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# Two Simple Residual Bounds for the Eigenvalues of Hermitian Matrices

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ABSTRACT

Let $A$ be Hermitian and let the orthonormal columns of $X$ span an approximate invariant subspace of $X$. Then the residual $R = AX - XM$ ($M = X^{\mathrm{H}}AX$) will be small. The theorems of this paper bound the distance of the spectrum of $M$ from the spectrum of $A$ in terms of appropriate norms of $R$.

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# Two Simple Residual Bounds for the Eigenvalues of Hermitian Matrices

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Let $A$ be a Hermitian matrix with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. If $X$ is a matrix with orthonormal columns that spans an invariant subspace of $A$ and

$$M = X^{\mathrm{H}}AX, \tag{1}$$

then $AX - XM = 0$.

Now suppose that the columns of $X$ span an *approximate* invariant subspace of $A$. Then the matrix

$$R = AX - XM$$

will be small, say in the spectral norm $\|\cdot\|$ defined by $\|R\| = \max_{\|x\|=1} \|Rx\|$, where $\|x\|$ is the Euclidean norm of $x$.[1] If the eigenvalues of $M$ are $\mu_1 \geq \cdots \geq \mu_k$, then we should expect the $\mu_i$ to be near $k$ of the $\lambda_i$. The problem treated in this note is to derive a bound in terms of the matrix $R$.

An important result, due to Kahan [3] (see also [6, p.219]) states that there are eigenvalues $\lambda_{j_1}, \ldots, \lambda_{j_k}$ of $A$ such that

$$|\mu_i - \lambda_{j_i}| \leq \|R\|, \qquad i = 1, \ldots, k. \tag{2}$$

If nothing further is known about the spectrum of $A$, this bound is generally satisfactory, although it can be improved somewhat [5]. However, it frequently happens (e.g., in the Lanczos algorithm or simultaneous iteration [6, Ch.13-14]) that we know that $n-k$ of the eigenvalues of $A$ are well separated from the eigenvalues of $M$: specifically, if we know that

> there is a number $\delta > 0$ such that exactly $n-k$ of the eigenvalues of $A$ lie outside the interval $[\mu_k - \delta, \mu_1 + \delta]$, (3)

then the bound in (2) can be replaced by a bound of order $\|R\|^2$. Bounds of the kind have been given by Temple, Kato, and Lehman (see [6, Ch.10] and [1, §6.5]).

[1] In fact, the choice (1) of $M$ minimizes $\|R\|$, although we will not make use of this fact here.

Early bounds of this kind, dealt only with a single eigenvalue and eigenvector. Lehman's bounds are in some sense optimal, but are quite complicated.

The purpose of this note is to give two other bounds derived from bounds on the accuracy of the column space of $X$ as an invariant subspace of $A$. They are very simple to state and yet are asymptotically sharp. In addition they can be established by appealing to results readily available in the literature.

**Theorem 1.** *With the above definitions, assume that $A$ and $M$ satisfy (3). If*

$$\rho \equiv \frac{\|R\|}{\delta} < 1,$$

*then there is an index $j$ such that $\lambda_j, \ldots, \lambda_{j+k-1} \in (\mu_k - \delta, \mu_1 + \delta)$ and*

$$|\mu_i - \lambda_{j+i-1}| \le \frac{1}{1-\rho^2}\frac{\|R\|^2}{\delta}, \qquad i = 1, \ldots, k.$$

**Proof.** Let $(X\ Y)$ be unitary. Then

$$\begin{pmatrix} X^{\mathrm{H}} \\ Y^{\mathrm{H}} \end{pmatrix} A(X\ Y) = \begin{pmatrix} M & S^{\mathrm{H}} \\ S & N \end{pmatrix}$$

where $\|S\| = \|R\|$. By the "$\sin\Theta$" theorem of Davis and Kahan [2] there is a matrix $P$ satisfying

$$\|P(I + P^{\mathrm{H}}P)^{\frac{1}{2}}\| \le \rho. \tag{4}$$

such that the columns of

$$\hat{X} = (X + YP)(I + P^{\mathrm{H}}P)^{-\frac{1}{2}}$$

(which are are orthonormal) span an invariant subspace of $A$. From (4) it follows that

$$\frac{\|P\|}{\sqrt{1 + \|P\|^2}} \le \rho,$$

and since $\rho < 1$

$$\|P\| \le \frac{\rho}{\sqrt{1-\rho^2}}. \tag{5}$$

Let $\hat{Y} = (Y - XP^{\mathrm{H}})(I + PP^{\mathrm{H}})^{-\frac{1}{2}}$. Then $(\hat{X}\ \hat{Y})$ is unitary. Since the columns of $\hat{X}$ span an invariant subspace of $A$, we have $\hat{Y}^{\mathrm{H}}A\hat{X} = 0$. Hence

$$\begin{pmatrix} \hat{X}^{\mathrm{H}} \\ \hat{Y}^{\mathrm{H}} \end{pmatrix} A(\hat{X}\ \hat{Y}) = \begin{pmatrix} \hat{M} & 0 \\ 0 & \hat{N} \end{pmatrix}.$$

In [7] it is shown that

$$\hat{M} = (I + P^{\mathrm{H}}P)^{\frac{1}{2}}(M + S^{\mathrm{H}}P)(I + P^{\mathrm{H}}P)^{-\frac{1}{2}}.$$

The eigenvalues of $\hat{M}$ are eigenvalues of $A$. Since $\rho < 1$ it follows from (2), they lie in the interval $(\mu_k - \delta, \mu_1 + \delta)$, and hence are $\lambda_j, \ldots, \lambda_{j+k-1}$ for some index $j$. By a result of Kahan [4] on non-Hermitian perturbations of Hermitian matrices,

$$|\mu_i - \lambda_{j+i-1}| \leq \|(I + P^{\mathrm{H}}P)^{\frac{1}{2}}\|\|(I + P^{\mathrm{H}}P)^{-\frac{1}{2}}\|\|S\|\|P\|, \qquad i = 1, \ldots, k.$$

The theorem now follows on noting that $\|(I + P^{\mathrm{H}}P)^{-\frac{1}{2}}\| \leq 1$ and inserting the bound (5) for $\|P\|$. ■

Two remarks. First, the theorem extends to operators in Hilbert space, provided $X$ (now itself an operator) has a finite dimensional domain. Second, the bound is asymptotically sharp, as may be seen by letting $X = (1\ 0)^{\mathrm{T}}$ and

$$A = \begin{pmatrix} 0 & \epsilon \\ \epsilon & 1 \end{pmatrix}$$

(the eigenvalues of $A$ are asymptotic to $\epsilon^2$ and $1 - \epsilon^2$).

The requirement (3) unfortunately does not allow the eigenvalues of $M$ to be scattered through the spectrum of $A$. If we pass to the Frobenius norm defined by $\|X\|_{\mathrm{F}}^2 = \mathrm{trace}(X^{\mathrm{H}}X)$, then we can obtain a Hoffman-Wielandt type residual bound. Specifically, if

$$\delta = \min\{|\lambda_i - \mu_j| : \lambda_i \in \lambda(A), \mu_j \in \lambda(M)\} > 0, \tag{6}$$

then a variant of the $\sin\Theta$ theorem shows that there is a matrix $P$ satisfying

$$\|P(I + P^{\mathrm{H}}P)^{\frac{1}{2}}\| \leq \|P(I + P^{\mathrm{H}}P)^{\frac{1}{2}}\|_{\mathrm{F}} \leq \frac{\|R\|_{\mathrm{F}}}{\delta}$$

such that the columns of

$$\hat{X} = (X + YP)(I + P^{\mathrm{H}}P)^{-\frac{1}{2}}$$

span an invariant subspace of $A$. By a variant of Kahan's theorem due to Sun [9, 8], the eigenvalues $\lambda_{j_1}, \ldots, \lambda_{j_k}$ of $\hat{M}$ may be ordered so that

$$\sqrt{\sum_{i=1}^{k}(\mu_i - \lambda_{j_i})^2} \leq \|(I + P^{\mathrm{H}}P)^{\frac{1}{2}}\|\|(I + P^{\mathrm{H}}P)^{-\frac{1}{2}}\|\|S\|_{\mathrm{F}}\|P\|.$$

Hence we have the following theorem.

**Theorem 2.** *With the above definitions, assume that $A$ and $M$ satisfy (6). If*

$$\rho_{\mathrm{F}} \equiv \frac{\|R\|_{\mathrm{F}}}{\delta} < 1,$$

*then there are eigenvalues $\lambda_{j_1}, \ldots, \lambda_{j_k}$ of $A$ such that*

$$\sqrt{\sum_{i=1}^{k}(\mu_i - \lambda_{j_i})^2} \leq \frac{1}{1-\rho_{\mathrm{F}}^2}\frac{\|R\|_{\mathrm{F}}^2}{\delta}.$$

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