_\epsilon$ has finite Lebesgue measure, it holds that $\lim_{i\to\infty} \mathcal{L}\left(\mathbb{I}_i\right) = 0$.

[1] Notice that under the configurations as in **(CL1)** or **(CL2)**, it holds that

$$\nabla_x^T S_{ED}(p(t), x(t)) V(p(t), x(t)) + \nu \cdot V^T(p(t), x(t)) V(p(t), x(t)) \leq 0$$

In what follows, we show that for each $\epsilon > 0$, there exists $T_\epsilon > 0$ for which $S_{ED}(p(t), x(t)) < \epsilon$ holds for all $t \geq T_\epsilon$, and we conclude that $\lim_{t\to\infty} S_{ED}(p(t), x(t)) = 0$. To achieve this, by contradiction, suppose that there exists an infinite subset $\mathbb{J}$ of $\mathbb{N}$ for which

$$\max_{t \in \mathrm{cl}(\mathbb{I}_j)} S_{ED}(p(t), x(t)) \geq \epsilon$$

for each j in $\mathbb{J}$, where $\mathrm{cl}\,(\mathbb{I}_j)$ is the closure of $\mathbb{I}_j$. Let $\bar{t}_j \in \mathrm{cl}\,(\mathbb{I}_j)$ be for which

$$S_{ED}(p(\bar{t}_j), x(\bar{t}_j)) = \max_{t \in \mathrm{cl}(\mathbb{I}_j)} S_{ED}(p(t), x(t))$$

holds. By letting $\mathbb{I}_j = (a_j, b_j)$, we can derive the following:

$$\begin{aligned}
&S_{ED}(p(\bar{t}_j), x(\bar{t}_j)) - S_{ED}(p(a_j), x(a_j)) \\
&= \int_{a_j}^{\bar{t}_j} \frac{\mathrm{d}}{\mathrm{d}\tau} S_{ED}(p(\tau), x(\tau))\, \mathrm{d}\tau \\
&\overset{(1)}{\leq} \int_{a_j}^{\bar{t}_j} \dot{p}^T(\tau) V(p(\tau), x(\tau))\, \mathrm{d}\tau \\
&\overset{(2)}{<} M \cdot \delta_2 \cdot \mathcal{L}\,(\mathbb{I}_j)
\end{aligned} \tag{C.24}$$

The inequality (1) can be derived using Theorem 4.2.3. To see that (2) holds, recall that from (C.22), $S_{ED}(p(t), x(t)) \leq S_{ED}(p(0), x(0)) + \alpha$ holds for all $t \geq 0$, and that $\dot{p}$ is bounded, i.e, there is a positive real M for which $\|\dot{p}(t)\| < M$ holds for all t in $\mathbb{R}_+$. Hence, according to the contrapositive of **(A1)**, we can derive $\dot{p}^T(\tau) V(p(\tau), x(\tau)) < M \cdot \delta_2$ for some $\delta_2 > 0$, which yields (2).

Since $S_{ED}(p(a_j), x(a_j)) \leq \frac{\epsilon}{2}$ and $\lim_{j\to\infty} \mathcal{L}\,(\mathbb{I}_j) = 0$, from (C.24), we can see that $S_{ED}(p(\bar{t}_j), x(\bar{t}_j)) < \epsilon$ for sufficiently large j in $\mathbb{J}$. This contradicts the hypothesis that $S_{ED}(p(\bar{t}_j), x(\bar{t}_j)) \geq \epsilon$ holds for all j in $\mathbb{J}$. Since this holds for every $\epsilon > 0$, we can infer that for each $\epsilon > 0$, there exists $T_\epsilon > 0$ for which $S_{ED}(p(t), x(t)) < \epsilon$ holds for all $t \geq T_\epsilon$.

□

C.10 Proof of Proposition 4.3.2

Suppose that $\nu_2 > 0$. We proceed by showing that the following relations hold:

$$\begin{aligned}
&\int_0^t \dot{p}^T(\tau)\dot{x}(\tau)\,\mathrm{d}\tau \\
&= \int_0^t \dot{x}^T(\tau)DF_1^T(x(\tau))\dot{x}(\tau)\,\mathrm{d}\tau + \int_0^t \dot{x}^T(\tau-d)DF_2^T(x(\tau-d))\dot{x}(\tau)\,\mathrm{d}\tau \\
&\overset{(1)}{\leq} \nu_1 \int_0^t \dot{x}^T(\tau)\dot{x}(\tau)\,\mathrm{d}\tau + \frac{\nu_2}{2}\int_0^t \dot{x}^T(\tau)\dot{x}(\tau)\,\mathrm{d}\tau \\
&\quad + \frac{1}{2\nu_2}\int_0^t \dot{x}^T(\tau-d)DF_2^T(x(\tau-d))DF_2(x(\tau-d))\dot{x}(\tau-d)\,\mathrm{d}\tau \\
&\overset{(2)}{\leq} \nu_1 \int_0^t \dot{x}^T(\tau)\dot{x}(\tau)\,\mathrm{d}\tau + \frac{\nu_2}{2}\int_0^t \dot{x}^T(\tau)\dot{x}(\tau)\,\mathrm{d}\tau + \frac{\nu_2}{2}\int_0^t \dot{x}^T(\tau-d)\dot{x}(\tau-d)\,\mathrm{d}\tau \\
&\leq (\nu_1+\nu_2)\int_0^t \dot{x}^T(\tau)\dot{x}(\tau)\,\mathrm{d}\tau + \frac{\nu_2}{2}\int_{-d}^0 \dot{x}^T(\tau)\dot{x}(\tau)\,\mathrm{d}\tau
\end{aligned} \tag{C.25}$$

To show (1), we use (4.25a) and the following fact:

$$\begin{aligned}
&\dot{x}^T(\tau)DF_2(x(\tau-d))\dot{x}(\tau-d) \\
&\leq \frac{\nu_2}{2}\dot{x}^T(\tau)\dot{x}(\tau) + \frac{1}{2\nu_2}\dot{x}^T(\tau-d)DF_2^T(x(\tau-d))DF_2(x(\tau-d))\dot{x}(\tau-d)
\end{aligned}$$

To see (2), we use (4.25b). Then, by defining $\alpha = \frac{\nu_2}{2} \cdot \int_{-d}^0 \dot{x}^T(\tau)\dot{x}(\tau)\mathrm{d}\tau$, we can see that the payoff function (4.24) satisfies (4.22) with $\nu = \nu_1 + \nu_2$.

If $\nu_2 = 0$ then we can see that $DF_2(x)z = \mathbf{0}$ for all x in $\mathbb{X}$ and z in $T\mathbb{X}$. Based on the above analysis, by defining $\alpha = 0$, we can conclude that (4.24) satisfies (4.22) with $\nu = \nu_1$. □

C.11 Proof of Proposition 4.3.3

First of all, we note that with a choice of P as $\hat{\mathbf{p}}(s) = s \cdot \hat{p}$, according to (**A**) of the EPT dynamics, we can see that $S\left(\hat{p}\right) = \int_0^1 \hat{p}^T \varrho(s \cdot \hat{p}) \, \mathrm{d}s \geq 0$. We proceed the proof by choosing a storage function for the dynamics as $S_{EPT}(p, x) = S\left(\hat{p}\right)$, and show that **(A1)** and **(A2)** of Theorem 4.3.1 hold. To start with, we verify that the contrapositive of **(A1)** is true. Notice that

$$\begin{aligned} \|V(p,x)\| &= \left\|\varrho(\hat{p}) - x\mathbf{1}^T \varrho(\hat{p})\right\| \\ &\leq \|\varrho(\hat{p})\| + \|x\| \left|\mathbf{1}^T \varrho(\hat{p})\right| \\ &\leq (\sqrt{n} + 1) \cdot \|\varrho(\hat{p})\| \end{aligned} \tag{C.26}$$

where the last inequality follows from the Cauchy-Schwarz inequality. Suppose that there exists $\epsilon > 0$ for which $S_{EPT}(p, x) < \epsilon$ holds. By the contrapositive of **(C1)**, there exists $\delta' > 0$ for which $\|\varrho(\hat{p})\| < \delta'$ holds. By (C.26), it holds that $\|V(p,x)\| < (\sqrt{n}+1) \cdot \delta' = \delta$; hence the contrapositive of **(A1)** holds.

To verify that **(A2)** of Theorem 4.3.1 holds, notice that

$$\nabla_x^T S_{EPT}(p,x) V(p,x) = -\left(\mathbf{1}^T \varrho(\hat{p})\right) \left(\hat{p}^T \varrho(\hat{p})\right)$$

We claim that if $S_{EPT}(p, x) \geq \epsilon$ holds for a positive real ϵ, then there exist $\delta_1, \delta_2 > 0$ for which $\mathbf{1}^T \varrho(\hat{p}) \geq \delta_1$ and $\hat{p}^T \varrho(\hat{p}) \geq \delta_2$ hold. Hence we have that

$$\nabla_x^T S_{EPT}(p,x) V(p,x) \leq -\delta_1 \cdot \delta_2 = -\delta$$

This proves that the contrapositive of **(A2)** is true. We prove the claim by contradiction. Suppose that there exists a sequence $\left\{\left(p^{(l)}, x^{(l)}\right)\right\}_{l \in \mathbb{N}} \subset \mathbb{R}^n \times \mathbb{X}$ for which either

$\lim_{l\to\infty} \mathbf{1}^T \varrho\left(\hat{p}^{(l)}\right) = 0$ or $\lim_{i\to\infty} \left(\hat{p}^{(l)}\right)^T \varrho\left(\hat{p}^{(l)}\right) = 0$ or both and $S_{EPT}(p^{(l)}, x^{(l)}) \geq \epsilon$ hold for all $l \in \mathbb{N}$. Then it follows from **(C2)** that $\lim_{l\to\infty} S_{EPT}\left(p^{(l)}, x^{(l)}\right) = 0$, which contradicts our hypothesis that $S_{EPT}(p^{(l)}, x^{(l)}) \geq \epsilon$ holds for all $l \in \mathbb{N}$. $\square$

C.12 Proof of Proposition 4.3.5

According to Proposition 4.2.6, we can choose a storage function for the pairwise comparison dynamics as

$$S_{PC}(p, x) = \sum_{i=1}^{n} \sum_{j=1}^{n} S_j(p_j - p_i, x_i) \tag{C.27}$$

The partial derivative of (C.27) with respect to x satisfies[2]

$$\nabla_x^T S_{PC}(p, x) V(p, x) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left[x_i \varrho_j(p_j - p_i) \sum_{k=1}^{n} \int_{p_k - p_i}^{p_k - p_j} \varrho_k(s)\,\mathrm{d}s \right] \tag{C.28}$$

To prove the Proposition, we show that **(A1)** and **(A2)** of Theorem 4.3.1 hold. We first show that the contrapositive of **(A1)** of Theorem 4.3.1 is true. Suppose that for some $\epsilon > 0$, it holds that $S_{PC}(p, x) < \epsilon$. Then $S_j(p_j - p_i, x_i) < \epsilon$ holds for all i, j in $\{1, \cdots, n\}$, and by **(C1)**, there exists $\delta' > 0$ for which $x_i \varrho_j(p_j - p_i) < \delta'$ holds for all $i, j \in \{1, \cdots, n\}$. From the following inequality

$$|V_i(p, x)| \leq \sum_{j=1}^{n} x_j \varrho_i(p_i - p_j) + \sum_{j=1}^{n} x_i \varrho_j(p_j - p_i) \tag{C.29}$$

we have that $|V_i(p, x)| < 2n \cdot \delta'$ which implies that $\|V(p, x)\| < 2n^{3/2} \cdot \delta' = \delta$. This shows that the contrapositive of **(A1)** is true.

Next, we consider the contrapositive of **(A2)**. Suppose that $S_{PC}(p, x) \geq \epsilon$ for some

[2] This fact directly follows from the analysis in the proof of Theorem 7.1 in [14]

$\epsilon > 0$. We note that according to (C.27), there exists $\delta_1 > 0$, e.g., $\delta_1 = \frac{\epsilon}{n^2}$, for which

$$\max_{i,j\in\{1,\cdots,n\}} S_j\left(p_j - p_i, x_i\right) = \max_{i,j\in\{1,\cdots,n\}} x_i \int_0^{p_j - p_i} \varrho_j(s)\,\mathrm{d}s \geq \delta_1 \tag{C.30}$$

holds. Let $i^*, j^* \in \{1, \cdots, n\}$ be indices that satisfy

$$x_{i^*} \int_0^{p_{j^*} - p_{i^*}} \varrho_{j^*}(s)\,\mathrm{d}s = \max_{i,j\in\{1,\cdots,n\}} x_i \int_0^{p_j - p_i} \varrho_j(s)\,\mathrm{d}s \tag{C.31}$$

Then by the contrapositive of **(C2)**, there exists $\delta_2 > 0$ for which it holds that

$$\varrho_{j^*}(p_{j^*} - p_{i^*}) \geq \delta_2 \tag{C.32}$$

Since $x_i \varrho_j(p_j - p_i) \int_{p_k - p_i}^{p_k - p_j} \varrho_k(s)\,\mathrm{d}s \leq 0$ holds, according to (C.28), we can derive the following:

$$\nabla_x^T S_{PC}(p,x) V(p,x) \leq x_{i^*} \varrho_{j^*}(p_{j^*} - p_{i^*}) \int_{p_{k^*} - p_{i^*}}^{p_{k^*} - p_{j^*}} \varrho_{k^*}(s)\,\mathrm{d}s$$

for any i^*, j^*, k^* in $\{1, \cdots, n\}$. In particular, we choose i^*, j^* that satisfy (C.31) and $k^* = j^*$. Then, in conjunction with (C.30) and (C.32), we have that

$$\nabla_x^T S_{PC}(p,x) V(p,x) \leq x_{i^*} \varrho_{j^*}(p_{j^*} - p_{i^*}) \int_{p_{j^*} - p_{i^*}}^{0} \varrho_{j^*}(s)\,\mathrm{d}s \leq -\delta_1 \cdot \delta_2 = -\delta$$

This verifies **(A2)** of Theorem 4.3.1. This completes the proof of the Proposition. $\square$

C.13 Proof of Proposition 4.3.7

From Proposition 4.2.7, we have seen that

$$S_{PBR}(p,x) = \max_{y \in \mathrm{int}(\mathbb{X})} \left(p^T y - v(y)\right) - \left(p^T x - v(x)\right)$$

is a storage function of the PBR dynamics. We verify that the assumptions **(A1)** and **(A2)** of Theorem 4.3.1 hold.

The verification of **(A1)** directly follows from the facts that $\|V(p,x)\| \leq \|C(p)\| + \|x\|$ and $C(p)$ belongs to $\mathbb{X}$ for every p in $\mathbb{R}^n$. To show **(A2)**, consider the following relations:

$$\begin{aligned} S_{PBR}(p,x) &= \left(p^T C(p) - v(C(p))\right) - \left(p^T x - v(x)\right) \\ &= p^T \left(C(p) - x\right) - \left(v(C(p)) - v(x)\right) \\ &\overset{(1)}{\leq} \left(p - \nabla_x v(x)\right)^T \left(C(p) - x\right) \\ &\overset{(2)}{=} -\nabla_x^T S_{PBR}(p,x) V(p,x) \end{aligned} \tag{C.33}$$

To obtain (1), we use the fact that v is a convex function; and to show (2), we use (C.8) in the proof of Proposition 4.2.7. Therefore, from (C.33), we conclude that **(A2)** is true. $\square$

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