ABSTRACT

Title of dissertation: THE TAYLOR-COUETTE PROBLEM FOR FLOW IN A DEFORMABLE CYLINDER

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The Taylor-Couette problem is a fundamental example in bifurcation theory and hydrodynamic stability, and has been the subject of over 1500 papers. This thesis treats a generalization of this problem in which the rigid outer cylinder is replaced by a deformable (nonlinearly viscoelastic) cylinder whose motion is not prescribed, but responds to the forces exerted on it by the moving liquid. The inner cylinder is rigid and rotates at a prescribed angular velocity, driving the liquid, which in turn drives the deformable cylinder. The motion of the outer cylinder is governed by a geometrically exact theory of shells and the motion of the liquid by the Navier-Stokes equations, where the domain occupied by the liquid depends on the deformation of the outer cylinder.

This thesis treats the stability of Couette flow, a steady solution of the nonlinear fluid-solid system that can be found analytically, first with respect to perturbations that are independent of z, then with respect to axisymmetic perturbations. The linearized stability problems are governed by quadratic eigenvalue problems. For each problem, this thesis gives a detailed characterization of how the spectrum

of the linearized operator depends on the control parameter, which is the angular velocity of the rigid inner cylinder. In particular, there are theorems detailing how the eigenvalues cross the imaginary axis. The spectrum is computed by a mixed Fourier-finite element method. The spectral properties determine the conditions under which the system loses its linearized stability. The same conditions support theorems on nonlinear stability. New physical phenomena are discovered that are not observed in the classical Taylor-Couette problem. The fluid-solid interaction models that are developed have applications in structural engineering and human physiology.

THE TAYLOR-COUETTE PROBLEM FOR FLOW IN A DEFORMABLE CYLINDER

by

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Table of Contents

Notation

Linear Algebra. We employ Gibbs notation (see Antman (2005, Chapter 11)) for vectors and tensors: Vectors, which are elements of Euclidean n-space $\mathbb{E}^n$, and vector-valued functions are denoted by lower-case, italic, bold-face symbols. The dot product of (vectors) $\boldsymbol{v}$ and $\boldsymbol{u}$ is denoted by $\boldsymbol{v} \cdot \boldsymbol{u}$. A (second-order) tensor is just a linear transformation from $\mathbb{E}^n$ to itself. The value of tensor $\boldsymbol{A}$ at vector $\boldsymbol{u}$ is denoted $\boldsymbol{A} \cdot \boldsymbol{u}$ or $\boldsymbol{A}\boldsymbol{u}$ and the product of $\boldsymbol{A}$ and $\boldsymbol{B}$ is denoted $\boldsymbol{A} \cdot \boldsymbol{B}$ or $\boldsymbol{A}\boldsymbol{B}$. The transpose of $\boldsymbol{A}$ is denoted $\boldsymbol{A}^*$. We write $\boldsymbol{v} \cdot \boldsymbol{A} = \boldsymbol{A}^* \cdot \boldsymbol{v}$. The inner product of $\boldsymbol{A}$ and $\boldsymbol{B}$ (which equals the trace of $\boldsymbol{A} \cdot \boldsymbol{B}^*$) is denoted $\boldsymbol{A} : \boldsymbol{B}$. The identity tensor is denoted by $\boldsymbol{I}$. The dyadic product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is the tensor denoted $\boldsymbol{a}\boldsymbol{b}$ (in place of the more usual $\boldsymbol{a} \otimes \boldsymbol{b}$), which is defined by $(\boldsymbol{a}\boldsymbol{b}) \cdot \boldsymbol{v} = (\boldsymbol{b} \cdot \boldsymbol{v})\boldsymbol{a}$ for all $\boldsymbol{v}$. If $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is an orthonormal basis for $\mathbb{E}^3$, then these definitions imply that $\boldsymbol{I} = \boldsymbol{e}_1\boldsymbol{e}_1 + \boldsymbol{e}_2\boldsymbol{e}_2 + \boldsymbol{e}_3\boldsymbol{e}_3$. The trace $\operatorname{tr} \boldsymbol{A} := \boldsymbol{I} : \boldsymbol{A}$.

Calculus. If $\boldsymbol{u} \mapsto \boldsymbol{f}(\boldsymbol{u})$ is (Fréchet) differentiable at $\boldsymbol{v}$, then its differential in the direction $\boldsymbol{h}$ is

$$\frac{d}{dt}\boldsymbol{f}(\boldsymbol{v} + t\boldsymbol{h})\bigg|_{t=0} =: \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}(\boldsymbol{v}) \cdot \boldsymbol{h} \equiv [\partial \boldsymbol{f}(\boldsymbol{v})/\partial \boldsymbol{u}] \cdot \boldsymbol{h} \equiv \boldsymbol{f}_{\boldsymbol{u}}(\boldsymbol{v}) \cdot \boldsymbol{h}.$$

The tensor $\partial \boldsymbol{f}(\boldsymbol{v})/\partial \boldsymbol{u}$ is the (Fréchet) derivative or the transposed gradient of $\boldsymbol{f}$ at $\boldsymbol{u}$. The partial derivative of a function f with respect to a scalar argument such as t is denoted by either f_t or $\partial_t f$. The operator ∂_t is assumed to apply only to the term immediately following it. We sometimes denote the function $\boldsymbol{u} \mapsto \boldsymbol{f}(\boldsymbol{u})$ by $\boldsymbol{f}(\cdot)$. The divergence of $\boldsymbol{f}$ at $\boldsymbol{x}$ is $\operatorname{tr} \boldsymbol{f}_{\boldsymbol{x}}(\boldsymbol{x}) \equiv \boldsymbol{I} : \boldsymbol{f}_{\boldsymbol{x}}(\boldsymbol{x})$.

Complex Variables. We denote the real part of a complex number z by $\operatorname{Re}(z)$. The complex conjugate of z is denoted $\bar{z}$.

Bases. Let $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ be a right-handed orthonormal basis for Euclidean 3-space. For any angle χ we define the vectors

$$\begin{aligned}
\boldsymbol{e}_1(\chi) &:= \cos\chi\, \boldsymbol{i} + \sin\chi\, \boldsymbol{j}, \\
\boldsymbol{e}_2(\chi) &:= -\sin\chi\, \boldsymbol{i} + \cos\chi\, \boldsymbol{j} \equiv \boldsymbol{k} \times \boldsymbol{e}_1(\chi), \\
\boldsymbol{e}_3(\chi) &:= \boldsymbol{k}.
\end{aligned}$$

Symbols. The following tables list the bilinear forms and function spaces that are used and where they are introduced (equation number). Some symbols have a different meaning in different parts of the thesis, in which case there is more than one equation listed for where they are defined.

Bilinear forms:

a_f	(2.10.32)	a_s	(2.10.39)	a_0	(2.11.9), (5.10.3)
$\hat{a}_0$	(2.11.19)	$\tilde{a}_0$	(2.11.42), (5.10.12)	a_0^k	(2.11.52), (5.10.24)
$\hat{a}_0^k$	(2.11.65)	a_1	(2.11.10), (5.10.4)	$\tilde{a}_1$	(2.11.43), (5.10.13)
a_1^k	(2.11.53), (5.10.25)	a_2	(2.11.11), (5.10.5)	$\tilde{a}_2$	(2.11.44), (5.10.14)
a_2^k	(2.11.54), (5.10.26)	b	(2.11.11), (5.10.6)	$\tilde{b}$	(2.11.45), (5.10.15)
b^k	(2.11.55), (5.10.27)	b_1^k	(2.12.49)	b_2^k	(2.12.49)
c^k	(2.12.66)	d_1	(2.11.23)	d_1^k	(2.11.69)
d_2	(2.11.23)	d_2^k	(2.11.69)	d_3^k	(2.11.69)

Function spaces:

$H_0^m(\mathit{\Omega};\mathrm{div})$	(2.10.22)	$H_a^m(\mathit{\Omega};\mathrm{div})$	(2.10.22)
$H_a^1(\mathit{\Omega})$	(2.11.6), (5.10.1)	$H_\mathrm{S}^m(\mathbb{T}_{2\pi})$	(2.10.22)
$H_\mathrm{S}^1(\mathit{\Gamma}_R)$	(5.10.1)	Π	(2.11.6), (5.10.1)
Π^k	(2.11.49), (5.10.20)	Π_h^k	(2.12.29)
$\Pi^m(\mathit{\Omega})$	(2.10.22)	$\tilde{\Pi}$	(2.11.40), (5.10.10)
$\mathcal{V}_1$	(2.11.6), (5.10.1)	$\mathcal{V}_2$	(2.11.6), (5.10.1)
V_1	(2.11.38), (5.10.10)	V_2	(2.11.39), (5.10.10)
V_1^k	(2.11.49), (5.10.20)	V_2^k	(2.11.49), (5.10.20)
$V_{1,h}^k$	(2.12.29)	$V_{2,h}^k$	(2.12.29)
V_h	(2.12.31)	$\mathcal{Z}_1$	(2.11.17)
$\mathcal{Z}_2$	(2.11.17)	$\mathcal{Z}$	(2.11.26)
$\mathcal{Z}_0$	(2.11.26)	Z_1^k	(2.11.63)
Z_2^k	(2.11.63)	Z^k	(2.11.72)
Z_0^k	(2.11.72)		

The following tables list the principal symbols used, their meanings, and where they are introduced (section or equation number). Some symbols have a different meaning in different parts of the thesis. This is indicated in the table.

γ	kinematic viscosity of the fluid	(2.3.1)
γ_0	trace operator	Sec. 2.10
γ_R	restricted trace operator	Sec. 2.10
η	strain variable for the ring	(3.2.4)
	strain variable for the shell	(5.2.5)
θ	characterizes orientation of material fibers	(3.2.3), (5.2.4)
λ	eigenvalue representing the perturbation growth rate	Sec. 2.9, 3.8, 5.8
μ	strain variable for the ring	(3.2.4)
	strain variable for the shell	(5.2.6)
$\tilde{\mu}$	dynamic viscosity of the fluid	(2.3.1)
ν	stretch of the string	(2.2.3)
	strain variable for the ring	(3.2.4)
	strain variable for the shell	(5.2.5)
ρ	density of the fluid	(2.3.1)
ϱ	density of the 2-dimensional elastic body	Sec. 4.2
$2\varrho h$	density of the shell	(5.2.8)
ϱA	density of the string	(2.2.6)
	density of the ring	(3.2.9)
ϱI	first moment of mass of the ring & shell	(3.2.9), (5.2.8)
ϱJ	second moment of mass of the ring & shell	(3.2.12), (5.2.8)
σ	strain variable for the shell	(5.2.6)
$\varSigma$	length of the vector $\boldsymbol{m}_2$	Sec. 5.2
$\hat{\varSigma}$	constitutive function for the shell	(5.2.12)
$\boldsymbol{\Sigma}$	Cauchy stress tensor	(2.3.3)
τ	strain variable for the shell	(5.2.6)
ω	angular velocity of the inner cylinder	Ch. 1
ω_{crit}	critical values of ω : $\text{Re}(\lambda(\omega_{\text{crit}})) = 0$	Sec. 2.10
$\varOmega$	the annulus $\{a < \lvert\boldsymbol{x}\rvert < R\}$	Sec. 2.10
	period cell for the fluid	(5.9.1)

a	radius of the rigid cylinder	Sec. 2.1
$\boldsymbol{a}$	unit vector orthogonal to $\boldsymbol{d}$	(3.2.2), (5.2.3)
$\boldsymbol{C}$	Cauchy-Green deformation tensor	Sec. 4.2
$\boldsymbol{d}$	characterizes orientation of material fibers	Sec. 3.2, 5.2
$\boldsymbol{D}$	symmetric part of $\partial\boldsymbol{v}/\partial\boldsymbol{x}$	(2.3.2)
$\boldsymbol{f}$	force of the fluid on the deformable cylinder	(2.2.6), (3.2.9), (5.2.8)
$\boldsymbol{F}$	deformation gradient	Sec. 4.2
g	Lagrange multiplier	(5.2.8)
h	mesh size	Sec. 2.12
	thickness of the 2-dimensional annulus	Sec. 4.2

$\hat{H}$	constitutive function for the ring	(3.2.21)
	constitutive function for the shell	(5.2.12)
k	Fourier wave number	(2.10.16)
$\boldsymbol{m}_1$	internal contact couple in the shell	Sec. 5.2
$\boldsymbol{m}_2$	internal contact couple in the shell	Sec. 5.2
M	internal contact couple in the ring	(3.2.12)
	component of the vector $\boldsymbol{m}_1$	Sec. 5.2
$\hat{M}$	constitutive function for the ring	(3.2.21)
	constitutive function for the shell	(5.2.12)
$\boldsymbol{n}$	internal contact force in the string	(2.2.6)
	internal contact force in the ring	(3.2.9)
$\boldsymbol{n}_1$	internal contact force in the shell	Sec. 5.2
$\boldsymbol{n}_2$	internal contact force in the shell	Sec. 5.2
$\boldsymbol{n}_\Omega$	unit outer normal vector to $\partial\Omega$	Sec. 2.10
N	length of the vector $\boldsymbol{n}$ (for the string)	(2.2.7)
(N, H)	components of the vector $\boldsymbol{n}$ (for the ring)	(3.2.10)
	components of the vector $\boldsymbol{n}_1$	Sec. 5.2
$\hat{N}$	constitutive function for the string	(2.2.8)
	constitutive function for the ring	(3.2.21)
	constitutive function for the shell	(5.2.12)
p	fluid pressure (normalized by density)	(2.3.1)
$\boldsymbol{p}$	position of the 2-dimensional elastic body	Sec. 4.2
(q, ψ)	polar coordinates for $\boldsymbol{r}$ (for the string)	(2.2.5)
(q, ζ)	polar coordinates for $\boldsymbol{r}$ (for the shell)	(5.2.2)
R	radius of deformable cylinder, Couette solution	Sec. 2.5
$\boldsymbol{r}$	position of the string	Sec. 2.2
	position of the ring	Sec. 3.2
	position of the shell	Sec. 5.2
s	identifies material points of the string & ring	Sec. 2.2, 3.2
(s, ϕ)	identifies material points of the shell	Sec. 5.2
$\boldsymbol{S}$	second Piola-Kirchhoff stress tensor	Sec. 4.2
$\hat{\boldsymbol{S}}$	constitutive function for the 2-dimensional body	Sec. 4.2
t	time	Sec. 2.2
T	length of the vector $\boldsymbol{n}_2$	Sec. 5.2
$\hat{T}$	constitutive function for the shell	(5.2.12)
$\boldsymbol{T}$	first Piola-Kirchhoff stress tensor	Sec. 4.2
$\hat{\boldsymbol{T}}$	constitutive function for the 2-dimensional body	Sec. 4.2
(u, v)	polar coordinates for $\boldsymbol{v}$	(2.3.11)
(u, v, w)	polar coordinates for $\boldsymbol{v}$	(5.3.1)
$\boldsymbol{v}$	velocity of the fluid	(2.3.1)
$\boldsymbol{x}$	point in the fluid domain	(2.3.1)
Z	axial period	(5.5.1)

Chapter 1

Introduction

Background and Motivation. The classical Taylor-Couette problem concerns the motion of a viscous incompressible fluid in the region between two rigid coaxial cylinders, which rotate at constant angular velocities. If, for example, the outer cylinder is held fixed and angular velocity of the inner cylinder is small, then laminar flow is observed. As the angular velocity of the inner cylinder is slowly increased past a critical value, the laminar flow destabilizes into a secondary steady flow. Increasing the angular velocity further produces a rich family of bifurcations and flows, e.g., periodic solutions, quasiperiodic solutions, and turbulence. Since Couette first performed this experiment 100 years ago, this fundamental problem in bifurcation theory has been the subject of over 1500 papers. See Chandrasekhar (1981) and Chossat & Iooss (1993) for introductions to the Taylor-Couette problem and Tagg (1992) for an extensive bibliography. Taylor (1923) conducted the first serious mathematical analysis of the problem.

This thesis treats a generalization of this problem in which the outer cylinder is a deformable (nonlinearly viscoelastic) shell. The motion of the shell is not prescribed, but responds to the forces exerted on it by the moving liquid. The inner cylinder is rigid and rotates at a prescribed angular velocity ω, driving the liquid, which in turn drives the deformable shell.

The motivation for this project, in addition to its connection with the classical Taylor-Couette problem, is to develop new modelling and analysis techniques for fluid-structure interaction problems, which are of great importance in structural engineering and biomechanics. For example, the fluid-solid interaction models derived here could be used to model blood flow in arteries or biomembranes. Dynamical problems for the interaction of fluids with deformable solids undergoing large displacements are notoriously difficult to analyze. This thesis treats a rare instance of such an interaction in which the geometry is simple, the physics is interesting, and the behavior of solutions can be determined by mathematical analysis, without an immediate recourse to numerical computation.

There has been great interest in fluid-structure problems in the past few years. Existence theorems for the interaction of fluids with deformable bodies are given in Chambolle et al. (2005), Coutand & Shkoller (2005, 2006), and Cheng et al. (to appear). Numerical methods and eigenvalue problems governing linear stability of coupled fluid-structure systems have been studied by Planchard & Thomas (1991), Conca & Durán (1995), and Bermúdez & Rodríguez (2002). Applications are given in Quarteroni & Formaggia (2005), Shelley et al. (2005), and Čanić et al. (2006). This represents just a small sample of recent work.

Modelling. We limit our attention to two types of motion of the coupled fluid-solid system: cylindrical motions and axisymmetric motions. See Figure 1.1.1.

In Chapters 2–4 we consider cylindrical motions, where the deformable cylinder remains cylindrical, although not necessarily a circular cylinder, and there is no

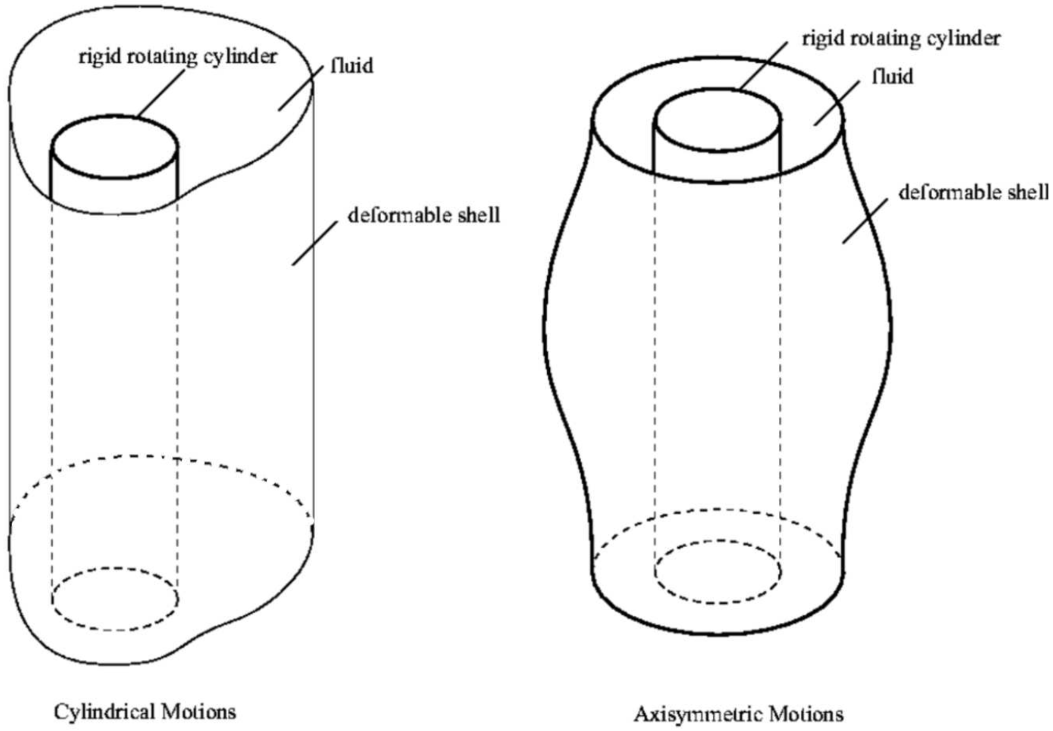

Figure 1.1.1: In Chapters 2–4 we consider cylindrical motions of the deformable cylinder (pictured left) and in Chapter 5 we consider axisymmetric motions of the deformable cylinder (pictured right).

motion of the fluid in the axial direction. Thus we can represent the system by a horizontal cross section, reducing the problem to two dimensions. We present three different models for a horizontal cross section of the deformable cylinder: We model a cross section as a viscoelastic string in Chapter 2, a viscoelastic ring in Chapter 3 (using a Cosserat rod theory), and a 2-dimensional elastic body in Chapter 4 (using 2-dimensional continuum mechanics).

In Chapter 5 we consider axisymmetric motions of the fluid-solid system, where the base surface of the deformable cylinder is axisymmetric and the components of the fluid velocity with respect to a polar basis are independent of the angle variable. Axisymmetric motions were the starting point for studying the classical Taylor-Couette problem. The deformable cylinder is modelled using a geometrically exact shell theory, the special Cosserat theory of shells, which accounts for flexure, base surface extension, and shear. The motion of the shell is governed by a quasilinear parabolic-hyperbolic system of partial differential equations. We assume that the shell is viscoelastic and work with a broad class of nonlinear constitutive functions.

For both the cylindrical and axisymmetric problems we assume that the fluid is viscous, incompressible, and Newtonian, so that its motion is described by the Navier-Stokes equations. These are coupled to the equations for the deformable body through the adherence boundary condition, which states that the velocity of the fluid at the boundary equals the velocity of the boundary, and through the traction condition, which states that the force of the fluid on the deformable cylinder is equal and opposite to the force of the deformable cylinder on the fluid. For the string, ring, and shell models, the traction condition is included as a body force term in the linear momentum equation for the deformable body. For the 2-dimensional elasticity model, the traction condition is included as a boundary condition. In total this thesis includes four different fluid-solid interaction models.

The Couette Steady Solution. For both the cylindrical and axisymmetric problems there exists a rigid Couette steady solution analogous to the Couette solution of the classical Taylor-Couette problem: The fluid streamlines are concentric circles and the deformable cylinder rotates rigidly with the same angular velocity ω as the rigid inner cylinder. See Sections 2.7, 3.6, 4.6, and 5.6.

We study the stability of the rigid Couette solution with respect to the prescribed angular velocity ω, which is taken to be the bifurcation parameter.

Linearization and the Quadratic Eigenvalue Problem. For both the cylindrical and axisymmetric problems problems, linearizing the equations of motion about the Couette steady solution yields a quadratic eigenvalue problem, which can be thought of as a perturbation of the Stokes eigenvalue problem with complicated boundary conditions (governing the motion of the shell); the eigenvalue parameter λ appears quadratically in the boundary conditions for the fluid. See Sections 2.9, 3.8, and 5.8.

Typically in hydrodynamic stability problems, the eigenvalues of the linearized problem migrate towards the right as a bifurcation parameter (ω in our case) is increased, and the way the eigenvalues cross the imaginary axis can yield important information about the stability and structure of solutions to the fully nonlinear problem.

Analytical Results: Cylindrical Motions of the Shell. Due to the complexity of the quadratic eigenvalue problem it is not possible to compute the eigenvalues analytically. Much information can be obtained, however, before turning to numerics. For the string model we prove that the eigenvalues λ cross the imaginary axis through the origin (Theorem (2.10.4)) and find an explicit formula for the critical values of ω at which the eigenvalues cross (Theorem (2.10.21)), highlighting the role of the material properties. Unlike the classical Taylor-Couette problem, it turns out that the Couette solution is unstable for all $\omega > 0$, and so not observable: We prove that $\lambda = 0$ is an eigenvalue when $\omega = 0$. (This is shown for the string, ring, and 2-dimensional elasticity models. See Sections 2.10, 3.9, and 4.7.) Numerical results in Section 2.13 indicate that for the string problem these eigenvalues move into the right half-plane as ω is increased. This instability occurs via a drift of the deformable string off-center, breaking axisymmetry. If this mode is stabilized by a suitable feedback control to keep the center of mass of the string at the origin, then we can study other bifurcations; numerical results show that as the control parameter ω is increased, complex-conjugate pairs of eigenvalues in the left half-plane move towards the origin. Such pairs meet at the origin and then move into the right half-plane, whereupon they cease to be real. This is known as a Takens-Bogdanov bifurcation. Mathematically the unstable mode can be stabilized for the 3-dimensional problem by seeking axisymmetric solutions, which is what we do in Chapter 5.

For the string model we also characterize the spectrum of the linear operator. See Theorems (2.10.23) and (2.12.69).

Analytical Results: Axisymmetric Motions of the Shell. As for the string problem, we prove that the eigenvalues λ of the quadratic eigenvalue problem cross the imaginary axis through the origin. See Theorem (5.9.18). We also find a shear instability, Theorem (5.9.19), and a cubic equation for the critical values of ω^2 at which the eigenvalues cross, equation (5.9.25). To determine whether the eigenvalues λ cross through the origin along the real axis (suggesting a steady state bifurcation) or in complex conjugate pairs (as for the string problem) requires a numerical study, which we perform in Section (5.11) and describe below.

Numerical Results: Cylindrical Motions of the Shell. Due to the coupling terms between the fluid and deformable solid, deriving a well-posed weak formulation of the quadratic eigenvalue problem is tricky. The pressure terms must be treated with care and different function spaces must used for the eigenfunctions and the test functions. See Section 2.11.

For the string problem the eigenvalues of this non-selfadjoint problem are computed using the Fourier-finite element method: Fourier series in the angle variable are used to reduce the partial differential equations on an annulus to ordinary differential equations in the radial variable r, which are discretized using the 1-dimensional finite element method. Employing the direct QZ eigensolver to solve the matrix eigenvalue problem leads to a fast algorithm. For stability of this mixed problem we used the Taylor-Hood finite element. (Only very recently were discrete inf-sup conditions proved for the Fourier-finite element discretization of Stokes problem in axisymmetric domains. See Belhachmi et al. (2006a, 2006b).) Our discretized eigenvalue problem, like the discretized Stokes eigenvalue problem, has many infinite eigenvalues (corresponding to the zero eigenvalues of a related problem), which must be computed in a stable manner, using the shift and invert transformation, or factored out by performing the linear algebra on the divergence-free subspace. See Cliffe, Garratt, and Spence (1994). We verify our computations with COMSOL Multiphyics, using the exact formula for the critical values of ω (Theorem (2.10.21)), and using Bessel functions to obtain a transcendental equation for the eigenvalues for the case $k = 0$. See Section 2.13.

We prove continuous and discrete inf-sup conditions for the bilinear forms appearing in the weak formulation (Theorems (2.11.56), (2.11.64), (2.12.47), and (2.12.63)). Rate of convergence estimates for the eigenvalues then follow from results in Babuška & Osborn (1991) and Kolata (1976). See Section 2.12.

Numerical Results: Axisymmetric Motions of the Shell. The eigenvalues for the axisymmetric problem are computed using the Fourier-finite element method with the QZ eigensolver, as for the string problem. We find that the first eigenvalue to cross the imaginary axis is real, indicating a steady state bifurcation. See Section (5.11). As ω crosses its critical value ω_{crit}, the Couette steady solution destabilizes into a new steady solution where the deformable shell is buckled, but the fluid steamlines are still concentric circles. This is a new phenomenon not observed in

Chapters 2–4 or in the classical Taylor-Couette probelm.

Organization of the Thesis. In Chapters 2–4 we study cylindrical motions of the shell, modelling a horizontal cross section as a string (Chapter 2), a ring (Chapter 3), and a 2-dimensional body (Chapter 4). In Chapter 5 we study axisymmetric motions of the shell. The most important results are in Chapters 2 and 5. In Chapter 2 we carefully characterize the spectrum of the quadratic eigenvalue problem, derive a well-posed weak formulation, prove continuous and discrete inf-sup conditions, and uses these to prove convergence of the numerical method. In subsequent chapters we do not pause to use the techniques of Chapter 2 to check all of these details.

Chapter 2

Cylindrical Motions of the Shell: The String Model

2.1 Introduction

In this chapter we begin our study of cylindrical motions of the deformable shell, where there is no motion of the fluid in the axial direction and the deformable shell remains cylindrical, but not necessarily a circular cylinder. We can represent this system by a horizontal cross section, which reduces the problem to two dimensions. In this chapter we model a cross section of the deformable shell as a deformable string. This is the simplest model. In Chapters 3 and 4 we use more refined models; we model a cross section of the shell as a ring and a 2-dimensional elastic body. By definition, a string offers no resistance to bending, only to stretching.

Thus we study the motion of a viscous incompressible liquid in the region between a rigid circular disk of radius $a < 1$ rotating at a prescribed angular velocity ω and a viscoelastic string whose natural state is circle of radius 1. The motion of the string is not prescribed, but responds to the forces exerted on it by the moving liquid; the rigid disk drives the liquid, which in turn drives the deformable string.

We find a rigid Couette steady solution of this coupled system and analyze its stability with respect to the bifurcation parmeter ω. By linearizing the governing equations about this steady solution and seeking normal modes we arrive at a quadratic eigenvalue problem. In this chapter we adhere to high standards of mathematical hygiene; we characterize the spectrum of the quadratic eigenvalue problem, find a well-posed weak formulation, prove continuous and discrete inf-sup conditions, and apply the Galerkin approximation theory for polynomial eigenvalue problems to derive a convergent numerical scheme. In subsequent chapters we do not pause to check all these details.

The main results of this chapter are theorems detailing how the spectrum of the linearized operator depends on the control parameter, which is the angular velocity ω of the rigid inner cylinder, and the computation of the spectrum using a mixed Fourier-finite element method.

2.2 Formulation of the Equations for the String

In this section we summarize the theory of deformable strings from Antman (2005, Chapter 2).

Geometry of Deformation

Let $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ be a right-handed orthonormal basis for Euclidean 3-space. For any angle ψ we define the vectors

$$\boldsymbol{e}_1(\chi) := \cos\chi\, \boldsymbol{i} + \sin\chi\, \boldsymbol{j}, \quad \boldsymbol{e}_2(\chi) := -\sin\chi\, \boldsymbol{i} + \cos\chi\, \boldsymbol{j} \equiv \boldsymbol{k} \times \boldsymbol{e}_1(\chi). \tag{2.2.1}$$

The reference configuration of the string is a circle of radius 1, given parametrically by

$$\boldsymbol{r}^\circ(s) = \boldsymbol{e}_1(s). \tag{2.2.2}$$

The arc-length parameter $s \in [0, 2\pi]$ identifies material points of the string, with the points 0 and 2π identified. The position of material point s at time t is $\boldsymbol{r}(s,t)$. The curve $\boldsymbol{r}(\cdot, t)$ is assumed to lie in the $\{\boldsymbol{i}, \boldsymbol{j}\}$-plane for each t. The stretch $\nu(s,t)$ is defined by

$$\nu(s,t) := |\boldsymbol{r}_s(s,t)| \equiv \sqrt{\boldsymbol{r}_s \cdot \boldsymbol{r}_s}. \tag{2.2.3}$$

Note that $\nu = 1$ in the reference configuration. Since ν measures the stretch of the string, i.e., the local ratio of deformed to reference length of the string, we stipulate that $\nu > 0$. We require that the configuration satisfy the periodicity conditions

$$\boldsymbol{r}(2\pi, t) = \boldsymbol{r}(0,t), \qquad \boldsymbol{r}_s(2\pi, t) = \boldsymbol{r}_s(0,t). \tag{2.2.4}$$

Sometimes it will be convenient to work in polar coordinates. We define functions $q(s,t) := |\boldsymbol{r}(s,t)|$ and $\psi(s,t) \in [0, 2\pi)$ by

$$\boldsymbol{r}(s,t) =: q(s,t)\boldsymbol{e}_1(\psi(s,t) + \omega t). \tag{2.2.5}$$

Mechanics

Let $\boldsymbol{n}(\xi, t)$ be the internal contact force exerted at time t by the material of the string with $s \in (\xi, \xi + \varepsilon]$ on the material of the string with $s \in [\xi - \varepsilon, \xi]$ where ε is a sufficiently small positive number and this interpretation is independent of ε. Let $\boldsymbol{f}(s,t)$ be the force per unit reference length exerted by the fluid on material point s of the string at time t. We give an expression for the force $\boldsymbol{f}$ in Section 2.4. The string obeys the Linear Momentum Law

$$\varrho A \boldsymbol{r}_{tt} = \boldsymbol{n}_s + \boldsymbol{f} \tag{2.2.6}$$

where $(\varrho A)(s)$ is the mass density of the string per reference length. We assume that the string is uniform so that ϱA is constant and $2\pi\varrho A$ is the mass of the string. Equation (2.2.6) is derived by adding up all the forces on a segment $[s_1, s_2]$ of the string, $\boldsymbol{n}(s_2, t) - \boldsymbol{n}(s_1, t) + \int_{s_1}^{s_2} \boldsymbol{f}(s, t)\, ds$, and setting this equal to the rate of change of linear momentum of the segment, $\partial_t \int_{s_1}^{s_2} \varrho A \boldsymbol{r}_t(s, t)\, ds$. Differentiating the resulting equation with respect to s_2 gives (2.2.6).

We assume that the string can bend and stretch, but that it offers no resistance to bending, only to stretching. Thus the internal contact force $\boldsymbol{n}(s, t)$ is tangent to the string and has the form

$$\boldsymbol{n} =: N(s, t)\frac{\boldsymbol{r}_s}{|\boldsymbol{r}_s|}. \tag{2.2.7}$$

Constitutive Equations

We assume that the string is uniform and viscoelastic of strain-rate type, i.e., there is a function

$$\nu, \dot{\nu} \mapsto \hat{N}(\nu, \dot{\nu}) \tag{2.2.8}$$

such that

$$N(s, t) = \hat{N}(\nu(s, t), \nu_t(s, t)). \tag{2.2.9}$$

The superposed dot on the last argument of (2.2.8) has no operational significance; it merely identifies the argument of the constitutive function that is to be occupied by the time derivative of ν. This form of $\hat{N}$ is derived by starting with a general constitutive function of the form $N = \hat{N}(\boldsymbol{r}, \boldsymbol{r}_s, \boldsymbol{r}_t, t)$ and applying the Principle of Frame-Indifference, which requires that material properties be invariant under rigid motions. We assume that the constitutive function $\hat{N}$ is as smooth as necessary for our analysis.

We assume that the monotonicity conditions hold:

$$\hat{N}_\nu > 0 \quad \text{and} \quad \hat{N}_{\dot{\nu}} > 0. \tag{2.2.10}$$

These mean that increases in strain and strain-rate are each accompanied by an increase in stress. Similarly, it is expected that an extreme strain be accompanied by an extreme stress. Therefore we stipulate that the constitutive function satisfies the growth conditions

$$\hat{N}(\nu, \dot{\nu}) \longrightarrow \left\{ \begin{array}{c} +\infty \\ \\ -\infty \end{array} \right\} \quad \text{as} \quad \nu \longrightarrow \left\{ \begin{array}{c} +\infty \\ \\ 0 \end{array} \right\} \tag{2.2.11}$$

for fixed values of $\dot{\nu}$. Finally, we make the nonrestrictive assumption that the resultant vanishes when the body is at rest in the reference configuration:

$$\hat{N}(1, 0) = 0. \tag{2.2.12}$$

This means that the reference configuration of the string is a natural configuration for it.

2.3 Formulation of the Equations for the Fluid

Coordinate-free Equations

Let $\mathcal{D}(t)$ be the domain occupied by the fluid at time t. This is the region between the rigid disk of radius $a < 1$ and the curve $\boldsymbol{r}(\cdot, t)$. We assume that the fluid is viscous, incompressible, and homogeneous. We denote by

ρ	the density of the fluid,
$\tilde{\mu}$	the dynamic viscosity of the fluid,
γ	the kinemetic viscosity of the fluid, $\gamma = \tilde{\mu}/\rho$,
$\boldsymbol{v}(\boldsymbol{x}, t)$	the velocity of the fluid particle occupying position $\boldsymbol{x} \in \mathcal{D}(t)$ at time t,
$\rho p(\boldsymbol{x}, t)$	the pressure on the fluid particle occupying position $\boldsymbol{x} \in \mathcal{D}(t)$ at time t,
$\boldsymbol{\Sigma}(\boldsymbol{v}, p)$	the Cauchy stress tensor,
$\boldsymbol{D}(\boldsymbol{v})$	the symmetric part of the velocity gradient, which is defined by

(2.3.1)

$$\boldsymbol{D}(\boldsymbol{v}) = \frac{1}{2}\left[\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}} + \left(\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}}\right)^*\right] \tag{2.3.2}$$

where the asterisk denotes the transpose. (Note that the ρ for the density of the fluid differs from the ϱ appearing in the ϱA for the density per unit reference length of the string.) We assume that the fluid is Newtonian, so that the Cauchy stress $\boldsymbol{\Sigma}$ has the Navier-Stokes form

$$\boldsymbol{\Sigma}(\boldsymbol{v}, p) = -\rho p\, \boldsymbol{I} + 2\tilde{\mu}\boldsymbol{D}(\boldsymbol{v}). \tag{2.3.3}$$

The requirement that the fluid be incompressible is that $\nabla \cdot \boldsymbol{v} = 0$. In this case,

$$\tfrac{1}{\rho} \operatorname{div} \boldsymbol{\Sigma} = -\nabla p + \gamma \Delta \boldsymbol{v}, \tag{2.3.4}$$

so that the momentum equation is the Navier-Stokes equation

$$\begin{aligned} \boldsymbol{v}_t + \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}} \cdot \boldsymbol{v} = \tfrac{1}{\rho} \operatorname{div} \boldsymbol{\Sigma} = -\nabla p + \gamma \Delta \boldsymbol{v} \qquad &\text{in } \mathcal{D}(t), \\ \nabla \cdot \boldsymbol{v} = 0 \qquad \text{in } \mathcal{D}(t). & \end{aligned} \tag{2.3.5}$$

Polar Coordinates

We assign polar coordinates (r, ϕ) to a typical point $\boldsymbol{x}$ in the $\{\boldsymbol{i}, \boldsymbol{j}\}$-plane with respect to the basis $\{\boldsymbol{e}_1(\omega t), \boldsymbol{e}_2(\omega t)\}$ rotating with the rigid inner disk by

$$\boldsymbol{x} = r\,\boldsymbol{e}_1(\phi + \omega t) = r[\cos\phi\,\boldsymbol{e}_1(\omega t) + \sin\phi\,\boldsymbol{e}_2(\omega t)]. \tag{2.3.6}$$

The virtue of such a rotating basis is explained in Section 2.7. We denote by $\tilde{r}(\boldsymbol{x}) := |\boldsymbol{x}|$ and $\tilde{\phi}(\boldsymbol{x}, t) \in [0, 2\pi)$ the unique solution of (2.3.6) for r and ϕ in terms of $\boldsymbol{x}$ and t. Thus the functions satisfy the identity

$$\boldsymbol{x} =: \tilde{r}(\boldsymbol{x})\; \boldsymbol{e}_1(\tilde{\phi}(\boldsymbol{x}, t) + \omega t). \tag{2.3.7}$$

Differentiate (2.3.7) with respect to t to obtain

$$\frac{\partial \tilde{\phi}}{\partial t} = -\omega. \tag{2.3.8}$$

The Chain Rule yields

$$\begin{aligned}
\boldsymbol{I} &\equiv \boldsymbol{e}_1(\phi + \omega t)\boldsymbol{e}_1(\phi + \omega t) + \boldsymbol{e}_2(\phi + \omega t)\boldsymbol{e}_2(\phi + \omega t) \\
&= \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{x}} = \frac{\partial \boldsymbol{x}}{\partial r}\frac{\partial \tilde{r}}{\partial \boldsymbol{x}} + \frac{\partial \boldsymbol{x}}{\partial \phi}\frac{\partial \tilde{\phi}}{\partial \boldsymbol{x}} = \boldsymbol{e}_1(\phi + \omega t)\frac{\partial \tilde{r}}{\partial \boldsymbol{x}} + r\,\boldsymbol{e}_2(\phi + \omega t)\frac{\partial \tilde{\phi}}{\partial \boldsymbol{x}},
\end{aligned} \tag{2.3.9}$$

whence

$$\frac{\partial \tilde{r}}{\partial \boldsymbol{x}} = \boldsymbol{e}_1(\phi + \omega t), \quad \frac{\partial \tilde{\phi}}{\partial \boldsymbol{x}} = \frac{1}{r}\boldsymbol{e}_2(\phi + \omega t). \tag{2.3.10}$$

We write the fluid velocity $\boldsymbol{v}$ in the form

$$\boldsymbol{v}(r\,\boldsymbol{e}_1(\phi + \omega t), t) =: u(r, \phi, t)\boldsymbol{e}_1(\phi + \omega t) + v(r, \phi, t)\boldsymbol{e}_2(\phi + \omega t). \tag{2.3.11}$$

Therefore

$$\boldsymbol{v}(\boldsymbol{x}, t) = u(\tilde{r}(\boldsymbol{x}), \tilde{\phi}(\boldsymbol{x}, t), t)\; \boldsymbol{e}_1(\tilde{\phi}(\boldsymbol{x}, t) + \omega t) + v(\tilde{r}(\boldsymbol{x}), \tilde{\phi}(\boldsymbol{x}, t), t)\; \boldsymbol{e}_2(\tilde{\phi}(\boldsymbol{x}, t) + \omega t). \tag{2.3.12}$$

Differentiating (2.3.12) with respect to t, and using (2.3.8), we find that

$$\boldsymbol{v}_t = (u_\phi \tilde{\phi}_t + u_t)\boldsymbol{e}_1 + (v_\phi \tilde{\phi}_t + v_t)\boldsymbol{e}_2 = (u_t - \omega u_\phi)\boldsymbol{e}_1 + (v_t - \omega v_\phi)\boldsymbol{e}_2. \tag{2.3.13}$$

Here and through the rest of this subsection the argument of $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ is $\phi + \omega t$. The (transposed) gradient of $\boldsymbol{v}$ is given by the Chain Rule:

$$\begin{aligned} \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}} &= \frac{\partial [u\boldsymbol{e}_1 + v\boldsymbol{e}_2]}{\partial r}\frac{\partial \tilde{r}}{\partial \boldsymbol{x}} + \frac{\partial [u\boldsymbol{e}_1 + v\boldsymbol{e}_2]}{\partial \phi}\frac{\partial \tilde{\phi}}{\partial \boldsymbol{x}} \\ &= [u_r \boldsymbol{e}_1 + v_r \boldsymbol{e}_2]\,\boldsymbol{e}_1 + \frac{1}{r}\,[u_\phi \boldsymbol{e}_1 + v_\phi \boldsymbol{e}_2 + u\boldsymbol{e}_2 - v\boldsymbol{e}_1]\,\boldsymbol{e}_2. \end{aligned} \tag{2.3.14}$$

Substitute (2.3.14) into (2.3.2) to derive

$$\boldsymbol{D}(\boldsymbol{v}) = u_r \boldsymbol{e}_1\boldsymbol{e}_1 + \frac{1}{2}\left(v_r + \tfrac{1}{r}u_\phi - \tfrac{1}{r}v\right)(\boldsymbol{e}_1\boldsymbol{e}_2 + \boldsymbol{e}_2\boldsymbol{e}_1) + \frac{1}{r}(v_\phi + u)\boldsymbol{e}_2\boldsymbol{e}_2. \tag{2.3.15}$$

Substituting (2.3.11), (2.3.13), and (2.3.14) into (2.3.5) gives the Navier-Stokes equations in rotating polar coordinates:

$$\begin{aligned} u_t - \omega u_\phi + u u_r + \frac{v u_\phi}{r} - \frac{v^2}{r} &= -p_r + \gamma\left[u_{rr} + \frac{u_r}{r} + \frac{u_{\phi\phi}}{r^2} - \frac{2v_\phi}{r^2} - \frac{u}{r^2}\right], \\ v_t - \omega v_\phi + u v_r + \frac{v v_\phi}{r} + \frac{uv}{r} &= -\frac{p_\phi}{r} + \gamma\left[v_{rr} + \frac{v_r}{r} + \frac{v_{\phi\phi}}{r^2} + \frac{2u_\phi}{r^2} - \frac{v}{r^2}\right], \\ (ru)_r + v_\phi &= 0. \end{aligned} \tag{2.3.16}$$

2.4 The Coupling Between the Fluid and the String Equations

The equations for the fluid and the string are coupled through the adherence boundary condition (no-slip) and the body force term $\boldsymbol{f}$ in the equation for the string, equation (2.2.6).

We adopt the standard requirement for viscous fluids that the fluid adhere to solid surfaces, here the disk and the string, with which it is in contact. Thus

$$u(a,\phi,t) = 0, \quad v(a,\phi,t) = a\omega \quad \forall\, \phi, t, \tag{2.4.1}$$

$$\boldsymbol{v}(\boldsymbol{r}(s,t),t) = \boldsymbol{r}_t(s,t) \quad \forall\, s, t. \tag{2.4.2}$$

Now we derive an expression for the body force $\boldsymbol{f}$ exerted by the fluid on the string. The outward pointing unit normal to $\boldsymbol{r}(\cdot,t)$ is $\boldsymbol{r}_s \times \boldsymbol{k}/|\boldsymbol{r}_s|$. The definition of the Cauchy stress tensor says that the force per unit (actual) length exerted by the ring on the fluid at $\boldsymbol{r}(s,t)$ is thus $\boldsymbol{\Sigma}\cdot(\boldsymbol{r}_s \times \boldsymbol{k})/|\boldsymbol{r}_s|$. Therefore the force per unit reference length exerted by the fluid on the ring at $\boldsymbol{r}(s,t)$ is

$$\begin{aligned}\boldsymbol{f} &= -\boldsymbol{\Sigma}\cdot(\boldsymbol{r}_s \times \boldsymbol{k}) = \boldsymbol{\Sigma}\cdot(\boldsymbol{k}\times\boldsymbol{r}_s) = [-\rho p \boldsymbol{I} + 2\tilde{\mu}\boldsymbol{D}(\boldsymbol{v})]\cdot(\boldsymbol{k}\times\boldsymbol{r}_s) \\ &= -\rho p(\boldsymbol{k}\times\boldsymbol{r}_s) \\ &\quad + \tilde{\mu}\left[2u_r\boldsymbol{e}_1\boldsymbol{e}_1 + \left(v_r + \tfrac{1}{q}u_\phi - \tfrac{1}{q}v\right)(\boldsymbol{e}_1\boldsymbol{e}_2 + \boldsymbol{e}_2\boldsymbol{e}_1) + \tfrac{2}{q}(v_\phi + u)\boldsymbol{e}_2\boldsymbol{e}_2\right]\cdot(\boldsymbol{k}\times\boldsymbol{r}_s),\end{aligned} \tag{2.4.3}$$

where we have used (2.3.3) and (2.3.15). The components u and v of the fluid velocity are evaluated at $(r,\phi) = (q(s,t),\psi(s,t))$, which are the polar coordinates for $\boldsymbol{r}(s,t)$ (see equation (2.2.5)), and the argument of $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ is $\psi(s,t)+\omega t$. Substituting (2.2.3), (2.2.7) and (2.4.3) into the linear momentum equation for the string (2.2.6) gives

$$\varrho A\boldsymbol{r}_{tt} = \left(\hat{N}_\nu\,\nu_s + \hat{N}_{\dot\nu}\,\nu_{st} - \hat{N}\,\frac{\nu_s}{\nu}\right)\frac{\boldsymbol{r}_s}{\nu} + \hat{N}\,\frac{\boldsymbol{r}_{ss}}{\nu} + \boldsymbol{\Sigma}\cdot(\boldsymbol{k}\times\boldsymbol{r}_s). \tag{2.4.4}$$

2.5 The Area Side Condition

The fluid in our problem is incompressible and so it has constant area. Fix $R > 1$, and fix the area of the fluid to be $\pi(R^2 - a^2)$, i.e., the area of fluid is the same as the area of the annulus $\{a < |\boldsymbol{x}| < R\}$. This choice is motivated by the form of the Couette steady solution. See Section 2.7. The area of the rigid disk is πa^2. Therefore in order to prohibit the formation of cavities in the fluid we must enforce the side condition that the area enclosed by the string is πR^2. By Green's Theorem in the Plane

$$\pi R^2 = \tfrac{1}{2}\,\boldsymbol{k} \cdot \int_0^{2\pi} \boldsymbol{r}(s,t) \times \boldsymbol{r}_s(s,t)\, ds. \tag{2.5.1}$$

The parameters R and ω are at our disposal, and the choice of R and ω uniquely determines the pressure of the fluid in the Couette steady solution (see Section 2.7). This situation is different from that for problems involving an incompressible fluid in a domain with fixed boundary, where the pressure of the fluid is determined only up to a constant. It is easy to see why the pressure constant must be determined uniquely: Suppose that we add a constant to the pressure. This will have the effect of inflating the string. But inflating the string increases the area it encloses. This cannot happen without the formation of cavities in the fluid.

Recall that our 2-dimensional problem can be thought of as a horizontal cross section of the 3-dimensional problem of a fluid confined to the region between a rotating rigid cylinder and a membrane where the there is no motion of the fluid in the z-direction, the fluid variables are independent z, and the membrane remains cylindrical, but not necessarily a circular cylinder (see Section 2.1). Equivalent to prescribing the area of the fluid in each cross section (which is equivalent to prescribing R), we could prescribe the pressure on the fluid at the ends of the cylinder $z = \pm\infty$. This pressure dictates how much fluid is squeezed into each cross section, i.e., determines R. In this thesis, however, we will always prescribe the area of the fluid rather than the pressure at the ends of the cylinder.

While our equations make sense for all values of $R > 1$ and ω, we limit ourselves to those values that lead to a physical pressure $p > 0$ everywhere. (We do not explicitly calculate the values of R and ω that lead to a positive pressure p, but it will be implicitly understood that we only work with values of R and ω that satisfy this property. Thus our problem has the character of a semi-inverse problem.)

2.6 Summary of the Initial-Boundary-Value Problem

In this section we summarize the equations that were derived in Sections 2.2–2.5. Given constants ω, γ, $\tilde{\mu}$, ρ, a, R, and ϱA, constitutive function $\hat{N}$, and initial data $\boldsymbol{v}_0$, p_0, and $\boldsymbol{r}_0$, we seek functions

$$\mathcal{D}(t) \times [0,\infty) \ni (\boldsymbol{x}, t) \mapsto \boldsymbol{v}(\boldsymbol{x}, t) \in \mathbb{R}^2, \tag{2.6.1}$$

$$\mathcal{D}(t) \times [0,\infty) \ni (\boldsymbol{x}, t) \mapsto p(\boldsymbol{x}, t) \in \mathbb{R}, \tag{2.6.2}$$

$$\mathbb{R}/2\pi\mathbb{Z} \times [0,\infty) \ni (s, t) \mapsto \boldsymbol{r}(s, t) \in \mathbb{R}^2, \tag{2.6.3}$$

satisfying the initial conditions $\boldsymbol{v}(\boldsymbol{x}, 0) = \boldsymbol{v}_0(\boldsymbol{x})$, $p(\boldsymbol{x}, 0) = p_0(\boldsymbol{x})$, $\boldsymbol{r}(s, 0) = \boldsymbol{r}_0(s)$ and the following equations, where $\mathcal{D}(t)$ is the domain between the circle of radius $a < 1$ centred at the origin and the curve $\boldsymbol{r}(\cdot, t)$:

Navier-Stokes equations.

$$\begin{aligned} \boldsymbol{v}_t + \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}} \cdot \boldsymbol{v} = -\nabla p + \gamma \Delta \boldsymbol{v} \qquad \text{in } \mathcal{D}(t) \times [0,\infty), \\ \nabla \cdot \boldsymbol{v} = 0 \qquad \text{in } \mathcal{D}(t) \times [0,\infty). \end{aligned} \tag{2.6.4}$$

The equation for the string.

$$\varrho A \boldsymbol{r}_{tt} = \left(\hat{N}_\nu \,\nu_s + \hat{N}_{\dot\nu}\,\nu_{st} - \hat{N}\,\frac{\nu_s}{\nu}\right)\frac{\boldsymbol{r}_s}{\nu} + \hat{N}\,\frac{\boldsymbol{r}_{ss}}{\nu} + \boldsymbol{\Sigma}\cdot(\boldsymbol{k}\times\boldsymbol{r}_s) \qquad \text{in } [0,2\pi]\times[0,\infty), \tag{2.6.5}$$

where $\nu(s,t) := |\boldsymbol{r}_s(s,t)|$, and where $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}(\boldsymbol{v}(\boldsymbol{r}(s,t),t), p(\boldsymbol{r}(s,t),t))$. $\boldsymbol{\Sigma}$ was defined in equation (2.3.3). The constitutive function $\hat{N}(\nu,\dot\nu)$ was introduced in Section 2.2.

The adherence boundary condition.

$$\boldsymbol{v}(a\boldsymbol{e}_1(\phi+\omega t),t) = a\omega\,\boldsymbol{e}_2(\phi+\omega t) \qquad \forall\,\phi,t, \tag{2.6.6}$$

$$\boldsymbol{v}(\boldsymbol{r}(s,t),t) = \boldsymbol{r}_t(s,t) \qquad \forall\,s,t. \tag{2.6.7}$$

The area side condition.

$$\pi R^2 = \tfrac{1}{2}\,\boldsymbol{k}\cdot\int_0^{2\pi}\boldsymbol{r}(s,t)\times\boldsymbol{r}_s(s,t)\,ds \qquad \forall\,t. \tag{2.6.8}$$

2.7 The Couette Steady Solution

In this section we show that our problem admits a rigid Couette steady solution. The symmetry of our problem suggests that we seek a steady solution in which the string is circular and rotates rigidly with constant angular velocity $\mathit{\Omega}$, and the fluid streamlines are concentric circles. Thus we seek solutions of the following form (using the polar coordinates introduced in Section 2.3):

$$u(r,\phi,t) = 0, \qquad v(r,\phi,t) = V(r), \qquad p(r,\phi,t) = P(r), \tag{2.7.1}$$

$$\boldsymbol{r}(s,t) = R\boldsymbol{e}_1(s+\mathit{\Omega} t). \tag{2.7.2}$$

Note that $\boldsymbol{r}$ satisfies the side condition (2.5.1). $R > 1$ is the radius of the circle formed by the string. (Recall that the natural state of the string is a circle of radius 1.) In the notation introduced in equations (2.2.3) and (2.2.5),

$$\nu = |\boldsymbol{r}_s| = R, \qquad q = |\boldsymbol{r}| = R, \qquad \psi(s,t) = s + (\Omega - \omega)t. \tag{2.7.3}$$

The substitution of (2.7.1) into the Navier-Stokes equations (2.3.16) yields

$$P_r = \frac{V^2}{r}, \qquad V_{rr} + \frac{V_r}{r} - \frac{V}{r^2} \equiv \left[V_r + \frac{V}{r}\right]_r \equiv \left[\frac{1}{r}(rV)_r\right]_r = 0. \tag{2.7.4}$$

Thus there are constants B, C, D such that

$$V(r) = Br + \frac{C}{r}, \tag{2.7.5}$$

$$P(r) = \frac{B^2r^2}{2} + 2BC\ln r - \frac{C^2}{2r^2} + D. \tag{2.7.6}$$

The adherence conditions (2.4.1), (2.4.2) imply that

$$a\omega = Ba + \frac{C}{a}, \qquad R\Omega = BR + \frac{C}{R} \iff B = \frac{R^2\Omega - a^2\omega}{R^2 - a^2}, \quad C = \frac{R^2a^2(\omega - \Omega)}{R^2 - a^2}. \tag{2.7.7}$$

We must obtain equations for the unknown constants Ω and D in terms of parameters ω, a, R and $\hat{N}$. By substituting (2.7.1) and (2.7.2) into the equation for the string (2.6.5) we find that

$$-\varrho A\Omega^2 R\boldsymbol{e}_1(s + \Omega t) = -\hat{N}(R,0)\boldsymbol{e}_1(s + \Omega t) + \rho RP(R)\boldsymbol{e}_1(s + \Omega t) + \frac{2C\tilde{\mu}}{R}\boldsymbol{e}_2(s + \Omega t). \tag{2.7.8}$$

Taking the inner product of (2.7.8) with $\boldsymbol{e}_2(s + \Omega t)$ yields

$$C = 0 \implies \Omega = \omega, \qquad B = \omega. \tag{2.7.9}$$

Therefore the fluid and the elastic string rotate rigidly with the same angular velocity as the rigid disk. The system behaves like a rigid body. We call this the rigid Couette solution.

Note that this rigid Couette solution is not time independent; the position of the string $\boldsymbol{r}$ depends on t. However, the coordinates of $\boldsymbol{r}$ with respect to the rotating basis $\{\boldsymbol{e}_1(\omega t), \boldsymbol{e}_2(\omega t)\}$ are time independent. This explains why the rotating basis was introduced in Section 2.3 and why we refer to the rigid Couette solution as a steady solution.

An expression for D can be obtained by taking the inner product of equation (2.7.8) with $\boldsymbol{e}_1(s + \Omega t)$. By (2.7.9), formulas (2.7.5) and (2.7.6) for V and P reduce to the simple forms

$$V = \omega r, \qquad P(r) = \tfrac{1}{2}\omega^2 r^2 + D, \tag{2.7.10}$$

where

$$D = D(R, \omega^2) = \frac{\hat{N}(R,0)}{\rho R} - \frac{\varrho A \omega^2}{\rho} - \tfrac{1}{2}\omega^2 R^2. \tag{2.7.11}$$

Observe that

$$P(R) = P(R; \omega^2) = \frac{\hat{N}(R,0)}{\rho R} - \frac{\varrho A \omega^2}{\rho}, \tag{2.7.12}$$

i.e., the pressure at the liquid-solid interface is the balance of the tension in the string and the centrifugal force.

Other steady solutions. The rigid Couette solution is not the only steady solution. It is easy to check that for any $\alpha \in \mathbb{R}$, $\boldsymbol{r}(s,t) = R\boldsymbol{e}_1(s + \alpha + \omega t)$ is also a steady solution with the same fluid velocity and pressure as above. This steady solution can be obtained from the rigid Couette solution by a rotation or by relabelling the material points, and therefore is not essentially different. Consequently we will factor it out in Section 2.10. When $\omega = 0$, $\boldsymbol{r}(s) = R\boldsymbol{e}_1(s) + \boldsymbol{c}$ is a steady solution for any $\boldsymbol{c} \in \mathbb{R}^2$ with $|\boldsymbol{c}| < R - a$. This is an off-center solution: the rigid disk and circular string are not concentric. We will see in Section 2.10 that this gives rise to an instability, which could be factored out if we had a feedback control to keep the center of mass of the string at the origin.

2.8 Linearization

We are interested in the stability of the rigid Couette solution with respect to the parameter ω. For what range of angular velocities ω is the rigid Couette solution observable? First we consider linear stability. To do this we must linearize the equations of motion about the rigid Couette solution.

Coordinate-free Equations

Let $(\boldsymbol{v}^0, p^0, \boldsymbol{r}^0, \nu^0)$ be the coordinate-free representation of the rigid Couette solution found in Section 2.7:

$$\boldsymbol{v}^0(\boldsymbol{x}) = \omega\,|\boldsymbol{x}|\,\boldsymbol{e}_2(\tilde{\phi}(\boldsymbol{x},t) + \omega t), \qquad p^0(\boldsymbol{x}) = \tfrac{1}{2}\omega^2\,|\boldsymbol{x}|^2 + D, \tag{2.8.1}$$

$$\boldsymbol{r}^0(s,t) = R\boldsymbol{e}_1(s + \omega t), \qquad \nu^0 = R, \tag{2.8.2}$$

where $\tilde{\phi}$ was defined in equation (2.3.7). To linearize our equations of motion about the rigid Couette solution, we first introduce the small parameter ε and perturbation variables, decorated with a superscript 1, by

$$
\begin{aligned}
\boldsymbol{v}(\boldsymbol{x},t,\varepsilon) &= \boldsymbol{v}^0(\boldsymbol{x}) + \varepsilon \boldsymbol{v}^1(\boldsymbol{x},t) + \mathcal{O}(\varepsilon^2),\\
p(\boldsymbol{x},t,\varepsilon) &= p^0(\boldsymbol{x}) + \varepsilon p^1(\boldsymbol{x},t) + \mathcal{O}(\varepsilon^2),\\
\boldsymbol{r}(s,t,\varepsilon) &= \boldsymbol{r}^0(s,t) + \varepsilon \boldsymbol{r}^1(s,t) + \mathcal{O}(\varepsilon^2),\\
\nu(s,t,\varepsilon) &= \nu^0 + \varepsilon \nu^1(s,t) + \mathcal{O}(\varepsilon^2).
\end{aligned}
\tag{2.8.3}
$$

We linearize the evolution equations by substituting (2.8.3) into them, differentiating the resulting equations with respect to ε, and then setting $\varepsilon = 0$. The domain of the linearized fluid equations is the annulus $\{a < |\boldsymbol{x}| < R\}$ rather than the time-dependent domain $\mathcal{D}(t)$.

The Navier-Stokes equations. From (2.3.14) and (2.8.1) we obtain

$$
\frac{\partial \boldsymbol{v}^0}{\partial \boldsymbol{x}}(r\boldsymbol{e}_1(\phi+\omega t)) = \omega[\boldsymbol{e}_2(\phi+\omega t)\boldsymbol{e}_1(\phi+\omega t) - \boldsymbol{e}_1(\phi+\omega t)\boldsymbol{e}_2(\phi+\omega t)] \equiv \omega \boldsymbol{k}\times. \tag{2.8.4}
$$

Therefore the tensor $\partial \boldsymbol{v}^0/\partial \boldsymbol{x}$ is constant and skew; in particular $\boldsymbol{D}(\boldsymbol{v}^0) = \boldsymbol{0}$. Linearizing the Navier-Stokes equations (2.3.5) by the method described above yields

$$
\begin{aligned}
\boldsymbol{v}_t^1 + \omega \boldsymbol{k}\times \boldsymbol{v}^1 + \left(\frac{\partial \boldsymbol{v}^1}{\partial \boldsymbol{x}}\right)\cdot \boldsymbol{v}^0 &= -\nabla p^1 + \gamma \Delta \boldsymbol{v}^1,\\
\nabla\cdot \boldsymbol{v}^1 &= 0.
\end{aligned}
\tag{2.8.5}
$$

The equation for the string. Linearizing the forcing term $\boldsymbol{\Sigma}\cdot(\boldsymbol{k}\times\boldsymbol{r}_s)$ in (2.4.4) requires care:

$$
\begin{aligned}
\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \boldsymbol{\Sigma}\cdot(\boldsymbol{k}\times\boldsymbol{r}_s) &= \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \boldsymbol{\Sigma}(\boldsymbol{v}(\boldsymbol{r}(s,t,\varepsilon),t,\varepsilon),\, p(\boldsymbol{r}(s,t,\varepsilon),t,\varepsilon))\cdot(\boldsymbol{k}\times\boldsymbol{r}_s(s,t,\varepsilon))\\
&= -\rho P(R)(\boldsymbol{k}\times\boldsymbol{r}_s^1) + \rho R^2\omega^2(\boldsymbol{e}_1\cdot\boldsymbol{r}^1)\boldsymbol{e}_1 - R\,\boldsymbol{\Sigma}(\boldsymbol{v}^1,p^1)\cdot\boldsymbol{e}_1,
\end{aligned}
\tag{2.8.6}
$$

where $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ have argument $s + \omega t$. Therefore by linearizing (2.4.4) we obtain

$$\varrho A \boldsymbol{r}^1_{tt} = \left[\left(N^\circ_\nu - \frac{N^\circ}{R}\right)\nu^1_s + N^\circ_{\dot\nu}\,\nu^1_{st}\right]\boldsymbol{e}_2 - \left[\left(N^\circ_\nu - \frac{N^\circ}{R}\right)\nu^1 + N^\circ_{\dot\nu}\,\nu^1_t\right]\boldsymbol{e}_1 + \frac{N^\circ}{R}\boldsymbol{r}^1_{ss}$$
$$- \rho P(R)(\boldsymbol{k} \times \boldsymbol{r}^1_s) + \rho R^2\omega^2(\boldsymbol{e}_1 \cdot \boldsymbol{r}^1)\boldsymbol{e}_1 - R\,\boldsymbol{\Sigma}(\boldsymbol{v}^1, p^1) \cdot \boldsymbol{e}_1, \quad (2.8.7)$$

where N°, N°_ν and $N^\circ_{\dot\nu}$ denote $\hat{N}(R, 0)$, $\hat{N}_\nu(R, 0)$ and $\hat{N}_{\dot\nu}(R, 0)$. The fluid variables $\boldsymbol{v}^1$ and p^1 are evaluated at $(\boldsymbol{x}, t) = (R\boldsymbol{e}_1(s + \omega t), t)$.

The adherence boundary condition. The linearization of the adherence condition for the disk (2.4.1) yields

$$\boldsymbol{v} = 0, \quad \text{on } |\boldsymbol{x}| = a. \quad (2.8.8)$$

Next we linearize the adherence condition for the string. Differentiate (2.4.2) with respect to ε, then set $\varepsilon = 0$ to obtain

$$\frac{\partial \boldsymbol{v}^0}{\partial \boldsymbol{x}}(R\boldsymbol{e}_1(s + \omega t), t) \cdot \boldsymbol{r}^1(s, t) + \boldsymbol{v}^1(R\boldsymbol{e}_1(s + \omega t), t) = \boldsymbol{r}^1_t(s, t). \quad (2.8.9)$$

(Note that $\partial \boldsymbol{v}^0/\partial \boldsymbol{x} = (\partial \boldsymbol{v}/\partial \boldsymbol{x})|_{\varepsilon=0}$ as a consequence of (2.8.3).) Now substitute (2.8.4) into (2.8.9) to derive

$$\boldsymbol{r}^1_t = \omega \boldsymbol{k} \times \boldsymbol{r}^1 + \boldsymbol{v}^1, \quad (2.8.10)$$

where $\boldsymbol{v}^1$ has arguments $(R\boldsymbol{e}_1(s + \omega t), t)$.

The strain-configuration relation. Linearizing (2.2.3) yields

$$\nu^1 = \boldsymbol{r}^1_s \cdot \boldsymbol{e}_2(s + \omega t). \quad (2.8.11)$$

The area side condition. By linearizing (2.5.1) we obtain

$$\int_0^{2\pi} \boldsymbol{r}^1(s,t) \cdot \boldsymbol{e}_1(s+\omega t)\, ds = 0. \tag{2.8.12}$$

Polar Coordinates

In the following section we seek solutions of the linearized equations with exponential time dependence $\exp(\lambda t)$, $\lambda \in \mathbb{C}$, which replaces every time derivative in the linearized equations with λ, to obtain an eigenvalue problem for the perturbation growth rate λ. Since the rigid Couette solution is steady (time-independent) in polar coordinates with respect to the rotating basis, but not steady in coordinate-free form, it is necessary to introduce polar coordinates before seeking solutions with exponential time dependence.

The Navier-Stokes equations. By substituting equations (2.3.11), (2.3.13), (2.3.14), and (2.8.1) into (2.8.5), or alternatively by linearizing (2.3.16), we obtain the linearized Navier-Stokes equations in rotating polar coordinates:

$$\begin{aligned}
u_t^1 - 2\omega v^1 &= -p_r^1 + \gamma\left(u_{rr}^1 + \frac{u_r^1}{r} + \frac{u_{\phi\phi}^1}{r^2} - \frac{2v_\phi^1}{r^2} - \frac{u^1}{r^2}\right), \\
v_t^1 + 2\omega u^1 &= -\frac{p_\phi^1}{r} + \gamma\left(v_{rr}^1 + \frac{v_r^1}{r} + \frac{v_{\phi\phi}^1}{r^2} + \frac{2u_\phi^1}{r^2} - \frac{v^1}{r^2}\right), \\
(ru^1)_r + v_\phi^1 &= 0,
\end{aligned} \tag{2.8.13}$$

where the fluid variables have arguments (r,ϕ,t).

The equations for the string. By linearizing equation (2.2.5) we obtain

$$\boldsymbol{r}^1(s,t) = q^1(s,t)\,\boldsymbol{e}_1(s+\omega t) + R\,\psi^1(s,t)\,\boldsymbol{e}_2(s+\omega t). \tag{2.8.14}$$

Substituting equations (2.3.11), (2.3.14), (2.8.1), and (2.8.14) into equation (2.8.7), then taking the inner products of the resulting equation with $\boldsymbol{e}_1(s+\omega t)$ and $\boldsymbol{e}_2(s+\omega t)$ yields the linearized string equations in polar coordinates. The expression for $\boldsymbol{\Sigma}$ in polar coordinates can be read off from equation (2.4.3).

$$\varrho A\,(q^1_{tt} - \omega^2 q^1 - 2\omega R\,\psi^1_t) = -\left(N^\circ_\nu - \tfrac{1}{R}N^\circ\right)\nu^1 - N^\circ_{\dot\nu}\,\nu^1_t + \tfrac{1}{R}N^\circ(q^1_{ss} - q^1 - 2R\,\psi^1_s)$$
$$+\,\rho P(R)(R\,\psi^1_s + q^1) + \rho R^2\omega^2 q^1 + R\rho p^1 - 2R\tilde\mu u^1_r, \quad (2.8.15)$$

$$\varrho A\,(R\,\psi^1_{tt} - \omega^2 R\,\psi^1 + 2\omega q^1_t) = \left(N^\circ_\nu - \tfrac{1}{R}N^\circ\right)\nu^1_s + N^\circ_{\dot\nu}\,\nu^1_{st} + \tfrac{1}{R}N^\circ(R\,\psi^1_{ss} - R\,\psi^1 + 2q^1_s)$$
$$-\,\rho P(R)(q^1_s - R\,\psi^1) - R\,\tilde\mu(v^1_r - \tfrac{1}{R}v^1 + \tfrac{1}{R}u^1_\phi), \quad (2.8.16)$$

where the fluid variables u^1, v^1, p^1 have arguments (R, s, t).

The adherence boundary condition. Writing the adherence boundary conditions (2.8.8) and (2.8.10) in polar coordinates gives

$$u^1(a,\phi,t) = 0, \qquad v^1(a,\phi,t) = 0, \quad (2.8.17)$$

$$q^1_t(s,t) = u^1(R,s,t), \qquad R\,\psi^1_t(s,t) = v^1(R,s,t). \quad (2.8.18)$$

The strain-configuration relation. In polar coordinates equation (2.8.11) becomes

$$\nu^1 = q^1 + R\,\psi^1_s. \quad (2.8.19)$$

The area side condition. In polar coordinates equation (2.8.12) becomes

$$\int_0^{2\pi} q^1(s,t)\,ds = 0. \quad (2.8.20)$$

2.9 The Quadratic Eigenvalue Problem

Polar Coordinates

We seek solutions of the linearized equations with exponential time-dependence:

$$u^1(r, \phi, t) = u(r, \phi)\, e^{\lambda t}, \quad v^1(r, \phi, t) = v(r, \phi)\, e^{\lambda t}, \quad p^1(r, \phi, t) = p(r, \phi)\, e^{\lambda t}, \quad (2.9.1)$$

$$q^1(s, t) = q(s)\, e^{\lambda t}, \qquad \psi^1(s, t) = \psi(s)\, e^{\lambda t}, \qquad \nu^1(s, t) = \nu(s)\, e^{\lambda t}, \qquad (2.9.2)$$

where $\lambda \in \mathbb{C}$ is the perturbation growth rate. Note that the letters u, v, p, q, ψ, and ν have a meaning here different from those they had in the previous sections. We substitute the normal mode decompositions (2.9.1) and (2.9.2) into the linearized equations (2.8.13)–(2.8.20) to obtain a quadratic eigenvalue problem (note that each time derivative in (2.8.13)–(2.8.20) has been replaced by a power of λ):

The Navier-Stokes equations.

$$\begin{aligned} \lambda u - 2\omega v &= -p_r + \gamma \left(u_{rr} + \frac{u_r}{r} + \frac{u_{\phi\phi}}{r^2} - \frac{2v_\phi}{r^2} - \frac{u}{r^2} \right), \\ \lambda v + 2\omega u &= -\frac{p_\phi}{r} + \gamma \left(v_{rr} + \frac{v_r}{r} + \frac{v_{\phi\phi}}{r^2} + \frac{2u_\phi}{r^2} - \frac{v}{r^2} \right), \qquad (2.9.3) \\ (ru)_r + v_\phi &= 0. \end{aligned}$$

The equations for the string. Substituting (2.9.2) into the strain-configuration relation (2.8.19) yields

$$\nu = q + R\,\psi_s. \qquad (2.9.4)$$

We use (2.9.4) to eliminate ν from the string equations. Substituting (2.9.1), (2.9.2), (2.9.4) into the string equations (2.8.15), (2.8.16) gives

$$
\begin{aligned}
&\lambda^2 \varrho A q + \lambda(N^\circ_\nu q - 2\omega\varrho AR\,\psi + N^\circ_\nu R\,\psi_s) \\
&= (\varrho A\omega^2 - N^\circ_\nu + \rho P(R) + \rho R^2\omega^2)\,q + \tfrac{1}{R}N^\circ q_{ss} + (\rho P(R)R - N^\circ - RN^\circ_\nu)\,\psi_s \\
&\quad + R\rho\,p(R,s) - 2R\tilde{\mu}u_r(R,s),
\end{aligned}
\tag{2.9.5}
$$

$$
\begin{aligned}
&\lambda^2 \varrho AR\,\psi + \lambda(2\varrho A\omega q - N^\circ_\nu q_s - N^\circ_\nu R\,\psi_{ss}) \\
&= (N^\circ_\nu + \tfrac{1}{R}N^\circ - \rho P(R))\,q_s + (\varrho A\omega^2 R - N^\circ + \rho RP(R))\,\psi + N^\circ_\nu R\,\psi_{ss} \\
&\quad + \tilde{\mu}[-Rv_r(R,s) + v(R,s) - u_\phi(R,s)].
\end{aligned}
\tag{2.9.6}
$$

The adherence boundary condition.

$$u(a,\phi) = 0, \qquad v(a,\phi) = 0, \tag{2.9.7}$$

$$\lambda q(s) = u(R,s), \qquad \lambda R\,\psi(s) = v(R,s). \tag{2.9.8}$$

The area side condition.

$$\int_0^{2\pi} q(s)\,ds = 0. \tag{2.9.9}$$

Coordinate-free Equations

Now that the time dependence has been removed from our equations, we rewrite them in coordinate-free form. This will be convenient for the analysis in Section 2.10. (It is easier to derive energy estimates in coordinate-free form). Define $\tilde{\boldsymbol{v}}(\boldsymbol{x})$, $\tilde{p}(\boldsymbol{x})$ and $\tilde{\boldsymbol{r}}(s)$ by

$$\tilde{\boldsymbol{v}}(r\,\boldsymbol{e}_1(\phi)) := u(r,\phi)\,\boldsymbol{e}_1(\phi) + v(r,\phi)\,\boldsymbol{e}_2(\phi), \qquad \tilde{p}(r\,\boldsymbol{e}_1(\phi)) := p(r,\phi), \tag{2.9.10}$$

$$\tilde{\boldsymbol{r}}(s) := q(s)\,\boldsymbol{e}_1(s) + R\,\psi(s)\,\boldsymbol{e}_2(s). \tag{2.9.11}$$

We now drop the tilde from these variables. Equations (2.9.10) and (2.9.11) can be used to write eigenvalue problem (2.9.3)–(2.9.9) in the coordinate-free form (2.9.12), given below. It is easy to check that the substitution of (2.9.10) and (2.9.11) into (2.9.12) yields (2.9.3)–(2.9.9). Note that it is not easy to derive eigenvalue problem (2.9.12) directly from equations (2.8.5)–(2.8.12) because these equations are the linearization of the governing equations about a solution that is not time independent.

Eigenvalue problem (2.9.12) is a perturbation of the Stokes eigenvalue problem (obtained by setting $\omega = 0$ in (2.9.13)), but with complicated boundary conditions: The eigenvalue parameter λ appears in the boundary terms and appears quadratically.

2.9.12 Classical formulation of the quadratic eigenvalue problem.

Find eigenvalue-eigenvector pairs $(\lambda, (\boldsymbol{v}, \boldsymbol{r}, p))$ satisfying

The Navier-Stokes equations. For $a < |\boldsymbol{x}| < R$,

$$\begin{aligned} &\lambda \boldsymbol{v} = -\nabla p + \gamma \Delta \boldsymbol{v} - 2\omega\, \boldsymbol{k} \times \boldsymbol{v} = \tfrac{1}{\rho} \operatorname{div} \boldsymbol{\Sigma}(p, \boldsymbol{v}) - 2\omega\, \boldsymbol{k} \times \boldsymbol{v}, \\ &\nabla \cdot \boldsymbol{v} = 0. \end{aligned} \tag{2.9.13}$$

The string equation. For $s \in [0, 2\pi)$,

$$\begin{aligned} \lambda^2 \varrho A \boldsymbol{r} = \lambda[N_\nu^\circ(\boldsymbol{e}_2\boldsymbol{e}_2 \cdot \boldsymbol{r}_s)_s + 2\varrho A \omega \boldsymbol{r} \times \boldsymbol{k}] + \tfrac{1}{R} N^\circ \boldsymbol{r}_{ss} + (N_\nu^\circ - \tfrac{1}{R} N^\circ)(\boldsymbol{e}_2\boldsymbol{e}_2 \cdot \boldsymbol{r}_s)_s \\ - \rho P(R) \boldsymbol{k} \times \boldsymbol{r}_s + \varrho A \omega^2 \boldsymbol{r} + \rho R^2 \omega^2 (\boldsymbol{r} \cdot \boldsymbol{e}_1)\boldsymbol{e}_1 - R\, \boldsymbol{\Sigma}(\boldsymbol{v}, p) \cdot \boldsymbol{e}_1. \end{aligned} \tag{2.9.14}$$

The adherence boundary condition.

$$\begin{aligned} \boldsymbol{v} = 0 \quad \text{for } |\boldsymbol{x}| = a, \\ \lambda \boldsymbol{r}(s) = \boldsymbol{v}(R\boldsymbol{e}_1(s)). \end{aligned} \tag{2.9.15}$$

The area side condition.

$$\int_0^{2\pi} \boldsymbol{r}(s) \cdot \boldsymbol{e}_1(s)\, ds = 0. \tag{2.9.16}$$

2.10 Analysis of the Spectrum

If all the eigenvalues λ of problem (2.9.12) satisfy $\mathrm{Re}(\lambda) < 0$, then the perturbations $\boldsymbol{v}^1$, $\boldsymbol{r}^1$, p^1 decay exponentially in time and we say that the rigid Couette solution is *linearly stable*. On the other hand, if one eigenvalue satisfies $\mathrm{Re}(\lambda) > 0$, then the perturbation corresponding to this eigenvalue will grow exponentially in time and we say that the rigid Couette solution is *linearly unstable*.

The eigenvalues are a function of the angular velocity ω, $\lambda = \lambda(\omega)$, and we are interested in how the eigenvalues move as ω is increased from 0. Typically in hydrodynamic stability problems all the eigenvalues are in the left half-plane when some parameter is small, and they migrate towards the right as the parameter is increased. If the eigenvalues cross the imaginary axis, then the way that they cross can provide valuable information about solutions to the fully nonlinear problem. For example, if the leading eigenvalues cross the imaginary axis in complex conjugate pairs, then (under some additional mild assumptions) periodic solutions of the nonlinear problem have a (Hopf) bifurcation from the trivial solution. In Sections 2.10–2.13 we study the eigenvalue problem (2.9.12).

Critical Values of ω and Eigenvalue Crossings

Due to the complicated form of the eigenvalue problem (2.9.12) it is not possible to compute the eigenvalues analytically. Much information can be obtained, however, before turning to numerics. We prove that the eigenvalues cross the imaginary axis through the origin (Theorem (2.10.4)) and compute explicitly the critical values of ω at which the eigenvalues cross (Theorem (2.10.21)), obtaining a formula that highlights the role of the material properties. Since the eigenvalue problem has real coefficients and the eigenvalues cross the imaginary axis through the origin, then either the eigenvalues cross along the real axis, suggesting a steady-state bifurcation, or they cross through the origin in complex-conjugate pairs (meaning that complex-conjugate pairs of eigenvalues in the left half-plane move towards the origin, meet at the origin, and then move into the right half-plane, whereupon they cease to be real) suggesting a Takens-Bogdanov bifurcation. The computational results in Section 2.13 indicate that a Takens-Bogdanov bifurcation takes place.

Let Ω denote the annulus $\{\boldsymbol{x} : a < |\boldsymbol{x}| < R\}$.

Lemma 2.10.1 (Energy equality). *Let* $(\lambda, (\boldsymbol{v}, \boldsymbol{r}, p))$ *be a smooth eigenpair of* (2.9.12). *Then*

$$\begin{aligned}
\operatorname{Re}(\lambda) \Big(&||\boldsymbol{v}||^2_{L^2(\Omega)} + \tfrac{N^\circ}{\rho R}||\boldsymbol{r}_s \cdot \boldsymbol{e}_1||^2_{L^2(0,2\pi)} + \tfrac{N^\circ_\nu}{\rho}||\boldsymbol{r}_s \cdot \boldsymbol{e}_2||^2_{L^2(0,2\pi)} \\
&+ \tfrac{\varrho A}{\rho}|\lambda|^2||\boldsymbol{r}||^2_{L^2(0,2\pi)} - \tfrac{\varrho A\omega^2}{\rho}||\boldsymbol{r}||^2_{L^2(0,2\pi)} - R^2\omega^2||\boldsymbol{r}\cdot\boldsymbol{e}_1||^2_{L^2(0,2\pi)} \Big) \\
= -\tfrac{N^\circ_{\tilde\nu}}{\rho}|\lambda|^2||\boldsymbol{r}_s \cdot \boldsymbol{e}_2||^2_{L^2(0,2\pi)} &- \tfrac{2\tilde\mu}{\rho}||\boldsymbol{D}(\boldsymbol{v})||^2_{L^2(\Omega)} - P(R)\operatorname{Re}(\lambda)\int_0^{2\pi}(\boldsymbol{k}\times\boldsymbol{r}_s)\cdot\bar{\boldsymbol{r}}\,ds.
\end{aligned}$$

Proof. Let $\boldsymbol{n}_\Omega$ be the unit outer normal to $\partial\Omega$. Take the inner product of (2.9.13) with $\bar{\boldsymbol{v}}$, where a bar denotes complex conjugation, and integrate by parts to obtain

$$\begin{aligned}
\lambda||\boldsymbol{v}||^2_{L^2(\Omega)} &= \tfrac{1}{\rho}\int_\Omega \operatorname{div} \boldsymbol{\Sigma}\cdot\bar{\boldsymbol{v}}\ d\boldsymbol{x} - 2\omega\int_\Omega (\boldsymbol{k}\times\boldsymbol{v})\cdot\bar{\boldsymbol{v}}\ dx \\
&= \tfrac{1}{\rho}\int_{\{|\boldsymbol{x}|=R\}} \boldsymbol{n}_\Omega\cdot\boldsymbol{\Sigma}\cdot\bar{\boldsymbol{v}}\ dS - \tfrac{1}{\rho}\int_\Omega \boldsymbol{\Sigma} : \frac{\partial\bar{\boldsymbol{v}}}{\partial\boldsymbol{x}}\ d\boldsymbol{x} - 2\omega\int_\Omega (\boldsymbol{k}\times\boldsymbol{v})\cdot\bar{\boldsymbol{v}}\ dx \\
&= \tfrac{1}{\rho}\int_0^{2\pi} \boldsymbol{e}_1\cdot\boldsymbol{\Sigma}\cdot\bar{\boldsymbol{v}}\ Rds - \tfrac{2\tilde{\mu}}{\rho}||\boldsymbol{D}(\boldsymbol{v})||^2_{L^2(\Omega)} - 2\omega\int_\Omega (\boldsymbol{k}\times\boldsymbol{v})\cdot\bar{\boldsymbol{v}}\ dx,
\end{aligned} \tag{2.10.2}$$

where we have used the adherence boundary condition $\boldsymbol{v} = 0$ on $\{|\boldsymbol{x}| = a\}$, the divergence-free constraint $\boldsymbol{I} : \partial\bar{\boldsymbol{v}}/\partial\boldsymbol{x} \equiv \operatorname{div}\bar{\boldsymbol{v}} = 0$, and the identity $\boldsymbol{D}(\boldsymbol{v}) : \partial\bar{\boldsymbol{v}}/\partial\boldsymbol{x} = |\boldsymbol{D}(\boldsymbol{v})|^2$.

In the first term on the right-hand side of (2.10.2) substitute for $\boldsymbol{\Sigma}\cdot\boldsymbol{e}_1$ from (2.9.14), substitute for $\bar{\boldsymbol{v}}$ with $\bar{\lambda}\bar{\boldsymbol{r}}$, and integrate by parts to find that

$$\begin{aligned}
&\tfrac{1}{\rho}\int_0^{2\pi} \boldsymbol{e}_1\cdot\boldsymbol{\Sigma}\cdot\bar{\boldsymbol{v}}\ Rds \\
&= -\tfrac{N^\circ_{\dot{\nu}}}{\rho}|\lambda|^2||\boldsymbol{r}_s\cdot\boldsymbol{e}_2||^2_{L^2(0,2\pi)} + \tfrac{2\varrho A\omega}{\rho}|\lambda|^2\int_0^{2\pi}(\boldsymbol{r}\times\boldsymbol{k})\cdot\bar{\boldsymbol{r}}\ ds - \tfrac{N^\circ}{\rho R}\bar{\lambda}||\boldsymbol{r}_s||^2_{L^2(0,2\pi)} \\
&- \tfrac{(N^\circ_\nu - R^{-1}N^\circ)}{\rho}\bar{\lambda}||\boldsymbol{r}_s\cdot\boldsymbol{e}_2||^2_{L^2(0,2\pi)} - P(R)\,\bar{\lambda}\int_0^{2\pi}(\boldsymbol{k}\times\boldsymbol{r}_s)\cdot\bar{\boldsymbol{r}}\ ds + \tfrac{\varrho A\omega^2}{\rho}\bar{\lambda}||\boldsymbol{r}||^2_{L^2(0,2\pi)} \\
&+ R^2\omega^2\bar{\lambda}||\boldsymbol{r}\cdot\boldsymbol{e}_1||^2_{L^2(0,2\pi)} - \tfrac{\varrho A}{\rho}\lambda|\lambda|^2||\boldsymbol{r}||^2_{L^2(0,2\pi)}.
\end{aligned} \tag{2.10.3}$$

Observe that $(\boldsymbol{r}\times\boldsymbol{k})\cdot\bar{\boldsymbol{r}}$ and $(\boldsymbol{k}\times\boldsymbol{v})\cdot\bar{\boldsymbol{v}}$ are purely imaginary. It can be shown that the term

$$\int_0^{2\pi}(\boldsymbol{k}\times\boldsymbol{r}_s)\cdot\bar{\boldsymbol{r}}\ ds$$

is real by taking its complex conjugate and integrating by parts. By substituting (2.10.3) into (2.10.2), taking the real part of the resulting equation, and simplifying, we complete the proof. □

Theorem 2.10.4 (Eigenvalue crossings). *Let* $(\lambda, (\boldsymbol{v}, \boldsymbol{r}, p))$ *be a smooth eigenpair of problem* (2.9.12)*. Suppose that* $\mathrm{Re}(\lambda) = 0$*. Then* $\lambda = 0$*. Therefore any eigenvalues that cross the imaginary axis cross through the origin.*

Proof. By substituting $\mathrm{Re}(\lambda) = 0$ into the energy equality in Lemma (2.10.1) we find that $||\boldsymbol{D}(\boldsymbol{v})||_{L^2(\Omega)} = 0$, which implies that $\boldsymbol{v} = 0$ by the Korn and Poincaré inequalities. But $\boldsymbol{v} = 0$ implies $\lambda = 0$ by equation $(2.9.15)_2$. □

Theorem 2.10.5 (Eigenvalue-free regions of $\mathbb{C}$). *Define*

$$M := \tfrac{1}{\varrho A}\left(\varrho A\omega^2 + \rho R^2\omega^2 + \tfrac{1}{4}\rho P(R)^2 \max\{R/N^\circ, 1/N_\nu^\circ\}\right).$$

Then problem (2.9.12) *has no eigenvalues in the set*

$$\{\lambda \in \mathbb{C} : |\lambda|^2 \geq M,\ \mathrm{Re}(\lambda) > 0\} \cup \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) = 0,\ \lambda \neq 0\}.$$

Remark. Note that M increases as ω^2 increases. Also note that when $\omega = 0$,

$$M = \frac{\rho P(R)^2 \max\{R/N^\circ, 1/N_\nu^\circ\}}{4\varrho A} = \frac{N^{\circ 2} \max\{R/N^\circ, 1/N_\nu^\circ\}}{4\varrho A R^2} \to 0 \text{ as } R \to 1$$

since $N^\circ = N(R, 0) \to N(1, 0) = 0$ and $N_\nu > 0$.

Proof of Theorem (2.10.5). By rearranging the energy equality in Lemma (2.10.1) we obtain

$$\begin{aligned} 0 \geq \mathrm{Re}(\lambda) \Big(&||\boldsymbol{v}||^2_{L^2(\Omega)} + \tfrac{N^\circ}{\rho R}||\boldsymbol{r}_s \cdot \boldsymbol{e}_1||^2_{L^2(0,2\pi)} + \tfrac{N_\nu^\circ}{\rho}||\boldsymbol{r}_s \cdot \boldsymbol{e}_2||^2_{L^2(0,2\pi)} + \tfrac{\varrho A}{\rho}|\lambda|^2||\boldsymbol{r}||^2_{L^2(0,2\pi)} \\ &- \tfrac{\varrho A\omega^2}{\rho}||\boldsymbol{r}||^2_{L^2(0,2\pi)} - R^2\omega^2||\boldsymbol{r} \cdot \boldsymbol{e}_1||^2_{L^2(0,2\pi)} + P(R)\int_0^{2\pi} (\boldsymbol{k} \times \boldsymbol{r}_s) \cdot \bar{\boldsymbol{r}}\, ds \Big). \end{aligned} \tag{2.10.6}$$

Let $\epsilon > 0$. Using the Cauchy-Bunyakovskiĭ-Schwarz inequality and the Young inequality $ab \leq \epsilon a^2 + b^2/4\epsilon$ we can bound

$$\int_0^{2\pi} (\boldsymbol{k} \times \boldsymbol{r}_s) \cdot \bar{\boldsymbol{r}}\, ds \geq -\epsilon||\boldsymbol{r}_s||^2_{L^2(0,2\pi)} - \tfrac{1}{4\epsilon}||\boldsymbol{r}||^2_{L^2(0,2\pi)}. \tag{2.10.7}$$

By substituting (2.10.7) into (2.10.6) and writing

$$||\boldsymbol{r}||^2_{L^2(0,2\pi)} = ||\boldsymbol{r}\cdot\boldsymbol{e}_1||^2_{L^2(0,2\pi)} + ||\boldsymbol{r}\cdot\boldsymbol{e}_2||^2_{L^2(0,2\pi)}, \tag{2.10.8}$$

we obtain the estimate

$$\begin{aligned}0 \geq \operatorname{Re}(\lambda)\Big(||\boldsymbol{v}||^2_{L^2(\Omega)} + c_1||\boldsymbol{r}_s\cdot\boldsymbol{e}_1||^2_{L^2(0,2\pi)} + c_2||\boldsymbol{r}_s\cdot\boldsymbol{e}_2||^2_{L^2(0,2\pi)} + c_3||\boldsymbol{r}\cdot\boldsymbol{e}_1||^2_{L^2(0,2\pi)}\\ + c_4||\boldsymbol{r}\cdot\boldsymbol{e}_2||^2_{L^2(0,2\pi)}\Big),\end{aligned} \tag{2.10.9}$$

where

$$c_1 = \frac{N^\circ}{\rho R} - \epsilon|P(R)|, \qquad c_2 = \frac{N^\circ_\nu}{\rho} - \epsilon|P(R)|,$$
$$c_3 = \frac{\varrho A|\lambda|^2}{\rho} - \frac{\varrho A\omega^2}{\rho} - R^2\omega^2 - \frac{|P(R)|}{4\epsilon}, \qquad c_4 = \frac{\varrho A|\lambda|^2}{\rho} - \frac{\varrho A\omega^2}{\rho} - \frac{|P(R)|}{4\epsilon}.$$

Choose $\epsilon = \min\{R^{-1}N^\circ, N^\circ_\nu\}/\rho|P(R)|$ so that c_1 and c_2 are nonnegative. With this choice of ϵ it is easy to check that $c_3 \geq 0$ if and only if $|\lambda|^2 \geq M$, where M was defined in the theorem statement. Note that $c_4 > c_3$.

Let $(\lambda, (\boldsymbol{v}, \boldsymbol{r}, p))$ be a smooth eigenpair of (2.9.12) with $|\lambda|^2 \geq M$. (We will prove that the eigenfunctions of (2.9.12) are smooth in Theorem (2.10.56).) Then c_1, c_2, c_3, and c_4 are nonnegative and so $\operatorname{Re}(\lambda) \leq 0$ by inequality (2.10.9). Therefore problem (2.9.12) has no eigenvalues in the set $\{\lambda \in \mathbb{C} : |\lambda|^2 \geq M,\ \operatorname{Re}(\lambda) > 0\}$. We proved that there are no eigenvalues in the set $\{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) = 0,\ \lambda \neq 0\}$ in Theorem (2.10.4). This completes the proof. □

Next we find explicit formulas for the values of ω at which eigenvalues cross the imaginary axis. We denote these critical values of ω by ω_{crit}. Since the eigenvalues cross the imaginary axis through the origin (Theorem (2.10.4)) we can substitute $\lambda = 0$ into eigenvalue problem (2.9.12) to obtain a new eigenvalue problem for ω_{crit}:

2.10.10 Eigenvalue problem for ω_{crit}.

Find eigenvalue-eigenvector pairs $(\omega_{\text{crit}}, (\boldsymbol{v}, \boldsymbol{r}, p))$ satisfying

Navier-Stokes equations. For $a < |\boldsymbol{x}| < R$,

$$
\begin{aligned}
0 &= -\nabla p + \gamma \Delta \boldsymbol{v} - 2\omega_{\text{crit}}\, \boldsymbol{k} \times \boldsymbol{v} = \tfrac{1}{\rho} \operatorname{div} \boldsymbol{\Sigma}(p, \boldsymbol{v}) - 2\omega_{\text{crit}}\, \boldsymbol{k} \times \boldsymbol{v}, \\
\nabla \cdot \boldsymbol{v} &= 0.
\end{aligned} \tag{2.10.11}
$$

The string equation. For $s \in [0, 2\pi)$,

$$
\begin{aligned}
0 = \tfrac{1}{R} N^{\circ} \boldsymbol{r}_{ss} + (N^{\circ}_{\nu} - \tfrac{1}{R} N^{\circ})(\boldsymbol{e}_2 \boldsymbol{e}_2 \cdot \boldsymbol{r}_s)_s - \rho P(R) \boldsymbol{k} \times \boldsymbol{r}_s + \varrho A \omega_{\text{crit}}^2 \boldsymbol{r} \\
+ \rho R^2 \omega_{\text{crit}}^2 (\boldsymbol{r} \cdot \boldsymbol{e}_1) \boldsymbol{e}_1 - R\, \boldsymbol{\Sigma}(\boldsymbol{v}, p) \cdot \boldsymbol{e}_1.
\end{aligned} \tag{2.10.12}
$$

The adherence boundary condition.

$$
\boldsymbol{v} = 0 \quad \text{for } |\boldsymbol{x}| = a, R. \tag{2.10.13}
$$

The area side condition.

$$
\int_0^{2\pi} \boldsymbol{r}(s) \cdot \boldsymbol{e}_1(s)\, ds = 0. \tag{2.10.14}
$$

The fluid equations are now uncoupled from the string equation. Clearly $\boldsymbol{v} = 0$ and $p = \textit{constant}$ satisfy (2.10.11) and (2.10.13). To see that this is the only solution, multiply $(2.10.11)_1$ by $\bar{\boldsymbol{v}}$, integrate by parts, and take the real part of the resulting equation to obtain $||\nabla \boldsymbol{v}||^2_{L^2(\Omega)} = 0$, which implies that $\boldsymbol{v} = 0$, and so p is constant.

Next we substitute $\boldsymbol{v} = 0$ and $p = \textit{constant}$ into the string equation. It is convenient to return to polar coordinates. Substitute $\lambda = 0$, $\omega = \omega_{\text{crit}}$, $P(R) =$

$(\frac{1}{R}N^\circ - \varrho A\omega^2)/\rho$, $\boldsymbol{v} = 0$, and $p(R,s) = p = \textit{constant}$ into the string equations (2.9.5) and (2.9.6) in polar coordinates to obtain

$$\begin{aligned} \tfrac{1}{R}N^\circ q_{ss} + (\tfrac{1}{R}N^\circ - N_\nu^\circ + \rho R^2\omega_{\text{crit}}^2)q - R(N_\nu^\circ + \varrho A\omega_{\text{crit}}^2)\psi_s &= -R\rho p, \\ RN_\nu^\circ\psi_{ss} + (N_\nu^\circ + \varrho A\omega_{\text{crit}}^2)q_s &= 0. \end{aligned} \tag{2.10.15}$$

Integrate $(2.10.15)_1$ over $[0, 2\pi]$ and use (2.9.9) and the periodicity of q_s and ψ to see that $p = 0$. Let q_k and ψ_k be the Fourier coefficients of q and ψ:

$$q(s) = \sum_{k\in\mathbb{Z}} q_k e^{iks}, \qquad \psi(s) = \sum_{k\in\mathbb{Z}} \psi_k e^{iks}. \tag{2.10.16}$$

For each $k \in \mathbb{Z}$ take the L^2-inner product of (2.10.15) with e^{iks} and use Parseval's Theorem to obtain the linear system

$$\begin{bmatrix} \frac{1}{R}N^\circ(1-k^2) - N_\nu^\circ + \rho R^2\omega_{\text{crit}}^2 & -ikR(N_\nu^\circ + \varrho A\omega_{\text{crit}}^2) \\ ik(N_\nu^\circ + \varrho A\omega_{\text{crit}}^2) & -k^2RN_\nu^\circ \end{bmatrix} \begin{bmatrix} q_k \\ \psi_k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{2.10.17}$$

The matrix on the left-hand side has determinant

$$\textit{det} = -Rk^2[(\varrho A)^2\omega_{\text{crit}}^4 + N_\nu^\circ(2\varrho A + \rho R^2)\omega_{\text{crit}}^2 - \tfrac{1}{R}N^\circ N_\nu^\circ(k^2-1)]. \tag{2.10.18}$$

If $\textit{det} = 0$, then (2.10.17) has nontrivial solutions and so ω_{crit} is an eigenvalue of (2.10.10) and $\lambda = 0$ is an eigenvalue of (2.9.12) when $\omega = \omega_{\text{crit}}$. If $k = 0$, then $\textit{det} = 0$ for all $\omega_{\text{crit}} \in \mathbb{R}$. If $|k| = 1$, then $\textit{det} = 0$ if and only if $\omega_{\text{crit}} = 0$. For $|k| \geq 2$, $\textit{det} = 0$ if and only if

$$\begin{aligned} \omega_{\text{crit}}^2 &= \omega_{\text{crit}}^2(k) \\ &= -\tfrac{1}{2(\varrho A)^2}N_\nu^\circ(2\varrho A + \rho R^2) + \tfrac{1}{2(\varrho A)^2}\sqrt{[N_\nu^\circ(2\varrho A + \rho R^2)]^2 + \tfrac{4}{R}(\varrho A)^2N^\circ N_\nu^\circ(k^2-1)}. \end{aligned} \tag{2.10.19}$$

We analyze each Fourier mode in turn.

For $k = 0$, $det = 0$ for all $\omega_{\text{crit}} \in \mathbb{R}$ and so $\lambda = 0$ is an eigenvalue of (2.9.12) for all $\omega \in \mathbb{R}$, with corresponding eigenvector $(\boldsymbol{v}, p, \boldsymbol{r}) = (0, 0, q_0\boldsymbol{e}_1 + R\psi_0\boldsymbol{e}_2)$. This appears to be a very undesirable situation: no matter what the value of our control parameter ω, $\lambda = 0$ is an eigenvalue. We will demonstrate, however, that this eigenvalue does not have any physical significance and can be factored out.

Equation (5.10.11) implies that the constant $q_0 = 0$. Therefore $\lambda = 0$ has eigenvector $(\boldsymbol{v}, p, \boldsymbol{r}) = (0, 0, R\psi_0\boldsymbol{e}_2)$ and the perturbation $\boldsymbol{r}^1$ has the form $\boldsymbol{r}^1(s,t) = C\boldsymbol{e}_2(s+\omega t)$, where C is a constant. (This follows from (2.8.14), (2.9.2) and (2.9.11).) Since the perturbed solution $\boldsymbol{r} \approx \boldsymbol{r}^0 + \varepsilon\boldsymbol{r}^1 = R\boldsymbol{e}_1(s+\omega t) + \varepsilon C\boldsymbol{e}_2(s+\omega t)$ (note that $\boldsymbol{r}^0$ and $\boldsymbol{r}^1$ are orthogonal) and since ε is small, then $|\boldsymbol{r}| \approx |\boldsymbol{r}^0| = R$. This suggests that $\boldsymbol{r}(s,t) = R\boldsymbol{e}_1(s+\alpha+\omega t)$ for some $\alpha \in \mathbb{R}$, which can be obtained from the rigid Couette solution by a rotation or by relabelling the material points. See Section 2.7. These steady solutions arise from the symmetry of the problem and are not essentially different from the rigid Couette solution; the two strings $\boldsymbol{r}(s,t) = R\boldsymbol{e}_1(s+\omega t)$ and $\boldsymbol{r}(s,t) = R\boldsymbol{e}_1(s+\alpha+\omega t)$ would look the same to an observer. Thus we factor out these solutions and the eigenvalue $\lambda = 0$ corresponding to $k = 0$ by supplementing (2.10.10) with the side condition

$$\int_0^{2\pi} \boldsymbol{r}(s) \cdot \boldsymbol{e}_2(s)\, ds = 0. \tag{2.10.20}$$

Now we consider the case $|k| = 1$. We saw that $det = 0$ if $\omega_{\text{crit}} = 0$, which suggests that the rigid Couette solution is linearly unstable for all $\omega > 0$ and so not observable. (We expect the eigenvalue $\lambda = 0$ to move into the right half-plane when ω is increased from 0. Numerical results in Section 2.13 confirm this.) To understand how this instability occurs we compute the eigenvector of (2.10.10) corresponding to eigenvalue $\omega_{\text{crit}} = 0$. Substitute $\omega_{\text{crit}} = 0$ and $k = \pm 1$ into (2.10.17) to obtain

$$\begin{bmatrix} -N_\nu^\circ & \mp iRN_\nu^\circ \\ \pm iN_\nu^\circ & -RN_\nu^\circ \end{bmatrix} \begin{bmatrix} q_{\pm 1} \\ \psi_{\pm 1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which has nontrivial solutions $[q_{\pm 1}, \psi_{\pm 1}] = c[\mp iR, 1]$, c constant. Take $c = 1$ for now. Therefore the unstable perturbations have the form

$$\boldsymbol{r}^1_{\pm 1}(s) = [q_{\pm 1}\boldsymbol{e}_1(s) + R\psi_{\pm 1}\boldsymbol{e}_2(s)]e^{\pm is} = [\mp iR\boldsymbol{e}_1(s) + R\boldsymbol{e}_2(s)]e^{\pm is},$$

which have real parts

$$\mathrm{Re}(\boldsymbol{r}^1_{\pm 1}) = R\sin(s)\boldsymbol{e}_1 + R\cos(s)\boldsymbol{e}_2 = R\boldsymbol{j}.$$

Thus the unstable perturbation is a translation of the circular string in the $\boldsymbol{j}$ direction. (Any other direction can be achieved by choosing the eigenvector scaling c appropriately.) This corresponds to the off-center steady solution that we found at the end of Section 2.7 and suggests that the rigid Couette solution becomes unstable through the following mechanism: Experimentally it is not possible to exactly align the center of the rigid disk with the center of the circular string. So when $\omega = 0$ we observe an off-center solution and not the rigid Couette solution. As ω is increased from 0 the misalignment of the centers will cause the string to move eccentrically and deform. Even if the string was rigid we still expect the rigid Couette solution to be unstable; in this case it would move eccentrically, but not deform. This type of instability could be avoided by introducing a suitable feedback control to fix the center of mass of the string at the origin. We assume that this is done, and thereby give meaning to our subsequent analysis.

Finally we turn to the case $|k| \geq 2$. Equation (2.10.19) gives all the values of ω at which the eigenvalues of (2.9.12) cross the imaginary axis. Observe that each Fourier mode k gives rise to exactly one unstable perturbation (only one eigenvalue crosses the imaginary axis for each k) and that the Fourier modes become unstable in order, i.e.,

$$0 = \omega_{\mathrm{crit}}^2(\pm 1) < \omega_{\mathrm{crit}}^2(\pm 2) < \omega_{\mathrm{crit}}^2(\pm 3) < \cdots .$$

Also, the critical values of ω do not depend on the viscosities μ and N_ν°. Since $N^\circ \to 0$ as $R \to 1$ (by equation (2.2.12)), we see from formula (2.10.19) that $\omega_{\mathrm{crit}} \to 0$ as $R \to 1$ for all k. We also see that the behavior of ω_{crit} for large R depends on the behavior of $R^{-1}N^\circ N_\nu^\circ$. If $R^{-1}N^\circ N_\nu^\circ \to \infty$ as $R \to \infty$ (the material response in tension is asymptotically strictly superlinear), then $\omega_{\mathrm{crit}} \to \infty$ as $R \to \infty$. If $R^{-1}N^\circ N_\nu^\circ \to 0$ as $R \to \infty$ (the material response in tension is asymptotically strictly sublinear), then $\omega_{\mathrm{crit}} \to 0$ as $R \to \infty$.

Recall that we only consider values of ω for which the pressure is positive (see Section 2.5). In particular we must have $0 < \rho P(R) = R^{-1}N^\circ - \varrho A \omega^2$. For some materials and some Fourier modes, $\omega_{\mathrm{crit}}(k)$ is greater than the physically admissible values.

We summarize our results in the following theorem.

Theorem 2.10.21 (Critical values of ω). *Let ω_{crit} denote the critical values of ω at which eigenvalues λ of* (2.9.12) *cross the imaginary axis, i.e.,* $\mathrm{Re}(\lambda(\omega_{\mathrm{crit}})) = 0$ (*which implies that $\lambda(\omega_{\mathrm{crit}}) = 0$ by Theorem* (2.10.4)). *Then $\{\omega_{\mathrm{crit}}\}$ are eigenvalues of problem* (2.10.10). *The eigenvalue problem obtained by supplementing* (2.10.10) *with the side condition* (2.10.20) *has eigenvalues*

$$\begin{aligned}
&\omega_{\text{crit}}(k) \equiv \omega_{\text{crit}}(k, R) \\
&= -\tfrac{1}{2(\varrho A)^2} N_\nu^\circ (2\varrho A + \rho R^2) + \tfrac{1}{2(\varrho A)^2} \sqrt{[N_\nu^\circ (2\varrho A + \rho R^2)]^2 + \tfrac{4}{R} (\varrho A)^2 N^\circ N_\nu^\circ (k^2 - 1)}
\end{aligned}$$

for $|k| = 1, 2, 3, \ldots,$ *which satisfy*

- $0 = \omega_{\text{crit}}^2(\pm 1) < \omega_{\text{crit}}^2(\pm 2) < \omega_{\text{crit}}^2(\pm 3) < \cdots.$
- *For* $|k| \geq 2,$ $\lim_{R \to 1} \omega_{\text{crit}}(k, R) = 0.$
- *For* $|k| \geq 2,$

$$\lim_{R \to \infty} \omega_{\text{crit}}(k, R) = \begin{cases} \infty & \textit{if} \quad R^{-1} N^\circ N_\nu^\circ \to \infty \textit{ as } R \to \infty \ (\textit{superlinear}), \\ 0 & \textit{if} \quad R^{-1} N^\circ N_\nu^\circ \to 0 \textit{ as } R \to \infty \ (\textit{sublinear}). \end{cases}$$

Characterization of the Spectrum for Elastic Strings

We shall characterize the spectrum of the quadratic eigenvalue problem (2.9.12), considering separately the cases $N_{\dot{\nu}} \equiv 0$ (elastic strings) and $N_{\dot{\nu}} \not\equiv 0$ (viscoelastic strings). It is easier to characterize the spectrum for elastic strings, which is what we do in this section (Theorem (2.10.23)). Viscoelastic strings are considered in the following section. Before stating the spectral theorem we need to introduce some more notation.

Notation. Recall that $\Omega = \{\boldsymbol{x} \in \mathbb{R}^2 : a < |\boldsymbol{x}| < R\}$. Let $\mathbb{T}_{2\pi}$ be the torus $\mathbb{R}/2\pi\mathbb{Z}$. For $0 \le m \in \mathbb{R}$, define function spaces

$$\begin{aligned} H_a^m(\Omega; \operatorname{div}) &:= \{\boldsymbol{v} \in H^m(\Omega; \mathbb{C}^2) : \boldsymbol{v} = 0 \text{ on } |\boldsymbol{x}| = a,\ \operatorname{div} \boldsymbol{v} = 0\}, \\ H_0^m(\Omega; \operatorname{div}) &:= H_0^m(\Omega; \mathbb{C}^2) \cap H_a^m(\Omega; \operatorname{div}), \\ H_{\mathrm{s}}^m(\mathbb{T}_{2\pi}) &:= \Big\{\boldsymbol{r} \in H^m(\mathbb{T}_{2\pi}; \mathbb{C}^2) : \int_0^{2\pi} \boldsymbol{r} \cdot \boldsymbol{e}_1 \, ds = 0\Big\}, \\ \Pi^m(\Omega) &:= \Big\{p \in H^m(\Omega; \mathbb{C}) : \int_\Omega p \, d\boldsymbol{x} = 0\Big\}. \end{aligned} \tag{2.10.22}$$

Let γ_0 denote the usual trace operator; $\gamma_0 : H^m(\Omega; \mathbb{C}^2) \to H^{m-1/2}(\partial\Omega; \mathbb{C}^2)$ for $m \ge 1$; see Temam (1977, p. 9). Let γ_R denote the restriction of γ_0 to $\{|\boldsymbol{x}| = R\}$, the outer boundary of the annulus Ω.

Theorem 2.10.23 (Spectral characterization of eigenvalue problem (2.9.12)**).** *Set* $N_\nu^\circ = 0$ *in problem* (2.9.12)*. The resulting problem has nontrivial solutions*

$$(\lambda, (\boldsymbol{v}, p, \boldsymbol{r})) \in \mathbb{C} \times H_a^2(\Omega; \operatorname{div}) \times H^1(\Omega; \mathbb{C}) \times H_{\mathrm{s}}^2(\mathbb{T}_{2\pi}).$$

The set of eigenvalues $\{\lambda\}$ *is countable with its only possible accumulation point at infinity, and each eigenvalue has finite multiplicity.*

Since the proof of this theorem requires a lot of preliminary lemmas we first sketch the main idea.

Idea of the proof of Theorem (2.10.23)**.** The quadratic eigenvalue problem (2.9.12) can be written in the form

$$(\lambda^2 T_2 + \lambda T_1 + L)(\boldsymbol{v}, p, \boldsymbol{r}) = 0, \tag{2.10.24}$$

where T_i are bounded linear operators and L is an elliptic operator defined on appropriate function spaces (for the precise definition of T_2, T_1, and L see equations (2.10.49) and (2.10.50)). The main tool for characterizing the spectrum of (2.10.24) is the following theorem.

Theorem 2.10.25 (Spectral theorem for compact polynomial operator pencils). *Let*

$$A(\lambda) = I + \sum_{k=0}^{n} \lambda^k T_k,$$

where $T_k : H \to H$ are compact operators on a Hilbert space H. Define the spectrum of A to be the set of points $\lambda \in \mathbb{C}$ such that $A(\lambda)$ lacks a bounded inverse. Then either the spectrum of A is the entire complex plane or it consists of eigenvalues of finite multiplicity with infinity as their only possible accumulation point.

For a proof see Markus (1988, Theorem 12.9). We would like to show that L has a compact inverse since then (2.10.24) could be written in the form

$$A(\lambda)(\boldsymbol{v}, p, \boldsymbol{r}) := (\lambda^2 L^{-1} T_2 + \lambda L^{-1} T_1 + I)(\boldsymbol{v}, p, \boldsymbol{r}) = 0, \tag{2.10.26}$$

with $A(\lambda)$ satisfying the hypotheses of theorem (2.10.25). This would prove Theorem (2.10.23). Therefore proving Theorem (2.10.23) reduces to proving an existence and regularity theorem for our steady linearized problem with nonhomogeneous term, $L(\boldsymbol{v}, p, \boldsymbol{r}) = (\boldsymbol{g}, \boldsymbol{h}, \Phi)$. (An existence theorem would imply that L is invertible, and a regularity theorem would imply that $L^{-1}T_i$ are compact.)

It turns out, however, that L is not invertible, but there exists a bounded linear operator T_0 such that $\tilde{L} := L - T_0$ is invertible and $\tilde{L}^{-1}$ is compact. Then

$$\tilde{A}(\lambda) := \lambda^2 \tilde{L}^{-1} T_2 + \lambda \tilde{L}^{-1} T_1 + \tilde{L}^{-1} T_0 + I \tag{2.10.27}$$

satisfies the hypotheses of Theorem (2.10.25), and $\tilde{A}(\lambda)(\boldsymbol{v}, p, \boldsymbol{r}) = 0$ is equivalent to equation (2.10.24), which proves Theorem (2.10.23).

Before giving the full proof of Theorem (2.10.23) we state some preliminary results. In many cases these are slight variations of well-known results and so we just sketch the proof.

Lemma 2.10.28 (Corollary of the Fundamental Lemma of the Calculus of Variations). *Let* $\boldsymbol{f} \in L^2(\mathbb{T}_{2\pi}, \mathbb{C}^2)$*. If*

$$\int_0^{2\pi} \boldsymbol{f} \cdot \boldsymbol{q} \, ds = 0 \quad \textit{for all} \quad \boldsymbol{q} \in L^2(\mathbb{T}_{2\pi}, \mathbb{C}^2) \quad \textit{such that} \quad \int_0^{2\pi} \boldsymbol{q} \cdot \boldsymbol{e}_1 \, ds = 0,$$

then $\boldsymbol{f}(s) = c\, \boldsymbol{e}_1(s)$*, where* c *is a constant.*

Proof. Write $\boldsymbol{f}(s) = f_1(s)\boldsymbol{e}_1(s) + f_2(s)\boldsymbol{e}_2(s)$ and $\boldsymbol{q}(s) = q_1(s)\boldsymbol{e}_1(s) + q_2(s)\boldsymbol{e}_2(s)$. Then

$$0 = \int_0^{2\pi} \boldsymbol{f} \cdot \boldsymbol{q} \, ds = \int_0^{2\pi} (f_1\, q_1 + f_2\, q_2) \, ds.$$

Set $q_1 = 0$. Then

$$\int_0^{2\pi} f_2\, q_2 \, ds = 0 \quad \text{for all} \quad q_2 \in L^2(\mathbb{T}_{2\pi}, \mathbb{C}),$$

and so $f_2 = 0$ by the Fundamental Lemma of the Calculus of Variations. Therefore

$$\int_0^{2\pi} f_1\, q_1 \, ds = 0 \quad \text{for all} \quad q_1 \in L^2(\mathbb{T}_{2\pi}, \mathbb{C}) \quad \text{such that} \quad \int_0^{2\pi} q_1 \, ds = 0. \quad (2.10.29)$$

Recall the Du Bois-Reymond Lemma:

$$\text{If } f \in L^1(a, b) \text{ satisfies } \int_a^b f(x)\varphi'(x) \, dx = 0 \quad \forall\, \varphi \in C_c^\infty(a, b), \text{ then } f = \text{constant}.$$

(See Giaquinta and Hildebrandt (1996, p. 32, Lemma 4) for a proof.)

Since all $\varphi \in C_c^\infty((0,2\pi);\mathbb{C})$ satisfy $\int_0^{2\pi} \varphi_s(s)\, ds = 0$, equation (2.10.29) with $q_1 = \varphi_s$ implies that

$$\int_0^{2\pi} f_1 \varphi_s \, ds = 0 \quad \text{for all} \quad \varphi \in C_c^\infty((0,2\pi),\mathbb{C}),$$

and so f_1 is a constant by the Du Bois-Reymond Lemma. This completes the proof. $\square$

Theorem 2.10.30 (Equivalent formulations of the Stokes-like system). *Let* $\boldsymbol{g} \in L^2(\Omega;\mathbb{C}^2)$ *and* $\varphi \in H^{1/2}(\{|\boldsymbol{x}| = R\};\mathbb{C}^2)$ *with* $\int_0^{2\pi} \varphi(R\boldsymbol{e}_1(s)) \cdot \boldsymbol{e}_1(s)\, ds = 0$. *Let* $\boldsymbol{u}_0 \in H_a^1(\Omega;\mathbb{C}^2)$ *satisfy* $\boldsymbol{u}_0 = \varphi$ *on* $\{|\boldsymbol{x}| = R\}$ *and* $\boldsymbol{u}_1 \in H_0^1(\Omega;\mathbb{C}^2)$ *satisfy* $\operatorname{div} \boldsymbol{u}_1 = -\operatorname{div} \boldsymbol{u}_0$. *(The existence of* $\boldsymbol{u}_0$ *and* $\boldsymbol{u}_1$ *is shown in Temam (1977, pp. 31-32, Section 2.4.).) Then the following are equivalent:*

(i) $\boldsymbol{v} \in H_a^1(\Omega;\operatorname{div})$ *satisfies* $\boldsymbol{v} = \varphi$ *on* $\{|\boldsymbol{x}| = R\}$, *and there exists* $p \in L^2(\Omega,\mathbb{C})$, *unique up to a constant, such that*

$$-\gamma\Delta \boldsymbol{v} + 2\omega \boldsymbol{k} \times \boldsymbol{v} + \nabla p = \boldsymbol{g} \quad \textit{in } \Omega$$

in the sense of distributions.

(ii) $\boldsymbol{v} \in H_a^1(\Omega;\operatorname{div})$ *and* $\boldsymbol{u} := \boldsymbol{v} - \boldsymbol{u}_0 - \boldsymbol{u}_1 \in H_0^1(\Omega;\operatorname{div})$ *satisfies*

$$a_f(\boldsymbol{u},\boldsymbol{w}) = (\boldsymbol{g},\boldsymbol{w})_{L^2(\Omega)} - a_f(\boldsymbol{u}_0 + \boldsymbol{u}_1, \boldsymbol{w}) \quad \textit{for all} \quad \boldsymbol{w} \in H_0^1(\Omega;\operatorname{div}), \tag{2.10.31}$$

where

$$a_f(\boldsymbol{u},\boldsymbol{v}) := \gamma \int_\Omega \left\{ \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}} : \frac{\partial \bar{\boldsymbol{w}}}{\partial \boldsymbol{x}} + 2\omega(\boldsymbol{u} \times \bar{\boldsymbol{w}}) \cdot \boldsymbol{k} \right\} d\boldsymbol{x}. \tag{2.10.32}$$

Proof. We just sketch the proof since it is very similar to the proof for the standard Stokes equations (the case $\omega = 0$), which is given in Temam (1977, p. 22, Lemma 2.1). First we show that (i) implies (ii). From (i) it follows that equation (2.10.31) holds for all $\boldsymbol{w} \in \mathcal{D}$. Now use a density argument to show that (2.10.31) holds for all $\boldsymbol{w} \in H_0^1(\Omega; \mathrm{div})$. To prove the converse, integrate by parts in (2.10.31) and use De Rham's Theorem (see Temam (1977, p. 14, Propositions 1.1,1.2)) to obtain (i). $\square$

Theorem 2.10.33 (Existence for the Stokes-like system). *Equation* (2.10.31) *has a unique solution* $\boldsymbol{u} \in H_0^1(\Omega; \mathrm{div})$.

Proof. This is also a standard result. Note that $\mathrm{Re}(\boldsymbol{u} \times \bar{\boldsymbol{u}}) = 0$. Then the theorem follows easily from the complex version of the Lax-Milgram Theorem. $\square$

Theorem 2.10.34 (Regularity for the Stokes-like system). *Let* $m \in (0, \infty)$ *and let* $\boldsymbol{v} \in H^1(\Omega; \mathbb{C}^2)$, $p \in L^2(\Omega; \mathbb{C})$ *be solutions of the Stokes-like system*

$$-\gamma \Delta \boldsymbol{v} + 2\omega \boldsymbol{k} \times \boldsymbol{v} + \nabla p = \boldsymbol{g} \quad \text{in } \Omega,$$

$$\nabla \cdot \boldsymbol{v} = 0 \quad \text{in } \Omega,$$

$$\gamma_0 \boldsymbol{v} = \Phi, \ \text{i.e.,} \ \boldsymbol{v} = \Phi \quad \text{on } \partial\Omega.$$

If $\boldsymbol{g} \in H^m(\Omega; \mathbb{C}^2)$ *and* $\Phi \in H^{m+3/2}(\partial\Omega; \mathbb{C}^2)$, *then*

$$\boldsymbol{v} \in H^{m+2}(\Omega; \mathbb{C}^2), \quad p \in H^{m+1}(\Omega; \mathbb{C}). \tag{2.10.35}$$

Proof. The proof of this regularity theorem is very similar to the proof of the regularity theorem for the Stokes equation, which can be found in Temam (1984, p. 33, Proposition 2.2), and which follows from the elliptic regularity results of Agmon, Douglis, and Nirenberg (1964) . $\square$

Theorem 2.10.36 (Gårding inequality for the string equation). *Given* $\boldsymbol{h} \in L^2(\mathbb{T}_{2\pi};\mathbb{C}^2)$*, consider the following equation for* $\boldsymbol{r} \in H^1_{\mathrm{S}}(\mathbb{T}_{2\pi})$*:*

$$-R^{-1}N^\circ \boldsymbol{r}_{ss} - (N^\circ_\nu - R^{-1}N^\circ)(\boldsymbol{e}_2\boldsymbol{e}_2\cdot\boldsymbol{r}_s)_s + \rho P(R)\boldsymbol{k}\times\boldsymbol{r}_s - \varrho A\omega^2\boldsymbol{r} - \rho R^2\omega^2(\boldsymbol{r}\cdot\boldsymbol{e}_1)\boldsymbol{e}_1 = \boldsymbol{h}, \tag{2.10.37}$$

which has the following weak formulation: Find $\boldsymbol{r} \in H^1_{\mathrm{S}}(\mathbb{T}_{2\pi})$ *such that*

$$a_s(\boldsymbol{r},\boldsymbol{q}) = (\boldsymbol{h},\boldsymbol{q})_{L^2(0,2\pi)} \quad \text{for all} \quad \boldsymbol{q} \in H^1_{\mathrm{S}}(\mathbb{T}_{2\pi}), \tag{2.10.38}$$

where

$$\begin{aligned} a_s(\boldsymbol{r},\boldsymbol{q}) := \int_0^{2\pi} &\{R^{-1}N^\circ \boldsymbol{r}_s\cdot\bar{\boldsymbol{q}}_s + (N^\circ_\nu - R^{-1}N^\circ)(\boldsymbol{r}_s\cdot\boldsymbol{e}_2)(\bar{\boldsymbol{q}}_s\cdot\boldsymbol{e}_2) \\ &+ \rho P(R)(\boldsymbol{r}_s\times\bar{\boldsymbol{q}})\cdot\boldsymbol{k} - \varrho A\omega^2\boldsymbol{r}\cdot\bar{\boldsymbol{q}} - \rho R^2\omega^2(\boldsymbol{r}\cdot\boldsymbol{e}_1)(\bar{\boldsymbol{q}}\cdot\boldsymbol{e}_1)\}\, ds. \end{aligned} \tag{2.10.39}$$

Then for all $\boldsymbol{r} \in H^1_{\mathrm{S}}(\mathbb{T}_{2\pi})$

$$a_s(\boldsymbol{r},\boldsymbol{r}) = \mathrm{Re}[a_s(\boldsymbol{r},\boldsymbol{r})] \geq C_1||\boldsymbol{r}_s||^2_{L^2(0,2\pi)} - C_2||\boldsymbol{r}||^2_{L^2(0,2\pi)}, \tag{2.10.40}$$

where

$$2C_1 = \min\{R^{-1}N^\circ, N^\circ_\nu\}, \quad C_2 = \varrho A\omega^2 + \rho R^2\omega^2 + \frac{\rho^2|P(R)|^2}{2\min\{R^{-1}N^\circ, N^\circ_\nu\}}.$$

Proof. In the proof of Theorem (2.10.2) we saw that the expression

$$\int_0^{2\pi} (\boldsymbol{r}_s\times\bar{\boldsymbol{r}})\cdot\boldsymbol{k}\, ds$$

is real. Therefore

$$
\begin{aligned}
\mathrm{Re}[a_s(\boldsymbol{r},\boldsymbol{r})] &= a_s(\boldsymbol{r},\boldsymbol{r}) \\
&= R^{-1}N^\circ ||\boldsymbol{r}_s||^2_{L^2(0,2\pi)} + (N^\circ_\nu - R^{-1}N^\circ)||\boldsymbol{r}_s\cdot\boldsymbol{e}_2||^2_{L^2(0,2\pi)} \\
&\quad + \rho P(R)\int_0^{2\pi} (\boldsymbol{r}_s \times \bar{\boldsymbol{r}})\cdot\boldsymbol{k}\; ds - \varrho A\omega^2 ||\boldsymbol{r}||^2_{L^2(0,2\pi)} - \rho R^2\omega^2||\boldsymbol{r}\cdot\boldsymbol{e}_1||^2_{L^2(0,2\pi)} \\
&= R^{-1}N^\circ ||\boldsymbol{r}_s\cdot\boldsymbol{e}_1||^2_{L^2(0,2\pi)} + N^\circ_\nu ||\boldsymbol{r}_s\cdot\boldsymbol{e}_2||^2_{L^2(0,2\pi)} \\
&\quad + \rho P(R)\int_0^{2\pi} (\boldsymbol{r}_s \times \bar{\boldsymbol{r}})\cdot\boldsymbol{k}\; ds - (\varrho A\omega^2 + \rho R^2\omega^2)||\boldsymbol{r}\cdot\boldsymbol{e}_1||^2_{L^2(0,2\pi)} \\
&\quad - \varrho A\omega^2 ||\boldsymbol{r}\cdot\boldsymbol{e}_2||^2_{L^2(0,2\pi)} \\
&\geq \min\{R^{-1}N^\circ, N^\circ_\nu\}||\boldsymbol{r}_s||^2_{L^2(0,2\pi)} - \rho|P(R)|\;||\boldsymbol{r}_s||^2_{L^2(0,2\pi)}||\boldsymbol{r}||^2_{L^2(0,2\pi)} \\
&\qquad - (\varrho A\omega^2 + \rho R^2\omega^2)||\boldsymbol{r}||^2_{L^2(0,2\pi)}.
\end{aligned}
$$

We can estimate the right-hand side using the Young inequality $ab \leq \epsilon a^2 + b^2/4\epsilon$:

$$
\begin{aligned}
\mathrm{Re}[a_s(\boldsymbol{r},\boldsymbol{r})] &\geq \min\{R^{-1}N^\circ, N^\circ_\nu\}||\boldsymbol{r}_s||^2_{L^2(0,2\pi)} - \rho|P(R)|\,\left(\epsilon||\boldsymbol{r}_s||^2_{L^2(0,2\pi)} + \tfrac{1}{4\epsilon}||\boldsymbol{r}||^2_{L^2(0,2\pi)}\right) \\
&\qquad - (\varrho A\omega^2 + \rho R^2\omega^2)||\boldsymbol{r}||^2_{L^2(0,2\pi)}.
\end{aligned}
$$

By choosing $\epsilon = \min\{R^{-1}N^\circ, N^\circ_\nu\}/(2\rho|P(R)|)$, we obtain the desired estimate (2.10.40). □

Theorem 2.10.41 (Existence for the string equation). *Let* $\boldsymbol{h} \in L^2(\mathbb{T}_{2\pi};\mathbb{C}^2)$. *Then there exists a unique* $\boldsymbol{r} \in H^1_{\mathrm{S}}(\mathbb{T}_{2\pi})$ *such that*

$$
a_s(\boldsymbol{r},\boldsymbol{q}) + 2C_2(\boldsymbol{r},\boldsymbol{q})_{L^2(0,2\pi)} = (\boldsymbol{h},\boldsymbol{q})_{L^2(0,2\pi)} \quad \textit{for all} \quad \boldsymbol{q} \in H^1_{\mathrm{S}}(\mathbb{T}_{2\pi}), \tag{2.10.42}
$$

where a_s *and* C_2 *were introduced in Theorem* (2.10.36).

Proof. This follows easily from the Gårding inequality (2.10.40) and the Lax-Milgram Theorem. □

Theorem 2.10.43 (Regularity for the string equation). *Let $m \geq 0$ be an integer and $\boldsymbol{h} \in H^m(\mathbb{T}_{2\pi};\mathbb{C}^2)$. Suppose that $\boldsymbol{r} \in H^1_{\mathrm{S}}(\mathbb{T}_{2\pi})$ satisfies* (2.10.42). *Then $\boldsymbol{r} \in H^{m+2}_{\mathrm{S}}(\mathbb{T}_{2\pi})$.*

Proof. Consider the unconstrained version of problem (2.10.42): Find $\tilde{\boldsymbol{r}} \in H^1(\mathbb{T}_{2\pi};\mathbb{C}^2)$ such that

$$a_s(\tilde{\boldsymbol{r}}, \boldsymbol{q}) + 2C_2(\tilde{\boldsymbol{r}}, \boldsymbol{q})_{L^2(0,2\pi)} = (\boldsymbol{h}, \boldsymbol{q})_{L^2(0,2\pi)} \quad \text{for all} \quad \boldsymbol{q} \in H^1(\mathbb{T}_{2\pi};\mathbb{C}^2). \quad (2.10.44)$$

This problem has a unique solution $\tilde{\boldsymbol{r}}$ by the Lax-Milgram Theorem. First we show that if $\tilde{\boldsymbol{r}} \in H^{m+2}(\mathbb{T}_{2\pi};\mathbb{C}^2)$, then $\boldsymbol{r} \in H^{m+2}_{\mathrm{S}}(\mathbb{T}_{2\pi})$. We complete the proof by showing that $\tilde{\boldsymbol{r}} \in H^{m+2}(\mathbb{T}_{2\pi};\mathbb{C}^2)$.

If $\tilde{\boldsymbol{r}} \in H^{m+2}(\mathbb{T}_{2\pi};\mathbb{C}^2)$, then we can integrate by parts in (2.10.44) to obtain

$$-R^{-1}N^\circ\tilde{\boldsymbol{r}}_{ss} - (N^\circ_\nu - R^{-1}N^\circ)(\boldsymbol{e}_2\boldsymbol{e}_2\cdot\tilde{\boldsymbol{r}}_s)_s + \rho P(R)\boldsymbol{k}\times\tilde{\boldsymbol{r}}_s - \varrho A\omega^2\tilde{\boldsymbol{r}} - \rho R^2\omega^2(\tilde{\boldsymbol{r}}\cdot\boldsymbol{e}_1)\boldsymbol{e}_1 + 2C_2\tilde{\boldsymbol{r}} = \boldsymbol{h}. \quad (2.10.45)$$

We modify $\tilde{\boldsymbol{r}}$ to obtain a function $\hat{\boldsymbol{r}} \in H^{m+2}_{\mathrm{S}}(\mathbb{T}_{2\pi})$:

$$\hat{\boldsymbol{r}} := \tilde{\boldsymbol{r}} - \frac{1}{2\pi}\left(\int_0^{2\pi} \tilde{\boldsymbol{r}}\cdot\boldsymbol{e}_1\, ds\right)\boldsymbol{e}_1.$$

Observe that $\hat{\boldsymbol{r}}$ satisfies the same equation as $\tilde{\boldsymbol{r}}$, equation (2.10.45), but with a modified right-hand side where $\boldsymbol{h}$ is replaced by $\boldsymbol{h} + c\boldsymbol{e}_1$, for some constant c. Therefore $\hat{\boldsymbol{r}}$ satisfies the constrained problem (2.10.42). But (2.10.42) has a unique solution $\boldsymbol{r}$. Thus $\boldsymbol{r} = \hat{\boldsymbol{r}} \in H^{m+2}_{\mathrm{S}}(\mathbb{T}_{2\pi})$, as required.

Now we show that $\tilde{\boldsymbol{r}} \in H^{m+2}(\mathbb{T}_{2\pi};\mathbb{C}^2)$. Integrate by parts in equation (2.10.44) to replace $\boldsymbol{q}$ wherever it appears in the volume terms with $\boldsymbol{q}_s$:

$$\int_0^{2\pi} \{(R^{-1}N^\circ\boldsymbol{e}_1\boldsymbol{e}_1 + N^\circ_\nu\boldsymbol{e}_2\boldsymbol{e}_2)\tilde{\boldsymbol{r}}_s - \rho P(R)(\boldsymbol{k}\times\tilde{\boldsymbol{r}}) - \zeta\}\cdot\bar{\boldsymbol{q}}_s\, ds = -\zeta(2\pi)\cdot\bar{\boldsymbol{q}}(2\pi), \quad (2.10.46)$$

where

$$\zeta(s) = \int_0^s \{-\varrho A\omega^2 \tilde{\boldsymbol{r}} - \rho R^2\omega^2(\tilde{\boldsymbol{r}} \cdot \boldsymbol{e}_1)\boldsymbol{e}_1 + 2C_2\tilde{\boldsymbol{r}} - \boldsymbol{h}\}\, ds.$$

Let $\boldsymbol{e}$ be a constant vector and $0 < s_1 < s_2 < 2\pi$ be Lebesgue points of the function

$$(R^{-1}N^\circ \boldsymbol{e}_1\boldsymbol{e}_1 + N_\nu^\circ \boldsymbol{e}_2\boldsymbol{e}_2)\tilde{\boldsymbol{r}}_s - \rho P(R)(\boldsymbol{k} \times \tilde{\boldsymbol{r}}) - \zeta.$$

Define H_ϵ to be the piecewise linear (pre-Heaviside) function

$$H_\epsilon := \begin{cases} 0 & s \in [0, s_1 - \frac{\epsilon}{2}], \\ \frac{s-(s_1-\frac{\epsilon}{2})}{\epsilon} & s \in [s_1 - \frac{\epsilon}{2}, s_1 + \frac{\epsilon}{2}], \\ 1 & s \in [s_1 + \frac{\epsilon}{2}, s_2 - \frac{\epsilon}{2}], \\ -\frac{s-(s_2+\frac{\epsilon}{2})}{\epsilon} & s \in [s_2 - \frac{\epsilon}{2}, s_2 + \frac{\epsilon}{2}], \\ 0 & s \in [s_2 + \frac{\epsilon}{2}, 2\pi]. \end{cases}$$

Substitute $\boldsymbol{q} = H_\epsilon \boldsymbol{e}$ into (2.10.46) to obtain

$$\begin{aligned} &\frac{1}{\epsilon}\int_{s_1-\epsilon/2}^{s_1+\epsilon/2} \{(R^{-1}N^\circ \boldsymbol{e}_1\boldsymbol{e}_1 + N_\nu^\circ \boldsymbol{e}_2\boldsymbol{e}_2)\tilde{\boldsymbol{r}}_s - \rho P(R)(\boldsymbol{k} \times \tilde{\boldsymbol{r}}) - \zeta\}\, ds \cdot \bar{\boldsymbol{e}} \\ &\qquad = \frac{1}{\epsilon}\int_{s_2-\epsilon/2}^{s_2+\epsilon/2} \{(R^{-1}N^\circ \boldsymbol{e}_1\boldsymbol{e}_1 + N_\nu^\circ \boldsymbol{e}_2\boldsymbol{e}_2)\tilde{\boldsymbol{r}}_s - \rho P(R)(\boldsymbol{k} \times \tilde{\boldsymbol{r}}) - \zeta\}\, ds \cdot \bar{\boldsymbol{e}}. \end{aligned}$$

Now apply the Lebesgue Differentiation Theorem to pass to the limit $\epsilon \to 0$ to yield

$$\begin{aligned} &\{[R^{-1}N^\circ \boldsymbol{e}_1(s_1)\boldsymbol{e}_1(s_1) + N_\nu^\circ \boldsymbol{e}_2(s_1)\boldsymbol{e}_2(s_1)]\tilde{\boldsymbol{r}}_s(s_1) - \rho P(R)[\boldsymbol{k} \times \tilde{\boldsymbol{r}}(s_1)] - \zeta(s_1)\} \cdot \bar{\boldsymbol{e}} \\ &= \{[R^{-1}N^\circ \boldsymbol{e}_1(s_2)\boldsymbol{e}_1(s_2) + N_\nu^\circ \boldsymbol{e}_2(s_2)\boldsymbol{e}_2(s_2)]\tilde{\boldsymbol{r}}_s(s_2) - \rho P(R)[\boldsymbol{k} \times \tilde{\boldsymbol{r}}(s_2)] - \zeta(s_2)\} \cdot \bar{\boldsymbol{e}}. \end{aligned} \tag{2.10.47}$$

Denote the right-hand side by $C(s_2)$. Since (2.10.47) holds for all $\boldsymbol{e}$ we obtain

$$[R^{-1}N^\circ \boldsymbol{e}_1(s_1)\boldsymbol{e}_1(s_1) + N_\nu^\circ \boldsymbol{e}_2(s_1)\boldsymbol{e}_2(s_1)]\tilde{\boldsymbol{r}}_s(s_1) - \rho P(R)[\boldsymbol{k} \times \tilde{\boldsymbol{r}}(s_1)] - \zeta(s_1) = C(s_2).$$

Therefore

$$\tilde{\boldsymbol{r}}_s(s_1) = [R^{-1}N^\circ \boldsymbol{e}_1(s_1)\boldsymbol{e}_1(s_1) + N^\circ_\nu \boldsymbol{e}_2(s_1)\boldsymbol{e}_2(s_1)]^{-1}\{\rho P(R)[\boldsymbol{k}\times\tilde{\boldsymbol{r}}(s_1)] + \zeta(s_1) + C(s_2)\}. \tag{2.10.48}$$

This holds for almost every $s_1 \in (0, 2\pi)$. Note that the right-hand side of (2.10.48) is absolutely continuous and its derivative with respect to s_1 belongs to $L^2(0, 2\pi)$. Therefore $\tilde{r}_s \in H^1(\mathbb{T}_{2\pi}; \mathbb{C}^2)$ and so $\tilde{\boldsymbol{r}} \in H^2(\mathbb{T}_{2\pi}; \mathbb{C}^2)$. Since $\boldsymbol{h} \in H^m(\mathbb{T}_{2\pi}; \mathbb{C}^2)$, by repeatedly differentiating (2.10.48) we find that $\tilde{\boldsymbol{r}} \in H^{m+2}(\mathbb{T}_{2\pi}; \mathbb{C}^2)$, as required. □

Now we are in a position to prove Theorem (2.10.23).

Proof of Theorem (2.10.23). Define $\mathcal{H}_1 := H^2_a(\Omega; \mathrm{div}) \times H^1(\Omega; \mathbb{C}) \times H^2_{\mathrm{S}}(\mathbb{T}_{2\pi})$. For $j \in \{0, 1, 2\}$, define linear operators

$$T_j : \mathcal{H}_1 \to H^2(\Omega; \mathrm{div}) \times H^2(\mathbb{T}_{2\pi}; \mathbb{C}^2) \times H^2_{\mathrm{S}}(\mathbb{T}_{2\pi}),$$

$$L : \mathcal{H}_1 \to L^2(\Omega; \mathbb{C}^2) \times L^2(\mathbb{T}_{2\pi}; \mathbb{C}^2) \times H^{3/2}_{\mathrm{S}}(\mathbb{T}_{2\pi}) =: \mathcal{H}_2$$

by

$$T_0(\boldsymbol{v}, p, \boldsymbol{r}) = \begin{bmatrix} 0 \\ -2C_2\boldsymbol{r} \\ 0 \end{bmatrix}, \quad T_1(\boldsymbol{v}, p, \boldsymbol{r}) = \begin{bmatrix} \boldsymbol{v} \\ -2\varrho A\omega\boldsymbol{r}\times\boldsymbol{k} \\ \boldsymbol{r} \end{bmatrix}, \quad T_2(\boldsymbol{v}, p, \boldsymbol{r}) = \begin{bmatrix} 0 \\ \varrho A\boldsymbol{r} \\ 0 \end{bmatrix}, \tag{2.10.49}$$

$$L(\boldsymbol{v}, p, \boldsymbol{r}) = \begin{bmatrix} L_1(\boldsymbol{v}, p, \boldsymbol{r}) \\ L_2(\boldsymbol{v}, p, \boldsymbol{r}) \\ L_3(\boldsymbol{v}, p, \boldsymbol{r}) \end{bmatrix}, \tag{2.10.50}$$

where

$$L_1(\boldsymbol{v}, p, \boldsymbol{r}) = -\gamma\Delta\boldsymbol{v} + 2\omega\boldsymbol{k}\times\boldsymbol{v} + \nabla p,$$

$$L_2(\boldsymbol{v}, p, \boldsymbol{r}) = R^{-1}N^\circ\boldsymbol{r}_{ss} - (N^\circ_\nu - R^{-1}N^\circ)(\boldsymbol{e}_2\boldsymbol{e}_2\cdot\boldsymbol{r}_s)_s + \rho P(R)\boldsymbol{k}\times\boldsymbol{r}_s - \varrho A\omega^2\boldsymbol{r}$$
$$- \rho R^2\omega^2(\boldsymbol{r}\cdot\boldsymbol{e}_1)\boldsymbol{e}_1 + R\boldsymbol{\Sigma}(\boldsymbol{v}, p)\cdot\boldsymbol{e}_1,$$

$$L_3(\boldsymbol{v}, p, \boldsymbol{r}) = -(\gamma_R\boldsymbol{v})(R\boldsymbol{e}_1(s)). \tag{2.10.51}$$

Define $\tilde{L} := L - T_0 : \mathcal{H}_1 \to \mathcal{H}_2$. Then $(\lambda, (\boldsymbol{v}, p, \boldsymbol{r})) \in \mathcal{H}_1$ satisfies the quadratic eigenvalue problem (2.9.12) if and only if it satisfies

$$(\lambda^2 T_2 + \lambda T_1 + T_0 + \tilde{L})(\boldsymbol{v}, p, \boldsymbol{r}) = 0. \tag{2.10.52}$$

We characterize the spectrum of (2.10.52), and thus the spectrum of (2.9.12), using Theorem (2.10.25). To put (2.10.52) in a form that satisfies the hypotheses of Theorem (2.10.25) we need to show that $\tilde{L}$ is invertible and $\tilde{L}^{-1}T_j : \mathcal{H}_1 \to \mathcal{H}_1$ are compact.

First we show that $\tilde{L}$ is invertible, i.e., we show that given $(\boldsymbol{g}, \boldsymbol{h}, \varphi) \in \mathcal{H}_2$, there exists a unique $(\boldsymbol{v}, p, \boldsymbol{r}) \in \mathcal{H}_1$ such that $\tilde{L}(\boldsymbol{v}, p, \boldsymbol{r}) = (\boldsymbol{g}, \boldsymbol{h}, \varphi)$. The existence and regularity theorems (2.10.30), (2.10.33), and (2.10.34) imply that there is a unique $\boldsymbol{v} \in H^2_a(\Omega; \mathrm{div})$ and a unique $\tilde{p} \in \Pi^1(\Omega)$ such that

$$-\gamma\Delta\boldsymbol{v} + 2\omega\boldsymbol{k}\times\boldsymbol{v} + \nabla\tilde{p} = \boldsymbol{g}, \qquad -(\gamma_R\boldsymbol{v})(R\boldsymbol{e}_1(s)) = \varphi(s).$$

By the existence and regularity theorems (2.10.41) and (2.10.43), there exists a unique $\boldsymbol{r} \in H^2_{\mathrm{s}}(\mathbb{T}_{2\pi})$, such that

$$a_s(\boldsymbol{r}, \boldsymbol{q}) + 2C_2(\boldsymbol{r}, \boldsymbol{q})_{L^2(0,2\pi)} = (\boldsymbol{h} - R\boldsymbol{\Sigma}(\boldsymbol{v}, \tilde{p})\cdot\boldsymbol{e}_1, \boldsymbol{q})_{L^2(0,2\pi)} \quad \text{for all} \quad \boldsymbol{q} \in H^1_{\mathrm{S}}(\mathbb{T}_{2\pi}), \tag{2.10.53}$$

where a_s was defined in equation (2.10.39). Integrate by parts in equation (2.10.53) and use Lemma (2.10.28) to obtain

$$-R^{-1}N^{\circ}\boldsymbol{r}_{ss}-(N^{\circ}_{\nu}-R^{-1}N^{\circ})(\boldsymbol{e}_2\boldsymbol{e}_2\cdot\boldsymbol{r}_s)_s+\rho P(R)\boldsymbol{k}\times\boldsymbol{r}_s-\varrho A\omega^2\boldsymbol{r}-\rho R^2\omega^2(\boldsymbol{r}\cdot\boldsymbol{e}_1)\boldsymbol{e}_1+2C_2\boldsymbol{r}$$
$$=\boldsymbol{h}-R\boldsymbol{\Sigma}(\boldsymbol{v},\tilde{p})\cdot\boldsymbol{e}_1+c\boldsymbol{e}_1, \quad (2.10.54)$$

where c is the constant coming from Lemma (2.10.28). Equation (2.10.54) does not have the desired form $\tilde{L}_2(\boldsymbol{v},\tilde{p},\boldsymbol{r})=\boldsymbol{h}$ since we have the extra term $c\boldsymbol{e}_1$ on the right-hand side. Note, however, that the pressure $\tilde{p}\in\Pi^1(\Omega)$ is essentially only determined up to a constant. We fix the constant by defining a new pressure $p=\tilde{p}-c/(\rho R)$. Then $\tilde{L}(\boldsymbol{v},p,\boldsymbol{r})=(\boldsymbol{g},\boldsymbol{h},\varphi)$, as required, and so $\tilde{L}^{-1}:\mathcal{H}_2\to\mathcal{H}_1$ exists.

Next we show that $\tilde{L}^{-1}T_j:\mathcal{H}_1\to\mathcal{H}_1$ is compact for each j. Fix j and let $(\boldsymbol{g},\boldsymbol{h},\varphi)$ be in the range of T_j. Then $\boldsymbol{g}\in H^2(\Omega;\mathbb{C}^2)$, $\boldsymbol{h}\in H^2(\mathbb{T}_{2\pi};\mathbb{C}^2)$, and $\varphi\in H^2_{\mathrm{S}}(\mathbb{T}_{2\pi})$. Following the same arguments we used to show that $\tilde{L}$ is invertible, and setting $m=\frac{1}{2}$ in Theorem (2.10.34) and $m=1$ in Theorem (2.10.43), we find that $\tilde{L}^{-1}(\boldsymbol{g},\boldsymbol{h},\varphi)\in H^{5/2}_a(\Omega;\mathrm{div})\times H^{3/2}(\Omega;\mathbb{C})\times H^3_{\mathrm{S}}(\mathbb{T}_{2\pi})$, which is compactly embedded in $\mathcal{H}_1$. Therefore $\tilde{L}^{-1}T_j:\mathcal{H}_1\to\mathcal{H}_1$ is compact.

Multiplying equation (2.10.52) by $\tilde{L}^{-1}$ puts it in a form that satisfies the hypotheses of Theorem (2.10.25):

$$A(\lambda)(\boldsymbol{v},p,\boldsymbol{r}):=(\lambda^2\tilde{L}^{-1}T_2+\lambda\tilde{L}^{-1}T_1+\tilde{L}^{-1}T_0+I)(\boldsymbol{v},p,\boldsymbol{r})=0. \quad (2.10.55)$$

Theorem (2.10.5) implies that not every $\lambda\in\mathbb{C}$ in an eigenvalue of (2.10.55), which implies that the spectrum of A is not the whole complex plane. (Note that the alternative in Theorem (2.10.25) that the spectrum be the whole complex plane is equivalent to every point being an eigenvalue: $A(\mu)$ has a bounded inverse if and only if $I - (-\mu^N T_N - \ldots - \mu T_1 - T_0) =: I - K$ has a bounded inverse, where K is compact. By Fredholm's Alternative, $I - K$ has a bounded inverse if and only if the equation $0 = (I - K)u \equiv \mu^N T_N u + \ldots + \mu T_1 u + T_0 u + u$ has no nontrivial solutions., i.e., if and only if μ is not an eigenvalue of A.) Therefore we can apply Theorem (2.10.25) to complete the proof. □

In Section 2.12 we give an alternative proof that the spectrum is countable. The proof given above is more direct and avoids the tricky weak formulation of the coupled system and the inf-sup conditions, but instead relies on elliptic regularity results.

Theorem 2.10.56 (Regularity of the eigenfunctions). *The eigenfunctions* $(\boldsymbol{v}, p, \boldsymbol{r})$ *of* (2.9.12) *are smooth* (C^∞) *functions.*

Proof. Let $(\lambda, (\boldsymbol{v}, p, \boldsymbol{r})) \in \mathbb{C} \times H^2_a(\Omega; \mathrm{div}) \times H^1(\Omega; \mathbb{C}) \times H^2_{\mathrm{S}}(\mathbb{T}_{2\pi})$ satisfy (2.9.12). In the notation of the proof of Theorem (2.10.23),

$$(\lambda^2 T_2 + \lambda T_1 + T_0 + \tilde{L})(\boldsymbol{v}, p, \boldsymbol{r}) = 0,$$

which implies that

$$(\boldsymbol{v}, p, \boldsymbol{r}) = -\tilde{L}^{-1}(\lambda^2 T_2(\boldsymbol{v}, p, \boldsymbol{r}) + \lambda T_1(\boldsymbol{v}, p, \boldsymbol{r}) + T_0(\boldsymbol{v}, p, \boldsymbol{r})).$$

By the ellipticity of $\tilde{L}$, it can be shown using the same arguments as in the proof of Theorem (2.10.23) that

$$(\boldsymbol{v}, p, \boldsymbol{r}) \in H^{5/2}_a(\Omega; \mathrm{div}) \times H^{3/2}(\Omega; \mathbb{C}) \times H^3_{\mathrm{S}}(\mathbb{T}_{2\pi}).$$

By iterating this process we see that

$$(\boldsymbol{v}, p, \boldsymbol{r}) \in H_a^{7/2}(\Omega; \mathrm{div}) \times H^{5/2}(\Omega; \mathbb{C}) \times H_{\mathrm{S}}^4(\mathbb{T}_{2\pi})$$

$$(\boldsymbol{v}, p, \boldsymbol{r}) \in H_a^{9/2}(\Omega; \mathrm{div}) \times H^{7/2}(\Omega; \mathbb{C}) \times H_{\mathrm{S}}^5(\mathbb{T}_{2\pi})$$

$$\vdots$$

$$(\boldsymbol{v}, p, \boldsymbol{r}) \in C^\infty(\Omega; \mathbb{C}^2) \times C^\infty(\Omega; \mathbb{C}) \times C^\infty(\mathbb{T}_{2\pi}; \mathbb{C}^2).$$

□

Characterization of the Spectrum for Viscoelastic Strings

The proof of Theorem (2.10.23) given above does not apply if $N_\nu^\circ \neq 0$. In this case the operator T_1 defined in (2.10.49) is replaced by

$$T_1(\boldsymbol{v}, p, \boldsymbol{r}) = \begin{bmatrix} \boldsymbol{v} \\ N_\nu^\circ(\boldsymbol{e}_2\boldsymbol{e}_2 \cdot \boldsymbol{r}_s)_s - 2\varrho A\omega \boldsymbol{r} \times \boldsymbol{k} \\ \boldsymbol{r} \end{bmatrix}, \tag{2.10.57}$$

which is regularity decreasing. It follows that $\tilde{L}^{-1}T_1$ is not compact and so the spectral theorem for compact polynomial operator pencils, Theorem (2.10.25), cannot be applied. In Section 2.12 we prove that for $N_\nu^\circ \neq 0$ the spectrum of (2.9.12) is countable (although not necessarily without finite accumulation points) by introducing a Fourier decomposition. This has the effect of removing the troublesome derivatives in (2.10.57). See the remark following Theorem (2.12.69).

2.11 Weak Formulation of the Quadratic Eigenvalue Problem

We derive a mixed weak formulation of the quadratic eigenvalue problem (2.9.12), which will be used in Sections 2.12 and 2.13 to compute the eigenvalues numerically. Our weak formulation is motivated by weak formulations of other fluid-structure interaction problems. See Planchard & Thomas (1991) and Chambolle et al. (2005). We use a mixed formulation because it is inconvenient to apply the finite element method with incompressible shape functions.

Derivation

Recall that $\Omega = \{\boldsymbol{x} \in \mathbb{R}^2 : a < |\boldsymbol{x}| < R\}$ and $\boldsymbol{n}_\Omega$ is the unit outer normal to $\partial\Omega$. Let $\boldsymbol{w} \in H^1(\Omega; \mathbb{C}^2)$ satisfy $\boldsymbol{w} = 0$ on $\{|\boldsymbol{x}| = a\}$. Take the inner product of $(2.9.13)_1$ with $\bar{\boldsymbol{w}}$, where a superscript bar denotes complex conjugation, and integrate over Ω to obtain

$$
\begin{aligned}
\lambda \int_\Omega \boldsymbol{v} \cdot \bar{\boldsymbol{w}} \, d\boldsymbol{x} &= \int_\Omega \left\{ \tfrac{1}{\rho} \operatorname{div} \boldsymbol{\Sigma}(\boldsymbol{v}, p) \cdot \bar{\boldsymbol{w}} - 2\omega \, (\boldsymbol{k} \times \boldsymbol{v}) \cdot \bar{\boldsymbol{w}} \right\} \, d\boldsymbol{x} \\
&= - \int_\Omega \left\{ \tfrac{1}{\rho} \, \boldsymbol{\Sigma}(\boldsymbol{v}, p) : \frac{\partial \bar{\boldsymbol{w}}}{\partial \boldsymbol{x}} + 2\omega \, (\boldsymbol{k} \times \boldsymbol{v}) \cdot \bar{\boldsymbol{w}} \right\} \, d\boldsymbol{x} \\
&\quad + \tfrac{1}{\rho} \int_{|\boldsymbol{x}|=R} \boldsymbol{n}_\Omega \cdot \boldsymbol{\Sigma}(\boldsymbol{v}, p) \cdot \bar{\boldsymbol{w}} \, dS \\
&= -2 \int_\Omega \{ \gamma \, \boldsymbol{D}(\boldsymbol{v}) : \boldsymbol{D}(\bar{\boldsymbol{w}}) + \omega \, (\boldsymbol{k} \times \boldsymbol{v}) \cdot \bar{\boldsymbol{w}} \} \, d\boldsymbol{x} \; + \; \int_\Omega p \operatorname{div} \bar{\boldsymbol{w}} \, d\boldsymbol{x} \\
&\quad + \tfrac{1}{\rho} \int_0^{2\pi} \boldsymbol{e}_1(s) \cdot \boldsymbol{\Sigma}(\boldsymbol{v}, p) \cdot \bar{\boldsymbol{w}} \; R \, ds,
\end{aligned}
\tag{2.11.1}
$$

where $\boldsymbol{v}$, p, and $\bar{\boldsymbol{w}}$ are evaluated at $\boldsymbol{x} = R\boldsymbol{e}_1(s)$ in the boundary term in the last line.

Let $q \in \Pi^0(\Omega)$, where $\Pi^0(\Omega)$ was defined in equation (2.10.22). Multiply $(2.9.13)_2$ by $\bar{q}$ and integrate over Ω to obtain

$$\int_{\Omega} \bar{q} \operatorname{div} \boldsymbol{v}\, d\boldsymbol{x} = 0. \tag{2.11.2}$$

Let $\boldsymbol{q} \in H^1_{\mathrm{S}}(\mathbb{T}_{2\pi})$, where $H^1_{\mathrm{S}}(\mathbb{T}_{2\pi})$ was defined in equation (2.10.22). Take the inner product of (2.9.14) with $\bar{\boldsymbol{q}}$ and integrate by parts over $[0, 2\pi]$ to derive

$$\begin{aligned}
&\lambda^2 \varrho A \int_0^{2\pi} \boldsymbol{r} \cdot \bar{\boldsymbol{q}}\, ds + \lambda \int_0^{2\pi} \left\{ N^\circ_\nu \, (\boldsymbol{r}_s \cdot \boldsymbol{e}_2)(\bar{\boldsymbol{q}}_s \cdot \boldsymbol{e}_2) - 2\varrho A \omega (\boldsymbol{r} \times \boldsymbol{k}) \cdot \bar{\boldsymbol{q}} \right\}\, ds \\
&= - \int_0^{2\pi} \Big\{ R^{-1} N^\circ \boldsymbol{r}_s \cdot \bar{\boldsymbol{q}}_s + \left(N^\circ_\nu - R^{-1} N^\circ\right) (\boldsymbol{r}_s \cdot \boldsymbol{e}_2)(\bar{\boldsymbol{q}}_s \cdot \boldsymbol{e}_2) + \rho P(R)(\boldsymbol{k} \times \boldsymbol{r}_s) \cdot \bar{\boldsymbol{q}} \\
&- \varrho A \omega^2 \boldsymbol{r} \cdot \bar{\boldsymbol{q}} - \rho R^2 \omega^2 (\boldsymbol{r} \cdot \boldsymbol{e}_1)(\bar{\boldsymbol{q}} \cdot \boldsymbol{e}_1) \Big\}\, ds - R \int_0^{2\pi} \boldsymbol{e}_1(s) \cdot \boldsymbol{\Sigma}(\boldsymbol{v}, p) \cdot \bar{\boldsymbol{q}}\, ds.
\end{aligned} \tag{2.11.3}$$

Recall that $\gamma_R \boldsymbol{v}$ denotes the trace of $\boldsymbol{v}$ restricted to $\{|\boldsymbol{x}| = R\}$. Taking the $H^{1/2}$ inner product of the adherence condition $(2.9.15)_2$ with $\boldsymbol{t} \in H^{1/2}(\mathbb{T}_{2\pi}; \mathbb{C}^2)$ gives

$$\lambda \langle \boldsymbol{r}, \boldsymbol{t} \rangle_{H^{1/2}} = \langle \gamma_R \boldsymbol{v}, \boldsymbol{t} \rangle_{H^{1/2}}, \tag{2.11.4}$$

where $\langle \cdot, \cdot \rangle_{H^{1/2}}$ is the inner product on $H^{1/2}(\mathbb{T}_{2\pi}; \mathbb{C}^2)$. The importance of enforcing the adherence boundary condition in the $H^{1/2}$-inner product rather than just the L^2-inner product will become clear in the proof of Theorem (2.11.18). If $\boldsymbol{r} = r^1(s)\boldsymbol{e}_1(s) + r^2(s)\boldsymbol{e}_2(s)$, $\boldsymbol{t} = t^1(s)\boldsymbol{e}_1(s) + t^2(s)\boldsymbol{e}_2(s)$, and if r^j and t^j have Fourier coefficients $\{r^j_k\}_{k \in \mathbb{Z}}$ and $\{t^j_k\}_{k \in \mathbb{Z}}$, for $j \in \{1, 2\}$, then the $H^{1/2}$-inner product can be defined by

$$\langle \boldsymbol{r}, \boldsymbol{t} \rangle_{H^{1/2}} = \sum_{k=-\infty}^{\infty} (1 + |k|)\, r^1_k\, \overline{t^1_k} + \sum_{k=-\infty}^{\infty} (1 + |k|)\, r^2_k\, \overline{t^2_k}.$$

Observe that if we choose test functions $\boldsymbol{w}$ and $\boldsymbol{q}$ with $\boldsymbol{q} = \gamma_R \boldsymbol{w}$, then multiplying equation (2.11.1) by ρ and adding it to equation (2.11.3) eliminates the boundary terms involving $\boldsymbol{\Sigma}(\boldsymbol{v}, p)$. This is an important trick: if the boundary terms are not eliminated, then we must seek solutions with high enough regularity ($\boldsymbol{v} \in H^2(\Omega; \mathbb{C})$, $p \in H^1(\Omega; \mathbb{C})$) so that the restriction of $\boldsymbol{\Sigma}$ to $\partial\Omega$ makes sense.

The sum of equation (2.11.3), equation (2.11.4), and ρ times equation (2.11.1) is

$$\begin{aligned}
&\lambda^2 \varrho A \int_0^{2\pi} \boldsymbol{r} \cdot \bar{\boldsymbol{q}}\, ds + \lambda \Bigg(\int_0^{2\pi} \{ N_\nu^\circ (\boldsymbol{r}_s \cdot \boldsymbol{e}_2)(\bar{\boldsymbol{q}}_s \cdot \boldsymbol{e}_2) - 2\varrho A \omega (\boldsymbol{r} \times \boldsymbol{k}) \cdot \bar{\boldsymbol{q}} \}\, ds \\
&\qquad\qquad + \rho \int_\Omega \boldsymbol{v} \cdot \bar{\boldsymbol{w}}\, d\boldsymbol{x} - \langle \boldsymbol{r}, \boldsymbol{t} \rangle_{H^{1/2}} \Bigg) \\
&= -\int_0^{2\pi} \{ R^{-1} N^\circ \boldsymbol{r}_s \cdot \bar{\boldsymbol{q}}_s + (N_\nu^\circ - R^{-1} N^\circ)(\boldsymbol{r}_s \cdot \boldsymbol{e}_2)(\bar{\boldsymbol{q}}_s \cdot \boldsymbol{e}_2) + \rho P(R)(\boldsymbol{r}_s \times \bar{\boldsymbol{q}}) \cdot \boldsymbol{k} - \varrho A \omega^2 \boldsymbol{r} \cdot \bar{\boldsymbol{q}} \\
&\qquad\qquad - \rho R^2 \omega^2 (\boldsymbol{r} \cdot \boldsymbol{e}_1)(\bar{\boldsymbol{q}} \cdot \boldsymbol{e}_1) \}\, ds \\
&\quad - 2 \int_\Omega \{ \mu \boldsymbol{D}(\boldsymbol{v}) : \boldsymbol{D}(\bar{\boldsymbol{w}}) + \rho\omega (\boldsymbol{k} \times \boldsymbol{v}) \cdot \bar{\boldsymbol{w}} \}\, d\boldsymbol{x} + \rho \int_\Omega p \operatorname{div} \bar{\boldsymbol{w}}\, d\boldsymbol{x} - \langle \gamma_R \boldsymbol{v}, \boldsymbol{t} \rangle_{H^{1/2}}.
\end{aligned} \tag{2.11.5}$$

Notation. Define

$$H_a^1(\Omega) := \{ \boldsymbol{v} \in H^1(\Omega; \mathbb{C}^2) : \boldsymbol{v} = 0 \text{ on } \{|\boldsymbol{x}| = a\} \}, \tag{2.11.6}$$

$$\mathcal{V}_1 := \{ (\boldsymbol{v}, \boldsymbol{r}) \in H_a^1(\Omega) \times H_{\mathrm{S}}^1(\mathbb{T}_{2\pi}) \},$$

$$\mathcal{V}_2 := \{ (\boldsymbol{w}, \boldsymbol{q}, \boldsymbol{t}) \in H_a^1(\Omega) \times H_{\mathrm{S}}^1(\mathbb{T}_{2\pi}) \times H^{1/2}(\mathbb{T}_{2\pi}; \mathbb{C}^2) : \gamma_R \boldsymbol{w}(R\boldsymbol{e}_1(s)) = \boldsymbol{q}(s) \},$$

$$\Pi := \Pi^0(\Omega) = \left\{ p \in L^2(\Omega; \mathbb{C}) : \int_\Omega p\, d\boldsymbol{x} = 0 \right\}.$$

Lemma 2.11.7 (Norms on $\mathcal{V}_1$ and $\mathcal{V}_2$). *Define*

$$\langle (\boldsymbol{v}_1, \boldsymbol{r}_1), (\boldsymbol{v}_2, \boldsymbol{r}_2) \rangle_{\mathcal{V}_1} := \langle \boldsymbol{D}(\boldsymbol{v}_1), \boldsymbol{D}(\boldsymbol{v}_2) \rangle_{L^2(\Omega)} + \langle \boldsymbol{r}_1, \boldsymbol{r}_2 \rangle_{H^1(0,2\pi)},$$

$$\|(\boldsymbol{v}, \boldsymbol{r})\|_{\mathcal{V}_1} := \left(\|\boldsymbol{D}(\boldsymbol{v})\|_{L^2(\Omega)}^2 + \|\boldsymbol{r}\|_{H^1(0,2\pi)}^2 \right)^{1/2},$$

and

$$\langle(\boldsymbol{w}_1,\boldsymbol{q}_1,\boldsymbol{t}_1),(\boldsymbol{w}_2,\boldsymbol{q}_2,\boldsymbol{t}_2)\rangle_{\mathcal{V}_2} := \langle \boldsymbol{D}(\boldsymbol{w}_1),\boldsymbol{D}(\boldsymbol{w}_2)\rangle_{L^2(\Omega)} + \langle \boldsymbol{q}_1,\boldsymbol{q}_2\rangle_{H^1(0,2\pi)} + \langle \boldsymbol{t}_1,\boldsymbol{t}_2\rangle_{H^{1/2}(0,2\pi)},$$

$$||\,(\boldsymbol{w},\boldsymbol{q},\boldsymbol{t})\,||_{\mathcal{V}_2} := \left(||\boldsymbol{D}(\boldsymbol{w})||^2_{L^2(\Omega)} + ||\boldsymbol{q}||^2_{H^1(0,2\pi)} + ||\boldsymbol{t}||^2_{H^{1/2}(0,2\pi)}\right)^{1/2}.$$

Then $(\mathcal{V}_1,\langle\cdot\,,\cdot\rangle_{\mathcal{V}_1})$ *and* $(\mathcal{V}_2,\langle\cdot\,,\cdot\rangle_{\mathcal{V}_2})$ *are complex Hilbert spaces.*

Proof. This is left as an easy exercise. Note that $||\boldsymbol{D}(\cdot)||_{L^2(\Omega)}$ is a norm on $H^1_a(\Omega)$ by Korn's inequality and Poincaré's inequality. □

Equations (2.11.2) and (2.11.5) constitute the

Weak formulation of the quadratic eigenvalue problem. Find $\lambda \in \mathbb{C}$ and $0 \neq (\boldsymbol{v},\boldsymbol{r},p) \in \mathcal{V}_1 \times \Pi$ such that for all $(\boldsymbol{w},\boldsymbol{q},\boldsymbol{t},q) \in \mathcal{V}_2 \times \Pi$

$$\begin{aligned}\lambda^2 a_2(\boldsymbol{r},\boldsymbol{q}) + \lambda a_1((\boldsymbol{v},\boldsymbol{r}),(\boldsymbol{w},\boldsymbol{q},\boldsymbol{t})) + a_0((\boldsymbol{v},\boldsymbol{r}),(\boldsymbol{w},\boldsymbol{q},\boldsymbol{t})) + b(\boldsymbol{w},p) &= 0,\\ b(\boldsymbol{v},q) &= 0,\end{aligned} \tag{2.11.8}$$

where

$$\begin{aligned}&a_0((\boldsymbol{v},\boldsymbol{r}),(\boldsymbol{w},\boldsymbol{q},\boldsymbol{t})) :=\\ &\int_0^{2\pi} \left\{R^{-1}N^\circ \boldsymbol{r}_s\cdot\bar{\boldsymbol{q}}_s + (N^\circ_\nu - R^{-1}N^\circ)(\boldsymbol{r}_s\cdot\boldsymbol{e}_2)(\bar{\boldsymbol{q}}_s\cdot\boldsymbol{e}_2) + \rho P(R)(\boldsymbol{k}\times\boldsymbol{r}_s)\cdot\bar{\boldsymbol{q}}\right.\\ &\qquad\qquad \left.-\varrho A\omega^2\boldsymbol{r}\cdot\bar{\boldsymbol{q}} - \rho R^2\omega^2(\boldsymbol{r}\cdot\boldsymbol{e}_1)(\bar{\boldsymbol{q}}\cdot\boldsymbol{e}_1)\right\}\,ds\\ &+2\int_\Omega \{\tilde{\mu}\boldsymbol{D}(\boldsymbol{v}):\boldsymbol{D}(\bar{\boldsymbol{w}}) + \rho\omega(\boldsymbol{k}\times\boldsymbol{v})\cdot\bar{\boldsymbol{w}}\}\,d\boldsymbol{x} + \langle\gamma_R\boldsymbol{v},\boldsymbol{t}\rangle_{H^{1/2}},\end{aligned} \tag{2.11.9}$$

$$\begin{aligned}&a_1((\boldsymbol{v},\boldsymbol{r}),(\boldsymbol{w},\boldsymbol{q},\boldsymbol{t})) :=\\ &\int_0^{2\pi}\{N^\circ_{\dot{\nu}}(\boldsymbol{r}_s\cdot\boldsymbol{e}_2)(\bar{\boldsymbol{q}}_s\cdot\boldsymbol{e}_2) - 2\varrho A\omega(\boldsymbol{r}\times\boldsymbol{k})\cdot\bar{\boldsymbol{q}}\}\,ds + \rho\int_\Omega \boldsymbol{v}\cdot\bar{\boldsymbol{w}}\,d\boldsymbol{x} - \langle\boldsymbol{r},\boldsymbol{t}\rangle_{H^{1/2}},\end{aligned} \tag{2.11.10}$$

$$a_2(\boldsymbol{r},\boldsymbol{q}) := \varrho A\int_0^{2\pi}\boldsymbol{r}\cdot\bar{\boldsymbol{q}}\,ds, \qquad b(\boldsymbol{w},p) := -\rho\int_\Omega p\,\mathrm{div}\,\bar{\boldsymbol{w}}\,d\boldsymbol{x}. \tag{2.11.11}$$

Lemma 2.11.12 (Smooth solutions of the weak problem satisfy the classical problem). *Let* $(\lambda, (\boldsymbol{v}, \boldsymbol{r}, p))$ *be a smooth solution of the weak problem* (2.11.8). *Then there exists a unique constant* Q *such that* $(\lambda, (\boldsymbol{v}, \boldsymbol{r}, p + Q))$ *satisfies the classical problem* (2.9.12).

Proof.

(i) Setting $\boldsymbol{w} = 0$, $\boldsymbol{q} = 0$, and $\boldsymbol{t} = \lambda \boldsymbol{r} - \gamma_R \boldsymbol{v}$ in $(2.11.8)_1$ yields $||\lambda \boldsymbol{r} - \gamma_R \boldsymbol{v}||^2_{H^{1/2}} = 0$, which implies that $\gamma_R \boldsymbol{v} = \lambda \boldsymbol{r}$, i.e., $\boldsymbol{v} = \lambda \boldsymbol{r}$ on $\{|\boldsymbol{x}| = R\}$.

(ii) By the Divergence Theorem and (i),

$$\int_\Omega \operatorname{div} \boldsymbol{v} \; d\boldsymbol{x} = \int_{\partial\Omega} \boldsymbol{v} \cdot \boldsymbol{n} \; d\boldsymbol{x} = \lambda \int_0^{2\pi} \boldsymbol{r} \cdot \boldsymbol{e}_1 \; ds = 0. \tag{2.11.13}$$

Therefore we can substitute $q = \operatorname{div} \boldsymbol{v}$ into $(2.11.8)_2$ to obtain $||\operatorname{div} \boldsymbol{v}||^2_{L^2(\Omega)} = 0$. Thus $\operatorname{div} \boldsymbol{v} = 0$.

(iii) Set $\boldsymbol{q} = \boldsymbol{t} = 0$ in $(2.11.8)_1$ to obtain the following (note that $\boldsymbol{q} = 0$ implies $\boldsymbol{w} = 0$ on $\partial\Omega$):

$$\begin{aligned}
\lambda\rho \int_\Omega \boldsymbol{v} \cdot \bar{\boldsymbol{w}} \; d\boldsymbol{x} &= \int_\Omega \{-2\tilde{\mu} \boldsymbol{D}(\boldsymbol{v}) : \boldsymbol{D}(\bar{\boldsymbol{w}}) - 2\rho\omega(\boldsymbol{k} \times \boldsymbol{v}) \cdot \bar{\boldsymbol{w}}\} \; d\boldsymbol{x} \\
&\quad + \int_\Omega \rho \, p \operatorname{div} \bar{\boldsymbol{w}} \; d\boldsymbol{x} \\
&= \int_\Omega \{2\tilde{\mu} \operatorname{div} \boldsymbol{D}(\boldsymbol{v}) - 2\rho\omega(\boldsymbol{k} \times \boldsymbol{v}) - \rho\nabla p\} \cdot \bar{\boldsymbol{w}} \; d\boldsymbol{x}
\end{aligned} \tag{2.11.14}$$

for all $\boldsymbol{w} \in H^1_0(\Omega, \mathbb{C}^2)$. Since $\boldsymbol{v}$ is divergence-free, $2 \operatorname{div} \boldsymbol{D}(\boldsymbol{v}) = \Delta \boldsymbol{v}$. Therefore equation (2.11.14) implies that the Stokes-like equation $(2.9.13)_1$ is satisfied.

(iv) Set $\boldsymbol{t} = 0$ in $(2.11.8)_1$, then integrate by parts and use $(2.9.13)_1$ and $\gamma_R \boldsymbol{w} = \boldsymbol{q}$

to obtain

$$0 = \int_0^{2\pi} \big\{\lambda^2 \varrho A \boldsymbol{r} - \lambda[N_\nu^\circ(\boldsymbol{e}_2\boldsymbol{e}_2 \cdot \boldsymbol{r}_s)_s + 2\varrho A\omega \boldsymbol{r} \times \boldsymbol{k}] - R^{-1}N^\circ \boldsymbol{r}_{ss}$$

$$-(N_\nu^\circ - R^{-1}N^\circ)(\boldsymbol{e}_2\boldsymbol{e}_2 \cdot \boldsymbol{r}_s)_s + \rho P(R)\boldsymbol{k} \times \boldsymbol{r}_s - \varrho A\omega^2 \boldsymbol{r}$$

$$-\rho R^2\omega^2(\boldsymbol{r} \cdot \boldsymbol{e}_1)\boldsymbol{e}_1 + R\,\boldsymbol{\Sigma}(\boldsymbol{v}, p) \cdot \boldsymbol{e}_1\big\} \cdot \bar{\boldsymbol{q}}\; ds$$

for all $\boldsymbol{q} \in H_\mathrm{s}^1(\mathbb{T}_{2\pi})$. Therefore by Lemma (2.10.28) there exists a constant Q_1 such that

$$Q_1\boldsymbol{e}_1 = \lambda^2 \varrho A \boldsymbol{r} - \lambda[N_\nu^\circ(\boldsymbol{e}_2\boldsymbol{e}_2 \cdot \boldsymbol{r}_s)_s + 2\varrho A\omega \boldsymbol{r} \times \boldsymbol{k}] - R^{-1}N^\circ \boldsymbol{r}_{ss}$$

$$-(N_\nu^\circ - R^{-1}N^\circ)(\boldsymbol{e}_2\boldsymbol{e}_2 \cdot \boldsymbol{r}_s)_s + \rho P(R)\boldsymbol{k} \times \boldsymbol{r}_s - \varrho A\omega^2 \boldsymbol{r} - \rho R^2\omega^2(\boldsymbol{r} \cdot \boldsymbol{e}_1)\boldsymbol{e}_1 + R\,\boldsymbol{\Sigma}(\boldsymbol{v}, p) \cdot \boldsymbol{e}_1.$$

The term on the left-hand side can be included in the pressure term:

$$0 = \lambda^2 \varrho A \boldsymbol{r} - \lambda[N_\nu^\circ(\boldsymbol{e}_2\boldsymbol{e}_2 \cdot \boldsymbol{r}_s)_s + 2\varrho A\omega \boldsymbol{r} \times \boldsymbol{k}] - R^{-1}N^\circ \boldsymbol{r}_{ss} - (N_\nu^\circ - R^{-1}N^\circ)(\boldsymbol{e}_2\boldsymbol{e}_2 \cdot \boldsymbol{r}_s)_s$$

$$+ \rho P(R)\boldsymbol{k} \times \boldsymbol{r}_s - \varrho A\omega^2 \boldsymbol{r} - \rho R^2\omega^2(\boldsymbol{r} \cdot \boldsymbol{e}_1)\boldsymbol{e}_1 + R\,\boldsymbol{\Sigma}(\boldsymbol{v}, p + Q_1/\rho R) \cdot \boldsymbol{e}_1.$$

If we define $Q := Q_1/\rho R$, then we see that $(\lambda, (\boldsymbol{v}, \boldsymbol{r}, p+Q))$ satisfies the string equation (2.9.14).

□

From the weak formulation (2.11.8) we could use the 2-dimensional finite element method to compute the eigenvalues. A more efficient method, however, is to write equation (2.11.8) in polar coordinates and then use Fourier series in the angle variable ϕ to reduce the partial differential equations in r and ϕ to ordinary differential equations in r. Then the 1-dimensional finite element method can be used. This is what we do in the following sections, but first we prove that the bilinear forms a_0 and b satisfy the inf-sup conditions. These inf-sup conditions could be used to give an alternative proof of the spectral theorem (2.10.23), and discrete versions of these inf-sup conditions could be used to prove convergence of the 2-dimensional finite element method. While we do not do this, it is illuminating to prove the inf-sup conditions anyway because it shows that our weak formulation is well-posed and shows why it is important to enforce the adherence boundary condition in $H^{1/2}$ and not just L^2.

The following result is well-known (see Girault & Raviart (1986) or Brenner & Scott (2002, Chapter 12, Section 2)) :

Theorem 2.11.15 (b satisfies an inf-sup condition). *Let b be the bilinear form defined in equation* (2.11.11)*. Then*

$$\inf_{p\in\Pi}\sup_{\boldsymbol{w}\in H_0^1(\Omega)}\frac{|b(\boldsymbol{w},p)|}{||p||_\Pi||\boldsymbol{w}||_{H^1}}=\beta>0.$$

It follows immediately that

$$\inf_{p\in\Pi}\sup_{(\boldsymbol{w},\boldsymbol{q},\boldsymbol{t})\in\mathcal{V}_2}\frac{|b(\boldsymbol{w},p)|}{||p||_\Pi||(\boldsymbol{w},\boldsymbol{q},\boldsymbol{t})||_{\mathcal{V}_2}}=\beta>0. \tag{2.11.16}$$

Define

$$\begin{aligned}\mathcal{Z}_1&:=\{(\boldsymbol{v},\boldsymbol{r})\in\mathcal{V}_1 : b(\boldsymbol{v},p)=0 \text{ for all } p\in\Pi\},\\ \mathcal{Z}_2&:=\{(\boldsymbol{w},\boldsymbol{q},\boldsymbol{t})\in\mathcal{V}_2 : b(\boldsymbol{w},p)=0 \text{ for all } p\in\Pi\}.\end{aligned} \tag{2.11.17}$$

Theorem 2.11.18 (a_0 satisfies a Gårding-type inf-sup condition). *There exists a constant $C_{\mathrm{g}}>0$ such that the bilinear form $\hat{a}_0:\mathcal{V}_1\times\mathcal{V}_2\to\mathbb{C}$ defined by*

$$\hat{a}_0((\boldsymbol{v},\boldsymbol{r}),(\boldsymbol{w},\boldsymbol{q},\boldsymbol{t})):=a_0((\boldsymbol{v},\boldsymbol{r}),(\boldsymbol{w},\boldsymbol{q},\boldsymbol{t}))+C_{\mathrm{g}}\int_0^{2\pi}\boldsymbol{r}\cdot\bar{\boldsymbol{q}}\,ds \tag{2.11.19}$$

satisfies the inf-sup conditions

$$\inf_{\substack{(\boldsymbol{v},\boldsymbol{r})\in\mathcal{Z}_1\\ \|(\boldsymbol{v},\boldsymbol{r})\|_{\mathcal{V}_1}=1}} \sup_{\substack{(\boldsymbol{w},\boldsymbol{q},\boldsymbol{t})\in\mathcal{Z}_2\\ \|(\boldsymbol{w},\boldsymbol{q},\boldsymbol{t})\|_{\mathcal{V}_2}=1}} |\hat{a}_0((\boldsymbol{v},\boldsymbol{r}),(\boldsymbol{w},\boldsymbol{q},\boldsymbol{t}))| = \alpha > 0, \tag{2.11.20}$$

$$(\boldsymbol{w},\boldsymbol{q},\boldsymbol{t}) = 0 \ \textit{ if } \ \hat{a}_0((\boldsymbol{v},\boldsymbol{r}),(\boldsymbol{w},\boldsymbol{q},\boldsymbol{t})) = 0 \ \textit{ for all } \ (\boldsymbol{v},\boldsymbol{r})\in\mathcal{Z}_1. \tag{2.11.21}$$

Will we use the following lemma to prove Theorem (2.11.18).

Lemma 2.11.22. *Decompose the bilinear form a_0 defined in* (2.11.9) *into three bilinear forms:*

$$a_0((\boldsymbol{v},\boldsymbol{r}),(\boldsymbol{w},\boldsymbol{q},\boldsymbol{t})) =: d_1(\boldsymbol{r},\boldsymbol{q}) + d_2(\boldsymbol{v},\boldsymbol{w}) + \langle\gamma_R\boldsymbol{v},\boldsymbol{t}\rangle_{H^{1/2}}, \tag{2.11.23}$$

where d_1 and d_2 correspond to the string terms and the fluid terms of a_0. (*Note that d_1 is the same as the bilinear form a_s defined in equation* (2.10.39).) *Then*

(i) d_1 satisfies a Gårding inequality: There exists a constant $C_{\mathrm{g}_1} > 0$ such that

$$d_1(\boldsymbol{r},\boldsymbol{r}) + C_{\mathrm{g}_1}\|\boldsymbol{r}\|^2_{L^2(0,2\pi)} \geq \alpha_1\|\boldsymbol{r}\|^2_{H^1(0,2\pi)} \ \textit{ for all } \ \boldsymbol{r}\in H^1_{\mathrm{S}}(\mathbb{T}_{2\pi});$$

(ii) d_2 is coercive:

$$\mathrm{Re}[d_2(\boldsymbol{v},\boldsymbol{v})] = 2\tilde{\mu}\|\boldsymbol{D}(\boldsymbol{v})\|^2_{L^2(\Omega)} \ \textit{ for all } \ \boldsymbol{v}\in H^1_a(\Omega).$$

Proof. Part (i) is just a restatement of Theorem (2.10.36). Part (ii) is clear. □

Proof of Theorem (2.11.18). Recall that the inf-sup conditions (2.11.20) and (2.11.21) are equivalent to the well-posedness of the following problem:

Find $(\boldsymbol{v},\boldsymbol{r})\in\mathcal{Z}_1$ such that $\hat{a}_0((\boldsymbol{v},\boldsymbol{r}),(\boldsymbol{w},\boldsymbol{q},\boldsymbol{t})) = F(\boldsymbol{w},\boldsymbol{q},\boldsymbol{t})$ for all $(\boldsymbol{w},\boldsymbol{q},\boldsymbol{t})\in\mathcal{Z}_2$,

(2.11.24)

where $F \in (\mathcal{Z}_2)^*$, the space of bounded linear functionals on $\mathcal{Z}_2$. See Ern & Guermond (2004, p. 85, Theorem 2.6). To prove Theorem (2.11.18) we will prove that problem (2.11.24) is well-posed.

Choose $C_\mathrm{g} = C_{\mathrm{g}_1}$, where C_g is the constant appearing in the definition of $\hat{a}_0$ and C_{g_1} was defined in Lemma (2.11.22)(i).

We start by constructing a candidate for the solution $(\boldsymbol{v}, \boldsymbol{r})$ of (2.11.24). Substitute $\boldsymbol{w} = 0$, $\boldsymbol{q} = 0$ into (2.11.24) to obtain the problem: find $\gamma_R \boldsymbol{v}(R\boldsymbol{e}_1(s)) = \varphi(s) \in H^{1/2}(\mathbb{T}_{2\pi}; \mathbb{C}^2)$ such that

$$\langle \varphi, \boldsymbol{t} \rangle_{H^{1/2}} = F(0, 0, \boldsymbol{t}) \quad \text{for all} \quad \boldsymbol{t} \in H^{1/2}(\mathbb{T}_{2\pi}; \mathbb{C}^2). \tag{2.11.25}$$

By the Riesz Representation Theorem there exists a unique solution φ to (2.11.25). Note that $||\varphi||_{H^{1/2}} = ||F(0, 0, \cdot)||$, where $||F(0, 0, \cdot)||$ denotes the norm of the bounded linear operator $F(0, 0, \cdot)$.

Let

$$\begin{aligned} \mathcal{Z} &= \{\boldsymbol{w} \in H^1(\Omega; \mathbb{C}^2) : b(\boldsymbol{w}, p) = 0 \text{ for all } p \in \Pi\}, \\ \mathcal{Z}_0 &= H_0^1(\Omega; \mathbb{C}^2) \cap \mathcal{Z}. \end{aligned} \tag{2.11.26}$$

Substitute $\boldsymbol{q} = 0$, $\boldsymbol{t} = 0$ in (2.11.24) to obtain the problem: Find $\boldsymbol{v} \in \mathcal{Z}$ with $\boldsymbol{v} = 0$ on $\{|\boldsymbol{x}| = a\}$ and $\gamma_R \boldsymbol{v} = \varphi$ such that

$$d_2(\boldsymbol{v}, \boldsymbol{w}) = F(\boldsymbol{w}, 0, 0) \quad \text{for all} \quad \boldsymbol{w} \in \mathcal{Z}_0. \tag{2.11.27}$$

There exists a $\boldsymbol{g} \in \mathcal{Z}$ such that $\boldsymbol{g} = 0$ on $\{|\boldsymbol{x}| = a\}$, $\gamma_R \boldsymbol{g} = \varphi$, and $||\boldsymbol{g}||_{H^1} \leq C||\varphi||_{H^{1/2}}$. (The existence of $\boldsymbol{g}$ is proved as follows. Since $\varphi \in H^{1/2}(\mathbb{T}_{2\pi}; \mathbb{C}^2)$ the Trace Theorem (see for example Ern & Guermond (2004, p. 488, Theorem B.52)) implies that there exists a function $\boldsymbol{h} \in H^1(\Omega; \mathbb{C}^2)$ such that $\boldsymbol{h} = 0$ on $\{|\boldsymbol{x}| = a\}$ and $\boldsymbol{h} = \varphi$ on $\{|\boldsymbol{x}| = R\}$ in the sense of trace, and $||\boldsymbol{h}||_{H^1} \leq ||\varphi||_{H^{1/2}}$. This highlights the importance of enforcing the adherence boundary condition in the $H^{1/2}$-inner product (see (2.11.4)). If we had used only the L^2-inner product, then φ need not belong to $H^{1/2}$, in which case the Trace Theorem would not apply, $\boldsymbol{g}$ would not exist, and our eigenvalue problem would not be well-posed. Let $q = -\operatorname{div} \boldsymbol{h} + \frac{1}{|\Omega|}\int_\Omega \operatorname{div} \boldsymbol{h}\, d\boldsymbol{x}$. Since $\int_\Omega q\, d\boldsymbol{x} = 0$, it is well-known that there exists an $\boldsymbol{f} \in H^1_0(\Omega; \mathbb{C}^2)$ satisfying $\operatorname{div} \boldsymbol{f} = q$ and $||\boldsymbol{f}||_{H^1} \leq C||q||_{L^2}$ (see Brenner & Scott (2002, p. 282, Lemma 11.2.3)). The function $\boldsymbol{g} := \boldsymbol{h} + \boldsymbol{f}$ has the desired properties.)

Define $\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{g}$. Then problem (2.11.27) is equivalent to: Find $\boldsymbol{u} \in \mathcal{Z}_0$ such that

$$d_2(\boldsymbol{u}, \boldsymbol{w}) = F(\boldsymbol{w}, 0, 0) - d_2(\boldsymbol{g}, \boldsymbol{w}) \text{ for all } \boldsymbol{w} \in \mathcal{Z}_0. \tag{2.11.28}$$

By (2.11.22)(ii), Korn's inequality, and the Lax-Milgram Theorem, there exists a unique $\boldsymbol{u} \in \mathcal{Z}_0$ satisfying (2.11.28). This determines $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{g}$. By substituting $\boldsymbol{w} = \boldsymbol{u}$ into (2.11.28) and using (2.11.22)(ii) we obtain the estimate

$$||\boldsymbol{D}(\boldsymbol{v})||_{L^2} \leq C(||F(\cdot, 0, 0)|| + ||\varphi||_{H^{1/2}}) = C(||F(\cdot, 0, 0)|| + ||F(0, 0, \cdot)||). \tag{2.11.29}$$

Set $\boldsymbol{t} = 0$ in (2.11.24) to obtain the problem: Find $\boldsymbol{r} \in H^1_{\mathrm{S}}(\mathbb{T}_{2\pi})$ such that

$$d_1(\boldsymbol{r}, \boldsymbol{q}) + C_{\mathrm{g}} \int_a^R \boldsymbol{r} \cdot \bar{\boldsymbol{q}}\, ds = F(\boldsymbol{w}, \boldsymbol{q}, 0) - d_2(\boldsymbol{v}, \boldsymbol{w}) \tag{2.11.30}$$

for all $(\boldsymbol{w}, \boldsymbol{q}, 0) \in \mathcal{Z}_2$. Note that, given $\boldsymbol{q} \in H^1_{\mathrm{S}}(\mathbb{T}_{2\pi})$, the right-hand side of (2.11.30) is independent of our choice of $\boldsymbol{w}$ by (2.11.27). By the Trace Theorem we can take $\boldsymbol{w} = \boldsymbol{h}$, where $\boldsymbol{h} \in H^1(\Omega; \mathbb{C}^2)$ satisfies $\boldsymbol{h} = 0$ on $\{|\boldsymbol{x}| = a\}$ and $\boldsymbol{h} = \boldsymbol{q}$ on $\{|\boldsymbol{x}| = R\}$

in the sense of trace, and $||\boldsymbol{h}||_{H^1} \leq ||\boldsymbol{q}||_{H^{1/2}}$. Substitute $\boldsymbol{w} = \boldsymbol{h}$ into (2.11.30) to obtain the problem: Find $\boldsymbol{r} \in \mathring{H}^1_{\mathrm{S}}(\mathbb{T}_{2\pi})$ such that

$$d_1(\boldsymbol{r}, \boldsymbol{q}) + C_{\mathrm{g}} \int_a^R \boldsymbol{r} \cdot \bar{\boldsymbol{q}} \, ds = F(\boldsymbol{h}, \boldsymbol{q}, 0) - d_2(\boldsymbol{v}, \boldsymbol{h}) \tag{2.11.31}$$

for all $\boldsymbol{q} \in \mathring{H}^1_{\mathrm{S}}(\mathbb{T}_{2\pi})$. By the Gårding inequality (2.11.22)(i), the bilinear form on the left-hand side of (2.11.31) is coercive, and so there exists a unique solution $\boldsymbol{r} \in \mathring{H}^1_{\mathrm{S}}(\mathbb{T}_{2\pi})$. Moreover, (2.11.30), (2.11.22)(ii), and (2.11.29) imply that

$$||\boldsymbol{r}||_{H^1} \leq C(||F(\cdot,\cdot,0)|| + ||F(\cdot,0,0)|| + ||F(0,0,\cdot)||) \leq C'||F||. \tag{2.11.32}$$

It follows from (2.11.25) and (2.11.30) that $(\boldsymbol{v}, \boldsymbol{r})$ satisfies (2.11.24).

If we have two solutions $(\boldsymbol{v}^1, \boldsymbol{r}^1)$, $(\boldsymbol{v}^2, \boldsymbol{r}^2)$ of (2.11.24), then the difference $(\boldsymbol{v}^1 - \boldsymbol{v}^2, \boldsymbol{r}^1 - \boldsymbol{r}^2)$ satisfies (2.11.24) with $F = 0$. By substituting into (2.11.24) $(\boldsymbol{w}, \boldsymbol{q}) = (0,0)$, then $(\boldsymbol{q}, \boldsymbol{t}) = 0$, and then $\boldsymbol{t} = 0$, as above, we see that $(\boldsymbol{v}^1 - \boldsymbol{v}^2, \boldsymbol{r}^1 - \boldsymbol{r}^2) = (0,0)$, and so (2.11.24) has a unique solution.

Finally, the continuous dependence of $(\boldsymbol{v}, \boldsymbol{r})$ on the data F follows from the estimates (2.11.29) and (2.11.32). $\square$

The Weak Formulation in Polar Coordinates

In this section we write the weak equations (2.11.8) in polar coordinates. Decompose the functions in $\mathcal{V}_1 \times \Pi$ as

$$\boldsymbol{v}(r\boldsymbol{e}_1(\phi)) = v^1(r,\phi)\boldsymbol{e}_1(\phi) + v^2(r,\phi)\boldsymbol{e}_2(\phi), \quad \boldsymbol{r}(s) = r^1(s)\boldsymbol{e}_1(s) + r^2\boldsymbol{e}_2(s), \tag{2.11.33}$$

$$p(r\boldsymbol{e}_1(\phi)) = \tilde{p}(r,\phi). \tag{2.11.34}$$

Note that, in the notation of Section 2.9, $v^1 = u$, $v^2 = v$, $r^1 = q$, $r^2 = R\psi$, and $\tilde{p} = p$. See equation (2.9.11). Decompose the functions in $\mathcal{V}_2 \times \Pi$ as

$$\boldsymbol{w}(r\boldsymbol{e}_1(\phi)) = w^1(r,\phi)\boldsymbol{e}_1(\phi) + w^2(r,\phi)\boldsymbol{e}_2(\phi), \quad \boldsymbol{q}(s) = q^1(s)\boldsymbol{e}_1(s) + q^2\boldsymbol{e}_2(s), \tag{2.11.35}$$

$$\boldsymbol{t}(s) = t^1(s)\boldsymbol{e}_1(s) + t^2\boldsymbol{e}_2(s), \quad q(r\boldsymbol{e}_1(\phi)) = \tilde{q}(r,\phi). \tag{2.11.36}$$

Now drop the tilde from $\tilde{p}$ and $\tilde{q}$. Define

$$(\mathbf{v},\mathbf{r}) := (v^1, v^2, r^1, r^2), \quad (\mathbf{w},\mathbf{q},\mathbf{t}) := (w^1, w^2, q^1, q^2, t^1, t^2). \tag{2.11.37}$$

We obtain new function spaces V_1, V_2, and $\tilde{\Pi}$ by substituting the polar coordinates for $(\boldsymbol{v},\boldsymbol{r})$, $(\boldsymbol{w},\boldsymbol{q},\boldsymbol{t})$, and p into $\mathcal{V}_1$, $\mathcal{V}_2$, and Π:

$$V_1 := \Big\{(\mathbf{v},\mathbf{r}) \in [H^1([a,R]\times\mathbb{T}_{2\pi};\mathbb{C})]^2 \times [H^1(\mathbb{T}_{2\pi};\mathbb{C})]^2 : \\ v^i(a,\phi) = 0 \ \forall\phi, \ \int_0^{2\pi} r^1 \, ds = 0\Big\}, \tag{2.11.38}$$

$$V_2 := \Big\{(\mathbf{w},\mathbf{q},\mathbf{t}) \in [H^1([a,R]\times\mathbb{T}_{2\pi};\mathbb{C})]^2 \times [H^1(\mathbb{T}_{2\pi};\mathbb{C})]^2 \times [H^{1/2}(\mathbb{T}_{2\pi};\mathbb{C})]^2 : \\ w^i(a,\phi) = 0 \ \forall\phi, \ w^i(R,s) = q^i(s) \ \forall s, \ \int_0^{2\pi} q^1 \, ds = 0\Big\}, \tag{2.11.39}$$

$$\tilde{\Pi} := \Big\{p \in L^2([a,R]\times\mathbb{T}_{2\pi};\mathbb{C}) : \int_0^{2\pi}\int_a^R p(r,\phi) \, r dr d\phi = 0\Big\}. \tag{2.11.40}$$

Now drop the tilde from $\tilde{\Pi}$. Note that we do not need to include weights such as r and $1/r$ in the Sobolev spaces for $\mathbf{v}$, $\mathbf{w}$, and p since $r \in [a,R]$, a bounded set that does not include the origin. Equip V_1 and V_2 with the norms

$$||(\mathbf{v},\mathbf{r})||_{V_1} := ||(v^1\boldsymbol{e}_1 + v^2\boldsymbol{e}_2\,,\, r^1\boldsymbol{e}_1 + r^2\boldsymbol{e}_2)||_{\mathcal{V}_1} = ||(\boldsymbol{v},\boldsymbol{r})||_{\mathcal{V}_1},$$

$$||(\mathbf{w},\mathbf{q},\mathbf{t})||_{V_2} := ||(w^1\boldsymbol{e}_1 + w^2\boldsymbol{e}_2\,,\, q^1\boldsymbol{e}_1 + q^2\boldsymbol{e}_2\,,\, t^1\boldsymbol{e}_1 + t^2\boldsymbol{e}_2)||_{\mathcal{V}_2} = ||(\boldsymbol{w},\boldsymbol{q},\boldsymbol{t})||_{\mathcal{V}_2}.$$

$(V_1, ||\cdot||_{V_1})$ and $(V_2, ||\cdot||_{V_2})$ are complex Banach spaces.

If we substitute (2.11.33)–(2.11.36) into (2.11.8), we obtain a

Weak formulation of the quadratic eigenvalue problem in polar coordinates. Find $\lambda \in \mathbb{C}$ and $0 \neq (\mathbf{v}, \mathbf{r}, p) \in V_1 \times \Pi$ such that for all $(\mathbf{w}, \mathbf{q}, \mathbf{t}, q) \in V_2 \times \Pi$

$$\begin{aligned} \lambda^2 \tilde{a}_2(\mathbf{r}, \mathbf{q}) + \lambda \tilde{a}_1((\mathbf{v}, \mathbf{r}), (\mathbf{w}, \mathbf{q}, \mathbf{t})) + \tilde{a}_0((\mathbf{v}, \mathbf{r}), (\mathbf{w}, \mathbf{q}, \mathbf{t})) + \tilde{b}(\mathbf{w}, p) &= 0, \\ \tilde{b}(\mathbf{v}, q) &= 0, \end{aligned} \tag{2.11.41}$$

where

$$\begin{aligned} &\tilde{a}_0((\mathbf{v}, \mathbf{r}), (\mathbf{w}, \mathbf{q}, \mathbf{t})) := a_0((\boldsymbol{v}, \boldsymbol{r}), (\boldsymbol{w}, \boldsymbol{q}, \boldsymbol{t})) \\ &= \int_0^{2\pi} \Big\{ R^{-1} N^\circ [(r_s^1 - r^2)(\overline{q_s^1} - \overline{q^2}) + (r^1 + r_s^2)(\overline{q^1} + \overline{q_s^2})] \\ &\qquad + (N_\nu^\circ - R^{-1} N^\circ)(r^1 + r_s^2)(\overline{q^1} + \overline{q_s^2}) + \rho P(R)[\overline{q^2}(r_s^1 - r^2) - \overline{q^1}(r^1 + r_s^2)] \\ &\qquad - \varrho A \omega^2 (r^1 \overline{q^1} + r^2 \overline{q^2}) - \rho R^2 \omega^2 r^1 \overline{q^1} \Big\} \, ds \\ &+ 2 \int_a^R \int_0^{2\pi} \Big\{ \tilde{\mu} [v_r^1 \overline{w_r^1} + \tfrac{1}{r^2}(v_\phi^2 + v^1)(\overline{w_\phi^2} + \overline{w^1}) + \tfrac{1}{2}(\tfrac{1}{r} v_\phi^1 - \tfrac{1}{r} v^2 + v_r^2)(\tfrac{1}{r}\overline{w_\phi^1} - \tfrac{1}{r}\overline{w^2} + \overline{w_r^2})] \\ &\qquad + \omega \rho (v^1 \overline{w^2} - v^2 \overline{w^1}) \Big\} \, r d\phi \, dr + \langle \gamma_R v^1 \boldsymbol{e}_1 + \gamma_R v^2 \boldsymbol{e}_2, t^1 \boldsymbol{e}_1 + t^2 \boldsymbol{e}_2 \rangle_{H^{1/2}}, \end{aligned} \tag{2.11.42}$$

$$\begin{aligned} &\tilde{a}_1((\mathbf{v}, \mathbf{r}), (\mathbf{w}, \mathbf{q}, \mathbf{t})) := a_1((\boldsymbol{v}, \boldsymbol{r}), (\boldsymbol{w}, \boldsymbol{q}, \boldsymbol{t})) \\ &= \int_0^{2\pi} \Big\{ N_\nu^\circ (r_s^2 + r^1)(\overline{q_s^2} + \overline{q^1}) + 2\varrho A \omega (r^1 \overline{q^2} - r^2 \overline{q^1}) \Big\} \, ds \\ &\qquad + \rho \int_a^R \int_0^{2\pi} \Big\{ v^1 \overline{w^1} + v^2 \overline{w^2} \Big\} \, r d\phi \, dr - \langle r^1 \boldsymbol{e}_1 + r^2 \boldsymbol{e}_2, t^1 \boldsymbol{e}_1 + t^2 \boldsymbol{e}_2 \rangle_{H^{1/2}}, \end{aligned} \tag{2.11.43}$$

$$\tilde{a}_2(\mathbf{r}, \mathbf{q}) := a_2(\boldsymbol{r}, \boldsymbol{q}) = \varrho A \int_0^{2\pi} \Big\{ r^1 \overline{q^1} + r^2 \overline{q^2} \Big\} \, ds, \tag{2.11.44}$$

$$\tilde{b}(\mathbf{w}, p) := b(\boldsymbol{w}, p) = -\rho \int_a^R \int_0^{2\pi} p(\overline{w_r^1} + \tfrac{1}{r}\overline{w^1} + \tfrac{1}{r}\overline{w_\phi^2}) \, r d\phi \, dr. \tag{2.11.45}$$

Fourier Decomposition and a Family of Weak Problems

In this section we expand the functions in V_1, V_2, and Π as Fourier series in the angle variable (ϕ or s) and use this to generate a family of weak problems indexed by the Fourier wave number.

For $j \in \{1,2\}$ decompose

$$v^j(r,\phi) = \sum_{k=-\infty}^{\infty} v_k^j(r) e^{ik\phi}, \quad r^j(s) = \sum_{k=-\infty}^{\infty} r_k^j e^{iks}, \tag{2.11.46}$$

$$w^j(r,\phi) = \sum_{k=-\infty}^{\infty} w_k^j(r) e^{ik\phi}, \quad q^j(s) = \sum_{k=-\infty}^{\infty} q_k^j e^{iks}, \quad t^j(s) = \sum_{k=-\infty}^{\infty} t_k^j e^{iks}, \tag{2.11.47}$$

$$p(r,\phi) = \sum_{k=-\infty}^{\infty} p_k(r) e^{ik\phi}. \tag{2.11.48}$$

Define

$$(\mathbf{v}_k, \mathbf{r}_k) := (v_k^1, v_k^2, r_k^1, r_k^2), \quad (\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k) := (w_k^1, w_k^2, q_k^1, q_k^2, t_k^1, t_k^2).$$

We define a family of spaces indexed by the Fourier wave number $k \in \mathbb{Z}$. For $k \neq 0$

$$\begin{aligned}
&V_1^k := \left\{ (\mathbf{v}_k, \mathbf{r}_k) \in [H^1([a,R];\mathbb{C})]^2 \times \mathbb{C}^2 : v_k^j(a) = 0 \right\}, \\
&V_2^k := \left\{ (\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k) \in [H^1([a,R];\mathbb{C})]^2 \times \mathbb{C}^2 \times \mathbb{C}^2 : w_k^j(a) = 0,\ w_k^j(R) = q_k^j \right\}, \\
&\Pi^k := L^2([a,R];\mathbb{C}).
\end{aligned} \tag{2.11.49}$$

(Note that these spaces are independent of k.) For $k = 0$

$$\begin{aligned}
&V_1^0 := \left\{ (\mathbf{v}_0, \mathbf{r}_0) \in [H^1([a,R];\mathbb{C})]^2 \times \mathbb{C}^2 : v_0^j(a) = 0,\ r_0^1 = 0 \right\}, \\
&V_2^0 := \left\{ (\mathbf{w}_0, \mathbf{q}_0, \mathbf{t}_0) \in [H^1([a,R];\mathbb{C})]^2 \times \mathbb{C}^2 \times \mathbb{C}^2 : w_0^j(a) = 0,\ w_0^j(R) = q_0^j,\ q_0^1 = 0 \right\}, \\
&\Pi^0 := \left\{ p_0 \in L^2([a,R];\mathbb{C}) : \int_0^{2\pi} p_0(r)\, r dr = 0 \right\}.
\end{aligned} \tag{2.11.50}$$

We equip V_1^k, V_2^k, and Π^k with the norms

$$||(\mathbf{v}_k, \mathbf{r}_k)||_{V_1^k}^2 = ||\mathbf{v}_k||_{H^1([a,R])}^2 + |\mathbf{r}_k|^2,$$

$$||(\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k)||_{V_1^k}^2 = ||\mathbf{w}_k||_{H^1([a,R])}^2 + |\mathbf{q}_k|^2 + |\mathbf{t}_k|^2,$$

$$||p_k||_{\Pi^k}^2 = \int_a^R |p_k|^2\, r dr.$$

Note that as before we do not need to include weights r and $1/r$ in the Sobolev spaces V_1^k and V_2^k because $r \in [a, R]$, a bounded set that does not include the origin. We shall see that it is convenient to include the weight r in the pressure space.

Let $k \in \mathbb{Z}$, $(\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k) \in V_2^k$, $q_k \in \Pi^k$. Substitute into (2.11.41) the Fourier decompositions (2.11.46) and (2.11.48) and

$$w^j(r,\phi) = w_k^j(r)e^{ik\phi}, \quad q^j(s) = q_k^j e^{iks}, \quad t^j(s) = t_k^j e^{iks}, \quad q(r,\phi) = q_k(r)e^{ik\phi}$$

to obtain the following family of weak problems:

A family of weak problems indexed by the Fourier wave number. For each $k \in \mathbb{Z}$, find $\lambda \in \mathbb{C}$ and $0 \neq (\mathbf{v}_k, \mathbf{r}_k, p_k) \in V_1^k \times \Pi^k$ such that for all $(\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k, q_k) \in V_2^k \times \Pi^k$

$$\lambda^2 a_2^k(\mathbf{r}_k, \mathbf{q}_k) + \lambda a_1^k((\mathbf{v}_k, \mathbf{r}_k), (\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k)) + a_0^k((\mathbf{v}_k, \mathbf{r}_k), (\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k)) + b^k(\mathbf{w}_k, p_k) = 0,$$

$$b^k(\mathbf{v}_k, q_k) = 0 \tag{2.11.51}$$

where

$$
\begin{aligned}
& a_0^k((\mathbf{v}_k, \mathbf{r}_k), (\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k)) \\
& := R^{-1} N^\circ [(ikr_k^1 - r_k^2)(-ik\overline{q_k^1} - \overline{q_k^2}) + (r_k^1 + ikr_k^2)(\overline{q_k^1} - ik\overline{q_k^2})] \\
& \quad + (N_\nu^\circ - R^{-1} N^\circ)(r_k^1 + ikr_k^2)(\overline{q_k^1} - ik\overline{q_k^2}) + \rho P(R)[\overline{q_k^2}(ikr_k^1 - r_k^2) - \overline{q_k^1}(r_k^1 + ikr_k^2)] \\
& \quad - \varrho A \omega^2 (r_k^1 \overline{q_k^1} + r_k^2 \overline{q_k^2}) - \rho R^2 \omega^2 r_k^1 \overline{q_k^1} \\
& \quad + 2 \int_a^R \Big\{ \tilde{\mu} [(v_k^1)_r (\overline{w_k^1})_r + \tfrac{1}{r^2}(ikv_k^2 + v_k^1)(-ik\overline{w_k^2} + \overline{w_k^1}) \\
& \qquad + \tfrac{1}{2}(\tfrac{ik}{r} v_k^1 - \tfrac{1}{r} v_k^2 + (v_k^2)_r)(-\tfrac{ik}{r}\overline{w_k^1} - \tfrac{1}{r}\overline{w_k^2} + (\overline{w_k^2})_r)] + \omega \rho (v_k^1 \overline{w_k^2} - v_k^2 \overline{w_k^1}) \Big\} \, r dr \\
& \quad + (1 + |k|)(v_k^1(R)\overline{t_k^1} + v_k^2(R)\overline{t_k^2}),
\end{aligned}
\tag{2.11.52}
$$

$$
\begin{aligned}
& a_1^k((\mathbf{v}_k, \mathbf{r}_k), (\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k)) \\
& := N_\nu^\circ (ikr_k^2 + r_k^1)(-ik\overline{q_k^2} + \overline{q_k^1}) + 2\varrho A \omega (r_k^1 \overline{q_k^2} - r_k^2 \overline{q_k^1}) + \rho \int_a^R \Big\{ v_k^1 \overline{w_k^1} + v_k^2 \overline{w_k^2} \Big\} \, r dr \\
& \quad - (1 + |k|)(r_k^1 \overline{t_k^1} + r_k^2 \overline{t_k^2}),
\end{aligned}
\tag{2.11.53}
$$

$$
a_2^k(\mathbf{r}_k, \mathbf{q}_k) := \varrho A (r_k^1 \overline{q_k^1} + r_k^2 \overline{q_k^2}), \tag{2.11.54}
$$

$$
b^k(\mathbf{w}_k, p_k) := -\rho \int_a^R p_k [(\overline{w_k^1})_r + \tfrac{1}{r}\overline{w_k^1} - \tfrac{ik}{r}\overline{w_k^2}] \, r dr. \tag{2.11.55}
$$

Now we prove that the bilinear forms a_0^k and b^k satisfy the inf-sup conditions. These conditions are used in section 2.12 to characterize the spectrum of problem (2.11.51). By proving discrete versions of these inf-sup conditions in section 2.12 we will also be able to construct a convergent numerical scheme for computing the eigenvalues.

Theorems (2.11.56), (2.11.64), and (2.11.68) are analogous to Theorems (2.11.15), (2.11.18), and (2.11.22).

Theorem 2.11.56 (b^k satisfies an inf-sup condition). *Let $k \in \mathbb{Z}$. Let b^k be the bilinear form defined in equation* (2.11.55)*. Then*

$$\inf_{p_k \in \Pi^k} \sup_{\mathbf{w}_k \in [H_0^1]^2} \frac{|b^k(\mathbf{w}_k, p_k)|}{||p_k||_{\Pi^k} ||\mathbf{w}_k||_{H^1}} = \beta > 0. \tag{2.11.57}$$

The constant β is independent of k.

***Proof*.** Bernardi et al. (1999, Prop. IX.1.1) prove a very similar and more general result (for general 3-dimensional axisymmetric domains rather than just the 2-dimensional annulus). Their proof follows from the well-known inf-sup condition for the Stokes equation, Theorem (2.11.15), which in turn is proved by inverting the divergence operator. Since we are only interested in a 2-dimensional annular domain, inverting the divergence operator amounts to solving an ordinary differential equation in r, and so Theorem (2.11.56) has an elementary proof, which we give here.

First consider the case $k = 0$. Given $p_0 \in \Pi^0$, we wish to construct a $u_0^1 \in H_0^1([a, R]; \mathbb{C})$ satisfying

$$(ru_0^1)_r = p_0 r \tag{2.11.58}$$

and $||u_0^1||_{H^1} \leq C||p_0||_{\Pi^0}$ since then

$$\sup_{\mathbf{w}_0 \in [H_0^1]^2} \frac{|b^0(\mathbf{w}_0, p_0)|}{||\mathbf{w}_0||_{H^1}} \geq \frac{|b^0((u_0^1, 0), p_0)|}{||u_0^1||_{H^1}} = \frac{\rho \int_a^R |p_0|^2 \, r dr}{||u_0^1||_{H^1}} \geq \beta \, ||p_0||_{\Pi^0} \tag{2.11.59}$$

for $\beta = \rho a / C$, which implies (2.11.57). The unique solution of (2.11.58) with $u_0^1(a) = u_0^1(R) = 0$ is

$$u_0^1(r) = \frac{1}{r} \int_a^r p_0(\varrho) \varrho \, d\varrho.$$

Note that $u_0^1(R) = 0$ because $p_0 \in \Pi^0$. It is easy to check the estimate $||u_0^1||_{H^1} \leq C||p_0||_{\Pi^0}$.

Now consider the case $k \neq 0$. Given $p_k \in \Pi^k$, we wish to construct a $\mathbf{u}_k \in [H_0^1([a,R];\mathbb{C})]^2$ satisfying

$$(ru_k^1)_r + iku_k^2 = p_k r \tag{2.11.60}$$

and $||\mathbf{u}_k||_{H^1} \leq C||p_k||_{\Pi^k}$. Then we can argue as in (2.11.59) to prove (2.11.57). Given u_k^2, equation (2.11.60) and the boundary condition $u_k^1(a) = 0$ determines u_k^1:

$$u_k^1(r) = \frac{1}{r}\int_a^r (p_k(\varrho)\varrho - iku_k^2(\varrho))\, d\varrho.$$

Now we must choose u_k^2 such that

$$u_k^2(a) = u_k^2(R) = u_k^1(R) = 0 \tag{2.11.61}$$

and $||\mathbf{u}_k||_{H^1} \leq C||p_k||_{\Pi^k}$. We try $u_k^2(r) = c_2(r-a)^2 + c_1(r-a)$. Equation (2.11.61) determines c_1 and c_2:

$$c_1 = -c_2(R-a), \quad c_2 = \frac{6i}{k(R-a)^3}\int_a^R p_k\, rdr.$$

Once again, it is easy to check the estimates $||u_k^1||_{H^1} \leq C||p_k||_{\Pi^k}$, $||u_k^2||_{H^1} \leq \frac{C'}{|k|}||p_k||_{\Pi^k}$, where C and C' are independent of k. Therefore $||\mathbf{u}_k||_{H^1} \leq C(1 + 1/|k|)||p_k||_{\Pi^k} \leq 2C||p_k||_{\Pi^k}$, as required. $\square$

It follows immediately from Theorem (2.11.56) that

$$\inf_{p_k \in \Pi^k} \sup_{(\mathbf{w}_k,\mathbf{q}_k,\mathbf{t}_k)\in V_2^k} \frac{|b^k(\mathbf{w}_k, p_k)|}{||p_k||_{\Pi^k}||(\mathbf{w}_k,\mathbf{q}_k,\mathbf{t}_k)||_{V_2^k}} = \beta > 0. \tag{2.11.62}$$

For $k \in \mathbb{Z}$, define

$$\begin{aligned} Z_1^k &:= \{(\mathbf{v}_k, \mathbf{r}_k) \in V_1^k : b^k(\mathbf{v}_k, p_k) = 0 \text{ for all } p_k \in \Pi^k\}, \\ Z_2^k &:= \{(\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k) \in V_2^k : b^k(\mathbf{w}_k, p_k) = 0 \text{ for all } p_k \in \Pi^k\}. \end{aligned} \tag{2.11.63}$$

Theorem 2.11.64 (a_0^k satisfies a Gårding-type inf-sup condition). *Let $k \in \mathbb{Z}$. There exists a constant $C_{\rm g}^k > 0$ such that the bilinear form $\hat{a}_0^k : V_1^k \times V_2^k \to \mathbb{C}$ defined by*

$$\begin{aligned}\hat{a}_0^k((\mathbf{v}_k, \mathbf{r}_k), (\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k)) &:= a_0^k((\mathbf{v}_k, \mathbf{r}_k), (\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k)) \\ &\quad + C_{\rm g}^k \left(\int_a^R \{ v_k^1 \overline{w_k^1} + v_k^2 \overline{w_k^2} \} \, dr + r_k^1 \overline{q_k^1} + r_k^2 \overline{q_k^2} \right)\end{aligned} \tag{2.11.65}$$

satisfies the inf-sup conditions

$$\inf_{\substack{(\mathbf{v}_k, \mathbf{r}_k) \in Z_1^k \\ ||(\mathbf{v}_k, \mathbf{r}_k)||_{V_1^k} = 1}} \sup_{\substack{(\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k) \in Z_2^k \\ ||(\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k)||_{V_2^k} = 1}} |\hat{a}_0^k((\mathbf{v}_k, \mathbf{r}_k), (\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k))| = \alpha > 0, \tag{2.11.66}$$

$$(\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k) = 0 \;\; \textit{if} \;\; \hat{a}_0^k((\mathbf{v}_k, \mathbf{r}_k), (\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k)) = 0 \;\; \textit{for all} \;\; (\mathbf{v}_k, \mathbf{r}_k) \in Z_1^k. \tag{2.11.67}$$

Will we use the following lemma to prove Theorem (2.11.64).

Lemma 2.11.68. *Let $k \in \mathbb{Z}$. Decompose the bilinear form a_0^k defined in* (2.11.52) *into three bilinear forms:*

$$a_0^k((\mathbf{v}_k, \mathbf{r}_k), (\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k)) =: d_1^k(\mathbf{r}_k, \mathbf{q}_k) + d_2^k(\mathbf{v}_k, \mathbf{w}_k) + d_3^k(\mathbf{v}_k(R), \mathbf{t}_k), \tag{2.11.69}$$

where d_1^k, d_2^k, and d_3^k correspond to the string terms, the fluid terms, and the adherence boundary condition terms of a_0^k. Then

(i) d_1^k satisfies a Gårding inequality: There exists a constant $C_{\rm g_1}^k > 0$ such that

$$d_1^k(\mathbf{r}_k, \mathbf{r}_k) + C_{\rm g_1}^k |\mathbf{r}_k|^2 \geq \alpha_1 |\mathbf{r}_k|^2 \;\; \textit{for all} \;\; \mathbf{r}_k \in \mathbb{C}^2;$$

(ii) d_2^k satisfies a Gårding inequality: There exists a constant $C_{\rm g_2}^k > 0$ such that

$$\mathrm{Re}[d_2^k(\mathbf{v}_k, \mathbf{v}_k)] + C_{\rm g_2}^k ||\mathbf{v}_k||^2_{L^2(a,R)} \geq \alpha_2 ||\mathbf{v}_k||^2_{H^1(a,R)} \;\; \textit{for all} \;\; \mathbf{v}_k \in [H^1([a,R];\mathbb{C})]^2;$$

(iii) d_3^k *is coercive (positive definite):*

$$d_3^k(\mathbf{t}_k, \mathbf{t}_k) = (1 + |k|)|\mathbf{t}_k|^2 \quad \textit{for all} \quad \mathbf{t}_k \in \mathbb{C}^2.$$

Proof. These are easy estimates. □

Proof of Theorem (2.11.64). We just prove the case $k \neq 0$. The proof for $k = 0$ is similar. Recall that the inf-sup conditions (2.11.66) and (2.11.67) are equivalent to the well-posedness of the following problem:

$$\text{Find } (\mathbf{v}_k, \mathbf{r}_k) \in Z_1^k \text{ such that } \hat{a}_0^k((\mathbf{v}_k, \mathbf{r}_k), (\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k)) = F(\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k) \tag{2.11.70}$$

for all $(\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k) \in Z_2^k$, where $F \in (Z_2^k)^*$, the space of bounded linear functionals on Z_2^k. See Ern & Guermond (2004, p. 85, Theorem 2.6). To prove Theorem (2.11.64) we will prove that problem (2.11.70) is well-posed.

Choose $C_{\mathrm{g}}^k \geq \max\{C_{\mathrm{g}1}^k, C_{\mathrm{g}2}^k\}$, where C_{g}^k is the constant appearing in the definition of $\hat{a}_0^k$ and $C_{\mathrm{g}_1}^k$ and $C_{\mathrm{g}_2}^k$ were defined in Lemma (2.11.68).

We start by constructing a candidate for the solution $(\mathbf{v}_k, \mathbf{r}_k)$ of (2.11.70). Substitute $\mathbf{w}_k = 0$, $\mathbf{q}_k = 0$ into (2.11.70) to obtain the problem: Find $\mathbf{v}_k(R) \in \mathbb{C}^2$ such that

$$d_3^k(\mathbf{v}_k(R), \mathbf{t}_k) = F(0, 0, \mathbf{t}_k) \quad \text{for all} \quad \mathbf{t}_k \in \mathbb{C}^2. \tag{2.11.71}$$

Since d_3^k is coercive (positive definite), see (2.11.68)(iii), there exists a unique solution $\mathbf{v}_k(R) = \varphi$ to (2.11.71). Note that $|\varphi| \leq C||F(0, 0, \cdot)||$, where $||F(0, 0, \cdot)||$ denotes the norm of the bounded linear operator $F(0, 0, \cdot)$.

Let

$$\begin{aligned} Z^k &= \{\mathbf{w}_k \in [H^1([a,r];\mathbb{C})]^2 : b^k(\mathbf{w}_k, p_k) = 0 \text{ for all } p_k \in \Pi^k\}, \\ Z_0^k &= [H_0^1([a,R];\mathbb{C})]^2 \cap Z^k. \end{aligned} \tag{2.11.72}$$

Substitute $\mathbf{q}_k = 0$, $\mathbf{t}_k = 0$ in (2.11.70) to obtain the problem: Find $\mathbf{v}_k \in Z^k$ with $\mathbf{v}_k(a) = 0$ and $\mathbf{v}_k(R) = \varphi$ such that

$$d_2^k(\mathbf{v}_k, \mathbf{w}_k) + C_{\mathrm{g}}^k \int_a^R \{v_k^1 \overline{w_k^1} + v_k^2 \overline{w_k^2}\}\, rdr = F(\mathbf{w}_k, 0, 0) \;\text{ for all }\; \mathbf{w}_k \in Z_0^k. \tag{2.11.73}$$

There exists a $\mathbf{g}_k \in Z^k$ such that $\mathbf{g}_k(a) = 0$, $\mathbf{g}_k(R) = \varphi$, and $||\mathbf{g}_k||_{H^1} \leq C|\varphi|$. (The existence of $\mathbf{g}_k$ is proved as follows. Define $\mathbf{h}_k = \varphi(r-a)/(R-a)$. Then $\mathbf{h}_k(a) = 0$, $\mathbf{h}_k(R) = \varphi$, and $||\mathbf{h}_k||_{H^1} \leq C|\varphi|$. By the same method used to prove the inf-sup condition (2.11.57) there exists a function $\mathbf{f}_k \in [H_0^1([a,R];\mathbb{C})]^2$ such that $(f_k^1)_r + \frac{1}{r} f_k^1 + \frac{ik}{r} f_k^2 = -((h_k^1)_r + \frac{1}{r} h_k^1 + \frac{ik}{r} h_k^2)$ and $||\mathbf{f}_k||_{H^1} \leq C||(h_k^1)_r + \frac{1}{r} h_k^1 + \frac{ik}{r} h_k^2||_{L^2} \leq C|\varphi|$. The function $\mathbf{g}_k := \mathbf{h}_k + \mathbf{f}_k$ has the desired properties.) Define $\mathbf{u}_k = \mathbf{v}_k - \mathbf{g}_k$. Then problem (2.11.73) is equivalent to: Find $\mathbf{u}_k \in Z_0^k$ such that

$$\begin{aligned} d_2^k(\mathbf{u}_k, \mathbf{w}_k) &+ C_{\mathrm{g}}^k \int_a^R \{u_k^1 \overline{w_k^1} + u_k^2 \overline{w_k^2}\}\, dr \\ &= F(\mathbf{w}_k, 0, 0) - d_2^k(\mathbf{g}_k, \mathbf{w}_k) - C_{\mathrm{g}}^k \int_a^R \{g_k^1 \overline{w_k^1} + g_k^2 \overline{w_k^2}\}\, dr \end{aligned} \tag{2.11.74}$$

for all $\mathbf{w}_k \in Z_0^k$. By the Gårding inequality (2.11.68)(ii) and the Lax-Milgram Theorem, there exists a unique $\mathbf{u}_k \in Z_0^k$ satisfying (2.11.74). This determines $\mathbf{v}_k = \mathbf{u}_k + \mathbf{g}_k$. By substituting $\mathbf{w_k} = \mathbf{u}_k$ into (2.11.74) and using (2.11.68)(ii) we obtain the estimate

$$||\mathbf{v}_k||_{H^1} \leq C(||F(\cdot,0,0)|| + |\varphi|) \leq C'(||F(\cdot,0,0)|| + ||F(0,0,\cdot)||). \tag{2.11.75}$$

Set $\mathbf{t}_k = 0$ in (2.11.70) to obtain the problem: Find $\mathbf{r}_k \in \mathbb{C}^2$ such that

$$d_1^k(\mathbf{r}_k, \mathbf{q}_k) + C_{\mathrm{g}}^k(r_k^1\overline{q_k^1} + r_k^2\overline{q_k^2}) = F(\mathbf{w}_k, \mathbf{q}_k, 0) - d_2^k(\mathbf{v}_k, \mathbf{w}_k) - C_{\mathrm{g}}^k \int_a^R \{v_k^1\overline{w_k^1} + v_k^2\overline{w_k^2}\}\, dr \tag{2.11.76}$$

for all $(\mathbf{w}_k, \mathbf{q}_k, 0) \in Z_2^k$. Note that, given $\mathbf{q}_k \in \mathbb{C}^2$, the right-hand side of (2.11.76) is independent of our choice of $\mathbf{w}_k$ by (2.11.73). Substitute $\mathbf{w}_k = \mathbf{q}_k(r-a)/(R-a) =: \mathbf{h}_k$ into (2.11.76) to obtain the problem: Find $\mathbf{r}_k \in \mathbb{C}^2$ such that

$$d_1^k(\mathbf{r}_k, \mathbf{q}_k) + C_{\mathrm{g}}^k(r_k^1\overline{q_k^1} + r_k^2\overline{q_k^2}) = F(\mathbf{h}_k, \mathbf{q}_k, 0) - d_2^k(\mathbf{v}_k, \mathbf{h}_k) - C_{\mathrm{g}}^k \int_a^R \{v_k^1\overline{h_k^1} + v_k^2\overline{h_k^2}\}\, dr \tag{2.11.77}$$

for all $\mathbf{q}_k \in \mathbb{C}^2$. By the Gårding inequality (2.11.68)(i), the bilinear form on the the left-hand side of (2.11.77) is coercive (positive definite), and so there exists a unique solution $\mathbf{r}_k \in \mathbb{C}^2$. Moreover, (2.11.76), (2.11.68)(ii), and (2.11.75) imply that

$$|\mathbf{r}_k| \leq C(||F(\cdot,\cdot,0)|| + ||F(\cdot,0,0)|| + ||F(0,0,\cdot)||) \leq C'||F||. \tag{2.11.78}$$

It follows from (2.11.71) and (2.11.76) that $(\mathbf{v}_k, \mathbf{r}_k)$ satisfies (2.11.70).

If we have two solutions $(\mathbf{v}_k^1, \mathbf{r}_k^1)$, $(\mathbf{v}_k^2, \mathbf{r}_k^2)$ of (2.11.70), then the difference $(\mathbf{v}_k^1 - \mathbf{v}_k^2, \mathbf{r}_k^1 - \mathbf{r}_k^2)$ satisfies (2.11.70) with $F = 0$. By substituting into (2.11.70) $(\mathbf{w}_k, \mathbf{q}_k) = (0,0)$, then $(\mathbf{q}_k, \mathbf{t}_k) = 0$, and then $\mathbf{t}_k = 0$, as above, we see that $(\mathbf{v}_k^1 - \mathbf{v}_k^2, \mathbf{r}_k^1 - \mathbf{r}_k^2) = (0,0)$, and so (2.11.70) has a unique solution.

Finally, the continuous dependence of $(\mathbf{v}_k, \mathbf{r}_k)$ on the data F follows from the estimates (2.11.75) and (2.11.78). □

2.12 Numerical Analysis of the Spectrum

Galerkin Approximation of Polynomial Eigenvalue Problems

In this section we summarize the spectral approximation theory for polynomial eigenvalue problems of the form

$$\text{Find } \lambda \in \mathbb{C} \text{ and } 0 \neq u \in V_1 \text{ such that for all } v \in V_2$$

$$A(u,v) = \lambda^N B_N(u,v) + \lambda^{N-1} B_{N-1}(u,v) + \ldots + \lambda B_1(u,v) + B_0(u,v) \qquad (2.12.1)$$

where V_1 and V_2 are Hilbert spaces and A, $B_0, \ldots, B_N$ are bilinear forms. We apply this theory in the following sections to design a convergent numerical scheme for computing the eigenvalues of (2.11.51). The spectral approximation theory of standard eigenvalue problems (the case $N = 1$ in (2.12.1)), which was developed in the 1970s, is described in Babuška and Osborn (1991). Kolata (1976) showed how to extend this theory to polynomial eigenvalue problems. We give a slightly different presentation; we weaken one of the hypotheses (see the paragraph preceding Theorem (2.12.15)) and simplify some of the arguments. We explain the main difference before going into details. It is possible to write eigenvalue problem (2.12.1) as an operator eigenvalue problem of the form

$$\lambda^N T_N u + \lambda^{N-1} T_{N-1} u + \ldots + \lambda T_1 u + T_0 u + u = 0, \qquad (2.12.2)$$

where $T_j : V_1 \to V_2$ are linear operators. To study the spectrum of (2.12.1), or equivalently (2.12.2), we must reduce the problem to a standard eigenvalue problem of the form

$$a(u,v) = \lambda b(u,v) \quad \text{for all } v, \qquad (2.12.3)$$

or equivalently an operator eigenvalue problem of the form

$$Tu = \lambda u. \tag{2.12.4}$$

Ultimately we must arrive at a problem of the form (2.12.4) so that we can apply the Spectral Theorem for Compact Operators. There are two ways to arrive at (2.12.4) from (2.12.1). The first choice, used by Kolata (1976), is to reduce equation (2.12.1) to another bilinear form equation of the form (2.12.3), and then derive an operator equation of the form (2.12.4). The second choice, used here, is to write (2.12.1) as an operator equation of the form (2.12.2), and then reduce this to another operator equation of the form (2.12.4). We believe that the second choice simplifies the presentation.

Let V_1, V_2, and W be complex Hilbert spaces with norms $||\cdot||_{V_1}$, $||\cdot||_{V_2}$, and $||\cdot||_W$, and with V_1 compactly embedded in W. Let $A : V_1 \times V_2 \to \mathbb{C}$, $B_0 : W \times V_2 \to \mathbb{C}$, ..., $B_N : W \times V_2 \to \mathbb{C}$ be continuous bilinear forms satisfying

$$\begin{aligned} |A(u,v)| &\le C||u||_{V_1}||v||_{V_2} \quad \text{for all } u \in V_1,\ v \in V_2, \\ |B_j(u,v)| &\le C_j||u||_W||v||_{V_2} \quad \text{for all } u \in W,\ v \in V_2, \quad \text{for } j \in \{0 \dots N\}. \end{aligned} \tag{2.12.5}$$

We assume that A satisfies the inf-sup conditions

$$\inf_{\substack{u \in V_1 \\ ||u||_{V_1}=1}} \sup_{\substack{v \in V_2 \\ ||v||_{V_2}=1}} |A(u,v)| = \alpha > 0, \tag{2.12.6}$$

$$v = 0 \text{ if } A(u,v) = 0 \text{ for all } u \in V_1. \tag{2.12.7}$$

Under the assumption that A is continuous, it can be shown that (2.12.6) and

(2.12.7) hold if and only if

$$\inf_{\substack{v \in V_2 \\ ||v||_{V_2} = 1}} \sup_{\substack{u \in V_1 \\ ||u||_{V_1} = 1}} |A(u, v)| = \alpha > 0, \tag{2.12.8}$$

$$u = 0 \text{ if } A(u, v) = 0 \text{ for all } v \in V_2. \tag{2.12.9}$$

See Babuška and Osborn (1991, p. 692). We consider the spectral approximation of the following problem:

The Continuous Problem. Find $\lambda \in \mathbb{C}$ and $0 \neq u \in V_1$ such that for all $v \in V_2$

$$A(u, v) = \lambda^N B_N(u, v) + \lambda^{N-1} B_{N-1}(u, v) + \ldots + \lambda B_1(u, v) + B_0(u, v). \tag{2.12.10}$$

If (λ, u) satisfies (2.12.10), then we call λ an eigenvalue and u an eigenvector of (2.12.10).

The continuity and inf-sup conditions (2.12.5)–(2.12.7) imply that there exists unique bounded linear operators $T_0 : V_1 \to V_1, \ldots, T_N : V_1 \to V_1$ satisfying

$$A(T_j u, v) = -B_j(u, v) \quad \text{for all } v \in V_2, \quad \text{for } j \in \{0, \ldots, N\}.$$

See Ern & Guermond (2004, p. 85, Theorem 2.6). Moreover, $T_0, \ldots, T_N$ are compact: Let $\{u_n\}$ be a bounded sequence in V_1. Then it has a subsequence $\{u_{n_k}\}$ converging strongly in W by the compact embedding $V_1 \subset\subset W$. For each $j \in \{0, \ldots, N\}$, the inf-sup conditions imply that

$$||T_j u||_{V_1} \leq \frac{C_j}{\alpha} ||u||_W.$$

Therefore

$$||T_j(u_{n_k} - u_{n_l})||_{V_1} \leq \frac{C_j}{\alpha} ||u_{n_k} - u_{n_l}||_W \to 0$$

and $\{T_j u_{n_k}\}$ is a Cauchy sequence in V_1. This shows that T_j is compact.

For $\lambda \in \mathbb{C}$ define $T(\lambda) : V_1 \to V_1$ by

$$T(\lambda) := \lambda^N T_N + \ldots + \lambda T_1 + T_0 + I, \tag{2.12.11}$$

where $I : V_1 \to V_1$ is the identity operator on V_1.

Lemma 2.12.12 (An equivalent formulation of the continuous eigenvalue problem). *The pair* $(\lambda, u) \in \mathbb{C} \times (V_1 \setminus 0)$ *satisfies problem* (2.12.10) *if and only if it is an eigenpair of* T, *i.e.,*

$$T(\lambda)\, u \equiv \lambda^N T_N u + \ldots + \lambda T_1 u + T_0 u + u = 0. \tag{2.12.13}$$

Proof. First we assume that (λ, u) satisfies (2.12.13). Then for all $v \in V_2$

$$\begin{aligned} 0 &= A(\lambda^N T_n u + \ldots + \lambda T_1 u + T_0 u + u, v) \\ &= -\lambda^N B_N(u, v) - \ldots - \lambda B_1(u, v) - B_0(u, v) + A(u, v), \end{aligned}$$

which implies that (λ, u) satisfies (2.12.10).

Now assume that (λ, u) satisfies (2.12.10). Then for all $v \in V_2$

$$\begin{aligned} 0 &= -\lambda^N B_N(u, v) - \ldots - \lambda B_1(u, v) - B_0(u, v) + A(u, v) \\ &= \lambda^N A(T_N u, v) + \ldots + \lambda A(T_1 u, v) + A(T_0 u, v) + A(u, v) \\ &= A(\lambda^N T_N u + \ldots + \lambda T_1 u + T_0 u + u, v), \end{aligned}$$

which implies that (λ, u) satisfies (2.12.13) by (2.12.9). $\square$

We make the additional hypothesis that

$$\text{There exists a } \xi \in \mathbb{C} \text{ such that } T(\xi) : V_1 \to V_1 \text{ has a bounded inverse.} \tag{2.12.14}$$

Note that hypothesis (2.12.14) holds if and only if there exists a $\xi \in \mathbb{C}$ that is not an eigenvalue of (2.12.10). Kolata (1976) makes the stronger hypothesis that $T(0)$ has a bounded inverse, which is true if and only if -1 is not an eigenvalue of T_0.

Theorem 2.12.15 (Characterization of the Spectrum). *Assume that* (2.12.5)–(2.12.7) *and* (2.12.14) *hold. Then problem* (2.12.10) *has a countable set of eigenvalues with infinity as its only possible accumulation point. If hypothesis* (2.12.14) *does not hold, then every point in the complex plane is an eigenvalue.*

Proof. By Lemma (2.12.12) the set of eigenvalues of (2.12.10) equals the set of eigenvalues of (2.12.13), which is characterized by Theorem (2.10.25). This proves Theorem (2.12.15).

We give a second, longer proof. Instead of using Theorem (2.10.25) we will go through the steps of its proof explicitly because we will need to refer to one of these steps in the proof of Theorem (2.12.27). By hypothesis (2.12.14), there exists a $\xi \in \mathbb{C}$ such that $T(\xi)$ has a bounded inverse. Then

$$T(\lambda+\xi) = (\lambda+\xi)^N T_N + \ldots + (\lambda+\xi)T_1 + T_0 + I = \lambda^N T_N' + \ldots + \lambda T_1' + T(\xi),$$

for some compact operators $T_1', \ldots, T_N'$. Lemma (2.12.12) implies that (λ, u) is an eigenpair of (2.12.10) if and only if $T(\lambda)u = 0$. Define $\mu = \lambda - \xi$. Then

$$\begin{aligned} T(\lambda)u = 0 &\iff T(\mu+\xi)u = 0 \\ &\iff (\mu^N T_N' + \ldots + \mu T_1' + T(\xi))u = 0 \qquad (2.12.16) \\ &\iff (\mu^N T(\xi)^{-1} T_N' + \ldots + \mu T(\xi)^{-1} T_1' + I)u = 0. \end{aligned}$$

Define $(u_1, u_2, \ldots, u_N) := (u, \mu u, \ldots, \mu^{N-1}u)$. Then (2.12.16) holds if and only if

$$B\boldsymbol{u} := \begin{bmatrix} -T(\xi)^{-1}T_1' & \cdots & \cdots & -T(\xi)^{-1}T_N' \\ I & & & 0 \\ & \ddots & & \vdots \\ & & I & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix} = \frac{1}{\mu} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix}. \tag{2.12.17}$$

The operator B on the left-hand side of equation (2.12.17) is not compact since the identity operator is not compact on infinite dimensional spaces, but B^N is compact, and so B has a countable set of eigenvalues with zero as its only possible accumulation point. See Dunford & Schwartz (1957, Section VII.4, Theorem 6). This completes the proof. □

Let $V_{1,h}$ and $V_{2,h}$ be finite-dimensional subspaces of V_1 and V_2 parametrized by $h > 0$. We assume that A satisfies the discrete inf-sup conditions

$$\inf_{\substack{u \in V_{1,h} \\ ||u||_{V_1} = 1}} \sup_{\substack{v \in V_{2,h} \\ ||v||_{V_2} = 1}} |A(u, v)| = \alpha(h) > 0, \tag{2.12.18}$$

$$v = 0 \text{ if } A(u, v) = 0 \text{ for all } u \in V_{1,h}. \tag{2.12.19}$$

We make the approximability assumption that

$$\lim_{h \to 0} \alpha(h)^{-1} \inf_{\chi \in V_{1,h}} ||u - \chi||_{V_1} = 0 \quad \text{for all } u \in V_1. \tag{2.12.20}$$

The Discrete Problem. Find $\lambda \in \mathbb{C}$ and $0 \neq u_h \in V_{1,h}$ such that for all $v \in V_{2,h}$

$$A(u_h, v) = \lambda^N B_N(u_h, v) + \lambda^{N-1} B_{N-1}(u_h, v) + \ldots + \lambda B_1(u_h, v) + B_0(u_h, v). \tag{2.12.21}$$

We approximate the eigenvalues of the continuous problem (2.12.10) by the eigenvalues of the discrete problem (2.12.21).

The continuity and inf-sup conditions (2.12.5), (2.12.18), and (2.12.19) imply that there exists unique bounded linear operators $T_{0,h} : V_1 \to V_{1,h}, \ldots, T_{N,h} : V_1 \to V_{1,h}$ satisfying

$$A(T_{j,h}u, v) = -B_j(u, v) \quad \text{for all } v \in V_{2,h}, \quad \text{for } j \in \{0, \ldots, N\}.$$

Note that $T_{j,h}$ are finite rank operators and so are compact. Let $P_h : V_1 \to V_{1,h}$ be the projection defined by

$$A(P_h u, v) = A(u, v) \quad \text{for all } v \in V_{2,h}.$$

(P_h is well-defined by Babuška-Brezzi Theorem.) Then for all $j \in \{0, \ldots, N\}$, $T_{j,h} = P_h T_j$ since

$$A(T_{j,h}u - P_h T_j u, v) = -B_j(u, v) - A(T_j u, v) = 0 \tag{2.12.22}$$

for all $u \in V_1$ and $v \in V_{2,h}$. It is well-known that $(2.12.5)_1$ and (2.12.18) imply the quasi-optimality estimate

$$||u - P_h u||_{V_1} \leq \left(1 + \frac{C}{\alpha(h)}\right) \inf_{\chi \in V_{1,h}} ||u - \chi||_{V_1}. \tag{2.12.23}$$

(This is Céa's Lemma. See Ern & Guermond (2004, Lemma 2.28).) Therefore $P_h \to I$ pointwise by (2.12.20) and (2.12.23). Thus, for all $j \in \{0, \ldots, N\}$, $T_{j,h} = P_h T_j \to T_j$ in norm since T_j are compact.

For $\lambda \in \mathbb{C}$ define $T_h(\lambda) : V_1 \to V_1$ by

$$T_h(\lambda) := \lambda^N T_{N,h} + \ldots + \lambda T_{1,h} + T_{0,h} + I. \tag{2.12.24}$$

Lemma 2.12.25 (An equivalent formulation of the discrete eigenvalue problem). *The pair* $(\lambda, u_h) \in \mathbb{C} \times (V_1 \setminus 0)$ *satisfies problem* (2.12.21) *if and only if it is an eigenpair of the operator* $T_h : V_1 \to V_{1,h}$*, i.e.,*

$$T_h(\lambda)u_h \equiv \lambda^N T_{N,h}u_h + \ldots + \lambda T_{1,h}u_h + T_{0,h}u_h + u_h = 0. \tag{2.12.26}$$

Proof. Note that if (λ, u_h) is an eigenpair of T_h, then $u_h \in V_{1,h}$ since

$$u_h = -\lambda^N T_{N,h}u_h - \ldots - \lambda T_{1,h}u_h - T_{0,h}u_h.$$

The rest of the proof is analogous to the proof of Lemma (2.12.12). □

Observe that $T_h(\lambda) \to T(\lambda)$ in norm for all λ since $T_{j,h} \to T_j$ in norm for $j \in \{0, \ldots, N\}$. Therefore, for h sufficiently small, $T_h(\xi)$ has a bounded inverse (because $T(\xi)$ has a bounded inverse), and $T_h(\xi)^{-1} \to T(\xi)^{-1}$ in norm. See Kato (1980, p. 196, Theorem 1.16).

Theorem 2.12.27 (Convergence of the Eigenvalues). *Assume that* (2.12.5)–(2.12.7), (2.12.14), (2.12.18)–(2.12.20) *hold. Then the eigenvalues of problem* (2.12.21) *converge to the eigenvalues of problem* (2.12.10) *as* $h \to 0$.

Proof. Recall from the proof of Theorem (2.12.15) that (λ, u) is an eigenpair of (2.12.10) if and only if $(1/\mu, u)$ is an eigenpair of the operator B, where $\mu = \lambda - \xi$. The same method as in the proof of Theorem (2.12.15) shows that (λ, u_h) is an eigenpair of (2.12.21) if and only if $(1/\mu, u_h)$ is an eigenpair of the operator

$$B_h = \begin{bmatrix} -T_h(\xi)^{-1}T'_{1,h} & \cdots & \cdots & -T_h(\xi)^{-1}T'_{N,h} \\ I & & & 0 \\ & \ddots & & \vdots \\ & & I & 0 \end{bmatrix}. \tag{2.12.28}$$

But $B_h \to B$ in norm and so the eigenvalues of B_h converge to the eigenvalues of B. (Recall that if noncompact operators $L_h \to L$ in norm, then the isolated points of the spectrum of L_h converge. See Descloux, Nassif, and Rappaz (1978). In our case $L_h = B_h$ are polynomially compact and so every point of the spectrum is isolated.) □

Rate of convergence estimates. Kolata (1976) applied the spectral theory for compact operators from Osborn (1975) to obtain rate of convergence estimates for polynomial eigenvalue problems. We do no repeat these estimates here, but will specialize them to problem (2.11.51) in a remark following Theorem (2.12.69).

Finite Element Discretization and Discrete Inf-Sup Conditions

In this section we discretize the eigenvalue problem (2.11.51) using finite elements and prove discrete inf-sup conditions for the bilinear forms b^k and a_0^k.

Let $a = r_0 < r_1 < \ldots < r_N = R$ be a uniform partition of $[a, R]$ with $R - a = Nh$, so that $r_n = a + nh$, $n \in \{0, \ldots, N\}$. Let $V_{1,h}^k$, $V_{2,h}^k$, and Π_h^k be the finite dimensional subspaces of V_1^k, V_2^k, and Π^k defined by

$$\begin{aligned}
V_{1,h}^k &:= \left\{(\mathbf{v}_k, \mathbf{r}_k) \in V_1^k : \text{for } j \in \{1,2\}, v_k^j \text{ is continuous }, v_k^j|_{[r_n, r_{n+1}]} \text{ is quadratic}\right\}, \\
V_{2,h}^k &:= \left\{(\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k) \in V_2^k : \text{for } j \in \{1,2\}, w_k^j \text{ is continuous }, w_k^j|_{[r_n, r_{n+1}]} \text{ is quadratic}\right\}, \\
\Pi_h^k &:= \left\{p_k \in \Pi^k : p_k \text{ is continuous }, p_k|_{[r_n, r_{n+1}]} \text{ is linear}\right\},
\end{aligned} \tag{2.12.29}$$

for all $k \in \mathbb{Z}$. We will approximate the eigenpairs $(\lambda, (\mathbf{v}_k, \mathbf{r}_k, p_k)) \in \mathbb{C} \times V_1^k \times \Pi^k$ of problem (2.11.51) by the eigenpairs $(\lambda, (\mathbf{v}_k^h, \mathbf{r}_k^h, p_k^h)) \in \mathbb{C} \times V_{1,h}^k \times \Pi_h^k$ of the following problem:

The discrete eigenvalue problem. For each $k \in \mathbb{Z}$, find $\lambda \in \mathbb{C}$ and $0 \neq (\mathbf{v}_k^h, \mathbf{r}_k^h, p_k^h) \in V_{1,h}^k \times \Pi_h^k$ such that for all $(\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k) \in V_{2,h}^k$, $q_k \in \Pi_h^k$

$$\begin{aligned}
\lambda^2 a_2^k(\mathbf{r}_k^h, \mathbf{q}_k) + \lambda a_1^k((\mathbf{v}_k^h, \mathbf{r}_k^h), (\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k)) + a_0^k((\mathbf{v}_k^h, \mathbf{r}_k^h), (\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k)) + b^k(\mathbf{w}_k, p_k^h) &= 0, \\
b^k(\mathbf{v}_k^h, q_k) &= 0.
\end{aligned} \tag{2.12.30}$$

Define

$$V_h = \left\{\mathbf{v} \in [H_0^1([a,R];\mathbb{C})]^2 : \text{for } j \in \{1,2\}, v^j \text{ is continuous }, v^j|_{[r_n, r_{n+1}]} \text{ is quadratic}\right\}. \tag{2.12.31}$$

We prove two discrete inf-sup theorems for b^k, Theorems (2.12.32) and (2.12.47). Theorem (2.12.32) is a weak version of Theorem (2.12.47) because it requires h to be sufficiently small. We remove this condition in Theorem (2.12.47).

Discrete inf-sup conditions of the form (2.12.47) for the Fourier-finite element discretization of the Stokes equations in axisymmetric domains have been proved by Belhachmi et al. (2006a, 2006b), among others. In these papers general

3-dimensional axisymmetric domains are considered. The Fourier decomposition reduces the problem to a PDE on a 2-dimensional domain, which is discretized using finite elements. The discrete inf-sup condition is proved by Belhachmi et al. (2006a) for the $\mathbb{P}_1$-iso-$\mathbb{P}_2/\mathbb{P}_1$ element for $k = 0$ and by Belhachmi et al. (2006b) for the $\mathbb{P}_2$-bubble/$\mathbb{P}_1$-discontinuous element for all k. The inf-sup condition (2.12.47) for the Taylor-Hood $\mathbb{P}_2/\mathbb{P}_1$ element can be proved using the same techniques. Since we are only interested in a 2-dimensional annular domain, however, there is a more elementary proof, which we present here and which does not require the use of weighted Clément interpolant operators and other technical tools used by Belhachmi et al. (2006a, 2006b), although the basic idea is the same. The proof for the case $k = 0$ is analogous to the standard proof of the discrete inf-sup condition for the Taylor-Hood element.

Theorem 2.12.32 (b^k satisfies a discrete inf-sup condition). *Let $k \in \mathbb{Z}$, $h > 0$. Let b^k be the bilinear form defined in equation* (2.11.55)*. Then for h sufficiently small ($h \leq \frac{1}{2|k|}$ suffices)*

$$\inf_{p_k \in \Pi_h^k} \sup_{\mathbf{w}_k \in V_h} \frac{|b^k(\mathbf{w}_k, p_k)|}{||p_k||_{\Pi^k} ||\mathbf{w}_k||_{H^1}} = \beta > 0, \tag{2.12.33}$$

where β is independent of k and h. If $k = 0$ there is no restriction on h.

We will need the following lemma.

Lemma 2.12.34. *Let $k \in \mathbb{Z}$ and $p_k \in \Pi_h^k$. Then*

$$\sup_{\mathbf{w}_k \in V_h} \frac{|b^k(\mathbf{w}_k, p_k)|}{||\mathbf{w}_k||_{H^1}} \geq Ch||\partial_r p_k||_{\Pi^k}, \tag{2.12.35}$$

where C is independent of k and h.

Proof. Let $\mathbf{w}_k \in V_h$. Since p_k is continuous and piecewise linear we can integrate by parts in the expression for b^k to obtain

$$\begin{aligned} b^k(\mathbf{w}_k, p_k) &= -\rho \int_a^R p_k[\tfrac{1}{r}(r\overline{w_k^1})_r - \tfrac{ik}{r}\overline{w_k^2}]\, rdr = \rho \int_a^R \partial_r p_k \overline{w_k^1}\, rdr + \rho ik \int_a^R p_k \overline{w_k^2}\, dr \\ &= \rho \sum_{n=0}^{N-1} \int_{r_n}^{r_{n+1}} \partial_r p_k \overline{w_k^1}\, rdr + \rho ik \int_a^R p_k \overline{w_k^2}\, dr. \end{aligned} \tag{2.12.36}$$

We define a function $\mathbf{v}_k \in V_h$ as follows. Let $v_k^2(r) \equiv 0$ and let $v_k^1(r)$ be the continuous piecewise quadratic function that is uniquely determined by

$$v_k^1 = \begin{cases} 0 & \text{at vertices } r_n, \text{ for } n \in \{0, \dots, N\} \\ h\, \partial_r p_k & \text{at midpoints } \frac{r_n + r_{n+1}}{2}, \text{ for } n \in \{0, \dots, N-1\}. \end{cases} \tag{2.12.37}$$

We can use Simpson's quadrature rule, which is exact for cubic polynomials, to write (2.12.36) as

$$\begin{aligned} b^k(\mathbf{v}_k, p_k) &= \rho \sum_{n=0}^{N-1} \frac{4h}{6} \partial_r p_k \left(\tfrac{r_n + r_{n+1}}{2}\right) \overline{v_k^1} \left(\tfrac{r_n + r_{n+1}}{2}\right) \tfrac{r_n + r_{n+1}}{2} \\ &= \rho \sum_{n=0}^{N-1} \frac{2h^2}{3} \left|\partial_r p_k \left(\tfrac{r_n + r_{n+1}}{2}\right)\right|^2 \tfrac{r_n + r_{n+1}}{2} \end{aligned} \tag{2.12.38}$$

by the definition of v_k^1. Since p_k is piecewise linear

$$\int_{r_n}^{r_{n+1}} |\partial_r p_k|^2\, rdr = \left|\partial_r p_k \left(\tfrac{r_n + r_{n+1}}{2}\right)\right|^2 \int_{r_n}^{r_{n+1}} r\, dr = h \left|\partial_r p_k \left(\tfrac{r_n + r_{n+1}}{2}\right)\right|^2 \tfrac{r_n + r_{n+1}}{2}. \tag{2.12.39}$$

Combining (2.12.38) and (2.12.39) we find that

$$b^k(\mathbf{v}_k, p_k) = \frac{2\rho h}{3} \int_a^R |\partial_r p_k|^2\, rdr = \frac{2\rho h}{3} ||\partial_r p_k||_{\Pi^k}^2. \tag{2.12.40}$$

It is easy but tedious to check that

$$||v_k^1||_{H^1(a,R)} \leq C\, ||\partial_r p_k||_{\Pi^k}, \tag{2.12.41}$$

where C is independent of k and h. Therefore, by (2.12.40) and (2.12.41),

$$\sup_{\mathbf{w}_k \in V_h} \frac{|b^k(\mathbf{w}_k, p_k)|}{||\mathbf{w}_k||_{H^1}} \geq \frac{|b^k(\mathbf{v}_k, p_k)|}{||\mathbf{v}_k||_{H^1}} \geq \frac{2\rho h}{3C} ||\partial_r p_k||_{\Pi^k}. \tag{2.12.42}$$

□

Proof of Theorem (2.12.32). Recall that, given $p_k \in \Pi^k$, the proof of the continuous inf-sup condition (2.11.57) relied upon the construction of functions $\mathbf{u}_k \in [H_0^1([a, R]; \mathbb{C})]^2$ satisfying

$$(ru_k^1)_r + iku_k^2 = p_k r, \quad ||\mathbf{u}_k||_{H^1} \leq C||p_k||_{\Pi^k}. \tag{2.12.43}$$

These functions are not piecewise quadratic, however, and so we cannot prove the discrete inf-sup condition in the same way. Instead we approximate $\mathbf{u}_k$ by its continuous piecewise linear Lagrange interpolant $\mathcal{I}_h \mathbf{u}_k$. Let us recall some properties of the Lagrange interpolant $\mathcal{I}_h$. For all $f \in H^1(a, R)$

$$\begin{aligned} ||\mathcal{I}_h f||_{H^1(a,R)} &\leq C||f||_{H^1(a,R)} \\ ||\mathcal{I}_h f - f||_{L^2(a,R)} &\leq Ch||f||_{H^1(a,R)}. \end{aligned} \tag{2.12.44}$$

See, for example, Ern & Guermond (2004, pp. 11-12, Propositions 1.11, 1.12). Then for all $p_k \in \Pi^k$

$$\begin{aligned} \sup_{\mathbf{w}_k \in V_h} \frac{|b^k(\mathbf{w}_k, p_k)|}{||\mathbf{w}_k||_{H^1}} &\geq \frac{|b^k(\mathcal{I}_h \mathbf{u}_k, p_k)|}{||\mathcal{I}_h \mathbf{u}_k||_{H^1}} \geq \frac{|b^k(\mathcal{I}_h \mathbf{u}_k, p_k)|}{C||\mathbf{u}_k||_{H^1}} \qquad \text{(by (2.12.44))} \\ &= \frac{|b^k(\mathbf{u}_k, p_k) + b^k(\mathcal{I}_h \mathbf{u}_k - \mathbf{u}_k, p_k)|}{C||\mathbf{u}_k||_{H^1}} \\ &\geq \frac{|b^k(\mathbf{u}_k, p_k)|}{C||\mathbf{u}_k||_{H^1}} - \frac{|b^k(\mathcal{I}_h \mathbf{u}_k - \mathbf{u}_k, p_k)|}{C||\mathbf{u}_k||_{H^1}} \\ &\geq c\,||p_k||_{\Pi^k} - \frac{|b^k(\mathcal{I}_h \mathbf{u}_k - \mathbf{u}_k, p_k)|}{C||\mathbf{u}_k||_{H^1}} \qquad \text{(by (2.12.43))}. \end{aligned} \tag{2.12.45}$$

Now we bound the last term on the right-hand side of (2.12.45).

$$\begin{aligned}
|b^k(\mathcal{I}_h\mathbf{u}_k - \mathbf{u}_k, p_k)| &= \left|\rho\int_a^R \left\{p_k[r(\mathcal{I}_h\overline{u_k^1} - \overline{u_k^1})]_r - ikp_k(\mathcal{I}_h\overline{u_k^2} - \overline{u_k^2})\right\} dr\right| \\
&= \left|\rho\int_a^R \left\{-\partial_r p_k[r(\mathcal{I}_h\overline{u_k^1} - \overline{u_k^1})] - ikp_k(\mathcal{I}_h\overline{u_k^2} - \overline{u_k^2})\right\} dr\right| \\
&\leq \rho R||\partial_r p_k||_{L^2}||\mathcal{I}_h u_k^1 - u_k^1||_{L^2} + \rho|k|\,||p_k||_{L^2}||\mathcal{I}_h u_k^2 - u_k^2||_{L^2} \\
&\leq Ch||\partial_r p_k||_{\Pi^k}||u_k^1||_{H^1} + C'|k|h\,||p_k||_{\Pi^k}||u_k^2||_{H^1} \qquad \text{(by (2.12.44))} \\
&\leq C\left(\sup_{\mathbf{w}_k\in V_h}\frac{|b^k(\mathbf{w}_k, p_k)|}{||\mathbf{w}_k||_{H^1}} + |k|h\,||p_k||_{\Pi^k}\right)||\mathbf{u}_k||_{H^1}
\end{aligned} \tag{2.12.46}$$

by Lemma (2.12.34). By substituting for $|b^k(\mathcal{I}_h\mathbf{u}_k - \mathbf{u}_k, p_k)|$ from (2.12.46) into (2.12.45) we obtain

$$\sup_{\mathbf{w}_k\in V_h}\frac{|b^k(\mathbf{w}_k, p_k)|}{||\mathbf{w}_k||_{H^1}} \geq C(1 - |k|h)||p_k||_{\Pi^k} \quad \text{for all } p_k \in \Pi_h^k.$$

For $h \leq \frac{1}{2|k|}$, $C(1 - |k|h) \geq C/2 =: \beta > 0$. This completes the proof. □

Theorem 2.12.47 (b^k satisfies a discrete inf-sup condition). *Let $k \in \mathbb{Z}$, $h > 0$. Let b^k be the bilinear form defined in equation* (2.11.55)*. Then*

$$\inf_{p_k\in\Pi_h^k}\sup_{\mathbf{w}_k\in V_h}\frac{|b^k(\mathbf{w}_k, p_k)|}{||p_k||_{\Pi^k}||\mathbf{w}_k||_{H^1}} = \beta_k > 0, \tag{2.12.48}$$

where β_k is independent of h and is uniformly bounded from below by a positive constant β.

Proof. For the case $k = 0$, Theorem (2.12.47) is equivalent to Theorem (2.12.32), which we have already proved. Now we consider the case $k \neq 0$. Decompose the bilinear form b^k into two bilinear forms b_1^k and b_2^k:

$$b^k((w_k^1, w_k^2), p_k) = b_1^k(w_k^1, p_k) + b_2^k(w_k^2, p_k) = -\rho\int_a^R p_k(r\overline{w_k^1})_r\, dr + ik\rho\int_a^R p_k\overline{w_k^2}\, dr. \tag{2.12.49}$$

We will prove inf-sup conditions for the bilinear forms b_1^k and b_2^k and then combine the results to obtain the inf-sup condition for the bilinear form b^k. Define

$$\tilde{\Pi} = \{\tilde{p} : [a, R] \to \mathbb{C} : \tilde{p} \text{ is constant}\}. \tag{2.12.50}$$

Each $p_k \in \Pi_h^k$ can be decomposed as $p_k = p_0 + \tilde{p}$, where

$$\tilde{p} = \frac{\int_a^R p^k \, rdr}{\int_a^R r \, dr} \in \tilde{\Pi}, \quad p_0 = p_k - \tilde{p} \in \Pi_h^0. \tag{2.12.51}$$

Therefore $\Pi_h^k = \Pi_h^0 \bigoplus \tilde{\Pi}$. Moreover, Π_h^0 is orthogonal to $\tilde{\Pi}$ with respect to the weighted inner product $\langle \cdot, \cdot \rangle_{L^2([a,R],rdr)}$.

Lemma 2.12.52 (b_1^k satisfies an inf-sup condition). *Let $0 \neq k \in \mathbb{Z}$, $h > 0$. Let b_1^k be the bilinear form defined in equation* (2.12.49)*. Then*

$$\inf_{p_0 \in \Pi_h^0} \sup_{(w^1, 0) \in V_h} \frac{|b_1^k(w^1, p_0)|}{||p_0||_{\Pi^0} ||w^1||_{H^1}} = \beta_1 > 0, \tag{2.12.53}$$

where β_1 is independent of h and k.

Proof. This lemma is just a restatement of Theorem (2.12.47) for the case $k = 0$, which we have already proved. □

Lemma 2.12.54 (b_2^k satisfies an inf-sup condition). *Let $0 \neq k \in \mathbb{Z}$, $h > 0$. Let b_2^k be the bilinear form defined in equation* (2.12.49)*. Then*

$$\inf_{\tilde{p} \in \tilde{\Pi}} \sup_{(0, w^2) \in V_h} \frac{|b_2^k(w^2, \tilde{p})|}{||\tilde{p}||_{\Pi^k} ||w^2||_{H^1}} = |k| \beta_2 > 0, \tag{2.12.55}$$

where β_2 is independent of h and k.

Proof. Let $\tilde{p} \in \tilde{\Pi}$. Define $v_k^2(r) = -\frac{i}{k} \tilde{p} \, \psi^h(r)$, where $\psi^h(r)$ is a continuous piecewise linear function that equals 1 on a (more or less) fixed part of the interval $[a, R]$. To be precise, let $\lfloor x \rfloor$ denote the largest integer that is less than or equal to x and let $\lceil x \rceil$ denote the smallest integer that is greater than or equal to x. Then we define the continuous piecewise linear function $\psi^h(r)$ by

$$\psi^h(r) := \begin{cases} 1 & \text{in the interval } h\lfloor \frac{R-a}{3h} \rfloor \leq r \leq h\lceil \frac{2(R-a)}{3h} \rceil \\ 0 & \text{at the endpoints } r = a, R \\ \text{is linear} & \text{in the intervals } a \leq r \leq h\lfloor \frac{R-a}{3h} \rfloor,\ h\lceil \frac{2(R-a)}{3h} \rceil \leq r \leq R. \end{cases}$$

Therefore

$$b_2^k(v_k^2, \tilde{p}) = \rho|\tilde{p}|^2 \int_a^R \psi^h(r)\, dr \geq \rho|\tilde{p}|^2(R-a)/3 = \frac{2(R-a)\rho}{3(R^2-a^2)}||\tilde{p}||_{\Pi^k}^2. \tag{2.12.56}$$

It is easy to check that

$$||v_k^2||_{H^1} \leq \frac{C}{|k|}||\tilde{p}||_{\Pi^k}, \tag{2.12.57}$$

where C is independent of k. By combining (2.12.56) and (2.12.57) we complete the proof:

$$\sup_{(0,w^2)\in V_h} \frac{|b_2^k(w^2,\tilde{p})|}{||w^2||_{H^1}} \geq \frac{|b_2^k(v_k^2,\tilde{p})|}{||v_k^2||_{H^1}} \geq |k|\beta_2||\tilde{p}||_{\Pi^k} \quad \text{for all } \tilde{p} \in \tilde{\Pi}.$$

□

Final step of the proof of Theorem (2.12.47)**.** Let $p_k \in \Pi_h^k$. Decompose $p_k = p_0 + \tilde{p}$, where $p_0 \in \Pi_h^0$ and $\tilde{p} \in \tilde{\Pi}$ are defined in equation (2.12.51). By the inf-sup conditions (2.12.52) and (2.12.54) there exists $(v^1, v^2) \in V_h$ satisfying

$$\begin{aligned} b_1^k(v^1, p_0) &= ||p_0||_{\Pi^k}^2, \quad ||v^1||_{H^1} \leq \tfrac{1}{\beta_1}||p_0||_{\Pi^k}, \\ b_2^k(v^2, \tilde{p}) &= ||\tilde{p}||_{\Pi^k}^2, \quad ||v^2||_{H^1} \leq \tfrac{1}{|k|\beta_2}||\tilde{p}||_{\Pi^k}. \end{aligned} \tag{2.12.58}$$

(The existence of v^1 is shown as follows. By Lemma (2.12.52)

$$\sup_{\substack{(w^1,0)\in V_h \\ ||w^1||_{H^1}=1}} |b_1^k(w^1,p_0)| \geq \beta_1 ||p_0||_{\Pi^k}.$$

The supremum is clearly obtained. Let $(w,0)\in V_h$ be a maximizing function. Then $v^1 := w||p_0||^2_{\Pi^k}/b(w,p_0)$ has the desired properties. The existence of v^2 is shown similarly.) Let $\eta > 0$ be a constant that is to be determined. Then

$$\begin{aligned} b^k((\eta v^1, v^2), p_k) &= b_1^k(\eta v^1, p_k) + b_2^k(v^2, p_k) \\ &= b_1^k(\eta v^1, p_0) + b_1^k(\eta v^1, \tilde{p}) + b_2^k(v^2, p_0) + b_2^k(v^2, \tilde{p}) \\ &= \eta ||p_0||^2_{\Pi^k} + b_2^k(v^2, p_0) + ||\tilde{p}||^2_{\Pi^k}. \end{aligned}$$

(Note that $b_1^k(\eta v^1, \tilde{p}) = 0$ since $\tilde{p}$ is constant and $v^1(a) = v^1(R) = 0$.) Therefore

$$\begin{aligned} |b^k((\eta v^1, v^2), p_k)| &\geq \eta ||p_0||^2_{\Pi^k} + ||\tilde{p}||^2_{\Pi^k} - \frac{\rho|k|}{a} ||v^2||_{H^1} ||p_0||_{\Pi^k} \\ &\geq \eta ||p_0||^2_{\Pi^k} + ||\tilde{p}||^2_{\Pi^k} - \frac{\rho}{a\beta_2} ||\tilde{p}||_{\Pi^k} ||p_0||_{\Pi^k} \\ &\geq \eta ||p_0||^2_{\Pi^k} + ||\tilde{p}||^2_{\Pi^k} - \frac{\rho}{a\beta_2} \left(\frac{\epsilon}{2} ||\tilde{p}||^2_{\Pi^k} + \frac{1}{2\epsilon} ||p_0||^2_{\Pi^k} \right) \\ &= \left(\eta - \frac{\rho}{2a\beta_2\epsilon} \right) ||p_0||^2_{\Pi^k} + \left(1 - \frac{\rho\epsilon}{2a\beta_2} \right) ||\tilde{p}||^2_{\Pi^k}. \end{aligned}$$

Choose $\epsilon = \frac{a\beta_2}{\rho}$ and $\eta = \frac{1}{2} + \frac{\rho}{2a\beta_2\epsilon} = \frac{1}{2} + \frac{\rho^2}{2a^2\beta_2^2}$. Then we obtain the bound

$$|b^k((\eta v^1, v^2), p_k)| \geq \frac{1}{2}(||p_0||^2_{\Pi^k} + ||\tilde{p}||^2_{\Pi^k}) = \tfrac{1}{2} ||p_k||^2_{\Pi^k} \tag{2.12.59}$$

since p_0 is orthogonal to $\tilde{p}$ in $\Pi^k = L^2([a,R]; rdr)$. Finally, we need to estimate $||(\eta v^1, v^2)||_{H^1}$.

$$\begin{aligned} ||(\eta v^1, v^2)||^2_{H^1} = ||\eta v^1||^2_{H^1} + ||v^2||^2_{H^1} &\leq \frac{\eta^2}{\beta_1^2} ||p_0||^2_{\Pi^k} + \frac{1}{|k|^2\beta_2^2} ||\tilde{p}||^2_{\Pi^k} \\ &\leq \max\left\{ \frac{\eta^2}{\beta_1^2}, \frac{1}{|k|^2\beta_2^2} \right\} (||p_0||^2_{\Pi^k} + ||\tilde{p}||^2_{\Pi^k}) = \max\left\{ \frac{\eta^2}{\beta_1^2}, \frac{1}{|k|^2\beta_2^2} \right\} ||p_k||^2_{\Pi^k}. \end{aligned} \tag{2.12.60}$$

Define

$$\beta_k = \frac{1}{2\max\left\{\frac{\eta}{\beta_1}, \frac{1}{|k|\beta_2}\right\}}. \tag{2.12.61}$$

Combining (2.12.59)–(2.12.61) we obtain the desired result: for $k \neq 0$, for all $p_k \in \Pi_h^k$

$$\sup_{\mathbf{w}_k \in V_h} \frac{|b^k(\mathbf{w}_k, p_k)|}{||\mathbf{w}_k||_{H^1}} \geq \frac{|b^k((\eta v^1, v^2), p_k)|}{||(\eta v^1, v^2)||_{H^1}} = \beta_k ||p_k||_{\Pi^k}.$$

Observe that for $|k|$ large enough

$$\beta_k = \frac{\beta_1}{2\eta} =: \beta,$$

which is independent of k. □

It follows immediately from Theorem (2.12.47) that

$$\inf_{p_k \in \Pi_h^k} \sup_{(\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k) \in V_{2,h}^k} \frac{|b^k(\mathbf{w}_k, p_k)|}{||p_k||_{\Pi^k} ||(\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k)||_{V_{2,h}^k}} = \beta_k > 0. \tag{2.12.62}$$

For $k \in \mathbb{Z}$, define

$$Z_{1,h}^k := \{(\mathbf{v}_k, \mathbf{r}_k) \in V_{1,h}^k : b^k(\mathbf{v}_k, p_k) = 0 \text{ for all } p_k \in \Pi_h^k\},$$

$$Z_{2,h}^k := \{(\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k) \in V_{2,h}^k : b^k(\mathbf{w}_k, p_k) = 0 \text{ for all } p_k \in \Pi_h^k\}.$$

Theorem 2.12.63 (a_0^k satisfies a discrete Gårding-type inf-sup condition). *Let $k \in \mathbb{Z}$. The bilinear form $\hat{a}_0^k$ defined in Theorem* (2.11.64) *satisfies the discrete inf-sup conditions*

$$\inf_{\substack{(\mathbf{v}_k, \mathbf{r}_k) \in Z_{1,h}^k \\ ||(\mathbf{v}_k, \mathbf{r}_k)||_{V_1^k} = 1}} \sup_{\substack{(\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k) \in Z_{2,h}^k \\ ||(\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k)||_{V_2^k} = 1}} |\hat{a}_0^k((\mathbf{v}_k, \mathbf{r}_k), (\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k))| = \alpha > 0, \tag{2.12.64}$$

$$(\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k) = 0 \textit{ if } \hat{a}_0^k((\mathbf{v}_k, \mathbf{r}_k), (\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k)) = 0 \textit{ for all } (\mathbf{v}_k, \mathbf{r}_k) \in Z_{1,h}^k. \tag{2.12.65}$$

Proof. This is similar to the proof of Theorem (2.11.64) and is left as an exercise. (In fact the proof is simpler. We just need to check uniqueness of solutions to the finite-dimensional version of equation (2.11.70).) $\square$

Convergence of the Numerical Method

In this section we apply the abstract spectral approximation theory to eigenvalue problem (2.11.51) and its discretization (2.12.30). We prove another spectral theorem of the form (2.10.23) and show that the finite element approximation of the eigenvalues converges.

Define the bilinear form

$$c^k((\mathbf{v}_k, \mathbf{r}_k, p_k), (\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k, q_k)) := \hat{a}_0^k((\mathbf{v}_k, \mathbf{r}_k), (\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k)) + b^k(\mathbf{w}_k, p_k) + b^k(\mathbf{v}_k, q_k), \tag{2.12.66}$$

where $\hat{a}_0^k$ and b^k were defined in equations (2.11.65) and (2.11.55). Let C_{g}^k be the constant introduced in Theorem (2.11.64). Then the weak formulations (2.11.51) and (2.12.30) can be written in the form:

Equivalent formulation of the continuous eigenvalue problem (2.11.51)**.** For each $k \in \mathbb{Z}$, find $\lambda \in \mathbb{C}$ and $0 \neq (\mathbf{v}_k, \mathbf{r}_k, p_k) \in V_1^k \times \Pi^k$ such that for all $(\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k, q_k) \in V_2^k \times \Pi^k$

$$\begin{aligned} c^k((\mathbf{v}_k, \mathbf{r}_k, p_k), (\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k, q_k)) &= -\lambda^2 a_2^k(\mathbf{r}_k, \mathbf{q}_k) - \lambda a_1^k((\mathbf{v}_k, \mathbf{r}_k), (\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k)) \\ &\quad + C_{\mathrm{g}}^k \left(\int_a^R \{v_k^1 \overline{w_k^1} + v_k^2 \overline{w_k^2}\}\, dr + r_k^1 \overline{q_k^1} + r_k^2 \overline{q_k^2} \right). \end{aligned} \tag{2.12.67}$$

Equivalent formulation of the discrete eigenvalue problem (2.12.30)**.** For each $k \in \mathbb{Z}$, find $\lambda \in \mathbb{C}$ and $0 \neq (\mathbf{v}_k^h, \mathbf{r}_k^h, p_k^h) \in V_{1,h}^k \times \Pi_h^k$ such that for all $(\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k, q_k) \in V_{2,h}^k \times \Pi_h^k$.

$$\begin{aligned} c^k((\mathbf{v}_k^h, \mathbf{r}_k^h, p_k^h), (\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k, q_k)) &= -\lambda^2 a_2^k(\mathbf{r}_k^h, \mathbf{q}_k) - \lambda a_1^k((\mathbf{v}_k^h, \mathbf{r}_k^h), (\mathbf{w}_k, \mathbf{q}_k, \mathbf{t}_k)) \\ &+ C_{\mathrm{g}}^k \left(\int_a^R \{v_k^{1,h}\overline{w_k^1} + v_k^{2,h}\overline{w_k^2}\} \, dr + r_k^{1,h}\overline{q_k^1} + r_k^{2,h}\overline{q_k^2} \right) \quad (2.12.68) \end{aligned}$$

For each $k \in \mathbb{Z}$, we will apply the abstract spectral approximation theory from section (2.12) to the eigenvalue problem (2.12.67) and its discretization (2.12.68) with

$$A = c^k, \quad B_2 = -a_2^k, \quad B_1 = -a_1^k, \quad B_0 = C_{\mathrm{g}}^k \left(\int_a^R \{v_k^1\overline{w_k^1} + v_k^2\overline{w_k^2}\} \, dr + r_k^1\overline{q_k^1} + r_k^2\overline{q_k^2} \right)$$

It is well-known that the continuous and discrete inf-sup conditions for c^k follow from those for $\hat{a}_0^k$ and b^k, which we proved in Theorems (2.11.56), (2.11.64), (2.12.47) and (2.12.63). See Ern & Guermond (2004, p. 101, Proposition 2.36) or Brezzi & Fortin (1991). It follows from Theorem (2.10.5) that not every complex number is an eigenvalue of (2.12.67). Therefore hypotheses (2.12.6), (2.12.7), (2.12.14), (2.12.18), and (2.12.19) of Theorem (2.12.27) are satisfied and we have proved

Theorem 2.12.69 (Characterization of the spectrum of (2.12.67) **and convergence of the finite element approximation of the eigenvalues.).** *The problem* (2.12.67) *has a countable set of eigenvalues with infinity as its only possible accumulation point. The eigenvalues of problem* (2.12.68) *converge to the eigenvalues of problem* (2.12.67) *as* $h \to 0$.

Characterization of the spectrum for viscoelastic strings. Theorem (2.12.69) applies to both the elastic ($N_\nu^\circ = 0$) and viscoelastic ($N_\nu^\circ \neq 0$) cases. (Recall that Theorem (2.10.23) applied onto to the elastic case.) In particular, for $N_\nu^\circ \neq 0$ and each k, the spectrum of (2.12.67) is countable. It follows that for $N_\nu^\circ \neq 0$ the spectrum of the original problem (2.9.12) is the countable union of countable sets and so is countable. This Fourier method does not, however, eliminate the possibility of a finite accumulation point.

Rate of convergence estimates. Since the eigenfunctions of (2.12.67) are smooth and we are using $\mathbb{P}_2/\mathbb{P}_1$ elements, applying the results of Osborn (1975) and Kolata (1976) yields the rate of convergence estimate $|\lambda - \lambda_h| \leq Ch^4$ for simple eigenvalues.

2.13 Computation of the Spectrum

The eigenvalues of the discrete problem (2.12.30) were computed using MATLAB. In this section we present our results and discuss some of the computational issues.

Constitutive functions and material constants. Up until now we have been working with a broad class of constitutive functions. To compute the spectrum we must choose a constitutive function $\hat{N}$. We choose

$$\hat{N}(\nu) = E\hbar(\nu - 1), \tag{2.13.1}$$

where E is the modulus of elasticity and $\hbar$ is the thickness the string. Note that $\hat{N}$ is linear in the strain variable ν, but not in the displacement $\boldsymbol{r}$. Equation (2.13.1) is sometimes referred to as the generalized Hooke's law. This constitutive relation does not penalize compression. Since we only consider the linearization of $\hat{N}$ about a stretched state $\nu = R$, however, we do not need an accurate model of the tension for materials under compression.

In addition to choosing a constitutive function we must also choose values for all the numerical constants. These are listed in Table (2.13.1). We chose the fluid to be water and the deformable body to be either steel or a soft, rubber-like material. The ratio of the radius of the inner cylinder to the radius of the outer cylinder is close to the value used by G.I. Taylor in his experiments on the classical Taylor-Couette problem in the 1920s.

Radius of Rigid Cylinder	a	0.75 m
Radius of Deformable Cylinder	R	1.01 m
Density of Water	ρ	1000 kg/m^3
Dynamic Viscosity of Water	$\tilde{\mu}$	1.002×10^3 kg/ms
Thickness of Deformable Cylinder	$\hbar$	$2\pi/1000$ m
Density of Steel	ϱ_{s}	7850 kg/m^3
Density of Rubber	ϱ_{r}	920 kg/m^3
Modulus of Elasticity of Steel	E_{s}	207 GPa
Modulus of Elasticity of Rubber	E_{r}	0.01 GPa

Table 2.13.1: Values of the numerical constants used for the computation.

Recall that the constant (ϱA) is the mass density of the string per reference length. Since $\hbar$ is small we take $(\varrho A) := \varrho\hbar$.

Results. Figures (2.13.1)–(2.13.3) show plots of the eigenvalues of (2.12.30) moving around the complex plane as ω is varied. In Section 2.10 we proved that all the eigenvalues that cross the imaginary axis must cross through the origin, but we did not prove anything about the way that they crossed. Our numerical results show that, for the constitutive function and numerical constants given above, the eigenvalues cross through the origin in complex conjugate pairs, signaling a Takens-Bogdanov bifurcation. We exhibit convergence rates for the eigenvalues in Figure (2.13.4).

Eigensolver. By introducing basis functions for the finite dimensional spaces appearing in equation (2.12.30) we obtain a matrix quadratic eigenvalue problem of the form $\lambda^2 C_2\boldsymbol{x} + \lambda C_1\boldsymbol{x} + C_0\boldsymbol{x} = 0$, which we solve using the MATLAB function *polyeig*. Polyeig first reduces the quadratic eigenvalue problem to a generalized eigenvalue problem of the form $A\boldsymbol{y} = \lambda B\boldsymbol{y}$, where the matrices A and B are twice the dimension of the matrices C_0, C_1, and C_2. This reduction is done by introducing a new variable $\boldsymbol{w} = \lambda\boldsymbol{x}$ and by defining $\boldsymbol{y}^T = (\boldsymbol{w}^T, \boldsymbol{x}^T)$. Then the generalized eigenvalue problem is solved using the direct QZ algorithm of Moler & Stewart (1973). The number of operations is $\mathcal{O}(M^3)$, where M is the dimension of the matrices A and B. In our case M is of the order of 250 (for $N = 25$ mesh points) and the eigenvalues are returned within less than half a second on a 2.80GHz Intel Pentium 4 with 512MB of memory (this includes the time to build the matrices).

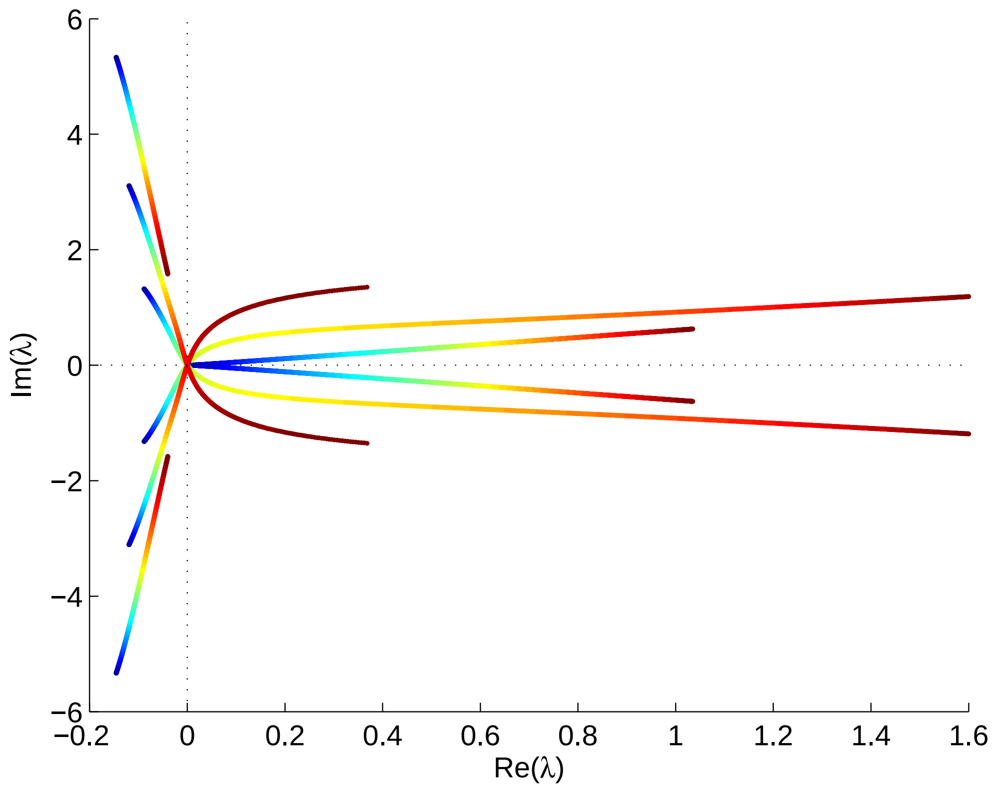

Figure 2.13.1: Trajectories of the leading eigenvalues λ for a rubber string, Fourier modes $|k| \in \{1, 2, 3, 4\}$, and angular velocities $\omega \in [0, 2.5]$. The color of each trajectory changes from blue to red as ω changes from 0 to 2.5. The eigenvalues cross the imaginary axis in order of Fourier mode: If $\omega_{\text{crit}}(k)$ denotes the critical value of ω for Fourier mode k, then $0 = \omega_{\text{crit}}(\pm 1) < \omega_{\text{crit}}(\pm 2) < \omega_{\text{crit}}(\pm 3) < \cdots$. The domain $[a, R]$ of the fluid velocity and pressure was partitioned with $N = 25$ equally spaced mesh points.

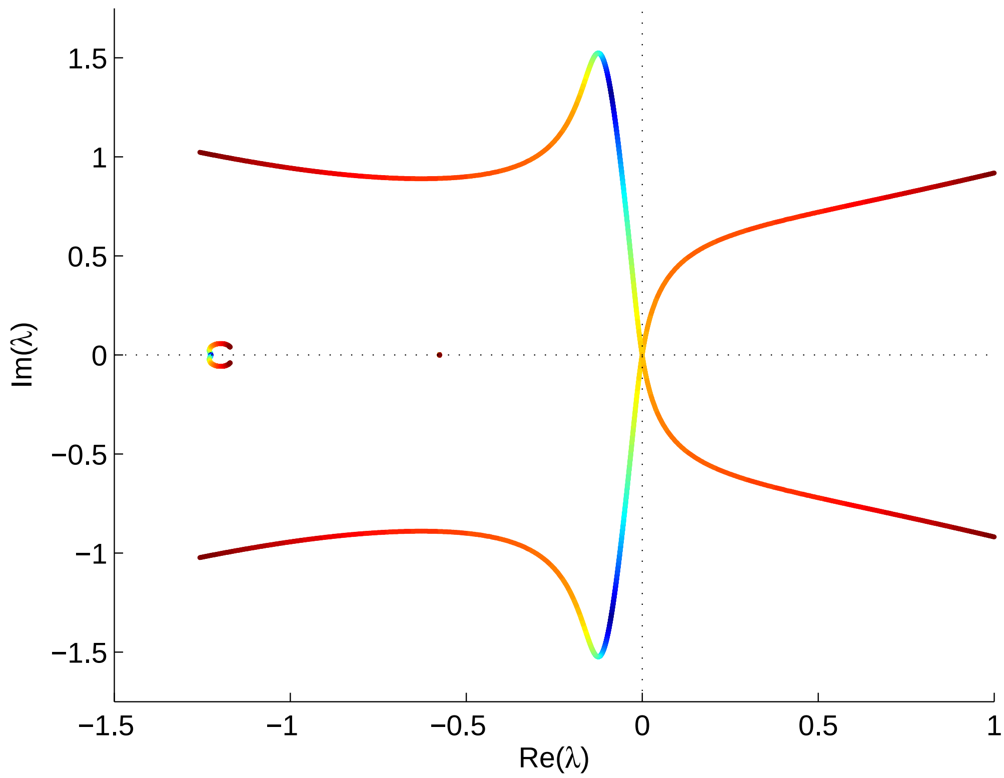

Figure 2.13.2: Trajectories of the top 4 eigenvalues λ (sorted by decreasing real part) for a rubber string, Fourier modes $|k| = 2$, and angular velocities $\omega \in [0, 2]$. The color of each trajectory changes from blue to red as ω changes from 0 to 2. The blue region in the 2nd quadrant shows that two eigenvalues start from the same point when $\omega = 0$. As ω is increased one of these eigenvalues moves toward the origin and the other moves up and then left. $N = 25$ mesh points were used.

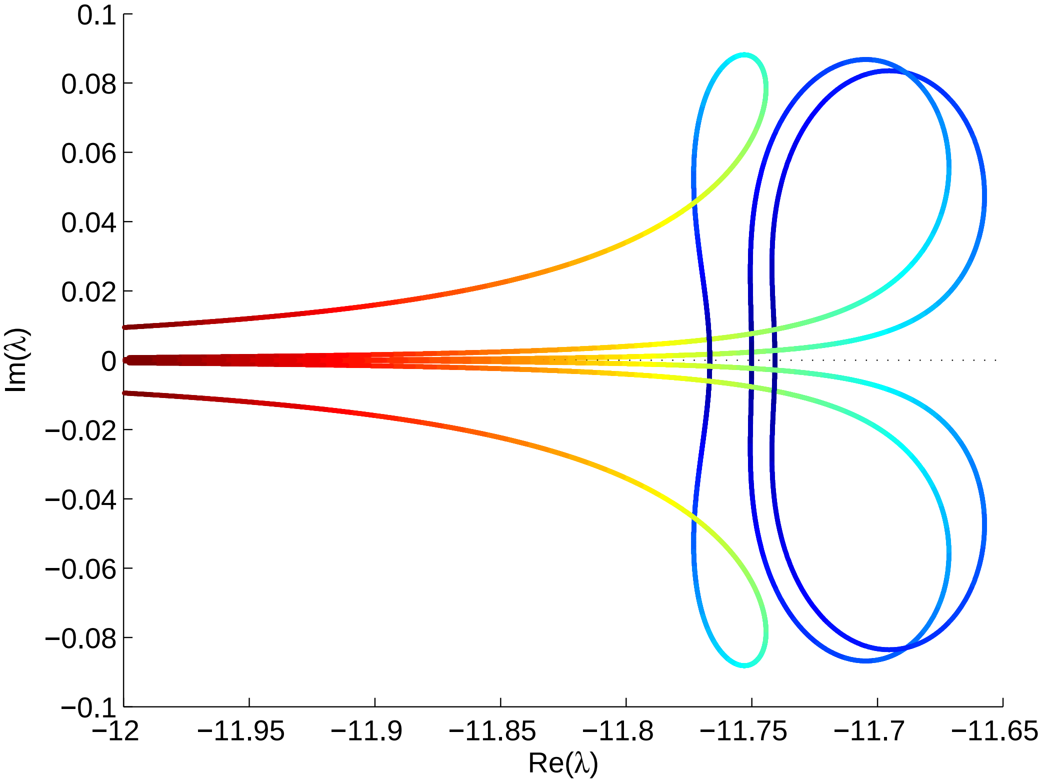

Figure 2.13.3: Eigenvalue trajectories for Fourier modes $|k| \in \{1, 2, 3\}$ and angular velocities $\omega \in [0, 50]$. The 10th eigenvalue of each Fourier mode is plotted (where the eigenvalues are ordered by decreasing real part). The color of each trajectory changes from blue to red as ω changes from 0 to 50. These results are for a rubber string. $N = 25$ mesh points were used.

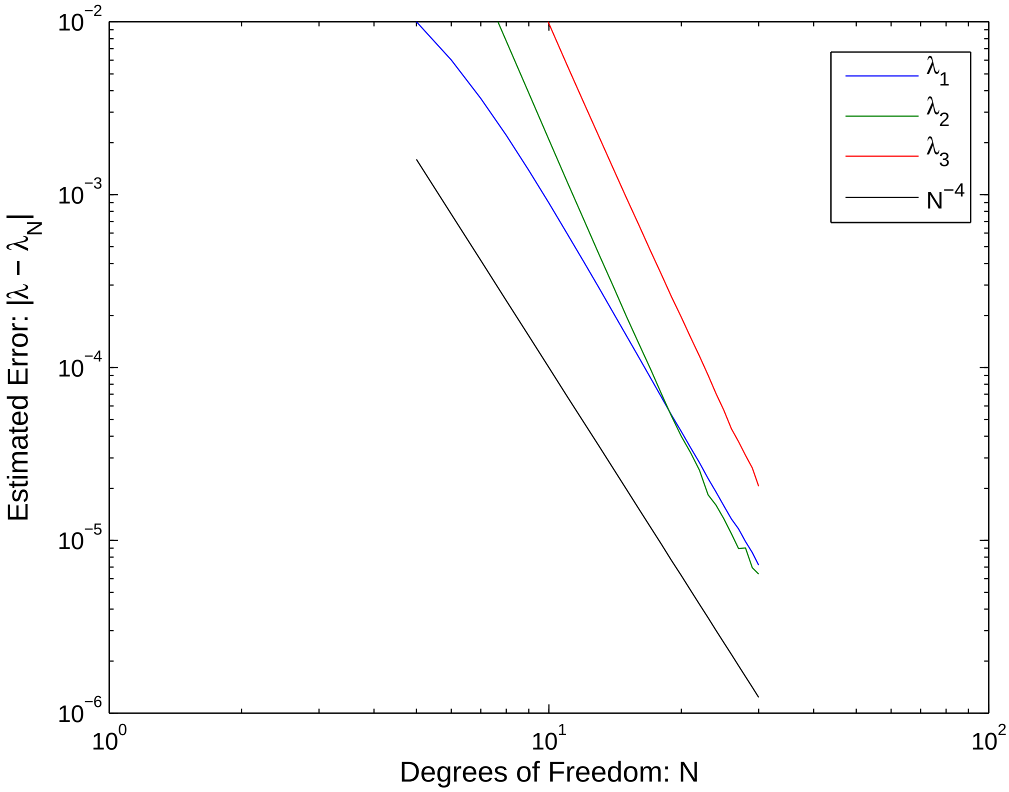

Figure 2.13.4: This log-log plot exhibits the fourth order convergence rate of the first three eigenvalues λ_1, λ_2, and λ_3 for a rubber string, $k = 1$, and $\omega = 5$. The true value of each eigenvalue was approximated using $N = 100$ mesh points.

The QZ algorithm returns all of the eigenvalues of a given matrix. For larger problems, such as those arising from the discretization of partial differential equations on 2- or 3-dimensional domains, an iterative method is needed and not all the eigenvalues can be computed. Popular iterative methods include the implicitly restarted Arnoldi method (see Lehoucq et al. (1998)) and inexact inverse iteration (the inverse power method with updated shifts and inexact linear solves). The Arnoldi method returns just a few of the eigenvalues of largest magnitude. Thus to solve stability problems we must first transform the eigenvalue problem so that the eigenvalues of largest real part are mapped to eigenvalues of largest magnitude. This can be achieved using the exponential map or a shift-and-invert transformation. Inexact inverse iteration returns just one eigenvalue, the eigenvalue closest to a given point. In either case, whether we use the Arnoldi method or inexact inverse iteration, a large sparse linear system must be solved. Solving the system with a Krylov subspace method requires an effective preconditioner. The numerical solution of large scale eigenvalue problems is an active area of research.

Basis for Π_h^0. Recall that Π_h^0 is the space of continuous piecewise linear functions $p(r)$ on the uniform grid $a = r_0 < r_1 < \ldots < r_N = R$ satisfying the zero mean condition $\int_a^R p(r) r\, dr = 0$. See (2.11.50) and (2.12.29). Let $\{\hat{\varphi}_i\}_{i=0}^N$ be the usual hat functions, which satisfy $\hat{\varphi}_i(r_j) = \delta_{ij}$. These do not satisfy the zero mean condition and so do not form a basis for Π_h^0. We modify them as follows. Define

$$\varphi_i := \hat{\varphi}_i - \frac{\int_a^R \hat{\varphi}_i \, r dr}{\int_a^R r dr} =: \hat{\varphi}_i - c_i.$$

Then $\varphi_i \in \Pi_h^0$. It is easy to check that $\{\varphi_i\}_{i=0}^N$ is not linearly independent and so not a basis for Π_h^0, but that a basis can be obtained by dropping any one of the functions φ_i. We drop φ_N to obtain the basis $\{\varphi_i\}_{i=0}^{N-1}$. Note that for all $i \in \{0, \ldots, N\}$ and $(\mathbf{w}_0, \mathbf{q}_0, \mathbf{t}_0) \in V_2^0$

$$\begin{aligned} b^0(\mathbf{w}_0, \varphi_i) &= b^0(\mathbf{w}_0, \hat{\varphi}_i) - c_i b^0(\mathbf{w}_0, 1) \\ &= b^0(\mathbf{w}_0, \hat{\varphi}_i) + c_i \rho \int_a^R (\overline{w_0^1} r)_r \, dr \\ &= b^0(\mathbf{w}_0, \hat{\varphi}_i) + c_i \rho [\overline{w_0^1}(R) R - \overline{w_0^1}(a) a] \\ &= b^0(\mathbf{w}_0, \hat{\varphi}_i) \end{aligned} \tag{2.13.2}$$

since $w_0^1(a) = w_0^1(R) = 0$. Similarly, for all $i \in \{0, \ldots, N\}$ and $(\mathbf{v}_0, \mathbf{r}_0) \in V_1^0$

$$b^0(\mathbf{v}_0, \varphi_i) = b^0(\mathbf{v}_0, \hat{\varphi}_i) + c_i \rho \, \overline{v_0^1}(R) R. \tag{2.13.3}$$

Note that $v_0^1(R)$ is not necessarily equal to zero a priori. However, $r_0^1 = 0$ and the weak formulation (2.12.30) enforces the adherence condition $v_0^1(R) = \lambda r_0^1 = 0$. Therefore by (2.13.2) and (2.13.3) we see that the discretization matrices for grad and div do not change if we use the modified basis $\{\varphi_i\}_{i=0}^{N-1}$ instead of the original hat functions $\{\hat{\varphi}_i\}_{i=0}^N$ (except that we have one less basis function and so one less column or row in the matrices).

Accuracy check. The accuracy of the code was verified using the following four methods:

(i) We checked that the Laplacian, divergence, and gradient operators for the fluid had been discretized correctly by using the discretization matrices to solve the eigenvalue problems

$$-\mathrm{div}\boldsymbol{D}(\boldsymbol{v}) = \lambda \boldsymbol{v} \quad \text{in } \Omega = \{\boldsymbol{x} : a < |\boldsymbol{x}| < R\},$$

$$\boldsymbol{v} = 0 \qquad \text{on } \partial\Omega,$$

and

$$\lambda \boldsymbol{v} = -\nabla p + \Delta \boldsymbol{v} \quad \text{in } \Omega,$$

$$\mathrm{div}\boldsymbol{v} = 0 \quad \text{in } \Omega,$$

$$\boldsymbol{v} = 0 \quad \text{on } \partial\Omega.$$

(Note that the second eigenvalue problem is Stokes eigenvalue problem.) We compared our results to those produced by the commercial software COMSOL Multiphysics. They were in good agreement.

(ii) Theorem (2.10.21) gives an exact formula for the critical values of ω, ω_{crit}, which satisfy $\lambda(\omega_{\text{crit}}) = 0$. The computed values of $\lambda(\omega_{\text{crit}})$ are reported in Table (2.13.2). We see that in the worst case the computed value of $\lambda(\omega_{\text{crit}})$ is of the order 10^{-8}.

(iii) In Section 2.10 we showed that the Fourier mode $k = 0$ has eigenvalue $\lambda = 0$ for all ω. The MATLAB program exhibits this property. In fact, for $k = 0$, the leading eigenvalue that is returned is exactly equal to zero (to machine precision).

(iv) For the case $k = 0$ it is possible to use Bessel functions to reduce the quadratic eigenvalue problem to a nonlinear scalar equation for λ. This algebraic equation can then be solved using the MATLAB function *fsolve*. Table (2.13.3) displays the eigenvalues computed with the finite element method with $N = 100$

k	steel		rubber	
	ω_{crit}	computed value of $\lambda(\omega_{\text{crit}})$	ω_{crit}	computed value of $\lambda(\omega_{\text{crit}})$
1	0	5.2828×10^{-8}	0	1.3604×10^{-12}
2	185.8218	$(0.0009 - 7.7706i) \times 10^{-8}$	1.3450	$(-0.0126 + 6.6488i) \times 10^{-11}$
3	303.4312	$(0.8547 - 5.6934i) \times 10^{-8}$	2.1964	$(3.6084 - 4.3216i) \times 10^{-12}$
4	415.4623	$(0.3408 - 5.1650i) \times 10^{-8}$	3.0075	$(-3.9973 - 8.1133i) \times 10^{-12}$

Table 2.13.2: Accuracy check. Critical values of ω (computed using formula (2.10.19)) tabulated against the computed values of $\lambda(\omega_{\text{crit}})$. The exact value of $\lambda(\omega_{\text{crit}})$ is zero. The eigenvalues were computed with $N = 50$ mesh points. In fact, the same order of accuracy can be achieved with only $N = 2$ mesh points since the eigenvalue $\lambda = 0$ has corresponding eigenvector $(\mathbf{v}_k, \mathbf{r}_k, p_k) = (0, \mathbf{r}_k, 0) \in V_1^k \times \Pi^k$, which belongs to the finite-dimensional subspace $V_{1,h}^k \times \Pi_h^k$ for all h.

mesh points against those computed using the Bessel method. We see that, except in one case, the eigenvalues agree to six decimal places (using chopping). Since it is highly unlikely that the two different numerical methods would agree on unconverged digits, we conclude that the Fourier-finite element algorithm with $N = 100$ mesh points and QZ eigensolver produces eigenvalues that are (in general) accurate to six decimal places. This is the best accuracy that we could hope for because the discretization matrices were constructed using a quadrature rule with a tolerance of 10^{-6}.

We briefly outline how to use Bessel functions to obtain the nonlinear equation for λ. We return to the classical form of the eigenvalue problem in polar coordinates, equations (2.9.3)–(2.9.9). Since $k = 0$, all the fluid variables are independent of ϕ and all the string variables are independent of s. The incompressibility condition $(2.9.3)_3$ implies that ru is constant. But $u(a) = 0$ by (2.9.7). Therefore $u = 0$. Substituting $u = 0$ into the Navier-Stokes equation $(2.9.3)_1$ determines p in terms of v up to a constant. The Navier-Stokes equation $(2.9.3)_2$ reduces to an equation for v:

$$\lambda v = \gamma \left(v_{rr} + \frac{v_r}{r} - \frac{v}{r^2} \right). \tag{2.13.4}$$

This has general solution

$$v(r) = jJ_1(\beta r) + yY_1(\beta r), \quad \beta^2 = -\frac{\lambda}{\gamma}, \tag{2.13.5}$$

where j and y are constants, and J_1 and Y_1 are Bessel functions. See Jahnke et al. (1960). Note that $q = 0$ by the area side condition (2.9.9). The pressure constant can then be found in terms of ψ and λ from the linear momentum equation (2.9.5):

$$\rho p(R) = -2\omega \varrho A \lambda \psi. \tag{2.13.6}$$

It follows that $\psi \neq 0$ else $v = p = 0$ and so λ is not an eigenvalue. Since $\psi \neq 0$ we can choose it to equal 1 (we are just choosing the eigenvector scaling). Substituting $\psi = 1$ into the boundary conditions (2.9.7) and (2.9.8) gives a pair of linear equations for the constants j and y in terms of β:

$$\begin{aligned} v(a) &= jJ_1(\beta a) + yY_1(\beta a) = 0, \\ v(R) &= jJ_1(\beta R) + yY_1(\beta R) = \lambda R = -\gamma\beta^2 R. \end{aligned} \tag{2.13.7}$$

Differentiating $(2.13.5)_1$ with respect to r, setting $r = R$, and using $(2.13.7)_2$ yields

$$v_r(R) = \beta[jJ_0(\beta R) + yY_0(\beta R)] + \gamma\beta^2. \tag{2.13.8}$$

We have applied the standard differentiation formula for Bessel functions: $Z_1'(z) = Z_0(z) - \frac{1}{z}Z_1(z)$, where $Z = J$ or Y. Finally, substitute $(2.13.7)_2$, (2.13.8), $u = q = 0$, $\psi = 1$, and $\lambda = -\gamma\beta^2$ into the linear momentum equation (2.9.6) to arrive at a nonlinear equation for β:

$$\gamma^2\beta^4\varrho A = -2\tilde{\mu}\gamma\beta^2 - \tilde{\mu}\beta[j(\beta)J_0(\beta R) + y(\beta)Y_0(\beta R)]. \tag{2.13.9}$$

After solving this equation numerically for β we can recover $\lambda = -\gamma\beta^2$. Notice that ω does not appear in equation (2.13.9) (neither does the constitutive function). Therefore the eigenvalues λ are independent of ω when $k = 0$.

λ (steel)		λ (rubber)	
FEM	Bessel	FEM	Bessel
-0.01701808	-0.01703582	-0.02361765	-0.02361795
-0.23705989	-0.23705961	-0.30262021	-0.30262018
-0.70478409	-0.70478465	-0.85894234	-0.85894250
-1.44478884	-1.44478838	-1.69386639	-1.69386600

Table 2.13.3: Accuracy check. The first few eigenvalues for Fourier mode $k = 0$ (the eigenvalue $\lambda = 0$ is omitted). The eigenvalues were computed using both the finite element method (FEM) and by using Bessel functions (Bessel) to obtain a nonlinear equation for λ, which was solved using the MATLAB function *fsolve*. $N = 100$ mesh points were used for the finite element method. As an initial guess for the function *fsolve* we took the value returned by the finite element method, rounded to one significant figure (except for in the last row of the steel column where, in order for *fsolve* to find the correct zero, we took initial approximation -1.4). Since for $k = 0$ the eigenvalues are independent of ω, we chose $\omega = 0$. Note that in practice the eigenvalues vary with ω due to round-off error.

Infinite eigenvalues. By introducing basis functions for the finite dimensional spaces appearing in equation (2.12.30) we obtain a matrix quadratic eigenvalue problem of the form

$$\lambda^2 C_2^k \boldsymbol{x} + \lambda C_1^k \boldsymbol{x} + C_0^k \boldsymbol{x} = 0, \tag{2.13.10}$$

where

$$C_0^k = \begin{bmatrix} A_0^k & B^k \\ B^{k*} & 0 \end{bmatrix}, \quad C_1^k = \begin{bmatrix} A_1^k & 0 \\ 0 & 0 \end{bmatrix}, \quad C_2^k = \begin{bmatrix} A_2^k & 0 \\ 0 & 0 \end{bmatrix}. \tag{2.13.11}$$

The matrices A_0^k, A_1^k, A_2^k, and B^k correspond to the bilinear forms a_0^k, a_1^k, a_2^k, and b^k. Let N be the number of mesh points, which satisfies $N = (R-a)/h$. For $k \neq 0$ the matrices A_i^k have dimensions $(4N+2) \times (4N+2)$. B^k has dimension $(4N+2) \times (N+1)$. We do not present the case $k = 0$, which is similar. The vector $\boldsymbol{x}$ has the form

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{z} \\ p_k \end{bmatrix}, \text{ where } \boldsymbol{z} = \begin{bmatrix} \mathbf{v}_k \\ \mathbf{r}_k \end{bmatrix}. \tag{2.13.12}$$

We can rewrite equation (2.13.10) in the form

$$C_2^k \boldsymbol{x} + \lambda^{-1} C_1^k \boldsymbol{x} + \lambda^{-2} C_0^k \boldsymbol{x} = 0. \tag{2.13.13}$$

Setting $\boldsymbol{z} = 0$ in (2.13.13) yields

$$\lambda^{-2} C_0^k \begin{bmatrix} 0 \\ p_k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{2.13.14}$$

But C_0^k is nonsingular. Thus equation (2.13.14) can only be satisfied if $\lambda = \infty$. We say that (2.13.10) has an infinite eigenvalue. Infinite eigenvalues occur for any polynomial eigenvalue problem of the from $\lambda^n C_n \boldsymbol{x} + \cdots + \lambda C_1 \boldsymbol{x} + C_0 \boldsymbol{x}$ provided that C_0 is nonsingular and at least one of the C_i is singular, for $i \neq 0$. (Stewart & Sun (1990) elegantly avoid the use of infinity: Instead of considering the generalized eigenvalue problem $A\boldsymbol{x} = \lambda B\boldsymbol{x}$, they consider the problem $\beta A\boldsymbol{x} = \alpha B\boldsymbol{x}$, where the eigenvalues are defined to be the pairs $[\alpha, \beta] \in \mathbb{CP}^1$. The case $\beta = 0$ corresponds to $\lambda = \infty$.)

Counting the number of infinite eigenvalues (and therefore the number of finite eigenvalues) is tricky. For example, consider the generalized eigenvalue problems $A\boldsymbol{x} = \lambda B_i\boldsymbol{x}$ with A upper triangular and

$$B_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{2.13.15}$$

In each case the eigenvalue problem $A\boldsymbol{x} = \lambda B_i\boldsymbol{x}$ has 3 infinite eigenvalues and no finite eigenvalues. Note that $\dim(\ker(B_1)) = 1$, $\dim(\ker(B_2)) = 2$, $\dim(\ker(B_3)) = 3$, so the dimension of the kernel of B_i does not determine the number of infinite eigenvalues. Moreover, if A is not upper triangular, then all we can say is that the number of infinite eigenvalues is greater than or equal to the dimension of the kernel of B_i.

Cliffe et al. (1994) consider eigenvalue problems of the form

$$C\boldsymbol{x} = \lambda D\boldsymbol{x} \tag{2.13.16}$$

with

$$C = \begin{bmatrix} A_0 & B \\ B^* & 0 \end{bmatrix}, \quad D = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} \tag{2.13.17}$$

where A_0 is nonsingular of dimension $n \times n$, A_1 is symmetric positive definite of dimension $n \times n$, and B has rank m and dimension $n \times m$. Eigenvalues problems of this form arise in the finite element discretization of Navier-Stokes equations. It is shown that eigenvalue problem (2.13.16) has $n-m$ finite eigenvalues and $2m$ infinite eigenvalues (note that C and D have dimensions $(n+m) \times (n+m)$). This result can be extended to eigenvalue problem (2.13.10). Recall that B^k has dimension $(4N+2) \times (N+1)$. Denote the QR factorization of B^k by

$$B^k = QR = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} R_1 \\ 0 \end{bmatrix}, \tag{2.13.18}$$

where Q is orthogonal, R is upper triangular, R_1 is upper triangle and nonsingular of dimension $(N+1) \times (N+1)$, Q_1 has dimension $(4N+2) \times (N+1)$, and Q_2 has dimension $(4N+2) \times (3N+1)$. Using the same methods as in Cliffe et al. (1994) we can reduce equation (2.13.10) to the $(3N+1) \times (3N+1)$ system

$$\lambda^2 Q_2^* A_2^k Q_2 + \lambda Q_2^* A_1^k Q_2 + Q_2^* A_0^k Q_2 = 0. \tag{2.13.19}$$

The eigenvalues of (2.13.19) are eigenvalues of (2.13.10), and the finite eigenvalues of (2.13.10) are eigenvalues of (2.13.19). We have eliminated $2(N+1)$ infinite eigenvalues by reducing (2.13.10) to (2.13.19). Since A_2^k is singular, however, equation (2.13.19) still has some infinite eigenvalues. The dimension of the kernel of A_2^k equals $4N$. Numerical results show that $\dim(\ker(Q_2^* A_2^k Q_2)) = 3N-1$ and that equation (2.13.19) has $3N+1$ infinite eigenvalues and so $3N+1$ finite eigenvalues ($6N+2$ eigenvalues in total). Therefore eigenvalue problem (2.13.10) has $5N+3$ infinite eigenvalues and $3N+1$ finite eigenvalues ($8N+4$ eigenvalues in total).

Why are we interested in infinite eigenvalues? It is well-known that the perturbation of infinite eigenvalues due to round-off errors can give rise to spurious eigenvalues, which are values returned by the QZ algorithm that do not satisfy the given eigenvalue problem. Moreover, spurious eigenvalues may not be large in magnitude and so can be difficult to distinguish from finite eigenvalues. We describe techniques for the stable computation of infinite eigenvalues.

Consider the generalized eigenvalue problem $A\boldsymbol{x} = \lambda B\boldsymbol{x}$, where A is non-singular and B is singular. (Note that any polynomial eigenvalue problem can be reduced to a generalized eigenvalue problem.) This problem has infinite eigenvalues (the number being greater than or equal to $\dim(\ker(B))$). A stable way to compute the infinite eigenvalues is so solve the equivalent system $B\boldsymbol{x} = \mu A\boldsymbol{x}$, where $\mu = 1/\lambda$. The infinite eigenvalues $\lambda = \infty$ are mapped to zero eigenvalues $\mu = 0$.

The reduction of eigenvalue problem (2.13.10) to (2.13.19) described above eliminates some, but not all, of the infinite eigenvalues, and so reduces the likelihood of spurious eigenvalues. This reduction technique, derived by Cliffe et al. (1994) for eigenvalue problems of the form (2.13.16), was originally intended as an analytical tool for counting the number of eigenvalues rather than as a computational tool (as we use it). For larger matrices it is not practical to perform the QR decomposition that is necessary to obtain reduced equations of the form (2.13.19). Instead Cliffe et al. (1994) introduce a three-parameter family of shifted eigenvalue problems, which allows the finite eigenvalues of (2.13.16) to be shifted and the infinite eigenvalues to be mapped to any desired location. Thus if one is only interested in the eigenvalues of largest real part, as for stability problems, then the infinite eigenvalues can be mapped far away from the origin so that their perturbations will not be mistaken for finite eigenvalues. We are not able to apply this technique to our problem (2.13.10), however, since the matrix A_2^k is singular and so an additional source of infinite eigenvalues (in addition to the infinite eigenvalues arising from the saddle-point structure of the equations).

Chapter 3

Cylindrical Motions of the Shell: The Ring Model

3.1 Introduction

In this chapter we continue our study of cylindrical motions of the deformable shell. Instead of modelling a horizontal cross section of the shell as a deformable string, which has only stretching stiffness, we model it as a deformable ring, which has stretching, bending, and shearing stiffness. We do not repeat all the steps of the previous chapter; we do not characterize the spectrum of the quadratic eigenvalue problem, prove theorems about eigenvalue crossings, design a convergent numerical scheme by proving inf-sup conditions, or compute the spectrum, but we do enough to recover the same behavior observed in Chapter 2: We find that the rigid Couette solution is unstable for all $\omega > 0$.

We study the motion of a viscous incompressible liquid in the region between a rigid circular disk of radius $a < 1$ rotating at a prescribed angular velocity ω and a viscoelastic ring whose natural state is circle of radius 1. The motion of the ring is not prescribed, but responds to the forces exerted on it by the moving liquid; the rigid disk drives the liquid, which in turn drives the deformable ring. We find a rigid Couette steady solution of this coupled system and analyze its stability with respect to the bifurcation parmeter ω.

3.2 Formulation of the Equations for the Ring

In this section we briefly specialize the planar rod theory from Antman (2005, Chapter 4) to rings. This geometrically exact theory, known as the special Cosserat theory for rods, accounts for flexure, extension, and shear.

Geometry of deformation

The reference configuration of the inner bounding curve of the ring, the part in contact with the fluid, is a circle of radius 1, given parametrically by

$$\boldsymbol{r}^\circ(s) = \boldsymbol{e}_1(s). \tag{3.2.1}$$

We refer to $\boldsymbol{r}^\circ$ as the base curve of the ring. The arc-length parameter $s \in [0, 2\pi]$ identifies material points of the ring, with the points 0 and 2π identified. The position of material point s at time t is $\boldsymbol{r}(s,t)$. The curve $\boldsymbol{r}(\cdot,t)$ is assumed to lie in the $\{\boldsymbol{i},\boldsymbol{j}\}$-plane for each t. Its parameter s need no longer be arc-length; we allow the base curve to stretch.

The ring represents a thin 2-dimensional annulus. Consider the material fiber (or cross section) of the annulus that is normal to the base curve at $\boldsymbol{r}^\circ$. This lies on the line spanned by $\boldsymbol{d}^\circ(s) := -\boldsymbol{e}_1(s)$. At time t the fiber will have deformed and may no longer be straight. We introduce a unit vector $\boldsymbol{d}(s,t)$ to characterize some average orientation of this material fiber at time t. Alternatively, we can think of the rod theory as describing thin bodies undergoing motions in which the material fibers are always straight, with the orientation given by $-\boldsymbol{d}(s,t)$. We call $\boldsymbol{d}$ the director. Define

$$\boldsymbol{a}(s,t) = \boldsymbol{d}(s,t) \times \boldsymbol{k}. \tag{3.2.2}$$

The basis $\{\boldsymbol{a}(s,t), \boldsymbol{d}(s,t)\}$, since it corresponds to material properties, is the most natural basis for our problems. (In Antman (2005, Chapter 4) this basis is denoted by $\{\boldsymbol{a}(s,t), \boldsymbol{b}(s,t)\}$.) Since this basis is orthonormal we can introduce a function $\theta(s,t)$ by

$$\begin{aligned}\boldsymbol{a}(s,t) &= \cos\theta(s,t)\boldsymbol{i} + \sin\theta(s,t)\boldsymbol{j} = \boldsymbol{e}_1(\theta(s,t)),\\ \boldsymbol{d}(s,t) &= -\sin\theta(s,t)\boldsymbol{i} + \cos\theta(s,t)\boldsymbol{j} = \boldsymbol{e}_2(\theta(s,t)).\end{aligned} \tag{3.2.3}$$

The configuration of the rod at time t is the pair $\{\boldsymbol{r}(\cdot,t), \boldsymbol{d}(\cdot,t)\}$, or equivalently $\{\boldsymbol{r}(\cdot,t), \theta(\cdot,t)\}$, a parametrized curve equipped with a unit vector at each point. The advantage of using a rod theory rather than the 2-dimensional theory of continuum mechanics is that there is only one spatial variable, s, rather than two. This is at the cost of introducing the extra unknown function $\boldsymbol{d}$.

All geometrical quantities in the reference configuration are denoted by the superscript $\circ$. We choose s so that $\theta^\circ(s) = s + \frac{\pi}{2}$.

The strains $\nu(s,t)$, $\eta(s,t)$, $\mu(s,t)$ are defined by

$$\boldsymbol{r}_s =: \nu\boldsymbol{a} + \eta\boldsymbol{d}, \qquad \theta_s =: \mu. \tag{3.2.4}$$

In the reference configuration they take values

$$\nu^\circ = 1, \quad \eta^\circ = 0, \quad \mu^\circ = 1. \tag{3.2.5}$$

Although imprecise, it is useful to think of ν and μ as measuring stretching and bending of the base curve, and η as measuring shearing of the material fibers. If there is no shearing, $\eta = 0$, then ν really does measure stretching: $\nu = |\boldsymbol{r}_s|$. If the material is not stretched or sheared, $\nu = 1$ and $\eta = 0$, then $\mu = \kappa$, the (signed) curvature of $\boldsymbol{r}(\cdot, t)$. If $\eta = 0$, then $\kappa = \mu/\nu$. The strains $\{\nu, \eta, \mu\}$ are invariant under rigid motions and determine the configuration of the ring $\{\boldsymbol{r}, \boldsymbol{d}\}$ up to a rigid motion. The set of strains could be any set of functions satisfying these two properties. The choice of $\{\nu, \eta, \mu\}$ has proved convenient for analysis.

Let h be the thickness of the 2-dimensional annulus represented by the ring. To ensure that distinct cross sections of the ring never intersect and that the local ratio of deformed to reference length of the ring be everywhere positive, we stipulate that

$$\nu(s) > \max\{-h\mu(s), 0\} \equiv \begin{cases} -h\mu(s) & \text{for} \quad \mu(s) \leq 0, \\ 0 & \text{for} \quad \mu(s) \geq 0. \end{cases} \tag{3.2.6}$$

This condition is derived from 2-dimensional continuum mechanics. See equation (3.2.14).

We require that the configuration satisfy the periodicity conditions

$$\boldsymbol{r}(2\pi, t) = \boldsymbol{r}(0, t), \qquad \theta(2\pi, t) = \theta(0, t) + 2\pi. \tag{3.2.7}$$

Sometimes it will be convenient to work in polar coordinates. We define functions $q(s, t) := |\boldsymbol{r}(s, t)|$ and $\psi(s, t) \in [0, 2\pi)$ by

$$\boldsymbol{r}(s, t) =: q(s, t)\boldsymbol{e}_1(\psi(s, t) + \omega t). \tag{3.2.8}$$

Mechanics

Let $\boldsymbol{n}(\xi, t)$ be the internal contact force exerted at time t by the material of the ring with $s \in (\xi, \xi + \varepsilon]$ on the material of the ring with $s \in [\xi - \varepsilon, \xi]$ where ε is a sufficiently small positive number and this interpretation is independent of ε. Let $\boldsymbol{f}(s, t)$ be the force per unit reference length exerted by the fluid on material point s of the ring at time t. We give an expression for the force $\boldsymbol{f}$ in Section 3.4. The ring obeys the Balance of Linear Momentum Law

$$\boldsymbol{n}_s + \boldsymbol{f} = \varrho A \boldsymbol{r}_{tt} + \varrho I \boldsymbol{d}_{tt}, \tag{3.2.9}$$

where $(\varrho A)(s)$ and $(\varrho I)(s)$ may be regarded as the mass and first moment of mass of the ring per unit reference length. These along with the linear momentum terms on the right-hand side of (3.2.9) require motivation from the theory of 2-dimensional continuum mechanics and are derived below. The left-hand side of equation (3.2.9) can be derived from a free-body diagram without relying on the 2-dimensional theory.

We introduce the resultants $N(s, t)$, $H(s, t)$ by

$$\boldsymbol{n} = N \boldsymbol{a} + H \boldsymbol{d}. \tag{3.2.10}$$

N and H may be thought of, somewhat imprecisely, as the tension and shear force in the ring. Substituting (3.2.10) into (3.2.9) and taking the dot product with $\boldsymbol{a}$ and $\boldsymbol{d}$ we obtain the componential form of the linear momentum law:

$$\begin{aligned} N_s - \mu H + \boldsymbol{f} \cdot \boldsymbol{a} &= \varrho A \boldsymbol{r}_{tt} \cdot \boldsymbol{a} + \varrho I \boldsymbol{d}_{tt} \cdot \boldsymbol{a}, \\ H_s + \mu N + \boldsymbol{f} \cdot \boldsymbol{d} &= \varrho A \boldsymbol{r}_{tt} \cdot \boldsymbol{d} + \varrho I \boldsymbol{d}_{tt} \cdot \boldsymbol{d}. \end{aligned} \tag{3.2.11}$$

Let $M(\xi, t)\,\boldsymbol{k}$ be the internal contact couple exerted at time t by the material of the ring with $s \in (\xi, \xi + \varepsilon]$ on the material of the ring with $s \in [\xi - \varepsilon, \xi]$ where ε is a sufficiently small positive number and this interpretation is independent of ε. The Balance of Angular Momentum Law for the ring is

$$M_s + \nu H - \eta N = \varrho J \theta_{tt} - \varrho I\, \boldsymbol{r}_{tt} \cdot \boldsymbol{a}, \tag{3.2.12}$$

where $(\varrho J)(s)$ may be regarded as the second moment of mass of the ring per unit reference length.

Motivation from 2-Dimensional Continuum Mechanics

We consider a 2-dimensional body whose reference configuration is an annulus of inner radius 1 and outer radius $1 + h$, which consists of all material points of the form $\boldsymbol{p}^\circ = (1 - \xi)\boldsymbol{e}_1(s)$ with $s \in [0, 2\pi]$, $-h \le \xi \le 0$. We consider motions of this annulus in which the material point with coordinates (s, ξ) is constrained so that its position at time t has the form

$$\boldsymbol{p}(s, \xi, t) = \boldsymbol{r}(s, t) + \xi\, \boldsymbol{d}(s, t). \tag{3.2.13}$$

The Jacobian of the transformation $\boldsymbol{p}$ is

$$(\boldsymbol{p}_s \times \boldsymbol{p}_\xi) \cdot \boldsymbol{k} = \{[(\nu \boldsymbol{a} + \eta \boldsymbol{d} - \xi \mu \boldsymbol{a})] \times \boldsymbol{d}\} \cdot \boldsymbol{k} = \nu - \xi\mu. \tag{3.2.14}$$

The requirement that this be positive for all ξ in $[-h, 0]$ gives (3.2.6).

The time-derivatives of the linear and angular momenta per unit of s for such an annular body of constant reference mass density ϱ undergoing a motion of the form (3.2.13) are

$$
\begin{aligned}
&\frac{d}{dt}\int_{-h}^{0} \varrho \boldsymbol{p}_t(s,\xi,t)(1-\xi)\,d\xi && (3.2.15)\\
&= \int_{-h}^{0} \varrho[\boldsymbol{r}_{tt}(s,t) + \xi \boldsymbol{d}_{tt}(s,t)](1-\xi)\,d\xi\\
&=: \varrho A \boldsymbol{r}_{tt}(s,t) + \varrho I \boldsymbol{d}_{tt}(s,t),\\
&\frac{d}{dt}\int_{-h}^{0} \varrho \boldsymbol{k}\cdot[\boldsymbol{p}(s,\xi,t)\times \boldsymbol{p}_t(s,\xi,t)](1-\xi)\,d\xi && (3.2.16)\\
&= \int_{-h}^{0} \varrho \boldsymbol{k}\cdot[\boldsymbol{p}(s,\xi,t)\times \boldsymbol{p}_{tt}(s,\xi,t)](1-\xi)\,d\xi\\
&= \int_{-h}^{0} \varrho \boldsymbol{k}\cdot\{[\boldsymbol{r}(s,t)+\xi\boldsymbol{d}(s,t)]\times[\boldsymbol{r}_{tt}(s,t)+\xi\boldsymbol{d}_{tt}(s,t)]\}(1-\xi)\,d\xi\\
&=: \varrho A \boldsymbol{k}\cdot[\boldsymbol{r}(s,t)\times\boldsymbol{r}_{tt}(s,t)] + \varrho I \boldsymbol{k}\cdot[\boldsymbol{r}(s,t)\times\boldsymbol{d}_{tt}(s,t)]\\
&\quad + \varrho I \boldsymbol{k}\cdot[\boldsymbol{d}(s,t)\times\boldsymbol{r}_{tt}(s,t)] + \varrho J \boldsymbol{k}\cdot[\boldsymbol{d}(s,t)\times\boldsymbol{d}_{tt}(s,t)]
\end{aligned}
$$

with

$$
\begin{aligned}
\varrho A &= \int_{-h}^{0} \varrho(1-\xi)\,d\xi = \varrho h(1+\tfrac{1}{2}h),\\
\varrho I &= \int_{-h}^{0} \varrho\xi(1-\xi)\,d\xi - \tfrac{1}{2}\varrho h^2(1+\tfrac{2}{3}h), && (3.2.17)\\
\varrho J &= \int_{-h}^{0} \varrho\xi^2(1-\xi)\,d\xi = \tfrac{1}{3}\varrho h^3(1+\tfrac{3}{4}h).
\end{aligned}
$$

The factor $1-\xi$ in the integrals is the Jacobian of $\boldsymbol{p}^\circ$. Note that $\boldsymbol{d}\times\boldsymbol{d}_{tt} = \theta_{tt}\boldsymbol{k}$.

Let $\boldsymbol{\tau}(s_0,\xi,t)$ denote the force per unit reference length of the material line $\{(1-\xi)\boldsymbol{e}_1(s_0)\}$ at (s_0,ξ) exerted by the material with coordinates $s\in(s_0,s_0+\varepsilon^+)$ on the material with $s\in(s_0-\varepsilon^-,s_0]$ for all small positive $\varepsilon^\pm$. Then the internal contact force exerted across this section is

$$
\boldsymbol{n}(s_0,t) = \int_{-h}^{0} \boldsymbol{\tau}(s_0,\xi,t)\,(1-\xi)\,d\xi, \qquad (3.2.18)
$$

and the resultant torque of this force about $\boldsymbol{r}(s_0, t)$ is

$$\begin{aligned} M(s_0,t)\boldsymbol{k} &= \int_{-h}^{0} [\boldsymbol{p}(s_0,\xi,t) - \boldsymbol{r}(s_0,t)] \times \boldsymbol{\tau}(s_0,\xi,t)\,(1-\xi)\,d\xi \\ &\equiv \boldsymbol{d}(s_0,t) \times \int_{-h}^{0} \xi\boldsymbol{\tau}(s_0,\xi,t)\,(1-\xi)\,d\xi. \end{aligned} \tag{3.2.19}$$

Suppose that the ring is subjected to an external force in the $\{\boldsymbol{i},\boldsymbol{j}\}$-plane acting on the base curve $\boldsymbol{r}(\cdot,t)$ with intensity $\boldsymbol{f}$ per unit of s. Then its contribution to the total force on the material segment $\{(1-\xi)\boldsymbol{e}_1(s) : s_1 \le s \le s_2,\ -h \le \xi \le 0\}$ is $\int_{s_1}^{s_2} \boldsymbol{f}(s,t)\,ds$ and its contribution to the total torque about $\boldsymbol{0}$ is $\int_{s_1}^{s_2} \boldsymbol{r}(s,t)\times\boldsymbol{f}(s,t)\,ds$. We assume that there are no other external forces acting on the ring.

The requirement that the resultant force on any segment of the ring equal the time-derivative of the linear momentum gives the integral version of (3.2.9). The requirement that the resultant torque on the segment $[s_1, s_2]$ equal the time-derivative of the angular momentum gives

$$\begin{aligned} &M(s_2,t) + \boldsymbol{k}\cdot[\boldsymbol{r}(s_2,t)\times\boldsymbol{n}(s_2,t)] - M(s_1,t) - \boldsymbol{k}\cdot[\boldsymbol{r}(s_1,t)\times\boldsymbol{n}(s_1,t)] \\ &\quad + \boldsymbol{k}\cdot\int_{s_1}^{s_2} \boldsymbol{r}(s,t)\times\boldsymbol{f}(s,t)\,ds \\ &= \int_{s_1}^{s_2} \boldsymbol{k}\cdot\{\varrho A[\boldsymbol{r}\times\boldsymbol{r}_{tt}(s,t)] + \varrho I[\boldsymbol{r}(s,t)\times\boldsymbol{d}_{tt}(s,t)] \\ &\quad + \varrho I[\boldsymbol{d}(s,t)\times\boldsymbol{r}_{tt}(s,t)] + \varrho J[\boldsymbol{d}(s,t)\times\boldsymbol{d}_{tt}(s,t)]\}\,ds. \end{aligned} \tag{3.2.20}$$

Differentiating this equation with respect to s_2 and then using (3.2.9) yields (3.2.12). Note that $\boldsymbol{f}$ makes no contribution to (3.2.12) because it is applied to the image $\boldsymbol{r}(s,t)$ of the base curve.

The Constitutive Equations

We assume that the ring is uniform in which case ϱA, ϱI and ϱJ are independent of s, and the constitutive functions are independent of s. The ring is said to be viscoelastic of strain-rate type if there are functions

$$\nu, \eta, \mu, \dot{\nu}, \dot{\eta}, \dot{\mu} \mapsto \hat{N}(\nu, \eta, \mu, \dot{\nu}, \dot{\eta}, \dot{\mu}),\ \hat{H}(\nu, \eta, \mu, \dot{\nu}, \dot{\eta}, \dot{\mu}),\ \hat{M}(\nu, \eta, \mu, \dot{\nu}, \dot{\eta}, \dot{\mu}) \quad (3.2.21)$$

such that

$$N(s,t) = \hat{N}(\nu(s,t), \eta(s,t), \mu(s,t), \nu_t(s,t), \eta_t(s,t), \mu_t(s,t)), \quad \text{etc.} \quad (3.2.22)$$

The superposed dots on the last three arguments of (3.2.21) have no operational significance; they merely identify the arguments of the constitutive functions that are to be occupied by the time derivatives of ν, η, μ. This form of the constitutive functions can be derived by starting with constitutive functions of the form $N = \hat{N}(\boldsymbol{r}, \boldsymbol{r}_s, \boldsymbol{r}_t, \boldsymbol{d}, \boldsymbol{d}_s, \boldsymbol{d}_t, t)$ and applying the Principle of Frame-Indifference. We assume that these constitutive functions are sufficiently smooth for our purposes.

We assume that the monotonicity conditions hold:

$$\text{the matrices} \quad \frac{\partial(\hat{N}, \hat{H}, \hat{M})}{\partial(\nu, \eta, \mu)} \quad \text{and} \quad \frac{\partial(\hat{N}, \hat{H}, \hat{M})}{\partial(\dot{\nu}, \dot{\eta}, \dot{\mu})} \quad \text{are positive-definite.} \quad (3.2.23)$$

(These follow from the Strong Ellipticity Condition of 2-dimensional nonlinear elasticity.) It is expected that an extreme strain be accompanied by an extreme stress.

Therefore we stipulate that the constitutive functions satisfy the growth conditions

$$\begin{aligned}
&\hat{N}(\nu,\eta,\mu,\dot{\nu},\dot{\eta},\dot{\mu}) \longrightarrow \left\{\begin{matrix} +\infty \\ \\ -\infty \end{matrix}\right\} && \text{as} \quad \nu \longrightarrow \left\{\begin{matrix} +\infty \\ \\ \max\{-\mu h, 0\} \end{matrix}\right\}, \\
&\hat{H}(\nu,\eta,\mu,\dot{\nu},\dot{\eta},\dot{\mu}) \longrightarrow \pm\infty && \text{as} \quad \eta \longrightarrow \pm\infty, \\
&\hat{M}(\nu,\eta,\mu,\dot{\nu},\dot{\eta},\dot{\mu}) \longrightarrow \pm\infty && \text{as} \quad \mu \longrightarrow \left\{\begin{matrix} \infty \\ \\ -h^{-1}\nu \end{matrix}\right\}
\end{aligned} \tag{3.2.24}$$

for fixed values of the arguments not entering the limit process. We require that the effects of shearing in one sense be the same as in the opposite sense:

$$\hat{N}(-\eta,-\dot{\eta}) = \hat{N}(\eta,\dot{\eta}), \qquad \hat{H}(-\eta,-\dot{\eta}) = -\hat{H}(\eta,\dot{\eta}), \qquad \hat{M}(-\eta,-\dot{\eta}) = \hat{M}(\eta,\dot{\eta}). \tag{3.2.25}$$

Here we have suppressed the arguments ν, μ, $\dot{\nu}$, $\dot{\mu}$ of the constitutive functions. Relations (3.2.25) imply that

$$\hat{H} = \hat{H}_\nu = \hat{H}_\mu = \hat{H}_{\dot{\nu}} = \hat{H}_{\dot{\mu}} = \hat{N}_\eta = \hat{N}_{\dot{\eta}} = \hat{M}_\eta = \hat{M}_{\dot{\eta}} = 0 \quad \text{when} \quad (\eta,\dot{\eta}) = (0,0). \tag{3.2.26}$$

Finally, we make the assumption that the reference configuration of the ring is its natural configuration, which implies that the resultants vanish when the body is at rest in the reference configuration:

$$\hat{N}(1,0,1,0,0,0) = \hat{H}(1,0,1,0,0,0) = \hat{M}(1,0,1,0,0,0) = 0. \tag{3.2.27}$$

3.3 Formulation of the Equations for the Fluid

At time t the fluid occupies the region between the disk of radius $a < 1$ and the curve $\boldsymbol{r}(\cdot,t)$. The equations for the fluid are identical to those given in Section

2.3 and so we do not repeat them here.

3.4 The Coupling Between the Fluid and the Ring Equations

We adopt the standard requirement for viscous fluids that the fluid adhere to solid surfaces with which it is in contact, which are here the disk and the ring. Thus

$$u(a,\phi,t) = 0, \quad v(a,\phi,t) = a\omega \quad \forall\, \phi, \tag{3.4.1}$$

$$\boldsymbol{v}(\boldsymbol{r}(s,t),t) = \boldsymbol{r}_t(s,t). \tag{3.4.2}$$

The outward pointing unit normal to $\boldsymbol{r}(\cdot,t)$ at $\boldsymbol{r}(s,t)$ is $\boldsymbol{r}_s \times \boldsymbol{k}/|\boldsymbol{r}_s|$. The force per unit (actual) length exerted by the ring on the fluid at $\boldsymbol{r}(s,t)$ is thus $\boldsymbol{\Sigma} \cdot (\boldsymbol{r}_s \times \boldsymbol{k})/|\boldsymbol{r}_s|$. Therefore the force per unit reference length exerted by the fluid on the ring at $\boldsymbol{r}(s,t)$ is

$$\begin{aligned}
\boldsymbol{f} &= -\boldsymbol{\Sigma} \cdot (\boldsymbol{r}_s \times \boldsymbol{k}) = \boldsymbol{\Sigma} \cdot (\boldsymbol{k} \times \boldsymbol{r}_s) = \boldsymbol{\Sigma} \cdot (-\eta \boldsymbol{a} + \nu \boldsymbol{d}) \\
&= [-\rho p \boldsymbol{I} + 2\tilde{\mu} \boldsymbol{D}(\boldsymbol{v})] \cdot (-\eta \boldsymbol{a} + \nu \boldsymbol{d}) \\
&= \rho p(\eta \boldsymbol{a} - \nu \boldsymbol{d})) \\
&\quad + \tilde{\mu} \left[2u_r \boldsymbol{e}_1 \boldsymbol{e}_1 + \left(v_r + \tfrac{1}{q} u_\phi - \tfrac{1}{q} v \right) (\boldsymbol{e}_1 \boldsymbol{e}_2 + \boldsymbol{e}_2 \boldsymbol{e}_1) + \tfrac{2}{q}(v_\phi + u) \boldsymbol{e}_2 \boldsymbol{e}_2 \right] \cdot (-\eta \boldsymbol{a} + \nu \boldsymbol{d}),
\end{aligned} \tag{3.4.3}$$

where we have used (2.3.3) and (2.3.15). The components u and v of the fluid velocity are evaluated at $(r,\phi) = (q(s,t), \psi(s,t))$, which are the polar coordinates for $\boldsymbol{r}(s,t)$ (see equation (3.2.8)), and the argument of $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ is $\psi(s,t) + \omega t$.

Since we are taking the base curve, given by $\boldsymbol{r}(\cdot,t)$, to be in contact with the fluid, the fluid exerts no body couple on the ring (see the discussion in Section 3.2).

By using the expression for $\boldsymbol{f}$ given in (3.4.3), we may rewrite the $\boldsymbol{a}$- and $\boldsymbol{d}$-components of the linear momentum equation (3.2.11) and rewrite the angular momentum equation (3.2.12) as

$$\begin{aligned}
&N_s - \mu H + \rho p \eta + \tilde{\mu}(v_r + \tfrac{1}{q}u_\phi - \tfrac{1}{q}v)[\eta \sin 2(\psi - \theta + \omega t) + \nu \cos 2(\psi - \theta + \omega t)] \\
&+ \tilde{\mu}(u_r - \tfrac{1}{q}v_\phi - \tfrac{1}{q}u)[-2\eta \cos^2(\psi - \theta + \omega t) + \nu \sin 2(\psi - \theta + \omega t)] \\
&\qquad = \varrho A \boldsymbol{r}_{tt} \cdot \boldsymbol{a} - \varrho I \theta_{tt} \\
&\qquad \equiv \varrho A\{[q_{tt} - q(\psi_t + \omega)^2] \cos(\psi - \theta + \omega t) \\
&\qquad\qquad - [q\psi_{tt} + 2q_t(\psi_t + \omega)] \sin(\psi - \theta + \omega t)\} - \varrho I \theta_{tt},
\end{aligned} \tag{3.4.4}$$

$$\begin{aligned}
&H_s + \mu N - \rho p \nu + \tilde{\mu}(v_r + \tfrac{1}{q}u_\phi - \tfrac{1}{q}v)[\nu \sin 2(\psi - \theta + \omega t) - \eta \cos 2(\psi - \theta + \omega t)] \\
&- \tilde{\mu}(u_r - \tfrac{1}{q}v_\phi - \tfrac{1}{q}u)[2\nu \sin^2(\psi - \theta + \omega t) + \eta \sin 2(\psi - \theta + \omega t)] + \tilde{\mu}\nu(u_r + \tfrac{1}{q}v_\phi + \tfrac{1}{q}u) \\
&\quad = \varrho A \boldsymbol{r}_{tt} \cdot \boldsymbol{d} - \varrho I \theta_t^2 \\
&\quad \equiv \varrho A\{[q_{tt} - q(\psi_t + \omega)^2] \sin(\psi - \theta + \omega t) + [q\psi_{tt} + 2q_t(\psi_t + \omega)] \cos(\psi - \theta + \omega t)\} \\
&\qquad - \varrho I \theta_t^2,
\end{aligned} \tag{3.4.5}$$

$$\begin{aligned}
M_s + \nu H - \eta N &= \varrho J \theta_{tt} - \varrho I \boldsymbol{r}_{tt} \cdot \boldsymbol{a} \\
&\equiv \varrho J \theta_{tt} - \varrho I\{[q_{tt} - q(\psi_t + \omega)^2] \cos(\psi - \theta + \omega t) \\
&\qquad - [q\psi_{tt} + 2q_t(\psi_t + \omega)] \sin(\psi - \theta + \omega t)\}
\end{aligned} \tag{3.4.6}$$

where the arguments of u and v are $(r, \phi, t) = (q(s,t), \psi(s,t), t)$.

3.5 The Area Side Condition

As for the string problem, we prescribe the area of the fluid to be $\pi(R^2 - a^2)$ for some $R > 1$. See Section 2.5. To ensure that the fluid occupies the entire region between the rigid disk and the ring we specify that the area enclosed by the ring equal πR^2. Therefore

$$\pi R^2 = \tfrac{1}{2}\, \boldsymbol{k} \cdot \int_0^{2\pi} \boldsymbol{r}(s,t) \times \boldsymbol{r}_s(s,t)\, ds \tag{3.5.1}$$

by Green's Theorem in the Plane.

3.6 The Couette Steady Solution

In this section we find a rigid Couette solution similar to the one found in Chapter 2. The symmetry of our problem suggests that we seek steady solutions in which the ring is circular and rotates rigidly with constant angular velocity Ω, and the fluid streamlines are concentric circles. Thus we seek solutions of the form

$$u(r,\phi,t) = 0, \qquad v(r,\phi,t) = V(r), \qquad p(r,\phi,t) = P(r), \tag{3.6.1}$$

$$\boldsymbol{r}(s,t) = R\boldsymbol{e}_1(s + \Omega t), \tag{3.6.2}$$

$$\nu = \text{const}, \qquad \eta = \text{const}, \qquad \mu = \text{const}. \tag{3.6.3}$$

Thus, in the notation introduced in equation (3.2.8),

$$q = R, \qquad \psi(s,t) = s + (\Omega - \omega)t. \tag{3.6.4}$$

The constant $R > 1$ denotes the radius of the ring.

Equations (3.2.4), (3.6.2) and (3.6.3) imply that

$$\nu = R\cos\beta, \qquad \eta = -R\sin\beta, \qquad \theta = s + \beta + \Omega t + \tfrac{\pi}{2}, \qquad \mu = 1, \tag{3.6.5}$$

where the shear angle β is a constant to be determined. Since ν must be positive, β must lie in $(-\frac{\pi}{2}, \frac{\pi}{2})$. The constancy of the strains ensure that the resultants N, H, M are constants.

The substitution of (3.6.1) into the Navier-Stokes equations (2.3.16) yields

$$P_r = \frac{V^2}{r}, \qquad V_{rr} + \frac{V_r}{r} - \frac{V}{r^2} \equiv \left[V_r + \frac{V}{r}\right]_r \equiv \left[\frac{1}{r}(rV)_r\right]_r = 0. \tag{3.6.6}$$

Thus there are constants B, C, D such that

$$V(r) = Br + \frac{C}{r}. \tag{3.6.7}$$

$$P(r) = \frac{1}{2}B^2r^2 + 2BC\ln r - \frac{C^2}{2r^2} + D. \tag{3.6.8}$$

The adherence conditions $(3.4.1)_2$, (3.4.2) imply that

$$a\omega = Ba + \frac{C}{a}, \qquad R\Omega = BR + \frac{C}{R} \iff B = \frac{R^2\Omega - a^2\omega}{R^2 - a^2}, \quad C = \frac{R^2a^2(\omega - \Omega)}{R^2 - a^2}. \tag{3.6.9}$$

We must determine the constants β, Ω, and D. Substituting (3.6.1), (3.6.2), (3.6.5), (3.6.7) and (3.6.8) into (3.4.4), (3.4.5),(3.4.6), and using the symmetry condition (3.2.25) we obtain

$$\hat{H}(R\cos\beta, R\sin\beta, 1, 0, 0, 0) - \rho P(R) R\sin\beta + \frac{2C\tilde{\mu}\cos\beta}{R} = \varrho A R \Omega^2 \sin\beta, \quad (3.6.10)$$

$$\hat{N}(R\cos\beta, R\sin\beta, 1, 0, 0, 0) - \rho P(R) R\cos\beta - \frac{2C\tilde{\mu}\sin\beta}{R} = \varrho A R \Omega^2 \cos\beta - \varrho I \Omega^2, \quad (3.6.11)$$

$$- R\hat{H}(R\cos\beta, R\sin\beta, 1, 0, 0, 0)\cos\beta + R\hat{N}(R\cos\beta, R\sin\beta, 1, 0, 0, 0)\sin\beta$$
$$= -\varrho I R \Omega^2 \sin\beta. \quad (3.6.12)$$

Multiplying equation (3.6.10) by $-R\cos\beta$ and equation (3.6.11) by $R\sin\beta$, adding the resulting equations together and then subtracting equation (3.6.12) we discover that

$$C = 0 \quad \Longrightarrow \quad \Omega = \omega, \qquad B = \omega. \quad (3.6.13)$$

Therefore the fluid and the elastic ring rotate rigidly with the same angular velocity as the rigid disk. The system behaves like a rigid body. We call this the rigid Couette solution.

By (3.6.13), formulas (3.6.7) and (3.6.8) for V and P reduce to the simple forms

$$V = \omega r, \qquad P(r) = \tfrac{1}{2}\omega^2 r^2 + D. \quad (3.6.14)$$

Substituting (3.6.13) and (3.6.14) into (3.6.10), (3.6.11), (3.6.12) yields

$$\hat{H}(R\cos\beta, R\sin\beta, 1, 0, 0, 0) - \rho(\tfrac{1}{2}\omega^2 R^2 + D) R\sin\beta = \varrho A R \omega^2 \sin\beta, \quad (3.6.15)$$

$$\hat{N}(R\cos\beta, R\sin\beta, 1, 0, 0, 0) - \rho(\tfrac{1}{2}\omega^2 R^2 + D) R\cos\beta = \varrho A R \omega^2 \cos\beta - \rho I \omega^2, \quad (3.6.16)$$

$$-\hat{H}(R\cos\beta, R\sin\beta, 1, 0, 0, 0)\cos\beta + \hat{N}(R\cos\beta, R\sin\beta, 1, 0, 0, 0)\sin\beta = -\varrho I \omega^2 \sin\beta. \quad (3.6.17)$$

Note that equation (3.6.17) is the sum of $\sin\beta$ times equation (3.6.16) minus $\cos\beta$ times equation (3.6.15). The symmetry conditions (3.2.25) imply that if (β, ω) satisfies (3.6.15)–(3.6.17), then so does $(-\beta, \omega)$. Thus the sign of the shear angle β does not depend on the sign of ω. This indicates that a nonzero β represents a shear instability induced by tension in the ring. The symmetry property (3.2.26)

$$\hat{H}|_{\eta=\dot{\eta}=0} = 0 \tag{3.6.18}$$

implies that $\beta = 0$ satisfies equations (3.6.15) and (3.6.17). Substituting $\beta = 0$ into (3.6.16) determines D:

$$D = \tfrac{1}{\rho R}[\hat{N}(R, 0, 1, 0, 0, 0) - \varrho A R \omega^2 + \varrho I \omega^2] - \tfrac{1}{2}\omega^2 R^2. \tag{3.6.19}$$

We shall not study steady solutions with $\beta \neq 0$.

3.7 Linearization

We linearize our equations of motion about the rigid Couette solution. Introduce the small parameter ε and perturbation variables, decorated with a superscript

1, in the following way:

$$
\begin{aligned}
u(r,\phi,t,\varepsilon) &= \varepsilon u^1(r,\phi,t) + \mathcal{O}(\varepsilon^2),\\
v(r,\phi,t,\varepsilon) &= \omega r + \varepsilon v^1(r,\phi,t) + \mathcal{O}(\varepsilon^2),\\
p(r,\phi,t,\varepsilon) &= \tfrac{1}{2}\omega^2 r^2 + D + \varepsilon p^1(r,\phi,t) + \mathcal{O}(\varepsilon^2),\\
q(s,t,\varepsilon) &= R + \varepsilon q^1(s,t) + \mathcal{O}(\varepsilon^2),\\
\psi(s,t,\varepsilon) &= s + \varepsilon \psi^1(s,t) + \mathcal{O}(\varepsilon^2),\\
\theta(s,t,\varepsilon) &= s + \omega t + \tfrac{\pi}{2} + \varepsilon \theta^1(s,t) + \mathcal{O}(\varepsilon^2),\\
\nu(s,t,\varepsilon) &= R + \varepsilon \nu^1(s,t) + \mathcal{O}(\varepsilon^2),\\
\eta(s,t,\varepsilon) &= \varepsilon \eta^1(s,t) + \mathcal{O}(\varepsilon^2),\\
\mu(s,t,\varepsilon) &= 1 + \varepsilon \mu^1(s,t) + \mathcal{O}(\varepsilon^2).
\end{aligned}
\tag{3.7.1}
$$

We linearize the evolution equations by substituting (3.7.1) into them, differentiating the resulting equations with respect to ε, and then setting $\varepsilon = 0$. We obtain the following perturbation equations.

The Navier-Stokes equations.

$$
\begin{aligned}
u^1_t - 2\omega v^1 &= -p^1_r + \gamma\left(\frac{1}{r}(r u^1_r)_r + \frac{u^1_{\phi\phi}}{r^2} - \frac{2}{r^2}v^1_\phi - \frac{u^1}{r^2}\right),\\
v^1_t + 2\omega u^1 &= -\frac{p^1_\phi}{r} + \gamma\left(\frac{1}{r}(r v^1_r)_r + \frac{v^2_{\phi\phi}}{r^2} + \frac{2u^1_\phi}{r^2} - \frac{v^1}{r^2}\right)\\
(r u^1)_r + v^1_\phi &= 0.
\end{aligned}
\tag{3.7.2}
$$

The angular momentum equation.

$$
\begin{aligned}
M^\circ_\nu \nu^1_s + M^\circ_\mu \mu^1_s + M^\circ_{\dot\nu} \nu^1_{st} + M^\circ_{\dot\mu} \mu^1_{st} + R H^\circ_\eta \eta^1 + R H^\circ_{\dot\eta} \eta^1_t - N^\circ \eta^1&\\
= \varrho J \theta^1_{tt} - \varrho I (R\omega^2\theta^1 - R\omega^2\psi^1 + 2\omega q^1_t + R\psi^1_{tt}),&
\end{aligned}
\tag{3.7.3}
$$

where $M_\nu^\circ := \hat{M}_\nu(R,0,1,0,0,0)$, $M_\mu^\circ := \hat{M}_\mu(R,0,1,0,0,0)$, etc.

The linear momentum equation (*a*- and *d*-components).

$$N_\nu^\circ \nu_s^1 + N_\mu^\circ \mu_s^1 + N_{\dot\nu}^\circ \nu_{st}^1 + N_{\dot\mu}^\circ \mu_{st}^1 - H_\eta^\circ \eta^1 - H_{\dot\eta}^\circ \eta_t^1 + \rho(\tfrac{1}{2}\omega^2 R^2 + D)\eta^1 - \tilde{\mu}(Rv_r^1 + u_\phi^1 - v^1)$$
$$= \varrho A(R\omega^2\theta^1 - R\omega^2\psi^1 + 2\omega q_t^1 + R\psi_{tt}^1) - \varrho I\theta_{tt}^1, \quad (3.7.4)$$

$$H_\eta^\circ \eta_s^1 + H_{\dot\eta}^\circ \eta_{st}^1 + N_\nu^\circ \nu^1 + N_\mu^\circ \mu^1 + N_{\dot\nu}^\circ \nu_t^1 + N_{\dot\mu}^\circ \mu_t^1 + N^\circ \mu^1 - 2\rho R^2\omega^2 \hat{r}^1 - \rho R p^1$$
$$- \rho(\tfrac{1}{2}\omega^2 R^2 + D)\nu^1 + 2\tilde{\mu} R u_r^1 = \varrho A(-q_{tt}^1 + \omega^2 q^1 + 2R\omega\psi_t^1) - 2\varrho I\omega\theta_t^1, \quad (3.7.5)$$

where the fluid variables are evaluated at $(r,\phi,t) = (R,s,t)$.

The adherence boundary conditions.

$$u^1(R,s,t) = q_t^1(s,t), \qquad v^1(R,s,t) = R\,\psi_t^1(s,t), \quad (3.7.6)$$

$$u^1(a,\phi,t) = 0, \qquad v^1(a,\phi,t) = 0. \quad (3.7.7)$$

The strain-configuration relations.

$$\nu^1 = q^1 + R\,\psi_s^1, \qquad \eta^1 = -q_s^1 + R\,\psi^1 - R\,\theta^1, \qquad \mu^1 = \theta_s^1. \quad (3.7.8)$$

The spatial periodicity conditions.

$$q^1(2\pi,t) = q^1(0,t), \qquad \theta^1(2\pi,t) = \theta^1(0,t). \quad (3.7.9)$$

The area side condition.

$$\int_0^{2\pi} q^1(s,t)\,ds = 0. \quad (3.7.10)$$

3.8 The Quadratic Eigenvalue Problem

We seek solutions of the linear perturbation equations with exponential time dependence, $u^1(r,\phi,t) = u(r,\phi)\exp(\lambda t)$, $v^1(r,\phi,t) = v(r,\phi)\exp(\lambda t)$, etc. Substituting these expressions into the linearized equations yields a quadratic eigenvalue problem:

The Navier-Stokes equations.

$$\begin{aligned}
\lambda u - 2\omega v &= -p_r + \gamma\left(\frac{1}{r}(ru_r)_r + \frac{u_{\phi\phi}}{r^2} - \frac{2v_\phi}{r^2} - \frac{u}{r^2}\right), \\
\lambda v + 2\omega u &= -\frac{p_\phi}{r} + \gamma\left(\frac{1}{r}(rv_r)_r + \frac{v_{\phi\phi}}{r^2} + \frac{2u_\phi}{r^2} - \frac{v}{r^2}\right), \\
(ru)_r + v_\phi &= 0.
\end{aligned} \tag{3.8.1}$$

The angular momentum equation.

$$\begin{aligned}
&(M^\circ_\nu + \lambda M^\circ_{\dot\nu})\nu_s + (M^\circ_\mu + \lambda M^\circ_{\dot\mu})\mu_s + (RH^\circ_\eta + R\lambda H^\circ_{\dot\eta} - N^\circ)\eta \\
&\qquad = (\lambda^2 \varrho J - \varrho I R\omega^2)\theta + \varrho I R(\omega^2 - \lambda^2)\psi - 2\lambda\omega\varrho I q.
\end{aligned} \tag{3.8.2}$$

The linear momentum equation ($\boldsymbol{a}$- and $\boldsymbol{d}$-components).

$$\begin{aligned}
&(N^\circ_\nu + \lambda N^\circ_{\dot\nu})\nu_s + (N^\circ_\mu + \lambda N^\circ_{\dot\mu})\mu_s + (\rho(\tfrac{1}{2}\omega^2 R^2 + D) - H^\circ_\eta - \lambda H^\circ_{\dot\eta})\eta - \tilde\mu(Rv_r + u_\phi - v) \\
&\qquad = (\varrho A R\omega^2 - \varrho I\lambda^2)\theta + 2\varrho A\lambda\omega q + \varrho A R(\lambda^2 - \omega^2)\psi,
\end{aligned} \tag{3.8.3}$$

$$\begin{aligned}
&(H^\circ_\eta + \lambda H^\circ_{\dot\eta})\eta_s + (N^\circ_\nu + \lambda N^\circ_{\dot\nu} - \rho(\tfrac{1}{2}\omega^2 R^2 + D))\nu + (N^\circ_\mu + \lambda N^\circ_{\dot\mu} + N^\circ)\mu - \rho R p + 2\tilde\mu R u_r \\
&\qquad = (\varrho A\omega^2 - \varrho A\lambda^2 + 2\rho R^2\omega^2)q - 2\varrho I\lambda\omega\theta + 2\varrho A\lambda R\omega\psi.
\end{aligned} \tag{3.8.4}$$

The adherence boundary conditions.

$$u(R,s) = \lambda q(s), \qquad v(R,s) = \lambda R\psi(s). \tag{3.8.5}$$

The strain-configuration relations.

$$\nu = q + R\,\psi_s, \qquad \eta = -q_s + R\,\psi - R\,\theta^1, \qquad \mu = \theta_s. \tag{3.8.6}$$

The area side condition.

$$\int_0^{2\pi} q(s)\;ds = 0. \tag{3.8.7}$$

3.9 Analysis of the Spectrum

In Chapter 2 we found that the rigid Couette solution for the string problem is unstable for all $\omega > 0$. Is the same true for the ring problem?

Motivated by the results of Chapter 2 we seek solutions of the quadratic eigenvalue problem with

$$\lambda = \omega = u = v = p = 0. \tag{3.9.1}$$

Since the unstable mode found in Chapter 2 was a rigid translation of the Couette solution, we also set the strain perturbations equal to zero:

$$\nu = \eta = \mu = 0. \tag{3.9.2}$$

This leaves only three unknowns, $q(s)$, $\psi(s)$, and $\theta(s)$. Equation (3.8.6) implies that θ is constant. We take $\theta = 0$. Substituting $\theta = 0$ and (3.9.1) and (3.9.2) into the quadratic eigenvalue problem (3.8.1)–(3.8.6) yields

$$q + R\psi_s = 0, \qquad -q_s + R\psi = 0, \tag{3.9.3}$$

which has nontrivial solutions $\psi(s) = A\sin s + B\cos s$, $q(s) = -RA\cos s + RB\sin s$, where A and B are constants. Therefore when $\omega = 0$, $\lambda = 0$ is an eigenvalue of

(3.8.1)–(3.8.7) of geometric multiplicity two with eigenvectors

$$(u, v, p, q, \psi, \theta, \nu, \eta, \mu) = (0, 0, 0, -RA\cos s + RB\sin s, A\sin s + B\cos s, 0, 0, 0). \tag{3.9.4}$$

Since these eigenvectors correspond to a rigid motion of the Couette steady solution (because the strain perturbations are zero) and have Fourier mode $|k| = 1$, like the unstable modes found in Chapter 2, we expect that the corresponding eigenvalue $\lambda = 0$ will move into the right half-plane as ω is increased from 0, which would imply that the rigid Couette solution is unstable for all $\omega > 0$. This could be checked numerically. We do not stop to do so, however, but move on to a brief look at a 2-dimensional model for ring before arriving at the most important chapter of the thesis, concerning axisymmetric motions of the shell, in which the Couette solution cannot suffer the instabilities found here.

Chapter 4

Cylindrical Motions of the Shell: The 2-Dimensional Elasticity Model

4.1 Introduction

In Chapters 2 and 3 we studied cylindrical motions of the deformable shell, using string and rod theories to model a horizontal cross section of the shell, and found that the rigid Couette solution is unstable for all $\omega > 0$. In this chapter we show that the sting and rod theories were sufficient to capture the physics of the problem; we model a cross section of the shell using the full theory of 2-dimensional nonlinear elasticity and show that, once again, the rigid Couette solution is unstable for all $\omega > 0$.

We study the motion of a viscous incompressible liquid in the region between a rigid circular disk of radius $a < 1$ rotating at a prescribed angular velocity ω and a 2-dimensional elastic body whose natural state is an annulus of inner radius 1 and outer radius $1 + h$ (for our analysis h need not be small). The motion of the elastic body is not prescribed, but responds to the forces exerted on it by the moving liquid; the rigid disk drives the liquid, which in turn drives the deformable body. We find a rigid Couette steady solution of this coupled system and analyze its stability with respect to the bifurcation parmeter ω.

4.2 Formulation of the Equations for the 2-Dimensional Elastic Body

We summarize the theory of 2-dimensional nonlinear elasticity from Antman (2005, Chapters 12 & 13).

Geometry of Deformation

Let the reference configuration of the deformable body be an annulus of inner radius 1 and outer radius $1 + h$, $h > 0$, which consists of material points of the form $\boldsymbol{x} = (1 - \xi)\boldsymbol{e}_1(s)$ for $(s, \xi) \in [0, 2\pi) \times [-h, 0]$. (These coordinates agree with those introduced in Section 3.2 to derive the ring equations from 2-dimensional continuum mechanics.) The position of material point $\boldsymbol{x}$ at time t is denoted by $\boldsymbol{p}(x, t)$. Let $\boldsymbol{F} = \partial\boldsymbol{p}/\partial\boldsymbol{x}$ be the deformation gradient and $\boldsymbol{C} = \boldsymbol{F}^* \cdot \boldsymbol{F}$ be the Cauchy-Green deformation tensor. We ask that the deformation $\boldsymbol{p}$ be orientation preserving: $\det\boldsymbol{F} > 0$.

Mechanics

Let $\varrho(\boldsymbol{x})$ denote the mass density of the deformable body and $\boldsymbol{T}(\boldsymbol{x}, t)$ denote the first Piola-Kirchhoff stress tensor. The vector $\boldsymbol{T} \cdot \boldsymbol{n}$ is the internal contact force exerted across a material surface with unit outer normal $\boldsymbol{n}$ in the reference configuration. The Linear Momentum Law for the deformable body states that

$$\operatorname{div} \boldsymbol{T}^* = \varrho\, \boldsymbol{p}_{tt}. \tag{4.2.1}$$

Observe that, unlike in Chapters 2 and 3, there is no body force term $\boldsymbol{f}$ for the force of the fluid on the deformable body. The fluid only exerts a force on the boundary of the deformable body. We derive these boundary conditions in Section 4.4.

The Angular Momentum Law for the deformable body implies that $\boldsymbol{T} \cdot \boldsymbol{F}^{*}$ is symmetric. It follows that the second Piola-Kirchhoff stress tensor $\boldsymbol{S} := \boldsymbol{F}^{-1} \cdot \boldsymbol{T}$ is also symmetric.

Constitutive Equations

We assume that the material is uniform and elastic so that ϱ is constant and there exists a function $\boldsymbol{F} \mapsto \hat{\boldsymbol{T}}(\boldsymbol{F})$ such that $\boldsymbol{T}(\boldsymbol{x}, t) = \hat{\boldsymbol{T}}(\boldsymbol{F}(\boldsymbol{x}, t))$. The Principle of Frame-Indifference implies that there exists a function $\boldsymbol{C} \mapsto \hat{\boldsymbol{S}}(\boldsymbol{C})$ such that $\boldsymbol{S}(\boldsymbol{x}, t) = \hat{\boldsymbol{S}}(\boldsymbol{C}(\boldsymbol{x}, t))$. We take the reference configuration of the deformable body to be natural so that stresses $\hat{\boldsymbol{T}}$ and $\hat{\boldsymbol{S}}$ vanish in the reference configuration. We let $\hat{\boldsymbol{T}}$ and $\hat{\boldsymbol{S}}$ be as smooth as necessary for our analysis to hold.

We also assume that the material is isotropic. Let

$$\iota(\boldsymbol{C}) = (\operatorname{tr} \boldsymbol{C}, \tfrac{1}{2}((\operatorname{tr} \boldsymbol{C})^{2} - \operatorname{tr}(\boldsymbol{C}^{2})), \det \boldsymbol{C}) \tag{4.2.2}$$

denote the principal invariants of $\boldsymbol{C}$. By the Representation Theorem for Isotropic Materials there exists scalar-valued functions ϕ_0, ϕ_1, and ϕ_2 of $\iota(\boldsymbol{C})$ such that

$$\hat{\boldsymbol{S}}(\boldsymbol{C}) = \phi_0(\iota(\boldsymbol{C}))\boldsymbol{I} + \phi_1(\iota(\boldsymbol{C}))\,\boldsymbol{C} + \phi_2(\iota(\boldsymbol{C}))\,\boldsymbol{C}^{2}. \tag{4.2.3}$$

Finally, we assume that the function $\hat{\boldsymbol{T}}$ satisfies the Strong Ellipticity Condition of nonlinear elasticity. Strong Ellipticity corresponds to rank-one convexity of the stored energy potential for hyperelastic materials. See Antman (2005, Chapter 13) for more details. We will use the following consequence of the Strong Ellipticity

Condition:

$$\text{The map } \boldsymbol{a} \cdot \boldsymbol{F} \cdot \boldsymbol{b} \mapsto \boldsymbol{a} \cdot \hat{\boldsymbol{T}}(\boldsymbol{F}) \cdot \boldsymbol{b} \text{ is strictly increasing} \tag{4.2.4}$$

for all $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^2$ and $\boldsymbol{F}$ with $\det \boldsymbol{F} > 0$. This means that the $\boldsymbol{ab}$-component of the first Piola-Kirchhoff stress tensor is an increasing function of the corresponding component of $\boldsymbol{F}$.

4.3 Formulation of the Equations for the Fluid

We assume that the fluid is viscous, incompressible, and Newtonian so that it is governed by the Navier-Stokes equations. These equations were given in Chapters 2 and 3. We use the same notation here.

4.4 The Coupling Between the Fluid and the Elasticity Equations

We require that the fluid adhere to any solid surfaces with which it is in contact, which are here the elastic body and the rigid disk:

$$\boldsymbol{v}(\boldsymbol{p}(\boldsymbol{x}, t), t) = \boldsymbol{p}_t(\boldsymbol{x}, t), \tag{4.4.1}$$

$$\boldsymbol{v}(a\boldsymbol{e}_1(\phi + \omega t), t) = a\omega \boldsymbol{e}_2(\phi + \omega t). \tag{4.4.2}$$

The inner boundary of the elastic body has outer normal $-\boldsymbol{e}_1(s)$ in the reference configuration and $-\partial_s \boldsymbol{p}(\boldsymbol{e}_1(s), t) \times \boldsymbol{k}$ at time t. We assume that the force exerted by the elastic body on the fluid equals minus the force exerted by the fluid on the elastic body (per reference length of the inner boundary of the elastic body):

$$\boldsymbol{\Sigma} \cdot (\partial_s \boldsymbol{p}(\boldsymbol{e}_1(s), t) \times \boldsymbol{k}) = \boldsymbol{T} \cdot \boldsymbol{e}_1(s), \tag{4.4.3}$$

where $\boldsymbol{\Sigma}$ has arguments $(\boldsymbol{v}, p) = (\boldsymbol{v}(\boldsymbol{p}(\boldsymbol{e}_1(s), t), t), p(\boldsymbol{p}(\boldsymbol{e}_1(s), t), t))$ and $\boldsymbol{T}$ has arguments $(\boldsymbol{x}, t) = (\boldsymbol{e}_1(s), t)$. We also assume that there is no force on the outer boundary of the elastic body:

$$\boldsymbol{T}((1+h)\boldsymbol{e}_1(s), t) \cdot \boldsymbol{e}_1(s) = 0. \tag{4.4.4}$$

4.5 The Area Side Condition

As in Chapters 2 and 3 we prescribe the area of the fluid to be $\pi(R^2 - a^2)$, for some choice of $R > 1$. We enforce that the deformable body enclose an area of πR^2 at all times so that the fluid occupies the whole region between the disk and the deformable body.

4.6 The Couette Steady Solution

Motivated by Chapters 2 and 3 we seek a rigid Couette steady solution of the form

$$u = w = 0, \quad v(r) = \omega r, \quad p(r) = \tfrac{1}{2}\omega^2 r^2 + D, \tag{4.6.1}$$

$$\boldsymbol{p}(\boldsymbol{x}, t) = Q(\xi)\boldsymbol{e}_1(s + \omega t), \tag{4.6.2}$$

where $\boldsymbol{x} = (1 - \xi)\boldsymbol{e}_1(s)$. The area side condition implies that $Q(0) = R$. The constant D can be found by substituting (4.6.1) and (4.6.2) into the boundary condition (4.4.3).

It is not possible to write down a formula for $Q(\xi)$. Instead we will derive a nonlinear two-point boundary-value problem for Q and prove that it has a solution.

We introduce the shorthand notation $\boldsymbol{e}_j := \boldsymbol{e}_j(s+\omega t)$ and $\boldsymbol{e}_j^\circ := \boldsymbol{e}_j(s)$ for $j \in \{1,2\}$.

From equation (4.6.2) we can compute

$$\boldsymbol{F} = -Q_\xi \boldsymbol{e}_1 \boldsymbol{e}_1^\circ + \frac{Q}{1-\xi} \boldsymbol{e}_2 \boldsymbol{e}_2^\circ, \tag{4.6.3}$$

$$\boldsymbol{C} = Q_\xi^2 \boldsymbol{e}_1^\circ \boldsymbol{e}_1^\circ + \left(\frac{Q}{1-\xi}\right)^2 \boldsymbol{e}_2^\circ \boldsymbol{e}_2^\circ, \tag{4.6.4}$$

$$\varrho \boldsymbol{p}_{tt} = -\omega^2 Q \boldsymbol{e}_1. \tag{4.6.5}$$

Decompose the Piola-Kirchhoff stress tensor with respect to the basis $\{\boldsymbol{e}_i \boldsymbol{e}_j^\circ\}_{i,j\in\{1,2\}}$: $\hat{\boldsymbol{T}} = \hat{T}_{ij} \boldsymbol{e}_i \boldsymbol{e}_j^\circ$ (using summation convention). Then

$$\begin{aligned}
\operatorname{div} \boldsymbol{T}^* &= \left(-\boldsymbol{e}_1^\circ \frac{\partial}{\partial \xi} + \frac{1}{1-\xi} \boldsymbol{e}_2^\circ \frac{\partial}{\partial s}\right) \cdot (\hat{T}_{ij} \boldsymbol{e}_j^\circ \boldsymbol{e}_i) \\
&= \left[-\frac{\partial \hat{T}_{11}}{\partial \xi} + \frac{1}{1-\xi}\left(\frac{\partial \hat{T}_{12}}{\partial s} + \hat{T}_{11} - \hat{T}_{22}\right)\right] \boldsymbol{e}_1 \\
&+ \left[-\frac{\partial \hat{T}_{21}}{\partial \xi} + \frac{1}{1-\xi}\left(\frac{\partial \hat{T}_{22}}{\partial s} + \hat{T}_{21} + \hat{T}_{12}\right)\right] \boldsymbol{e}_2.
\end{aligned} \tag{4.6.6}$$

Substituting (4.6.5) and (4.6.6) into the linear momentum equation (4.2.1) yields

$$-\frac{\partial \hat{T}_{11}}{\partial \xi} + \frac{1}{1-\xi}\left(\frac{\partial \hat{T}_{12}}{\partial s} + \hat{T}_{11} - \hat{T}_{22}\right) = -\omega^2 Q \tag{4.6.7}$$

$$-\frac{\partial \hat{T}_{21}}{\partial \xi} + \frac{1}{1-\xi}\left(\frac{\partial \hat{T}_{22}}{\partial s} + \hat{T}_{21} + \hat{T}_{12}\right) = 0. \tag{4.6.8}$$

We use the isotropy constitutive assumption to simplify these equations. By substituting the expression for $\boldsymbol{C}$ from (4.6.4) into the representation for $\hat{\boldsymbol{S}}$, equation (4.2.3), we see that $\hat{\boldsymbol{S}}$ has no $\boldsymbol{e}_1^\circ \boldsymbol{e}_2^\circ$- or $\boldsymbol{e}_2^\circ \boldsymbol{e}_1^\circ$-component. Then

$$\begin{aligned}
\hat{\boldsymbol{T}} &= \boldsymbol{F} \cdot \hat{\boldsymbol{S}} \\
&= \left(-Q_\xi \boldsymbol{e}_1 \boldsymbol{e}_1^\circ + \frac{Q}{1-\xi} \boldsymbol{e}_2 \boldsymbol{e}_2^\circ\right) \cdot (\hat{S}_{11} \boldsymbol{e}_1^\circ \boldsymbol{e}_1^\circ + \hat{S}_{22} \boldsymbol{e}_2^\circ \boldsymbol{e}_2^\circ),
\end{aligned} \tag{4.6.9}$$

and we can read off that

$$\hat{T}_{11} = -Q_\xi \hat{S}_{11}, \quad \hat{T}_{22} = \frac{Q}{1-\xi} \hat{S}_{22}, \quad \hat{T}_{12} = \hat{T}_{21} = 0. \tag{4.6.10}$$

Observe that $\hat{S}_{22} = \hat{S}_{22}(\boldsymbol{C}) \equiv \hat{S}_{22}(C_{11}, C_{22}) = \hat{S}_{22}(Q_\xi^2, \frac{Q^2}{(1-\xi)^2})$. Therefore $\partial \hat{S}_{22}/\partial s = 0$ and so $\partial \hat{T}_{22}/\partial s = 0$ by (4.6.10). Substituting $\hat{T}_{12} = \hat{T}_{21} = \partial \hat{T}_{22}/\partial s = 0$ into the linear momentum equations (4.6.7) and (4.6.8) gives

$$-[(1-\xi)\hat{T}_{11}]_\xi - \hat{T}_{22} = -(1-\xi)\omega^2 Q \quad \text{for } -h < \xi < 0. \tag{4.6.11}$$

This differential equation for $Q(\xi)$ is supplemented with boundary conditions

$$\begin{aligned} \hat{T}_{11} = 0 \quad &\text{for } \xi = -h, \\ Q = R \quad &\text{for } \xi = 0. \end{aligned} \tag{4.6.12}$$

where the first boundary condition is found by specializing equation (4.4.4). We wish to prove that the quasilinear two-point boundary-value problem (4.6.11), (4.6.12) is well-posed. By the monotonicity constitutive assumption (4.2.4) we see that $\hat{T}_{11}$ is a strictly increasing function of its first argument, which is the position occupied by $-Q_\xi$ (to see this replace $\boldsymbol{a}$ and $\boldsymbol{b}$ by $\boldsymbol{e}_1$ and $\boldsymbol{e}_1^\circ$ in equation (4.2.4)). If we assume certain growth rates on $\hat{T}_{11}$, then it may be possible to apply standard methods in PDEs, such as the direct method of the calculus of variations or the monotonicity method of Browder and Minty, to show that the boundary-value problem (4.6.11), (4.6.12) has a unique weak solution. We use another approach, the Poincaré shooting method, which yields classical solutions and does not require growth rates on $\hat{T}_{11}$, but which gives well-posedness only for ω small and R close to 1.

Since the function $(q', q) \mapsto \hat{T}_{11}(q', q)$ is strictly increasing in q' we can define an inverse function $(T_{11}, q) \mapsto \hat{q}'(T_{11}, q)$ by

$$\hat{q}'(T_{11}, q) = q' \iff \hat{T}_{11}(q', q) = T_{11}. \tag{4.6.13}$$

We can rewrite the second order equation for $Q(\xi)$, equation (4.6.11), as a first order system for $Q(\xi)$ and the new variable $T_{11}(\xi)$:

$$\begin{aligned} -[(1-\xi)T_{11}]_\xi &= \hat{T}_{22} - (1-\xi)\omega^2 Q, \\ Q_\xi &= -\hat{q}'(T_{11}, \tfrac{Q}{1-\xi}), \end{aligned} \tag{4.6.14}$$

for $-h < \xi < 0$, where $\hat{T}_{22} = \hat{T}_{22}(\hat{q}'(T_{11}, \frac{Q}{1-\xi}), \frac{Q}{1-\xi})$. We equip (4.6.14) with the boundary conditions

$$\begin{aligned} T_{11} &= 0 \quad \text{for } \xi = -h, \\ Q &= R \quad \text{for } \xi = 0. \end{aligned} \tag{4.6.15}$$

Systems $\{(4.6.11), (4.6.12)\}$ and $\{(4.6.14), (4.6.15)\}$ are equivalent.

Let $\alpha \in \mathbb{R}$. Consider the initial-value problem obtained by supplementing equation (4.6.14) with the initial conditions

$$\begin{aligned} T_{11}(-h) &= 0, \\ Q(-h) &= 1 + h + \alpha. \end{aligned} \tag{4.6.16}$$

For $\omega = \alpha = 0$ equations (4.6.14), (4.6.16) have a unique solution $Q(\xi) = 1 - \xi$, $T_{11} = \hat{T}_{11}(-Q_\xi, \frac{Q}{1-\xi}) = \hat{T}_{11}(1,1) = 0$ (since the stress variables vanish in the reference configuration). If $R = 1$, then Q and T_{11} also satisfy the boundary-value problem (4.6.14) and (4.6.15). By standard methods in ordinary differential equations, for $|\alpha|$ and $|\omega|$ small, there exists unique functions $Q(\xi;\omega,\alpha)$ and $T_{11}(\xi;\omega,\alpha)$ satisfying (4.6.14) and (4.6.16). We wish to choose α so that $Q(0;\omega,\alpha) = R$. We have reduced the differential equation (4.6.11), (4.6.12) to the algebraic equation

$$F(R,\omega,\alpha) \equiv Q(0;\omega,\alpha) - R = 0. \tag{4.6.17}$$

We know that this equation has a solution $(R,\omega,\alpha) = (1,0,0)$. If we can prove that $F_\alpha(1,0,0) \neq 0$, then by the Implicit Function Theorem there exists a neighborhood of $(R,\omega) = (1,0)$ and a function $(R,\omega) \mapsto \hat{\alpha}(R,\omega)$ such that $F(R,\omega,\hat{\alpha}(R,\omega)) = 0$ and $\hat{\alpha}(1,0) = \omega$. Thus $Q(\xi;\omega,\hat{\alpha}(R,\omega))$ and $T_{11}(\xi;\omega,\hat{\alpha}(R,\omega))$ satisfy (4.6.11) and (4.6.12).

Now we prove that $0 \neq F_\alpha(1,0,0) \equiv Q_\alpha(0;0,0)$. For convenience we introduce the notation $\hat{T}^{11} := \hat{T}_{11}$ and $\hat{T}^{22} := \hat{T}_{22}$. Differentiate (4.6.11) and $(4.6.16)_2$ with respect to α and then set $(\omega,\alpha) = (0,0)$ to find that $Q_\alpha(\xi;0,0)$ satisfies the linear elliptic equation

$$-(1-\xi)\hat{T}^{11}_{q'}(1,1)\,\partial_{\xi\xi}Q_\alpha + [\hat{T}^{11}_{q}(1,1) + \hat{T}^{11}_{q'}(1,1) - \hat{T}^{22}_{q'}(1,1)]\partial_\xi Q_\alpha + \frac{\hat{T}^{22}_{q}(1,1)}{1-\xi}Q_\alpha = 0 \quad (4.6.18)$$

and the boundary condition

$$Q_\alpha(-h;0,0) = 1. \quad (4.6.19)$$

The coefficients $\hat{T}^{11}_{q'}(1,1)$ and $\hat{T}^{22}_{q}(1,1)$ are positive by the monotonicity condition (4.2.4). Therefore the weak maximum principle and the boundary condition (4.6.19) imply that

$$\begin{aligned}\max_{\xi\in[-h,0]} |Q_\alpha(\xi;0,0)| &= \max\{|Q_\alpha(-h;0,0)|, |Q_\alpha(0;0,0)|\} \\ &= \max\{1, |Q_\alpha(0;0,0)|\}.\end{aligned} \quad (4.6.20)$$

We find a condition under which $\partial_\xi Q_\alpha(-h;0,0) > 0$. In this case Q_α is increasing at the end point $\xi = -h$ and so the maximum value of $|Q_\alpha(\xi;0,0)|$ is greater than 1. Therefore $|Q_\alpha(0;0,0)|$ is greater than 1 by (4.6.20), and $F_\alpha(1,0,0) = Q_\alpha(0;0,0) \neq 0$, as desired.

To find out when $\partial_\xi Q_\alpha(-h;0,0) > 0$, differentiate $(4.6.16)_1$ with respect to α

and set $\omega = \alpha = 0$ to obtain

$$
\begin{aligned}
&-\hat{T}^{11}_{q'}(1,1)\partial_\xi Q_\alpha(-h;0,0) + \hat{T}^{11}_{q}(1,1)\frac{Q_\alpha(-h;0,0)}{1+h} = 0 \\
&\qquad\qquad \iff \hat{T}^{11}_{q'}(1,1)\partial_\xi Q_\alpha(-h;0,0) = \hat{T}^{11}_{q}(1,1)\frac{1}{1+h}. \quad (4.6.21)
\end{aligned}
$$

But $\hat{T}^{11}_{q'}(1,1) > 0$ and so $\partial_\xi Q_\alpha(-h;0,0) > 0$ provided that $\hat{T}^{11}_{q}(1,1) > 0$. We have proved that under the constitutive assumption $\hat{T}^{11}_{q}(1,1) > 0$, the boundary-value problem (4.6.11), (4.6.12) has a classical solution for R close to 1 and ω close to 0. (The regularity of this solution depends on the regularity of $\hat{T}_{11}$ and $\hat{T}_{22}$.) Therefore there exists a rigid Couette solution of the form (4.6.1), (4.6.2) for R close to 1 and ω close to 0.

4.7 Analysis of the Spectrum

In this section we analyze the stability of the rigid Couette steady solution (4.6.1), (4.6.2) with respect to ω. The important observation is that when $\omega = 0$ there is a branch of steady solutions bifurcating (or branching) from the rigid Couette steady solution: When $\omega = 0$ the functions

$$
\boldsymbol{v} = 0, \quad \boldsymbol{p}(\boldsymbol{x}, t) = Q(\xi)\boldsymbol{e}_1(s) + \boldsymbol{d}, \quad p = D \qquad (4.7.1)
$$

satisfy the coupled fluid-solid equations for all constant vectors $\boldsymbol{d}$ with $|\boldsymbol{d}| < R - a$. It follows from a standard theorem in bifurcation theory that $\omega = 0$ is an eigenvalue of the equations that are obtained by linearizing the steady state fluid-solid equations about the Couette steady solution. See Antman (2005, Chapter 5, Theorem 4.1). Therefore, when $\omega = 0$, $\lambda = 0$ is an eigenvalue of the equations that are obtained by linearizing the time dependent fluid-solid equations about the Couette steady solution. While we cannot prove it analytically, it is expected that this eigenvalue will move into the right half-plane as ω is increased from 0, and so the rigid Couette steady solution will be unstable for all $\omega > 0$. This is the same behavior that we observed in Chapters 2 and 3. This shows that the reduced string and ring models are sufficient to capture the important physics of the problem.

Chapter 5

Axisymmetric Motions of the Shell

5.1 Introduction

In Chapters 2–4 we considered cylindrical motions of the deformable shell. We found that the rigid Couette solution is unstable for all $\omega > 0$ via a drift instability. This unstable mode can be stabilized by fixing the center of mass of the deformable cylinder at the origin. Mathematically, this can be achieved by seeking axisymmetric motions of the deformable cylinder, which is what we do in this chapter. Axisymmetric motions were the starting point for studying the classical Taylor-Couette problem.

We study the motion of a viscous incompressible liquid in the region between a rigid circular cylinder of radius $a < 1$ rotating at a prescribed angular velocity ω and a viscoelastic shell whose natural state is a circular cylinder of radius 1. Both cylinders have infinite length. We limit our attention to motions in which the fluid velocity and the shell are axisymmetric, and the meridian curves of the shell rotate at angular velocity ω. Other than this, the motion of the shell is not prescribed, but responds to the forces exerted on it by the moving liquid; the inner cylinder drives the liquid, which in turn drives the deformable shell.

We find a rigid Couette steady solution of this coupled system, similar to the steady solutions found in Chapters 2–4, and analyze its stability with respect to the bifurcation parameter ω and perturbations that are periodic in the axial direction. We discover new phenomena not observed in Chapters 2–4 or in the classical Taylor-Couette problem.

5.2 Formulation of the Equations for the Shell

In this section we summarize the theory of deformable axisymmetric shells from Antman (2005, Chapter 10, Section 1; Chapter 17, Section 4). These shells can suffer flexure, base surface extension, and shear.

Geometry of Deformation

The reference configuration of the inner bounding surface of the deformable shell, the part in contact with the fluid, is a circular cylinder of radius of 1, given parametrically by

$$\boldsymbol{r}^{\circ}(s,\phi) = \boldsymbol{e}_1(\phi) + s\boldsymbol{k}, \tag{5.2.1}$$

where $(s,\phi) \in (-\infty,\infty) \times [0,2\pi)$ identify material points of the shell. The position of material point (s,ϕ) at time t is $\boldsymbol{r}(s,\phi,t)$. We assume that the surface $\boldsymbol{r}(\cdot,\cdot,t)$ is axisymmetric and of the form

$$\boldsymbol{r}(s,\phi,t) = q(s,t)\boldsymbol{e}_1(\phi+\omega t) + \zeta(s,t)\boldsymbol{k}. \tag{5.2.2}$$

Unlike the string problem (Chapter 2), where the motion of the string in not prescribed, here we partially prescribe the motion of the shell; we specify that it remains axisymmetric and rotates with angular velocity ω.

The configuration of the shell at time t is the pair $\{\boldsymbol{r}(\cdot,\cdot,t), \boldsymbol{d}(\cdot,\cdot,t)\}$, where the unit vector $\boldsymbol{d}(s,\phi,t)$ characterizes the deformed configuration of the material fiber whose reference configuration is on the normal to the base surface $\boldsymbol{r}^\circ(s,\phi)$. (The reference value of $\boldsymbol{d}$ is therefore $\boldsymbol{d}^\circ = \boldsymbol{r}_s^\circ \times \boldsymbol{r}_\phi^\circ = -\boldsymbol{e}_1$.) We define

$$\boldsymbol{a}(s,\phi,t) = \boldsymbol{e}_2(\phi+\omega t) \times \boldsymbol{d}(s,\phi,t). \tag{5.2.3}$$

We assume that $\boldsymbol{d}(s,\phi,t)$ is confined to the plane spanned by $\{\boldsymbol{e}_1(\phi+\omega t), \boldsymbol{k}\}$ so that the shell has $O(2)$-symmetry. Thus we can define a function $\theta(s,t)$ by

$$\begin{aligned} \boldsymbol{a}(s,\phi,t) &= \cos\theta(s,t)\boldsymbol{e}_1(\phi+\omega t) + \sin\theta(s,t)\boldsymbol{k}, \\ \boldsymbol{d}(s,\phi,t) &= -\sin\theta(s,t)\boldsymbol{e}_1(\phi+\omega t) + \cos\theta(s,t)\boldsymbol{k}. \end{aligned} \tag{5.2.4}$$

We have limited ourselves to motions in which there is no shearing in the $\boldsymbol{e}_2$-direction, the direction of rotation, due to the unavailability at the time of writing of a rotationally symmetric shell theory (for shells possessing $SO(2)$-symmetry). Antman and Bourne (in preparation) are currently developing such a theory.

We introduce strains $\mathbf{q} = (\tau, \nu, \eta, \sigma, \mu)$ by

$$\boldsymbol{r}_s(s,\phi) =: \nu(s)\boldsymbol{a}(s,\phi) + \eta(s)\boldsymbol{d}(s,\phi), \tag{5.2.5}$$

$$\tau := q, \quad \sigma := \sin\theta, \quad \mu := \theta_s. \tag{5.2.6}$$

The geometric interpretations of these strains are given in Antman (2005, Chapter 10, Section 1). Roughly speaking, ν and τ measure stretching, η measures shearing, and σ and μ measure bending. (The definitions of ν, η and μ given here are not to be confused with those given in earlier chapters for the string and the ring.) The set of strains $\{\tau, \nu, \eta, \sigma, \mu\}$ is sufficient to determine the configuration of the shell

$\{\boldsymbol{r}, \boldsymbol{d}\}$ up to a rigid motion. In the reference configuration the strains equal

$$(\tau^\circ, \nu^\circ, \eta^\circ, \sigma^\circ, \mu^\circ) = (1, 1, 0, 1, 0). \tag{5.2.7}$$

The requirement that distinct cross sections of the shell never intersect and that the local ratio of deformed to reference area of the shell be everywhere positive leads to restrictions on the strains, for example, $\nu > 0$. See Antman (2005, Chapter 10, Section 1).

Mechanics

Let $\boldsymbol{n}_1(s, \phi, t)$ and $\boldsymbol{m}_1(s, \phi, t)$ denote the internal contact force and contact couple per unit reference length exerted across circles of latitude of the shell, and $\boldsymbol{n}_2(s, \phi, t)$ and $\boldsymbol{m}_2(s, \phi, t)$ denote the internal contact force and contact couple per unit reference length exerted across meridian curves of the shell. For more explanation see Antman (2005, Chapter 10, Section 1). Since we seek axisymmetric configurations of the shell, we require that these stresses have the form

$$\boldsymbol{n}_1(s, \phi, t) = N(s,t)\boldsymbol{a}(s, \phi, t) + H(s,t)\boldsymbol{d}(s, \phi, t), \qquad \boldsymbol{n}_2(s, \phi, t) = T(s,t)\boldsymbol{e}_2(\phi + \omega t),$$
$$\boldsymbol{m}_1(s, \phi, t) = -M(s)\boldsymbol{e}_2(\phi + \omega t), \qquad \boldsymbol{m}_2(s, \phi) = \Sigma(s,t)\boldsymbol{a}(s, \phi, t).$$

Let $\boldsymbol{f}(s, \phi, t)$ be the force per unit reference area exerted by the fluid on material point (s, ϕ) at time t. We give an expression for $\boldsymbol{f}$ in Section 5.4. The Linear and Angular Momentum Laws for the shell are

$$\begin{gathered}(N\boldsymbol{a} + H\boldsymbol{d})_s - T\boldsymbol{e}_1 + \boldsymbol{f} + g\boldsymbol{e}_2 = 2\varrho h \boldsymbol{r}_{tt} + \varrho I \boldsymbol{d}_{tt} \\ M_s - \Sigma\cos\theta + \nu H - \eta N = -(\varrho I \boldsymbol{d} \times \boldsymbol{r}_{tt} + \varrho J \boldsymbol{d} \times \boldsymbol{d}_{tt}) \cdot \boldsymbol{e}_2.\end{gathered} \tag{5.2.8}$$

(These equations are obtained by substituting $\boldsymbol{h} = \rho J \boldsymbol{d} \times \boldsymbol{d}_{tt}$, $\boldsymbol{q} = \boldsymbol{d}$, and $r^{\circ} = 1$ into equations (17.4.15) and (17.4.16) of Antman (2005, Chapter 17, Section 4).) Here $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ here have argument $\phi + \omega t$. The force terms on the left-hand side of equation (5.2.8) can be derived from a free-body diagram by adding up all the forces on the shell. The linear and angular momentum terms on the right-hand side of (5.2.8) and the functions $(\varrho h)(s, \phi)$, $(\varrho I)(s, \phi)$, and $(\varrho J)(s, \phi)$ require motivation from the 3-dimensional theory of continuum mechanics. The derivation is similar to that given for the ring in Section 3.2. We can think of $2\varrho h$, ϱI, and ϱJ as being the mass, first moment of mass, and second moment of mass of the shell per reference area.

The term $g(s)\boldsymbol{e}_2(\phi+\omega t)$ in (5.2.8) is the force required to keep the shell rotating at angular velocity ω. Note that without this force the system of equations for the shell is overdetermined: We have three functions describing the shell, q, ζ, and θ, but four equations that they must satisfy, the angular momentum equation and three components of the linear momentum equation. By including the force $g\boldsymbol{e}_2$ as an unknown we obtain a square system, i.e., we have the same number of equations as unknowns.

There is also a physical explanation for using the force $g\boldsymbol{e}_2$. From the form of $\boldsymbol{r}$, equation (5.2.2), we are asking that each generator of the cylinder lie in a vertical plane rotating at angular velocity ω no matter what force is exerted on it by the fluid. This cannot happen (for an unconstrained shell) unless we provide a force $g\boldsymbol{e}_2$ to maintain the motion. We can think of g as a feedback control.

It can be shown that the use of g is not artificial. Antman and Bourne (in preparation) formulate a general theory of shells that are rotationally symmetric, but not necessarily axisymmetric: All components of the deformation and stress resultants are independent of ϕ, and the shell is invariant under the group $SO(2)$, but not necessarily under the group $O(2)$. If the deformations are *constrained* to be axisymmetric, then such constraints are maintained by Lagrange multipliers. These Lagrange multipliers intervene in the equation corresponding to (5.2.8), and g accounts for their contribution.

In Section 5.6 we will see that the function g is not needed for the existence of a steady solution, i.e., we can take $g = 0$ in this case.

The $\boldsymbol{e}_1$–, $\boldsymbol{e}_2$–, and $\boldsymbol{k}$–components of equation $(5.2.8)_1$ are

$$
\begin{aligned}
&(N_s - H)\cos\theta - (N + H_s)\sin\theta - T + \boldsymbol{f}\cdot\boldsymbol{e}_1 \\
&\qquad = 2\varrho h(q_{tt} - \omega^2 q + \zeta_{tt}) + \varrho I(\sin\theta\,\theta_t^2 - \cos\theta\,\theta_{tt} + \sin\theta\,\omega^2), \quad (5.2.9)
\end{aligned}
$$

$$\boldsymbol{f}\cdot\boldsymbol{e}_2 + g = 4\varrho h\omega q_t - 2\varrho I\cos\theta\,\theta_t\omega, \tag{5.2.10}$$

$$(N_s - H)\sin\theta + (N + H_s)\cos\theta + \boldsymbol{f}\cdot\boldsymbol{k} = 2\varrho h\zeta_{tt} - \varrho I(\cos\theta\,\theta_t^2 + \sin\theta\,\theta_{tt}). \tag{5.2.11}$$

Constitutive Equations

We assume that the shell is uniform so that ϱh, ϱI, and ϱJ are constants and the constitutive functions (defined below) are independent of s and ϕ. Recall that $\mathbf{q} = (\tau, \nu, \eta, \sigma, \mu)$. The shell is said to be viscoelastic of strain-rate type if there are functions

$$\mathbf{q}, \dot{\mathbf{q}} \mapsto \hat{N}(\mathbf{q}, \dot{\mathbf{q}}),\ \hat{H}(\mathbf{q}, \dot{\mathbf{q}}),\ \hat{M}(\mathbf{q}, \dot{\mathbf{q}}),\ \hat{T}(\mathbf{q}, \dot{\mathbf{q}}),\ \hat{\Sigma}(\mathbf{q}, \dot{\mathbf{q}}) \tag{5.2.12}$$

such that

$$N(s,t) = \hat{N}(\mathbf{q}(s,t), \mathbf{q}_t(s,t)), \quad \text{etc.} \tag{5.2.13}$$

The superposed dot on $\mathbf{q}$ has no operational significance; it merely identifies the argument of the constitutive functions that is to be occupied by the time derivative of $\mathbf{q}$. This form of the constitutive functions follows from the Principle of Frame-Indifference by starting with general constitutive functions of the form $N(s,t) = \hat{N}(\boldsymbol{r}, \boldsymbol{r}_s, \boldsymbol{r}_t, \boldsymbol{d}, \boldsymbol{d}_s, \boldsymbol{d}_t, s, t)$.

The Strong Ellipticity Condition from 3-dimensional nonlinear elasticity implies that the following monotonicity conditions hold:

$$\text{The matrices} \quad \frac{\partial(\hat{N}, \hat{H}, \hat{M})}{\partial(\nu, \eta, \mu)} \quad \text{and} \quad \frac{\partial(\hat{T}, \hat{\Sigma})}{\partial(\tau, \sigma)} \quad \text{are positive-definite.} \tag{5.2.14}$$

These ensure that an increase in tension accompanies an increase in strain. We also require that an extreme tension accompanies an extreme strain. This leads to blow-up conditions for $\hat{T}$, $\hat{N}$, $\hat{H}$, $\hat{\Sigma}$, and $\hat{M}$. See Antman (2005, Chapter 10, Section 1).

We require that the effects of shearing in one sense be the same as in the opposite sense:

$$\hat{N}(-\eta, -\dot{\eta}) = \hat{N}(\eta, \dot{\eta}), \qquad \hat{H}(-\eta, -\dot{\eta}) = -\hat{H}(\eta, \dot{\eta}), \qquad \hat{M}(-\eta, -\dot{\eta}) = \hat{M}(\eta, \dot{\eta}),$$
$$\hat{T}(-\eta, -\dot{\eta}) = \hat{T}(\eta, \dot{\eta}), \qquad \hat{\Sigma}(-\eta, -\dot{\eta}) = \hat{\Sigma}(\eta, \dot{\eta}).$$

Here we have suppressed the arguments τ, ν, σ, μ, $\dot{\tau}$, $\dot{\nu}$, $\dot{\sigma}$, $\dot{\mu}$ of the constitutive functions. It follows that if $(\eta, \dot{\eta}) = (0, 0)$, then

$$\hat{H} = \hat{H}_\tau = \hat{H}_\nu = \hat{H}_\sigma = \hat{H}_\mu = \hat{H}_{\dot{\tau}} = \hat{H}_{\dot{\nu}} = \hat{H}_{\dot{\sigma}} = \hat{H}_{\dot{\mu}} = 0, \tag{5.2.15}$$

$$\hat{N}_\eta = \hat{N}_{\dot{\eta}} = \hat{M}_\eta = \hat{M}_{\dot{\eta}} = \hat{T}_\eta = \hat{T}_{\dot{\eta}} = \hat{\Sigma}_\eta = \hat{\Sigma}_{\dot{\eta}} = 0. \tag{5.2.16}$$

We also take the reference configuration of the shell to be natural so that the constitutive functions vanish in the reference configuration: $\hat{N}((1, 1, 0, 1, 0), (0, 0, 0, 0, 0)) = 0$, etc.

5.3 Formulation of the Equations for the Fluid

We use the same notation for the fluid variables as in Chapter 2 except that the dynamic viscosity is now denoted by $\tilde{\mu}$ instead of μ (in this chapter μ denotes a strain variable). The fluid occupies the region between the rigid inner cylinder of radius $a < 1$ and the deformable shell. We assume that the fluid velocity $\boldsymbol{v}$ is axisymmetric. Therefore it has the following decomposition into rotating cylindrical polar coordinates (u, v, w):

$$\boldsymbol{v}(r\boldsymbol{e}_1(\phi+\omega t)+z\boldsymbol{k}, t) = u(r, z, t)\boldsymbol{e}_1(\phi+\omega t)+v(r, z, t)\boldsymbol{e}_2(\phi+\omega t)+w(r, z, t)\boldsymbol{k}. \tag{5.3.1}$$

The (transposed) gradient of $\boldsymbol{v}$ is

$$\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}} = (u_r\boldsymbol{e}_1 + v_r\boldsymbol{e}_2 + w_r\boldsymbol{k})\boldsymbol{e}_1 + \tfrac{1}{r}(u\boldsymbol{e}_2 - v\boldsymbol{e}_1)\boldsymbol{e}_2 + (u_z\boldsymbol{e}_1 + v_z\boldsymbol{e}_2 + w_z\boldsymbol{k})\boldsymbol{k}. \tag{5.3.2}$$

It can be shown that the Navier-Stokes equations for axisymmetric flow in rotating cylindrical polar coordinates are

$$\begin{aligned} u_t + uu_r + wu_z - \frac{v^2}{r} &= -p_r + \gamma \left(u_{rr} + \frac{u_r}{r} + u_{zz} - \frac{u}{r^2} \right), \\ v_t + uv_r + wv_z + \frac{uv}{r} &= \gamma \left(v_{rr} + \frac{v_r}{r} + v_{zz} - \frac{v}{r^2} \right), \\ w_t + uw_r + ww_z &= -p_z + \gamma \left(w_{rr} + \frac{w_r}{r} + w_{zz} \right), \\ u_r + \frac{u}{r} + w_z &= 0. \end{aligned} \tag{5.3.3}$$

5.4 The Coupling Between the Fluid and the Shell Equations

The equations for the fluid and the shell are coupled through the adherence boundary condition and the body force term $\boldsymbol{f}$ in the linear momentum equation for the shell. The adherence boundary condition states that

$$v(a, z, t) = a\omega, \quad u(a, z, t) = w(a, z, t) = 0, \tag{5.4.1}$$

$$\boldsymbol{v}(\boldsymbol{r}(s, \phi, t), t) = \boldsymbol{r}_t(s, \phi, t). \tag{5.4.2}$$

Now we derive an expression for the body force $\boldsymbol{f}$ exerted by the fluid on the shell. The outward pointing unit normal to the surface $\boldsymbol{r}(\cdot, \cdot, t)$ is $-(\boldsymbol{r}_s \times \boldsymbol{r}_\phi)/|\boldsymbol{r}_s \times \boldsymbol{r}_\phi|$. The force per unit (actual) area exerted by the shell on the fluid at $\boldsymbol{r}(s, \phi, t)$ is thus $-\boldsymbol{\Sigma} \cdot (\boldsymbol{r}_s \times \boldsymbol{r}_\phi)/|\boldsymbol{r}_s \times \boldsymbol{r}_\phi|$. Therefore the force per unit reference area exerted by the fluid on the shell at $\boldsymbol{r}(s,\phi,t)$ is

$$\begin{aligned}
\boldsymbol{f} &= \boldsymbol{\Sigma}\cdot(\boldsymbol{r}_s\times\boldsymbol{r}_\phi)\\
&= \left\{-\rho p\boldsymbol{I}+\tilde{\mu}\left[\frac{\partial\boldsymbol{v}}{\partial\boldsymbol{x}}+\left(\frac{\partial\boldsymbol{v}}{\partial\boldsymbol{x}}\right)^*\right]\right\}\cdot(\boldsymbol{r}_s\times q\boldsymbol{e}_2)\\
&= -\rho p(\boldsymbol{r}_s\times q\boldsymbol{e}_2)+\\
&\qquad \tilde{\mu}\Big[2u_r\boldsymbol{e}_1\boldsymbol{e}_1+(v_r-\tfrac{1}{q}v)(\boldsymbol{e}_1\boldsymbol{e}_2+\boldsymbol{e}_2\boldsymbol{e}_1)+(w_r+u_z)(\boldsymbol{e}_1\boldsymbol{k}+\boldsymbol{k}\boldsymbol{e}_1)\\
&\qquad\qquad +v_z(\boldsymbol{e}_2\boldsymbol{k}+\boldsymbol{k}\boldsymbol{e}_2)+\tfrac{2}{q}u\boldsymbol{e}_2\boldsymbol{e}_2+2w_z\boldsymbol{k}\boldsymbol{k}\Big]\cdot(\boldsymbol{r}_s\times q\boldsymbol{e}_2)
\end{aligned} \tag{5.4.3}$$

where u, v, and w have arguments (q,ζ,t) and $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ have argument $\phi+\omega t$.

5.5 Axial Periodicity and the Volume Side Condition

We seek solutions with axial periodicity. Recall that the Taylor vortex flow in the classical Taylor-Couette problem has this property. We assume that there exists constants S and Z such that

$$q(s+S,t)=q(s,t),\quad \zeta(s+S,t)=\zeta(s,t)+Z,\quad \theta(s+S,t)=\theta(s,t), \tag{5.5.1}$$

$$\boldsymbol{v}(\boldsymbol{x}+Z\boldsymbol{k},t)=\boldsymbol{v}(\boldsymbol{x},t),\quad p(\boldsymbol{x}+Z\boldsymbol{k},t)=p(\boldsymbol{x},t). \tag{5.5.2}$$

For simplicity we choose $S=Z$. This permits the existence of a steady solution that is unstretched in the vertical direction. See Section 5.6.

Since the fluid is incompressible, the volume of fluid in each period cell must be constant. Call this constant V. In order to prohibit the formation of cavities in the fluid we require that the volume of the period cell enclosed by the shell = volume of fluid in the period cell + volume of the rigid inner cylinder in the period cell = $V+\pi a^2 Z$. By applying the Divergence Theorem to the vector field $z\boldsymbol{k}$, this volume-preserving condition can be written as

$$V + \pi a^2 Z = \pi q^2 Z - 2\pi \int_0^Z \zeta q q_s \, ds. \tag{5.5.3}$$

Equation (5.5.3) can also be derived from the compatibility condition

$$\int_{\Omega(t)} \mathrm{div}\boldsymbol{v} \, d\boldsymbol{x} = 0, \tag{5.5.4}$$

where $\Omega(t)$ is the domain occupied by the fluid at time t, although once again it is necessary to assume that the fluid occupies the whole region between the deformable and rigid cylinders.

5.6 The Couette Steady Solution

We seek a rigid Couette steady solution of the form

$$u = w = 0, \quad v(r) = \omega r, \quad p(r) = P(r) \tag{5.6.1}$$

$$\boldsymbol{r}(s,\phi,t) = R\boldsymbol{e}_1(\phi + \omega t) + \zeta(s)\boldsymbol{k}, \quad \boldsymbol{d} = -\sin\theta\,\boldsymbol{e}_1 + \cos\theta\boldsymbol{k}, \tag{5.6.2}$$

where R and θ are constants. Therefore the strains are

$$\nu = \boldsymbol{r}_s \cdot \boldsymbol{a} = \zeta_s \sin\theta, \quad \eta = \boldsymbol{r}_s \cdot \boldsymbol{d} = \zeta_s \cos\theta, \quad \mu = 0, \quad \tau = R, \quad \sigma = \sin\theta, \tag{5.6.3}$$

and the force exerted by the fluid on the shell is

$$\boldsymbol{f} = \rho R P(R) \zeta_s \, \boldsymbol{e}_1. \tag{5.6.4}$$

Note that

$$4\varrho h \omega q_t - 2\varrho I \cos\theta\,\theta_t \omega - \boldsymbol{f} \cdot \boldsymbol{e}_2 = 0 \tag{5.6.5}$$

and so $g = 0$ (see equation (5.2.10)). Substituting (5.6.2)–(5.6.4) and $g = 0$ into the linear momentum equation $(5.2.8)_1$ yields

$$\begin{aligned}
&\zeta_{ss}(\hat{N}_\nu \sin\theta + \hat{N}_\eta \cos\theta)(\cos\theta\boldsymbol{e}_1 + \sin\theta\boldsymbol{k}) \\
&+ \zeta_{ss}(\hat{H}_\nu \sin\theta + \hat{H}_\eta \cos\theta)(-\sin\theta\boldsymbol{e}_1 + \cos\theta\boldsymbol{k}) - \hat{T}\boldsymbol{e}_1 + \rho RP(R)\zeta_s\boldsymbol{e}_1 \\
&= -2\varrho hR\omega^2\boldsymbol{e}_1 + \varrho I\sin\theta\omega^2\boldsymbol{e}_1,
\end{aligned} \tag{5.6.6}$$

where $\hat{N}_\nu$, $\hat{N}_\eta$, $\hat{H}_\nu$, $\hat{H}_\eta$, and $\hat{T}$ have arguments

$$(\mathbf{q}, \dot{\mathbf{q}}) = ((R, \zeta_s\sin\theta, \zeta_s\cos\theta, \sin\theta, 0), (0,0,0,0,0)). \tag{5.6.7}$$

Take the inner product of (5.6.6) with $\boldsymbol{k}$ to obtain

$$\begin{aligned}
0 &= \zeta_{ss}[\sin\theta(\hat{N}_\nu \sin\theta + \hat{N}_\eta\cos\theta) + \cos\theta(\hat{H}_\nu\sin\theta + \hat{H}_\eta\cos\theta)] \\
&= \zeta_{ss}[\sin\theta, \cos\theta]\begin{bmatrix}\hat{N}_\nu & \hat{N}_\eta \\ \hat{H}_\nu & \hat{H}_\eta\end{bmatrix}\begin{bmatrix}\sin\theta \\ \cos\theta\end{bmatrix}.
\end{aligned} \tag{5.6.8}$$

The matrix on the right-hand side is positive definite by the monotonicity constitutive assumption (5.2.14). Therefore

$$\zeta_{ss} = 0 \iff \zeta(s) = cs + d \tag{5.6.9}$$

for constants c and d. The periodicity condition $(5.5.1)_1$ implies that $c = 1$. Therefore the shell is unstretched in the vertical direction. The $\boldsymbol{e}_1$-component of the linear momentum equation (5.6.6) and the angular momentum equation $(5.2.8)_2$ simplify to

$$\begin{aligned}
-\hat{T} + \rho RP(R) &= -2\varrho hR\omega^2 + \varrho I\sin\theta\omega^2, \\
-\hat{\Sigma}\cos\theta + \sin\theta\hat{H} - \cos\theta\hat{N} &= \varrho IR\omega^2\cos\theta - \varrho J\sin\theta\cos\theta\omega^2,
\end{aligned} \tag{5.6.10}$$

where $\hat{T}$, $\hat{\Sigma}$, $\hat{H}$, $\hat{N}$ have arguments (5.6.7). $P(r)$ can be found from the Navier-Stokes equation $(5.3.3)_1$:

$$P(r) = \tfrac{1}{2}r^2\omega^2 + D, \tag{5.6.11}$$

where D is a constant. Now we have two algebraic equations (5.6.10) for four constants: d, R, D, θ. This leaves us with some choice. Since the constant d just corresponds to a rigid motion of the shell, we choose $d = 0$. Prescribing R or D can be thought of as prescribing conditions at the ends of the cylinder $z = \pm\infty$. As in the string problem, Chapter 2, we choose to prescribe R rather than D. Then we can solve (5.6.10) for D and θ. Note that if $\theta = \pi/2$ (no shearing), then $(5.6.10)_2$ reduces to

$$\hat{H}((R, 1, 0, 1, 0), (0, 0, 0, 0, 0)) = 0, \tag{5.6.12}$$

which is an identity by constitutive assumption (5.2.15). Then D can be read off from equations $(5.6.10)_1$ and (5.6.11):

$$\begin{aligned} P(R) = P(R;\omega) &= \frac{1}{\rho R}(\hat{T} - 2\varrho hR\omega^2 + \varrho I\omega^2), \\ D &= \frac{1}{\rho R}(\hat{T} - 2\varrho hR\omega^2 + \varrho I\omega^2) - \tfrac{1}{2}R^2\omega^2. \end{aligned} \tag{5.6.13}$$

Note that if we had chosen to prescribe D, rather than R, then $(5.6.10)_1$ with $\theta = \pi/2$ is a nonlinear equation for R, which may have no or many solutions depending on the constitutive function $\hat{T}$. Note also that there may be nontrivial solutions $\theta \neq \pi/2$. We find these shear instabilities in Section 5.9.

When we refer to the Couette steady solution we mean the solution found in this section with $d = 0$, $\theta = \pi/2$, R a prescribed constant, and D given by $(5.6.13)_2$. We analyze the stability of this solution with respect to the angular velocity ω.

5.7 Linearization

We linearize the equations of motion about the Couette steady solution using the same procedure and notation as in Section 2.8.

The strain-configuration relations. Recall the geometry and strain-configuration relations:

$$\boldsymbol{r} = q\boldsymbol{e}_1 + \zeta\boldsymbol{k}, \quad \boldsymbol{d} = -\sin\theta\,\boldsymbol{e}_1 + \cos\theta\,\boldsymbol{k}, \quad \boldsymbol{a} = \cos\theta\,\boldsymbol{e}_1 + \sin\theta\,\boldsymbol{k}, \tag{5.7.1}$$

$$\boldsymbol{r}_s = \nu\boldsymbol{a} + \eta\boldsymbol{d}, \quad \sigma = \sin\theta, \quad \tau = q, \quad \mu = \theta_s. \tag{5.7.2}$$

Linearizing these equations yields

$$\boldsymbol{r}^1 = q^1\boldsymbol{e}_1 + \zeta^1\boldsymbol{k}, \quad \boldsymbol{d}^1 = -\theta^1\boldsymbol{k}, \quad \boldsymbol{a}^1 = -\theta^1\boldsymbol{e}_1, \tag{5.7.3}$$

$$\boldsymbol{r}_s^1 = -(\theta^1 + \eta^1)\boldsymbol{e}_1 + \nu^1\boldsymbol{k}, \quad \sigma^1 = 0, \quad \tau^1 = q^1, \quad \mu^1 = \theta_s^1. \tag{5.7.4}$$

We can solve (5.7.3) and (5.7.4) for the strain perturbations in terms of q^1, ζ^1, and θ^1:

$$\tau^1 = q^1, \quad \nu^1 = \zeta_s^1, \quad \eta^1 = -q_s^1 - \theta^1, \quad \sigma^1 = 0, \quad \mu^1 = \theta_s^1. \tag{5.7.5}$$

The linear momentum equation. Let a superscript $\circ$ on a constitutive function denote the function value at the Couette steady solution, e.g.,

$$N^\circ := \hat{N}((R, 1, 0, 1, 0), (0, 0, 0, 0, 0)). \tag{5.7.6}$$

Equations (5.3.2) and (5.6.1) imply that

$$\frac{\partial \boldsymbol{v}^0}{\partial \boldsymbol{x}}(r\boldsymbol{e}_1(\phi + \omega t) + z\boldsymbol{k}) = \omega[\boldsymbol{e}_2(\phi + \omega t)\boldsymbol{e}_1(\phi + \omega t) - \boldsymbol{e}_1(\phi + \omega t)\boldsymbol{e}_2(\phi + \omega t)] \equiv \omega\boldsymbol{k}\times, \tag{5.7.7}$$

a skew-symmetric, constant tensor. Linearizing the force term in the linear momentum equation $(5.2.8)_1$ requires care:

$$
\begin{aligned}
\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \boldsymbol{f} &= \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \boldsymbol{\Sigma} \cdot (\boldsymbol{r}_s \times q\boldsymbol{e}_2) \\
&= \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \boldsymbol{\Sigma}(\boldsymbol{v}(\boldsymbol{r}(s,\phi,t;\varepsilon),t;\varepsilon), p(\boldsymbol{r}(s,\phi,t;\varepsilon),t;\varepsilon)) \cdot (\boldsymbol{r}_s(s,\phi,t;\varepsilon) \times q(s,t;\varepsilon)\boldsymbol{e}_2) \\
&= \rho R^2\omega^2(\boldsymbol{e}_1 \cdot \boldsymbol{r}^1)\boldsymbol{e}_1 - R\boldsymbol{\Sigma}(\boldsymbol{v}^1, p^1) \cdot \boldsymbol{e}_1 - \rho R P(R)(\boldsymbol{r}_s^1 \times \boldsymbol{e}_2) + \rho P(R) q^1 \boldsymbol{e}_1 \\
&= \rho R^2\omega^2 q^1 \boldsymbol{e}_1 + \rho R p^1 \boldsymbol{e}_1 - R\tilde{\mu}[2u_r^1 \boldsymbol{e}_1 + (v_r^1 - \tfrac{1}{R}v^1)\boldsymbol{e}_2 + (w_r^1 + u_z^1)\boldsymbol{k}] \\
&\quad - \rho R P(R)(q_s^1 \boldsymbol{k} - \zeta_s^1 \boldsymbol{e}_1) + \rho P(R) q^1 \boldsymbol{e}_1,
\end{aligned}
\tag{5.7.8}
$$

where the fluid variables u_r^1, v_r^1, etc., are evaluated at $(r,z,t) = (R,s,t)$. Therefore by linearizing the linear momentum equation $(5.2.8)_1$ we obtain

$$
\begin{aligned}
2\varrho h \boldsymbol{r}_{tt}^1 + \varrho I \boldsymbol{d}_{tt}^1 = N_s^1 \boldsymbol{k} - H_s^1 \boldsymbol{e}_1 - \mu^1 N^\circ \boldsymbol{e}_1 - T^1 \boldsymbol{e}_1 + \rho R^2\omega^2 q^1 \boldsymbol{e}_1 + \rho R p^1 \boldsymbol{e}_1 \\
- R\tilde{\mu}[2u_r^1 \boldsymbol{e}_1 + (v_r^1 - \tfrac{1}{R}v^1)\boldsymbol{e}_2 + (w_r^1 + u_z^1)\boldsymbol{k}] \\
- \rho R P(R)(q_s^1 \boldsymbol{k} - \zeta_s^1 \boldsymbol{e}_1) + \rho P(R) q^1 \boldsymbol{e}_1 + g^1 \boldsymbol{e}_2,
\end{aligned}
\tag{5.7.9}
$$

where

$$
\begin{aligned}
N_s^1 &= N_\tau^\circ \tau_s^1 + N_\nu^\circ \nu_s^1 + N_\mu^\circ \mu_s^1 + N_{\dot\tau}^\circ \tau_{st}^1 + N_{\dot\nu}^\circ \nu_{st}^1 + N_{\dot\mu}^\circ \mu_{st}^1, \\
H_s^1 &= H_\eta^\circ \eta_s^1 + H_{\dot\eta}^\circ \eta_{st}^1, \\
T^1 &= T_\tau^\circ \tau^1 + T_\nu^\circ \nu^1 + T_\mu^\circ \mu^1 + T_{\dot\tau}^\circ \tau_t^1 + T_{\dot\nu}^\circ \nu_t^1 + T_{\dot\mu}^\circ \mu_t^1.
\end{aligned}
\tag{5.7.10}
$$

Note that we have used (5.7.5) to replace σ^1 by 0 and the constitutive assumptions (5.2.15) and (5.2.16) to identify many of the coefficients of N_s^1, H_s^1, and T^1 as 0.

The $\boldsymbol{e}_1$–, $\boldsymbol{e}_2$–, and $\boldsymbol{k}$–components of equation (5.7.9) are

$$\begin{aligned} 2\varrho h(q^1_{tt} - \omega^2 q^1) &= -H^1_s - \mu^1 N^\circ - T^1 + \rho R^2\omega^2 q^1 + \rho R p^1 - 2R\tilde{\mu} u^1_r \\ &\quad + \rho P(R) R \zeta^1_s + \rho P(R) q^1, \\ 4\varrho h\omega q^1_t &= -R\tilde{\mu}(v^1_r - \tfrac{1}{R} v^1) + g^1, \\ 2\varrho h \zeta^1_{tt} - \varrho I \theta^1_{tt} &= N^1_s - R\tilde{\mu}(w^1_r + u^1_z) - \rho R P(R) q^1_s. \end{aligned} \tag{5.7.11}$$

Equation $(5.7.11)_2$ can be solved for g^1. Thus we can drop equation $(5.7.11)_2$ along with the variable g^1.

The angular momentum equation. By linearizing equation $(5.2.8)_2$ we find

$$\begin{aligned} M^1_s + \Sigma^\circ \theta^1 + H^1 - \eta^1 N^\circ &= -[\varrho I(\boldsymbol{r}^1_{tt} - R\omega^2 \boldsymbol{d}^1) + \varrho J(\boldsymbol{d}^1_{tt} + \omega^2 \boldsymbol{d}^1)] \cdot \boldsymbol{k} \\ &= -\varrho I(\zeta^1_{tt} + R\omega^2\theta^1) + \varrho J(\theta^1_{tt} + \omega^2\theta^1), \end{aligned} \tag{5.7.12}$$

where

$$\begin{aligned} M^1_s &= M^\circ_\tau \tau^1_s + M^\circ_\nu \nu^1_s + M^\circ_\mu \mu^1_s + M^\circ_{\dot\tau} \tau^1_{st} + M^\circ_{\dot\nu} \nu^1_{st} + M^\circ_{\dot\mu} \mu^1_{st}, \\ H^1 &= H^\circ_\eta \eta^1 + H^\circ_{\dot\eta} \eta^1_t. \end{aligned} \tag{5.7.13}$$

Once again we have used (5.7.5) and constitutive assumptions (5.2.15) and (5.2.16).

The Navier-Stokes equations. Linearizing (5.3.3) about the Couette solution gives

$$\begin{aligned} u^1_t - 2\omega v^1 &= -p^1_r + \gamma\left(u^1_{rr} + \frac{u^1_r}{r} + u^1_{zz} - \frac{u^1}{r^2}\right), \\ v^1_t + 2\omega u^1 &= \gamma\left(v^1_{rr} + \frac{v^1_r}{r} + v^1_{zz} - \frac{v^1}{r^2}\right), \\ w^1_t &= -p^1_z + \gamma\left(w^1_{rr} + \frac{w^1_r}{r} + w^1_{zz}\right), \\ u^1_r + \frac{u^1}{r} + w^1_z &= 0. \end{aligned} \tag{5.7.14}$$

The domain of the linearized equations (5.7.14) is constant and equals $\{(r,\phi,Z)\in[a,R]\times[0,2\pi]\times(-\infty,\infty)\}$.

The adherence boundary condition. Substitute (5.2.2) and (5.3.1) into (5.4.2) to obtain

$$u(q,\zeta,t)\boldsymbol{e}_1+v(q,\zeta,t)\boldsymbol{e}_2+w(q,\zeta,t)\boldsymbol{k}=q_t\boldsymbol{e}_1+\omega q\boldsymbol{e}_2+\zeta_t\boldsymbol{k}. \tag{5.7.15}$$

Taking the inner product with $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{k}$, and then linearizing, we obtain

$$u^1(R,s,t)=q^1_t(s,t),\qquad v^1(R,s,t)=0,\qquad w^1(R,s,t)=\zeta^1_t(s,t). \tag{5.7.16}$$

Linearizing (5.4.1) gives

$$u^1(a,z,t)=v^1(a,z,t)=w^1(a,z,t)=0. \tag{5.7.17}$$

The periodicity condition. Linearizing (5.5.1) yields

$$q^1(s+Z,t)=q^1(s,t),\quad \zeta^1(s+Z,t)=\zeta^1(s,t),\quad \theta^1(s+Z,t)=\theta^1(s,t), \tag{5.7.18}$$

$$\boldsymbol{v}^1(\boldsymbol{x}+Z\boldsymbol{k},t)=\boldsymbol{v}^1(\boldsymbol{x},t),\quad p^1(\boldsymbol{x}+Z\boldsymbol{k},t)=p^1(\boldsymbol{x},t). \tag{5.7.19}$$

The volume side condition. Equation (5.5.3) has linearization

$$\int_0^Z q^1\,ds=0. \tag{5.7.20}$$

5.8 The Quadratic Eigenvalue Problem

Polar Coordinates

We seek solutions of the linearized equations with exponential time-dependence:

$$u^1(r,z,t) = u(r,z)e^{\lambda t}, \quad v^1(r,z,t) = v(r,z)e^{\lambda t}, \quad w^1(r,z,t) = w(r,z)e^{\lambda t},$$

$$q^1(s,t) = q(s)e^{\lambda t}, \quad \zeta^1(s,t) = \zeta(s)e^{\lambda t}, \quad \theta^1(s,t) = \theta(s)e^{\lambda t},$$

$$\tau^1(s,t) = \tau^1(s)e^{\lambda t}, \quad \nu^1(s,t) = \nu^1(s)e^{\lambda t}, \quad \eta^1(s,t) = \eta^1(s)e^{\lambda t}, \quad \mu^1(s,t) = \mu^1(s)e^{\lambda t}.$$

Note that the symbols u, v, w, q, ζ, θ, τ, ν, η, and μ have a different meaning here than they did in the previous sections. Substituting these expressions into the linearized equations yields the following:

The strain-configuration relations.

$$\tau = q, \qquad \nu = \zeta_s, \qquad \eta = -q_s - \theta, \qquad \mu = \theta_s. \tag{5.8.1}$$

The linear momentum equations.

$$\begin{aligned} 2\varrho h(\lambda^2 - \omega^2)q = &-(H^\circ_\eta + \lambda H^\circ_{\dot\eta})\eta_s - N^\circ\mu - (T^\circ_\tau + \lambda T^\circ_{\dot\tau})\tau - (T^\circ_\nu + \lambda T^\circ_{\dot\nu})\nu - (T^\circ_\mu + \lambda T^\circ_{\dot\mu})\mu \\ &+ \rho R^2\omega^2 q + \rho R p - 2R\tilde{\mu} u_r + \rho P(R) R\zeta_s + \rho P(R) q, \end{aligned} \tag{5.8.2}$$

$$\begin{aligned} 2\varrho h\lambda^2\zeta - \varrho I\lambda^2\theta = &(N^\circ_\tau + \lambda N^\circ_{\dot\tau})\tau_s + (N^\circ_\nu + \lambda N^\circ_{\dot\nu})\nu_s + (N^\circ_\mu + \lambda N^\circ_{\dot\mu})\mu_s \\ &- R\tilde{\mu}(w_r + u_z) - \rho R P(R) q_s. \end{aligned} \tag{5.8.3}$$

The angular momentum equation.

$$\begin{aligned} -\varrho I(\lambda^2\zeta + R\omega^2\theta) + \varrho J(\lambda^2 + \omega^2)\theta = &(M^\circ_\tau + \lambda M^\circ_{\dot\tau})\tau_s + (M^\circ_\nu + \lambda M^\circ_{\dot\nu})\nu_s + (M^\circ_\mu + \lambda M^\circ_{\dot\mu})\mu_s \\ &+ \Sigma^\circ\theta + (H^\circ_\eta + \lambda H^\circ_{\dot\eta})\eta - N^\circ\eta. \end{aligned} \tag{5.8.4}$$

The Navier-Stokes equations.

$$
\begin{aligned}
\lambda u - 2\omega v &= -p_r + \gamma\left(u_{rr} + \frac{u_r}{r} + u_{zz} - \frac{u}{r^2}\right), \\
\lambda v + 2\omega u &= \gamma\left(v_{rr} + \frac{v_r}{r} + v_{zz} - \frac{v}{r^2}\right), \\
\lambda w &= -p_z + \gamma\left(w_{rr} + \frac{w_r}{r} + w_{zz}\right), \\
u_r + \frac{u}{r} + w_z &= 0.
\end{aligned} \tag{5.8.5}
$$

The adherence boundary condition.

$$u(R,s) = \lambda q(s), \qquad v(R,s) = 0, \qquad w(R,s) = \lambda\zeta(s), \tag{5.8.6}$$

$$u(a,z) = v(a,z) = w(a,z) = 0. \tag{5.8.7}$$

The periodicity condition.

$$q(s+Z) = q(s), \quad \zeta(s+Z) = \zeta(s), \quad \theta(s+Z) = \theta(s), \tag{5.8.8}$$

$$u(r,z+Z) = u(r,z), \quad v(r,z+Z) = v(r,z), \quad w(r,z+Z) = w(r,z), \tag{5.8.9}$$

$$p(r,z+Z) = p(r,z). \tag{5.8.10}$$

The volume side condition.

$$\int_0^Z q\, ds = 0. \tag{5.8.11}$$

Coordinate-free Equations

In order to write the quadratic eigenvalue problem in coordinate-free form we define

$$\begin{aligned}
\tilde{\boldsymbol{v}}(r\boldsymbol{e}_1(\phi) + z\boldsymbol{k}) &:= u(r,z)\boldsymbol{e}_1(\phi) + v(r,z)\boldsymbol{e}_2(\phi) + w(r,z)\boldsymbol{k}, \\
\tilde{p}(r\boldsymbol{e}_1(\phi) + z\boldsymbol{k}) &:= p(r,z), \\
\tilde{\boldsymbol{r}}(s,\phi) &:= q(s)\boldsymbol{e}_1(\phi) + \zeta(s)\boldsymbol{k}, \\
\tilde{\boldsymbol{d}}(s) &:= -\theta\boldsymbol{k}.
\end{aligned} \tag{5.8.12}$$

Now we drop the tilde from these variables. The new variables $\boldsymbol{v}$, p, $\boldsymbol{r}$, and $\boldsymbol{d}$ satisfy the quadratic eigenvalue problem (5.8.13)–(5.8.18) given below. This can be checked by substituting (5.8.12) into (5.8.13)–(5.8.18) to obtain (5.8.2)–(5.8.11). Note that we have eliminated the strain variables τ, ν, η, and μ in favor of $\boldsymbol{r}$ and $\boldsymbol{d}$. The vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ in (5.8.13) and (5.8.14) have argument ϕ.

The linear momentum equation.

$$\begin{aligned}
&2\varrho h\lambda^2\boldsymbol{r} - 2\varrho h\omega^2(\boldsymbol{r}\cdot\boldsymbol{e}_1)\boldsymbol{e}_1 + \lambda^2\varrho I\boldsymbol{d} \\
&= (N_\tau^\circ + \lambda N_{\dot\tau}^\circ)(\boldsymbol{r}_s\cdot\boldsymbol{e}_1)\boldsymbol{k} + (N_\nu^\circ + \lambda N_{\dot\nu}^\circ)(\boldsymbol{r}_{ss}\cdot\boldsymbol{k})\boldsymbol{k} - (N_\mu^\circ + \lambda N_{\dot\mu}^\circ)\boldsymbol{d}_{ss} + (H_\eta^\circ + \lambda H_{\dot\eta}^\circ)(\boldsymbol{r}_{ss}\cdot\boldsymbol{e}_1)\boldsymbol{e}_1 \\
&+ (T_\mu^\circ + \lambda T_{\dot\mu}^\circ + N^\circ - H_\eta^\circ - \lambda H_{\dot\eta}^\circ)(\boldsymbol{e}_2\times\boldsymbol{d}_s) - (T_\tau^\circ + \lambda T_{\dot\tau}^\circ)(\boldsymbol{r}\cdot\boldsymbol{e}_1)\boldsymbol{e}_1 - (T_\nu^\circ + \lambda T_{\dot\nu}^\circ)(\boldsymbol{r}_s\cdot\boldsymbol{k})\boldsymbol{e}_1 \\
&\quad + (\rho R^2\omega^2 + \rho P(R))(\boldsymbol{r}\cdot\boldsymbol{e}_1)\boldsymbol{e}_1 + \rho R P(R)\boldsymbol{e}_2\times\boldsymbol{r}_s - R\boldsymbol{\Sigma}\cdot\boldsymbol{e}_1 + R\tilde{\mu}(v_r - \tfrac{1}{R}v)\boldsymbol{e}_2.
\end{aligned} \tag{5.8.13}$$

The angular momentum equation.

$$
\begin{aligned}
&-\varrho I\lambda^2 \boldsymbol{r}\cdot\boldsymbol{k} - (\varrho J\lambda^2 + \varrho J\omega^2 - \varrho IR\omega^2)\boldsymbol{d}\cdot\boldsymbol{k} \\
&= (M_\tau^\circ + \lambda M_{\dot\tau}^\circ)(\boldsymbol{r}_s\times\boldsymbol{e}_2)\cdot\boldsymbol{k} + (M_\nu^\circ + \lambda M_{\dot\nu}^\circ)\boldsymbol{r}_{ss}\cdot\boldsymbol{k} - (M_\mu^\circ + \lambda M_{\dot\mu}^\circ)\boldsymbol{d}_{ss}\cdot\boldsymbol{k} - \Sigma^\circ\boldsymbol{d}\cdot\boldsymbol{k} \\
&\qquad + (H_\eta^\circ + \lambda H_{\dot\eta}^\circ - N^\circ)[\boldsymbol{d}\cdot\boldsymbol{k} - (\boldsymbol{r}_s\times\boldsymbol{e}_2)\cdot\boldsymbol{k}]. \qquad (5.8.14)
\end{aligned}
$$

The Navier-Stokes equations.

$$
\begin{aligned}
\lambda\boldsymbol{v} &= \tfrac{1}{\rho}\,\mathrm{div}\boldsymbol{\Sigma}(p,\boldsymbol{v}) - 2\omega\boldsymbol{k}\times\boldsymbol{v}, \\
\nabla\cdot\boldsymbol{v} &= 0.
\end{aligned} \qquad (5.8.15)
$$

The adherence boundary condition.

$$
\boldsymbol{v} = 0 \quad \text{for } r = a, \qquad \boldsymbol{v}(R\boldsymbol{e}_1(\phi) + s\boldsymbol{k}) = \lambda\boldsymbol{r}(s,\phi). \qquad (5.8.16)
$$

The periodicity condition.

$$
\begin{aligned}
\boldsymbol{v}(\boldsymbol{x}+Z\boldsymbol{k}) &= \boldsymbol{v}(\boldsymbol{x}), \qquad p(\boldsymbol{x}+Z\boldsymbol{k}) = p(\boldsymbol{x}), \\
\boldsymbol{r}(s+Z,\phi) &= \boldsymbol{r}(s,\phi), \qquad \boldsymbol{d}(s+Z) = \boldsymbol{d}(s).
\end{aligned} \qquad (5.8.17)
$$

The volume side condition.

$$
\int_0^Z \boldsymbol{r}(s,\phi)\cdot\boldsymbol{e}_1(\phi)\, ds = 0. \qquad (5.8.18)
$$

5.9 Analysis of the Spectrum

Eigenvalue Crossings

For the string problem we were able to prove that the eigenvalues λ cross the imaginary axis through the origin by using a simple energy estimate. See Section

2.10. What information can we learn about the shell problem by performing a similar estimate?

Let $\mathit{\Omega}$ denote the period cell

$$\mathit{\Omega} = \{\boldsymbol{x} = r\boldsymbol{e}_1(\phi) + z\boldsymbol{k} : a < r < R,\ 0 \le \phi \le 2\pi,\ 0 < z < Z\} \tag{5.9.1}$$

and $\boldsymbol{n}_{\mathit{\Omega}}$ the unit outer normal to $\partial\mathit{\Omega}$. Take the inner product of equation $(5.8.15)_1$ with $\bar{\boldsymbol{v}}$ and integrate by parts over $\mathit{\Omega}$ to obtain

$$\begin{aligned}\rho\,\lambda||\boldsymbol{v}||^2_{L^2(\mathit{\Omega})} &= \int_{\partial\mathit{\Omega}} \boldsymbol{n}_{\mathit{\Omega}}\cdot\boldsymbol{\Sigma}\cdot\bar{\boldsymbol{v}}\,d\boldsymbol{x} - \int_{\mathit{\Omega}} \boldsymbol{\Sigma} : \frac{\partial\bar{\boldsymbol{v}}}{\partial\boldsymbol{x}}\,d\boldsymbol{x} - 2\rho\omega\int_{\mathit{\Omega}}(\boldsymbol{k}\times\boldsymbol{v})\cdot\bar{\boldsymbol{v}}\,d\boldsymbol{x}\\ &= \int_0^Z\int_0^{2\pi} \boldsymbol{e}_1\cdot\boldsymbol{\Sigma}\cdot\bar{\boldsymbol{v}}\,R\,d\phi ds - 2\tilde{\mu}||\boldsymbol{D}(\boldsymbol{v})||^2_{L^2(\mathit{\Omega})} - 2\rho\omega\int_{\mathit{\Omega}}(\boldsymbol{k}\times\boldsymbol{v})\cdot\bar{\boldsymbol{v}}\,d\boldsymbol{x},\end{aligned} \tag{5.9.2}$$

where we have used the adherence boundary condition $\boldsymbol{v} = 0$ on $\{r = a\}$, the periodicity condition (5.8.17), the divergence free constraint $\boldsymbol{I} : \partial\bar{\boldsymbol{v}}/\partial\boldsymbol{x} \equiv \operatorname{div}\bar{\boldsymbol{v}} = 0$, and the identity $\boldsymbol{D}(\boldsymbol{v}) : \partial\bar{\boldsymbol{v}}/\partial\boldsymbol{x} = |\boldsymbol{D}(\boldsymbol{v})|^2$. In the boundary term in (5.9.2) substitute for $R\boldsymbol{\Sigma}\cdot\boldsymbol{e}_1$ from equation (5.8.13), substitute for $\bar{\boldsymbol{v}}$ from equation $(5.8.16)_2$, and integrate by parts to find that

$$\begin{aligned}&\int_0^Z\int_0^{2\pi} \boldsymbol{e}_1\cdot\boldsymbol{\Sigma}\cdot\bar{\boldsymbol{v}}\,R\,d\phi\,ds =\\ &(\rho R^2\omega^2 + \rho P(R))\bar{\lambda}||\boldsymbol{r}\cdot\boldsymbol{e}_1||^2_{L^2} - (N^\circ_\nu + \lambda N^\circ_{\dot{\nu}})\bar{\lambda}||\boldsymbol{r}_s\cdot\boldsymbol{k}||^2_{L^2} - (H^\circ_\eta + \lambda H^\circ_{\dot{\eta}})\bar{\lambda}||\boldsymbol{r}_s\cdot\boldsymbol{e}_1||^2_{L^2}\\ &\quad - (T^\circ_\tau + \lambda T^\circ_{\dot{\tau}})\bar{\lambda}||\boldsymbol{r}\cdot\boldsymbol{e}_1||^2_{L^2} - 2\varrho h\lambda|\lambda|^2||\boldsymbol{r}||^2_{L^2} + 2\varrho h\omega^2\bar{\lambda}||\boldsymbol{r}\cdot\boldsymbol{e}_1||^2_{L^2}\\ &+ \rho R P(R)\bar{\lambda}\int_0^Z\int_0^{2\pi}(\boldsymbol{e}_2\times\boldsymbol{r}_s)\cdot\bar{\boldsymbol{r}}\,d\phi\,ds + (N^\circ_\tau + \lambda N^\circ_{\dot{\tau}})\bar{\lambda}\int_0^Z\int_0^{2\pi}(\boldsymbol{r}_s\cdot\boldsymbol{e}_1)(\bar{\boldsymbol{r}}\cdot\boldsymbol{k})\,d\phi\,ds\\ &\quad - (N^\circ_\mu + \lambda N^\circ_{\dot{\mu}})\bar{\lambda}\int_0^Z\int_0^{2\pi}\boldsymbol{d}_{ss}\cdot\bar{\boldsymbol{r}}\,d\phi\,ds\\ &\quad + (T^\circ_\mu + \lambda T^\circ_{\dot{\mu}} + N^\circ - H^\circ_\eta - \lambda H^\circ_{\dot{\eta}})\bar{\lambda}\int_0^Z\int_0^{2\pi}(\boldsymbol{e}_2\times\boldsymbol{d}_s)\cdot\bar{\boldsymbol{r}}\,d\phi\,ds\\ &- (T^\circ_\nu + \lambda T^\circ_{\dot{\nu}})\bar{\lambda}\int_0^Z\int_0^{2\pi}(\boldsymbol{r}_s\cdot\boldsymbol{k})(\bar{\boldsymbol{r}}\cdot\boldsymbol{e}_1)\,d\phi ds - \varrho I\lambda|\lambda|^2\int_0^Z\int_0^{2\pi}\boldsymbol{d}\cdot\bar{\boldsymbol{r}}\,d\phi\,ds.\end{aligned} \tag{5.9.3}$$

We can eliminate some of the bad terms in this equation by combining it with the angular momentum equation. Multiply (5.8.14) by $\bar{\lambda}\bar{\theta}$, integrate by parts over $(s, \phi) \in (0, Z) \times (0, 2\pi)$, and use the identity $\boldsymbol{d} = -\theta \boldsymbol{k}$ to obtain

$$\begin{aligned}
&\varrho I \lambda |\lambda|^2 \int_0^Z \int_0^{2\pi} \boldsymbol{r} \cdot \bar{\boldsymbol{d}} \, d\phi \, ds + (\varrho J \lambda^2 + \varrho J \omega^2 - \varrho I R \omega^2) \bar{\lambda} ||\boldsymbol{d}||_{L^2}^2 = \\
&- (M_\tau^\circ + \lambda M_{\dot{\tau}}^\circ) \bar{\lambda} \int_0^Z \int_0^{2\pi} (\boldsymbol{r}_s \times \boldsymbol{e}_2) \cdot \bar{\boldsymbol{d}} \, d\phi \, ds - (M_\nu^\circ + \lambda M_{\dot{\nu}}^\circ) \bar{\lambda} \int_0^Z \int_0^{2\pi} \boldsymbol{r}_{ss} \cdot \bar{\boldsymbol{d}} \, d\phi \, ds \\
&- (M_\mu^\circ + \lambda M_{\dot{\mu}}^\circ) \bar{\lambda} ||\boldsymbol{d}_s||_{L^2}^2 + \bar{\lambda} \varSigma^\circ ||\boldsymbol{d}||_{L^2}^2 + (H_\eta^\circ + \lambda H_{\dot{\eta}}^\circ - N^\circ) \bar{\lambda} \int_0^Z \int_0^{2\pi} (\boldsymbol{r}_s \times \boldsymbol{e}_2) \cdot \bar{\boldsymbol{d}} \, d\phi \, ds \\
&- (H_\eta^\circ + \lambda H_{\dot{\eta}}^\circ - N^\circ) \bar{\lambda} ||\boldsymbol{d}||_{L^2}^2. \qquad (5.9.4)
\end{aligned}$$

Finally, add (5.9.4) to (5.9.2) and substitute for the boundary term in (5.9.2) from (5.9.3) to find

$$\begin{aligned}
&\rho \lambda ||\boldsymbol{v}||_{L^2(\varOmega)}^2 = -2\tilde{\mu} ||\boldsymbol{D}(\boldsymbol{v})||_{L^2(\varOmega)}^2 - 2\rho\omega \int_\varOmega (\boldsymbol{k} \times \boldsymbol{v}) \cdot \bar{\boldsymbol{v}} \, d\boldsymbol{x} \\
&+ (\rho R^2 \omega^2 + \rho P(R)) \bar{\lambda} ||\boldsymbol{r} \cdot \boldsymbol{e}_1||_{L^2}^2 - (N_\nu^\circ + \lambda N_{\dot{\nu}}^\circ) \bar{\lambda} ||\boldsymbol{r}_s \cdot \boldsymbol{k}||_{L^2}^2 - (H_\eta^\circ + \lambda H_{\dot{\eta}}^\circ) \bar{\lambda} ||\boldsymbol{r}_s \cdot \boldsymbol{e}_1||_{L^2}^2 \\
&- (T_\tau^\circ + \lambda T_{\dot{\tau}}^\circ) \bar{\lambda} ||\boldsymbol{r} \cdot \boldsymbol{e}_1||_{L^2}^2 - 2\varrho h \lambda |\lambda|^2 ||\boldsymbol{r}||_{L^2}^2 + 2\varrho h \omega^2 \bar{\lambda} ||\boldsymbol{r} \cdot \boldsymbol{e}_1||_{L^2}^2 \\
&+ \rho R P(R) \bar{\lambda} \int_0^Z \int_0^{2\pi} (\boldsymbol{e}_2 \times \boldsymbol{r}_s) \cdot \bar{\boldsymbol{r}} \, d\phi \, ds + (N_\tau^\circ + \lambda N_{\dot{\tau}}^\circ) \bar{\lambda} \int_0^Z \int_0^{2\pi} (\boldsymbol{r}_s \cdot \boldsymbol{e}_1)(\bar{\boldsymbol{r}} \cdot \boldsymbol{k}) \, d\phi \, ds \\
&- (N_\mu^\circ + \lambda N_{\dot{\mu}}^\circ) \bar{\lambda} \int_0^Z \int_0^{2\pi} \boldsymbol{d}_{ss} \cdot \bar{\boldsymbol{r}} \, d\phi \, ds + (T_\mu^\circ + \lambda T_{\dot{\mu}}^\circ) \bar{\lambda} \int_0^Z \int_0^{2\pi} (\boldsymbol{e}_2 \times \boldsymbol{d}_s) \cdot \bar{\boldsymbol{r}} \, d\phi \, ds \\
&- (T_\nu^\circ + \lambda T_{\dot{\nu}}^\circ) \bar{\lambda} \int_0^Z \int_0^{2\pi} (\boldsymbol{r}_s \cdot \boldsymbol{k})(\bar{\boldsymbol{r}} \cdot \boldsymbol{e}_1) \, d\phi \, ds - \varrho I \lambda |\lambda|^2 \int_0^Z \int_0^{2\pi} (\boldsymbol{r} \cdot \bar{\boldsymbol{d}} + \bar{\boldsymbol{r}} \cdot \boldsymbol{d}) \, d\phi \, ds \\
&- (M_\tau^\circ + \lambda M_{\dot{\tau}}^\circ) \bar{\lambda} \int_0^Z \int_0^{2\pi} (\boldsymbol{r}_s \times \boldsymbol{e}_2) \cdot \bar{\boldsymbol{d}} \, d\phi \, ds - (M_\nu^\circ + \lambda M_{\dot{\nu}}^\circ) \bar{\lambda} \int_0^Z \int_0^{2\pi} \boldsymbol{r}_{ss} \cdot \bar{\boldsymbol{d}} \, d\phi \, ds \\
&- (M_\mu^\circ + \lambda M_{\dot{\mu}}^\circ) \bar{\lambda} ||\boldsymbol{d}_s||_{L^2}^2 - (H_\eta^\circ + \lambda H_{\dot{\eta}}^\circ - N^\circ) \bar{\lambda} ||\boldsymbol{d}||_{L^2}^2 - (\varrho J \lambda^2 + \varrho J \omega^2 - \varrho I R \omega^2) \bar{\lambda} ||\boldsymbol{d}||_{L^2}^2 \\
&+ \bar{\lambda} \varSigma^\circ ||\boldsymbol{d}||_{L^2}^2 + (H_\eta^\circ + \lambda H_{\dot{\eta}}^\circ - N^\circ) \bar{\lambda} \int_0^Z \int_0^{2\pi} [(\bar{\boldsymbol{r}}_s \times \boldsymbol{e}_2) \cdot \boldsymbol{d} + (\boldsymbol{r}_s \times \boldsymbol{e}_2) \cdot \bar{\boldsymbol{d}}] \, d\phi \, ds.
\end{aligned}$$

(5.9.5)

The term

$$-2\rho\omega \int_{\Omega} (\boldsymbol{k} \times \boldsymbol{v}) \cdot \bar{\boldsymbol{v}} \; d\boldsymbol{x},$$

is purely imaginary. The integrals

$$\int_0^Z \int_0^{2\pi} (\boldsymbol{e}_2 \times \boldsymbol{r}_s) \cdot \bar{\boldsymbol{r}} \; d\phi \, ds, \quad \int_0^Z \int_0^{2\pi} (\boldsymbol{r} \cdot \bar{\boldsymbol{d}} + \bar{\boldsymbol{r}} \cdot \boldsymbol{d}) \; d\phi \, ds,$$
$$\int_0^Z \int_0^{2\pi} [(\bar{\boldsymbol{r}}_s \times \boldsymbol{e}_2) \cdot \boldsymbol{d} + (\boldsymbol{r}_s \times \boldsymbol{e}_2) \cdot \bar{\boldsymbol{d}}] \; d\phi \, ds.$$

are real. There are still many terms in equation (5.9.5) that cannot be immediately identified as real or imaginary. To make further progress with the analysis we must make some additional constitutive assumptions. We assume that the following symmetry conditions hold:

$$\hat{N}_\tau = \hat{T}_\nu, \quad \hat{T}_\mu = \hat{M}_\tau, \quad \hat{N}_\mu = \hat{M}_\nu. \tag{5.9.6}$$

We also require some monotonicity:

$$\hat{H}_{\dot{\eta}} \geq 0, \quad \text{the matrix} \begin{bmatrix} \hat{N}_{\dot{\nu}} & \hat{N}_{\dot{\tau}} & \hat{N}_{\dot{\mu}} \\ \hat{T}_{\dot{\nu}} & \hat{T}_{\dot{\tau}} & \hat{T}_{\dot{\mu}} \\ \hat{M}_{\dot{\nu}} & \hat{M}_{\dot{\tau}} & \hat{M}_{\dot{\mu}} \end{bmatrix} \text{is positive-semidefinite.} \tag{5.9.7}$$

Conditions (5.9.6) and (5.9.7) are satisfied by hyperelastic materials, for example. Recall that the shell is hyperelastic if there exists a stored energy potentital $W(\mathbf{q})$ such that

$$\hat{N} = W_\nu, \quad \hat{H} = W_\eta, \quad \hat{M} = W_\mu, \quad \hat{T} = W_\tau, \quad \hat{\Sigma} = W_\sigma. \tag{5.9.8}$$

The symmetry conditions (5.9.6) allow us to combine several of the terms of equation

(5.9.5):

$$N_\tau^\circ\bar{\lambda}\int_0^Z\int_0^{2\pi}(\boldsymbol{r}_s\cdot\boldsymbol{e}_1)(\bar{\boldsymbol{r}}\cdot\boldsymbol{k})\,d\phi\,ds - T_\nu^\circ\bar{\lambda}\int_0^Z\int_0^{2\pi}(\boldsymbol{r}_s\cdot\boldsymbol{k})(\bar{\boldsymbol{r}}\cdot\boldsymbol{e}_1)\,d\phi\,ds \tag{5.9.9}$$
$$= N_\tau^\circ\,\bar{\lambda}\int_0^Z\int_0^{2\pi}[(\boldsymbol{r}_s\cdot\boldsymbol{e}_1)(\bar{\boldsymbol{r}}\cdot\boldsymbol{k})+(\bar{\boldsymbol{r}}_s\cdot\boldsymbol{e}_1)(\boldsymbol{r}\cdot\boldsymbol{k})]\,d\phi\,ds,$$
$$T_\mu^\circ\bar{\lambda}\int_0^Z\int_0^{2\pi}(\boldsymbol{e}_2\times\boldsymbol{d}_s)\cdot\bar{\boldsymbol{r}}\,d\phi\,ds - M_\tau^\circ\bar{\lambda}\int_0^Z\int_0^{2\pi}(\boldsymbol{r}_s\times\boldsymbol{e}_2)\cdot\bar{\boldsymbol{d}}\,d\phi\,ds$$
$$= T_\mu^\circ\,\bar{\lambda}\int_0^Z\int_0^{2\pi}[(\boldsymbol{e}_2\times\boldsymbol{d}_s)\cdot\bar{\boldsymbol{r}}+(\boldsymbol{e}_2\times\bar{\boldsymbol{d}}_s)\cdot\boldsymbol{r}]\,d\phi\,ds,$$
$$-N_\mu^\circ\bar{\lambda}\int_0^Z\int_0^{2\pi}\boldsymbol{d}_{ss}\cdot\bar{\boldsymbol{r}}\,d\phi\,ds - M_\nu^\circ\bar{\lambda}\int_0^Z\int_0^{2\pi}\boldsymbol{r}_{ss}\cdot\bar{\boldsymbol{d}}\,d\phi\,ds$$
$$= -N_\mu^\circ\,\bar{\lambda}\int_0^Z\int_0^{2\pi}[\boldsymbol{d}_{ss}\cdot\bar{\boldsymbol{r}}+\bar{\boldsymbol{d}}_{ss}\cdot\boldsymbol{r}]\,d\phi\,ds.$$

Recall that $\boldsymbol{d} = -\theta\boldsymbol{k}$. The terms in (5.9.5) containing the factor $H_{\dot\eta}^\circ$ can be grouped together:

$$-|\lambda|^2H_{\dot\eta}^\circ||\boldsymbol{r}_s\cdot\boldsymbol{e}_1||_{L^2}^2-|\lambda|^2H_{\dot\eta}^\circ||\boldsymbol{d}||_{L^2}^2+|\lambda|^2H_{\dot\eta}^\circ\int_0^Z\int_0^{2\pi}[(\bar{\boldsymbol{r}}_s\times\boldsymbol{e}_2)\cdot\boldsymbol{d}+(\boldsymbol{r}_s\times\boldsymbol{e}_2)\cdot\bar{\boldsymbol{d}}]\,d\phi\,ds$$
$$= -|\lambda|^2H_{\dot\eta}^\circ||\boldsymbol{r}_s\cdot\boldsymbol{e}_1+\theta||_{L^2}^2. \tag{5.9.10}$$

All the remaining viscoelastic terms in (5.9.5) can be written as a quadratic form:

$$-|\lambda|^2N_{\dot\nu}^\circ||\boldsymbol{r}_s\cdot\boldsymbol{k}||_{L^2}^2-|\lambda|^2T_{\dot\tau}^\circ||\boldsymbol{r}\cdot\boldsymbol{e}_1||_{L^2}^2+|\lambda|^2N_{\dot\tau}^\circ\int_0^Z\int_0^{2\pi}(\boldsymbol{r}_s\cdot\boldsymbol{e}_1)(\bar{\boldsymbol{r}}\cdot\boldsymbol{k})\,d\phi\,ds$$
$$-|\lambda|^2N_{\dot\mu}^\circ\int_0^Z\int_0^{2\pi}\boldsymbol{d}_{ss}\cdot\bar{\boldsymbol{r}}\,d\phi\,ds+|\lambda|^2T_{\dot\mu}^\circ\int_0^Z\int_0^{2\pi}(\boldsymbol{e}_2\times\boldsymbol{d}_s)\cdot\bar{\boldsymbol{r}}\,d\phi\,ds$$
$$-|\lambda|^2T_{\dot\nu}^\circ\int_0^Z\int_0^{2\pi}(\boldsymbol{r}_s\cdot\boldsymbol{k})(\bar{\boldsymbol{r}}\cdot\boldsymbol{e}_1)\,d\phi\,ds-|\lambda|^2M_{\dot\tau}^\circ\int_0^Z\int_0^{2\pi}(\boldsymbol{r}_s\times\boldsymbol{e}_2)\cdot\bar{\boldsymbol{d}}\,d\phi\,ds$$
$$-|\lambda|^2M_{\dot\nu}^\circ\int_0^Z\int_0^{2\pi}\boldsymbol{r}_{ss}\cdot\bar{\boldsymbol{d}}\,d\phi\,ds-|\lambda|^2M_{\dot\mu}^\circ||\boldsymbol{d}_s||_{L^2}^2$$
$$= -|\lambda|^2\int_0^Z\int_0^{2\pi}\begin{bmatrix}\bar{\boldsymbol{r}}_s\cdot\boldsymbol{k} & \bar{\boldsymbol{r}}\cdot\boldsymbol{e}_1 & \bar\theta_s\end{bmatrix}\begin{bmatrix}N_{\dot\nu}^\circ & N_{\dot\tau}^\circ & N_{\dot\mu}^\circ\\ T_{\dot\nu}^\circ & T_{\dot\tau}^\circ & T_{\dot\mu}^\circ\\ M_{\dot\nu}^\circ & M_{\dot\tau}^\circ & M_{\dot\mu}^\circ\end{bmatrix}\begin{bmatrix}\boldsymbol{r}_s\cdot\boldsymbol{k}\\ \boldsymbol{r}\cdot\boldsymbol{e}_1\\ \theta_s\end{bmatrix}d\phi\,ds. \tag{5.9.11}$$

By substituting (5.9.9), (5.9.10), and (5.9.11) into (5.9.5), taking the real part of the resulting equation, and setting $\mathrm{Re}(\lambda) = 0$ we discover that

$$
\begin{aligned}
0 = &-2\tilde{\mu}||\boldsymbol{D}(\boldsymbol{v})||^2_{L^2(\Omega)} - |\lambda|^2 H^\circ_{\dot{\eta}}||\boldsymbol{r}_s \cdot \boldsymbol{e}_1 + \theta||^2_{L^2} \\
&- |\lambda|^2 \int_0^Z \int_0^{2\pi} \begin{bmatrix} \bar{\boldsymbol{r}}_s \cdot \boldsymbol{k} & \bar{\boldsymbol{r}} \cdot \boldsymbol{e}_1 & \bar{\theta}_s \end{bmatrix} \begin{bmatrix} N^\circ_{\dot{\nu}} & N^\circ_{\dot{\tau}} & N^\circ_{\dot{\mu}} \\ T^\circ_{\dot{\nu}} & T^\circ_{\dot{\tau}} & T^\circ_{\dot{\mu}} \\ M^\circ_{\dot{\nu}} & M^\circ_{\dot{\tau}} & M^\circ_{\dot{\mu}} \end{bmatrix} \begin{bmatrix} \boldsymbol{r}_s \cdot \boldsymbol{k} \\ \boldsymbol{r} \cdot \boldsymbol{e}_1 \\ \theta_s \end{bmatrix} d\phi\, ds. \qquad (5.9.12)
\end{aligned}
$$

By the monotonicity assumption (5.9.7), all the terms on the right-hand side of (5.9.12) are nonpositive. Therefore, if $\mathrm{Re}(\lambda) = 0$, then $||\boldsymbol{D}(\boldsymbol{v})||^2_{L^2(\Omega)} = 0$ and so $\boldsymbol{v} = 0$ by Korn's inequality and the boundary condition $\boldsymbol{v} = 0$ on $\{r = a\}$. But $\boldsymbol{v} = 0$ implies that $\lambda\boldsymbol{r} = 0$ by the adherence boundary condition (5.8.16). So either $\lambda = 0$ and eigenvalues cross the imaginary axis through the origin or $\boldsymbol{r} = 0$ and eigenvalues crossing the imaginary axis have eigenfunctions $(\boldsymbol{v}, p, \boldsymbol{r}, \boldsymbol{d}) = (0, constant, 0, \boldsymbol{d})$. We examine the second case. It is convenient to return to the equations in polar coordinates. Since $\boldsymbol{v} = \boldsymbol{r} = 0$, then $q = \zeta = u = v = w = 0$. Substituting these into equations (5.8.2)–(5.8.4) yields

$$
\begin{gathered}
H^\circ_\eta \theta_s - N^\circ\theta_s - T^\circ_\mu \theta_s + \rho R p = 0, \\
-\varrho I \lambda^2 \theta = N^\circ_\mu \theta_{ss}, \qquad (5.9.13) \\
-\varrho I R \omega^2 \theta + \varrho J(\lambda^2 + \omega^2)\theta = M^\circ_\mu \theta_{ss} + \Sigma^\circ\theta - (H^\circ_\eta - N^\circ)\theta.
\end{gathered}
$$

But p is constant. Therefore by differentiating $(5.9.13)_1$ with respect to s we find that

$$
(H^\circ_\eta - N^\circ - T^\circ_\mu)\theta_{ss} = 0. \qquad (5.9.14)
$$

So, unless we are in the non-generic case $H^\circ_\eta - N^\circ - T^\circ_\mu = 0$, then $\theta_{ss} = 0$. This and the periodicity condition (5.8.8) imply that $\theta = constant$. By substituting $\theta = constant$ back into $(5.9.13)_1$ we see that the pressure constant $p = 0$. Substituting $\theta = constant$ into $(5.9.13)_2$ yields $\lambda^2\theta = 0$. Thus either $\lambda = 0$ and we are back in the first case (eigenvalues crossing through the origin), or $\theta = 0$ and λ is not an eigenvalue since $(\boldsymbol{v}, p, \boldsymbol{r}, \boldsymbol{d}) = (0, 0, 0, 0)$. Therefore all eigenvalues λ that cross the imaginary axis must cross through the origin.

Observe that substituting $\lambda = 0$, $\theta = constant \neq 0$ into $(5.9.13)_3$ gives an equation for the critical values of ω^2:

$$(\varrho J - \varrho IR)\omega^2 = \Sigma^\circ - H^\circ_\eta + N^\circ. \tag{5.9.15}$$

The coefficient of ω^2 is positive. Therefore (5.9.15) has real solutions if and only if

$$\Sigma^\circ - H^\circ_\eta + N^\circ > 0. \tag{5.9.16}$$

Assume that (5.9.16) holds. Since the solution ω^2 of (5.9.15) has odd algebraic multiplicity, standard theorems in bifurcation theory imply that there exists a branch of steady solutions of the original nonlinear problem, which bifurcates from the Couette steady solution when

$$\omega^2 = \frac{\Sigma^\circ - H^\circ_\eta + N^\circ}{\varrho J - \varrho IR}. \tag{5.9.17}$$

These solutions correspond to shear instabilities of the Couette steady solution. We have proved the following:

Theorem 5.9.18 (Eigenvalue Crossings). *Assume that the following additional material properties hold:*

(i) The symmetry condition (5.9.6)*,*

(ii) The monotonicity condition (5.9.7)*.*

(These conditions are satisfied by hyperelastic materials, for example.) Furthermore, assume the generic condition $H_\eta^\circ - N^\circ - T_\mu^\circ \neq 0$*. (This condition can be dropped if either one of the statements in* (5.9.7) *is strict.) Let* $(\lambda, (\boldsymbol{v}, p, \boldsymbol{r}, \boldsymbol{d}))$ *be a smooth eigenpair of equations* (5.8.13)–(5.8.18)*. If* $\mathrm{Re}(\lambda) = 0$*, then* $\lambda = 0$*. Therefore any eigenvalues* λ *that cross the imaginary axis must cross through the origin.*

Theorem 5.9.19 (Shear Instabilities). *Assume that the hypotheses of Theorem* (5.9.18) *hold. Furthermore, assume that* $\Sigma^\circ - H_\eta^\circ + N^\circ > 0$*. Then eigenvalue problem* (5.8.13)–(5.8.18) *has an eigenpair* $(\lambda, (\boldsymbol{v}, p, \boldsymbol{r}, \boldsymbol{d})) = (0, (0, 0, 0, \boldsymbol{k}))$ *when*

$$\omega^2 = \frac{\Sigma^\circ - H_\eta^\circ + N^\circ}{\varrho J - \varrho I R}.$$

This steady state bifurcation corresponds to a shear instability of the Couette steady solution.

Critical Values of ω

In this section we assume that the hypotheses of Theorem (5.9.18) hold so that all the eigenvalues of (5.8.13)–(5.8.18) that cross the imaginary axis cross through the origin. This means that we can set $\lambda = 0$ in the quadratic eigenvalue problem to obtain an eigenvalue problem for the critical values of ω. Note that this is equivalent to linearising the steady state problem about the Couette solution.

By substituting $\lambda = 0$ in (5.8.5) and (5.8.6) we find that $u = v = w = 0$ and p is constant. By integrating equation (5.8.2) over $s \in [0, Z]$ and applying the periodicity conditions and the volume side condition we find that $p = 0$. Substituting $u = v = w = p = 0$ into (5.8.2)–(5.8.4) and using (5.8.1) to write the strains in terms of q, ζ, and θ yields

$$\begin{aligned} -2\varrho h\omega^2 q &= H_\eta^\circ(q_{ss} + \theta_s) - N^\circ\theta_s - T_\tau^\circ q - T_\nu^\circ\zeta_s - T_\mu^\circ\theta_s + \rho R^2\omega^2 q \\ &\quad + \rho P(R)R\zeta_s + \rho P(R)q, \\ 0 &= N_\tau^\circ q_s + N_\nu^\circ\zeta_{ss} + N_\mu^\circ\theta_{ss} - \rho RP(R)q_s, \\ -(\varrho IR\omega^2 + \varrho J\omega^2)\theta &= M_\tau^\circ q_s + M_\nu^\circ\zeta_{ss} + M_\mu^\circ\theta_{ss} + \Sigma^\circ\theta + (H_\eta^\circ - N^\circ)(-q_s - \theta). \end{aligned} \tag{5.9.20}$$

Recall that q, ζ, and θ have period Z. Decompose them into Fourier series:

$$q(s) = \sum_{k\in\mathbb{Z}} q_k e^{2\pi iks/Z}, \quad \zeta(s) = \sum_{k\in\mathbb{Z}} \zeta_k e^{2\pi iks/Z}, \quad \theta(s) = \sum_{k\in\mathbb{Z}} \theta_k e^{2\pi iks/Z}. \tag{5.9.21}$$

By Parseval's Theorem we can write (5.9.20) as a family of matrix equations indexed by $k \in \mathbb{Z}$:

$$\begin{bmatrix} a_{11}^k & a_{12}^k & a_{13}^k \\ a_{21}^k & a_{22}^k & a_{23}^k \\ a_{31}^k & a_{32}^k & a_{33}^k \end{bmatrix} \begin{bmatrix} q_k \\ \zeta_k \\ \theta_k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{5.9.22}$$

where

$$
\begin{aligned}
a_{11}^k &= 2\varrho h\omega^2 - 4\pi^2k^2Z^{-2}H_\eta^\circ - T_\tau^\circ + \rho R^2\omega^2 + \rho P(R;\omega),\\
a_{12}^k &= 2\pi ikZ^{-1}(-T_\nu^\circ + \rho RP(R;\omega)),\\
a_{13}^k &= 2\pi ikZ^{-1}(H_\eta^\circ - N^\circ - T_\mu^\circ),\\
a_{21}^k &= 2\pi ikZ^{-1}(N_\tau^\circ - \rho RP(R;\omega)),\\
a_{22}^k &= -4\pi^2k^2Z^{-2}N_\nu^\circ,\\
a_{23}^k &= -4\pi^2k^2Z^{-2}N_\mu^\circ,\\
a_{31}^k &= 2\pi ikZ^{-1}(M_\tau^\circ - H_\eta^\circ + N^\circ),\\
a_{32}^k &= -4\pi^2k^2Z^{-2}M_\nu^\circ,\\
a_{33}^k &= \varrho IR\omega^2 - \varrho J\omega^2 - 4\pi^2k^2Z^{-2}M_\mu^\circ + \Sigma^\circ - H_\eta^\circ + N^\circ,\\
P(R;\omega) &= \frac{1}{\rho R}(T^\circ - 2\varrho hR\omega^2 + \varrho I\omega^2).
\end{aligned}
\tag{5.9.23}
$$

We wish to find $\omega \in \mathbb{R}$ such that equation (5.9.22) has nontrivial solutions. Observe that k is a factor of the middle row and column of the matrix above. Thus by setting $k = 0$ we see that $\lambda = 0$ is an eigenvalue for all ω. As for the string problem, this eigenvalue corresponds to a rigid motion of the Couette steady solution. In this case the rigid motion is a vertical translation of the deformable cylinder.

In the previous section we found a shear instability of the Couette steady solution. This also corresponds to the case $k = 0$ since the eigenfunction is

$$(u, v, w, p, q, \zeta, \theta) = (0, 0, 0, 0, 0, 0, constant).$$

Are there any other eigenfunctions for $k = 0$? Note that the volume side condition

(5.8.11) implies that $q_0 = 0$. Substituting $q_0 = k = 0$ into (5.9.22) implies that

$$\zeta_0 \text{ is arbitrary}, \qquad a_{33}\theta_0 = 0. \tag{5.9.24}$$

These are the two eigenfunctions that we have already found, the vertical translation and the shear instability.

Denote the matrix on the left-hand side of (5.9.22) by $A(k) \equiv A(k, \omega^2)$. Observe that $A(-k) = \overline{A(k)}$. For hyperelastic materials A is also self-adjoint.

Now we consider the case $k \neq 0$. By setting the determinant of $A(k, \omega^2)$ equal to zero we obtain a cubic equation for ω^2:

$$\det A(k, \omega^2) = 0. \tag{5.9.25}$$

(Note that this equation would be quadratic in ω^2 if we had modelled the deformable body using a membrane theory rather than a shell theory.) It can be shown that the cubic equation (5.9.25) has real coefficients and the coefficient of $(\omega^2)^3$ is negative. The constant term in the polynomial does not have a sign. For some materials and some k and Z the constant term will be positive and so the cubic equation will have at least one positive solution ω^2_{crit}, which implies that $\lambda = 0$ is an eigenvalue of (5.8.13)–(5.8.18) when $\omega = \omega_{\text{crit}}$. It can also be checked that the coefficients of the cubic polynomial are functions of k^2.

In this section we have proved that all the eigenvalues λ of (5.8.13)–(5.8.18) that cross the imaginary axis cross through the origin, but have not proved anything about the way that they cross. In Section 5.11 we compute the eigenvalues numerically to find out if the bifurcation is steady state or Takens-Bogdanov (as for the string problem, see Section 2.13). We also solve equation (5.9.25) numerically to find the critical values of ω.

5.10 Weak Formulation of the Quadratic Eigenvalue Problem

Derivation

The weak formulation of the quadratic eigenvalue problem (5.8.13)–(5.8.18) is derived in a similar way to the weak formulation of the quadratic eigenvalue problem for the string problem. See Section 2.11. So we just sketch the details. Recall that Ω is the period cell for the fluid: $\Omega = \{\boldsymbol{x} = r\boldsymbol{e}_1(\phi) + z\boldsymbol{k} : a < r < R,\ 0 \le \phi < 2\pi,\ 0 \le z < Z\}$. Also recall that $\boldsymbol{e}_3 \equiv \boldsymbol{k}$. Let Γ_R and $\mathbb{T}_Z$ be period cells for the shell: $\Gamma_R := \{(s,\phi) \in [0,Z) \times [0,2\pi)\}$, $\mathbb{T}_Z := \{s \in [0,Z)\}$. Let

$$
\begin{aligned}
H^1_a(\Omega) &:= \{\boldsymbol{v} \in H^1(\Omega;\mathbb{C}^3) : \boldsymbol{v}(a\boldsymbol{e}_1 + z\boldsymbol{k}) = 0,\ \boldsymbol{v}(R\boldsymbol{e}_1(\phi) + z\boldsymbol{k}) \cdot \boldsymbol{e}_2(\phi) = 0, \\
&\qquad \text{for } j \in \{1,2,3\}, \boldsymbol{v}(r\boldsymbol{e}_1(\phi) + z\boldsymbol{k}) \cdot \boldsymbol{e}_j(\phi) \text{ is independent of } \phi\}, \\
H^1_{\mathrm{s}}(\Gamma_R) &:= \left\{\boldsymbol{r}(s,\phi) \in H^1(\Gamma_R;\mathbb{C}^3) : \int_0^Z \boldsymbol{r}(s,\phi)\cdot\boldsymbol{e}_1(\phi)\,ds = 0,\ \boldsymbol{r}(s,\phi)\cdot\boldsymbol{e}_2(\phi) = 0,\right. \\
&\qquad \left. \text{for } l \in \{1,3\}, \boldsymbol{r}(s,\phi)\cdot\boldsymbol{e}_l(\phi) \text{ is independent of } \phi\right\}, \\
\Pi &:= \left\{p \in L^2(\Omega;\mathbb{C}) : \int_\Omega p\,d\boldsymbol{x} = 0,\ p(r\boldsymbol{e}_1(\phi) + z\boldsymbol{k}) \text{ is independent of } \phi\right\}, \\
\mathcal{V}_1 &:= \{(\boldsymbol{v},\boldsymbol{r},\theta) \in H^1_a(\Omega) \times H^1_{\mathrm{s}}(\Gamma_R) \times H^1(\mathbb{T}_Z;\mathbb{C})\} \\
\mathcal{V}_2 &:= \{(\boldsymbol{w},\boldsymbol{q},\psi,\boldsymbol{t}) \in H^1_a(\Omega) \times H^1_{\mathrm{s}}(\Gamma_R) \times H^1(\mathbb{T}_Z;\mathbb{C}) \times H^{1/2}(\Gamma_R;\mathbb{C}^3) : \\
&\qquad \boldsymbol{w}(R\boldsymbol{e}_1(\phi) + s\boldsymbol{k}) = \boldsymbol{q}(s,\phi),\ \boldsymbol{t}(s,\phi)\cdot\boldsymbol{e}_2(\phi) = 0, \\
&\qquad \text{for } l \in \{1,3\}, \boldsymbol{t}(s,\phi)\cdot\boldsymbol{e}_l(\phi) \text{ is independent of } \phi\}.
\end{aligned}
\tag{5.10.1}
$$

Now we derive a weak formulation of the coupled equations (5.8.13)–(5.8.18). Replace $\boldsymbol{d}$ in (5.8.13) and (5.8.14) by $-\theta\boldsymbol{k}$. Let $(\boldsymbol{w}, \boldsymbol{q}, \psi, \boldsymbol{t}) \in \mathcal{V}_2$, $q \in \Pi$. Take the L^2-inner product of $(5.8.15)_1$ with $\boldsymbol{w}$, the L^2-inner product of $(5.8.15)_2$ with q, the L^2-inner product of (5.8.13) with $\boldsymbol{q}$, the L^2-inner product of (5.8.14) with ψ, the $H^{1/2}$-inner product of $(5.8.16)_2$ with $\boldsymbol{t}$, use integration by parts, and add all the resulting equations together to obtain the

Weak formulation of the quadratic eigenvalue problem. Find $\lambda \in \mathbb{C}$ and $0 \neq (\boldsymbol{v}, \boldsymbol{r}, \theta, p) \in \mathcal{V}_1 \times \Pi$ such that for all $(\boldsymbol{w}, \boldsymbol{q}, \psi, \boldsymbol{t}, q) \in \mathcal{V}_2 \times \Pi$

$$
\begin{aligned}
0 &= \lambda^2 a_2((\boldsymbol{r}, \theta), (\boldsymbol{q}, \psi)) + \lambda a_1((\boldsymbol{v}, \boldsymbol{r}, \theta), (\boldsymbol{w}, \boldsymbol{q}, \psi, \boldsymbol{t})) \\
&\qquad + a_0((\boldsymbol{v}, \boldsymbol{r}, \theta), (\boldsymbol{w}, \boldsymbol{q}, \psi, \boldsymbol{t})) + b(\boldsymbol{w}, p), \\
0 &= b(\boldsymbol{v}, q),
\end{aligned} \tag{5.10.2}
$$

where

$$
\begin{aligned}
&a_0((\boldsymbol{v}, \boldsymbol{r}, \theta), (\boldsymbol{w}, \boldsymbol{q}, \psi, \boldsymbol{t})) := \\
&\int_{\Gamma_R} \Big\{ -2\varrho h\omega^2 (\boldsymbol{r}\cdot\boldsymbol{e}_1)(\bar{\boldsymbol{q}}\cdot\boldsymbol{e}_1) - N_\tau^\circ(\boldsymbol{r}_s\cdot\boldsymbol{e}_1)(\bar{\boldsymbol{q}}\cdot\boldsymbol{k}) + N_\nu^\circ(\boldsymbol{r}_s\cdot\boldsymbol{k})(\bar{\boldsymbol{q}}_s\cdot\boldsymbol{k}) + N_\mu^\circ\theta_s\bar{\boldsymbol{q}}_s\cdot\boldsymbol{k} \\
&\qquad + H_\eta^\circ(\boldsymbol{r}_s\cdot\boldsymbol{e}_1)(\bar{\boldsymbol{q}}_s\cdot\boldsymbol{e}_1) + (T_\mu^\circ + N^\circ - H_\eta^\circ)\theta_s\bar{\boldsymbol{q}}\cdot\boldsymbol{e}_1 + T_\tau^\circ(\boldsymbol{r}\cdot\boldsymbol{e}_1)(\bar{\boldsymbol{q}}\cdot\boldsymbol{e}_1) \\
&\qquad + T_\nu^\circ(\boldsymbol{r}_s\cdot\boldsymbol{k})(\bar{\boldsymbol{q}}\cdot\boldsymbol{e}_1) - (\rho R^2\omega^2 + \rho P(R))(\boldsymbol{r}\cdot\boldsymbol{e}_1)(\bar{\boldsymbol{q}}\cdot\boldsymbol{e}_1) \\
&\qquad - \rho R P(R)(\boldsymbol{e}_2\times\boldsymbol{r}_s)\cdot\bar{\boldsymbol{q}} + \varrho J\omega^2\theta\bar{\psi} - \varrho I R\omega^2\theta\bar{\psi} - M_\tau^\circ\bar{\psi}\boldsymbol{r}_s\cdot\boldsymbol{e}_1 \\
&\qquad + M_\nu^\circ\bar{\psi}_s\boldsymbol{r}_s\cdot\boldsymbol{k} + M_\mu^\circ\theta_s\bar{\psi}_s - \Sigma^\circ\theta\bar{\psi} + (H_\eta^\circ - N^\circ)(\theta\bar{\psi} + \bar{\psi}\boldsymbol{r}_s\cdot\boldsymbol{e}_1)\Big\}\, ds\, d\phi \\
&\qquad + 2\int_\Omega \{\tilde{\mu}\boldsymbol{D}(\boldsymbol{v}) : \boldsymbol{D}(\bar{\boldsymbol{w}}) + \rho\omega(\boldsymbol{k}\times\boldsymbol{v})\cdot\bar{\boldsymbol{w}}\}\, d\boldsymbol{x} + \langle \boldsymbol{v}, \boldsymbol{t}\rangle_{H^{1/2}(\Gamma_R)}
\end{aligned} \tag{5.10.3}
$$

$$a_1((\boldsymbol{v},\boldsymbol{r},\theta),(\boldsymbol{w},\boldsymbol{q},\psi,\boldsymbol{t})) :=$$

$$\int_{\Gamma_R} \Big\{ N^\circ_{\dot\nu}(\boldsymbol{r}_s\cdot\boldsymbol{k})(\bar{\boldsymbol{q}}_s\cdot\boldsymbol{k}) - N^\circ_{\dot\tau}(\boldsymbol{r}_s\cdot\boldsymbol{e}_1)(\bar{\boldsymbol{q}}\cdot\boldsymbol{k}) + N^\circ_{\dot\mu}\theta_s\bar{\boldsymbol{q}}_s\cdot\boldsymbol{k} + H^\circ_{\dot\eta}(\boldsymbol{r}_s\cdot\boldsymbol{e}_1)(\bar{\boldsymbol{q}}_s\cdot\boldsymbol{e}_1)$$

$$+ (T^\circ_{\dot\mu} - H^\circ_{\dot\eta})\theta_s\bar{\boldsymbol{q}}\cdot\boldsymbol{e}_1 + T^\circ_{\dot\tau}(\boldsymbol{r}\cdot\boldsymbol{e}_1)(\bar{\boldsymbol{q}}\cdot\boldsymbol{e}_1) + T^\circ_{\dot\nu}(\boldsymbol{r}_s\cdot\boldsymbol{k})(\bar{\boldsymbol{q}}\cdot\boldsymbol{e}_1)$$

$$- M^\circ_{\dot\tau}\bar\psi\boldsymbol{r}_s\cdot\boldsymbol{e}_1 + M^\circ_{\dot\nu}\bar\psi_s\boldsymbol{r}_s\cdot\boldsymbol{k} + M^\circ_{\dot\mu}\theta_s\bar\psi_s + H^\circ_{\dot\eta}\theta\bar\psi + H^\circ_{\dot\eta}\bar\psi\boldsymbol{r}_s\cdot\boldsymbol{e}_1\Big\}\, ds\, d\phi$$

$$+ \rho\int_\Omega \boldsymbol{v}\cdot\bar{\boldsymbol{w}}\; d\boldsymbol{x} - \langle\boldsymbol{r},\boldsymbol{t}\rangle_{H^{1/2}(\Gamma_R)} \quad (5.10.4)$$

$$a_2((\boldsymbol{r},\theta),(\boldsymbol{q},\psi)) := \int_{\Gamma_R} \{2\varrho h\boldsymbol{r}\cdot\bar{\boldsymbol{q}} - \varrho I(\theta\,\bar{\boldsymbol{q}}\cdot\boldsymbol{k} + \bar\psi\,\boldsymbol{r}\cdot\boldsymbol{k}) + \varrho J\theta\bar\psi\}\; ds\, d\phi, \quad (5.10.5)$$

$$b(\boldsymbol{w},p) := -\rho\int_\Omega p\,\mathrm{div}\bar{\boldsymbol{w}}\; d\boldsymbol{x}. \quad (5.10.6)$$

The Weak Formulation in Polar Coordinates

In this section we write the weak equation (5.10.2) in polar coordinates. Decompose the functions in $\mathcal{V}_1\times\Pi$ as

$$\begin{aligned}
\boldsymbol{v}(r\boldsymbol{e}_1(\phi)+z\boldsymbol{k}) &= v^1(r,z)\boldsymbol{e}_1(\phi) + v^2(r,z)\boldsymbol{e}_2(\phi) + v^3(r,z)\boldsymbol{k},\\
\boldsymbol{r}(s,\phi) &= r^1(s)\boldsymbol{e}_1(\phi) + r^3(s)\boldsymbol{k},\\
p(r\boldsymbol{e}_1(\phi)+z\boldsymbol{k}) &= \tilde p(r,z).
\end{aligned} \quad (5.10.7)$$

Observe that, in the notation of Section 5.8, $v^1 = u$, $v^2 = v$, $v^3 = w$, $r^1 = q$, $r^3 = \zeta$, and $\tilde p = p$. Decompose the functions in $\mathcal{V}_2\times\Pi$ as

$$\begin{aligned}
\boldsymbol{w}(r\boldsymbol{e}_1(\phi)+z\boldsymbol{k}) &= w^1(r,z)\boldsymbol{e}_1(\phi) + w^2(r,z)\boldsymbol{e}_2(\phi) + w^3(r,z)\boldsymbol{k},\\
\boldsymbol{q}(s,\phi) &= q^1(s)\boldsymbol{e}_1(\phi) + q^3(s)\boldsymbol{k},\\
\boldsymbol{t}(s,\phi) &= t^1(s)\boldsymbol{e}_1(\phi) + t^3(s)\boldsymbol{k},\\
q(r\boldsymbol{e}_1(\phi)+z\boldsymbol{k}) &= \tilde q(r,z).
\end{aligned} \quad (5.10.8)$$

Now drop the tilde from $\tilde{p}$ and $\tilde{q}$. Define

$$(\mathbf{v},\mathbf{r},\theta) := (v^1, v^2, v^3, r^1, r^3, \theta), \quad (\mathbf{w},\mathbf{q},\psi,\mathbf{t}) := (w^1, w^2, w^3, q^1, q^3, \psi, t^1, t^3). \tag{5.10.9}$$

Recall that $\mathbb{T}_Z$ denotes the periodic domain $\mathbb{R}/Z\mathbb{Z}$. We obtain new function spaces V_1, V_2, and $\tilde{\Pi}$ by substituting the polar coordinates for $(\boldsymbol{v},\boldsymbol{r})$, $(\boldsymbol{w},\boldsymbol{q},\boldsymbol{t})$, and p into $\mathcal{V}_1$, $\mathcal{V}_2$, and Π:

$$\begin{aligned}
V_1 :=& \Big\{ (\mathbf{v},\mathbf{r},\theta) \in [H^1([a,R]\times\mathbb{T}_Z;\mathbb{C})]^3 \times [H^1(\mathbb{T}_Z;\mathbb{C})]^2 \times H^1(\mathbb{T}_Z;\mathbb{C}) : \\
& \qquad v^j(a,z) = 0 \text{ for } j\in\{1,2,3\},\ v^2(R,z)=0,\ \int_0^Z r^1(s)\,ds = 0 \Big\}, \\
V_2 :=& \Big\{ (\mathbf{w},\mathbf{q},\psi,\mathbf{t}) \in [H^1([a,R]\times\mathbb{T}_Z;\mathbb{C})]^3 \times [H^1(\mathbb{T}_Z;\mathbb{C})]^2 \times H^1(\mathbb{T}_Z;\mathbb{C}) \\
& \qquad \times [H^{1/2}(\mathbb{T}_Z;\mathbb{C})]^2 : \ w^j(a,z) = 0 \text{ for } j\in\{1,2,3\},\ w^2(R,z)=0, \\
& \qquad w^l(R,s) = q^l(s) \text{ for } l\in\{1,3\},\ \int_0^Z q^1(s)\,ds = 0 \Big\}, \\
\tilde{\Pi} :=& \Big\{ p\in L^2([a,R]\times\mathbb{T}_Z;\mathbb{C}) : \int_a^R \int_0^Z p\, r dr\, dz = 0 \Big\}.
\end{aligned} \tag{5.10.10}$$

Now drop the tilde from $\tilde{\Pi}$. If we substitute (5.10.7) and (5.10.8) into (5.10.2) we obtain a

Weak formulation of the quadratic eigenvalue problem in polar coordinates. Find $\lambda\in\mathbb{C}$ and $0\neq(\mathbf{v},\mathbf{r},\theta,p)\in V_1\times\Pi$ such that for all $(\mathbf{w},\mathbf{q},\psi,\mathbf{t},q)\in V_2\times\Pi$

$$\begin{aligned}
0 &= \lambda^2\tilde{a}_2((\mathbf{r},\theta),(\mathbf{q},\psi)) + \lambda\tilde{a}_1((\mathbf{v},\mathbf{r},\theta),(\mathbf{w},\mathbf{q},\psi,\mathbf{t})) \\
&\qquad + \tilde{a}_0((\mathbf{v},\mathbf{r},\theta),(\mathbf{w},\mathbf{q},\psi,\mathbf{t})) + \tilde{b}(\mathbf{w},p), \qquad (5.10.11) \\
0 &= \tilde{b}(\mathbf{v},q),
\end{aligned}$$

where

$$
\begin{aligned}
&\tilde{a}_0((\mathbf{v},\mathbf{r},\theta),(\mathbf{w},\mathbf{q},\psi,\mathbf{t})) = a_0((\boldsymbol{v},\boldsymbol{r},\theta),(\boldsymbol{w},\boldsymbol{q},\psi,\boldsymbol{t}))\\
&= 2\pi\int_0^Z \Big\{ -2\varrho h\omega^2 r^1\overline{q^1} - N^\circ_\tau r^1_s\overline{q^3} + N^\circ_\nu r^3_s\overline{q^3_s} + N^\circ_\mu\theta_s\overline{q^3_s} + (T^\circ_\mu + N^\circ - H^\circ_\eta)\theta_s\overline{q^1}\\
&\qquad + H^\circ_\eta r^1_s\overline{q^1_s} + T^\circ_\tau r^1\overline{q^1} + T^\circ_\nu r^3_s\overline{q^1} - (\rho R^2\omega^2 + \rho P(R))r^1\overline{q^1}\\
&\qquad - \rho RP(R)(r^3_s\overline{q^1} - r^1_s\overline{q^3}) + \varrho J\omega^2\theta\overline{\psi} - \varrho IR\omega^2\theta\overline{\psi} - M^\circ_\tau r^1_s\overline{\psi}\\
&\qquad + M^\circ_\nu r^3_s\overline{\psi}_s + M^\circ_\mu\theta_s\overline{\psi}_s - \varSigma^\circ\theta\overline{\psi} + (H^\circ_\eta - N^\circ)(\theta\overline{\psi} + r^1_s\overline{\psi})\Big\}\,ds\\
&\quad + 4\pi\int_a^R\int_0^Z \Big\{\tilde{\mu}[v^1_r\overline{w^1_r} + \tfrac{1}{2}(v^2_r - \tfrac{1}{r}v^2)(\overline{w^2_r} - \tfrac{1}{r}\overline{w^2}) + \tfrac{1}{2}(v^3_r + v^1_z)(\overline{w^3_r} + \overline{w^1_z})\\
&\qquad + \tfrac{1}{2}v^2_z\overline{w^2_z} + \tfrac{1}{r^2}v^1\overline{w^1} + v^3_z\overline{w^3_z}] + \rho\omega(v^1\overline{w^2} - v^2\overline{w^1})\Big\}\,rdr\,dz\\
&\qquad + \langle v^1(R,s)\boldsymbol{e}_1 + v^3(R,s)\boldsymbol{k},\ t^1(s)\boldsymbol{e}_1 + t^3(s)\boldsymbol{k}\rangle_{H^{1/2}(\mathbb{T}_Z)},
\end{aligned}
\tag{5.10.12}
$$

$$
\begin{aligned}
&\tilde{a}_1((\mathbf{v},\mathbf{r},\theta),(\mathbf{w},\mathbf{q},\psi,\mathbf{t})) = a_1((\boldsymbol{v},\boldsymbol{r},\theta),(\boldsymbol{w},\boldsymbol{q},\psi,\boldsymbol{t}))\\
&= 2\pi\int_0^Z \Big\{N^\circ_{\dot\nu} r^3_s\overline{q^3_s} - N^\circ_{\dot\tau} r^1_s\overline{q^3} + N^\circ_{\dot\mu}\theta_s\overline{q^3_s} + H^\circ_{\dot\eta} r^1_s\overline{q^1_s} + (T^\circ_{\dot\mu} - H^\circ_{\dot\eta})\theta_s\overline{q^1} + T^\circ_{\dot\tau} r^1\overline{q^1}\\
&\qquad + T^\circ_{\dot\nu} r^3_s\overline{q^1} - M^\circ_{\dot\tau} r^1_s\overline{\psi} + M^\circ_{\dot\nu} r^3_s\overline{\psi_s} + M^\circ_{\dot\mu}\theta_s\overline{\psi_s} + H^\circ_{\dot\eta}\theta\overline{\psi} + H^\circ_{\dot\eta} r^1_s\overline{\psi}\Big\}\,ds\\
&\quad + 2\pi\rho\int_a^R\int_0^Z\{v^1\overline{w^1} + v^2\overline{w^2} + v^3\overline{w^3}\}\,rdr\,dz - \langle r^1\boldsymbol{e}_1 + r^3\boldsymbol{k}, t^1\boldsymbol{e}_1 + t^3\boldsymbol{e}_3\rangle_{H^{1/2}(\mathbb{T}_Z)},
\end{aligned}
\tag{5.10.13}
$$

$$
\begin{aligned}
\tilde{a}_2((\mathbf{r},\theta),(\mathbf{q},\psi)) &= a_2((\boldsymbol{r},\theta),(\boldsymbol{q},\psi))\\
&= 2\pi\int_0^Z\{2\varrho h(r^1\overline{q^1} + r^3\overline{q^3}) - \varrho I(\theta\overline{q^3} + r^3\overline{\psi}) + \varrho J\theta\overline{\psi}\}\,ds,
\end{aligned}
\tag{5.10.14}
$$

$$
\tilde{b}(\mathbf{w},p) = b(\boldsymbol{w},p) = -2\pi\rho\int_a^R\int_0^Z p(\overline{w^1_r} + \tfrac{1}{r}\overline{w^1} + \overline{w^3_z})\,rdr\,dz. \tag{5.10.15}
$$

Fourier Decomposition and a Family of Weak Problems

In this section we expand the functions in V_1, V_2, and Π as Fourier series in the axial variable (z or s) and use this to generate a family of weak problems indexed by the Fourier wave number.

For $j \in \{1,2,3\}$, $l \in \{1,3\}$ decompose

$$v^j(r,z) = \sum_{k=-\infty}^{\infty} v_k^j(r) e^{2\pi ikz/Z}, \qquad r^l(s) = \sum_{k=-\infty}^{\infty} r_k^l e^{2\pi iks/Z}, \tag{5.10.16}$$

$$\theta(s) = \sum_{k=-\infty}^{\infty} \theta_k e^{2\pi iks/Z}, \qquad p(r,z) = \sum_{k=-\infty}^{\infty} p_k(r) e^{2\pi ikz/Z} \tag{5.10.17}$$

$$w^j(r,z) = \sum_{k=-\infty}^{\infty} w_k^j(r) e^{2\pi ikz/Z}, \qquad q^l(s) = \sum_{k=-\infty}^{\infty} q_k^l e^{2\pi iks/Z}, \tag{5.10.18}$$

$$t^l(s) = \sum_{k=-\infty}^{\infty} t_k^l e^{2\pi iks/Z}, \qquad \psi(s) = \sum_{k=-\infty}^{\infty} \psi_k e^{2\pi iks/Z}. \tag{5.10.19}$$

Define

$$(\mathbf{v}_k, \mathbf{r}_k, \theta_k) := (v_k^1, v_k^2, v_k^3, r_k^1, r_k^3, \theta_k),$$

$$(\mathbf{w}_k, \mathbf{q}_k, \psi_k, \mathbf{t}_k) := (w_k^1, w_k^2, w_k^3, q_k^1, q_k^3, \psi_k, t_k^1, t_k^3).$$

We define a family of spaces indexed by the Fourier wave number $k \in \mathbb{Z}$. For $k \neq 0$

$$\begin{aligned}
V_1^k &:= \{(\mathbf{v}_k, \mathbf{r}_k, \theta_k) \in [H^1([a,R];\mathbb{C})]^3 \times \mathbb{C}^2 \times \mathbb{C} : \\
&\qquad\qquad v_k^j(a) = 0 \text{ for } j \in \{1,2,3\},\ v_k^2(R) = 0\}, \\
V_2^k &:= \{(\mathbf{w}_k, \mathbf{q}_k, \psi_k, \mathbf{t}_k) \in [H^1([a,R];\mathbb{C})]^3 \times \mathbb{C}^2 \times \mathbb{C} \times \mathbb{C}^2 : \\
&\qquad w_k^j(a) = 0 \text{ for } j \in \{1,2,3\},\ w_k^l(R) = q_k^l \text{ for } l \in \{1,3\},\ w_k^2(R) = 0\}, \\
\Pi^k &:= L^2([a,R];\mathbb{C}).
\end{aligned} \tag{5.10.20}$$

(Note that these spaces are independent of k.) For $k = 0$

$$V_1^0 := \{(\mathbf{v}_0, \mathbf{r}_0, \theta_0) \in [H^1([a,R];\mathbb{C})]^3 \times \mathbb{C}^2 \times \mathbb{C} :$$

$$v_0^j(a) = 0 \text{ for } j \in \{1,2,3\},\ v_0^2(R) = 0,\ r_0^1 = 0\},$$

$$V_2^0 := \{(\mathbf{w}_0, \mathbf{q}_0, \psi_0, \mathbf{t}_0) \in [H^1([a,R];\mathbb{C})]^3 \times \mathbb{C}^2 \times \mathbb{C} \times \mathbb{C}^2 :$$

$$w_0^j(a) = 0 \text{ for } j \in \{1,2,3\},\ w_0^l(R) = q_0^l \text{ for } l \in \{1,2\},\ w_0^2(R) = 0,\ q_0^1 = 0\},$$

$$\Pi^0 := \left\{ p \in L^2([a,R];\mathbb{C}) : \int_a^R p(r)\, r dr = 0 \right\}.$$

Let $k \in \mathbb{Z}$, $(\mathbf{w}_k, \mathbf{q}_k, \psi_k, \mathbf{t}_k) \in V_2^k$, $q_k \in \Pi^k$. Substitute into (5.10.11) the Fourier decompositions (5.10.16) and (5.10.17) and

$$w^j(r,z) = w_k^j(r) e^{2\pi i k z/Z}, \quad q^l(s) = q_k^l e^{2\pi i k s/Z}, \quad \psi(s) = \psi_k e^{2\pi i k s/Z}, \tag{5.10.21}$$

$$t^l(s) = t_k^l e^{2\pi i k s/Z}, \quad q(r,z) = q_k(r) e^{2\pi i k z/Z} \tag{5.10.22}$$

to obtain the following family of weak problems:

A family of weak problems indexed by the Fourier wave number. For each $k \in \mathbb{Z}$, find $\lambda \in \mathbb{C}$ and $0 \neq (\mathbf{v}_k, \mathbf{r}_k, \theta_k, p_k) \in V_1^k \times \Pi^k$ such that for all $(\mathbf{w}_k, \mathbf{q}_k, \psi_k, \mathbf{t}_k, q_k) \in V_2^k \times \Pi^k$

$$\begin{aligned} 0 &= \lambda^2 a_2^k((\mathbf{r}_k, \theta_k), (\mathbf{q}_k, \psi_k)) + \lambda a_1^k((\mathbf{v}_k, \mathbf{r}_k, \theta_k), (\mathbf{w}_k, \mathbf{q}_k, \psi_k, \mathbf{t}_k)) \\ &\qquad + a_0^k((\mathbf{v}_k, \mathbf{r}_k, \theta_k), (\mathbf{w}_k, \mathbf{q}_k, \psi_k, \mathbf{t}_k)) + b^k(\mathbf{w}_k, p_k), \\ 0 &= b^k(\mathbf{v}_k, q_k), \end{aligned} \tag{5.10.23}$$

where

$$
\begin{aligned}
&a_0^k((\mathbf{v}_k, \mathbf{r}_k, \theta_k), (\mathbf{w}_k, \mathbf{q}_k, \psi_k, \mathbf{t}_k)) \\
&\quad = -2\varrho h\omega^2 r_k^1 \overline{q_k^1} - 2\pi ikZ^{-1} N_\tau^\circ r_k^1 \overline{q_k^3} + 4\pi^2k^2Z^{-2}N_\nu^\circ r_k^3\overline{q_k^3} + 4\pi^2k^2Z^{-2}N_\mu^\circ \theta_k \overline{q_k^3} \\
&\qquad + 4\pi^2k^2Z^{-2}H_\eta^\circ r_k^1\overline{q_k^1} + 2\pi ikZ^{-1}(T_\mu^\circ + N^\circ - H_\eta^\circ)\theta_k\overline{q_k^1} + T_\tau^\circ r_k^1 \overline{q_k^1} \\
&\qquad + 2\pi ikZ^{-1}T_\nu^\circ r_s^3 \overline{q_k^1} - (\rho R^2\omega^2 + \rho P(R)) r_k^1\overline{q_k^1} + \varrho J\omega^2\theta_k\overline{\psi_k} \\
&\qquad - 2\pi ikZ^{-1}\rho RP(R)(r_k^3\overline{q_k^1} - r_k^1\overline{q_k^3}) - \varrho IR\omega^2\theta_k\overline{\psi_k} - 2\pi ikZ^{-1}M_\tau^\circ r_k^1\overline{\psi_k} \\
&\qquad + 4\pi^2k^2Z^{-2}M_\nu^\circ r_k^3\overline{\psi}_k + 4\pi^2k^2Z^{-2}M_\mu^\circ\theta_k\overline{\psi}_k - \varSigma^\circ\theta_k\overline{\psi_k} \\
&\qquad + (H_\eta^\circ - N^\circ)(\theta_k\overline{\psi}_k + 2\pi ikZ^{-1}r_k^1\overline{\psi_k}) \\
&+ 2\int_a^R \Big\{\tilde{\mu}[(v_k^1)_r\overline{(w_k^1)_r} + \tfrac{1}{2}((v_k^2)_r - \tfrac{1}{r}v_k^2)(\overline{(w_k^2)_r} - \tfrac{1}{r}\overline{w_k^2}) \\
&\qquad + \tfrac{1}{2}((v_k^3)_r + 2\pi ikZ^{-1}v_k^1)(\overline{(w_k^3)_r} - 2\pi ikZ^{-1}\overline{w_k^1}) + 2\pi^2k^2Z^{-2}v_k^2\overline{w_k^2} \\
&\qquad + \tfrac{1}{r^2}v_k^1\overline{w_k^1} + 4\pi^2k^2Z^{-2}v_k^3\overline{w_k^3}] + \rho\omega(v_k^1\overline{w_k^2} - v_k^2\overline{w_k^1})\Big\}\, rdr \\
&\qquad + (1 + 2\pi|k|Z^{-1})(v_k^1(R)\overline{t_k^1} + v_k^3(R)\overline{t_k^3}),
\end{aligned}
\tag{5.10.24}
$$

$$
\begin{aligned}
&a_1^k((\mathbf{v}_k, \mathbf{r}_k, \theta_k), (\mathbf{w}_k, \mathbf{q}_k, \psi_k, \mathbf{t}_k)) \\
&\quad = 4\pi^2k^2Z^{-2}N_{\dot\nu}^\circ r_k^3\overline{q_k^3} - 2\pi ikZ^{-1}N_{\dot\tau}^\circ r_k^1\overline{q_k^3} + 4\pi^2k^2Z^{-2}N_{\dot\mu}^\circ\theta_k\overline{q_k^3} + 4\pi^2k^2Z^{-2}H_{\dot\eta}^\circ r_k^1\overline{q_k^1} \\
&\qquad + 2\pi ikZ^{-1}(T_{\dot\mu}^\circ - H_{\dot\eta}^\circ)\theta_k\overline{q_k^1} + T_{\dot\tau}^\circ r_k^1\overline{q_k^1} + 2\pi ikZ^{-1}T_{\dot\nu}^\circ r_k^3\overline{q_k^1} - 2\pi ikZ^{-1}M_{\dot\tau}^\circ r_k^1\overline{\psi_k} \\
&\qquad + 4\pi^2k^2Z^{-2}M_{\dot\nu}^\circ r_k^3\overline{\psi_k} + 4\pi^2k^2Z^{-2}M_{\dot\mu}^\circ\theta_k\overline{\psi_k} + H_{\dot\eta}^\circ\theta_k\overline{\psi_k} + 2\pi ikZ^{-1}H_{\dot\eta}^\circ r_k^1\overline{\psi_k} \\
&\qquad + \rho\int_a^R\{v_k^1\overline{w_k^1} + v_k^2\overline{w_k^2} + v_k^3\overline{w_k^3}\}\, rdr - (1 + 2\pi|k|Z^{-1})(r_k^1\overline{t_k^1} + r_k^3\overline{t_k^3}),
\end{aligned}
\tag{5.10.25}
$$

$$
a_2^k((\mathbf{r}_k, \theta_k), (\mathbf{q}_k, \psi_k)) \;=\; 2\varrho h(r_k^1\overline{q_k^1} + r_k^3\overline{q_k^3}) \;-\; \varrho I(\theta_k\overline{q_k^3} + r_k^3\overline{\psi_k}) \;+\; \varrho J\theta_k\overline{\psi_k}, \tag{5.10.26}
$$

$$
b^k(\mathbf{w}_k, p_k) = -\rho\int_a^R p_k(\overline{(w_k^1)_r} + \tfrac{1}{r}\overline{w_k^1} - 2\pi ikZ^{-1}\overline{w_k^3})\, rdr. \tag{5.10.27}
$$

In Chapter 2 we carefully characterized the spectrum of the quadratic eigenvalue problem, proved continuous and discrete inf-sup conditions for the bilinear forms appearing in the weak formulation, and proved that the numerical method converged. The same can be done the quadratic eigenvalue problem given in this chapter. We do not pause to do it, however, but move quickly on to the computation of the spectrum.

5.11 Computation of the Spectrum

Numerical Method. The quadratic eigenvalue problem (5.10.23) was discretized using the finite element method with Taylor-Hood elements for the fluid. The resulting matrix quadratic eigenvalue problem was solved using the MATLAB function *polyeig*. This is the same method that was used in Chapter 2 and the reader should refer there for more details.

Constitutive Functions. Up until now we have been working with a broad class of nonlinear constitutive functions. To compute the spectrum we must limit our attention to one set of constitutive functions $\{\hat{T}, \hat{N}, \hat{H}, \hat{\Sigma}, \hat{M}\}$. We use the following recipe to derive constitutive functions for the shell from an energy for a 3-dimensional hyperelastic body:

Step 1: We start by modelling the deformable cylinder as a 3-dimensional hyperelastic body rather than an axisymmetric shell. Let $\boldsymbol{p}(\boldsymbol{x}, t)$ denote the position of material point $\boldsymbol{x}$ at time t, $\boldsymbol{F} = \partial \boldsymbol{p}/\partial \boldsymbol{x}$ be the deformation gradient, $\boldsymbol{C} = \boldsymbol{F}^* \cdot \boldsymbol{F}$ be the Cauchy-Green deformation tensor, $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I})$ be the material strain tensor, and $\Phi(\boldsymbol{C}, \boldsymbol{x})$ be the stored energy potential (in place of the more usual notation W).

We must choose an energy Φ to make further progress. The function Φ cannot be strictly convex in $\boldsymbol{F}$ else the Principle of Frame-Indifference is violated, the convex function Φ blows up on the boundary of the nonconvex set $\{\boldsymbol{F} : \det(\boldsymbol{F}) > 0\}$ of orientation preserving deformations, and steady state problems in elasticity have unique solutions, which prohibits buckling. On the other hand, the function Φ should be rank-one convex so that the linear momentum equation for the 2-dimensional elastic body is hyperbolic. We could require Φ to be quasiconvex in $\boldsymbol{F}$ so that the energy functional for steady state problems has a minimizer. In practice the definition of quasiconvexity is too difficult to verify. Instead we could choose a polyconvex energy Φ, which implies quasiconvexity. Polyconvexity means that Φ is a convex function of the minors of $\boldsymbol{F}$, which are $\boldsymbol{F}$, $\det(\boldsymbol{F})$, and $\operatorname{cof}(\boldsymbol{F})$.

If we assume that the material is uniform and isotropic, then Φ can only depend on the principal invariants of $\boldsymbol{C}$: $\operatorname{tr}(\boldsymbol{C})$, $\det(\boldsymbol{C})$, and $\frac{1}{2}[(\operatorname{tr}(\boldsymbol{C})^2 - \operatorname{tr}(\boldsymbol{C}^2)]$. If we make the further assumption that the second Piola-Kirchhoff stress tensor $\boldsymbol{S} \equiv 2\,\partial\Phi/\partial\boldsymbol{C}$ is linear in $\boldsymbol{C}$, then Φ must have the form

$$\Phi(\boldsymbol{C}) = \tfrac{1}{2}\Lambda(\operatorname{tr}\boldsymbol{E})^2 + G\,\boldsymbol{E} : \boldsymbol{E} \tag{5.11.1}$$

for constants Λ and G. See Antman (2005, Chapters 12 & 13). We call Λ and G the Lamé constants. G is also known as the shear modulus. From experimental observations Λ and G have been related to the elastic modulus $E > 0$ and the

Poisson Ratio $0 < \upsilon < 1/2$ by

$$\Lambda = \frac{E\upsilon}{(1+\upsilon)(1-2\upsilon)}, \qquad G = \frac{E}{2(1+\upsilon)}. \tag{5.11.2}$$

For nearly incompressible materials υ is very close to 1/2.

For our computations we chose the energy given in equation (5.11.1). Note that this energy does not penalize compression, but since we only consider the linearization of the constitutive functions about a stretched state $(\tau, \nu, \eta, \sigma, \mu) = (R, 1, 0, 1, 0)$ we do not need an accurate model of the energy for materials under compression.

Step 2: Now we write the position $\boldsymbol{p}$ of the thin 3-dimensional hyperelastic cylinder in terms of the configration $\{\boldsymbol{r}, \boldsymbol{d}\}$ of the axisymmetric shell introducted in Section 5.2. Let the 3-dimensional body consist of material points of the form

$$\boldsymbol{x} = \boldsymbol{r}^\circ + \xi \boldsymbol{d}^\circ = (\boldsymbol{e}_1(\phi) + s\boldsymbol{k}) + \xi(-\boldsymbol{e}_1(\phi)) = (1-\xi)\boldsymbol{e}_1(\phi) + s\boldsymbol{k} \tag{5.11.3}$$

for $(s, \phi, \xi) \in (-\infty, \infty) \times [0, 2\pi) \times [-2h, 0]$. Thus the reference configuration of the body is a circular cylinder or inner radius 1, thickness $2h$, and infinite length. We consider motions of the body in which the material point with coordinates (s, ϕ, ξ) is constrained so that its position at time t has the form

$$\boldsymbol{p}(\boldsymbol{x}, t) = \boldsymbol{r}(s, \phi, t) + \xi \boldsymbol{d}(s, \phi, t), \tag{5.11.4}$$

where $\{\boldsymbol{r}, \boldsymbol{d}\}$ were introduced in Section 5.2. Thus

$$\boldsymbol{p}(\boldsymbol{x}, t) = (q\boldsymbol{e}_1 + \zeta\boldsymbol{k}) + \xi(-\sin\theta\,\boldsymbol{e}_1 + \cos\theta\,\boldsymbol{k}) = (q - \xi\sin\theta)\boldsymbol{e}_1 + (\zeta + \xi\cos\theta)\boldsymbol{k}, \tag{5.11.5}$$

where $\boldsymbol{e}_1$ has argument $\phi + \omega t$.

Step 3: Compute $\boldsymbol{F}$, $\boldsymbol{C}$, and $\boldsymbol{E}$ from equation (5.11.5). $\boldsymbol{E}$ can be written in terms of the strains τ, ν, η, σ and μ and the thickness variable ξ. By substituting $\boldsymbol{E}$ into equation (5.11.1) we obtain an expression for the stored energy potential Φ in terms of the strains and the thickness variable.

Step 4: The terms of Φ contain powers of the thickness variable ξ. Since the thickness is small, we neglect terms containing cubic and greater powers of ξ. We intergrate Φ through the thickness to obtain a stored energy potential $\tilde{W}$ for the shell:

$$\tilde{W}(\tau, \nu, \eta, \sigma, \mu) = \int_{-2h}^{0} \Phi(\tau, \nu, \eta, \sigma, \mu, \xi)\,(1-\xi)\,d\xi. \tag{5.11.6}$$

The factor $(1-\xi)$ in the integral is the Jacobian of the map $(s, \phi, \xi) \mapsto \boldsymbol{x} = (1-\xi)\boldsymbol{e}_1(\phi) + s\boldsymbol{k}$. To simplify the integration we replace the expression $(1-\xi)$ by 1 wherever it appears ($(1-\xi)$ appears in the denominator of several terms).

Step 5: We modify a few of the terms of $\tilde{W}$ to ensure that $\tilde{W} \geq 0$ and $\tilde{W} = 0$ in the reference configuration. (This ensures that the stresses $\hat{T}$, $\hat{N}$, $\hat{H}$, $\hat{\Sigma}$, and $\hat{M}$ vanish in the reference configuration.) We also modify $\tilde{W}$ so that that energy is isotropic, i.e., invariant under the transformation $(\tau, \nu, \eta, \sigma, \mu) \mapsto (\nu, \tau, \eta, \mu, \sigma)$. We arrive at the energy

$$
\begin{aligned}
W =& \tfrac{1}{2}\Lambda\{\tfrac{1}{2}h[(\tau^2-1)+(\nu^2-1)+\eta^2]^2 \\
&\quad + \tfrac{4}{3}h^3[\eta^2(\sigma^2+\mu^2)+\sigma^2(3(\tau-1)^2+(\nu-1)^2)+\mu^2(3(\nu-1)^2+(\tau-1)^2)]\} \\
&+ G\big\{\tfrac{1}{2}h[(\tau^2-1)^2+(\nu^2-1)^2+\eta^4+2\eta^2(\tau^2+\nu^2)] \\
&\quad + \tfrac{4}{3}h^3[\eta^2(\sigma^2+\mu^2)+3\sigma^2(\tau-1)^2+3\mu^2(\nu-1)^2]\big\}.
\end{aligned}
\tag{5.11.7}
$$

Step 6: Finally, we define constitutive functions

$$
\hat{T} := W_\tau, \quad \hat{N} := W_\nu, \quad \hat{H} := W_\eta, \quad \hat{\Sigma} := W_\sigma, \quad \hat{M} := W_\mu. \tag{5.11.8}
$$

These are the constitutive functions used in the computation.

Numerical Constants. In addition to choosing constitutive functions we must also choose values for all the numerical constants. These are listed in Table (5.11.1). We chose the fluid to be water and the deformable body to be a soft, nearly incompressible, rubber-like material. The ratio of the radius of the inner cylinder to the radius of the outer cylinder is close to the value used by G.I. Taylor in his experiments on the classical Taylor-Couette problem in the 1920s.

We must also assign values to ϱh, ϱI, and ϱJ. This requires motivation from 3-dimensional elasticity and is similar to the way that the constitutive functions were derived. (Also see Section 3.2.) As above, consider the deformable cylinder to be a 3-dimensional elastic body whose motion has the restricted form (5.11.4). Let the cylinder have mass density ϱ. Then the time derivatives of the linear and angular momenta per unit of s and ϕ of the cylinder are

$$
\begin{aligned}
&\frac{d}{dt}\int_{-2h}^{0} \varrho \boldsymbol{p}_t(s,\phi,\xi,t)(1-\xi)\,d\xi \\
&= \int_{-2h}^{0} \varrho[\boldsymbol{r}_{tt}(s,\phi,t) + \xi \boldsymbol{d}_{tt}(s,\phi,t)](1-\xi)\,d\xi \\
&=: 2\varrho h \boldsymbol{r}_{tt}(s,\phi,t) + \varrho I\,\boldsymbol{d}_{tt}(s,\phi,t), \\
&\frac{d}{dt}\int_{-2h}^{0} \varrho \boldsymbol{p}(s,\phi,\xi,t) \times \boldsymbol{p}_t(s,\phi,\xi,t)](1-\xi)\,d\xi \\
&= \int_{-2h}^{0} \varrho[\boldsymbol{p}(s,\phi,\xi,t) \times \boldsymbol{p}_{tt}(s,\phi,\xi,t)](1-\xi)\,d\xi \\
&= \int_{-2h}^{0} \varrho[\boldsymbol{r}(s,\phi,t) + \xi \boldsymbol{d}(s,\phi,t)] \times [\boldsymbol{r}_{tt}(s,\phi,t) + \xi \boldsymbol{d}_{tt}(s,\phi,t)](1-\xi)\,d\xi \\
&=: 2\varrho h[\boldsymbol{r}(s,\phi,t) \times \boldsymbol{r}_{tt}(s,\phi,t)] + \varrho I[\boldsymbol{r}(s,\phi,t) \times \boldsymbol{d}_{tt}(s,\phi,t)] \\
&\quad + \varrho I[\boldsymbol{d}(s,\phi,t) \times \boldsymbol{r}_{tt}(s,\phi,t)] + \varrho J[\boldsymbol{d}(s,\phi,t) \times \boldsymbol{d}_{tt}(s,\phi,t)]
\end{aligned}
$$

Radius of Rigid Cylinder	a	0.75 m
Radius of Deformable Cylinder	R	1.01 m
Density of Water	ρ	1000 kg/m^3
Dynamic Viscosity of Water	$\tilde{\mu}$	1.002×10^3 kg/ms
Thickness of Deformable Cylinder	h	$2\pi/1000$ m
Density of Rubber	ϱ	920 kg/m^3
Modulus of Elasticity of Rubber	E	0.01 GPa
Poisson's Ratio of Rubber	υ	0.49

Table 5.11.1: Values of the numerical constants used for the computation.

with

$$\varrho h = \varrho \int_{-2h}^{0} (1-\xi)\, d\xi, \quad \varrho I = \varrho \int_{-2h}^{0} \xi(1-\xi)\, d\xi, \quad \varrho J = \varrho \int_{-2h}^{0} \xi^2(1-\xi)\, d\xi. \quad (5.11.9)$$

The factor $(1-\xi)$ in the integrals is the Jacobian of the map $(s, \phi, \xi) \mapsto \boldsymbol{x} = (1-\xi)\boldsymbol{e}_1(\phi) + s\boldsymbol{k}$. When we derived the constitutive functions we replaced the expression $(1-\xi)$ with 1 everywhere it appeared. If we do the same in equation (5.11.9) we find that

$$\varrho h = \varrho\, h, \qquad \varrho I = 0, \qquad \varrho J = \tfrac{8}{3}\varrho h^3. \qquad (5.11.10)$$

These are the values that we use in the computation. Note that this technique for deriving ϱh, ϱI, and ϱJ is also used to derive the linear and angular momentum terms in the shell equations. (The body force terms can be derived from a free-body diagram without appealing to 3-dimensional elasticity.)

Results. In Section 5.9 we proved that all the eigenvalues λ that cross the imaginary axis cross through the origin, but we could not prove anything about the way that they crossed. By numerically solving equation (5.10.23) for a range of angular velocities using the method and constitutive functions described above we discover that the first eigenvalue to cross the imaginary axis is real. Therefore we have a steady state bifurcation. See Figures (5.11.1) and (5.11.2). We proved in Section 5.9 that the eigenfunction $\boldsymbol{v}$ equals 0 when $\lambda = 0$. Thus as ω crosses its critical value ω_{crit} the rigid Couette solution destabilizes into a new steady solution where the deformable shell is buckled, but the fluid steamlines are still concentric circles. This is a new phenomenon not observed in Chapters 2–4 or in the classical Taylor-Couette probelm.

For the classical Taylor-Couette problem, rigid Couette flow (where both cylinders have equal angular velocity) is linearly and nonlinearly stable for all angular velocities. This can be proved using the energy method. See Joseph (1976). Numerical observations suggest that increasing the elastic modulus E and shear modulus G of the deformable cylinder increases the critical value of ω. Thus by making the deformable cylinder rigid, by taking the limit $E, G \to \infty$, we recover the global linear stability of the classical rigid Couette flow.

In Section 5.9 we proved that a shear instability occurs if $\Sigma^\circ - H^\circ_\eta + N^\circ > 0$. For the constitutive functions and numerical constants defined above, this condition does not hold.

The first Fourier modes to become unstable are $|k| = 1$, and the Fourier modes become unstable in order, i.e., $\omega_{\text{crit}}(|k_2|) > \omega_{\text{crit}}(|k_1|)$ if $k_2 > k_1$. This can be seen from Figure (5.11.3) by fixing Z. (Note that this graph is not valid for $k = 0$, in which case $\lambda = 0$ is an eigenvalue for all ω, corresponding to a vertical translation of the shell. See Section 5.9.) Observe that, while Figure (5.11.3) plots ω_{crit} against k/Z, for our computations we fix $Z = 10$ and vary k.

By numerically solving equation (5.9.25), the bicubic equation for ω_{crit}, we find that each Fourier mode k has only one eigenvalue λ that crosses the imaginary axis. See Figure (5.11.4).

Finally, we remark that our results apply to the following finite length cylinder problem. Start with a rigid cylinder of radius $a < 1$ and finite length Z. To each end of the rigid cylinder weld a rigid disk of radius $R > 1$. Then attach to the edges of the rigid disks the ends of a deformable cylinder of natural radius 1 and natural height Z. Fill the annular region that has been created with a viscous incompressible fluid and rotate the rigid disks at angular velocity ω, which in turn cause the rigid inner cylinder and the ends of the deformable cylinder to rotate at angular velocity ω. This finite length cylinder problem can be described by the same equations of motion as the periodic, infinite length cylinder problem discussed in this chapter, except that the boundary conditions are slightly different. Both systems admit the same rigid Couette steady solution. By linearising the finite length problem about the rigid Couette solution, seeking normal modes, and writing in a weak formulation, we arrive at same weak formulation (5.10.23) as the periodic, infinite length problem, except that there is no constant Fourier mode $k = 0$. All our analytical and numerical results except those concerning the Fourier mode $k = 0$ apply to the finite length cylinder problem.

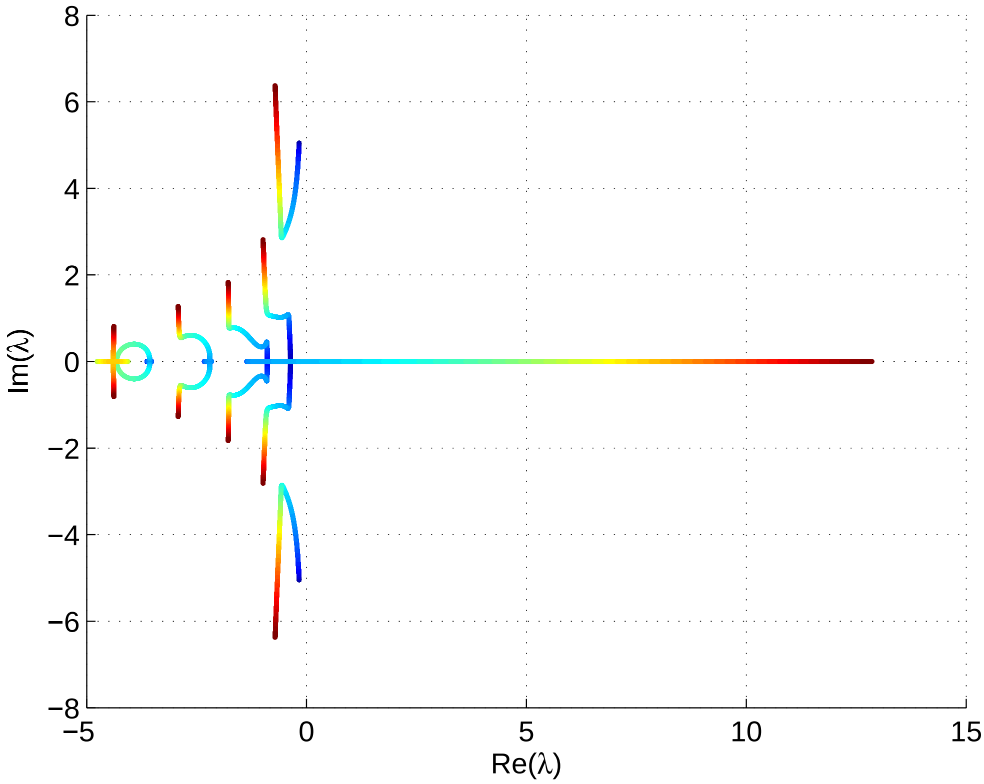

Figure 5.11.1: Trajectories of the top 11 eigenvalues λ (sorted by decreasing real part) for angular velocities $\omega \in [0, 50]$, Fourier mode $k = 1$, and axial period $Z = 10$. The color of each trajectory changes from blue to red as ω changes from 0 to 50 (in increments of 0.01). Observe that a steady state bifurcation takes place. The domain $[a, R]$ of the fluid velocity and pressure was partitioned with $N = 25$ equally spaced mesh points.

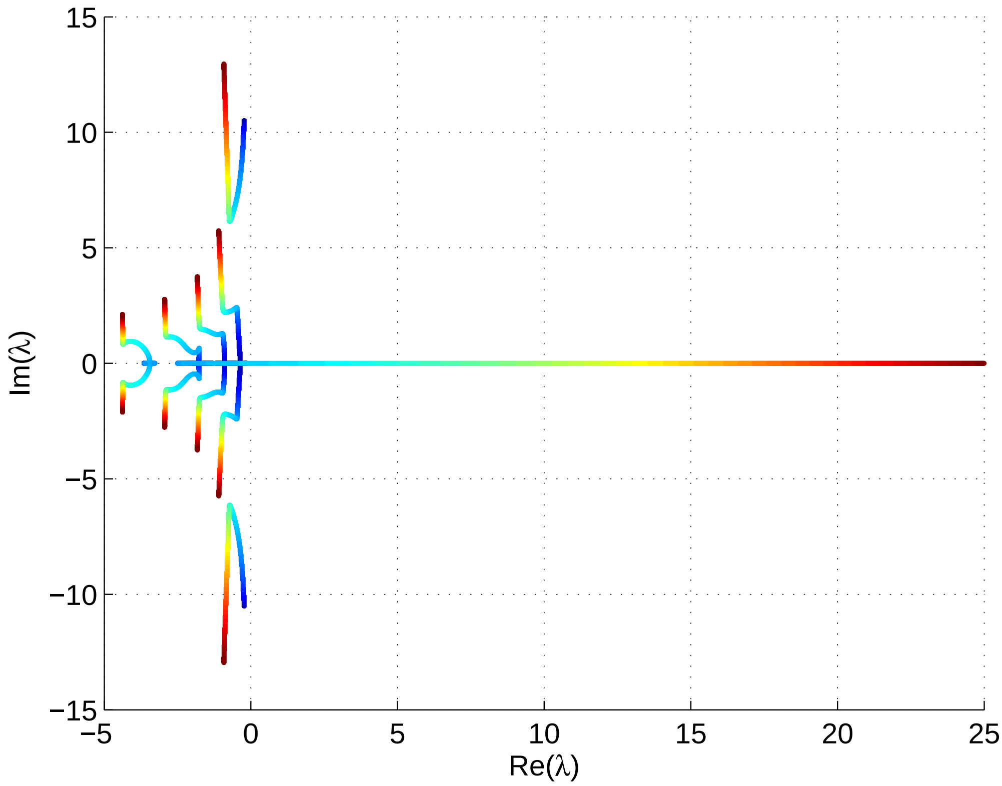

Figure 5.11.2: Trajectories of the top 11 eigenvalues λ (sorted by decreasing real part) for angular velocities $\omega \in [0, 50]$, Fourier mode $k = 2$, and axial period $Z = 10$. The color of each trajectory changes from blue to red as ω changes from 0 to 50.

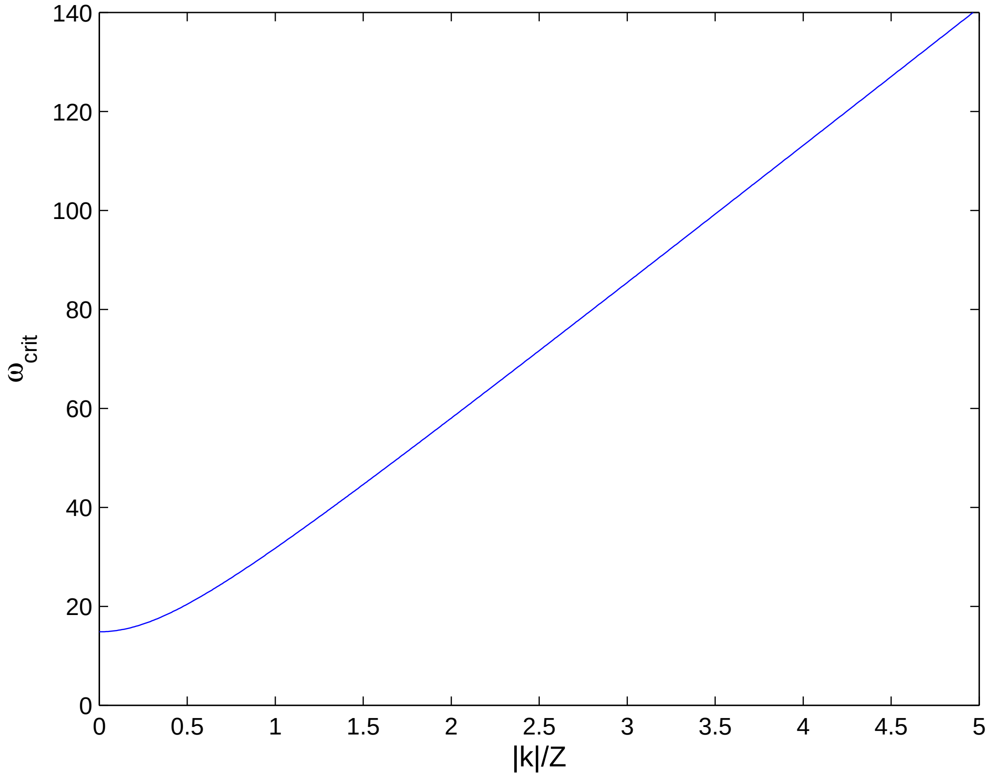

Figure 5.11.3: The critical value of ω, ω_{crit}, as a function of $|k|/Z$. We compute ω_{crit} by numerically solving the bicubic equation (5.9.25). By fixing Z we see from this graph that the Fourier modes become unstable in order. Also note that $\omega_{\text{crit}} \to \infty$ as $k/Z \to \infty$. This is to be expected since the bending of the shell increases and so the stored energy of the shell increases as $|k|/Z$ increases.

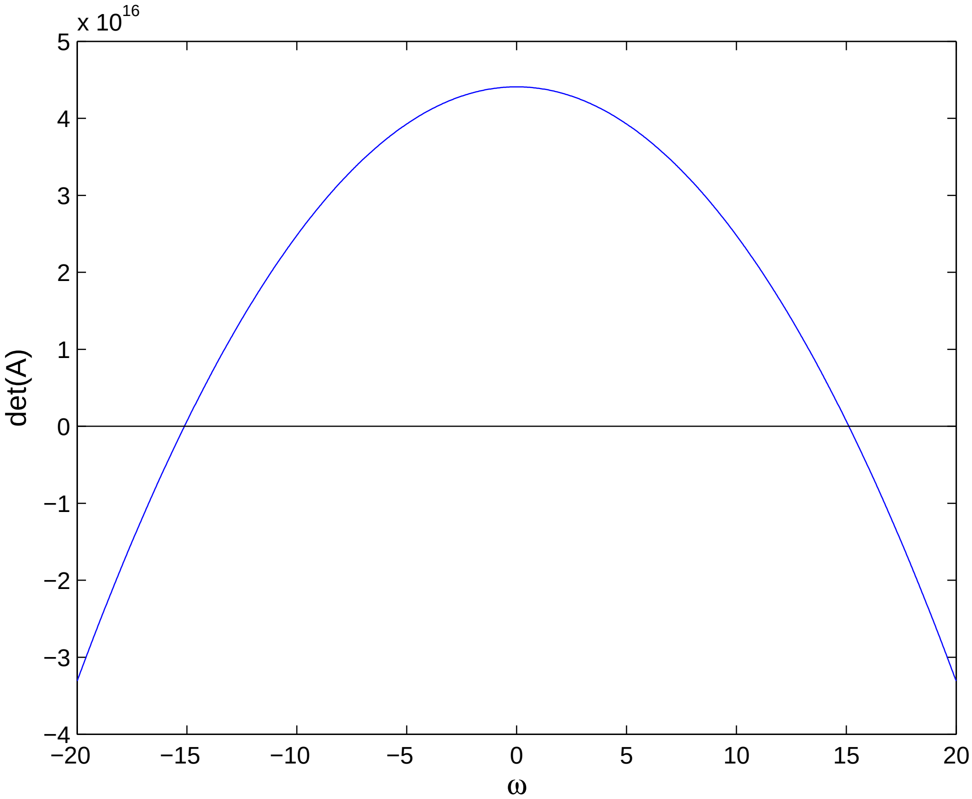

Figure 5.11.4: The determinant of A, the matrix on the left-hand side of equation (5.9.22), as a function of ω. The determinant of A is a bicubic polynomial in ω, i.e., a cubic polynomial in ω^2. The roots of $\det(A)$ are the critical values of ω. Numerical computation of the roots shows that the cubic equation for ω^2 has only one positive root and so each Fourier mode k has only one eigenvalue that crosses the imaginary axis as ω is increased from 0.

Chapter 6

Conclusions

Summary. In this thesis we studied the stability of Couette flow in a deformable cylinder with respect to cylindrical perturbations (Chapters 2–4) and axisymmetric perturbations (Chapter 5). We summarize our main results before discussing work in progress and open problems. Four different fluid-solid interaction models were developed, which have applications outside this thesis. For each model a rigid Couette steady solution was found. Understanding the stability of this solution with respect to the bifurcation parameter ω, the angular velocity of the inner cylinder, is the primary objective of this thesis. Linearizing the governing equations about the Couette steady solution and seeking normal mode solutions yields a quadratic eigenvalue problem. For the string problem (Chapter 2) we applied the spectral theorem for compact polynomial operator pencils to characterize the spectrum of the quadratic eigenvalue problem. For the both the string problem and the shell problem (Chapter 5) we proved that the only way for the eigenvalues to cross the imaginary axis is through the origin, and we derived an algebraic equation (a quadratic equation in Chapter 2 and a cubic equation in Chapter 5) for the critical values of ω at which the eigenvalues cross. Numerically computing the spectrum using a fast direct Fourier-finite element method shows that the rigid Couette solution loses its stability via a Takens-Bogdanov bifurcation for the string problem and a steady state bifurcation for the shell problem. In Chapter 2 the Galerkin approximation theory for polynomial eigenvalue problems was applied to prove convergence of the numerical method.

Work in progress and open problems. Up until now we have focused on linear stability of the rigid Couette solution, but said nothing about its nonlinear stability. It is expected that for small perturbations of the rigid Couette solution, the behavior of the linearized equations dictates the behavior of the nonlinear equations; if the rigid Couette solution is linearly stable for some ω, i.e., the spectrum of the linearized operator lies in the left half-plane, then we expect the rigid Couette solution to be asymptotically stable (in the sense of Lyapunov) for this ω. Similarly, if the rigid Couette solution is linearly unstable for some ω, i.e., the linearized operator has at least one eigenvalue in the right half-plane, then we expect the rigid Couette solution to be unstable (not Lyapunov stable) for this ω. Koch and Antman (2000) proved a stability theorem of this form for nonlinear parabolic-hyperbolic partial differential equations with complicated nonlinear boundary conditions. Their results, which extend those of Da Prato & Lunardi (1988) and Xu and Marsden (1996), apply to the equations describing nonlinearly viscoelastic strings, rods, and shells. It is not immediately clear that these results can be applied to the fluid-solid interaction problem discussed here. For example, Koch and Antman (2000) consider a scalar parabolic-hyperbolic partial differential equation with fixed domain, whereas the Taylor-Couette problem in a deformable cylinder is described by a system of partial differential equations, and the domain of the Navier-Stokes equations is time dependent. (Pulling back the Navier-Stokes equations to Lagrangian coordinates may fix the latter problem.) Extending the results of Koch and Antman (2000) so that they apply to the problem discussed here is work in progress.

We have not discussed the well-posedness of the coupled fluid-solid system. While there has been much effort to prove existence theorems for fluid-solid interaction problems in the last few years (see the references cited in Chapter 1), most of the existing results are only valid for short time intervals and for a particular choice of constitutive function.

In Chapter 5 we modelled the deformable shell using an axisymmetric shell theory. Antman and Bourne (in preparation) are currently developing a geometrically exact theory for rotationally symmetric shells, where the shell has $SO(2)$-symmetry rather than $O(2)$-symmetry. For the problem discussed here, this would allow material fibers of the deformable cylinder to shear in the direction of rotation. This extra degree of freedom significantly complicates the governing equations.

Further open problems include:

(i) Analyzing the global (energy) stability of the rigid Couette solution (in the sense of Joseph (1976) and Straughan (2004)).

(ii) Studying non-rigid Couette solutions, where the deformable cylinder rotates at a different angular velocity to the rigid cylinder.

(iii) The thin gap problem where the ratio of the radii of the cylinders is to sent to 1. (This method was employed by G.I. Taylor (1923).)

(iv) A full numerical simulation of the Taylor-Couette problem for flow in a deformable cylinder.

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