ABSTRACT

Title of Dissertation: METRIC GEOMETRY OF FINITE ENERGY CLASSES IN BIG COHOMOLOGY

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This thesis investigates the metric geometry of finite energy classes in big cohomology. These finite energy classes are made of functions that correspond to singular metrics on compact Kähler manifolds. These space of functions were introduced to find the canonical Kähler metrics. We extend their study to big cohomology classes.

On the space of finite energy potentials $\mathcal{E}^p(X,\theta)$ where θ represents a big cohomology class, we construct a complete geodesic metric d_p. We show that several metric properties of $(\mathcal{E}^p(X,\theta), d_p)$ are the same as in the Kähler setting.

In the end, we study the space of geodesic rays in $\mathcal{E}^p(X,\theta)$, $\mathcal{R}^p_\theta$, and construct a chordal metric d^c_p on it. We show that $(\mathcal{R}^p_\theta, d^c_p)$ is a complete geodesic metric as well.

COMPLETE METRICS ON FINITE ENERGY SPACES IN BIG COHOMOLOGY CLASSES

by

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Chapter 1: Introduction

Let X be a compact complex manifold. On such manifolds, we want to work with Riemannian metrics g that are invariant under the complex structure J. This means that for all $p \in X$ and any $u, v \in T_p(X)$, we want $g(Ju, Jv) = g(u, v)$. With the help of the complex structure J, we can construct a 2-form $\omega(\cdot, \cdot) = g(\cdot, J\cdot)$. We say that the metric g is Kähler if $d\omega = 0$. We will abuse the notation and call ω the Kähler metric as well. The Kähler condition is equivalent to the analytically useful condition that at each point p, we can find holomorphic normal coordinates. As $d\omega = 0$, it determines a cohomology class in $H^2(X, \mathbb{R})$. Throughout the thesis, we will work with a compact Kähler manifold (X, ω). If $\tilde{\omega}$ is another Kähler metric that determines the same cohomology class, then the $\partial\bar{\partial}$-lemma implies that $\tilde{\omega} = \omega + i\partial\bar{\partial}u$ for some $u \in C^\infty(X)$. See [1] for an introduction to the geometry of complex and Kähler manifolds. The space of Kähler potentials in the same cohomology class as ω is given by

$$\mathcal{H} = \{u \in C^\infty(X) \,|\, \omega + i\partial\bar{\partial}u > 0\}.$$

That is, any other Kähler metric $\tilde{\omega}$ for which $[\omega] = [\tilde{\omega}]$ satisfies $\tilde{\omega} = \omega + i\partial\bar{\partial}u =: \omega_u$ for some $u \in C^\infty(M)$. Moreover, this u is unique up to the addition of a constant $c \in \mathbb{R}$.

1.1 Background

To find a canonical metric on X in the same cohomology class as ω, Donaldson [2], Mabuchi [3], and Semmes[4], constructed a Riemannian structure on $\mathcal{H}$ as

$$\langle \phi, \psi \rangle_u = \frac{1}{V} \int_X \phi \bar{\psi} \omega_u^n, \tag{1.1}$$

where $V = \int_X \omega^n$ is the volume of the cohomology class of ω.

In [5] Chen showed that the path length metric obtained from this Riemannian structure is an honest metric d_2 on $\mathcal{H}$. Unfortunately, it is not a complete metric. Moreover, the geodesics induced by the Riemannian structure were not known to lie in $\mathcal{H}$. In [6] Darvas showed that the completion of this metric space can be identified with $\mathcal{E}^2(X, \omega)$, the space of potentials of finite energy, introduced by Guedj–Zeriahi [7]. Moreover, the geodesics induced by the Riemannian structure lie in $\mathcal{E}^2(X, \omega)$ as well. Darvas also showed in [8] that when $p \geq 1$, the space $\mathcal{E}^p(X, \omega)$ can be endowed with a complete geodesic metric with the same geodesics as in the case of $p = 2$. The metric space $(\mathcal{E}^1(X, \omega), d_1)$ was useful for studying the canonical metrics on (X, ω) [9], [10], [11], [12], [13].

The finite energy spaces $\mathcal{E}^p(X, \omega)$ were introduced by Guedj–Zeriahi in [7] to study the complex Monge–Ampére equation $(\omega + i\partial\bar{\partial}u)^n = \mu$ where μ is a measure with the same volume as that of ω. To solve this equation in this full generality, they associated a non-pluripolar Monge–Ampère measure to ω-plurisubharmonic function u. Recall that an upper semicontinuous function $u : X \to \mathbb{R} \cup \{-\infty\}$ is called ω-plurisubharmonic (ω-psh) if $\omega + i\partial\bar{\partial}u \geq 0$ as currents. We denote this set by $\mathrm{PSH}(X, \omega)$. When u is smooth, using Stokes' theorem we can see that the volume $\int_X \omega_u^n = \int_X \omega^n$. But for general $u \in \mathrm{PSH}(X,\omega)$, their construction holds that $\int_X \omega_u^n \leq \int_X \omega^n$. Thus to solve $\omega_u^n = \mu$ we look for the function in the space of potentials that have full mass, i.e., in the space

$$\mathcal{E}(X,\omega) = \left\{ u \in \mathrm{PSH}(X,\omega) \,|\, \int_X \omega_u^n = \omega^n \right\}.$$

Guedj–Zeriahi also introduced subsets of this space called the finite energy weight spaces. A weight is a function $\chi : \mathbb{R} \to \mathbb{R}$ that satisfies $\chi(0) = 0$, χ is even and on $(0,\infty)$, χ is increasing and smooth. Notice that for $p > 0$, the functions $\chi(t) = |t|^p$ satisfy these properties. Potentials with finite χ energy are denoted by

$$\mathcal{E}_\chi(X,\omega) = \left\{ u \in \mathcal{E}(X,\omega) \,|\, \int_X \chi(u)\omega_u^n < \infty \right\}.$$

These finite-energy subspaces have an interesting property that

$$\mathcal{E}(X,\omega) = \bigcup_\chi \mathcal{E}_\chi(X,\omega).$$

Thus, if we understand the metric structure on $\mathcal{E}_\chi(X,\omega)$, we could better understand the structure of $\mathcal{E}(X,\omega)$.

1.2 Quasi metrics in finite energy spaces

In [14] for any weight χ, Darvas showed that there is a metric d_χ on $\mathcal{H}$ whose completion can be identified with the space $\mathcal{E}_\chi(X,\omega)$. He showed that this metric is comparable to the functional

$$I_\chi(u,v) = \int_X \chi(u-v)(\omega_u^n + \omega_v^n),$$

for any $u, v \in \mathcal{E}_\chi(X,\omega)$.

This prompted the question: can one show that the functional I_χ satisfies a quasi-triangle inequality and consequently induces a complete metric structure without needing an actual metric on the space of smooth potentials? The first part of this thesis answers this question. This approach allows us to extend the results to a more general setting.

Although genuine Kähler metrics only exist in Kähler classes, these classes are not invariant under birational transformation. On the other hand, the set of big cohomology classes are invariant under birational transformation. In these classes, we do not have genuine Kähler metrics, but we have singular Kähler metrics. We can look for canonical metrics in this setting. Let θ be a smooth closed real $(1,1)$-form that represents a big cohomology class. Roughly, this means that there are plenty of θ-psh functions. See Chapter 2 for more precise definition of big cohomology classes. Moreover, there is a distinguished θ-psh function

$$V_\theta = \sup\{u \in \mathrm{PSH}(X,\theta) \,|\, u \leq 0\}$$

that satisfies $\int_X \theta_{V_\theta}^n \geq \int_X \theta_u^n$ for all $u \in \mathrm{PSH}(X,\theta)$. Moreover, if θ were smooth Kähler form, then $V_\theta = 0$. This allows us to write

$$\mathcal{E}(X,\theta) = \left\{ u \in \mathrm{PSH}(X,\theta) \,|\, \int_X \theta_u^n = \int_X \theta_{V_\theta}^n \right\}$$

and

$$\mathcal{E}_\chi(X,\theta) = \left\{ u \in \mathcal{E}(X,\theta) \mid \int_X \chi(u - V_\theta)\theta_u^n < \infty \right\}.$$

In [15], Gupta proved

Theorem 1.2.1. *Let θ be a closed smooth $(1,1)$-form whose cohomology class is big. Let χ be any finite energy weight, then for any $u, v \in \mathcal{E}_\chi(X,\theta)$,*

$$I_\chi(u,v) := \int_X \chi(u-v)(\theta_u^n + \theta_v^n) \tag{1.2}$$

is a quasi-metric. Moreover, the topology induced on $\mathcal{E}_\chi(X,\theta)$ by I_χ is completely metrizable.

As discussed in Section 3.1 a quasi-metric induces a metric that has the same convergence properties (and hence the same topology). The quasi-metric constructed in the theorem above gives us a genuine completely metrizable topology.

1.3 Geodesic metrics in big cohomology classes

One key aspect of the metric d_p introduced by Darvas in [8] is that any two points in $\mathcal{E}^p(X,\omega)$ can be joined by geodesic paths. This means that for any points $u_0, u_1 \in \mathcal{E}^p(X,\omega)$ there is a path $[0,1] \ni t \mapsto u_t \in \mathcal{E}^p(X,\omega)$ such that $d_p(u_t, u'_t) = |t-t'| d_p(u_0,u_1)$ for any $0 \le t, t' \le 1$. When $u_0, u_1 \in \mathcal{H}$, then the path minimizes the path length among all paths joining u_0, u_1 in $\mathcal{H}$. Thus, the geometry of $(\mathcal{E}^p(X,\omega), d_p)$ is naturally induced by the Riemannian structure in Equation (1.1).

The pluripotential theoretic methods alone used in constructing the pseudo-metric $(\mathcal{E}_\chi(X,\theta), I_\chi)$ in [15] were incapable of dealing with metric geodesics.

Other works had generalized the work of Darvas that accommodates geodesics but do not work in the full generality of what was done in [15]. In [16] Di Nezza–Lu constructed a complete geodesic metric d_p on $\mathcal{E}^p(X, \beta)$ where β represents a big and nef cohomology class. To construct d_p on $\mathcal{E}^p(X, \beta)$ they approximated β by nearby Kähler cohomology classes. In [17], Darvas–Di Nezza–Lu constructed the complete geodesic metric d_1 on $\mathcal{E}^1(X, \theta)$ where θ represents a big cohomology class. They defined d_1 by the formula

$$d_1(u, v) = I_\theta(u) + I_\theta(v) - 2I_\theta(P(u, v))$$

where I_θ is the Monge–Ampére energy and P is the envelope operator. See Chapter 2 for the definition of Monge–Ampère energy and the envelope operator and their properties.

Using very different methods, in [18], Xia constructed another metric d_p on $\mathcal{E}^p(X, \theta)$ which agreed with the previously constructed metrics, but the author could not prove that in the general setting that their metric is complete or has geodesic.

This leads to the second result of this thesis. In [19], Gupta proved

Theorem 1.3.1. *Let $p \geq 1$ and θ represent a big cohomology class. Then the finite energy space $\mathcal{E}^p(X, \theta)$ has a complete geodesic metric d_p. Moreover, the weak geodesics of $\mathcal{E}^p(X, \theta)$ are the same as the metric geodesics of $(\mathcal{E}^p(X, \theta), d_p)$.*

The weak geodesics in this theorem is the generalization of the geodesics path induced by the Riemannian structure of Equation (1.1). See Chapter 2 for more details.

When θ is big and nef, the metric agrees with the metric of [16], which in turn agrees with that of [8] when θ is Kähler. When $p = 1$, the metric d_1 constructed in Theorem 1.3.1 agrees with the one constructed in [17].

1.4 Space of rays in big cohomology classes

When $p = 1$ and θ represents a big cohomology class, the space of geodesic rays in $\mathcal{E}^1(X,\theta)$, denoted by $\mathcal{R}^1_\theta$, was introduced by Darvas–Di Nezza–Lu in [20] to study the space of singularity types. For geodesic rays $\{u_t\}_t$ and $\{v_t\}_t$ they introduced a chordal metric d_1^c on $\mathcal{R}^1_\theta$ as

$$d_1^c(\{u_t\}_t, \{v_t\}_t) = \lim_{t\to\infty} \frac{d_1(u_t, v_t)}{t}$$

and showed that it is a complete metric space. In the Kähler setting, in [11] Darvas–Lu studied $\mathcal{R}^p_\omega$, the space of finite energy rays in $\mathcal{E}^p(X,\omega)$ when ω is Kähler and $p \geq 1$. They showed that $(\mathcal{R}^p_\omega, d_p^c)$ is a complete geodesic metric space where for two geodesic rays $\{u_t\}_t$ and $\{v_t\}_t$, the chordal distance is defined by

$$d_p^c(\{u_t\}_t, \{v_t\}_t) = \lim_{t\to\infty} \frac{d_p(u_t, v_t)}{t}.$$

To show the existence of geodesic between two geodesic rays, Darvas–Lu had to study the space of rays in $\mathcal{E}^p(X,\omega)$ for $p > 1$.

The last part of the thesis deals with the construction of a complete geodesic metric d_p^c on $\mathcal{R}^p_\theta$ for $p \geq 1$. With the construction of the metric d_p on $\mathcal{E}^p(X,\theta)$ when θ represents a big class and $p > 1$ as in Theorem 1.3.1, in [21] the author proved

Theorem 1.4.1. *If θ represents a big cohomology class and $p \geq 1$, then $(\mathcal{R}^p_\theta, d_p^c)$ is a complete geodesic metric.*

1.5 Applications and future directions

The main tool in constructing the metric d_p in Theorem 1.3.1 was approximating the potentials of minimal singularity with potentials of analytic singularities. See Chapter 2 for definitions and Section 4.5 for the approximation process.

This approximation process has proved to be very insightful and has appeared in various other works. In [22], Di Nezza–Trapani–Trusiani use a similar approximation method to show that in the Mabuchi functional is convex in big cohomology classes. They also improve the scope of using the speed of weak geodesics to find the distance between two potentials in $\mathcal{E}^p(X, \theta)$. In particular, they prove that in Theorem 4.5.7 we could take minimal potentials of finite entropy, thus relaxing the the much stronger requirement that potentials be the envelope of a $C^{1,\bar{1}}$ function.

The thesis ends with discussing a Ross–Witt Nyström type correspondence between test curves and geodesic rays in Section 5.5. In particular, we can identify from the test curve, if the corresponding geodesic ray will lie in finite energy space $\mathcal{E}^p(X, \theta)$ or not.

Although not explicitly mentioned, when $p = 1$, methods in [23] by Darvas–Xia show that the d_1^c distance between two finite energy geodesic rays can be calculated in terms in terms of chordal distance. We expect that the methods used in this thesis could be extended to construct the chordal distance between rays in terms of the corresponding test curves.

1.6 Organization

In Chapter 2, we introduce the topic of pluripotential theory and recall results from the literature that will be useful in the rest of the thesis. We will also point out more detailed references on the subject there.

In Chapter 3, we will prove Theorem 1.2.1 using just pluripotential theoretic techniques. We will show that the result works for all finite energy weights.

In Chapter 4, we will prove Theorem 1.3.1. We will introduce the notion of Entropy of potentials, and explain how we can approximate minimal singularity potentials using analytic singularity potentials. We will use this approximation to construct the metric mentioned in the theorem. At the end of the chapter, we will prove some metric properties of $(\mathcal{E}^p(X,\theta), d_p)$ that show that the metric space is negatively curved.

In Chapter 5, we will prove some more properties of the metric space $(\mathcal{E}^p(X,\theta), d_p)$ and use those to study the space of geodesic rays in $\mathcal{E}^p(X,\theta)$ and prove Theorem 1.4.1.

Chapter 2: Pluripotential theory on compact Kähler manifolds

In this chapter, I will fix the notation and introduce necassary background for the thesis.

Throughout, we will work on a compact Kähler manifold (X, ω). Here ω is a smooth real closed $(1,1)$-form on X such that in local coordinates

$$\omega = \sqrt{-1} g_{ij} dz_i \wedge d\bar{z}^j$$

and the matrix $(g_{ij})_{ij}$ is a positive definite matrix at each point. For simplicity, we will assume that $\int_X \omega^n = 1$. We will also use the notation dd^c to denote $\sqrt{-1}\partial\bar{\partial}$.

2.1 Introduction

Here we give a quick introduction to the topic of pluripotential theory. For a more detailed study see [24], [25], and [26]. Also see [27] for a comprehensive study of pluripotential theory in the relative setting as discussed in this thesis.

Given a domain $\Omega \subset \mathbb{C}^n$, and an upper semicontinuous function $u : \Omega \to \mathbb{R} \cup \{-\infty\}$, we say u is plurisubharmonic (psh) if for every $a \in \Omega$ and $\xi \in \mathbb{C}^n$ such that the ball of radius $|\xi|$ around a is inside Ω,

$$u(a) \leq \frac{1}{2\pi} \int_0^{2\pi} u(a + e^{i\theta}\xi) d\theta.$$

The plurisubharmonicity property is invariant under holomorphic change of coordinates, so it makes sense to talk about plurisubharmonic functions on a complex manifold.

Given a smooth closed real $(1,1)$-form θ, we say that an upper semicontinuous function $u : X \to \mathbb{R} \cup \{-\infty\}$ is a θ-psh function if locally on $U \subset X$ where $\theta = dd^c g$, $u + g$ is plurisubharmonic. This implies that $\theta + dd^c u \geq 0$ as $(1,1)$-currents. We denote by PSH(X, θ) the set of all θ-psh functions that are not identically $-\infty$.

We denote by $\{\theta\}$ the $H^{1,1}(X, \mathbb{R})$ cohomology class of θ. We say that θ represents a Kähler class, if there exists a smooth θ-psh function u such that $\theta + dd^c u > 0$. θ represents a nef class if $\{\theta + \varepsilon\omega\}$ is a Kähler class for all $\varepsilon > 0$. We say θ represents a big class if there exists a potential $u \in$ PSH(X, θ) such that $\theta + dd^c u \geq \varepsilon\omega$ for some small enough $\varepsilon > 0$. If $u, v \in$ PSH(X, θ) satisfy $u \leq v + C$ for some constant C, then we say u is more singular than v and denote it by $u \preceq v$. If $u \preceq v$ and $v \preceq u$, then we say u and v have the same singularity type and denote that by $u \simeq v$. If θ represents a big cohomology class, then

$$V_\theta = \sup\{u \in \text{PSH}(X, \theta) : u \leq 0\}$$

is a θ-psh function that has minimal singularities. If θ is Kähler, then $V_\theta = 0$. From now on we fix a smooth closed real $(1,1)$-form θ that represents a big cohomology class.

We say that $\psi \in$ PSH(X, θ) has analytic singularities of type $(\mathcal{I}, c)$ if there exists a rational number $c > 0$ and a coherent ideal sheaf $\mathcal{I}$ such that for all $x \in X$, there exists a neighborhood $U \subset X$ of x such that $\mathcal{I}$ is generated on U by holomorphic functions $(f_1, \ldots, f_N)$ and

$$\psi_{|U} = c \log \left(\sum_{j=1}^{N} |f_j|^2 \right) + h$$

where h is a bounded function defined on U. If $\varphi \in \mathrm{PSH}(X,\theta)$ such that $\psi \simeq \varphi$, then φ has analytic singularities of type $(\mathcal{I}, c)$ as well. From [28, Lemma 2.4], we notice that analytic singularity is stable under max. This means that if $u, v \in \mathrm{PSH}(X,\theta)$ have analytic singularities, then $\max(u,v) \in \mathrm{PSH}(X,\theta)$ has analytic singularities as well.

A potential $\psi \in \mathrm{PSH}(X,\theta)$ is said to be a Kähler if $\theta + dd^c\psi > \varepsilon\omega$ for some $\varepsilon > 0$. By Demailly's regularization result, there exists a Kähler potential ψ that has analytic singularities.

In [29], the authors defined a non-pluripolar product of θ-psh functions. If $u_1, \ldots, u_n \in \mathrm{PSH}(X,\theta)$, they defined their non-pluripolar product $\langle\theta_{u_1} \wedge \cdots \wedge \theta_{u_n}\rangle$ as a non-pluripolar measure. For simplicity, we write $\theta_u^n := \langle\theta_u \wedge \cdots \wedge \theta_u\rangle$. The class represented by θ is big iff $\mathrm{Vol}(\theta) := \int_X \theta_{V_\theta}^n > 0$. From [30], and [31], we notice that for any $u_1, \ldots, u_n \in \mathrm{PSH}(X,\theta)$, $\int_X \langle\theta_{u_1} \wedge \cdots \wedge \theta_{u_n}\rangle \leq \mathrm{Vol}(\theta)$.

2.2 Finite energy classes

Finite energy classes in the Kähler setting were introduced by Guedj–Zeriahi [7] to solve the Complex Monge–Ampère equation on a compact Kähler manifold for a very general right-hand side. See [25] for a detailed study of finite energy classes in the Kähler setting. We define the space of potentials of full mass as

$$\mathcal{E}(X,\theta) = \left\{ u \in \mathrm{PSH}(X,\theta) : \int_X \theta_u^n = \int_X \theta_{V_\theta}^n \right\}.$$

For $p \geq 1$, the potentials with finite p-energy are defined as

$$\mathcal{E}^p(X,\theta) = \left\{ u \in \mathcal{E}(X,\theta) : \int_X |u - V_\theta|^p \theta_u^n < \infty \right\}.$$

2.3 Prescribed singularity setting

Darvas–Di Nezza–Lu developed pluripotential theory in the prescribed singularity setting in several papers including [31], [32], and [20]. See [27] to see the survey on this. Here we briefly recall the definitions of finite energy spaces in the prescribed singularity setting. We say that $\phi \in \mathrm{PSH}(X, \theta)$ with $\int_X \theta_\phi^n > 0$ is a model singularity type if

$$\phi = P_\theta[\phi] := \sup\{u \in \mathrm{PSH}(X, \theta) : u \preceq \phi, u \leq 0\}.$$

Darvas–Di Nezza–Lu introduced model singularities in [31] to solve the complex Monge–Ampère equation with prescribed singularities. We can also define the finite energy classes relative to ϕ. We denote by

$$\mathrm{PSH}(X, \theta, \phi) = \{u \in \mathrm{PSH}(X, \theta) : u \preceq \phi\}.$$

The space of potentials of full mass relative to ϕ is

$$\mathcal{E}(X, \theta, \phi) = \left\{u \in \mathrm{PSH}(X, \theta, \phi) : \int_X \theta_u^n = \int_X \theta_\phi^n\right\}.$$

Let χ be a weight i.e., $\chi : \mathbb{R} \to \mathbb{R}$ is an even function such that $\chi(0) = 0$, on $(0, \infty)$ χ is increasing and smooth and $\lim_{t\to\infty} \chi(t) = \infty$. We define the space of ϕ-relative finite χ-energy potentials by

$$\mathcal{E}_\chi(X, \theta, \phi) = \left\{u \in \mathcal{E}(X, \theta, \phi) : \int_X \chi(u - \phi)\theta_u^n < \infty\right\}.$$

If $u \in \mathrm{PSH}(X, \theta)$ has the same singularity type as ψ, we say u has relatively minimal singularities with respect to ψ. If u has relatively minimal singularities with respect to ψ, then $|u - \psi| \leq C$ for some constant C. Thus

$$\int_X |u - \psi|^p \theta_u^n \leq C^p \int_X \theta_u^n < \infty$$

Thus $u \in \mathcal{E}^p(X, \theta, \psi)$ for all $p \geq 1$.

The following results, described in detail in [27] will be used throughout the thesis.

Lemma 2.3.1 (Comparison Principle [27, Proposition 3.22])**.** *Let* $\phi \in PSH(X, \theta)$ *and* $u, v \in \mathcal{E}(X, \theta, \phi)$*. Then*

$$\int_{\{u<v\}} \theta_v^n \leq \int_{\{u<v\}} \theta_u^n.$$

Lemma 2.3.2 (The Fundamental inequality [27, Lemma 6.5])**.** *If* $u, v \in PSH(X, \theta, \phi)$ *are such that* $u \leq v \leq \phi$ *then*

$$\int_X \psi(v - \phi)\theta_v^n \leq 2^{n+1} \int_X \psi(u - \phi)\theta_u^n.$$

Thus for any $u \in \mathcal{E}_\psi(X, \theta, \phi)$ *and* $v \in PSH(X, \theta, \phi)$ *such that* $u \leq v$ *we have* $v \in \mathcal{E}_\psi(X, \theta, \phi)$*.*

Lemma 2.3.3 (Integrability [27, Lemma 6.4])**.** *Given* $u, v \in \mathcal{E}_\psi(X, \theta, \phi)$ *we have*

$$\int_X \psi(u - \phi)\theta_v^n < +\infty.$$

In particular, if $u, v \leq \phi$*, then*

$$\int_X \psi(u - \phi)\theta_v^n \leq 2 \int_X \psi(u - \phi)\theta_u^n + 2 \int_X \psi(v - \phi)\theta_v^n.$$

Corollary 2.3.4. *The proposed quasi-metric I_ψ (see Equation (1.2)) on $\mathcal{E}_\psi(X,\theta,\phi)$ is always finite.*

Lemma 2.3.5 (Domination Principle [27, Theorem 3.12])**.** *Let $\phi \in PSH(X,\theta)$ such that $\int_X \theta_\phi^n > 0$. Then for $u, v \in \mathcal{E}(X,\theta,\phi)$, if $\theta_u^n(\{u < v\}) = 0$, then $u \geq v$.*

Lemma 2.3.6 ([27, Lemma 6.7])**.** *Let $\phi \in PSH(X,\theta)$ such that $\int_X \theta_\phi^n > 0$. Let $u_j \in \mathcal{E}_\psi(X,\theta,\phi)$ such that*

$$\sup_j \int_X \psi(u_j - \phi)\theta_{u_j}^n < \infty.$$

If $u_j \to u$ in $L^1(\omega^n)$ for some $u \in PSH(X,\theta)$, then $u \in \mathcal{E}_\psi(X,\theta,\phi)$.

Theorem 2.3.7 ([27, Proposition 6.6])**.** *Let $u, v \in \mathcal{E}_\psi(X,\theta,\phi)$. Then*

$$P_\theta(u,v) := \sup\{w \in PSH(X,\theta) : w \leq \min(u,v)\} \in \mathcal{E}_\psi(X,\theta,\phi).$$

In particular, if $u, v \in \mathcal{E}(X,\theta,\phi)$, then $P_\theta(u,v) \in \mathcal{E}(X,\theta,\phi)$.

Lemma 2.3.8 ([27, Theorem 2.6])**.** *Let $\theta^1, \dots \theta^n$ be smooth real closed $(1,1)$-forms representing a big cohomology class and let $u_j, u_j^k \in PSH(X,\theta)$ be such that $u_j^k \to u_j$ in capacity as $k \to \infty$ for all $j \in \{1,\dots,n\}$. If $\chi_k, \chi \geq 0$ are quasi-continuous functions that are uniformly bounded and $\chi_k \to \chi$ in capacity, then*

$$\liminf_{k\to\infty} \int_X \chi_k \theta^1_{u_1^k} \wedge \dots \wedge \theta^n_{u_n^k} \geq \int_X \chi \theta^1_{u_1} \wedge \dots \wedge \theta^n_{u_n}.$$

Moreover if

$$\int_X \theta^1_{u_1} \wedge \dots \wedge \theta^n_{u_n} \geq \limsup_{k\to\infty} \int_X \theta^1_{u_1^k} \wedge \dots \wedge \theta^n_{u_n^k},$$

then the measures

$$\chi_k \theta^1_{u^k_1} \wedge \cdots \wedge \theta^n_{u^k_n} \to \chi \theta^1_{u_1} \wedge \cdots \wedge \theta^n_{u_n}$$

weakly.

We would need the following result about the $\mathcal{E}^p(X, \theta, \phi)$ spaces.

Theorem 2.3.9. *For $u, v \in \mathcal{E}^p(X, \theta, \phi)$, we define*

$$I_p(u, v) = \int_X |u - v|^p (\theta^n_u + \theta^n_v).$$

If $u^j_0, u^j_1, u_0, u_1 \in \mathcal{E}^p(X, \theta, \phi)$ such that $u^j_0 \searrow u_0$ and $u^j_1 \searrow u_1$, then $I_p(u^j_0, u^j_1) \to I_p(u_0, u_1)$ as $j \to \infty$.

Proof. The fact that $I_p(u, v) < \infty$ follows from the arguments in [15, Section 2] by modifying the proof for the weight $\chi(t) = |t|^p$.

The same proof as in [15, Theorem 4.1], shows that

$$I_p(u, v) = I_p(u, \max(u, v)) + I_p(v, \max(u, v)). \tag{2.1}$$

First, assume that $u^j_0 \leq u^j_1$, so consequently $u_0 \leq u_1$. Now we observe that the proof in [33, Proposition 2.20], works in the generality of prescribed singularity setting with big classes as well. Thus we obtain that in the case $u^j_0 \leq u^j_1$ and $u_0 \leq u_1$,

$$\int_X |u^j_0 - u^j_1|^p \theta^n_{u^j_0} \to \int_X |u_0 - u_1|^p \theta^n_{u_0}$$

and

$$\int_X |u_0^j - u_1^j|^p \theta_{u_1^j}^n \to \int_X |u_0^j - u_1^j|^p \theta_{u_1}^n$$

as $j \to \infty$. Adding the two we get

$$I_p(u_0^j, u_1^j) \to I_p(u_0, u_1)$$

as $j \to \infty$. More generally, if $u_0^j \searrow u_0$ and $u_1^j \searrow u_1$, then $\max(u_0^j, u_1^j) \searrow \max(u_0, u_1)$. Now the potentials $u_0^j \leq \max(u_0^j, u_1^j)$ and $u_1^j \leq \max(u_0^j, u_1^j)$. Thus

$$I_p(u_0^j, \max(u_0^j, u_1^j)) \to I_p(u_0, \max(u_0, u_1))$$

and

$$I_p(u_1^j, \max(u_0^j, u_1^j)) \to I_p(u_1, \max(u_0, u_1))$$

as $j \to \infty$. Adding the two, and using (2.1), we get

$$I_p(u_0^j, u_1^j) \to I_p(u_0, u_1)$$

as $j \to \infty$. $\square$

2.4 PSH Envelopes

Given a measurable function $f : X \to \mathbb{R} \cup \{\pm\infty\}$, and θ smooth real closed $(1,1)$-from representing a big cohomology class, we define

$$P_\theta(f) = (\sup\{u \in \text{PSH}(X,\theta) : u \leq f\})^*,$$

where $u^*(x) = \limsup_{y\to x} u(y)$ denotes the upper semicontinuous regularization. We say $P_\theta(f) = -\infty$, if the candidate set is empty, otherwise $P_\theta(f) \in \text{PSH}(X,\theta)$. In general, $P_\theta(f) \leq f$ away from a pluripolar set as the upper semicontinuous regularization only changes the function away from a pluripolar set. If f is upper semicontinuous, then $P_\theta(f) \leq f$ everywhere. If $f, g : X \to \mathbb{R} \cup \{\pm\infty\}$ are two measurable functions, we define the rooftop envelope $P_\theta(f,g) := P_\theta(\min\{f,g\})$.

Given f and θ as above, and $\phi \in \text{PSH}(X,\theta)$, we define the envelope with respect to the singularity type of ϕ by

$$P_\theta[\phi](f) := \left(\lim_{C\to\infty} P_\theta(\phi + C, f)\right)^*.$$

If f is bounded, then $P_\theta(\phi + C, f)$ is an increasing sequence of θ-psh functions that are bounded from above by f^*, thus the limit in the above equation exists. Moreover, $P_\theta[\phi](\cdot)$ depends only on the singularity type of ϕ. An elementary observation is

Lemma 2.4.1. *If f is a bounded function, then $P_\theta[\phi](f)$ has the same singularity type as ϕ.*

Proof. Let C be a constant such that $-C \leq f \leq C$. Then $P_\theta[\phi](-C) \leq P_\theta[\phi](f) \leq P_\theta[\phi](C)$. As $P_\theta[\phi](C) = P_\theta[\phi](0) + C = \phi + C$, we obtain, $\phi - C \leq P_\theta[\phi](f) \leq \phi + C$. Thus $P_\theta[\phi](f)$

has the same singularity type as ϕ. $\square$

A useful lemma about rooftop envelopes from [27] is the following.

Lemma 2.4.2 ([27, Theorem 3.10]). *Let $u, v \in PSH(X, \theta)$ such that $u \leq v$ and $\int_X \theta_u^n > 0$ and $b \in \left(1, \left(\frac{\int_X \theta_v^n}{\int_X \theta_v^n - \int_X \theta_u^n}\right)^{\frac{1}{n}}\right)$, then $P_\theta(bu + (1-b)v) \in PSH(X, \theta)$. Here,*

$$P_\theta(bu + (1-b)v) = (\sup\{h \in PSH(X, \theta) : h \leq bu + (1-b)v)\})^*$$

where $f^*(x) = \limsup_{y \to x} f(y)$ is the upper semicontinuous regularization of f.

The function $P_\theta[\phi](\cdot)$ also satisfies the following concavity property. We recall

Lemma 2.4.3 ([27, Lemma 2.12]). *Given a continuous function $f : X \to \mathbb{R}$, the operator $PSH(X, \theta) \ni u \mapsto P_\theta[u](f) \in PSH(X, \theta)$ is concave. This means for $t \in (0, 1)$,*

$$tP_\theta[u](f) + (1-t)P_\theta[v](f) \leq P_\theta[tu + (1-t)v](f).$$

If $f \in C^{1,\bar{1}}(X)$, which means that f has bounded distributional Laplacian, then we have good control on the Monge-Ampère measures of the envelopes $P_\theta[\phi](f)$. For that we recall,

Theorem 2.4.4 ([34]). *If θ represents a big cohomology class, $\phi \in PSH(X, \theta)$, and $f \in C^{1,\bar{1}}(X)$, then*

$$\theta^n_{P_\theta[\phi](f)} = \mathbb{1}_{\{P_\theta[\phi](f) = f\}} \theta^n_f.$$

In the same paper, the authors also prove

Theorem 2.4.5 ([34, Proposition 3.5])**.** *If $f_0, f_1 \in C^{1,\bar{1}}(X)$, and if we denote by $\Lambda_0 = \{P_\theta(f_0, f_1) = f_0\}$ and $\Lambda_1 = \{P_\theta(f_0, f_1) = f_1\}$, then*

$$\theta^n_{P_\theta(f_0,f_1)} = \mathbb{1}_{\Lambda_0}\theta^n_{f_0} + \mathbb{1}_{\Lambda_1\setminus\Lambda_0}\theta^n_{f_1}.$$

A corollary of this result is that

Corollary 2.4.6. *If $u_0 = P_\theta(f_0)$ and $u_1 = P_\theta(f_1)$ for $f_0, f_1 \in C^{1,\bar{1}}(X)$, then except for at most countably many $\tau \in \mathbb{R}$,*

$$\theta^n_{P_\theta(u_0,u_1+\tau)} = \mathbb{1}_{\{P_\theta(u_0,u_1+\tau)=u_0\}}\theta^n_{u_0} + \mathbb{1}_{\{P_\theta(u_0,u_1+\tau)=u_1+\tau\}}\theta^n_{u_1}.$$

Proof. Since the total measure of $\theta^n_{f_1}$ is finite, except for countably many $\tau \in \mathbb{R}$, $\theta^n_{f_1}(\{f_0 = f_1 + \tau\}) = 0$. Therefore, except for countably many $\tau \in \mathbb{R}$,

$$\theta^n_{P_\theta(f_0,f_1+\tau)} = \mathbb{1}_{\{P_\theta(f_0,f_1+\tau)=f_0\}}\theta^n_{f_0} + \mathbb{1}_{\{P_\theta(f_0,f_1+\tau)=f_1+\tau\}}\theta^n_{f_1}.$$

Notice that $P_\theta(f_0, f_1 + \tau) = P_\theta(u_0, u_1 + \tau)$ and Theorem 2.4.4 says that $\theta^n_{u_0} =$

$\mathbb{1}_{\{P_\theta(f_0)=f_0\}}\theta_{f_0}^n$ and $\theta_{u_1}^n = \mathbb{1}_{\{P_\theta(f_1)=f_1\}}\theta_{f_1}^n$. We use this to write

$$
\begin{aligned}
&\mathbb{1}_{\{P_\theta(u_0,u_1+\tau)=u_0\}}\theta_{u_0}^n + \mathbb{1}_{\{P_\theta(u_0,u_1+\tau)=u_1+\tau\}}\theta_{u_1}^n \\
=&\mathbb{1}_{\{P_\theta(u_0,u_1+\tau)=u_0\}}\mathbb{1}_{\{P_\theta(f_0)=f_0\}}\theta_{f_0}^n + \mathbb{1}_{\{P_\theta(u_0,u_1+\tau)=u_1+\tau\}}\mathbb{1}_{\{P_\theta(f_1)=f_1\}}\theta_{f_1}^n \\
=&\mathbb{1}_{\{P_\theta(f_0,f_1+\tau)=f_0\}}\theta_{f_0}^n + \mathbb{1}_{\{P_\theta(f_0,f_1+\tau)=f_1+\tau\}}\theta_{f_1}^n \\
=&\theta_{P_\theta(f_0,f_1+\tau)}^n \\
=&\theta_{P_\theta(u_0,u_1+\tau)}^n
\end{aligned}
$$

for all but countably many $\tau \in \mathbb{R}$. □

When η represents a Kähler class, we have a better control on the regularity of the envelope. In particular, we have

Lemma 2.4.7. *If η is a smooth closed real $(1,1)$-form that represents a Kähler class (and not necessarily be a Kähler form) and $f \in C^{1,\bar{1}}(X)$, then $P_\eta(f) \in C^{1,\bar{1}}(X)$.*

2.5 Weak geodesics and rooftop envelopes

Following Berndtsson [35] and Darvas–Di Nezza–Lu [36], we define weak geodesics as follows. Let X be a compact Kähler manifold and let θ represent a big cohomology class. Let $S = (0,1) \times \mathbb{R} \subset \mathbb{C}$ be the vertical strip in the complex plane. Let $\pi : X \times S \to X$ be the projection map. For $u_0, u_1 \in \mathrm{PSH}(X,\theta)$, a path $(0,1) \ni t \mapsto v_t \in \mathrm{PSH}(X,\theta)$ is a subgeodesic joining u_0 and u_1 if the map

$$X \times S \ni (x,z) \mapsto v_{\mathrm{Re}(z)}(x)$$

is a $\pi^*\theta$-psh function on $X \times S$ and $\limsup_{t\to 0,1} v_t \leq u_{0,1}$. We denote by

$$\mathcal{S} = \{(0,1) \ni t \mapsto v_t \in \text{PSH}(X,\theta) : v_t \text{ is a subgeodesic joining } u_0 \text{ and } u_1\}.$$

For arbitrary $u_0, u_1 \in \text{PSH}(X,\theta)$, there may not be any subgeodesics joining them. If $u_0, u_1 \in \mathcal{E}(X,\theta)$, then $P_\theta(u_0,u_1) := \sup\{u \in \text{PSH}(X,\theta) : u \leq u_0, u_1\} \in \mathcal{E}(X,\theta)$ (see [36, Theorem 2.10]), so the path $t \mapsto P_\theta(u_0,u_1)$ is a subgeodesic.

In the case $\mathcal{S}$ is not empty, we define the *weak geodesic* joining u_0 and u_1 by

$$u_t(x) = \sup_{v \in \mathcal{S}} v_t(x).$$

Each subgeodesic v_t is convex in the t-variable. Thus $v_t \leq (1-t)u_0 + tu_1$. Taking supremum over all $v \in \mathcal{S}$, we get $u_t \leq (1-t)u_0 + tu_1$. Now taking limit $t \to 0,1$ we get $\lim_{t\to 0,1} u_t \leq u_{0,1}$. Even if $X \times S \ni (x,z) \mapsto u_{\text{Re}(z)}(x)$ is not $\pi^*\theta$-psh, its upper semicontinuous regularization u^* is $\pi^*\theta$-psh. But for u_t^*, we observe that $u_t^* \leq ((1-t)u_0 + tu_1)^* = (1-t)u_0 + tu_1$. Taking limit to 0 or 1 we get $\lim_t u_t^* \leq u_{0,1}$. Thus u_t^* is a candidate for $\mathcal{S}$. Hence we do not take the upper semicontinuous regularization in the definition of weak geodesic u_t.

If $u_0, u_1 \in \mathcal{E}^p(X,\theta)$ then by Theorem 2.3.7 $P_\theta(u_0,u_1) \in \mathcal{E}^p(X,\theta)$. This means the weak geodesic u_t joining u_0, u_1 satisfy $u_t \in \mathcal{E}^p(X,\theta)$. The same result holds when $u_0, u_1 \in \mathcal{E}^p(X,\theta,\phi)$ due to [15, Theorem 2.9].

We recall the following useful lemmas from [33]. The results in op. cit. are for the Kähler case, but the proofs go through for the big case without change.

Lemma 2.5.1 ([33, Lemma 3.16]). *Let $u_0, u_1 \in PSH(X,\theta)$ and let u_t be the weak geodesic* *joining u_0 and u_1. Then for any $\tau \in \mathbb{R}$,*

$$\inf_{t\in(0,1)}(u_t - t\tau) = P_\theta(u_0, u_1 - \tau).$$

Proof. Since $t \mapsto v_t := u_t - t\tau$ is the weak geodesic joining u_0 and $u_1 - \tau$, it is enough to prove the result for $\tau = 0$.

Since $P_\theta(u_0, u_1) \leq u_0, u_1$, the map $t \mapsto w_t := P_\theta(u_0, u_1)$ is a weak subgeodesic joining u_0 and u_1, therefore $P_\theta(u_0, u_1) = w_t \leq u_t$ for all t. Therefore, $P_\theta(u_0, u_1) \leq \inf_{t\in(0,1)}(u_t)$.

For the other direction, we notice that Kiselman's minimum principle [24, Chapter 1, Theorem 7.5] implies $w := \inf_{t\in(0,1)}(u_t) \in \text{PSH}(X, \theta)$. Since $w \leq u_0, u_1$, we have $w \leq P_\theta(u_0, u_1)$. $\square$

Lemma 2.5.2 ([33, Lemma 3.17])**.** *Let $u_0, u_1 \in PSH(X, \theta)$ have minimal singularity and let u_t be the weak geodesic joining u_0 and u_1. Then for any $\tau \in \mathbb{R}$,*

$$\{\dot{u}_0 \geq \tau\} = \{P_\theta(u_0, u_1 - \tau) = u_0\}$$

on $X \setminus \{V_\theta = -\infty\}$.

Proof. Since u_0, u_1 have minimal singularity, $P_\theta(u_0, u_1 - \tau)$ and u_t have minimal singularity as well. Thus on $X \setminus \{V_\theta = -\infty\}$, $u_0, u_1, P_\theta(u_0, u_1 - \tau)$ are all finite. By the previous lemma, $\inf_{t\in(0,1)}(u_t - t\tau) = P_\theta(u_0, u_1 - \tau)$. Thus for $x \in X$, $P_\theta(u_0, u_1 - \tau)(x) = u_0(x)$ iff $\inf_{t\in(0,1)}(u_t - t\tau)(x) = u_0(x)$. Since $(u_t - t\tau)(x)$ is convex in t, this equality is possible iff $\dot{u}_0(x) \geq \tau$. $\square$

Combining Lemma 2.5.2 and Corollary 2.4.6 we get the following result.

Theorem 2.5.3. *Let $u_0 = P_\theta(f_0)$ and $u_1 = P_\theta(f_1)$ for $f_0, f_1 \in C^{1,\bar{1}}(X)$. If u_t is the weak geodesic joining u_0 and u_1, then for all $p \geq 1$,*

$$\int_X |\dot{u}_0|^p \theta_{u_0}^n = \int_X |\dot{u}_1|^p \theta_{u_1}^n.$$

Proof. The proof is the same as in [33, Lemma 3.30]. We will show that

$$\int_{\{\dot{u}_0 > 0\}} |\dot{u}_0|^p \theta_{u_0}^n = \int_{\{\dot{u}_1 > 0\}} |\dot{u}_1|^p \theta_{u_1}^n,$$

and a similar proof shows that

$$\int_{\{\dot{u}_0 < 0\}} |\dot{u}_0|^p \theta_{u_0}^n = \int_{\{\dot{u}_1 < 0\}} |\dot{u}_1|^p \theta_{u_1}^n.$$

$$\begin{aligned}
\int_{\{\dot{u}_0 > 0\}} |\dot{u}_0|^p \theta_{u_0}^n &= p \int_0^\infty \tau^{p-1} \theta_{u_0}^n(\{\dot{u}_0 \geq \tau\}) d\tau \\
&= p \int_0^\infty \tau^{p-1} \theta_{u_0}^n(\{P_\theta(u_0, u_1 - \tau)\} = u_0) d\tau.
\end{aligned}$$

Corollary 2.4.6 imply that $\mathrm{Vol}(\theta) = \theta_{u_0}^n(\{P_\theta(u_0, u_1 - \tau) = u_0\}) + \theta_{u_1}^n(\{P_\theta(u_0, u_1 - \tau) = u_1 - \tau\})$ which gives

$$\begin{aligned}
&= p \int_0^\infty \tau^{p-1} (\mathrm{Vol}(\theta) - \theta_{u_1}^n(\{P_\theta(u_0, u_1 - \tau) = u_1 - \tau\})) d\tau \\
&= p \int_0^\infty \tau^{p-1} \theta_{u_1}^n(\{P_\theta(u_0 + \tau, u_1) < u_1\}) d\tau.
\end{aligned}$$

Applying Lemma 2.5.2 to the reverse geodesic joining u_1 and u_0, we get $\{P_\theta(u_0+\tau, u_1) < u_1\} = \{\dot{u}_1 > \tau\}$. Thus

$$\begin{aligned}&= p\int_0^\infty \tau^{p-1}\theta_{u_1}^n(\{\dot{u}_1 > \tau\})d\tau \\ &= \int_{\{\dot{u}_1>0\}} |\dot{u}_1|^p \theta_{u_1}^n.\end{aligned}$$

□

In the Kähler setting, the geodesics joining "smooth" points have some regularity properties. If η is a smooth real closed $(1,1)$-form that represents a Kähler class (and not necessarily be a Kähler form), then we denote

$$\mathcal{H}_\eta^{1,\bar{1}} = \text{PSH}(X,\eta) \cap C^{1,\bar{1}}(X)$$

which will act as the space of "smooth" potentials for us.

Lemma 2.5.4. *If η is a smooth real closed $(1,1)$-form (not necessarily positive) that represents a Kähler class and $u_0, u_1 \in \mathcal{H}_\eta^{1,\bar{1}}$, then the geodesic u_t joining them is in $\mathcal{H}_\eta^{1,\bar{1}}$ as well.*

2.6 Geodesic Rays

Now we define geodesic rays. Again we assume that θ represents a big cohomology class. A path $(0,\infty) \ni t \mapsto v_t \in \text{PSH}(X,\theta)$ is a *sublinear subgeodesic ray* starting at $u_0 \in \text{PSH}(X,\theta)$ if $u_t \to u_0$ in $L^1(X)$ as $t \to 0$ and for any $0 < a < b < \infty$, the path $(a,b) \ni t \mapsto u_t$ is a subgeodesic between u_a and u_b and $u_t \leq u_0 + Ct$ for some constant C.

A sublinear subgeodesic ray $(0,\infty) \ni t \mapsto u_t \in \mathrm{PSH}(X,\theta)$ is a *geodesic ray* starting at $u_0 \in \mathrm{PSH}(X,\theta)$ if $u_t \to u_0$ in $L^1(X)$ as $t \to 0$ and for any $0 < a < b < \infty$, the path $(a,b) \ni t \mapsto u_t$ is the geodesic segment joining u_a and u_b. We recall a few useful results about the geodesic rays.

Lemma 2.6.1 ([37, Remark 3.3]). *If $[0,\infty) \ni t \mapsto u_t \in \mathcal{E}^p(X,\theta)$ is a geodesic ray starting from $u_0 = V_\theta$, then $t \mapsto \sup_X u_t = \sup_{Amp(\theta)}(u_t - V_\theta)$ is linear.*

Theorem 2.6.2 ([37, Proposition 3.8]). *If $[0,\infty) \ni t \mapsto u_t \in \mathcal{E}^p(X,\theta)$ is a geodesic ray starting at $u_0 = V_\theta$, then one can find geodesic rays of minimal singularity type $[0,\infty) \ni t \mapsto u_t^j \in \mathcal{E}^p(X,\theta)$ such that each u_t^j has minimal singularity and $u_t^j \searrow u_t$ as $j \to \infty$ for all $t \geq 0$.*

2.7 The Monge–Ampère energy

Among the finite energy spaces $\mathcal{E}^p(X,\theta)$, the case of $p = 1$ is special. It plays an important role in solving the complex Monge–Ampère equation on Kähler manifolds using the variational methods (see [38]). The Monge–Ampère energy functional I, given by

$$I(u) = \frac{1}{n+1}\sum_{j=0}^{n}\int_X (u - V_\theta)\theta_u^j \wedge \theta_{V_\theta}^{n-j},$$

characterizes $\mathcal{E}^1(X,\theta)$ as $u \in \mathcal{E}^1(X,\theta) \iff I(u) > -\infty$. The Monge–Ampère energy has many useful properties that we list below, that we will use in this thesis. For a comprehensive study of Monge–Ampère energy in relative setting, see [27, Section 5.1].

Lemma 2.7.1. *Let $u_0, u_1 \in \mathcal{E}^1(X,\theta)$, then*

1. *If $[0,1] \ni t \mapsto u_t$ is the geodesic joining u_0 and u_1, then $[0,1] \ni t \mapsto I(u_t)$ is affine.*

2. *If* $u_0 \leq u_1$, *then* $I(u_0) \leq I(u_1)$.

Chapter 3: A complete metric topology on relative low energy spaces

In this Chapter, we will prove Theorem 1.2.1.

We will also show that the metric topology on $\mathcal{E}_\chi(X,\theta)$ induced by I_χ has natural convergence property with respect to the Monge–Ampère measure. In particular,

Theorem 3.0.1. *If $u_k, u \in \mathcal{E}_\psi(X,\theta,\phi)$ such that $I_\psi(u_k,u) \to 0$ as $k \to \infty$ then $u_k \to u$ in capacity and hence $\theta^n_{u_k} \to \theta^n_u$ weakly as measures. In particular, $\int_X |u_k - u|\omega^n \to 0$ as $k \to \infty$ as well.*

At the end, we give an application to the Kähler-Ricci flow. Guedj–Zeriahi [39] and Di Nezza–Lu[40] showed that given a potential $u \in \mathrm{PSH}(X,\omega)$ such that u has zero Lelong numbers we have smooth function u_t for $t > 0$ such that

$$\frac{\partial u_t}{\partial t} = \log\left[\frac{(\omega + dd^c u_t)^n}{\omega^n}\right], u_t \to u \text{ in } L^1(\omega^n) \text{ as } t \to 0. \tag{3.1}$$

If $\omega_t := \omega + dd^c u_t$, then the above equation is equivalent to

$$\frac{\partial \omega_t}{\partial t} = -\mathrm{Ric}(\omega_t) + \mathrm{Ric}(\omega).$$

Moreover, these functions satisfy $u_t \to u$ in capacity as $t \to 0$. In case $u \in \mathcal{E}_\psi(X,\omega)$ we show a stronger convergence $u_t \to u$ as $t \to 0$ in the following theorem.

Theorem 3.0.2. *If $u \in \mathcal{E}_\chi(X,\omega)$ and u_t satisfy* (3.1) *then $I_\chi(u_t,u) \to 0$ and thus by Theorem 3.0.1, we recover that $u_t \to u$ in capacity and $\omega_{u_t}^n \to \omega_u^n$ weakly.*

In Section 3.2, we show that I_χ is a quasi-metric on $\mathcal{E}_\chi(X,\theta,\phi)$. In Section 3.3, we show that the induced topology on $\mathcal{E}_\chi(X,\theta,\phi)$ is completely metrizable thereby completing the proof of Theorem 1.2.1. In Section 3.4, we discuss some relevant properties of the new topology and prove Theorem 3.0.1. In Section 3.5, we discuss an application to Kähler Ricci flow and prove Theorem 3.0.2.

3.1 Metric from a quasi-metric

If we relax the condition of the triangle inequality from the definition of a metric space, we obtain what we call a quasi-metric space.

Definition 3.1.1 (Quasi-metric space)**.** Given a set X a function $\rho : X \times X \to [0,\infty)$ is a quasi-metric if it satisfies

1. (Non-degeneracy) For any $x, y \in X$, $\rho(x,y) = 0$ iff $x = y$.
2. (Symmetry) For all $x, y \in X$, $\rho(x,y) = \rho(y,x)$.
3. (Quasi-triangle inequality)There exists $C \geq 1$ such that $\rho(x,y) \leq C(\rho(x,z) + \rho(y,z))$ for all $x, y, z \in X$.

In [41] the authors show that given any quasi-metric, we can construct a metric comparable to the quasi-metric using a p-chain approach. In particular, if we consider $d_p : X \times X \to [0,\infty)$

given by

$$d_p(x,y) = \inf\left\{\sum_{i=0}^{n}\rho(x_n,x_{n+1})^p : x_0,\ldots,x_{n+1}\in X \text{ such that } x=x_0, y=x_{n+1}\right\}$$

then d_p is symmetric, satisfies triangle inequality, but in general $d_p(x,y)=0$ even if $x\neq y$. But if $0<p\leq 1$ is chosen such that $(2C)^p=2$, then d_p is non-degenerate and moreover,

$$d_p(x,y)\leq \rho(x,y)^p \leq 4d_p(x,y).$$

This shows that if for $x_n, x\in X$ we have $\rho(x_n,x)\to 0$ iff $d_p(x_n,x)\to 0$. Thus ρ and d_p induce the same topology.

3.2 Quasi-metric on $\mathcal{E}_\chi(X,\theta,\phi)$

In this section, we prove that the expression

$$I_\chi(u,v) = \int_X \chi(u-v)(\theta_u^n+\theta_v^n)$$

is a quasi-metric on $\mathcal{E}_\chi(X,\theta,\phi)$ where $\phi\in\mathrm{PSH}(X,\theta)$ is such that $\int_X\theta_\phi^n>0$.

Theorem 3.2.1 (Quasi-triangle inequality)**.** *Let $\phi\in PSH(X,\theta)$ be such that $\int_X\theta_\phi^n>0$, then for any $u,v,w\in\mathcal{E}_\chi(X,\theta,\phi)$, the functional*

$$I_\chi(u,v) = \int_X \chi(u-v)(\theta_u^n+\theta_v^n)$$

satisfies

$$I_\chi(u,v) \le C(I_\chi(u,w) + I_\chi(v,w))$$

for $C = 8 \cdot 3^{n+1}$.

Proof. The proof is inspired from the proof in [42, Theorem 1.6]. We start with

$$\int_X \chi(u-v)\theta_u^n = \int_0^\infty \theta_u^n(\{\chi(u-v) > t\})dt. \tag{3.2}$$

Changing the variable $t = \chi(2s)$, we get $dt = 2\chi'(2s)ds$. Thus

$$= 2\int_0^\infty \chi'(2s)\theta_u^n(\{\chi(u-v) > \chi(2s)\})ds \tag{3.3}$$

$$= 2\int_0^\infty \chi'(2s)\theta_u^n(\{|u-v| > 2s\})ds. \tag{3.4}$$

Observe that $\{w-s \le u < v-2s\} \subset \{w < \frac{u+2v}{3} - \frac{s}{3}\}$ and $\{u < v-2s\} \subset \{u < w-s\} \cup \{w < \frac{u+2v}{3} - \frac{s}{3}\}$ to obtain

$$\theta_u^n(\{u < v-2s\}) \le \theta_u^n(\{u < w-s\}) + \theta_u^n\left(\left\{w < \frac{u+2v}{3} - \frac{s}{3}\right\}\right). \tag{3.5}$$

Since $\theta_u^n \le 3^n \theta_{\frac{u+2v}{3}}^n$ and $\mathcal{E}(X,\theta,\phi)$ is convex [31, Corollary 3.15], we get

$$\le \theta_u^n(\{u < w-s\}) + 3^n\theta_{\frac{u+2v}{3}}^n\left(\left\{w < \frac{u+2v}{3} - \frac{s}{3}\right\}\right). \tag{3.6}$$

Now Lemma 2.3.1 gives

$$\leq \theta_u^n(\{u < w - s\}) + 3^n \theta_w^n\left(\left\{w < \frac{u+2v}{3} - \frac{s}{3}\right\}\right). \tag{3.7}$$

Again by the comparison principle we have

$$\theta_u^n(\{v < u - 2s\}) \leq \theta_v^n(\{v < u - 2s\}).$$

By a similar computation as in Equation (3.7) we get

$$\leq \theta_v^n(\{v < w - s\}) + 3^n \theta_w^n\left(\left\{w < \frac{v+2u}{3} - \frac{s}{3}\right\}\right). \tag{3.8}$$

Combining Equation (3.7) and (3.8) we get

$$\begin{aligned}
\theta_u^n(\{|u-v| > 2s\}) &\leq \theta_u^n(\{u < v - 2s\}) + \theta_u^n(\{v < u - 2s\}) \\
&\leq \theta_u^n(\{u < w - s\}) + 3^n \theta_w^n\left(\left\{w < \frac{u+2v}{3} - \frac{s}{3}\right\}\right) \\
&\quad + \theta_v^n(\{v < w - s\}) + 3^n \theta_w^n\left(\left\{w < \frac{v+2u}{3} - \frac{s}{3}\right\}\right) \\
&\leq \theta_u^n(\{|u-w| > s\}) + 3^n \theta_w^n\left(\left\{\left|w - \frac{u+2v}{3}\right| > \frac{s}{3}\right\}\right) \\
&+ \theta_v^n(\{|v-w| > s\}) + 3^n \theta_w^n\left(\left\{\left|w - \frac{v+2u}{3}\right| > \frac{s}{3}\right\}\right).
\end{aligned} \tag{3.9}$$

Combining Equations (3.4) and (3.9), we get

$$\begin{aligned}\int_X \chi(u-v)\theta_u^n \leq\, & 2\int_0^\infty \chi'(2s)\theta_u^n(\{|u-w|>s\})ds \\ & + 2\int_0^\infty \chi'(2s)\theta_v^n(\{|v-w|>s\})ds \\ & + 2\cdot 3^n\int_0^\infty \chi'(2s)\theta_w^n\left(\left\{\left|w-\frac{u+2v}{3}\right|>\frac{s}{3}\right\}\right)ds \\ & + 2\cdot 3^n\int_0^\infty \chi'(2s)\theta_w^n\left(\left\{\left|w-\frac{v+2u}{3}\right|>\frac{s}{3}\right\}\right)ds. \end{aligned} \tag{3.10}$$

Since χ is concave, therefore the slope is decreasing. Hence $\chi'(2s) \leq \chi'(s)$ and $\chi'(2s) < \chi'(s/3)$. Using this in Equation (3.10) we get

$$\begin{aligned} & \leq 2\int_0^\infty \chi'(s)\theta_u^n(\{|u-w|>s\})ds \\ & + 2\int_0^\infty \chi'(s)\theta_v^n(\{|v-w|>s\})ds \\ & + 2\cdot 3^n\int_0^\infty \chi'(s/3)\theta_w^n\left(\left\{\left|w-\frac{u+2v}{3}\right|>\frac{s}{3}\right\}\right)ds \\ & + 2\cdot 3^n\int_0^\infty \chi'(s/3)\theta_w^n\left(\left\{\left|w-\frac{v+2u}{3}\right|>\frac{s}{3}\right\}\right)ds. \end{aligned} \tag{3.11}$$

Changing the variable again $t = \chi(s)$ in the first two terms and $t = \chi(s/3)$ in the last two terms, we get

$$
\begin{aligned}
&= 2\int_0^\infty \theta_u^n(\{\chi(u-w) > t\})dt \\
&+ 2\int_0^\infty \theta_v^n(\{\chi(v-w) > t\})dt \\
&+ 2\cdot 3^{n+1}\int_0^\infty \theta_w^n\left(\left\{\chi\left(w - \frac{u+2v}{3}\right) > t\right\}\right)dt \\
&+ 2\cdot 3^{n+1}\int_0^\infty \theta_w^n\left(\left\{\chi\left(w - \frac{v+2u}{3}\right) > t\right\}\right)dt. \qquad (3.12)
\end{aligned}
$$

$$
\begin{aligned}
\int_X \chi(u-v)\theta_u^n &\leq 2\int_X \chi(u-w)\theta_u^n + 2\int_X \chi(v-w)\theta_v^n \\
&+ 2\cdot 3^{n+1}\int_X \chi\left(w - \frac{u+2v}{3}\right)\theta_w^n \\
&+ 2\cdot 3^{n+1}\int_X \chi\left(w - \frac{v+2u}{3}\right)\theta_w^n. \qquad (3.13)
\end{aligned}
$$

Using the fact that $\chi(a+b) \leq \chi(a) + \chi(b)$ for any $a, b \in \mathbb{R}$, (see [14, Lemma 2.6]) we get

$$
\begin{aligned}
&\leq 2\int_X \chi(u-w)\theta_u^n + 2\int_X \chi(v-w)\theta_v^n \\
&+ 2\cdot 3^{n+1}\int_X \left(\chi\left(\frac{w-u}{3}\right) + \chi\left(\frac{2(w-v)}{3}\right)\right)\theta_w^n \\
&+ 2\cdot 3^{n+1}\int_X \left(\chi\left(\frac{w-v}{3}\right) + \chi\left(\frac{2(w-u)}{3}\right)\right)\theta_w^n. \qquad (3.14)
\end{aligned}
$$

Since χ is increasing in $(0,\infty)$ and symmetric, we get that $\chi(x/3) \leq \chi(x)$ and $\chi(2x/3) \leq \chi(x)$ for any $x \in \mathbb{R}$, thus

$$\begin{aligned}
&\leq 2\int_X \chi(u-w)\theta_u^n + 2\int_X \chi(v-w)\theta_v^n \\
&+ 4\cdot 3^{n+1}\int_X \chi(w-u)\theta_w^n + 4\cdot 3^{n+1}\int_X \chi(w-v)\theta_w^n \qquad (3.15) \\
&\leq 4\cdot 3^{n+1}(I(u,w)+I(v,w)). \qquad (3.16)
\end{aligned}$$

A similar computation for $\int_X \chi(u-v)\theta_v^n$ gives

$$I(u,v) \leq 8\cdot 3^{n+1}(I(u,w)+I(v,w)). \qquad (3.17)$$

This finishes the proof of quasi-triangle inequality. □

Theorem 3.2.2 (Non-degeneracy). *If $u,v \in \mathcal{E}_\chi(X,\theta,\phi)$, such that $I_\chi(u,v)=0$, then $u=v$.*

Proof. Let $I_\chi(u,v)=0$. Then

$$\int_X \chi(u-v)\theta_u^n + \int_X \chi(u-v)\theta_v^n = 0.$$

Thus we have $u=v$ almost everywhere with respect to θ_u^n and θ_v^n. Using Lemma 2.3.5, we get that $u=v$. □

The symmetry of $I_\chi(u,v)$ follows from the fact that the weight function is an even function. This finishes the proof of the fact that I_χ is a quasi-metric on $\mathcal{E}_\chi(X,\theta,\phi)$.

3.3 Completeness

In this section, we will finish the proof of Theorem 1.2.1 by showing that the quasi-metric is complete.

We need the following lemma to construct a monotone sequence and show that it converges.

Theorem 3.3.1 (Pythagorean Identity)**.** *Let* $\phi \in PSH(X,\theta)$ *and let* $u, v \in \mathcal{E}_\chi(X,\theta,\phi)$*. We know that* $\max(u,v) \in \mathcal{E}_\chi(X,\theta,\phi)$*. Moreover, they satisfy*

$$I_\chi(u,v) = I_\chi(\max(u,v),u) + I_\chi(\max(u,v),v).$$

Proof.

$$\begin{aligned} I_\chi(\max(u,v),u) &= \int_X \chi(\max(u,v)-u)(\theta^n_{\max(u,v)} + \theta^n_u) \\ &= \int_{\{v>u\}} \chi(v-u)(\theta^n_{\max(u,v)} + \theta^n_u). \end{aligned}$$

Since $u \mapsto \theta^n_u$ is plurifine local, we have $\mathbb{1}_{\{v>u\}}\theta^n_{\max(u,v)} = \mathbb{1}_{\{v>u\}}\theta^n_v$

$$= \int_{\{v>u\}} \chi(v-u)(\theta^n_v + \theta^n_u).$$

Similarly,

$$I_\chi(\max(u,v),v) = \int_X \chi(\max(u,v)-v)(\theta^n_{\max(u,v)} + \theta^n_v)$$

and the same computation as before gives

$$= \int_{\{u>v\}} \chi(u-v)(\theta_u^n + \theta_v^n).$$

Adding the two gives

$$\begin{aligned} I_\chi(\max(u,v),u) + I_\chi(\max(u,v),v) &= \int_X \chi(u-v)(\theta_u^n + \theta_v^n) \\ &= I_\chi(u,v). \end{aligned}$$

□

Proposition 3.3.2. *If $u_j, u \in \mathcal{E}_\chi(X,\theta,\phi)$ such that $u_j \searrow u$ or $u_j \nearrow u$, then*

$$\int_X \chi(u_j - u)\theta_{u_j}^n \to 0.$$

Consequently, by the the monotone convergence theorem $I_\chi(u_j,u) \to 0$.

Proof. Without loss of generality, we can assume that $u_j, u \leq \phi \leq 0$. First, we assume that $\phi - L \leq u_j, u \leq \phi$. We also assume that $u_j \searrow u$ and the proof for $u_j \nearrow u$ is similar. Thus $0 \leq u_j - u \leq L$, which implies $0 \leq \chi(u_j - u) \leq \chi(L)$. For some large enough A, $\theta \leq A\omega$. Thus $\mathrm{PSH}(X,\theta) \subset \mathrm{PSH}(X, A\omega)$, and we get that $u_j, u \in \mathrm{PSH}(X, A\omega)$. Hence they are quasi-continuous (at least with respect to Cap_ω). So we can find an open set O such that u_j, u are continuous on $X \setminus O$ and $\mathrm{Cap}_\omega(O) < \varepsilon$. The next part follows the proof in [36, Corollary 2.12] adapted to our case. We have

$$\int_X \chi(u_j - u)\theta_{u_j}^n = \int_{X\setminus O} \chi(u_j - u)\theta_{u_j}^n + \int_O \chi(u_j - u)\theta_{u_j}^n.$$

Since $X\setminus O$ is compact and $\chi(u_j-u)$ are continuous functions decreasing to 0, by Dini's theorem, we get that the convergence is uniform. So for large enough j, $\chi(u_j - u) \le \varepsilon$ on $X \setminus O$.

Moreover,

$$\begin{aligned}\int_O \chi(u_j - u)\theta_{u_j}^n &\le \chi(L) \int_O \theta_{u_j}^n \\ &\le \chi(L)L^n \text{Cap}_{\theta,\phi}(O) \\ &\le \chi(L)L^n f(\text{Cap}_\omega(O)) \\ &\le \chi(L)L^n f(\varepsilon).\end{aligned}$$

Here we used [43, Theorem 1.1] with $(\theta_1, \chi_1) = (\theta, \phi)$ and $(\theta_2, \chi_2) = (\omega, 0)$ and thus $\text{Cap}_{\theta,\phi} \le f(\text{Cap}_\omega)$ where $f : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and $f(0) = 0$. Combining these two results we get that for large enough j,

$$\int_X \chi(u_j - u)\theta_{u_j}^n \le \varepsilon \text{vol}(\theta^n) + \tilde{C} f(\varepsilon).$$

Therefore, $\lim_{j\to\infty} \int_X \chi(u_j - u)\theta_{u_j}^n \to 0$.

Now we remove the assumption that $\phi - L \le u_j \le \phi$. Let $u_j^L = \max\{u_j, \phi - L\}$ and $u^L = \max\{u, \phi - L\}$. We want to show that

$$\left| \int_X \chi(u_j - u)\theta_{u_j}^n - \int_X \chi(u_j^L - u^L)\theta_{u_j^L}^n \right| \to 0 \text{ as } L \to \infty$$

uniformly with respect to j. Since $u_j \searrow u$, we have that $u_j \geq u$. On the set $\{u > \phi - L\}$, we have $u^L = u$ and $u_j^L = u_j$. Using the fact that Monge-Ampère measures are plurifine, we get that

$$\int_{\{u>\phi-L\}} \chi(u_j - u)\theta_{u_j}^n = \int_{\{u>\phi-L\}} \chi(u_j^L - u^L)\theta_{u_j^L}^n.$$

Thus we need to show that

$$\left| \int_{\{u\leq\phi-L\}} \chi(u_j - u)\theta_{u_j}^n - \int_{\{u\leq\phi-L\}} \chi(u_j^L - u^L)\theta_{u_j^L}^n \right| \to 0.$$

We will show that both the terms go to 0 as $L \to \infty$ independent of j.

Notice that since $u_j \geq u$ we have $u_j - u = (\phi - u) - (\phi - u_j) \leq \phi - u$. Since χ is increasing in $(0, \infty)$ we have $\chi(u_j - u) \leq \chi(\phi - u) = \chi(u - \phi)$. Now we can construct a weight $\tilde{\chi}$ such that $\tilde{\chi}$ is still even, $\tilde{\chi}(0) = 0$ and $\tilde{\chi}_{|(0,\infty)}$ is smooth increasing and concave such that $\chi(t) \leq \tilde{\chi}(t)$ and $\chi(t)/\tilde{\chi}(t) \searrow 0$ as $t \to \infty$. Moreover, we require that $u \in \mathcal{E}_{\tilde{\chi}}(X, \theta, \phi)$. Since $u_j \geq u$, we have $u_j \in \mathcal{E}_{\tilde{\chi}}(X, \theta, \phi)$ as well. Now

$$\begin{aligned}\int_{\{u\leq\phi-L\}} \chi(u_j - u)\theta_{u_j}^n &\leq \int_{\{u\leq\phi-L\}} \chi(u - \phi)\theta_{u_j}^n \\ &\leq \frac{\chi(L)}{\tilde{\chi}(L)} \int_{\{u\leq\phi-L\}} \tilde{\chi}(u - \phi)\theta_{u_j}^n.\end{aligned}$$

Since $u, u_j \in \mathcal{E}_{\tilde{\chi}}(X, \theta, \phi)$, Lemma 2.3.3 tells us

$$\leq 2\frac{\chi(L)}{\tilde{\chi}(L)} \left(\int_X \tilde{\chi}(u - \phi)\theta_u^n + \int_X \tilde{\chi}(u_j - \phi)\theta_{u_j}^n \right).$$

Using Lemma 2.3.2 and noticing that $u \le u_j \le \phi$ we get that

$$\le 2(2^{n+1}+1)\frac{\chi(L)}{\tilde{\chi}(L)}\int_X \tilde{\chi}(u-\phi)\theta_u^n.$$

Since u has finite $\tilde{\chi}$ energy, $\int_X \tilde{\chi}(u-\phi)\theta_u^n$ is finite and $\chi(L)/\tilde{\chi}(L) \to 0$ as $L \to \infty$ independent of j. The same proof shows that $\int_{\{u\le\phi-L\}} \chi(u_j^L - u^L)\theta_{u_j^L}^n \to 0$ independent of j.

We can also modify the proof to show that it works when $u_j \nearrow u$. $\square$

Lemma 3.3.3. *Let $u_k \in \mathcal{E}_\chi(X,\theta,\phi)$ be a decreasing that is I_χ-Cauchy: for any $\varepsilon > 0$, there exists N such that $I_\chi(u_j,u_k) \le \varepsilon$ for $j,k \ge N$. Then $\lim_{k\to\infty} u_k =: u \in \mathcal{E}_\chi(X,\theta,\phi)$ and $I_\chi(u_k,u) \to 0$.*

Proof. Find a subsequence of $\{u_k\}$ and still denote it by $\{u_k\}$ such that

$$I_\chi(u_k,u_{k+1}) \le C^{-2k}$$

where C is the same constant as in Theorem 3.2.1. Let $u = \lim_k u_k$. We have to show $u_k \not\equiv -\infty$. We have $I_\chi(u_k,\phi)$ is bounded as $k \to \infty$. To see this, repeated application of Theorem 3.2.1 gives

$$\begin{aligned} I_\chi(\phi,u_k) &\le C(I_\chi(\phi,u_1)+I_\chi(u_1,u_k)) \\ &\le CI_\chi(\phi,u_1)+C^2I_\chi(u_1,u_2)+C^2I_\chi(u_2,u_k). \end{aligned}$$

Doing this k times we get

$$
\begin{aligned}
&\leq CI_\chi(\phi, u_1) + \sum_{j=2}^{k} C^j I_\chi(u_{j-1}, u_j) \\
&\leq CI_\chi(\phi, u_1) + \sum_{j=2}^{k} C^j C^{-2j+2} \\
&\leq CI_\chi(\phi, u_1) + C^2 \sum_{j=2}^{\infty} C^{-j} \\
&= CI_\chi(\phi, u_1) + \frac{C}{C-1} = \tilde{C}.
\end{aligned}
$$

This gives us

$$\int_X \chi(u_k - \phi)\theta_\phi^n \leq I(\phi, u_k) \leq \tilde{C}.$$

Without loss of generality we can assue that $u_1 \leq \phi$ so $u_k \leq u_1 \leq \phi$. Since u_k is decreasing, we have $u_k - \phi$ is decreasing, thus $\chi(u_k - \phi)$ is increasing. Applying monotone convergence theorem we get that

$$\int_X \chi(u - \phi)\theta_\phi^n \leq \tilde{C}.$$

Thus $u \not\equiv -\infty$. Hence $u \in \text{PSH}(X, \theta)$. Since $u_j \searrow u$ and

$$\int_X \chi(u_k - \phi)\theta_{u_k}^n \leq I_\chi(u_k, \phi) \leq \tilde{C}$$

using Lemma 2.3.6 we get that $u \in \mathcal{E}_\chi(X, \theta)$. Now we need to show that $I_\chi(u_k, u) \to 0$. This means

$$\int_X \chi(u_k - u)\theta_u^n + \int_X \chi(u_k - u)\theta_{u_k}^n \to 0.$$

For the first integral, we notice that $u_k \geq u$, thus $u_k - u \geq 0$ and $u_k - u \searrow 0$ and thus $\chi(u_k - u) \searrow 0$. Thus we can apply the monotone convergence theorem to get

$$\int_X \chi(u_k - u)\theta_u^n \to 0.$$

To show that

$$\int_X \chi(u_k - u)\theta_{u_k}^n \to 0$$

we use Proposition 3.3.2. Thus $I_\chi(u_k, u) \to 0$. $\square$

Lemma 3.3.4. *Let $\{u_k\} \in \mathcal{E}_\chi(X, \theta, \phi)$ be such that*

$$I_\chi(u_k, u_{k+1}) \leq C^{-2k}.$$

Then u_k are uniformly bounded from above.

Proof. Let

$$v_k = \max\{u_1, \ldots, u_k\}.$$

Then repeated application of Theorem 3.2.1 and 3.3.1 gives

$$\begin{aligned}
I_\chi(\phi, v_k) &\leq C I_\chi(\phi, u_1) + C I_\chi(u_1, \max\{u_1, \ldots, u_k\}) \\
&\leq C I_\chi(\phi, u_1) + C I_\chi(u_1, \max\{u_2, \ldots, u_k\}) \\
&\leq C I_\chi(\phi, u_1) + C^2 I_\chi(u_1, u_2) + C^2 I_\chi(u_2, \max\{u_2, \ldots, u_k\}) \\
&\leq C I_\chi(\phi, u_1) + C^2 I_\chi(u_1, u_2) + C^2 I_\chi(u_2, \max\{u_3, \ldots, u_k\}).
\end{aligned}$$

Doing this k times we get

$$\begin{aligned}&\leq CI_\chi(\phi, u_1) + C^2 I_\chi(u_1, u_2) + \cdots + C^k I_\chi(u_{k-1}, u_k)\\&\leq \tilde{C}.\end{aligned}$$

Therefore,

$$\int_X \chi(v_k - \phi)\theta_\phi^n \leq \tilde{C}.$$

Let $w_k = \max\{\phi, v_k\} \in \mathcal{E}_\chi(X, \theta)$. Then $w_k - \phi \geq 0$ and as v_k is pointwise increasing, we get that $w_k - \phi$ is pointwise increasing. As χ is increasing on $[0, \infty)$, we get that $\chi(w_k - \phi)$ is pointwise increasing. Let $w := \lim_{k\to\infty} w_k$. Then $\chi(w - \phi) = \lim_{k\to\infty} \nearrow \chi(w_k - \phi)$. The monotone convergence theorem implies

$$\int_X \chi(w - \phi)\theta_\phi^n = \lim_{k\to\infty} \int_X \chi(w_k - \phi)\theta_\phi^n \leq \int_X \chi(v_k - \phi)\theta_\phi^n \leq \tilde{C}.$$

Thus the set $K = \{\chi(w - \phi) \leq (\tilde{C} + 1)/\int_X \theta_\phi^n\}$ has positive θ_ϕ^n-measure. Thus K is not $\mathrm{PSH}(X, \omega)$-polar. We can find a constant A such that $\theta \leq A\omega$. Then $\mathrm{PSH}(X, \theta) \subset \mathrm{PSH}(X, A\omega)$.

Consider the set

$$\mathcal{F}_K = \{\varphi \in \mathrm{PSH}(X, A\omega) : \sup_K \varphi = 0\}.$$

Since K is not $\mathrm{PSH}(X, A\omega)$-polar, we know that $\mathcal{F}_K$ is compact in $L^1(\omega^n)$-topology. This follows from [44, Theorem 4.7]. Let $\alpha_k = \sup_K v_k$. Let e be such that $\chi(e) = (\tilde{C} + 1)/\int_X \theta_\phi^n$.

Then $K = \{w - \phi \leq e\}$. Notice that $\phi \leq 0$ gives

$$\alpha_k = \sup_K v_k \leq \sup_K (v_k - \phi) \leq \sup_K (w - \phi) \leq e.$$

Define $\tilde{v}_k = v_k - \alpha_k$. Then $\tilde{v}_k \in \text{PSH}(X, \theta) \subset \text{PSH}(X, A\omega)$. Since $\sup_K \tilde{v}_k = 0$, we have $\tilde{v}_k \in \mathcal{F}_K$. Since $\mathcal{F}_K$ is relatively compact, we have $\{\tilde{v}_k\}$ is relatively compact in $L^1(X)$ topology and hence uniformly bounded from above. Therefore $v_k = \tilde{v}_k + \alpha_k$ are uniformly bounded from above and hence $u_k \leq v_k$ are uniformly bounded from above. $\square$

The following theorem is the last step in proving completeness of the quasi-metric.

Theorem 3.3.5. *Let $\{u_j\} \in \mathcal{E}_\chi(X, \theta, \phi)$ be an I_χ-Cauchy sequence. Then there exists $u \in \mathcal{E}_\chi(X, \theta, \phi)$ such that $I_\chi(u_j, u) \to 0$.*

Proof. Extract a subsequence and still denote it by $\{u_j\}$ such that

$$I_\chi(u_j, u_{j+1}) \leq C^{-2j}$$

where C is the same constant as in Theorem 3.2.1. By Lemma 3.3.4 $\{u_j\}$ are uniformly bounded from above. Thus

$$v_j := (\sup_{k \geq j} u_k)^* \in \text{PSH}(X, \theta).$$

Since $u_j \leq v_j$ we know using Lemma 2.3.2 that $v_j \in \mathcal{E}_\chi(X, \theta, \phi)$. Define

$$v_j^l := \max\{u_j, u_{j+1}, \ldots, u_{j+l}\}.$$

Then $v_j^l \in \mathcal{E}_\chi(X, \theta, \phi)$ and $v_j^l \nearrow v_j$ almost everywhere. Lemma 2.3.6 implies

$I_\chi(v_j^l, v_j) \to 0$ as $l \to \infty$. Using quasi-triangle inequality twice we get that $I_\chi(v_j, v_{j+1}) \leq C^2 \lim_{l\to\infty} I_\chi(v_j^{l+1}, v_{j+1}^l)$. Now,

$$I_\chi(v_j^{l+1}, v_{j+1}^l) = I_\chi(\max\{u_j, v_{j+1}^l\}, v_{j+1}^l).$$

Using Theroem 3.3.1, we get

$$\leq I_\chi(u_j, v_{j+1}^l).$$

Using Theorem 3.2.1, we get

$$\leq CI_\chi(u_j, u_{j+1}) + CI_\chi(u_{j+1}, \max\{u_{j+1}, v_{j+2}^{l-1}\}).$$

This again by Theorem 3.3.1 gives us

$$\leq CI_\chi(u_j, u_{j+1}) + CI_\chi(u_{j+1}, v_{j+2}^{l-1}).$$

Applying this l times we get

$$
\begin{aligned}
&\leq \sum_{k=1}^{l} C^k I_\chi(u_{j+k-1}, u_{j+k}) \\
&\leq \sum_{k=1}^{l} C^k C^{-2j-2k+2} \\
&\leq \sum_{k=1}^{\infty} C^{-2j-k+2} \\
&\leq C^{-2j+2} \frac{1}{C-1} = C^{-2j} \frac{C^2}{C-1}.
\end{aligned}
$$

Thus we obtain that

$$
I_\chi(v_j, v_{j+1}) \leq C^{-2j} \frac{C^4}{C-1}.
$$

This shows that $\{v_j\}$ is a decreasing I_χ-Cauchy sequence. Thus by Lemma 3.3.3, we get that there exists $v \in \mathcal{E}_\chi(X, \theta, \phi)$ such that $I_\chi(v_j, v) \to 0$.

Now we want to show that $\{u_j\}$ and $\{v_j\}$ are equivalent sequences, i.e., $I_\chi(u_j, v_j) \to 0$. Then $I_\chi(u_j, v) \leq C I_\chi(u_j, v_j) + C I_\chi(v_j, v)$. As both the terms go to 0, we get that $I_\chi(u_j, v) \to 0$.

Now,

$$
\begin{aligned}
I_\chi(u_j, v_j^l) &= I_\chi(u_j, \max\{u_j, v_{j+1}^{l-1}\}) \\
&\leq I_\chi(u_j, v_{j+1}^{l-1}) \\
&\leq C I_\chi(u_j, u_{j+1}) + C I_\chi(u_{j+1}, v_{j+1}^{l-1}) \\
&\leq C I_\chi(u_j, u_{j+1}) + C I_\chi(u_{j+1}, v_{j+2}^{l-2}) \\
&\leq C I_\chi(u_j, u_{j+1}) + C^2 I_\chi(u_{j+1}, u_{j+2}) + C^2 I_\chi(u_{j+2}, v_{j+2}^{l-2}).
\end{aligned}
$$

Doing this l times we get

$$\begin{aligned}
&\leq CI_\chi(u_j,u_{j+1})+C^2I_\chi(u_{j+1},u_{j+2})+\dots C^lI_\chi(u_{j+l-1},u_{j+l})\\
&\leq \sum_{k=1}^{l} C^kC^{-2k-2j+2}\\
&\leq C^{-2j}\frac{C^2}{C-1}.
\end{aligned}$$

Thus

$$I_\chi(u_j,v_j)\leq CI_\chi(u_j,v_j^l)+CI_\chi(v_j^l,v_j)\leq C^{-2j}\frac{C^3}{C-1}$$

by taking limit $l\to\infty$. Thus $I_\chi(u_j,v_j)\to 0$. Hence we have found $v\in\mathcal{E}_\chi(X,\theta,\phi)$ such that $I_\chi(u_j,v)\to 0$. Thus $(\mathcal{E}_\chi(X,\theta,\phi),I_\chi)$ is a completely metrizable topological space when topologized with the quasi-metric I_χ. □

This finishes the proof for completeness for of the quasi-metric I_χ on the space $\mathcal{E}_\chi(X,\theta,\phi)$.

3.4 Properties of the new topology

The following lemma generalizes [42, Lemma 1.5] and Lemma 2.3.6.

Lemma 3.4.1. *Let $u_j\in\mathcal{E}_\chi(X,\theta,\phi)$ be a decreasing sequence and $\varphi\in\mathcal{E}_\chi(X,\theta,\phi)$ be such that*

$$\sup_{j\in\mathbb{N}}\int_X \chi(u_j-\varphi)\theta_{u_j}^n<\infty.$$

Then $u=\lim_j u_j\in\mathcal{E}_\chi(X,\theta,\phi)$.

Proof. First we show that $u_j\not\to-\infty$ uniformly. If, on the contrary, $u\to\infty$ uniformly, up to choosing a subsequence and relabeling, we can assume that $u_j < -j$. Choose an A such that $\theta^n_\varphi\{\varphi > -A\} > 0$. Now

$$\begin{aligned}\int_X \chi(u_j - \varphi)\theta^n_{u_j} &= \int_0^\infty \theta^n_{u_j}\{\chi(u_j - \varphi) > t\}dt \\ &= \int_0^\infty \chi'(s)\theta^n_{u_j}\{|u_j - \varphi| > s\}ds \\ &= \int_0^\infty \chi'(s)\theta^n_{u_j}\{\varphi < u_j - s\}ds \\ &+ \int_0^\infty \chi'(s)\theta^n_{u_j}\{u_j < \varphi - s\}ds.\end{aligned}$$

Using comparison principle and the fact that $\chi'(s) \geq 0$, we get

$$\geq \int_0^\infty \chi'(s)\theta_\varphi\{u_j < \varphi - s\}ds.$$

Since $u_j < -j$, we get that $\{-j < \varphi - s\} \subset \{u_j < \varphi - s\}$. Thus,

$$\geq \int_0^\infty \chi'(s)\theta^n_\varphi\{-j < \varphi - s\}ds.$$

Choosing $j > A$, we can write

$$\geq \int_0^{j-A} \chi'(s)\theta^n_\varphi\{\varphi > s - j\}ds.$$

Again, we notice that $s < j - A$ implies that $\{\varphi > -A\} \subset \{\varphi > s - j\}$, which gives

$$\geq \int_0^{j-A} \chi'(s)\theta_\varphi^n\{\varphi > -A\}ds$$
$$= \theta_\varphi^n\{\varphi > -A\}\chi(j-A) \to \infty$$

as $j \to \infty$. This is a contradiction to the fact that $\int_X \chi(u_j - \varphi)\theta_{u_j}^n$ is bounded in j.

Moreover we observe that $I_\chi(u_j, \varphi)$ is bounded. Notice that, like in the previous argument, we have

$$\int_X \chi(u_j - \varphi)\theta_\varphi^n = \int_0^\infty \chi'(s)\theta_\varphi^n\{u_j < \varphi - s\}ds + \int_0^\infty \chi'(s)\theta_\varphi^n\{\varphi < u_j - s\}.$$

Using comparison principle for the first expression and the fact that $u_j \leq u_1$ in the second expression, we get that

$$\leq \int_0^\infty \chi'(s)\theta_{u_j}^n\{u_j < \varphi - s\}ds + \int_0^\infty \chi'(s)\theta_\varphi^n\{\varphi < u_1 - s\}ds$$
$$\leq \int_X \chi(u_j - \varphi)\theta_{u_j}^n + \int_X \chi(u_1 - \varphi)\theta_\varphi^n.$$

Thus $\int_X \chi(u_j - \varphi)\theta_\varphi^n$ is bounded from above independent of j. Now notice that

$$\int_X \chi(u_j - \phi)\theta_{u_j}^n \leq I_\chi(\phi, u_j).$$

By Quasi-triangle inequality for I_χ, we get

$$\begin{aligned} &\leq C(I_\chi(\phi,\varphi)+I_\chi(u_j,\varphi)) \\ &\leq C\left(I_\chi(\phi,\varphi)+\int_X \chi(u_j-\varphi)\theta^n_{u_j}+\int_X \chi(u_j-\varphi)\theta^n_{\varphi}\right). \end{aligned}$$

Thus $\int_X \chi(u_j-\phi)\theta^n_{u_j}$ is bounded from above independent of j and $u=\lim_j u_j \in \mathrm{PSH}(X,\theta)$, thus by Lemma 2.3.6 we get that $u \in \mathcal{E}_\chi(X,\theta,\phi)$. □

The following theorem is the generalization of [10, Proposition 2.6]. Also when $\chi(t)=|t|$, the following theorem appers in [45, Proposition 5.7].

Theorem 3.4.2. *Let $u_j, u \in \mathcal{E}_\chi(X,\theta,\phi)$ be such that $I_\chi(u_j,u)\to 0$. Then there exits a subsequence still denoted by u_j and $v_j, w_j \in \mathcal{E}_\chi(X,\theta,\phi)$ such that $v_j \leq u_j \leq w_j$ and v_j increase to u a.e. and w_j decrease to u. Thus, by Proposition 3.3.2 and monotone convergence theorem, $I_\chi(v_j,u)\to 0$ and $I_\chi(w_j,u)\to 0$.*

Proof. We can pass to a subsequence of (u_j) such that $I_\chi(u_j,u)\leq C^{-2j}$ where C is the same constant as in Theorem 3.2.1. By quasi-triangle inequality, $I_\chi(u_j,u_{j+1})\leq C^{-2j+2}$. By Lemma 3.3.4, u_j are uniformly bounded from above. Thus

$$w_j := (\sup_{k\geq j} u_k)^* \in \mathcal{E}_\chi(X,\theta,\phi).$$

Moreover, by the proof of Theorem 3.3.5 we get that w_j is a decreasing sequence in $\mathcal{E}_\chi(X,\theta)$ and is equivalent to u_j. Hence $w_j \geq u_j$ and $w_j \searrow u$ and thus $I_\chi(w_j,u)\to 0$.

For $j<k$, define $v_j^k = P_\theta(\min(u_j,\dots,u_k))$. By Theorem 2.3.7, $v_j^k \in \mathcal{E}_\chi(X,\theta,\phi)$. More-

over, by [31, Lemma 3.7],

$$\theta^n_{v_j^k} \leq \sum_{l=j}^{k} \mathbb{1}_{\{v_j^k = u_l\}} \theta^n_{u_l}.$$

Therefore

$$\int_X \chi(u - v_j^k)\theta^n_{v_j^k} \leq \sum_{l=j}^{k} \int_X \chi(u - u_l)\theta^n_{u_l} \leq \sum_{l=j}^{k} I_\chi(u, u_l) \leq C^{-2j+2}.$$

Also v_j^k is decreasing as $k \to \infty$, thus $v_j := \lim_k v_j^k \in \mathcal{E}_\chi(X, \theta, \phi)$ by Lemma 3.4.1 $v_j \in \mathcal{E}_\chi(X, \theta, \phi)$.

Since v_j^k decreases to v_j, we get that $v_j^k \to v_j$ in capacity. Moreover, the functions $\chi(u - v_j^k) \to \chi(u - v_j)$ in capacity. Using Lemma 2.3.8, we get that

$$\int_X \chi(u - v_j)\theta^n_{v_j} \leq \liminf_{k\to\infty} \int_X \chi(u - v_j^k)\theta^n_{v_j^k} \leq C^{-2j+2}.$$

Since $(\sup_{k\geq j} u_k)^* = w_j$, therefore, $\sup_{k\geq j} u_k = w_j$ a.e. As $w_j \searrow u$, we get that $\limsup_k u_k = u$ a.e. For any v_j, we have $v_j \leq u_k$ for $k \geq j$. Taking limsup we get $v_j \leq \limsup_{k\geq j} u_k = u$. Therefore, $v = (\lim_j v_j)^* \leq u$.

By the same argument as before, we have,

$$\int_X \chi(u - v)\theta^n_v \leq \liminf_{j\to\infty} \int_X \chi(u - v_j)\theta^n_{v_j} \leq \liminf_{j\to\infty} C^{-2j+2} = 0.$$

Thus $\theta^n_v(\{u \neq v\} = 0$. Again using the domination principle (see Lemma 2.3.5), we get that $u \leq v$ everywhere. This shows that $u = v$.

Thus we found an increasing sequence $v_j \leq u_j$ increasing to u. □

Corollary 3.4.3. *The I_χ-topology on $\mathcal{E}_\chi(X,\theta,\phi)$ is stronger than the $L^1(\omega^n)$ topology on $\mathcal{E}_\chi(X,\theta,\phi)$. More precisely, if $u_k, u \in \mathcal{E}_\chi(X,\theta,\phi)$ such that $I_\chi(u_k,u) \to 0$, then*

$$\int_X |u_k - u|\omega^n \to 0$$

as $k \to \infty$.

Proof. It is enough to show $L^1(\omega^n)$ convergence for a subsequence. Let u_{j_k} be the subsequence provided by Theorem 3.4.2 and v_{j_k} and w_{j_k} be corresponding monotone sequences. Then $v_{j_k} \le u_{j_k} \le w_{j_k}$ and $v_{j_k} \le u \le w_{j_k}$. Then

$$\begin{aligned}\int_X |u_{j_k} - u|\omega^n &\le \int_X (w_{j_k} - v_{j_k})\omega^n \\ &\le \int_X (w_{j_k} - u)\omega^n + \int_X (u - v_{j_k})\omega^n \\ &\to 0\end{aligned}$$

by the monotone convergence theorem. Thus the new topology is stronger and has more open, thus closed sets. □

Theorem 3.4.4. *If $u_j, u \in \mathcal{E}_\chi(X,\theta,\phi)$, such that $I_\chi(u_j,u) \to 0$ as $j \to \infty$, then $u_j \to u$ in capacity.*

Proof. It is enough to show the convergence in capacity for a subsequence. Let u_{j_k} be a subsequence as provided by Theorem 3.4.2 and v_{j_k} and w_{j_k} be the corresponding monotone sequences converging to u. We obtain $v_{j_k} \to u$ and $w_{j_k} \to u$ in capacity. We claim $u_{j_k} \to u$ in capacity.

Fix $\varepsilon > 0$.

$$\{|u_{j_k} - u| > \varepsilon\} \subset \{w_{j_k} - v_{j_k} > \varepsilon\} \subset \{w_{j_k} - u > \varepsilon/2\} \cup \{u - v_{j_k} > \varepsilon/2\}.$$

Taking capacity of above sets we get

$$\text{Cap}_\omega\{|u_{j_k} - u| > \varepsilon\} \leq \text{Cap}_\omega\{w_{j_k} - u > \varepsilon/2\} + \text{Cap}_\omega\{u - v_{j_k} > \varepsilon/2\}$$

and taking limit $k \to \infty$ we get $\lim_{k\to\infty} \text{Cap}_\omega\{|u_{j_k} - u| > \varepsilon\} = 0$.

Since for any subsequence u_{j_k} of u_j we can find another subsequence $u_{j_{k_l}}$ for which $\lim_{l\to\infty} \text{Cap}_\omega\{|u_{j_{k_l}} - u| > \varepsilon\} = 0$, we get that $\lim_{j\to\infty} \text{Cap}_\omega\{|u_j - u| > \varepsilon\} = 0$. Thus I_χ convergence implies convergence in capacity. □

Corollary 3.4.5 (Weak convergence of measures)**.** *If $u_k, u \in \mathcal{E}_\chi(X, \theta, \phi)$ such that $I_\chi(u_k, u) \to 0$, then $\theta^n_{u_k} \to \theta^n_u$ weakly as measures.*

Proof. Since $u_k \to u$ in I_χ, Theorem 3.4.4 shows that $u_k \to u$ in capacity. Since u_k, u have full mass, [31, Theorem 2.3] (see also [46, Theorem 1] and [47, Theorem 1]) implies that $\theta^n_{u_k} \to \theta^n_u$ weakly. □

3.5 Kähler Ricci flow

In [39], authors showed that we can start Kähler Ricci flow from any potential φ with zero Lelong numbers. They showed that if $\varphi \in \text{PSH}(X, \omega)$ has zero Lelong numbers then there exist smooth potentials φ_t for small time such that

$$\frac{\partial \varphi_t}{\partial t} = \log\left[\frac{(\omega + dd^c\varphi_t)^n}{\omega^n}\right], \varphi_t \to \varphi \text{ in } L^1(\omega^n) \text{ as } t \to 0. \tag{3.18}$$

Di Nezza–Lu [40, Corollary 5.2] further showed that $\varphi_j \to \varphi$ in capacity.

Without loss of generality, we can assume that $\varphi \leq -1$. In [39, Lemma 2.9], the authors showed that (for $\beta = 1$ and $\alpha = 1$) for a continuous ω-psh function u, satisfying

$$(\omega + dd^c u)^n = e^{u-2\varphi}\omega^n$$

we have

$$(1-2t)\varphi(x) + tu(x) + n(t\ln t - t) \leq \varphi_t.$$

Since $\varphi \leq -1$, we have $\varphi \leq (1-2t)\varphi$ for small t and since u is continuous there exists some constant C such that $C \leq u$. Combining these, we get

$$\varphi(x) + tC + n(t\ln t - t) \leq \varphi_t$$

or

$$\varphi(x) \leq \varphi_t - tC - n(t\ln t - t).$$

Notice that $-tC - n(t\ln t - t) = f(t)$ satisfies $f(t) \to 0$ as $t \to 0$.

Since $\varphi_t \to \varphi$ in $L^1(\omega^n)$ as $t \to 0$, we also obtain that $\varphi_t + f(t)$ converge to φ in $L^1(\omega^n)$ as $t \to 0$.

Lemma 3.5.1. *Given $\varphi \in \mathcal{E}_\chi(X,\omega)$ and a sequence $\varphi_j \in \mathcal{E}_\chi(X,\omega)$ such that*

$$\varphi \leq \varphi_j$$

and $\varphi_j \to \varphi$ in $L^1(\omega^n)$ as $j \to \infty$, then $I_\chi(\varphi_j, \varphi) \to 0$ as $j \to \infty$..

Proof. Since $\varphi_j \to \varphi$ in $L^1(\omega^n)$, we get a subsequence φ_{j_k} such that $\varphi_{j_k} \to \varphi$, a.e.. Consider the functions

$$v_j = \sup_{k \geq j} \varphi_{j_k}.$$

Then $v_j^* \in \mathrm{PSH}(X,\omega)$ and $v_j^* = v_j$ except on a pluripolar set. Moreover, since $v_j^* \searrow \varphi\, \omega^n$ almost everywhere, we get that $v_j^* \searrow \varphi$ everywhere.

This shows that $v_j \searrow \varphi$ except on a pluripolar set. This means $\lim_{j\to\infty} \sup_{k\geq j} \varphi_{k_j} = \varphi$ outside a pluripolar set. Therefore,

$$\limsup_{k\to\infty} \varphi_{j_k} = \varphi$$

except on a pluripolar set. Since $\varphi_{j_k} \geq \varphi$, we automatically have

$$\liminf_{k\to\infty} \varphi_{j_k} \geq \varphi$$

everywhere.

Therefore, $\varphi_{j_k} \to \varphi$ except on a pluripolar set. Using $\varphi \leq \varphi_{j_k} \leq v_k$ and that $v_k \searrow \varphi$ as $k \to \infty$, we get

$$\lim_{k\to\infty} \int_X \chi(\varphi_{j_k} - \varphi)\omega_\varphi^n \leq \lim_{k\to\infty} \int_X \chi(v_k - \varphi)\omega_\varphi^n = 0. \tag{3.19}$$

Moreover, since $\varphi_{j_k} \geq \varphi$, we have

$$\int_X \chi(\varphi_{j_k} - \varphi)\omega^n_{\varphi_{j_k}} = \int_0^\infty \chi'(s)\omega^n_{\varphi_{j_k}}(\varphi_{j_k} > \varphi + s)ds$$

Using comparison theorem, we get

$$\begin{aligned} &\leq \int_0^\infty \chi'(s)\omega^n_{\varphi}(\varphi_{j_k} > \varphi + s)ds \\ &= \int_X \chi(\varphi_{j_k} - \varphi)\omega^n_{\varphi}. \end{aligned}$$

Taking limit we get

$$\lim_{k\to\infty} \int_X \chi(\varphi_{j_k} - \varphi)\omega^n_{\varphi_{j_k}} \leq \lim_{k\to\infty} \int_X \chi(\varphi_{j_k} - \varphi)\omega^n_{\varphi} = 0.$$

Combining these two we get

$$\lim_{k\to\infty} I_\chi(\varphi_{j_k}, \varphi) = \lim_{k\to\infty} \int_X \chi(\varphi_{j_k} - \varphi)(\omega^n_{\varphi_{j_k}} + \omega^n_{\varphi}) = 0.$$

Since this holds for any subsequence of φ_j, we get that

$$I_\chi(\varphi_j, \varphi) \to 0.$$

$\square$

The following corollary generalizes the [39, Proposition 5.2] where they show that $\varphi_t \to \varphi$

in energy if $\varphi \in \mathcal{E}^1(X, \omega)$.

Corollary 3.5.2. *Let* $\varphi \in \mathcal{E}_\chi(X, \omega)$. *Let* φ_t *be the solution to Equation* (3.18). *Then* $I_\chi(\varphi_t, \varphi) \to 0$ *as* $t \to 0$.

Proof. We saw earlier that the assumptions imply that

$$\varphi \leq \varphi_t + f(t)$$

where $f(t) \to 0$ as $t \to 0$. Since $\varphi_t + f(t)$ are also θ-psh functions which converge in $L^1(\omega^n)$ to φ as $t \to 0$, using Lemma 3.5.1, we obtain $I_\chi(\varphi_t, \varphi) \to 0$ as $t \to 0$. $\square$

Note that using Theorem 3.4.4 and Corollary 3.4.5 we obtain that $\varphi_t \to \varphi$ in capacity and $\omega_{\varphi_t}^n \to \omega_\varphi^n$ as $t \to 0$, thus finishing the proof of Theorem 3.0.2.

Chapter 4: Complete Geodesic Metrics in big classes

In this chapter, we will prove Theorem 1.3.1. We will start with a discussion on Modifications in Section 4.1 as they play an important role in the construction of the metric. In Section 4.3, we will discuss the metric geometry of big and nef cohomology classes. In Section 4.4, we will discuss the construction of metric in the setting of analytic prescribed singularities. Finally, in Section 4.5, we will construct the metric in the big classes and prove Theorem 1.3.1. In Sections 4.6 and 4.8, we will prove various properties of the constructed metric.

4.1 Modifications

A holomorphic map $\mu : \tilde{X} \to X$ between compact Kähler manifolds $(\tilde{X}, \tilde{\omega})$ and (X, ω) is called a modification if outside a closed analytic set $E \subset \tilde{X}$, $\mu : \tilde{X} \setminus E \to X \setminus \mu(E)$ is a biholomorphism and $\mu(E) \subset X$ is also a closed analytic subset. We say that E is the exceptional set, and $\mu(E)$ is the center of the modification. In this paper, modifications arise from resolving singularities of quasi plurisubharmonic functions with analytic singularity type. See Section 4.4 for more details.

If θ is a smooth closed $(1,1)$-form on X representing a big class, then $\mu^*\theta$ is also a big class on $\tilde{X}$ (see [48, Proposition 4.12]). If $u \in \mathrm{PSH}(X, \theta)$, then $u \circ \mu \in \mathrm{PSH}(\tilde{X}, \mu^*\theta)$. In this particular case, the reverse is also true.

Lemma 4.1.1. *Let $\mu : \tilde{X} \to X$ be a modification with the exceptional set E and center $\mu(E)$. If θ represents a big cohomology class on X and $v \in PSH(\tilde{X}, \mu^*\theta)$, then there exists a unique $u \in PSH(X, \theta)$ such that $v = u \circ \mu$.*

Proof. Since $\mu : \tilde{X} \setminus E \to X \setminus \mu(E)$ is a biholomorphism, we know $v \circ \mu^{-1}$ is a θ-psh function on $X \setminus \mu(E)$. Since $\mu(E)$ is an analytic set and $v \circ \mu^{-1}$ is bounded from above, it extends over $\mu(E)$ to all of X. We call this extension u. Thus there exists $u \in \mathrm{PSH}(X, \theta)$ such that on $\tilde{X} \setminus E$, $u \circ \mu = v$. Since both $u \circ \mu$ and v are $\mu^*\theta$-psh functions that agree almost everywhere, they must agree everywhere. Thus $u \circ \mu = v$. By the same argument, u is unique as well. $\square$

In general, we can pullback smooth forms and pushforward currents. However, for positive $(1,1)$-currents, we can define the pullback as follows. If $u \in \mathrm{PSH}(X, \theta)$, then $\mu^*(\theta_u) := \mu^*\theta + dd^c u \circ \mu$. Moreover, it satisfies $\mu_*\mu^*\theta_u = \theta_u$. We recall

Theorem 4.1.2 ([49, Theorem 3.1])**.** *If $\mu : \tilde{X} \to X$ is a modification and $\theta^1 \dots \theta^n$ are real smooth closed $(1,1)$-forms on $\tilde{X}$ representing big cohomology classes and $u_j \in PSH(\tilde{X}, \theta_j)$, then*

$$\mu_* \langle \theta^1_{u_1} \wedge \cdots \wedge \theta^n_{u_n} \rangle = \langle \mu_* \theta^1_{u_1} \wedge \cdots \wedge \mu_* \theta^n_{u_n} \rangle.$$

Applying this theorem to $\mu^*\theta_u$, we obtain $\mu_*((\mu^*\theta)^n_{u \circ \mu}) = \theta^n_u$.

4.2 Spaces of finite entropy

Let θ represent a big class, and $u \in \mathrm{PSH}(X,\theta)$. Let ω^n be a background unit volume Kähler volume form. Assume $\int_X \theta_u^n > 0$. We define,

$$\mathrm{Ent}(\omega^n, \theta_u^n) = \int_X \log\left(\frac{\theta_u^n}{\omega^n}\right)\theta_u^n$$

if θ_u^n has a density with respect to ω^n and the entropy is $+\infty$ otherwise.

We denote by

$$\mathrm{Ent}(X,\theta) = \{u \in \mathrm{PSH}(X,\theta) : \mathrm{Ent}(\omega^n, \theta_u^n) < \infty\}.$$

The following lemma is an observation made in [50, Proposition 3.10]. It tells us that pulling back a potential of finite entropy under a modification still has finite entropy.

Lemma 4.2.1 ([50, Proposition 3.10])**.** *If $\mu : \tilde{X} \to X$ is a modification and $u \in Ent(X,\theta)$ has finite entropy, then $u \circ \mu \in Ent(\tilde{X}, \mu^*\theta)$.*

We recall another result from [50].

Lemma 4.2.2 ([50, Proposition 3.3])**.** *If $f \in C^{1,\bar{1}}(X)$, and $\phi \in PSH(X,\theta)$ is a model potential, then $P_\theta[\phi](f) \in Ent(X,\theta)$.*

4.3 Metric geometry in the big and nef case

The metric geometry of $\mathcal{E}^p(X,\beta)$, when β represents a big and nef cohomology class, was studied by Di Nezza-Lu in [16]. We will briefly describe how they defined the d_p metric on

$\mathcal{E}^p(X, \beta)$. They defined

$$\mathcal{H}_\beta = \{u \in \mathrm{PSH}(X, \beta) \,|\, u = P_\beta(f) \text{ for } f \in C(X) \text{ such that } dd^c f \leq C(f)\omega\}. \tag{4.1}$$

As β is big and nef class, for all $\varepsilon > 0$, $\omega_\varepsilon := \beta + \varepsilon\omega$ represents a Kähler class, although it may not be a Kähler form. The metric d_p on $\mathcal{H}_\beta$ is defined by approximation from $\mathcal{E}^p(X, \omega_\varepsilon)$. In particular, if $u_0, u_1 \in \mathcal{H}_\beta$, such that $u_0 = P_\beta(f_0)$ and $u_1 = P_\beta(f_1)$, then we define $u_{0,\varepsilon} = P_{\omega_\varepsilon}(f_0)$ and $u_{1,\varepsilon} = P_{\omega_\varepsilon}(f_1)$ and

$$d_p(u_0, u_1) := \lim_{\varepsilon \to 0} d_p(u_{0,\varepsilon}, u_{1,\varepsilon}).$$

More generally, on $\mathcal{E}^p(X, \beta)$, the metric d_p is defined by approximation from $\mathcal{H}_\beta$. In particular, if $u_0, u_1 \in \mathcal{E}^p(X, \beta)$, then we can find $u_0^j, u_1^j \in \mathcal{H}_\beta$ such that $u_0^j \searrow u_0$ and $u_1^j \searrow u_1$ and we define

$$d_p(u_0, u_1) := \lim_{j \to \infty} d_p(u_0^j, u_1^j).$$

In [16], Di Nezza–Lu proved

Theorem 4.3.1 ([16])**.** *If β represents a big and nef cohomology class, then the function d_p defined as above is a complete geodesic metric on $\mathcal{E}^p(X, \beta)$. They also showed in the proof of [16, Theorem 3.17] that the weak geodesic u_t joining $u_0, u_1 \in \mathcal{E}^p(X, \beta)$ are metric geodesics as well.*

We list some properties of $(\mathcal{E}^p(X, \beta), d_p)$ from their paper that we will frequently use.

Theorem 4.3.2 (Pythagorean identity, [16, Theorem 3.14])**.** *If $u, v \in \mathcal{E}^p(X, \beta)$, then*

$$d_p^p(u, v) = d_p^p(u, P_\beta(u, v)) + d_p^p(v, P_\beta(u, v)).$$

For $u_0, u_1 \in \mathcal{E}^p(X, \beta)$ we define

$$I_p(u, v) = \int_X |u - v|^p (\beta_u^n + \beta_v^n).$$

The following theorem shows that I_p controls the distance d_p.

Theorem 4.3.3 ([16, Proposition 3.12])**.** *Given $u_0, u_1 \in \mathcal{E}^p(X, \beta)$, there exists a constant $C > 1$ that depends only on the dimension, such that*

$$\frac{1}{C} I_p(u_0, u_1) \leq d_p^p(u_0, u_1) \leq C I_p(u_0, u_1).$$

We recall the following

Theorem 4.3.4 ([51, Theorem 1.2])**.** *If β represents a big and nef cohomology class, $u_0, u_1 \in$ Ent(X, β) have minimal singularity type, and u_t is the weak geodesic joining u_0 and u_1, then*

$$d_p(u_0, u_1) = \int_X |\dot{u}_t|^p \beta_{u_t}^n \qquad \textit{for all } t \in [0, 1].$$

If $p = 1$, we have some special properties for the distance d_1.

Theorem 4.3.5 ([16, Theorem 3.18])**.** *If $u_0, u_1 \in \mathcal{E}^1(X, \beta)$, then*

$$d_1(u_0, u_1) = I_\beta(u_0) + I_\beta(u_1) - 2I_\beta(P_\beta(u_0, u_1)).$$

This theorem implies the following stronger result when the potentials u_0 and u_1 are comparable.

Lemma 4.3.6. *If β represents a big and nef cohomology class, $u_0, u_1 \in$ Ent(X, β) having minimal singularity satisfy $u_0 \leq u_1$, and u_t is the weak geodesic joining u_0 and u_1, then*

$$I_\beta(u_1) - I_\beta(u_0) = \int_X \dot{u}_t \beta_{u_t}^n \qquad \textit{for all } t \in [0,1].$$

Proof. Since the path $w_t \mapsto u_0$ is a subgeodesic joining u_0 and u_1, therefore $u_t \geq u_0$. This means that $\dot{u}_0 \geq 0$. By convexity of u_t in the t variable, we get that $0 \leq \dot{u}_0 \leq \dot{u}_t$.

Since $u_0 \leq u_1$, we have $P_\beta(u_0, u_1) = u_0$. Thus Theorem 4.3.5 implies

$$d_1(u_0, u_1) = I_\beta(u_1) - I_\beta(u_0).$$

On the other hand, Theorem 4.3.4 along with the observation that $\dot{u}_t \geq 0$ imply that

$$d_1(u_0, u_1) = \int_X |\dot{u}_t| \beta_{u_t}^n = \int_X \dot{u}_t \beta_{u_t}^n \qquad \text{for all } t \in [0,1].$$

Combining the two expressions for $d_1(u_0, u_1)$ we get,

$$I_\beta(u_1) - I_\beta(u_0) = \int_X \dot{u}_t \beta_{u_t}^n \qquad \text{for all } t \in [0,1].$$

□

When θ represents a big, and not necessarily nef class, we can have the above result in a slightly restrictive setting as in the following lemma.

Lemma 4.3.7. *Let θ represent a big cohomology class and let $u_0 = P_\theta(f_0)$ and $u_1 = P_\theta(f_1)$ for*

$f_0, f_1 \in C^{1,\bar{1}}(X)$ *satisfy* $u_0 \leq u_1$. *If* u_t *is the weak geodesic joining* u_0 *and* u_1, *then*

$$I_\theta(u_1) - I_\theta(u_0) = \int_X \dot{u}_0 \theta_{u_0}^n = \int_X \dot{u}_1 \theta_{u_1}^n.$$

Proof. The proof extends the ideas in the proof of [16, Proposition 3.18] to the big case. The idea is to use Theorem 2.5.3, Lemma 2.7.1 along with [17, Theorem 2.4] which says that for $u, v \in \mathrm{PSH}(X, \theta)$ with minimal singularity type, $\int_X (u-v)\theta_u^n \leq I_\theta(u) - I_\theta(v) \leq \int_X (u-v)\theta_v^n$.

By convexity of the geodesic u_t in the t-direction, we have $0 \leq \dot{u}_0 \leq \dot{u}_t \leq \dot{u}_1$. Thus u_t is increasing with t. Thus we have

$$\begin{aligned}\int_X \dot{u}_0 \theta_{u_0}^n &= \int_X \lim_{t\to 0} \frac{u_t - u_0}{t} \theta_{u_0}^n \\ &= \lim_{t\to 0} \int_X \frac{u_t - u_0}{t} \theta_{u_0}^n \\ &\geq \lim_{t\to 0} \frac{I_\theta(u_t) - I_\theta(u_0)}{t} \\ &= \lim_{t\to 0} I_\theta(u_1) - I_\theta(u_0).\end{aligned}$$

In the second line, we could exchange limit with integral because of the convexity of u_t in the t variable and the monotone convergence theorem. In the third line, we used the inequality mentioned above, and in the last line, we used that I is affine along the weak geodesics. Similarly, we can show that

$$\begin{aligned}\int_X \dot{u}_1 \theta_{u_1}^n &= \int_X \lim_{t\to 1} \frac{u_1 - u_t}{1-t}\theta_{u_1}^n \\ &= \lim_{t\to 1}\int_X \frac{u_1 - u_t}{1-t}\theta_{u_1}^n \\ &\leq \lim_{t\to 1} \frac{I_\theta(u_1) - I_\theta(u_t)}{1-t} \\ &= I_\theta(u_1) - I_\theta(u_0).\end{aligned}$$

Combining these two we get $\int_X \dot{u}_0\theta_{u_0}^n \geq I_\theta(u_1) - I_\theta(u_0) \geq \int_X \dot{u}_1\theta_{u_1}^n$. Combining with Theorem 2.5.3 for $p = 1$, we get that

$$\int_X \dot{u}_0\theta_{u_0}^n = I_\theta(u_1) - I_\theta(u_0) = \int_X \dot{u}_1\theta_{u_1}^n.$$

$\square$

4.4 From the Big and Nef to the Prescribed Analytic Singularity

(X, ω) be a compact Kähler manifold and θ be a closed smooth $(1,1)$-form representing a big cohomology class. We fix $\psi \in \mathrm{PSH}(X, \theta)$ a model potential that has analytic singularities of type $(\mathcal{I}, c)$. By Hironaka's embedded desingularization theorem, we can find a modification $\mu : \tilde{X} \to X$ such that $\mu^*\mathcal{I} = \mathcal{O}(-E)$ where $E = \sum_i \lambda_i E_i$ is a simple normal crossing divisor. We can choose metrics h_i on $\mathcal{O}(E_i)$ and canonical sections s_i of $\mathcal{O}(E_i)$. Let R_{h_i} be the curvature for the metrics h_i on $\mathcal{O}(E_i)$. We denote

$$|s|_h^2 = \prod_{i=1}^{k} |s_i|_{h_i}^{2\lambda_i} \quad \text{and} \quad R_h = \sum_{i=1}^{k} \lambda_i R_{h_i}$$

Thus, for this modification, we have

$$\psi \circ \mu = c \log |s|_h^2 + g$$

where g is a bounded function. See [52, Section 5.9] for more details.

Now, $\mu^*\theta + dd^c\psi \circ \mu \geq 0$. Thus $\mu^*\theta + cdd^c \log |s|_h^2 + dd^c g \geq 0$. By the Poincaré–Lelong formula

$$[E] = R_h + dd^c \log |s|_h^2,$$

where $[E]$ is the current of integration along E, we can write $\mu^*\theta - cR_h + c[E] + dd^c g \geq 0$. Define

$$\tilde{\theta} = \mu^*\theta - cR_h, \tag{4.2}$$

so that

$$\mu^*\theta + dd^c(\psi \circ \mu) = \tilde{\theta} + c[E] + dd^c g. \tag{4.3}$$

On $\tilde{X} \setminus E$ (we abuse the notation to denote by E the analytic set on which the divisor E is supported), $\tilde{\theta} + dd^c g \geq 0$. As g is bounded from above, g extends uniquely to the whole $\tilde{X}$ to a $\tilde{\theta}$-psh function g. Thus $\tilde{\theta} + dd^c g \geq 0$ on all of $\tilde{X}$. Since g is a bounded $\tilde{\theta}$-psh function, $\tilde{\theta}$ represents a nef class. Soon in Corollary 4.4.3 we will see that $\tilde{\theta}$ is a big class as well. The following argument using Demailly's regularization theorem shows $\tilde{\theta}$ is nef.

Since $\tilde{\theta} + dd^c g \geq 0$, we have $\tilde{\theta} + \varepsilon\tilde{\omega} + dd^c g \geq \varepsilon\tilde{\omega}$, where $\tilde{\omega}$ is an arbitrary Kähler form on $\tilde{X}$. Demailly's regularization theorem implies there is a Kähler potential ϕ in the class $\{\theta + \varepsilon\tilde{\omega}\}$ with analytic singularities such that $\phi \geq g$ which is smooth outside its singular locus. Since g is bounded from below, ϕ has no singular locus, thus ϕ is a smooth Kähler potential in $\{\tilde{\theta} + \varepsilon\tilde{\omega}\}$, so $\tilde{\theta} + \varepsilon\tilde{\omega}$ is a Kähler class. This shows that $\tilde{\theta}$ is nef.

We can go back and forth between the spaces PSH(X, θ, ψ) and PSH$(\tilde{X}, \tilde{\theta})$ that preserves various pluripotential theoretic relationships. The following theorem describes the correspondence between PSH$(X, \theta, \psi) \leftrightarrow$ PSH$(\tilde{X}, \tilde{\theta})$. This correspondence is well known in the community (see [37, Lemma 4.3] and [53, Section 4.1]), but we write a proof here for completeness, as our definition of analytic singularities is slightly more general than in [53].

Theorem 4.4.1. *Let θ represent a big cohomology class on X and $\psi \in$ PSH(X, θ) has analytic singularities. Let $\mu : \tilde{X} \to X$ be the desingularization of the singularities of ψ and $\tilde{\theta}$ be a closed smooth $(1,1)$-form on $\tilde{X}$ as described above. Then the map PSH$(X, \theta, \psi) \ni u \mapsto \tilde{u} := (u - \psi) \circ \mu + g \in$ PSH$(\tilde{X}, \tilde{\theta})$ is an order-preserving bijection.*

Moreover, u has the same singularity type as ψ if and only if $\tilde{u}$ has minimal singularity type.

Proof. Let $u \in$ PSH(X, θ, ψ). On $X \setminus E$, $dd^c \log |s|_h^2 + R_h = 0$. Thus on $X \setminus E$, $u \circ \mu - c \log |s|_h^2$ is a $(\mu^*\theta - cR_h)$-psh function. As $u \circ \mu - c \log |s|_h^2 = u \circ \mu - \psi \circ \mu + g$ and $(u - \psi) \circ \mu$ is bounded from above, we get $(u - \psi) \circ \mu + g$ is bounded from above, so it extends to a $(\mu^*\theta - cR_h)$-psh function on all of $\tilde{X}$. As $\tilde{\theta} = \mu^*\theta - cR_h$, $(u - \psi) \circ \mu + g$ is $\tilde{\theta}$-psh.

For the reverse direction let $v \in \mathrm{PSH}(\tilde{X}, \tilde{\theta})$. So $\tilde{\theta} + dd^c v \geq 0$. From (4.3),

$$\mu^*\theta - c[E] + dd^c(\psi \circ \mu - g + v) \geq 0.$$

Thus

$$\mu^*\theta + dd^c(\psi \circ \mu - g + v) \geq 0.$$

Thus $(\psi \circ \mu - g + v)$ is a $\mu^*\theta$-psh function. From Lemma 4.1.1, we see that there exists a unique $u \in \mathrm{PSH}(X, \theta)$ such that $u \circ \mu = \psi \circ \mu - g + v$. On $X \setminus \mu(E)$, $u = \psi - g \circ \mu^{-1} + v \circ \mu^{-1} \leq \psi + C$. Thus this inequality holds everywhere. Thus $u \in \mathrm{PSH}(X, \theta, \psi)$.

Clearly, the map $u \mapsto (u - \psi) \circ \mu + g$ is order-preserving.

If u has the same singularity type as ψ, then for some C, $\psi - C \leq u$. Thus $-C \leq u - \psi$. Thus, $-C \leq (u - \psi) \circ \mu = \tilde{u} - g$. As g is bounded, we get $\tilde{u}$ has the minimal singularity type.

Similarly, if $\tilde{u}$ has the minimal singularity type, then $-C \leq (u - \psi) \circ \mu + g$. Thus, on $X \setminus \mu(E)$, $-C \leq u - \psi + g \circ \mu^{-1}$ that implies $\psi - C' \leq u$ as g is bounded. Since both are θ-psh functions, the inequality holds everywhere, therefore $\psi - C' \leq u$, hence u has the same singularity type as ψ. $\square$

Remark 4.4.2. We observe that the non-pluripolar product of $[E] = 0$ and Theorem 4.1.2 allows us to compute the non-pluripolar product of the modified potentials. Given $u_1, \ldots, u_n \in \mathrm{PSH}(X, \theta, \psi)$, and the corresponding $\tilde{u}_j := (u_j - \psi) \circ \mu + g \in \mathrm{PSH}(\tilde{X}, \tilde{\theta})$, their non-pluripolar product satisfies

$$\mu_* \langle \tilde{\theta}_{\tilde{u}_1} \wedge \cdots \wedge \tilde{\theta}_{\tilde{u}_n} \rangle = \langle \theta_{u_1} \wedge \cdots \wedge \theta_{u_n} \rangle.$$

A consequence of the above theorem is that the bijection $\mathrm{PSH}(X, \theta, \psi) \leftrightarrow \mathrm{PSH}(\tilde{X}, \tilde{\theta})$

preserves the mass and the finite energy classes of the potentials.

Corollary 4.4.3. *Under the bijection* $PSH(X,\theta,\psi) \ni u \mapsto \tilde{u} := (u-\psi)\circ\mu + g \in PSH(\tilde{X},\tilde{\theta})$, *we have* $\int_X \theta_u^n = \int_{\tilde{X}} \tilde{\theta}_{\tilde{u}}^n$ *and*

$$\int_X |u-\psi|^p \theta_u^n < \infty \iff \int_{\tilde{X}} |\tilde{u} - V_{\tilde{\theta}}|^p \tilde{\theta}_{\tilde{u}} < \infty$$

Thus the map $u \mapsto \tilde{u}$ *is also a bijection between* $\mathcal{E}^p(X,\theta,\psi)$ *and* $\mathcal{E}^p(\tilde{X},\tilde{\theta})$.

Proof. Applying Remark 4.4.2 to any potential $u \in \mathrm{PSH}(X,\theta,\psi)$, we get that $\mu_*\tilde{\theta}_{\tilde{u}}^n = \theta_u^n$. Upon integration, we find that

$$\int_X \theta_u^n = \int_X \mu_*\tilde{\theta}_{\tilde{u}}^n = \int_{\tilde{X}} \tilde{\theta}_{\tilde{u}}^n.$$

Thus, u and $\tilde{u}$ have the same mass. Similarly, integrating the function $|u-\psi|^p$ we get

$$\int_X |u-\psi|^p \theta_u^n = \int_X |u-\psi|^p \mu_*\tilde{\theta}_{\tilde{u}}^n = \int_{\tilde{X}} |(u-\psi)\circ\mu|^p \tilde{\theta}_{\tilde{u}}^n = \int_{\tilde{X}} |\tilde{u}-g|^p \tilde{\theta}_{\tilde{u}}^n.$$

Now if $\int_{\tilde{X}} |\tilde{u} - V_{\tilde{\theta}}|^p \tilde{\theta}_{\tilde{u}}^n < \infty$, then

$$\int_{\tilde{X}} |\tilde{u}-g|^p \tilde{\theta}_{\tilde{u}}^n = \int_{\tilde{X}} |\tilde{u} - V_{\tilde{\theta}} - (g - V_{\tilde{\theta}})|^p \tilde{\theta}_{\tilde{u}}^n \leq 2^{p-1}\left(\int_{\tilde{X}} |\tilde{u} - V_{\tilde{\theta}}|^p \tilde{\theta}_{\tilde{u}}^n + \int_{\tilde{X}} |g - V_{\tilde{\theta}}|^p \tilde{\theta}_{\tilde{u}}^n\right) < \infty.$$

Here we used the Minkowski's inequality ($|a+b|^p \leq 2^{p-1}(|a|^p + |b|^p)$ if $p \geq 1$), and the fact that g and $V_{\tilde{\theta}}$ are bounded functions. We can show the other side in the same manner. If $\int_{\tilde{X}} |\tilde{u}-g|^p \tilde{\theta}_{\tilde{u}}^n < \infty$, then

$$\int_{\tilde{X}} |\tilde{u} - V_{\tilde{\theta}}|^p \tilde{\theta}_{\tilde{u}}^n = \int_{\tilde{X}} |\tilde{u} - g + g - V_{\tilde{\theta}}|^p \tilde{\theta}_{\tilde{u}}^n \leq 2^{p-1}\left(\int_{\tilde{X}} |\tilde{u}-g|^p \tilde{\theta}_{\tilde{u}}^n + \int_{\tilde{X}} |g - V_{\tilde{\theta}}|^p \tilde{\theta}_{\tilde{u}}^n\right) < \infty.$$

$\square$

Lemma 4.2.1 shows that the bijection $\mathrm{PSH}(X, \theta, \psi) \leftrightarrow \mathrm{PSH}(\tilde{X}, \tilde{\theta})$ preserves entropy in one direction.

Lemma 4.4.4. *If $u \in PSH(X, \theta, \psi) \cap Ent(X, \theta)$. Then $\tilde{u} = (u - \psi) \circ \mu + g \in Ent(\tilde{X}, \tilde{\theta})$.*

Proof. Let $\tilde{\omega}^n$ be a unit Kähler volume form on $\tilde{X}$. Recall from Lemma 4.2.1 that $u \circ \mu \in \mathrm{Ent}(X, \mu^*\theta)$. Thus the measure $\langle(\mu^*\theta + dd^c u \circ \mu)^n\rangle$ has finite entropy with respect to $\tilde{\omega}^n$. As non-pluripolar measures, we know that $\tilde{\theta}^n_{\tilde{u}} = \langle(\mu^*\theta + dd^c u \circ \mu)^n\rangle$, we get that the measure $\tilde{\theta}^n_{\tilde{u}}$ has finite entropy with respect to $\tilde{\omega}^n$ as well. Thus $\tilde{u} \in \mathrm{Ent}(\tilde{X}, \tilde{\theta})$. $\square$

The bijective correspondence between $\mathrm{PSH}(\tilde{X}, \tilde{\theta}) \leftrightarrow \mathrm{PSH}(X, \theta, \psi)$ preserves the weak geodesics.

Theorem 4.4.5. *If $u_t \in PSH(X, \theta, \psi)$ is the weak geodesic joining $u_0, u_1 \in PSH(X, \theta, \psi)$, then $\tilde{u}_t \in PSH(\tilde{X}, \tilde{\theta})$ is the weak geodesic joining $\tilde{u}_0, \tilde{u}_1 \in PSH(\tilde{X}, \tilde{\theta})$.*

Proof. First, we will show that a subgeodesic $(0, 1) \ni t \to v_t \in \mathrm{PSH}(X, \theta, \psi)$ maps to a subgeodesic $(0, 1) \ni t \to \tilde{v}_t := (v_t - \psi) \circ \mu + g \in \mathrm{PSH}(\tilde{X}, \tilde{\theta})$ and vice versa.

We consider the following diagram of maps.

$$\begin{array}{ccc} \tilde{X} \times S & \xrightarrow{\tilde{\pi}} & \tilde{X} \\ {\scriptstyle \mu\times\mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle \mu} \\ X \times S & \xrightarrow{\pi} & X \end{array}$$

We will show that the map $\tilde{X} \times S \ni (x, z) \mapsto (v_{\mathrm{Re}(z)} - \psi) \circ \mu(x) + g(x)$ is $\tilde{\pi}^*\tilde{\theta}$-psh map. As earlier, we will show that it is true on $(\tilde{X} \setminus E) \times S$ and then use boundedness of $(v_{\mathrm{Re}(z)} - \psi) \circ \mu + g$ to conclude that it is true on all of $\tilde{X} \times S$.

Since $t \mapsto v_t \in \mathrm{PSH}(X, \theta, \psi)$ is a subgeodesic we get

$$\pi^*\theta + dd^c v_{\mathrm{Re}(z)}(x) \geq 0.$$

Pull it back by $\mu \times \mathrm{id}$ so that

$$(\mu \times \mathrm{id})^* \pi^* \theta + dd^c(v_{\mathrm{Re}(z)} \circ \mu(x)) \geq 0.$$

Using the fact that $\pi \circ (\mu \times \mathrm{id}) = \mu \circ \tilde{\pi}$ we get

$$\begin{aligned} &\tilde{\pi}^* \mu^* \theta + dd^c(v_{\mathrm{Re}(z)} \circ \mu(x)) \geq 0 \\ \implies &\tilde{\pi}^*(\mu^*\theta + dd^c\psi) + dd^c((v_{\mathrm{Re}(z)} - \psi) \circ \mu(x)) \geq 0. \end{aligned}$$

Since on $\tilde{X} \setminus E$, $\mu^*\theta + dd^c\psi = \tilde{\theta} + dd^c g$ (see (4.3)) we get

$$\tilde{\pi}^*\tilde{\theta} + dd^c((v_{\mathrm{Re}(z)} - \psi) \circ \mu(x) + g(x)) \geq 0.$$

Thus, we see that the function $(\tilde{X} \setminus E) \times S \ni (x, z) \mapsto (v_{\mathrm{Re}(z)} - \psi) \circ \mu(x) + g(x)$ is $\tilde{\pi}^*\tilde{\theta}$-psh function. Since the function is also bounded from above it extends to all of $\tilde{X} \times S$. Thus $(0, 1) \ni t \mapsto (v_t - \psi) \circ \mu + g \in \mathrm{PSH}(\tilde{X}, \tilde{\theta})$ is a subgeodesic.

Now we see the other direction. Let $(0, 1) \ni t \mapsto \tilde{v}_t \in \mathrm{PSH}(\tilde{X}, \tilde{\theta})$ is a subgeodesic. This means $\tilde{\pi}^*\tilde{\theta} + dd^c \tilde{v}_{\mathrm{Re}(z)}(x) \geq 0$. We saw earlier that for each $\tilde{v}_t$ there exists a unique $v_t \in \mathrm{PSH}(X, \theta, \psi)$ such that $(v_t - \psi) \circ \mu + g = \tilde{v}_t$. We have to show that $(0, 1) \ni t \mapsto v_t$ is a subgeodesic.

To see this notice

$$\tilde{\pi}^*\tilde{\theta} + dd^c\tilde{v}_{\mathrm{Re}(z)}(x) \geq 0.$$

Since $\tilde{\theta} = \mu^*\theta - [E] + dd^c\psi \circ \mu - dd^c g$ from (4.3), above equation implies

$$\tilde{\pi}^*\mu^*\theta + dd^c(\tilde{v}_{\mathrm{Re}(z)} + \psi \circ \mu - g)(x) \geq 0.$$

Now use that $v_t \circ \mu = \psi \circ \mu + \tilde{v}_t - g$ and commutation of the diagram, to see

$$(\mu \times \mathrm{id})^*\pi^*\theta + dd^c v_{\mathrm{Re}(z)} \circ \mu(x) \geq 0.$$

Now pushforward by $(\mu \times \mathrm{id})_*$ to $X \times S$ to see

$$\pi^*\theta + dd^c v_{\mathrm{Re}(z)}(x) \geq 0.$$

Hence $X \times S \ni (x, z) \mapsto v_{\mathrm{Re}(z)}(x)$ is a $\pi^*\theta$-psh function. Thus $(0, 1) \ni t \mapsto v_t$ is a subgeodesic.

Since subgeodesics correspond to subgeodesics under the correspondence PSH$(X, \theta, \psi) \leftrightarrow$ PSH$(\tilde{X}, \tilde{\theta})$, and geodesics are just supremum over subgeodesics, we get that the geodesics correspond to geodesics as well. In particular, if $u_0, u_1 \in$ PSH(X, θ, ψ) and $u_t \in$ PSH(X, θ, ψ) is a geodesic joining u_0 and u_1, then $\tilde{u}_t = (u - \psi) \circ \mu + g \in$ PSH$(\tilde{X}, \tilde{\theta})$ is the geodesic joining $\tilde{u}_0$ and $\tilde{u}_1$. $\square$

Our next theorem allows us to extend Lemma 2.7.1 in the case of prescribed singularity setting.

Proposition 4.4.6. *If $u_0, u_1 \in PSH(X, \theta, \psi)$ have the same singularity type as ψ, and if u_t is the weak geodesic joining u_0 and u_1, then*

$$I_\psi(u_t) = (1-t)I_\psi(u_0) + tI_\psi(u_1).$$

Proof. First, we will show that the correspondence between $\mathrm{PSH}(X, \theta, \psi)$ and $\mathrm{PSH}(\tilde{X}, \tilde{\theta})$ preserves the Monge–Ampère energy up to a constant. Second, we use Lemma 2.7.1 to obtain that the Monge–Ampère energy is linear along u_t.

Take $u \in \mathrm{PSH}(X, \theta, \psi)$ with the same singularity type as ψ and let $\tilde{u} := (u - \psi) \circ \mu + g \in \mathrm{PSH}(\tilde{X}, \tilde{\theta})$. Then Remark 4.4.2, tells us that

$$\mu_*(\tilde{\theta}^j_{\tilde{u}} \wedge \tilde{\theta}^{n-j}_g) = \theta^j_u \wedge \theta^{n-j}_\psi. \tag{4.4}$$

The Monge–Ampère energy of u is given by

$$I_\psi(u) = \frac{1}{n+1} \sum_{j=0}^{n} \int_X (u - \psi)\theta^j_u \wedge \theta^{n-j}_\psi.$$

From (4.4) we get

$$I_\psi(u) = \frac{1}{n+1} \sum_{j=0}^{n} \int_X (u - \psi)\mu_*(\tilde{\theta}^j_{\tilde{u}} \wedge \tilde{\theta}^{n-j}_g) = \frac{1}{n+1} \sum_{j=0}^{n} \int_{\tilde{X}} (u - \psi) \circ \mu \, \tilde{\theta}^j_{\tilde{u}} \wedge \tilde{\theta}^{n-j}_g.$$

Thus, we have,

$$I_\psi(u) = \frac{1}{n+1}\sum_{j=0}^{n}\int_{\tilde{X}}(\tilde{u}-g)\,\tilde{\theta}_{\tilde{u}}^{j}\wedge\tilde{\theta}_{g}^{n-j} = I_{\tilde{\theta}}(\tilde{u}) - I_{\tilde{\theta}}(g).$$

By Theorem 4.4.5, we know that $\tilde{u}_t$ is a geodesic joining $\tilde{u}_0$ and $\tilde{u}_1$ and by Theorem 4.4.1, $\tilde{u}_0$ and $\tilde{u}_1$ have minimal singularity in PSH$(\tilde{X}, \tilde{\theta})$. Thus we can use Lemma 2.7.1, to get that $I_{\tilde{\theta}}(\tilde{u}_t) = (1-t)I_{\tilde{\theta}}(\tilde{u}_0) + tI_{\tilde{\theta}}(\tilde{u}_0)$. From the calculation above, we have

$$I_\psi(u_t) = I_{\tilde{\theta}}(\tilde{u}_t) - I_{\tilde{\theta}}(g) = (1-t)I_{\tilde{\theta}}(\tilde{u}_0) + tI_{\tilde{\theta}}(\tilde{u}_1) - I_{\tilde{\theta}}(g) = (1-t)I_\psi(u_0) + tI_\psi(u_0)$$

as desired. □

4.4.1 Metric space structure on $\mathcal{E}^p(X, \theta, \psi)$

In this section, we will import the metric space structure on $\mathcal{E}^p(X, \theta, \psi)$ from the metric space structure in $\mathcal{E}^p(\tilde{X}, \tilde{\theta})$ when ψ is a model singularity with analytic singularity type.

We can define the distance between $u_0, u_1 \in \mathcal{E}^p(X, \theta, \psi)$ as follows. Let $\tilde{u}_0 = (u_0 - \psi) \circ \mu + g$ and $\tilde{u}_1 = (u_1 - \psi) \circ \mu + g$ be the corresponding potentials in $\mathcal{E}^p(\tilde{X}, \tilde{\theta})$. Corollary 4.4.3 tells us that $\tilde{u}_0, \tilde{u}_1 \in \mathcal{E}^p(\tilde{X}, \tilde{\theta})$. So we can define

$$d_p(u_0, u_1) := d_p(\tilde{u}_0, \tilde{u}_1). \tag{4.5}$$

Theorem 4.4.13 below, shows that the metric as defined above does not depend on the choice of the resolution of the singularities of ψ.

Theorem 4.4.7. *The map d_p as defined by (4.5) makes $\mathcal{E}^p(X,\theta,\psi)$ a complete geodesic metric space.*

Proof. $(\mathcal{E}^p(X,\theta,\psi),d_p)$ is a complete metric space because $(\mathcal{E}^p(\tilde{X},\tilde{\theta}),d_p)$ is a complete metric space. Moreover, if $u_t \in \mathcal{E}^p(X,\theta,\psi)$ is the weak geodesic joining $u_0,u_1 \in \mathcal{E}^p(X,\theta,\psi)$, we claim u_t is also the metric geodesic. This means that for $0 \leq t \leq s \leq 1$, we have $d_p(u_t,u_s) = |t-s|d_p(u_0,u_1)$.

We know $\tilde{u}_0,\tilde{u}_1 \in \mathcal{E}^p(\tilde{X},\tilde{\theta})$. In the proof of [16, Theorem 3.17], authors show that the weak geodesic $\tilde{u}_t$ joining $\tilde{u}_0$ and $\tilde{u}_1$ satisfies $d_p(\tilde{u}_t,\tilde{u}_s) = |t-s|d_p(\tilde{u}_0,\tilde{u}_1)$. Thus by the definition of d_p on $\mathcal{E}^p(X,\theta,\psi)$, we obtain that $d_p(u_t,u_s) = |t-s|d_p(u_0,u_1)$. Hence $(\mathcal{E}^p(X,\theta,\psi),d_p)$ is a complete geodesic metric space, with the weak geodesics being the metric geodesics as well. □

Now we prove some useful properties of the metric $(\mathcal{E}^p(X,\theta,\psi),d_p)$.

Lemma 4.4.8 (Pythagorean formula)**.** *For $u_0,u_1 \in \mathcal{E}^p(X,\theta,\psi)$, we have*

$$d_p^p(u_0,u_1) = d_p^p(u_0,P_\theta(u_0,u_1)) + d_p^p(u_1,P_\theta(u_0,u_1)).$$

Proof. The proof follows from Theorem 4.3.2, the Pythagorean identity for d_p in the big and nef case, and the fact that $\widetilde{P_\theta(u_0,u_1)} = P_{\tilde{\theta}}(\tilde{u}_0,\tilde{u}_1)$. This is true because the bijection $u \leftrightarrow \tilde{u} := (u-\psi)\circ\mu + g$ is order-preserving. □

The following lemma says that the d_p distance is controlled by the I_p "distance". We have

Lemma 4.4.9. *There is a constant $C > 1$, that depends only on n, such that for any $u_0,u_1 \in \mathcal{E}^p(X,\theta,\psi)$,*

$$\frac{1}{C}I_p(u_0,u_1) \leq d_p^p(u_0,u_1) \leq CI_p(u_0,u_1).$$

Proof. The proof follows from Theorem 4.3.3, and the fact that $I_p(u_0, u_1) = I_p(\tilde{u}_0, \tilde{u}_1)$. We observe

$$I_p(\tilde{u}_0, \tilde{u}_1) = \int_{\tilde{X}} |\tilde{u}_0 - \tilde{u}_1|^p (\tilde{\theta}^n_{\tilde{u}_0} + \tilde{\theta}^n_{\tilde{u}_1}) = \int_{\tilde{X}} |(u_0 - u_1) \circ \mu|^p (\tilde{\theta}^n_{\tilde{u}_0} + \tilde{\theta}^n_{\tilde{u}_1}).$$

Pushing forward to X by μ and using the fact that $\mu_* \tilde{\theta}^n_{\tilde{u}} = \theta^n_u$, we get

$$I_p(\tilde{u}_0, \tilde{u}_1) = \int_X |u_0 - u_1|^p (\theta^n_{u_0} + \theta^n_{u_1}) = I_p(u_0, u_1).$$

From Theorem 4.3.3, we know that there exists $C > 1$ such that

$$\frac{1}{C} I_p(\tilde{u}_0, \tilde{u}_1) \leq d_p^p(\tilde{u}_0, \tilde{u}_1) \leq C I_p(\tilde{u}_0, \tilde{u}_1).$$

Therefore, from the above calculation we obtain that for the same C, we have

$$\frac{1}{C} I_p(u_0, u_1) \leq d_p^p(u_0, u_1) \leq C I_p(u_0, u_1).$$

$\square$

Theorem 4.4.10. *Let $f_0, f_1 \in C^{1,\bar{1}}(X)$, $u_0 = P_\theta[\psi](f_0)$, $u_1 = P_\theta[\psi](f_1)$, and u_t be the Mabuchi geodesic joining u_0 and u_1. Then*

$$d_p^p(u_0, u_1) = \int_X |\dot{u}_t|^p \theta^n_{u_t} \qquad \forall t \in [0, 1].$$

Proof. From Lemma 4.2.2, $u_0, u_1 \in \text{Ent}(X, \theta, \psi)$. From Lemma 4.4.4, the potentials $\tilde{u}_0$ and

$\tilde{u}_1$ have finite entropy, and from Theorem 4.4.1 $\tilde{u}_0, \tilde{u}_1$ have minimal singularity. Thus, using Theorem 4.3.4,

$$d_p^p(\tilde{u}_0, \tilde{u}_1) = \int_{\tilde{X}} |\dot{\tilde{u}}_t|^p \tilde{\theta}^n_{\tilde{u}_t} \qquad \forall t \in [0,1]$$

where $\tilde{u}_t$ is the weak geodesic joining $\tilde{u}_0$ and $\tilde{u}_1$. We emphasize that Theorem 4.3.4, which is about the geodesic distance for potentials with finite entropy, plays a crucial role here. In our procedure for importing geometry from the big and nef setting to the analytic singularity setting, we lose the property that $\tilde{u}$ is of the form $P_{\tilde{\theta}}(\tilde{f})$ for some $\tilde{f} \in C^{1,\bar{1}}(\tilde{X})$ where $u = P_\theta[\psi](f)$ for some $f \in C^{1,\bar{1}}(X)$.

Since $\tilde{u}_t = (u_t - \psi) \circ \mu + g$ where u_t is the weak geodesic joining u_0 and u_1, we have $\dot{\tilde{u}}_t = \dot{u}_t \circ \mu$. We also have $\mu_* \tilde{\theta}^n_{\tilde{u}_t} = \theta^n_{u_t}$. Combining these we get

$$d_p^p(u_0, u_1) := d_p^p(\tilde{u}_0, \tilde{u}_1) = \int_{\tilde{X}} |\dot{\tilde{u}}_t|^p \theta^n_{\tilde{u}_t} = \int_{\tilde{X}} |\dot{u}_t \circ \mu|^p \theta^n_{\tilde{u}_t} = \int_X |\dot{u}_t|^p \theta^n_{u_t}$$

for all $t \in [0,1]$. □

In the special case of $p = 1$, we have

Lemma 4.4.11. *Let $f_0, f_1 \in C^{1,\bar{1}}(X)$ satisfy $f_0 \leq f_1$, and $u_0 = P_\theta[\psi](f_0)$ and $u_1 = P_\theta[\psi](f_1)$. If u_t is the weak geodesic joining u_0 and u_1, then*

$$I_\psi(u_1) - I_\psi(u_0) = \int_X \dot{u}_t \theta^n_{u_t} \qquad \textit{for all } t \in [0,1].$$

Proof. From the proof of Theorem 4.4.10, we know that $\dot{u}_t \circ \mu = \dot{\tilde{u}}_t$. From the proof of Proposition 4.4.6, we know that $I_{\tilde{\theta}}(\tilde{u}_0) - I_{\tilde{\theta}}(g) = I_\psi(u_0)$ and $I_{\tilde{\theta}}(\tilde{u}_1) - I_{\tilde{\theta}}(g) = I_\psi(u_1)$. Moreover,

$\mu_* \tilde{\theta}^n_{\tilde{u}_t} = \theta^n_{u_t}$. Combining these facts with Lemma 4.3.6, we get

$$I_\psi(u_1) - I_\psi(u_0) = I_{\tilde{\theta}}(\tilde{u}_0) - I_{\tilde{\theta}}(\tilde{u}_1) = \int_{\tilde{X}} \dot{\tilde{u}}_t \tilde{\theta}^n_{\tilde{u}_t} = \int_{\tilde{X}} \dot{u}_t \circ \mu \tilde{\theta}^n_{\tilde{u}_t} = \int_X \dot{u}_t \theta^n_{u_t}$$

as desired. □

Now we will show that the metric d_p as defined by (4.5) does not depend on the choice of resolution.

Lemma 4.4.12. *Let $u_0^k, u_1^k, u_0, u_1 \in \mathcal{E}^p(X, \theta, \psi)$ satisfy $u_0^k \searrow u_0$ and $u_1^k \searrow u_1$. Then $d_p(u_0^k, u_1^k) \to d_p(u_0, u_1)$ as $k \to \infty$.*

Proof. $u_0^k \searrow u_0$ implies that $\tilde{u}_0^k = (u_0^k - \psi) \circ \mu + g \searrow (u_0 - \psi) \circ \mu + g = \tilde{u}_0$. Similarly, $\tilde{u}_1^k \searrow \tilde{u}_1$. We claim that in the space $(\mathcal{E}^p(\tilde{X}, \tilde{\theta}), d_p)$ we have $d_p(\tilde{u}_0^k, \tilde{u}_1^k) \to d_p(\tilde{u}_0, \tilde{u}_1)$. To see this, we observe by the triangle inequality we have

$$d_p(\tilde{u}_0, \tilde{u}_1) - d_p(\tilde{u}_0^k, \tilde{u}_1^k) \leq d_p(\tilde{u}_0, \tilde{u}_0^k) + d_p(\tilde{u}_1^k, \tilde{u}_1).$$

As the other side is obtained similarly, we have

$$|d_p(\tilde{u}_0, \tilde{u}_1) - d_p(\tilde{u}_0^k, \tilde{u}_1^k)| \leq d_p(\tilde{u}_0, \tilde{u}_0^k) + d_p(\tilde{u}_1^k, \tilde{u}_1).$$

From [16, Proposition 3.12], we have $d_p(\tilde{u}_0^k, \tilde{u}_0) \to 0$ and $d_p(\tilde{u}_1^k, \tilde{u}_1) \to 0$ as $k \to \infty$. Thus $d_p(\tilde{u}_0^k, \tilde{u}_1^k) \to d_p(\tilde{u}_0, \tilde{u}_1)$ as $k \to \infty$.

Now from (4.5), we obtain that $d_p(u_0^k, u_1^k) \to d_p(u_0, u_1)$ as well. □

Theorem 4.4.13. *The metric d_p as defined by (4.5) on $\mathcal{E}^p(X,\theta,\psi)$ does not depend on the choice of resolution.*

Proof. If $u_0, u_1 \in \mathcal{E}^p(X,\theta,\psi)$ are of the form $u_0 = P_\theta[\psi](f_0)$ and $u_1 = P_\theta[\psi](f_1)$ for some functions $f_0, f_1 \in C^{1,\bar{1}}(X)$, then from Theorem 4.4.10, we know that $d_p(u_0,u_1)$ does not depend on the choice of resolution, it is determined by the weak geodesic joining them.

More generally, given any $u_0, u_1 \in \mathcal{E}^p(X,\theta,\psi)$, from [54], we can find smooth functions $f_0^k, f_1^k \in C^\infty(X)$ such that $f_0^k \searrow u_0$ and $f_1^k \searrow u_1$. Then $P_\theta[\psi](f_0^k) \searrow u_0$ and $P_\theta[\psi](f_1^k) \searrow u_1$ as well. From Lemma 4.4.12, $d_p(P_\theta[\psi](f_0^k), P_\theta[\psi](f_1^k)) \to d_p(u_0,u_1)$. From the discussion in the previous paragraph, $d_p(P_\theta[\psi](f_0^k), P_\theta[\psi](f_1^k))$ does not depend on the choice of resolution, thus the distance $d_p(u_0,u_1)$ can be determined without the choice of resolution as well. $\square$

4.5 Metric on $\mathcal{E}^p(X,\theta)$

In this section, we define a metric d_p on $\mathcal{E}^p(X,\theta)$ that makes $(\mathcal{E}^p(X,\theta), d_p)$ a complete geodesic metric space. The idea is to approximate the potentials in $\mathcal{E}^p(X,\theta)$ from the potentials in $\mathcal{E}^p(X,\theta,\psi)$ and use the metric structure on $\mathcal{E}^p(X,\theta,\psi)$ as described in Section 4.4.1.

In the big class represented by θ, we can find a Kähler potential $\varphi \in \mathrm{PSH}(X,\theta)$ with analytic singularities. We can also assume, by subtracting a constant if needed, that $\varphi \leq V_\theta$. Define $\varphi_j = \frac{1}{j}\varphi + \frac{j-1}{j}V_\theta$, so $\varphi_j \leq V_\theta$ and $\varphi_j \nearrow V_\theta$ outside a pluripolar set. Unfortunately, φ_j does not have analytic singularities. Since φ is a Kähler potential, φ_j is also a Kähler potential. By Demailly's regularization, we can find $\phi_j \geq \varphi_j$ such that ϕ_j is a Kähler potential with analytic singularities. But the sequence ϕ_j is not monotone.

Now, consider $P_\theta[\phi_j] := \sup\{u \in \mathrm{PSH}(X,\theta) : u \preceq \phi_j, u \leq V_\theta\}$. As $\varphi_j \leq \phi_j$ and $\varphi_j \leq V_\theta$, we have $\varphi_j \leq P_\theta[\phi_j] \leq V_\theta$. As $\varphi_j \nearrow V_\theta$ outside a pluripolar set, we find that $P_\theta[\phi_j] \to V_\theta$ pointwise outside a pluripolar set. Also, since ϕ_j has analytic singularities, and $[P_\theta[\phi_j]] = [\phi_j]$, which follows from [23, Proposition 2.20], we get $P_\theta[\phi_j]$ has analytic singularities as well. Now consider

$$\psi_j = \max\{P_\theta[\phi_1], \ldots, P_\theta[\phi_j]\}.$$

Then by [28, Lemma 2.4], ψ_j has analytic singularities, and $\psi_j \nearrow V_\theta$ except on a pluripolar set. We fix such a sequence $\psi_j \nearrow V_\theta$ for the rest of the paper.

Following [8] and [16], we will first define the d_p metric on the space of "smooth" potentials, and then extend it to the whole space $\mathcal{E}^p(X,\theta)$. In general, $\mathcal{E}^p(X,\theta)$ has no smooth potentials, but the space

$$\mathcal{H}_\theta = \{u \in \mathrm{PSH}(X,\theta) \,|\, u = P_\theta(f) \text{ for some } f \in C^{1,\bar{1}}(X)\}$$

will act as the space of "smooth" potentials for us.

Lemma 4.5.1. *If $u_0, u_1 \in \mathcal{H}_\theta$, then $P_\theta(u_0,u_1) \in \mathcal{H}_\theta$.*

Proof. Let $f_0, f_1 \in C^{1,\bar{1}}(X)$ and $u_0 = P_\theta(f_0)$ and $u_1 = P_\theta(f_1)$. Let C be such that $\theta \leq C\omega$. Then from [55, Theorem 2.5] $P_{C\omega}(f_0,f_1)$ is a $C^{1,\bar{1}}$ function. We claim that $P_\theta(f_0,f_1) = P_\theta(P_{C\omega}(f_0,f_1))$.

Since $P_{C\omega}(f_0,f_1) \leq \min\{f_0,f_1\}$, we have $P_\theta(P_{C\omega}(f_0,f_1)) \leq P_\theta(f_0,f_1)$. For the other direction, note that $0 \leq \theta + dd^c P_\theta(f_0,f_1) \leq C\omega + dd^c P_\theta(f_0,f_1)$. Thus $P_\theta(f_0,f_1)$ is a $C\omega$-psh as well. As $P_\theta(f_0,f_1) \leq \min\{f_0,f_1\}$, we have $P_\theta(f_0,f_1) \leq P_{C\omega}(f_0,f_1)$. Thus $P_\theta(f_0,f_1) \leq P_\theta(P_{C\omega}(f_0,f_1))$.

Also, $P_\theta(u_0, u_1) = P_\theta(f_0, f_1)$ by a similar argument. So $P_\theta(u_0, u_1) = P_\theta(P_{C\omega}(f_0, f_1))$ where $P_{C\omega}(f_0, f_1) \in C^{1,\bar{1}}(X)$. □

4.5.1 Metric on $\mathcal{H}_\theta$

In this subsection, we will define the metric d_p on $\mathcal{H}_\theta$. The idea is to approximate for $f_0, f_1 \in C^{1,\bar{1}}(X)$, potentials $u_0 = P_\theta(f_0), u_1 = P_\theta(f_1) \in \mathcal{H}_\theta$ via $u_0^k := P_\theta[\psi_k](f_0), u_1^k := P_\theta[\psi_k](f_1) \in \text{PSH}(X, \theta, \psi_k)$ for the increasing sequence of potentials with analytic singularity type $\psi_k \nearrow V_\theta$ as fixed in the beginning of the section. We fix the notation for $f_0, f_1, u_0, u_1, u_0^k, u_1^k$ for the rest of the section. Since u_0^k, u_1^k have the same singularity type as ψ_k, we get that u_0^k, $u_1^k \in \mathcal{E}^p(X, \theta, \psi_k)$. We wish to define

$$d_p(u_0, u_1) := \lim_{k\to\infty} d_p(u_0^k, u_1^k). \tag{4.6}$$

Here $d_p(u_0^k, u_1^k)$ is the distance defined in Section 4.4.1 on $\mathcal{E}^p(X, \theta, \psi_k)$.

In this subsection, we will first show that indeed u_0^k and u_1^k increase to u_0 and u_1 respectively. Moreover, the limit in (4.6) exists, is independent of the choice of the approximating sequence ψ_k, and defines a metric on $\mathcal{H}_\theta$.

The following lemma shows that envelopes with respect to an increasing sequence converge.

Lemma 4.5.2. *Let $\psi, \psi_k \in PSH(X, \theta)$ be an increasing sequence and $\psi = (\lim_{k\to\infty} \psi_k)^*$. Then for any continuous $f : X \to \mathbb{R}$, $P_\theta[\psi_k](f)$ is an increasing sequence and $(\lim_{k\to\infty} P_\theta[\psi_k](f))^* = P_\theta[\psi](f)$.*

Proof. If $k > l$, then $P_\theta(\psi_l + C, f) \leq P_\theta(\psi_k + C, f)$. Taking the limit $C \to \infty$, we get that $P_\theta[\psi_l](f) \leq P_\theta[\psi_k](f)$. Therefore, $P_\theta[\psi_k](f)$ is an increasing sequence of θ-psh functions. Thus, $(\lim_{k\to\infty} P_\theta[\psi_k](f))^*$ is a θ-psh function. Since $P_\theta[\psi_k](f) \leq P_\theta[\psi](f)$ for all k and is upper semicontinuous, $(\lim_{k\to\infty} P_\theta[\psi_k](f))^* \leq P_\theta[\psi](f)$.

To show the other direction we use Lemma 2.4.2. Since $\psi_k \nearrow \psi$, [31, Theorem 2.3] implies that $\int_X \theta^n_{\psi_k} \nearrow \int_X \theta^n_\psi$. From Lemma 2.4.2, we can find $\alpha_k \to 0$ such that

$$v_k = P_\theta\left(\frac{1}{\alpha_k}\psi_k + \left(1 - \frac{1}{\alpha_k}\right)\psi\right) \in \text{PSH}(X, \theta).$$

This implies

$$\alpha_k v_k + (1 - \alpha_k)\psi \leq \psi_k.$$

Using Lemma 2.4.3, we get

$$\alpha_k P_\theta[v_k](f) + (1 - \alpha_k)P_\theta[\psi](f) \leq P_\theta[\alpha_k v_k + (1 - \alpha_k)\psi](f) \leq P_\theta[\psi_k](f). \tag{4.7}$$

Now $\sup_X P_\theta[v_k](f)$ are bounded. As $P_\theta[v_k](f) \leq f$, so they are bounded from above. Also if $f \geq C$ for some C, then $\sup_X P_\theta[v_k](f) \geq C$. To see this first choose C_k such that such that $\sup_X(v_k + C_k) = C$. Then $P_\theta(v_k + C_k, f) = v_k + C_k$. As $P_\theta(v_k + C, f) \nearrow P_\theta[v_k](f)$ as $C \to \infty$, we get that $v_k + C_k \leq P_\theta[v_k](f)$. Therefore, $\sup_X P_\theta[v_k](f)$ is bounded. Hence after taking the weak L^1-limit in (4.7) we get

$$P_\theta[\psi](f) \leq \lim_{k\to\infty} P_\theta[\psi_k](f)$$

almost everywhere. Thus we get $(\lim_{k\to\infty} P_\theta[\psi_k](f))^* = P_\theta[\psi](f)$. □

In the following, we use Lemma 2.3.8 to prove that in the approximation scheme discussed above, Monge-Ampère energy and the I_p-"distance" converge.

Lemma 4.5.3. *Let $f \in C^{1,\bar{1}}$ and ψ_k are model potentials of analytic singularity type such that $\psi_k \nearrow V_\theta$ outside a pluripolar set. Let $u_k = P_\theta[\psi_k](f)$ and $u = P_\theta(f)$, then the ψ_k-relative Monge-Ampère energy of u_k converge to the Monge-Ampère energy of u.*

Proof. Let C be such that $\sup_X |f| \leq C$, then $|u_k - \psi_k| \leq C$. Thus $0 \leq u_k - \psi_k + C \leq 2C$. From Lemma 4.5.2 we know that $u_k \nearrow u$. Thus $u_k - \psi_k + C$ are uniformly bounded quasi-continuous functions that converge in capacity to $u - V_\theta + C$. Moreover, as u and V_θ have minimal singularity, from [30] and [31], we know that

$$\int_X \theta_u^j \wedge \theta_{V_\theta}^{n-j} \geq \limsup_{k\to\infty} \int_X \theta_{u_k}^j \wedge \theta_{\psi_k}^{n-j}.$$

Thus from Lemma 2.3.8, we know that the measures

$$(u_k - \psi_k + C)\theta_{u_k}^j \wedge \theta_{\psi_k}^{n-j} \to (u - V_\theta + C)\theta_u^j \wedge \theta_{V_\theta}^{n-j}$$

and

$$\theta_{u_k}^j \wedge \theta_{\psi_k}^{n-j} \to \theta_u^j \wedge \theta_{V_\theta}^{n-j}$$

weakly as $k \to \infty$. Thus the ψ_k-relative Monge-Ampère energy

$$I_{\psi_k}(u_k) = \frac{1}{n+1}\sum_{j=0}^n \int_X (u_k - \psi_k)\theta_{u_k}^j \wedge \theta_{\psi_k}^{n-j} \to \frac{1}{n+1}\sum_{j=0}^n \int_X (u - V_\theta)\theta_u^j \wedge \theta_{V_\theta}^{n-j} = I_\theta(u)$$

as $k \to \infty$. $\square$

Lemma 4.5.4. *Let u_0, u_1, u_0^k, u_1^k be as in the beginning of Section 4.5.1, then the I_p functional*

$$I_p(u_0^k, u_1^k) = \int_X |u_0^k - u_1^k|^p(\theta^n_{u_0^k} + \theta^n_{u_1^k}) \to \int_X |u_0 - u_1|^p(\theta^n_{u_0} + \theta^n_{u_1}) = I_p(u_0, u_1)$$

as $k \to \infty$.

Proof. We notice that

$$|u_0^k - u_1^k| \leq \sup_X |f_0 - f_1|.$$

This is true because if $C = \sup_X |f_0 - f_1|$, then $f_0 - C \leq f_1$, and therefore $P_\theta[\psi_k](f_0) - C$ is a candidate function for $P_\theta[\psi_k](f_1)$, therefore,

$$P_\theta[\psi_k](f_0) - C \leq P_\theta[\psi_k](f_1)$$

and the other direction is shown similarly. Thus

$$|P_\theta[\psi_k](f_0) - P_\theta[\psi_k](f_1)| \leq \sup_X |f_0 - f_1|.$$

Moreover, from Lemma 4.5.2 the functions the functions $u_0^k \nearrow u_0$ ad $u_1^k \nearrow u_1$ away from a pluripolar set, therefore $u_0^k \to u_0$ and $u_1^k \to u_1$ in capacity as $k \to \infty$. Moreover, $|u_0^k - u_1^k|^p$ are quasi-continuous and uniformly bounded, therefore from Lemma 2.3.8, we get that

$$\lim_{k\to\infty} \int_X |u_0^k - u_1^k|^p(\theta^n_{u_0^k} + \theta^n_{u_1^k}) = \int_X |u_0 - u_1|^p(\theta^n_{u_0} + \theta^n_{u_1}).$$

$\square$

The next theorem proves that the limit in (4.6) exists in the special setting when $u_0 \le u_1$.

Theorem 4.5.5. *If $f_0, f_1, u_0, u_1, u_0^k, u_1^k$ are as in the beginning of Section 4.5.1 along with the assumption that $f_0 \le f_1$, then*

$$\lim_{k\to\infty} d_p^p(u_0^k, u_1^k) = \int_X |\dot{u}_0|^p \theta_{u_0}^n = \int_X |\dot{u}_1|^p \theta_{u_1}^n,$$

where u_t is the weak geodesic joining u_0 and u_1. Thus the limit in (4.6) exists and is independent of the approximating sequence ψ_k if $f_0 \le f_1$.

Proof. Since $f_0 \le f_1$, we have $u_0 \le u_1$ and $u_0^k \le u_1^k$. Let u_t^k be the geodesic joining u_0^k and u_1^k. Since f_0, f_1 are bounded, u_0, u_1 have the minimal singularity type and u_0^k, u_1^k have the same singularity type as ψ_k.

It follows from Lemma 4.4.11 that

$$I_{\psi_k}(u_1^k) - I_{\psi_k}(u_0^k) = \int_X \dot{u}_0^k \theta_{u_0^k}^n.$$

From Lemma 4.5.3, we know that $I_{\psi_k}(u_0^k) \to I_\theta(u_0)$ and $I_{\psi_k}(u_1^k) \to I_\theta(u_1)$ as $k \to \infty$. Combining with Lemma 4.3.7, we get that

$$\lim_{k\to\infty} \int_X \dot{u}_0^k \theta_{u_0^k}^n = \int_X \dot{u}_0 \theta_{u_0}^n.$$

From Theorem 2.4.4 we obtain that $\theta_{u_0^k}^n = \mathbb{1}_{D_k} \theta_{f_0}^n$ where $D_k = \{P_\theta[\psi_k](f_0) = f_0\}$, and $\theta_{u_0}^n =$

$\mathbb{1}_D\theta^n_{u_0}$ where $D = \{P_\theta(f_0) = f_0\}$. So we can write that

$$\lim_{k\to\infty}\int_X \mathbb{1}_{D_k}\dot{u}_0^k\theta^n_{f_0} = \int_X \mathbb{1}_D\dot{u}_0\theta^n_{f_0}.$$

As $u_0^k \nearrow u_0$ and $u_1^k \nearrow u_1$, we find that the geodesics u_t^k joining u_0^k and u_1^k are also increasing. This holds because if $k < l$, then the geodesic u_t^k is a candidate subgeodesic joining u_0^l and u_1^l. Similarly, all geodesics u_t^k are candidate subgeodesics joining u_0 and u_1. Therefore u_t^k are increasing in k and $u_t^k \le u_t$.

Similarly, we can show that the contact sets D_k are increasing. If $k < l$, and $x \in D_k$, then $P_\theta[\psi_k](f_0)(x) = f_0(x)$ and since $P_\theta[\psi_k](f_0) \le P_\theta[\psi_l](f_0) \le f_0$, we find that $x \in D_l$ as well, so $D_k \subset D_l$. Moreover since $P_\theta[\psi_k](f_0) \le P_\theta(f_0) \le f_0$, we have $D_k \subset D$ for all D.

If $x \in D_k$ and $k < l$, then

$$\dot{u}_0^k(x) = \lim_{t\to 0}\frac{u_t^k(x) - u_0^k(x)}{t} = \lim_{t\to 0}\frac{u_t^k(x) - f_0(x)}{t} \le \lim_{t\to 0}\frac{u_t^l(x) - f_0(x)}{t} = \dot{u}_0^l(x).$$

Similarly, if $x \in D_k$, then

$$\dot{u}_0^k(x) = \lim_{t\to 0}\frac{u_t^k(x) - u_0^k(x)}{t} = \lim_{t\to 0}\frac{u_t^k(x) - f_0(x)}{t} \le \lim_{t\to 0}\frac{u_t(x) - f_0(x)}{t} = \dot{u}_0(x).$$

Also by assumption $u_0^k \le u_1^k$, so $\dot{u}_0^k, \dot{u}_0 \ge 0$. Therefore, $\mathbb{1}_{D_k}\dot{u}_0^k$ is an increasing sequence such that for each k, $\mathbb{1}_{D_k}\dot{u}_0^k \le \mathbb{1}_D\dot{u}_0$, and

$$\lim_{k\to\infty}\int_X \mathbb{1}_{D_k}\dot{u}_0^k\theta^n_{f_0} = \int_X \mathbb{1}_D\dot{u}_0\theta^n_{f_0}.$$

Therefore, $\mathbb{1}_{D_k}\dot{u}_0^k \nearrow \mathbb{1}_D\dot{u}_0$ pointwise $\theta_{f_0}^n$ almost everywhere.

Also $0 \leq \dot{u}_0^k \leq u_1^k - u_0^k \leq \sup_X |f_0 - f_1|$. Thus we have a uniform bound on $\dot{u}_0^k$. Therefore by Lebesgue's Dominated Convergence Theorem, we have

$$\int_X \mathbb{1}_{D_k}(\dot{u}_0^k)^p\theta_{f_0}^n \to \int_X \mathbb{1}_D(\dot{u}_0)^p\theta_{f_0}^n. \tag{4.8}$$

Now, from Theorem 4.4.10,

$$d_p^p(u_0^k, u_1^k) = \int_X |\dot{u}_0^k|^p\theta_{u_0^k}^n = \int_X \mathbb{1}_{D_k}(\dot{u}_0^k)^p\theta_{f_0}^n \to \int_X \mathbb{1}_D(\dot{u}_0)^p\theta_{f_0}^n = \int_X |\dot{u}_0|^p\theta_{u_0}^n$$

as $k \to \infty$.

The same proof works for $t = 1$ as well. $\square$

We now follow the proof of [33, Theorem 3.26] to get

Theorem 4.5.6. *Let $f_0, f_1 \in C^{1,\bar{1}}(X)$ and $u_0 = P_\theta(f_0)$ and $u_1 = P_\theta(f_1)$. Let u_t be the geodesic joining u_0 and u_1. Also assume that w_t is a geodesic joining $P_\theta(u_0, u_1)$ and u_0, and v_t is a geodesic joining $P_\theta(u_0, u_1)$ and u_1. Then*

$$\int_X |\dot{u}_0|^p\theta_{u_0}^n = \int_X |\dot{v}_0|^p\theta_{P_\theta(u_0,u_1)}^n + \int_X |\dot{w}_0|^p\theta_{P_\theta(u_0,u_1)}^n.$$

Proof. Just like in [33, Theorem 3.26], we will use Lemma 2.5.2 and Corollary 2.4.6 repeatedly to settle the claim.

We will prove that

$$\int_{\{\dot{u}_0>0\}} |\dot{u}_0|^p\theta_{u_0}^n = \int_X |\dot{v}_0|^p\theta_{P_\theta(u_0,u_1)}^n \tag{4.9}$$

and that

$$\int_{\{\dot{u}_0<0\}} |\dot{u}_0|^p \theta_{u_0}^n = \int_X |\dot{w}_0|^p \theta_{P_\theta(u_0,u_1)}^n \tag{4.10}$$

$$\begin{aligned}\int_{\{\dot{u}_0>0\}} |\dot{u}_0|^p \theta_{u_0}^n &= p\int_0^\infty \tau^{p-1}\theta_{u_0}^n(\{\dot{u}_0 \geq \tau\})d\tau \\ &= p\int_X \tau^{p-1}\theta_{u_0}^n(\{P_\theta(u_0, u_1-\tau) = u_0\})d\tau.\end{aligned}$$

On the other hand, as v_t is an increasing geodesic, $\dot{v}_0 \geq 0$. This shows

$$\begin{aligned}\int_X |\dot{v}_0|^p \theta_{P_\theta(u_0,u_1)}^n &= p\int_0^\infty \tau^{p-1}\theta_{P_\theta(u_0,u_1)}^n(\{\dot{v}_0 \geq \tau\})d\tau \\ &= p\int_0^\infty \tau^{p-1}\theta_{P_\theta(u_0,u_1)}^n(\{P_\theta(P_\theta(u_0,u_1), u_1-\tau) = P_\theta(u_0,u_1)\})d\tau \\ &= p\int_0^\infty \tau^{p-1}\theta_{u_0}^n(\{P_\theta(u_0, u_1-\tau) = u_0\})d\tau. \end{aligned}\tag{4.11}$$

For the last step, we used Corollary 2.4.6 and the fact that $P_\theta(P_\theta(u_0,u_1), u_1-\tau) = P_\theta(u_0, u_1-\tau)$, $\{P_\theta(u_0,u_1-\tau) = u_0\} = \{P_\theta(u_0,u_1-\tau) = P_\theta(u_0,u_1) = u_0\}$, and that the set $\{P_\theta(u_0,u_1-\tau) = P_\theta(u_0,u_1) = u_1\} = \emptyset$. Thus proving (4.9).

For (4.10), we observe from Corollary 2.4.6 that except for countably many τ, we have

$$\mathrm{Vol}(\theta) = \theta_{u_0}^n(\{P_\theta(u_0,u_1+\tau) = u_0\}) + \theta_{u_1}^n(\{P_\theta(u_0,u_1+\tau) = u_1+\tau\}).$$

Now,

$$\begin{aligned}
\int_{\{\dot{u}_0<0\}} |\dot{u}_0|^p \theta_{u_0}^n &= p\int_0^\infty \tau^{p-1}\theta_{u_0}^n(\{-\dot{u}_0>\tau\})d\tau \\
&= p\int_0^\infty \tau^{p-1}\theta_{u_0}^n(X\setminus\{-\dot{u}_0\le\tau\})d\tau \\
&= p\int_0^\infty \tau^{p-1}(\mathrm{Vol}(\theta)-\theta_{u_0}^n(\{P_\theta(u_0,u_1+\tau)=u_0\})d\tau \\
&= p\int_0^\infty \tau^{p-1}\theta_{u_1}^n(\{P_\theta(u_0,u_1+\tau)=u_1+\tau\})d\tau \\
&= p\int_0^\infty \tau^{p-1}\theta_{u_1}^n(\{P_\theta(u_0-\tau,u_1)=u_1\})d\tau.
\end{aligned}$$

This is the same expression as (4.11) with the roles of u_0 and u_1 reversed. Therefore,

$$\int_{\{\dot{u}_0<0\}} |\dot{u}_0|^p \theta_{u_0}^n = \int_X |\dot{w}_0|^p \theta_{P_\theta(u_0,u_1)}^n$$

proving (4.10). □

Now we can use this result to show that the limit in (4.6) exists without the monotone assumption.

Theorem 4.5.7. *Let u_0, u_1, u_0^k, u_1^k be as defined in the beginning of this section. Then the limit in* (4.6) *exists, and is independent of the choice of the approximating sequence ψ_k. Moreover, if u_t is the weak geodesic joining u_0 and u_1, then*

$$\lim_{k\to\infty} d_p^p(u_0^k, u_1^k) = \int_X |\dot{u}_0|^p\theta_{u_0}^n = \int_X |\dot{u}_1|^p\theta_{u_1}^n.$$

Proof. We know the result from Theorem 4.5.5 if $f_0 \le f_1$. To prove it in general, recall that

Lemma 4.5.1 shows that $P_\theta(u_0, u_1) \in \mathcal{H}_\theta$ as well. Here $P_\theta(u_0, u_1) = P_\theta(P_{C\omega}(f_0, f_1))$ and $h := P_{C\omega}(f_0, f_1) \in C^{1,\bar{1}}(X)$. Using the notation from the previous theorem, let w_t be the weak geodesic joining $P_\theta(u_0, u_1)$ and u_0 and v_t be the weak geodesic joining $P_\theta(u_0, u_1)$ and u_1.

Now, $h \le f_0, f_1$ are $C^{1,\bar{1}}$ functions. From Theorem 4.5.5,

$$\lim_{k\to\infty} d_p^p(P_\theta[\psi_k](h), u_0^k) = \int_X |\dot{w}_0|^p \theta^n_{P_\theta(u_0,u_1)}$$

and

$$\lim_{k\to\infty} d_p^p(P_\theta[\psi_k](h), u_1^k) = \int_X |\dot{v}_0|^p \theta^n_{P_\theta(u_0,u_1)}.$$

Lemma 4.4.8 says that the distance d_p on $\mathcal{E}^p(X, \theta, \psi_k)$ satisfies the Pythagorean formula. Observing that $P_\theta[\psi_k](h) = P_\theta(u_0^k, u_1^k) = P_\theta[\psi_k](\min(f_0, f_1))$, we get

$$\begin{aligned}
\lim_{k\to\infty} d_p^p(u_0^k, u_1^k) &= \lim_{k\to\infty} d_p^p(u_0^k, P_\theta[\psi_k](h)) + d_p^p(u_1^k, P_\theta[\psi_k](h)) \\
&= \int_X |\dot{w}_0|^p \theta^n_{P_\theta(u_0,u_1)} + \int_X |\dot{v}_0|^p \theta^n_{P_\theta(u_0,u_1)} \\
&= \int_X |\dot{u}_0|^p \theta^n_{u_0}.
\end{aligned}$$

Here, in the last line, we used Theorem 4.5.6. Similar proof shows that $\lim_{k\to\infty} d_p^p(u_0^k, u_1^k) = \int_X |\dot{u}_1|^p \theta^n_{u_1}$. □

With the help of Theorem 4.5.7, we see that the limit in (4.6) exists and does not depend on the choice of the approximating sequence. Thus we can define

Definition 4.5.8. Take $u_0, u_1 \in \mathcal{H}_\theta$ where $u_0 = P_\theta(f_0)$ and $u_1 = P_\theta(f_1)$ for $f_0, f_1 \in C^{1,\bar{1}}(X)$. Let $\psi_k \nearrow V_\theta$ outside a pluripolar set be an increasing sequence of θ-psh function with analytic singularities. Denote by $u_0^k = P_\theta[\psi_k](f_0)$ and $u_1^k = P_\theta[\psi_k](f_1)$. We define

$$d_p(u_0, u_1) := \lim_{k\to\infty} d_p(u_0^k, u_1^k).$$

The next theorem shows that (4.6) indeed defines a metric on $\mathcal{H}_\theta$.

Proposition 4.5.9. *If d_p is defined as in Definition 4.5.8, then $(\mathcal{H}_\theta, d_p)$ is a metric space and d_p is comparable to I_p. This means there exits $C > 1$, depending only on dimension such that for all $u_0, u_1 \in \mathcal{H}_\theta$,*

$$\frac{1}{C} I_p(u_0, u_1) \leq d_p^p(u_0, u_1) \leq C I_p(u_0, u_1).$$

Proof. From Lemma 4.5.4, we know that $\lim_{k\to\infty} I_p(u_0^k, u_1^k) = I_p(u_0, u_1)$. From Lemma 4.4.9, we know that there exists C such that

$$\frac{1}{C} I_p(u_0^k, u_1^k) \leq d_p^p(u_0, u_1) \leq C I_p(u_0, u_1)$$

Taking limit $k \to \infty$, we get

$$\frac{1}{C} I_p(u_0, u_1) \leq d_p^p(u_0, u_1) \leq C I_p(u_0, u_1).$$

Symmetry and triangle inequality for d_p follow from the definition and the fact that $(\mathcal{E}^p(X, \theta, \psi_k), d_p)$ satisfies these properties. Non-degeneracy of d_p follows from comparison with I_p. If $d_p(u_0, u_1) = 0$, then the above comparison shows that $I_p(u_0, u_1) = 0$. This implies that $u_0 = u_1$ from the domination principle (see [36, Proposition 2.4]). $\square$

4.5.2 Extending the metric to $\mathcal{E}^p(X,\theta)$

In this section, we will extend the metric d_p from $\mathcal{H}_\theta$ to all of $\mathcal{E}^p(X,\theta)$. We will do this by approximation. This process of approximation works identically to the one given in [16]. Given $u \in \mathcal{E}^p(X,\theta)$, from [54], we can find smooth functions f^j such that $f^j \searrow u$. By definition $u^j := P_\theta(f^j) \in \mathcal{H}_\theta$ and $u^j \searrow u$. Based on this we give a tentative definition:

Definition 4.5.10. Given $u_0, u_1 \in \mathcal{E}^p(X,\theta)$, we define

$$d_p(u_0,u_1) := \lim_{j\to\infty} d_p(u_0^j, u_1^j), \tag{4.12}$$

where $u_0^j, u_1^j \in \mathcal{H}_\theta$ satisfy $u_0^j \searrow u_0$ and $u_1^j \searrow u_1$.

Now we need to show the limit in (4.12) exists and is independent of the choice of the approximating sequence u_0^j and u_1^j.

Theorem 4.5.11. *The limit in (4.12) exists and is independent of the choice of the approximating sequence u_0^j and u_1^j.*

Proof. From Proposition 4.5.9 for $u, v \in \mathcal{H}_\theta$, there exists $C > 1$, depending only on n, such that

$$\frac{1}{C} I_p(u,v) \le d_p^p(u,v) \le C I_p(u,v).$$

We will show that $\{d_p(u_0^j, u_1^j)\}$ is a Cauchy sequence. By the triangle inequality we have

$$d_p(u_0^j,u_1^j) \le d_p(u_0^j,u_0^k) + d_p(u_0^k,u_1^k) + d_p(u_1^k,u_1^j)$$

$$\implies d_p(u_0^j,u_1^j) - d_p(u_0^k,u_1^k) \le C(I_p^{1/p}(u_0^j,u_0^k) + I_p^{1/p}(u_1^k,u_1^j)).$$

Since the other side is obtained identically, we get

$$|d_p(u_0^j, u_1^j) - d_p(u_0^k, u_1^k)| \leq C(I_p^{1/p}(u_0^j, u_0^k) + I_p^{1/p}(u_1^k, u_1^j)).$$

From Theorem 2.3.9 we get $I_p(u_0^j, u_0^k) \to 0$ and $I_p(u_1^j, u_1^k) \to 0$ as $j, k \to \infty$. Thus we have $|d_p(u_0^j, u_1^j) - d_p(u_0^k, u_1^k)| \to 0$. Thus the limit in (4.12) exists. Now we will show that the limit is unique. For that let $\tilde{u}_0^j, \tilde{u}_1^j \in \mathcal{H}_\theta$ be another sequence of functions decreasing to u_0 and u_1 respectively. To show that the definition of d_p does not depend on the choice of functions approximating u_0 and u_1, we will show that $|d_p(u_0^j, u_1^j) - d_p(\tilde{u}_0^j, \tilde{u}_1^j)| \to 0$ as $j \to \infty$. The proof is similar to the proof before.

$$\begin{aligned} & d_p(u_0^j, u_1^j) \leq d_p(u_0^j, \tilde{u}_0^j) + d_p(\tilde{u}_0^j, \tilde{u}_1^j) + d_p(\tilde{u}_1^j, u_1^j) \\ \implies & |d_p(u_0^j, u_1^j) - d_p(\tilde{u}_0^j, \tilde{u}_1^j)| \leq C(I_p^{1/p}(u_0^j, \tilde{u}_0^j) + I_p^{1/p}(u_1^j, \tilde{u}_1^j)). \end{aligned}$$

Since u_0^j and $\tilde{u}_0^j$ both decrease to u_0, Theorem 2.3.9 implies that $I_p(u_0^j, \tilde{u}_0^j) \to 0$. Similarly $I_p(u_1^j, \tilde{u}_1^j) \to 0$ as well. So d_p is well defined on $\mathcal{E}^p(X, \theta)$ by (4.12). □

Lemma 4.5.12. *There exists $C > 1$ that depends only on the dimension of X, such that for all $u_0, u_1 \in \mathcal{E}^p(X, \theta)$,*

$$\frac{1}{C} I_p(u_0, u_1) \leq d_p^p(u_0, u_1) \leq C I_p(u_0, u_1).$$

Proof. The statement is true for potentials in $\mathcal{H}_\theta$. Let $u_0^j \searrow u_0$ and $u_1^j \searrow u_1$. Then

$$\frac{1}{C} I_p(u_0^j, u_1^j) \leq d_p^p(u_0^j, u_1^j) \leq C I_p(u_0^j, u_1^j)$$

Taking the limit $j \to \infty$ and applying Theorem 2.3.9 and using (4.12) we get

$$\frac{1}{C} I_p(u_0, u_1) \leq d_p(u_0, u_1) \leq C I_p(u_0, u_1).$$

□

Theorem 4.5.13. (4.12) *defines a metric on* d_p *on* $\mathcal{E}^p(X, \theta)$.

Proof. Again, using approximation, we can show the triangle inequality. Let $u, v, w \in \mathcal{E}^p(X, \theta)$ and $u^j, v^j, w^j \in \mathcal{H}_\theta$ approximate u, v, and w respectively. Then

$$\begin{aligned} d_p(u, v) &= \lim_{j \to \infty} d_p(u^j, v^j) \\ &\leq \lim_{j \to \infty} (d_p(u^j, w^j) + d_p(w^j, v^j)) \\ &= d_p(u, w) + d_p(w, v). \end{aligned}$$

This shows the triangle inequality for d_p. Symmetry also follows from symmetry of d_p on $\mathcal{H}_\theta$. Non-degeneracy of d_p follows from Lemma 4.5.12. If $u, v \in \mathcal{E}^p(X, \theta)$ have satisfy $d_p(u, v) = 0$, then Lemma 4.5.12 says $I_p(u, v) = 0$, which implies that $u = v$ by the domination principle (see [36, Proposition 2.4]). □

4.6 Properties of the metric

In this section, we will show that the metric space $(\mathcal{E}^p(X,\theta), d_p)$ is a complete geodesic metric space.

Theorem 4.6.1. *The metric space $(\mathcal{E}^p(X,\theta), d_p)$ is a complete metric space.*

Proof. From Lemma 4.5.12, there exists a $C > 1$ such that for any $u_0, u_1 \in \mathcal{E}^p(X,\theta)$

$$\frac{1}{C} I_p(u_0, u_1) \leq d_p^p(u_0, u_1) \leq C I_p(u_0, u_1).$$

In [15], the author showed that the quasi-metric space $(\mathcal{E}^p(X,\theta), I_p)$ induces a complete metric topology. This means given a I_p-Cauchy sequence $\{u_k\}$, there exits $u \in \mathcal{E}^p(X,\theta)$ such that $I_p(u_k, u) \to 0$.

From the above inequality, a sequence $\{u_k\}$ is Cauchy in I_p iff it is Cauchy in d_p and similarly, a sequence u_k converges to u in I_p iff u_k converges to u in d_p.

This shows that any d_p-Cauchy sequence $\{u_k\}$ converges to some $u \in \mathcal{E}^p(X,\theta)$. $\square$

Now we want to show that the Mabuchi geodesics in $\mathcal{E}^p(X,\theta)$ are the metric geodesics as well. For that, we need to better understand the metric space structure of $\mathcal{E}^p(X,\theta)$.

Lemma 4.6.2. *If $u_0, u_1, u_0^j, u_1^j \in \mathcal{E}^p(X,\theta)$ satisfy $u_0^j \searrow u_0$ and $u_1^j \searrow u_1$, then $\lim_{j\to\infty} d_p(u_0^j, u_1^j) = d_p(u_0, u_1)$.*

Proof. Recall that [42, Propostion 1.9] implies that $I_p(u_0^j, u_0) \to 0$ and $I_p(u_1^j, u_1) \to 0$. As before, we use triangle inequality to write

$$d_p(u_0, u_1) \le d_p(u_0, u_0^j) + d_p(u_0^j, u_1^j) + d_p(u_1^j, u_1).$$

Using Lemma 4.5.12

$$d_p(u_0, u_1) - d_p(u_0^j, u_1^j) \le C\left(I_p^{1/p}(u_0^j, u_0) + I_p^{1/p}(u_1^j, u_1)\right).$$

Noticing that the other side is obtained similarly, and then we take the limit to obtain

$$\lim_{j\to\infty} |d_p(u_0, u_1) - d_p(u_0^j, u_1^j)| \le \lim_{j\to\infty} C\left(I_p^{1/p}(u_0^j, u_0) + CI_p^{1/p}(u_1^j, u_1)\right) = 0.$$

□

The following is the extension of [16, Lemma 3.13] to the big case. The proof is identical.

Lemma 4.6.3. *If $u_0 \in \mathcal{H}_\theta$, $u_1 \in \mathcal{E}^p(X, \theta)$, and u_t is the weak geodesic joining u_0 and u_1, then*

$$d_p^p(u_0, u_1) = \int_X |\dot{u}_0|^p \theta_{u_0}^n.$$

Proof. First, assume that $u_0 \ge u_1 + 1$. We can find $u_1^j \in \mathcal{H}_\theta$ such that $u_1^j \searrow u_1$ and $u_0 \ge u_1^j$. Let u_t^j be the weak geodesic joining u_0 and u_1^j. Moreover, $\dot{u}_0, \dot{u}_0^j \le 0$. We claim $\dot{u}_0^j \searrow \dot{u}_0$. Since the geodesics $u_t^j \searrow u_t$, and they start at the same point u_0, we know that $\dot{u}_0^j$ is decreasing. To see that they decrease to $\dot{u}_0$, notice that

$$\dot{u}_0 = \lim_{t\to 0} \frac{u_t - u_0}{t} \leq \lim_{t\to 0} \frac{u_t^j - u_0}{t} = \dot{u}_0^j \leq \frac{u_t^j - u_0}{t}.$$

Here in the last inequality, we used the convexity of the geodesic. Now, taking limit $j \to \infty$, we get

$$\dot{u}_0 \leq \lim_{j\to\infty} \dot{u}_0^j \leq \frac{u_t - u_0}{t}.$$

Now taking limit $t \to 0$, we get

$$\dot{u}_0 \leq \lim_{j\to\infty} \dot{u}_0^j \leq \dot{u}_0.$$

Thus $\lim_{j\to\infty} \dot{u}_0^j = \dot{u}_0$.

Now, by definition, $d_p(u_0, u_1) = \lim_{j\to\infty} d_p(u_0, u_1^j)$ and

$$d_p^p(u_0, u_1^j) = \int_X (-\dot{u}_0^j)^p \theta_{u_0}^n.$$

By the monotone convergence theorem,

$$d_p^p(u_0, u_1) = \lim_{j\to\infty} d_p^p(u_0, u_1^j) = \lim_{j\to\infty} \int_X (-\dot{u}_0^j)^p \theta_{u_0}^n = \int_X (-\dot{u}_0)^p \theta_{u_0}^n.$$

For the general case, let $C > 0$ satisfy $u_1 \leq u_0 + C$. Again choose $u_1^j \in \mathcal{H}_\theta$ such that $u_1^j \searrow u_1$. Consider $w_0 = u_0$ and $w_1 = u_1 - C - 1 \leq u_1 \leq u_1^j$. Now $w_0 \geq w_1 + 1$. If w_t is the geodesic joining w_0 and w_1 and if u_t^j are the geodesics joining u_0 and u_t^j, then we have

$$\dot{w}_0 \leq \dot{u}_0^j \leq u_1^j - u_0 \leq (u_1^j - V_\theta) + (V_\theta - u_0) \leq C,$$

where C is a uniform bound (independent of j). Thus, $|\dot{u}_0^j|^p \leq C_1 + |\dot{w}_0|^p$. By the same argument as before, $\dot{u}_0^j \to \dot{u}_0$ pointwise outside the pluripolar set $\{u_1 = -\infty\}$. Moreover, from the previous calculation

$$\int_X |\dot{w}_0|^p \theta_{u_0}^n = d_p^p(u_0, u_1 - C - 1).$$

Thus $|\dot{w}_0|^p$ is integrable with respect to $\theta_{u_0}^n$. Thus applying Lebesgue's Dominated Convergence Theorem, we obtain

$$d_p^p(u_0, u_1) = \lim_{j\to\infty} d_p^p(u_0, u_1^j) = \lim_{j\to\infty} \int_X |\dot{u}_j|^p \theta_{u_0}^n = \int_X |\dot{u}_0|^p \theta_{u_0}^n.$$

□

Proposition 4.6.4. *Take $u_0, u_1 \in \mathcal{E}^p(X,\theta)$ and let u_t be the weak geodesic joining u_0 and u_1. Then u_t is a metric geodesic for $(\mathcal{E}^p(X,\theta), d_p)$. This means that for any $0 \leq t \leq s \leq 1$, $d_p(u_t, u_s) = |t-s| d_p(u_0, u_1)$.*

Proof. The same proof as in [16, Theorem 3.17] works in our case as well. We will first show that given $0 \leq t \leq 1$, we have

$$d_p(u_0, u_t) = t d_p(u_0, u_1) \qquad \text{and} \qquad d_p(u_1, u_t) = (1-t) d_p(u_0, u_1).$$

First, assume that $u_0, u_1 \in \mathcal{H}_\theta$. Let $w_s = u_{ts}$ be the geodesic joining u_0 and u_t. By Lemma 4.6.3, we obtain that

$$d_p^p(u_0, u_t) = \int_X |\dot{w}_0|^p \theta_{u_0}^n = t^p \int_X |\dot{u}_0|^p \theta_{u_0}^n = t^p d_p^p(u_0, u_1).$$

The other equality is proved similarly.

For arbitrary $u_0, u_1 \in \mathcal{E}^p(X,\theta)$, find $u_0^j, u_1^j \in \mathcal{H}_\theta$ such that $u_0^j \searrow u_0$ and $u_1^j \searrow u_1$. If u_t^j is the weak geodesic joining u_0^j and u_1^j, then $u_t^j \searrow u_t$. As $d_p(u_0^j, u_t^j) = t d_p(u_0^j, u_1^j)$, taking limit $j \to \infty$ using Lemma 4.6.2, we obtain $d_p(u_0, u_t) = t d_p(u_0, u_1)$.

Now, for a more general case, if $0 < t < s < 1$, then applying the above result twice, we get

$$d_p(u_t, u_s) = \frac{s-t}{s} d_p(u_0, u_s) = (s-t) d_p(u_0, u_1).$$

□

Lastly, we prove that the metric d_p satisfies the Pythagorean identity.

Proposition 4.6.5. *If* $u_0, u_1 \in \mathcal{E}^p(X,\theta)$, *then*

$$d_p^p(u_0, u_1) = d_p^p(u_0, P_\theta(u_0, u_1)) + d_p^p(u_1, P_\theta(u_0, u_1)).$$

Proof. If $u_0, u_1 \in \mathcal{H}_\theta$, then this is the content of Theorem 4.5.6 when combined with Theorem 4.5.7.

More generally, if $u_0, u_1 \in \mathcal{E}^p(X,\theta)$, then we can find $u_0^k, u_1^k \in \mathcal{H}_\theta$ such that $u_0^k \searrow u_0$ and $u_1^k \searrow u_1$. Then $P_\theta(u_0^k, u_1^k) \searrow P_\theta(u_0, u_1)$ as well. Thus

$$\begin{aligned} d_p^p(u_0, u_1) &= \lim_{k\to\infty} d_p^p(u_0^k, u_1^k) \\ &= \lim_{k\to\infty} d_p^p(u_0^k, P_\theta(u_0^k, u_1^k)) + d_p^p(u_1^k, P_\theta(u_0^k, u_1^k)) \\ &= d_p^p(u_0, P_\theta(u_0, u_1)) + d_p^p(u_1, P_\theta(u_0, u_1)). \end{aligned}$$

□

4.6.1 Connection with the metric in the literature

In this subsection, we prove that when θ is big and nef, or when $p = 1$, then the metric d_p on $\mathcal{E}^p(X, \theta)$ constructed in Section 4.5 coincides with the metric d_p constructed in [16] and [17].

Theorem 4.6.6. *If β is a smooth closed real $(1,1)$-form representing a big and nef cohomology class, then the metric d_p constructed in Section 4.5 agrees with the one constructed in [16].*

Proof. Let us use D_p to represent the metric constructed in [16]. In case $u_0, u_1 \in \mathcal{H}_\beta$, then by Theorem 4.5.7 and by [16, Theorem 3.7]

$$d_p(u_0, u_1) = \int_X |\dot{u}_0|^p \beta_{u_0}^n = D_p(u_0, u_1),$$

where u_t is the weak geodesic joining u_0 and u_1.

By Definition 4.5.10 and the definition of D_p in [16, Equation above Proposition 3.12] when $u_0, u_1 \in \mathcal{E}^p(X, \beta)$, then

$$d_p(u_0, u_1) = \lim_{k\to\infty} d_p(u_0^k, u_1^k) = \lim_{k\to\infty} D_p(u_0^k, u_1^k) = D_p(u_0, u_1)$$

where $u_0^k, u_1^k \in \mathcal{H}_\beta$ such that $u_0^k \searrow u_0$ and $u_1^k \searrow u_1$. ∎

Theorem 4.6.7. *When $p = 1$, then $u_0, u_1 \in \mathcal{E}^1(X, \theta)$ satisfy*

$$d_1(u_0, u_1) = I_\theta(u_0) + I_\theta(u_1) - 2I_\theta(P_\theta(u_0, u_1)).$$

Thus d_1 agrees with the metric constructed in [17].

Proof. The proof is the same as in [16, Proposition 3.18]. We recall the steps for completeness. If $u_0, u_1 \in \mathcal{H}_\theta$ and $u_0 \le u_1$, then from Lemma 4.3.7 and Theorem 4.5.7,

$$d_1(u_0, u_1) = \int_X \dot{u}_0 \theta^n_{u_0} = \int_X \dot{u}_1 \theta^n_{u_1} = I_\theta(u_1) - I_\theta(u_0).$$

If $u_0, u_1 \in \mathcal{H}_\theta$ be arbitrary (we drop the condition that $u_0 \le u_1$), then by the Pythagorean identity (see Proposition 4.6.5),

$$\begin{aligned} d_1(u_0, u_1) &= d_1(u_0, P_\theta(u_0, u_1)) + d_1(u_1, P_\theta(u_0, u_1)) \\ &= I_\theta(u_0) + I_\theta(u_1) - 2I_\theta(P_\theta(u_0, u_1)). \end{aligned}$$

More generally, when $u_0, u_1 \in \mathcal{E}^p(X, \theta)$, then using $u_0^k, u_1^k \in \mathcal{H}_\theta$ such that $u_0^k \searrow u_0$ and $u_1^k \searrow u_1$, we can prove that

$$\begin{aligned} d_1(u_0, u_1) &= \lim_{k \to \infty} d_1(u_0^k, u_1^k) \\ &= \lim_{k \to \infty} I_{\psi_k}(u_0^k) + I_{\psi_k}(u_1^k) - 2I_{\psi_k}(P_\theta(u_0^k, u_1^k)) \\ &= I_\theta(u_0) + I_\theta(u_1) - 2I_\theta(P_\theta(u_0, u_1)). \end{aligned}$$

□

4.7 Contraction Property and a Consequence

Let θ be a smooth closed real $(1,1)$- form representing a big cohomology class, and $\psi \in \mathrm{PSH}(X, \theta)$ have analytic singularities. In this section, we will prove that the map $\mathcal{E}^p(X, \theta) \ni$

$u \mapsto P_\theta[\psi](u) \in \mathcal{E}^p(X,\theta,\psi)$ is well defined and is a contraction, i.e., $d_p(P_\theta[\psi](u), P_\theta[\psi](v)) \leq d_p(u,v)$. When $p = 1$, the results in this section were proved in [56, Section 4.1].

We need a technical lemma, whose proof can be obtained by modifying the proof in [15, Lemma 5.1] by changing the weight function.

Lemma 4.7.1. *If $u_j \in \mathcal{E}^p(X,\theta,\psi)$ is a decreasing sequence of functions such that for some $\varphi \in \mathcal{E}^p(X,\theta,\psi)$,*

$$\sup_j \int_X |u_j - \varphi|^p \theta_{u_j}^n < \infty,$$

then $u := \lim_{j\to\infty} u_j \in \mathcal{E}^p(X,\theta,\psi)$.

Lemma 4.7.2. *If ψ is a model potential, i.e., $P_\theta[\psi] = \psi$, then $u \in \mathcal{E}^p(X,\theta)$ implies $P_\theta[\psi](u) \in \mathcal{E}^p(X,\theta,\psi)$.*

Proof. If u has minimal singularity type, then $|V_\theta - u| \leq D$ for some constant $D > 0$. Therefore,

$$P_\theta(\psi + C, u) \leq P_\theta(\psi + C, V_\theta + D) = P_\theta(\psi + C - D, V_\theta) + D.$$

Taking the limit $C \to \infty$ we get

$$\lim_{C\to\infty} P_\theta(\psi + C, u) \leq \lim_{C\to\infty} P_\theta(\psi + C - D, V_\theta) + D = \lim_{C\to\infty} P_\theta(\psi + C, V_\theta) + D.$$

Taking upper semicontinuous regularization we get

$$P_\theta[\psi](u) \leq P_\theta[\psi](V_\theta) + D = \psi + D.$$

Similarly,

$$\psi \leq P_\theta[\psi](u) + D.$$

Thus $P_\theta[\psi](u)$ has the same singularity type as ψ, thus $P_\theta[\psi](u) \in \mathcal{E}^p(X, \theta, \psi)$.

More generally, if $u \in \mathcal{E}^p(X, \theta)$, then $u_j := \max(u, V_\theta - j)$ has the minimal singularity type. Then $P_\theta[\psi](u_j)$ has minimal singularity type and we claim that $P_\theta[\psi](u_j) \searrow P_\theta[\psi](u)$. Moreover, we will show that

$$\sup_j \int_X |P_\theta[\psi](u_j) - \psi|^p \theta^n_{P_\theta[\psi](u_j)} < \infty$$

concluding with Lemma 4.7.1 that $P_\theta[\psi](u) \in \mathcal{E}^p(X, \theta, \psi)$.

Let $u \leq K$. Then $u_j \leq K$ and $P_\theta[\psi](u_j) \leq K$ as well. Therefore, $P_\theta[\psi](u_j) - K \leq 0$ and $P_\theta[\psi](u_j)$ has the same singularity type as ψ, which is a model singularity, thus $P_\theta[\psi](u_j) - K \leq \psi$. Hence we get

$$P_\theta[\psi](u_j) - K - V_\theta \leq P_\theta[\psi](u_j) - K - \psi \leq 0.$$

We also need [31, Theorem 3.8] that says $\theta^n_{P_\theta[\psi](u_j)} \leq \mathbb{1}_{\{P_\theta[\psi](u_j) = u_j\}} \theta^n_{u_j}$. Using $(a+b)^p \leq 2^{p-1}(a^p + b^p)$, we get

$$\begin{aligned}
\int_X |P_\theta[\psi](u_j) - \psi|^p \theta^n_{P_\theta[\psi](u_j)} &\leq 2^{p-1} \int_X (|P_\theta[\psi](u_j) - K - \psi|^p + K^p) \theta^n_{P_\theta[\psi](u_j)} \\
&\leq 2^{p-1} \left(\int_X |P_\theta[\psi](u_j) - K - V_\theta|^p \theta^n_{P_\theta[\psi](u_j)} + K^p \int_X \theta^n_\psi \right) \\
&\leq 2^{p-1} \left(\int_{\{P_\theta[\psi](u_j) = u_j\}} |u_j - K - V_\theta|^p \theta^n_{u_j} + K^p \int_X \theta^n_\psi \right) \\
&\leq 2^{p-1} \left(\int_X |u_j - K - V_\theta|^p \theta^n_{u_j} + K^p \int_X \theta^n_\psi \right)
\end{aligned}$$

is uniformly bounded since $u \in \mathcal{E}^p(X,\theta)$ and by [29] $\sup \int_X |u_j - V_\theta|^p \theta^n_{u_j} < \infty$.

Since $P_\theta[\psi](u_j)$ is a decreasing sequence of potentials in $\mathcal{E}^p(X,\theta,\psi)$, the above calculation and Lemma 4.7.1 imply that $v := \lim_{j\to\infty} P_\theta[\psi](u_j) \in \mathcal{E}^p(X,\theta,\psi)$. As $v \leq u_j$ for all j, we get that $v \leq u$. Moreover, $v \preceq \psi$, thus v is a candidate for $P_\theta[\psi](u)$. Hence $P_\theta[\psi](u)$ exists and $v \leq P_\theta[\psi](u)$. Since $u_j \geq u$, we also have that $P_\theta[\psi](u_j) \geq P_\theta[\psi](u)$ for each j. Taking limit we get $v \geq P_\theta[\psi](u)$. Thus $\lim_{j\to\infty} P_\theta[\psi](u_j) = P_\theta[\psi](u) \in \mathcal{E}^p(X,\theta,\psi)$. □

Theorem 4.7.3. *If $\psi \in PSH(X,\theta)$ has analytic singularities, then the map $P_\theta[\psi](\cdot) : (\mathcal{E}^p(X,\theta), d_p) \to (\mathcal{E}^p(X,\theta,\psi), d_p)$ is a contraction. This means for any $u_0, u_1 \in \mathcal{E}^p(X,\theta)$,*

$$d_p(P_\theta[\psi](u_0), P_\theta[\psi](u_1)) \leq d_p(u_0,u_1). \tag{4.13}$$

Proof. First we assume that there are functions $f_0, f_1 \in C^{1,\bar{1}}(X)$ such that $u_0 = P_\theta(f_0)$ and $u_1 = P_\theta(f_1)$. Let $v_0 := P_\theta[\psi](u_0) = P_\theta[\psi](f_0)$ and $v_1 := P_\theta[\psi](u_1) = P_\theta[\psi](f_1)$. Moreover assume that $u_0 \leq u_1$. In this case, we know from Theorem 4.5.7 combined with Theorem 2.4.4 that

$$d_p^p(u_0,u_1) = \int_X |\dot{u}_0|^p \theta^n_{u_0} = \int_X \mathbb{1}_{\{P_\theta(f_0)=f_0\}} (\dot{u}_0)^p \theta^n_{f_0} \tag{4.14}$$

and from Theorem 4.4.10 combined with Theorem 2.4.4 that

$$d_p^p(v_0,v_1) = \int_X |\dot{v}_0|^p \theta^n_{v_0} = \int_X \mathbb{1}_{\{P_\theta[\psi](f_0)=f_0\}} (\dot{v}_0)^p \theta^n_{f_0}, \tag{4.15}$$

where u_t and v_t are the weak geodesics joining u_0, u_1 and v_0, v_1 respectively. Since $P_\theta[\psi](f_0) \leq P_\theta(f_0) \leq f_0$, we know $\{P_\theta[\psi](f_0) = f_0\} \subset \{P_\theta(f_0) = f_0\}$. As $u_0 \geq v_0$ and $u_1 \geq v_1$, we have

$u_t \geq v_t$. If $x \in \{P_\theta[\psi](f_0) = f_0\}$, then

$$\dot{v}_0(x) = \lim_{t\to\infty} \frac{v_t(x) - v_0(x)}{t} \leq \lim_{t\to\infty} \frac{u_t(x) - f_0(x)}{t} = \lim_{t\to\infty} \frac{u_t(x) - u_0(x)}{t} = \dot{u}_0(x).$$

Thus $\mathbb{1}_{\{P_\theta[\psi](f_0)=f_0\}}(\dot{v}_0)^p \leq \mathbb{1}_{P_\theta(f_0)=f_0\}}(\dot{u}_0)^p$. Now (4.14) and (4.15) give

$$d_p(P_\theta[\psi](u_0), P_\theta[\psi](u_1)) = d_p(v_0, v_1) \leq d_p(u_0, u_1).$$

Now we will remove the assumption that $u_0 \leq u_1$. We still assume that $u_0 = P_\theta(f_0)$ and $u_1 = P_\theta(f_1)$ for some $f_0, f_1 \in C^{1,\bar{1}}(X)$. We will use the Pythagorean formula for d_p metrics to establish (4.13) in this case. As before let $v_0 := P_\theta[\psi](u_0) = P_\theta[\psi](f_0)$ and $v_1 := P_\theta[\psi](u_1) = P_\theta[\psi](f_1)$. Let $C > 0$ be a constant such that $\theta \leq C\omega$. Then $h = P_{C\omega}(f_0, f_1) \in C^{1,\bar{1}}(X)$, and by Lemma 4.7.4, $P_\theta(u_0, u_1) = P_\theta(h)$, and $P_\theta(v_0, v_1) = P_\theta[\psi](h)$. Also observe that $P_\theta[\psi](P_\theta(u_0, u_1)) = P_\theta(v_0, v_1)$. Applying the result in the previous paragraph we obtain $d_p(u_0, P_\theta(u_0, u_1)) \geq d_p(v_0, P_\theta(v_0, v_1))$ and $d_p(u_1, P_\theta(u_0, u_1)) \geq d_p(v_1, P_\theta(v_0, v_1))$.

Using the Pythagorean formula, we write

$$\begin{aligned} d_p^p(u_0, u_1) &= d_p^p(u_0, P_\theta(u_0, u_1)) + d_p^p(u_1, P_\theta(u_0, u_1)) \\ &\geq d_p^p(v_0, P_\theta(v_0, v_1)) + d_p^p(v_1, P_\theta(v_0, v_1)) \\ &= d_p^p(v_0, v_1). \end{aligned}$$

Thus we have shown that (4.13) holds when $u_0 = P_\theta(f_0)$ and $u_1 = P_\theta(f_1)$ for $f_0, f_1 \in C^{1,\bar{1}}(X)$. We show it more generally by approximation.

If $u_0, u_1 \in \mathcal{E}^p(X, \theta)$, then we can find $u_0^j, u_1^j \in \mathcal{H}_\theta$ such that $u_0^j \searrow u_0$ and $u_1^j \searrow u_1$. Moreover, $P_\theta[\psi](u_0^j) \searrow P_\theta[\psi](u_0)$ and $P_\theta[\psi](u_1^j) \searrow P_\theta[\psi](u_1)$. The proof is the same as in Lemma 4.7.1. Thus from Lemma 4.4.12 we get that $\lim_{j\to\infty} d_p(P_\theta[\psi](u_0^j), P_\theta[\psi](u_1^j)) = d_p(P_\theta[\psi](u_0), P_\theta[\psi](u_1))$. Hence from the result in the previous paragraph we have

$$
\begin{aligned}
d_p(u_0, u_1) &= \lim_{j\to\infty} d_p(u_0^j, u_1^j) \\
&\geq \lim_{j\to\infty} d_p(P_\theta[\psi](u_0^j), P_\theta[\psi](u_1^j)) \\
&= d_p(P_\theta[\psi](u_0), P_\theta[\psi](u_1)).
\end{aligned}
$$

This proves (4.13) in the full generality as desired. □

Lemma 4.7.4. *Let $\psi, f_0, f_1, u_0, u_1, v_0, v_1$ and h be as in the proof of Theorem 4.7.3, then*

$$
P_\theta(u_0, u_1) = P_\theta(h) \qquad \textit{and} \qquad P_\theta(v_0, v_1) = P_\theta[\psi](h).
$$

Proof. We will prove $P_\theta(v_0, v_1) = P_\theta[\psi](h)$ as the proof of the other part is the same. By definition

$$
P_\theta(v_0, v_1) = \sup\{u \in \text{PSH}(X, \theta) \,|\, u \leq v_0, u \leq v_1\}
$$

and

$$
P_\theta[\psi](h) = \sup\{u \in \text{PSH}(X, \theta) \,|\, u \leq \psi + C, u \leq h\}.
$$

We will show that the candidate sets for both $P_\theta(v_0, v_1)$ and $P_\theta[\psi](h)$ are the same.

Let $u \in \text{PSH}(X, \theta)$ be a candidate for $P_\theta(v_0, v_1)$. As $v_0 = P_\theta[\psi](f_0)$, and $v_1 = P_\theta[\psi](f_1)$, we get $v_0 \leq f_0$ and $v_1 \leq f_1$. So $u \leq f_0$ and $u \leq f_1$. As $\theta \leq C\omega$, u is a $C\omega$-psh as well. Hence

u is a candidate for h. Therefore $u \leq h$. Moreover, since ψ is a model singularity type, v_0 and v_1 have the same singularity type as ψ, therefore, $u \leq \psi + D$ for some constant D. Thus u is a candidate for $P_\theta[\psi](h)$.

For the other direction, let $u \in \text{PSH}(X, \theta)$ be a candidate for $P_\theta[\psi](h)$. As $u \leq h$, we get that $u \leq f_0$ and $u \leq f_1$. Also $u \leq \psi + C$, so u is a candidate for v_0 and v_1. Thus $u \leq v_0$ and $u \leq v_1$. Thus u is a candidate for $P_\theta(v_0, v_1)$. □

A consequence of this contraction formula is that the approximation formula for d_p on $\mathcal{H}_\theta \subset \mathcal{E}^p(X, \theta)$ from potentials in analytic singularity type can be extended to any potentials in $\mathcal{E}^p(X, \theta)$. More precisely, we can prove

Theorem 4.7.5. *Let $\psi_k \in PSH(X, \theta)$ have analytic singularities and $\psi_k \nearrow V_\theta$ as described in the beginning of Section 4.5. If $u_0, u_1 \in \mathcal{E}^p(X, \theta)$, then*

$$d_p(u_0, u_1) = \lim_{k \to \infty} d_p(P_\theta[\psi_k](u_0), P_\theta[\psi_k](u_1)). \tag{4.16}$$

Proof. (4.16) is the definition of d_p when $u_0, u_1 \in \mathcal{H}_\theta$. Here we want to prove it more generally.

Let $u_0^j, u_1^j \in \mathcal{H}_\theta$ be such that $u_0^j \searrow u_0$ and $u_1^j \searrow u_1$. Then by definition of d_p on $\mathcal{E}^p(X, \theta)$, we have

$$d_p(u_0, u_1) = \lim_{j \to \infty} d_p(u_0^j, u_1^j)$$

and

$$d_p(u_0^j, u_1^j) = \lim_{k \to \infty} d_p(P_\theta[\psi_k](u_0^j), P_\theta[\psi_k](u_1^j)).$$

Combining the two we get

$$d_p(u_0, u_1) = \lim_{j\to\infty} \lim_{k\to\infty} d_p(P_\theta[\psi_k](u_0^j), P_\theta[\psi_k](u_1^j)). \tag{4.17}$$

We want to exchange the limit. First, observe that as $j \to \infty$, $P_\theta[\psi_k](u_0^j) \searrow P_\theta[\psi_k](u_0)$ and $P_\theta[\psi_k](u_1^j) \searrow P_\theta[\psi_k](u_1)$. Thus from Lemma 4.4.12,

$$\lim_{j\to\infty} d_p(P_\theta[\psi_k](u_0^j), P_\theta[\psi_k](u_1^j)) = d_p(P_\theta[\psi_k](u_0), P_\theta[\psi_k](u_1)). \tag{4.18}$$

Now we will show that the limit in (4.18) is uniform in k. For that, we observe by triangle inequality that

$$\begin{aligned}
&|d_p(P_\theta[\psi_k](u_0^j), P_\theta[\psi_k](u_1^j)) - d_p(P_\theta[\psi_k](u_0), P_\theta[\psi_k](u_1))| \\
\leq& d_p(P_\theta[\psi_k](u_0^j), P_\theta[\psi_k](u_0)) + d_p(P_\theta[\psi_k](u_1^j), P_\theta[\psi_k](u_1)) \\
\leq& d_p(u_0^j, u_0) + d_p(u_1^j, u_0),
\end{aligned}$$

where in the last line we used Theorem 4.7.3. Moreover, $\lim_{j\to\infty} d_p(u_0^j, u_0) = 0$ and $\lim_{j\to\infty} d_p(u_1^j, u_1) = 0$. Using the uniform convergence in k, we obtain that we can exchange the limits in (4.17). Thus

$$\begin{aligned}
d_p(u_0, u_1) &= \lim_{k\to\infty} \lim_{j\to\infty} d_p(P_\theta[\psi_k](u_0^j), P_\theta[\psi_k](u_1^j)) \\
&= \lim_{k\to\infty} d_p(P_\theta[\psi_k](u_0), P_\theta[\psi_k](u_1)),
\end{aligned}$$

as desired. □

4.8 Uniform Convexity

In [11] Darvas-Lu proved that for $p > 1$, $u, v_0, v_1 \in \mathcal{E}^p(X,\omega)$, and $(0,1) \ni \lambda \mapsto v_\lambda \in \mathcal{E}^p(X,\omega)$, the weak geodesic joining v_0 and v_1, satisfies

$$d_p(u,v_\lambda)^2 \leq (1-\lambda)d_p(u,v_0)^2 + \lambda d_p(u,v_1)^2 - (p-1)\lambda(1-\lambda)d_p(v_0,v_1)^2, \text{ if } 1 < p \leq 2 \text{ and} \tag{4.19}$$

$$d_p(u,v_\lambda)^p \leq (1-\lambda)d_p(u,v_0)^p + \lambda d_p(u,v_1)^p - \lambda^{p/2}(1-\lambda)^{p/2}d_p(v_0,v_1)^p, \text{ if } p \geq 2. \tag{4.20}$$

We will prove that this result holds for $(\mathcal{E}^p(X,\theta)$ where θ represents a big cohomology class.

4.8.1 For big and nef cohomology class

If β represents a big and nef cohomology class, using the approximation method used to construct the metric d_p on $\mathcal{E}^p(X,\beta)$, in this section we will extend these inequalities to $\mathcal{E}^p(X,\beta)$.

First, we will show the convexity property on $\mathcal{H}_\beta$ (see (4.1)). If $u, v_0, v_1 \in \mathcal{H}_\beta$ and $\lambda \mapsto v_\lambda$ is the weak geodesic joining v_0 and v_1, then we claim

$$d_p(u,v_\lambda)^2 \leq (1-\lambda)d_p(u,v_0)^2 + \lambda d_p(u,v_1)^2 - (p-1)\lambda(1-\lambda)d_p(v_0,v_1)^2, \text{ if } 1 < p \leq 2 \text{ and} \tag{4.21}$$

$$d_p(u,v_\lambda)^p \leq (1-\lambda)d_p(u,v_0)^p + \lambda d_p(u,v_1)^p - \lambda^{p/2}(1-\lambda)^{p/2}d_p(v_0,v_1)^p, \text{ if } p \geq 2. \tag{4.22}$$

We will show it by the approximation process. Let $u = P_\beta(f)$, $v_0 = P_\beta(f_0)$, and $v_1 =$

$P_\beta(f_1)$ for $f, f_0, f_1 \in C(X)$ such that $dd^c f, dd^c f_0, dd^c f_2 \leq C\omega$. Recall that we defined $\omega_\varepsilon := \beta + \varepsilon\omega$. Let $u_\varepsilon = P_{\omega_\varepsilon}(f)$, $v_{0,\varepsilon} = P_{\omega_\varepsilon}(f_0)$ and $v_{1,\varepsilon} = P_{\omega_\varepsilon}(f_1)$. Let $v_{\lambda,\varepsilon}$ be the geodesic joining $v_{0,\varepsilon}$ and $v_{1,\varepsilon}$. From the result in the Kähler case, we know that

$$\begin{aligned} d_p(u_\varepsilon, v_{\lambda,\varepsilon})^2 \leq &(1-\lambda) d_p(u_\varepsilon, v_{0,\varepsilon})^2 + \lambda d_p(u_\varepsilon, v_{1,\varepsilon})^2 \\ &- (p-1)\lambda(1-\lambda) d_p(v_{0,\varepsilon}, v_{1,\varepsilon})^2, \text{ if } 1 < p \leq 2 \text{ and} \end{aligned} \tag{4.23}$$

$$\begin{aligned} d_p(u_\varepsilon, v_{\lambda,\varepsilon})^p \leq &(1-\lambda) d_p(u_\varepsilon, v_{0,\varepsilon})^p + \lambda d_p(u_\varepsilon, v_{1,\varepsilon})^p \\ &- \lambda^{p/2}(1-\lambda)^{p/2} d_p(v_{0,\varepsilon}, v_{1,\varepsilon})^p, \text{ if } p \geq 2. \end{aligned} \tag{4.24}$$

By the definition of d_p on $\mathcal{H}_\beta$ (see Section 4.3), we know that the $\lim_{\varepsilon\to 0} d_p(u_\varepsilon, v_{0,\varepsilon}) = d_p(u, v_0)$, $\lim_{\varepsilon\to 0} d_p(u_\varepsilon, v_{1,\varepsilon}) = d_p(u, v_1)$, and $\lim_{\varepsilon\to 0} d_p(v_{0,\varepsilon}, v_{1,\varepsilon}) = d_p(v_0, v_1)$. So we are done if we can prove that

$$\lim_{\varepsilon\to 0} d_p(u_\varepsilon, v_{\lambda,\varepsilon}) = d_p(u, v_\lambda).$$

Let w_t be the weak geodesic joining u and v_λ and $w_{t,\varepsilon}$ be the weak geodesic joining u_ε and $v_{\lambda,\varepsilon}$. Since $u \in \mathcal{H}_\beta$ and $u_\varepsilon \in \mathcal{H}_{\omega_\varepsilon}$ from [16, Lemma 3.13] we get that

$$d_p(u, v_\lambda)^p = \int_X |\dot{w}_0|^p (\beta + dd^c u)^n,$$

and

$$d_p(u_\varepsilon, v_{\lambda,\varepsilon})^p = \int_X |\dot{w}_{0,\varepsilon}|^p (\omega_\varepsilon + dd^c u_\varepsilon)^n.$$

Using [16, Lemma 3.5], we get that if $(\beta + dd^c u)^n = \rho\omega^n$ and $(\omega_\varepsilon + dd^c u_\varepsilon)^n = \rho_\varepsilon \omega^n$, then $\varepsilon \mapsto \rho_\varepsilon$ is increasing, uniformly bounded and $\rho_\varepsilon \searrow \rho$ pointwise as $\varepsilon \to 0$. Moreover, from Theorem 2.4.4, the measure $(\beta + dd^c u)^n$ is supported on $D := \{P_\beta(f) = f\}$ and the measures $(\omega_\varepsilon + dd^c u_\varepsilon)^n$ are supported on $D_\varepsilon := \{P_{\omega_\varepsilon}(f) = f\}$. Moreover, $\cap_{\varepsilon>0} D_\varepsilon = D$.

We will show that

$$\lim_{\varepsilon\to 0} \int_X |\dot{w}_{0,\varepsilon}|^p (\omega_\varepsilon + dd^c u_\varepsilon)^n = \int_X |\dot{w}_0|^p (\beta + dd^c u)^n.$$

The proof is similar to [16, Lemma 3.6, and Theorem 3.7].

Lemma 4.8.1. *Let w_t and $w_{t,\varepsilon}$ be the weak geodesics joining u, v_λ and $u_\varepsilon, v_{\lambda,\varepsilon}$ respectively as described above. Then*

$$\lim_{\varepsilon\to 0} \mathbb{1}_{D_\varepsilon} |\dot{w}_{0,\varepsilon}|^p = \mathbb{1}_D |\dot{w}_0|^p.$$

Proof. First we observe that $u_\varepsilon \searrow u$ and $v_{\lambda,\varepsilon} \searrow v_\lambda$ as $\varepsilon \to 0$. We will explain why $v_{\lambda,\varepsilon} \searrow v_\lambda$. This follows because $v_{0,\varepsilon} \searrow v_0$ and $v_{1,\varepsilon} \searrow v_1$ as $\varepsilon \to 0$. If $\varepsilon_1 < \varepsilon_2$, then the geodesic $v_{\lambda,\varepsilon_1}$ is a candidate subgeodesic joining v_{0,ε_2} and v_{1,ε_2}. Therefore the geodesics $v_{\lambda,\varepsilon}$ are decreasing sequence of ω_ε-psh functions. If $v_{\lambda,\varepsilon} \searrow \phi_\lambda$, then $\phi_\lambda \geq v_\lambda$ because $v_{\lambda,\varepsilon} \geq v_\lambda$. But ϕ_λ is a candidate subgeodesic joining v_0 and v_1, therefore $\phi_\lambda \leq v_\lambda$. Thus $v_{\lambda,\varepsilon} \searrow v_\lambda$ as $\varepsilon \to 0$.

We can obtain that $w_{t,\varepsilon} \searrow w_t$ as well, by the same reasoning.

If $x \in D$, then

$$\dot{w}_0(x) = \lim_{t\to 0} \frac{w_t(x) - f(x)}{t} \leq \lim_{t\to 0} \frac{w_{t,\varepsilon}(x) - f(x)}{t} = \dot{w}_{0,\varepsilon} \leq \frac{w_{t,\varepsilon}(x) - w_{0,\varepsilon}(x)}{t}.$$

Here we used $w_{t,\varepsilon} \geq w_t$ and the convexity of the geodesic $w_{t,\varepsilon}$ for the last inequality. Taking

$\varepsilon \to 0$, and using $w_{t,\varepsilon} \searrow w_t$ we obtain

$$\dot{w}_0(x) \le \lim_{\varepsilon\to 0} \dot{w}_{0,\varepsilon}(x) \le \frac{w_t(x) - w_0(x)}{t}.$$

Taking $t \to 0$, we get

$$\dot{w}_0(x) \le \lim_{\varepsilon\to 0} \dot{w}_{0,\varepsilon}(x) \le \dot{w}_0(x).$$

Thus if $x \in D$, then $\lim_{\varepsilon\to 0} \dot{w}_{0,\varepsilon}(x) = \dot{w}_0(x)$. If $x \notin D$, then for ε small enough $x \notin D_\varepsilon$. Thus we get $\mathbb{1}_{D_\varepsilon}|\dot{w}_{0,\varepsilon}|^p = \mathbb{1}_D|\dot{w}_0|^p$ as $\varepsilon \to 0$ pointwise. □

Theorem 4.8.2.

$$\lim_{\varepsilon\to 0} \int_X |\dot{w}_{0,\varepsilon}|^p(\omega_\varepsilon + dd^c u_\varepsilon)^n = \int_X |\dot{w}_0|^p(\beta + dd^c u)^n.$$

Proof. We first notice that since $(\omega_\varepsilon + dd^c u_\varepsilon)^n = \rho_\varepsilon\omega^n$ and is supported on D_ε, therefore

$$\int_X |\dot{w}_{0,\varepsilon}|^p(\omega_\varepsilon + dd^c u_\varepsilon)^n = \int_X \mathbb{1}_{D_\varepsilon}|\dot{w}_{0,\varepsilon}|^p\rho_\varepsilon\omega^n.$$

Similarly

$$\int_X |\dot{w}_0|^p(\beta + dd^c u)^n = \int_X \mathbb{1}_D|\dot{w}_0|^p\rho\omega^n.$$

As $\rho_\varepsilon \to \rho$ pointwise and are uniformly bounded (from [16, Lemma 3.5]), we just need to show that $|\dot{w}_{0,\varepsilon}|$ are uniformly bounded in ε.

From convexity of $v_{\lambda,\varepsilon}$ in λ, we obtain that $v_{\lambda,\varepsilon} \le (1-\lambda)v_{0,\varepsilon} + \lambda v_{1,\varepsilon} \le \max(v_{0,\varepsilon}, v_{1,\varepsilon})$. Also $P_{\omega_\varepsilon}(v_{0,\varepsilon}, v_{1,\varepsilon}) \le v_{\lambda,\varepsilon}$. Combining we have $P_{\omega_\varepsilon}(v_{0,\varepsilon}, v_{1,\varepsilon}) \le v_{\lambda,\varepsilon} \le \max\{v_{0,\varepsilon}, v_{1,\varepsilon}\}$.

From [36, Lemma 3.1], we obtain

$$
\begin{aligned}
|\dot{w}_{0,\varepsilon}| &\leq |w_{1,\varepsilon} - w_{0,\varepsilon}| \\
&= |u_\varepsilon - v_{\lambda,\varepsilon}| \\
&\leq \max\{|u_\varepsilon - \max\{v_{0,\varepsilon}, v_{1,\varepsilon}\}|, |u_\varepsilon - P_{\omega_\varepsilon}(v_{0,\varepsilon}, v_{1,\varepsilon})|\} \\
&= \max\{|u_\varepsilon - v_{0,\varepsilon}|, |u_\varepsilon - v_{1,\varepsilon}|, |u_\varepsilon - P_{\omega_\varepsilon}(v_{0,\varepsilon}, v_{1,\varepsilon})|\}.
\end{aligned}
$$

Since $P_{\omega_\varepsilon}(v_{0,\varepsilon}, v_{1,\varepsilon}) = P_{\omega_\varepsilon}(\min\{f_0, f_1\})$ and $u_{0,\varepsilon} = P_{\omega_\varepsilon}(f)$, $v_{1,\varepsilon} = P_{\omega_\varepsilon}(f_1)$ and $v_{0,\varepsilon} = P_{\omega_\varepsilon}(f_0)$, and using that for any continuous $h_1, h_2 \in C(X)$, $|P_{\omega_\varepsilon}(h_1) - P_{\omega_\varepsilon}(h_2)| \leq \sup_X |h_1 - h_2|$, we obtain that

$$
|\dot{w}_{0,\varepsilon}| \leq \max\{\sup_X |f - f_0|, \sup_X |f - f_1|, \sup_X |f - \min\{f_0, f_1\}|\}.
$$

Therefore by Lebesgue's Dominated Convergence theorem, and Lemma 4.8.1,

$$
\lim_{\varepsilon \to 0} \int_X \mathbb{1}_{D_\varepsilon} |\dot{w}_{0,\varepsilon}|^p \rho_\varepsilon \omega^n = \int_X \mathbb{1}_D |\dot{w}_0|^p \rho \omega^n.
$$

□

Now the previous theorem proves

Theorem 4.8.3. *If $u, v_0, v_1 \in \mathcal{H}_\beta$, and v_λ is the weak geodesic joining v_0 and v_1, then*

$$
d_p(u, v_\lambda)^2 \leq (1-\lambda) d_p(u, v_0)^2 + \lambda d_p(u, v_1)^2 - (p-1)\lambda(1-\lambda) d_p(v_0, v_1)^2, \text{ if } 1 < p \leq 2 \text{ and}
$$

$$
d_p(u, v_\lambda)^p \leq (1-\lambda) d_p(u, v_0)^p + \lambda d_p(u, v_1)^p - \lambda^{p/2}(1-\lambda)^{p/2} d_p(v_0, v_1)^p, \text{ if } p \geq 2.
$$

Proof. We only needed to prove that $\lim_{\varepsilon\to 0} d_p(u_\varepsilon, v_{\lambda,\varepsilon}) = d_p(u, v_\lambda)$ which is proved by Theorem 4.8.2. Now taking the limit $\varepsilon \to 0$ in (4.23) and (4.24) proves the result. $\square$

Now we will extend this proof to all of $\mathcal{E}^p(X, \beta)$.

Theorem 4.8.4. *Let $u, v_0, v_1 \in \mathcal{E}^p(X, \beta)$ and v_λ be the weak geodesic joining v_0 and v_1, then*

$$d_p(u, v_\lambda)^2 \le (1-\lambda)d_p(u, v_0)^2 + \lambda d_p(u, v_1)^2 - (p-1)\lambda(1-\lambda)d_p(v_0, v_1)^2, \text{ if } 1 < p \le 2 \text{ and}$$

$$d_p(u, v_\lambda)^p \le (1-\lambda)d_p(u, v_0)^p + \lambda d_p(u, v_1)^p - \lambda^{p/2}(1-\lambda)^{p/2}d_p(v_0, v_1)^p, \text{ if } p \ge 2.$$

Proof. We give the proof by approximation from $\mathcal{H}_\beta$. Let $u^j, v_0^j, v_1^j \in \mathcal{H}_\beta$ satisfy $u^j \searrow u$, $v_0^j \searrow v_0$, and $v_1^j \searrow v_1$. If v_λ^j is the weak geodesic joining v_0^j and v_1^j, then by Theorem 4.8.3

$$\begin{aligned} d_p(u^j, v_\lambda^j)^2 &\le (1-\lambda)d_p(u^j, v_0^j)^2 + \lambda d_p(u^j, v_1^j)^2 \\ &\qquad - (p-1)\lambda(1-\lambda)d_p(v_0^j, v_1^j)^2, \qquad \text{if } 1 < p \le 2 \text{ and} \\ d_p(u^j, v_\lambda^j)^p &\le (1-\lambda)d_p(u^j, v_0^j)^p + \lambda d_p(u^j, v_1^j)^p \\ &\qquad - \lambda^{p/2}(1-\lambda)^{p/2}d_p(v_0^j, v_1^j)^p, \qquad \text{if } p \ge 2. \end{aligned}$$

By definition of d_p on $\mathcal{E}^p(X, \beta)$, we know that as we take the limit $j \to \infty$, $d_p(u^j, v_0^j) \to d_p(u, v_0)$, $d_p(u^j, v_1^j) \to d_p(u, v_1)$ and $d_p(v_0^j, v_1^j) \to d_p(v_0, v_1)$. So we are done if we could prove that $d_p(u^j, v_\lambda^j) \to d_p(u, v_\lambda)$.

By the same reasoning as in the proof of Lemma 4.8.1, we can see that $v_\lambda^j \searrow v_\lambda$. Since $u^j \searrow u$ and $v_\lambda^j \searrow v_\lambda$, from [16, Proposition 3.12], we get that $d_p(u^j, u) \to 0$ and $d_p(v_\lambda^j, v_\lambda) \to 0$. Combining with the triangle inequality we get $d_p(u_j, v_\lambda^j) \to d_p(u, v_\lambda)$. $\square$

4.8.2 Uniform Convexity in the big case

With the help of Theorem 4.7.5, we can prove uniform convexity in the big case as well. First, we see that the uniform convexity extends to the analytic singularity setting as well.

Theorem 4.8.5. *If θ represents a big cohomology class and $\psi \in PSH(X, \theta)$ has analytic singularities, then the metric space $(\mathcal{E}^p(X, \theta, \psi), d_p)$ for $p > 1$, as described in Section 4.4, is uniformly convex.*

Proof. Recall that we constructed the metric d_p on $\mathcal{E}^p(X, \theta, \psi)$ in Section 4.4.1 by resolving the singularities of ψ. Let $\mu : \tilde{X} \to X$ be the resolution as described in Section 4.4. Recall that from Theorem 4.4.1 there is a smooth closed real $(1,1)$-form $\tilde{\theta}$ on $\tilde{X}$ and a bounded function $g \in \text{PSH}(\tilde{X}, \tilde{\theta})$ such that the map $\text{PSH}(X, \theta, \psi) \ni u \mapsto \tilde{u} := (u - \psi) \circ \mu + g \in \text{PSH}(\tilde{X}, \tilde{\theta})$ is an order preserving bijection and from Corollary 4.4.3, $\mathcal{E}^p(X, \theta, \psi) \ni u \mapsto \tilde{u} \in \mathcal{E}^p(\tilde{X}, \tilde{\theta})$ is a bijection as well.

If $u, v_0, v_1 \in \mathcal{E}^p(X, \theta, \psi)$ and v_λ is the weak geodesic joining v_0 and v_1, then $\tilde{u}, \tilde{v}_0, \tilde{v}_1 \in \mathcal{E}^p(\tilde{X}, \tilde{\theta})$ and by Theorem 4.4.5 $\tilde{v}_\lambda := (v_\lambda - \psi) \circ \mu + g$ is the weak geodesic joining $\tilde{v}_0$ and $\tilde{v}_1$. From Theorem 4.8.4, $(\mathcal{E}^p(\tilde{X}, \tilde{\theta}), d_p)$ is uniformly convex, thus

$$d_p(\tilde{u}, \tilde{v}_\lambda)^2 \leq (1-\lambda) d_p(\tilde{u}, \tilde{v}_0)^2 + \lambda d_p(\tilde{u}, \tilde{v}_1)^2 - (p-1)\lambda(1-\lambda) d_p(\tilde{v}_0, \tilde{v}_1)^2, \text{ if } 1 < p \leq 2 \text{ and}$$

$$d_p(\tilde{u}, \tilde{v}_\lambda)^p \leq (1-\lambda) d_p(\tilde{u}, \tilde{v}_0)^p + \lambda d_p(\tilde{u}, \tilde{v}_1)^p - \lambda^{p/2}(1-\lambda)^{p/2} d_p(\tilde{v}_0, \tilde{v}_1)^p, \text{ if } p \geq 2.$$

For $u_0, u_1 \in \mathcal{E}^p(X, \theta, \psi)$, we defined $d_p(u_0, u_1) := d_p(\tilde{u}_0, \tilde{u}_1)$ in (4.5). Applying this we get

$$d_p(u, v_\lambda)^2 \le (1-\lambda)d_p(u, v_0)^2 + \lambda d_p(u, v_1)^2 - (p-1)\lambda(1-\lambda)d_p(v_0, v_1)^2, \text{ if } 1 < p \le 2 \text{ and}$$

$$d_p(u, v_\lambda)^p \le (1-\lambda)d_p(u, v_0)^p + \lambda d_p(u, v_1)^p - \lambda^{p/2}(1-\lambda)^{p/2}d_p(v_0, v_1)^p, \text{ if } p \le 2$$

implying uniform convexity of $(\mathcal{E}^p(X, \theta, \psi), d_p)$ for $p > 1$. □

We would also need the analytic singularity version of [11, Proposition 3.6] which holds true, because the proof in [11] only relies on the uniform convexity of Theorem 4.8.5.

Theorem 4.8.6. *Let $\psi \in PSH(X, \theta)$ have analytic singularities. Let $u_0, u_1 \in \mathcal{E}^p(X, \theta, \psi)$ for $p > 1$, and u_t be the weak geodesic joining u_0 and u_1. If $v \in \mathcal{E}^p(X, \theta, \psi)$ satisfies $d_p(u_0, v) \le (t+\varepsilon)d_p(u_0, u_1)$ and $d_p(u_1, v) \le (1-t+\varepsilon)d_p(u_0, u_1)$, for some $\varepsilon > 0$ and $t \in [0, 1]$, then for some constant $C(p) > 0$,*

$$d_p(v, u_t) \le \varepsilon^{1/r} C d_p(u_0, u_1)$$

where $r = \max\{2, p\}$.

Now we can prove one of our main results:

Theorem 4.8.7. *If θ represents a big cohomology class, then the metric space $(\mathcal{E}^p(X, \theta), d_p)$ for $p > 1$ is uniformly convex. This means for $u, v_0, v_1 \in \mathcal{E}^p(X, \theta)$, if v_λ is the geodesic joining v_0 and v_1, then*

$$d_p(u, v_\lambda)^2 \le (1-\lambda)d_p(u, v_0)^2 + \lambda d_p(u, v_1)^2 - (p-1)\lambda(1-\lambda)d_p(v_0, v_1)^2, \textit{ if } 1 < p \le 2 \textit{ and}$$

$$d_p(u, v_\lambda)^p \le (1-\lambda)d_p(u, v_0)^p + \lambda d_p(u, v_1)^p - \lambda^{p/2}(1-\lambda)^{p/2}d_p(v_0, v_1)^p, \textit{ if } p \ge 2.$$

Proof. Let $\psi_k \nearrow V_\theta$ be the increasing sequence of θ-psh functions with analytic singularities. Let $u^k = P_\theta[\psi_k](u)$, $v_0^k = P_\theta[\psi_k](v_0)$, and $v_1^k = P_\theta[\psi_k](v_1)$. Let v_λ^k be the weak geodesic joining v_0^k and v_1^k. From Theorem 4.8.5, we know that

$$\begin{aligned}
d_p(u^k, v_\lambda^k)^2 &\leq (1-\lambda)d_p(u^k, v_0^k)^2 + \lambda d_p(u^k, v_1^k)^2 \\
&\quad - (p-1)\lambda(1-\lambda)d_p(v_0^k, v_1^k)^2, \qquad \text{if } 1 < p \leq 2 \text{ and} \\
d_p(u^k, v_\lambda^k)^p &\leq (1-\lambda)d_p(u^k, v_0^k)^p + \lambda d_p(u^k, v_1^k)^p \\
&\quad - \lambda^{p/2}(1-\lambda)^{p/2}d_p(v_0^k, v_1^k)^p, \qquad \text{if } p \geq 2.
\end{aligned}$$

From Theorem 4.7.5 we know that $\lim_{k\to\infty} d_p(u^k, v_0^k) = d_p(u, v_0)$, $\lim_{k\to\infty} d_p(u^k, v_1^k) = d_p(u, v_1)$, and $d_p(v_0^k, v_1^k) = d_p(v_0, v_1)$. Thus to finish the proof by taking the limit $k \to \infty$, we need to show that $d_p(u^k, v_\lambda^k) \to d_p(u, v_\lambda)$. Unfortunately, it may not be true that $P_\theta[\psi_k](v_\lambda) = v_\lambda^k$. But by using Theorem 4.7.3, and Theorem 4.8.6, we can show that v_λ^k and $P_\theta[\psi_k](v_\lambda)$ are d_p-close.

From Theorem 4.7.3, and the fact that $(\mathcal{E}^p(X, \theta, \psi_k), d_p)$ is a geodesic metric space, we know that

$$d_p(v_0^k, P_\theta[\psi_k](v_\lambda)) \leq d_p(v_0, v_\lambda) = \lambda d_p(v_0, v_1)$$

and

$$d_p(v_1^k, P_\theta[\psi_k](v_\lambda)) \leq d_p(v_1, v_\lambda) = (1-\lambda)d_p(v_0, v_1).$$

Again by the contraction theorem $d_p(v_0^k, v_1^k) \leq d_p(v_0, v_1)$, moreover by Theorem 4.7.5, $\lim_{k\to\infty} d_p(v_0^k, v_1^k) = d_p(v_0, v_1)$. Thus we can write

$$\frac{d_p(v_0, v_1)}{d_p(v_0^k, v_1^k)} \leq 1 + \varepsilon_k$$

where $\varepsilon_k > 0$ and $\varepsilon_k \to 0$ as $k \to \infty$. Thus we have

$$d_p(v_0^k, P_\theta[\psi_k](v_\lambda)) \le (\lambda + \lambda\varepsilon_k) d_p(v_0^k, v_1^k) \le (\lambda + \varepsilon_k) d_p(v_0^k, v_1^k)$$

and

$$d_p(v_1^k, P_\theta[\psi_k](v_\lambda)) \le (1-\lambda)(1+\varepsilon_k) d_p(v_0^k, v_1^k) \le (1-\lambda+\varepsilon_k) d_p(v_0^k, v_1^k).$$

Applying Theorem 4.8.6 we get that

$$d_p(v_\lambda^k, P_\theta[\psi_k](v_\lambda)) \le (\varepsilon_k)^{1/r} C d_p(v_0^k, v_1^k) \le (\varepsilon_k)^{1/r} C d_p(v_0, v_1).$$

Taking the limit $k \to \infty$ and using that $\varepsilon_k \to 0$, we get

$$\lim_{k\to\infty} d_p(v_\lambda^k, P_\theta[\psi_k](v_\lambda)) = 0. \tag{4.25}$$

Now we will show that $d_p(u^k, v_\lambda^k) \to d_p(u, v_\lambda)$ as $k \to \infty$. By applying the triangle inequality twice we get

$$\begin{aligned} |d_p(u^k, v_\lambda^k) - d_p(u, v_\lambda)| &\le |d_p(u^k, v_\lambda^k) - d_p(u^k, P_\theta[\psi_k](v_\lambda))| + |d_p(u^k, P_\theta[\psi_k](v_\lambda)) - d_p(u, v_\lambda)| \\ &\le d_p(v_\lambda^k, P_\theta[\psi_k](v_\lambda)) + |d_p(u^k, P_\theta[\psi_k](v_\lambda)) - d_p(u, v_\lambda)| \end{aligned}$$

As $k \to \infty$, the first term goes to 0 due to (4.25), and the second term goes to 0 due to Theorem 4.7.5. $\square$

The same proofs as in [11, Theorem 3.5] gives

Corollary 4.8.8. *In the metric space* $(\mathcal{E}^p(X,\theta), d_p)$ *for* $p > 1$*, the weak geodesics are the only metric geodesics.*

The same proof as in [11, Theorem 3.6] proves that

Corollary 4.8.9. *Let* $u, v_0, v_1 \in \mathcal{E}^p(X,\theta)$ *for* $p > 1$*. Let* $t \in [0,1]$ *and* $\varepsilon > 0$ *such that* $d_p(u, v_0) \leq (t+\varepsilon)d_p(v_{0,1})$ *and* $d_p(u, v_1) \leq (1-t+\varepsilon)d_p(v_0, v_1)$*. If* v_s *is the weak geodesic joining* v_0 *and* v_1*, then there exists* $C(p) > 0$ *such that*

$$d_p(u, v_t) \leq \varepsilon^{\frac{1}{r}} C d_p(v_0, v_1)$$

where $r = \max\{2, p\}$.

Chapter 5: Finite energy geodesic rays in big cohomology classes

In this chapter, we will prove Theorem 1.4.1. To prove it we will need to prove some more properties of the metric d_p on $\mathcal{E}^p(X,\theta)$ as introduced in Chapter 4. In Section 5.1, we will point out some preliminary observations. In Section 5.2, we will prove a Lidskii type inequality. In Section 5.3, we will prove Buseman convexity of d_p metric. In particular we will prove

Theorem 5.0.1. *Let θ be a real, smooth, closed $(1,1)$-form representing a big cohomology class. If $p \geq 1$, then the metric space $(\mathcal{E}^p(X,\theta), d_p)$ is Buseman convex. This means if $u_0, u_1, v_0, v_1 \in \mathcal{E}^p(X,\theta)$, and $[0,1] \ni t \mapsto u_t$ and $[0,1] \ni t \mapsto v_t$ are the geodesics joining u_0, u_1, and v_0, v_1 respectively, then*

$$d_p(u_t, v_t) \leq (1-t)d_p(u_0, v_0) + td_p(u_1, v_1).$$

After this we will introduce the space of geodesic rays in $\mathcal{E}^p(X,\theta)$ and prove Theorem 1.4.1.

At the end we will give an application by proving a Ross–Witt Nyström type correspondence, which is new even in Kähler setting for $p > 1$. The notion of test curves was introduced in [57] by Ross–Witt Nyström in the Kähler setting, giving a potential theoretic framework to study stability.

Test curves are dual to (sub)geodesic rays via the Legendre transform. Ross–Witt Nyströms's work generalized the work of Phong–Sturm [58] where they associated geodesic rays to test configurations. The study of test curves was extended to the big setting in [17] by Darvas–Di Nezza–Lu. Furthermore, the notion of finite energy test curves was introduced in [23] (in the Kähler setting) and [37] (in the big setting). Test curves can detect several properties of the geodesic rays. Using the metric d_p on $\mathcal{E}^p(X,\theta)$, the following theorem proves that the geodesic rays lying in $\mathcal{E}^p(X,\theta)$ can be detected from the corresponding test curve.

If $(0,\infty) \ni t \mapsto u_t \in \mathrm{PSH}(X,\theta)$ is a θ-geodesic ray starting at V_θ, then its Legendre dual $\mathbb{R} \ni \tau \mapsto \hat{u}_\tau \in \mathrm{PSH}(X,\theta)$ is defined by

$$\hat{u}_\tau = \inf_{t>0}(u_t - t\tau).$$

For τ large enough, $\hat{u}_\tau \equiv -\infty$. Thus we define,

$$\tau^+_{\hat{u}} := \inf\{\tau \in \mathbb{R} \,:\, \hat{u}_\tau \equiv -\infty\} < \infty.$$

Theorem 5.0.2. *Let $\{u_t\}_t$ be a θ-geodesic ray starting from V_θ. Let $\{\hat{u}_\tau\}_\tau$ be the Legendre dual of $\{u_t\}_t$. Then $u_t \in \mathcal{E}^p(X,\theta)$ for all $t \geq 0$ iff*

$$\int_{-\infty}^{\tau^+_{\hat{u}}} (-\tau + \tau^+_{\hat{u}})^{p-1} \left(\int_X \theta^n_{V_\theta} - \int_X \theta^n_{\hat{u}_\tau} \right) d\tau < \infty \tag{5.1}$$

When $\tau^+_{\hat{u}} = 0$, the expression in Equation (5.1) equals $\frac{1}{p} d_p^p(V_\theta, u_1) = \frac{1}{p}\int_X |\dot{u}_0|^p \theta^n_{V_\theta}$. Thus this expression is closely related to the speed of the geodesic ray in $\mathcal{E}^p(X,\theta)$. When $p=1$, such characterization of finite energy geodesic rays was obtained in [23, Theorem 3.7] (in the Kähler setting) and [37, Theorem 3.9] (in the big setting). For $p>1$, this is new in the Kähler case as well.

5.1 Some preliminary observations

The following Lemma from [10] shows that the geodesics are stable under perturbations of the endpoints.

Lemma 5.1.1. *If $u_0^j, u_1^j, u_0, u_1 \in \mathcal{E}^p(X,\theta)$ satisfy $d_p(u_0^j, u_0) \to 0$ and $d_p(u_1^j, u_1) \to 0$ as $j \to \infty$, then $d_p(u_t^j, u_t) \to 0$ as $j \to \infty$ where u_t^j, u_t are the geodesics joining u_0^j, u_1^j and u_0, u_1 respectively.*

Proof. The proof is the same as in [10, Proposition 4.3]. We just need the corresponding results used in the proof in [10, Proposition 4.3] for the big setting. For that we can obtain the quasi-monotonicity for the d_p metric from [15, Theorem 5.2] and [19, Lemma 4.12]. The fact that for $u, v \in \mathcal{E}^1(X,\theta)$ satisfying $u \leq v$ and $I(u) = I(v)$ imply $u = v$ follows by combining [17, Theorem 2.4, Proposition 2.5, and Proposition 2.8]. Moreover, the fact that $u \leq v \leq w$ implies $d_p(u,v) \leq d_p(u,w)$ can be obtained from Theorem 5.2.3 (whose proof does not rely on this lemma). □

Lemma 5.1.2. *If $u_j \in \mathcal{E}^p(X,\theta)$ is a d_p-bounded decreasing sequence, then $\lim_{j\to\infty} u_j =: u \in \mathcal{E}^p(X,\theta)$.*

Proof. By [19, Lemma 4.12], the metric d_p is comparable to the quasi-metric I_p i.e., there exists $C > 1$, such that

$$\frac{1}{C} I_p(u_0, u_1) \leq d_p(u_0, u_1)^p \leq C I_p(u_0, u_1).$$

Here the functional I_p is given by

$$I_p(u_0, u_1) = \int_X |u_0 - u_1|^p (\theta_{u_0}^n + \theta_{u_1}^n)$$

for $u_0, u_1 \in \mathcal{E}^p(X, \theta)$. So, if the sequence u_j is d_p-bounded, then it is I_p-bounded as well. This means that

$$\sup_{j \in \mathbb{N}} \int_X |u_0 - u_j|^p \theta_{u_j}^n < \infty.$$

Thus by [42], the limit $\lim_{j\to\infty} u_j =: u \in \mathcal{E}^p(X, \theta)$. □

We will end this section by proving a d_p-contraction property for the projection into different cohomology classes. This result will play an important role in obtaining our main results. This is an analog of the contraction property in [19, Theorem 7.3], and [56, Theorem 4.3] for $p = 1$, where the projection operator maps to a different singularity type in the same cohomology class.

Lemma 5.1.3 (Contraction property)**.** *If θ and η are closed smooth real $(1,1)$-forms representing big cohomology classes such that $\theta \leq \eta$. Then the map $P_\theta : \mathcal{H}_\eta \mapsto \mathcal{H}_\theta$ is a contraction. This means that $v_0, v_1 \in \mathcal{H}_\eta$, then*

$$d_p(P_\theta(v_0), P_\theta(v_1)) \leq d_p(v_0, v_1).$$

Proof. First we will show that P_θ maps $\mathcal{H}_\eta$ to $\mathcal{H}_\theta$. If $v \in \mathcal{H}_\eta$, then $v = P_\eta(f)$ for some $f \in C^{1,\bar{1}}$. By a standard argument, we can show that $P_\theta(v) = P_\theta(f)$. Thus showing that $P_\theta(v) \in \mathcal{H}_\theta$.

Now, assume that $f_0, f_1 \in C^{1,\bar{1}}$ be such that $v_0 = P_\eta(f_0)$ and $v_1 = P_\eta(f_1)$. Also call $u_0 = P_\theta(f_0) = P_\theta(v_0)$ and $u_1 = P_\theta(f_1) = P_\theta(v_1)$. For now, we also assume that $f_0 \leq f_1$. By Theorem 4.5.7 and Theorem 2.4.4 the distances are given by

$$d_p(v_0, v_1)^p = \int_X |\dot{v}_0|^p \eta_{v_0}^n = \int_{\{v_0 = f_0\}} |\dot{v}_0|^p \eta_{f_0}^n$$

and

$$d_p(u_0,u_1)^p = \int_X |\dot{u}_0|^p \theta_{u_0}^n = \int_{\{u_0=f_0\}} |\dot{u}_0|^p \theta_{f_0}^n.$$

where v_t is the η-geodesic joining v_0 and v_1, and u_t is the θ-geodesic joining u_0 and u_1. Since $\theta \le \eta$, $u_0 \le v_0$ and $u_1 \le v_1$, u_t is an η-subgeodesic joining v_0 and v_1, therefore, $u_t \le v_t$.

Since $u_0 \le v_0 \le f_0$, the set $\{u_0 = f_0\} \subset \{v_0 = f_0\}$. Moreover, when $x \in \{u_0 = f_0\}$,

$$\begin{aligned}
\dot{u}_0(x) = \lim_{t\to 0} \frac{u_t(x)-u_0(x)}{t} &= \lim_{t\to 0} \frac{u_t(x)-f_0(x)}{t} \\
&\le \lim_{t\to 0} \frac{v_t(x)-f_0(x)}{t} = \lim_{t\to 0} \frac{v_t(x)-v_0(x)}{t} = \dot{v}_0(x).
\end{aligned}$$

Due to the assumption that $f_0 \le f_1$, $0 \le \dot{u}_0, \dot{v}_0$, thus we obtain that $|\dot{u}_0| \le |\dot{v}_0|$. Again due to $\theta \le \eta$, we have $\theta_{f_0}^n \le \eta_{f_0}^n$. Combining these we get

$$\int_{\{u_0=f_0\}} |\dot{u}_0|^p \theta_{f_0}^n \le \int_{\{v_0=f_0\}} |\dot{v}_0|^p \eta_{f_0}^n.$$

Therefore, $d_p(u_0,u_1) \le d_p(v_0,v_1)$.

More generally, if f_0 and f_1 are unrelated, we can use the Pythagorean identity (see Proposition 4.6.5) to prove the general result. We can do this because of [19, Lemma 4.1] implies that $P_\theta(u_0,u_1) \in \mathcal{H}_\theta$ and the fact that $P_\theta(P_\eta(v_0,v_1)) = P_\theta(u_0,u_1)$.

$$\begin{aligned}
d_p(u_0,u_1)^p &= d_p(u_0, P_\theta(u_0,u_1))^p + d_p(u_1, P_\theta(u_0,u_1))^p \\
&\le d_p(v_0, P_\eta(v_0,v_1))^p + d_p(v_1, P_\eta(v_0,v_1))^p \\
&= d_p(v_0,v_1)^p.
\end{aligned}$$

$\square$

Lemma 5.1.4. *If $1 \leq p' \leq p < \infty$ then for any $u_0, u_1 \in \mathcal{E}^p(X,\theta)$,*

$$d_{p'}(u_0,u_1) \leq d_p(u_0,u_1) \mathit{Vol}(\theta)^{\frac{1}{p'}-\frac{1}{p}}.$$

Proof. First we assume that $u_0, u_1 \in \mathcal{H}_\theta$ and u_t is the geodesic joining u_0 and u_1. Then by Theorem 4.5.7,

$$d_p(u_0,u_1)^p = \int_X |\dot{u}_0|^p \theta_{u_0}^n \qquad \text{and} \qquad d_{p'}(u_0,u_1)^{p'} = \int_X |\dot{u}_0|^{p'} \theta_{u_0}^n.$$

Applying Hölder inequality with factors p/p' and $p/(p-p')$ we can write,

$$\int_X |\dot{u}_0|^{p'} \theta_{u_0}^n \leq \left(\int_X |\dot{u}_0|^p \theta_{u_0}^n\right)^{p'/p} \left(\int_X \theta_{u_0}^n\right)^{\frac{p-p'}{p}}.$$

Raising both sides to $1/p'$ and noticing that $\int_X \theta_{u_0}^n = \mathrm{Vol}(\theta)$, we get that

$$d_{p'}(u_0,u_1) \leq d_p(u_0,u_1) \mathrm{Vol}(\theta)^{\frac{1}{p'}-\frac{1}{p}}.$$

More generally, if $u_0, u_1 \in \mathcal{E}^p(X,\theta)$, then we can find $u_0^j, u_1^j \in \mathcal{H}_\theta$ such that $u_0^j \searrow u_0$ and $u_1^j \searrow u_1$. By Lemma 4.6.2,

$$\lim_{j\to\infty} d_{p'}(u_0^j,u_1^j) = d_{p'}(u_0,u_1) \qquad \text{and} \qquad \lim_{j\to\infty} d_p(u_0^j,u_1^j) = d_p(u_0,u_1).$$

From the proof above we can write,

$$d_{p'}(u_0,u_1)=\lim_{j\to\infty} d_{p'}(u_0^j,u_1^j)\leq \lim_{j\to\infty} d_p(u_0^j,u_1^j)\mathrm{Vol}(\theta)^{\frac{1}{p'}-\frac{1}{p}}=d_p(u_0,u_1)\mathrm{Vol}(\theta)^{\frac{1}{p'}-\frac{1}{p}}.$$

□

5.2 Lidskii type inequality

In this section we will prove Lidskii type inequality. It will be useful in constructing a metric in the space of rays. The proof will follow the approximation process used to construct the metric d_p on $\mathcal{E}^p(X,\theta)$. First, we will prove the Lidskii type inequality for the big and nef cohomology classes and then prove it for the big cohomology classes. In the first

Lemma 5.2.1. *If β represents a big and nef cohomology class and $u,v,w\in\mathcal{E}^p(X,\beta)$ satisfy $u\geq v\geq w$, then*

$$d_p(v,w)^p\leq d_p(u,w)^p-d_p(u,v)^p.$$

Proof. First, we assume that $u,v,w\in\mathcal{H}_\beta$. Let $\omega_\varepsilon=\beta+\varepsilon\omega$ represent a Kähler class. Let $u=P_\beta(f)$, $v=P_\beta(g)$, and $w=P_\beta(h)$ where $f,g,h\in C^{1,\bar{1}}(X)$ satisfy $f\geq g\geq h$. We denote by $u_\varepsilon=P_{\omega_\varepsilon}(f)$, $v_\varepsilon=P_{\omega_\varepsilon}(g)$, and $w_\varepsilon=P_{\omega_\varepsilon}(h)$. Since $u_\varepsilon\geq v_\varepsilon\geq w_\varepsilon$ are in $\mathcal{E}^p(X,\omega_\varepsilon)$, by [11, Corollary 3.2],

$$d_p(v_\varepsilon,w_\varepsilon)^p\leq d_p(u_\varepsilon,w_\varepsilon)^p-d_p(u_\varepsilon,v_\varepsilon)^p.$$

By the definition of d_p on $\mathcal{H}_\beta$ (see Section 4.3), we know that $\lim_{\varepsilon\to 0} d_p(u_\varepsilon,v_\varepsilon)=d_p(u,v)$, $\lim_{\varepsilon\to 0} d_p(u_\varepsilon,w_\varepsilon)=d_p(u,w)$, and $\lim_{\varepsilon\to 0} d_p(v_\varepsilon,w_\varepsilon)=d_p(v,w)$. Thus taking the limit $\varepsilon\to 0$ in the above equation we get

$$d_p(v,w)^p \le d_p(u,w)^p - d_p(u,v)^p.$$

More generally, if $u, v, w \in \mathcal{E}^p(X,\beta)$, then we can find $u^j, v^j, w^j \in \mathcal{H}_\beta$ such that $u^j \ge v^j \ge w^j$ and $u^j \searrow u$, $v^j \searrow v$, and $w^j \searrow w$ as $j \to \infty$. We just proved that

$$d_p(v^j,w^j)^p \le d_p(u^j,w^j)^p - d_p(u^j,v^j)^p.$$

Taking the limit $j \to \infty$ and using Lemma 4.6.2, we get

$$d_p(v,w)^p \le d_p(u,w)^p - d_p(u,v)^p.$$

□

Before we can prove the Lidskii type inequality for the big cohomology class, we need to show the result for the analytic singularity type.

Lemma 5.2.2. *If θ represents a big cohomology class and ψ is a θ-psh function with analytic singularity type, then for $u, v, w \in \mathcal{E}^p(X,\theta,\psi)$ satisfying $u \ge v \ge w$, we have*

$$d_p(v,w)^p \le d_p(u,w)^p - d_p(u,v)^p.$$

Proof. Let $\mu : \tilde{X} \to X$ be the modification that resolves the singularities of ψ. As discussed in Section 4.4.1 there is a smooth closed real $(1,1)$-form $\tilde{\theta}$ representing a big and nef cohomology class, and a bounded $\tilde{\theta}$-psh function g on $\tilde{X}$ such that $\mathcal{E}^p(X,\theta,\psi) \ni u \mapsto \tilde{u} = (u-\psi)\circ\mu + g \in$

$\mathcal{E}^p(\tilde{X}, \tilde{\theta})$ is a bijection.

Since $\tilde{u}, \tilde{v}, \tilde{w} \in \mathcal{E}^p(\tilde{X}, \tilde{\theta})$ satisfy $\tilde{u} \geq \tilde{v} \geq \tilde{w}$, by Lemma 5.2.1 we have

$$d_p(\tilde{v}, \tilde{w})^p \leq d_p(\tilde{u}, \tilde{w})^p - d_p(\tilde{u}, \tilde{v})^p.$$

By Equation (4.5), we can write

$$d_p(v, w)^p \leq d_p(u, w)^p - d_p(u, v)^p.$$

□

Now we can prove the Lidskii type inequality for the big classes.

Theorem 5.2.3. *Let θ represent a big cohomology class. If $u, v, w \in \mathcal{E}^p(X, \theta)$ satisfy $u \geq v \geq w$, then*

$$d_p(v, w)^p \leq d_p(u, w)^p - d_p(u, v)^p.$$

Proof. As discussed in Section 4.5, let ψ_k be an increasing sequence of θ-psh functions with analytic singularities such that $\psi_k \nearrow V_\theta$ a.e. We denote by $u^k = P_\theta[\psi_k](u)$, $v^k = P_\theta[\psi_k](v)$, and $w^k = P_\theta[\psi_k](w)$. Since $u \geq v \geq w$, we have $u^k \geq v^k \geq w^k$. By Lemma 5.2.2,

$$d_p(v^k, w^k)^p \leq d_p(u^k, w^k)^p - d_p(u^k, v^k)^p.$$

Notice that the metric d_p applied to u^k, v^k, w^k is the metric on the space $\mathcal{E}^p(X, \theta, \psi_k)$, whereas the metric applied to u, v, w is the metric on the space $\mathcal{E}^p(X, \theta)$. Taking the limit $k \to \infty$ and using Equation (4.16) we get

$$d_p(v,w)^p \le d_p(u,w)^p - d_p(u,v)^p.$$

□

5.3 Buseman Convexity of $\mathcal{E}^p(X,\theta)$

In this section, we will prove Buseman convexity of the metric space $(\mathcal{E}^p(X,\theta), d_p)$ for $p \ge 1$ where θ represents a big cohomology class. This means that we will prove that if $u_0, u_1, v_0, v_1 \in \mathcal{E}^p(X,\theta)$, and u_t and v_t are geodesic joining u_0, u_1, and v_0, v_1 respectively, then

$$d_p(u_t, v_t) \le (1-t)d_p(u_0,v_0) + td_p(u_1,v_1).$$

5.3.1 Buseman convexity for $p = 1$

The Buseman convexity for $p = 1$ was proved in [20, Proposition 2.10]. We reproduce the arguments here for completeness.

Theorem 5.3.1. *If θ represents a big cohomology class, $u_0, u_1, v_0, v_1 \in \mathcal{E}^p(X,\theta)$, and u_t and v_t are the geodesics joining u_0,u_1, and v_0, v_1 respectively, then*

$$d_1(u_t, v_t) \le (1-t)d_1(u_0,v_0) + td_1(u_1,v_1).$$

Proof. Fix $a \le b \in [0,1]$. Let $w_a := P_\theta(u_a, v_a)$ and $w_b = P_\theta(u_b, v_b)$. Let $[a,b] \ni t \mapsto w_t \in \mathcal{E}^p(X,\theta)$ be the geodesic joining w_a and w_b. Since $w_a \le u_a, v_a$ and $w_b \le u_b, v_b$, the comparison principle for geodesics implies that $w_t \leq u_t, v_t$ for all $t \in [a,b]$. Thus $w_t \leq P_\theta(u_t, v_t)$. By Lemma 2.7.1, we get that $I(w_t) \leq I(P_\theta(u_t, v_t))$ and $I(P_\theta(u_t, v_t)) \geq I(w_t) = \frac{b-t}{b-a} I(P_\theta(u_a, v_a)) + \frac{t-a}{b-a} I(P_\theta(u_b, v_b))$ for all $t \in [a,b]$. Thus $t \mapsto I(P_\theta(u_t, v_t))$ is a concave map.

By Theorem 4.6.7, $d_1(u_t, v_t) = I(u_t) + I(v_t) - 2I(P_\theta(u_t, v_t))$. As $t \mapsto I(u_t)$ and $t \mapsto I(v_t)$ are linear and $t \mapsto I(P_\theta(u_t, v_t))$ is concave, we obtain that $t \mapsto d_1(u_t, v_t)$ is convex. □

Thus, we only need to prove Buseman convexity for $p > 1$. In this case we can use uniform convexity of $(\mathcal{E}^p(X,\theta), d_p)$ from Theorem 4.8.7. First, we will prove Buseman convexity for the big and nef case, and then prove the Buseman convexity for the big classes.

5.3.2 Buseman convexity for big and nef classes when $p > 1$

Fix a smooth closed real $(1,1)$-form β representing a big and nef cohomology class and fix a $p > 1$. Recall from Section 4.3 that $\omega_\varepsilon = \beta + \varepsilon\omega$ represents a Kähler class. Let $u_0, u_1, v_0, v_1 \in \mathcal{H}_\beta$ be given by $u_0 = P_\beta(f_0)$, $u_1 = P_\beta(f_1)$, $v_0 = P_\beta(g_0)$, and $v_1 = P_\beta(g_1)$ where $f_0, f_1, g_0, g_1 \in C^{1,\bar{1}}(X)$. Let $u_{0,\varepsilon} = P_{\omega_\varepsilon}(f_0)$, $u_{1,\varepsilon} = P_{\omega_\varepsilon}(f_1)$, $v_{0,\varepsilon} = P_{\omega_\varepsilon}(g_0)$, and $v_{1,\varepsilon} = P_{\omega_\varepsilon}(g_1)$. Let $u_{t,\varepsilon}$ and $v_{t,\varepsilon}$ be the ω_ε-geodesic joining $u_{0,\varepsilon}, u_{1,\varepsilon}$, and $v_{0,\varepsilon}, v_{1,\varepsilon}$ respectively.

By the Buseman convexity in the Kähler case, we obtain that

$$d_p(u_{t,\varepsilon}, v_{t,\varepsilon}) \leq (1-t) d_p(u_{0,\varepsilon}, v_{0,\varepsilon}) + t d_p(u_{1,\varepsilon}, v_{1,\varepsilon}).$$

By construction of d_p on $\mathcal{E}^p(X,\beta)$, we know that $\lim_{\varepsilon \to 0} d_p(u_{0,\varepsilon}, v_{0,\varepsilon}) = d_p(u_0, v_0)$ and $\lim_{\varepsilon \to 0} d_p(u_{1,\varepsilon}, v_{1,\varepsilon}) = d_p(u_1, v_1)$. Thus to show Buseman convexity for potentials in $\mathcal{H}_\beta$, it is enough to show that

$$d_p(u_t, v_t) \leq \lim_{\varepsilon \to 0} d_p(u_{t,\varepsilon}, v_{t,\varepsilon}),$$

where u_t and v_t are the β-geodesics joining u_0, u_1 and v_0, v_1 respectively.

Lemma 5.3.2. *Let $u_0, u_1 \in \mathcal{H}_\beta$ and u_t be the geodesic joining them. Let $u_{0,\varepsilon}, u_{1,\varepsilon}$ be as above and $u_{t,\varepsilon}$ be the geodesic joining $u_{0,\varepsilon}$ and $u_{1,\varepsilon}$. Then for $p > 1$,*

$$\lim_{\varepsilon \to 0} d_p(u_t, P_\beta(u_{t,\varepsilon})) = 0.$$

Proof. We claim that $P_\beta(u_{t,\varepsilon}) \searrow u_t$.

First notice that $u_{0,\varepsilon} \searrow u_0$ and $u_{1,\varepsilon} \searrow u_1$. Recall that by definition, $u_{0,\varepsilon} = P_{\omega_\varepsilon}(f_0)$ and $u_0 = P_\beta(f_0)$ for $f_0 \in C^{1,\bar{1}}$. If $\varepsilon_1 < \varepsilon_2$, then u_{0,ε_1} is ω_{ε_2}-psh as well and $u_{0,\varepsilon_1} \leq f_0$, therefore, $u_{0,\varepsilon_1} \leq u_{0,\varepsilon_2}$. By a similar argument $u_{0,\varepsilon} \geq u_0$ as well. Therefore, $u_{0,\varepsilon}$ are increasing sequence of functions, and $\lim_{\varepsilon \to 0} u_{0,\varepsilon} \geq u_0$. The limit $\lim_{\varepsilon \to 0} u_{0,\varepsilon}$ is a $\beta + \varepsilon\omega$-psh for every $\varepsilon > 0$, therefore, $\lim_{\varepsilon \to 0} u_{0,\varepsilon}$ is a β-psh function that satisfies $\lim_{\varepsilon \to 0} u_{0,\varepsilon} \leq f_0$, thus $\lim_{\varepsilon \to 0} u_{0,\varepsilon} \leq u_0$. Thus $\lim_{\varepsilon \to 0} u_{0,\varepsilon} = u_0$. Similarly, $u_{1,\varepsilon} \searrow u_1$.

We can make the same argument for the ω_ε-geodesics $u_{t,\varepsilon}$ joining $u_{0,\varepsilon}$ and $u_{1,\varepsilon}$. If $\varepsilon_1 \leq \varepsilon_2$, then the path $(0, 1) \ni t \mapsto u_{t,\varepsilon_1}$ is an ω_{ε_2}-subgeodesic joining u_{0,ε_2} and u_{1,ε_2}. Therefore, $u_{t,\varepsilon_1} \leq u_{t,\varepsilon_2}$. Similarly, $u_{t,\varepsilon} \geq u_t$. Therefore, $(0, 1) \ni t \mapsto u_{t,\varepsilon}$ is a decreasing sequence of paths and $\lim_{\varepsilon \to 0} u_{t,\varepsilon} \geq u_t$. The limit path $(0, 1) \ni t \mapsto \lim_{\varepsilon \to 0} u_{t,\varepsilon}$ is a $\beta + \varepsilon\omega$-subgeodesic joining u_0 and u_1 for each $\varepsilon > 0$, therefore it is a β-subgeodesic as well. Hence $\lim_{\varepsilon \to 0} u_{t,\varepsilon} \leq u_t$. So we get $\lim_{\varepsilon \to 0} u_{t,\varepsilon} = u_t$.

Since $u_{t,\varepsilon} \searrow u_t$ and $u_{t,\varepsilon} \geq P_\beta(u_{t,\varepsilon}) \geq u_t$, we have that $P_\beta(u_{t,\varepsilon}) \searrow u_t$ as well. Thus proving the claim. Now the proof of the lemma follows from Lemma 4.6.2.

□

Similarly for $p > 1$, $\lim_{\varepsilon\to 0} d_p(v_t, P_\beta(v_{t,\varepsilon})) = 0$. By the triangle inequality,

$$\lim_{\varepsilon\to 0} d_p(P_\beta(u_{t,\varepsilon}), P_\beta(v_{t,\varepsilon})) = d_p(u_t, v_t).$$

Since $f_0, f_1 \in C^{1,\bar{1}}(X)$, Lemma 2.4.7 implies that $u_{0,\varepsilon}, u_{1,\varepsilon} \in C^{1,\bar{1}}(X)$. Thus Lemma 2.5.4 implies that the geodesic $u_{t,\varepsilon} \in C^{1,\bar{1}}(X)$ as well. Similarly, the geodesic $v_{t,\varepsilon} \in C^{1,\bar{1}}(X)$. Thus we can apply Lemma 5.1.3, to get $d_p(P_\beta(u_{t,\varepsilon}), P_\beta(v_{t,\varepsilon})) \leq d_p(u_{t,\varepsilon}, v_{t,\varepsilon})$. Combining this with the above equation we get

$$d_p(u_t, v_t) = \lim_{\varepsilon\to 0} d_p(P_\beta(u_{t,\varepsilon}), P_\beta(v_{t,\varepsilon})) \leq \lim_{\varepsilon\to 0} d_p(u_{t,\varepsilon}, v_{t,\varepsilon})$$

as we needed to show. Thus we have proved

Lemma 5.3.3. *Let β represent a big and nef cohomology class and $p > 1$. Let $u_0, u_1, v_0, v_1 \in \mathcal{H}_\beta$. If u_t and v_t are the geodesics joining u_0, u_1 and v_0, v_1 respectively, then*

$$d_p(u_t, v_t) \leq (1-t)d_p(u_0, v_0) + t d_p(u_1, v_1).$$

Now we will extend this result to $\mathcal{E}^p(X, \beta)$ for $p > 1$.

Theorem 5.3.4. *Let β represent a big and nef cohomology class and $p > 1$. Let $u_0, u_1, v_0, v_1 \in$*

$\mathcal{E}^p(X, \beta)$. *If* u_t *and* v_t *are the geodesics joining* u_0, u_1 *and* v_0, v_1 *respectively, then*

$$d_p(u_t, v_t) \leq (1-t)d_p(u_0, v_0) + td_p(u_1, v_1).$$

Proof. As standard, we can find $u_0^j, u_1^j, v_0^j, v_1^j \in \mathcal{H}_\beta$ such that $u_0^j \searrow u_0$, $u_1^j \searrow u_1$, $v_0^j \searrow v_0$, and $v_1^j \searrow v_1$. If u_t^j and v_t^j are the geodesics joining u_0^j, u_1^j, and v_0^j, v_1^j respectively, then $u_t^j \searrow u_t$ and $v_t^j \searrow v_t$.

By the definition of d_p for $\mathcal{E}^p(X, \beta)$ in Section 4.3, we know that $\lim_{j\to\infty} d_p(u_0^j, v_0^j) = d_p(u_0, v_0)$ and that $\lim_{j\to\infty} d_p(u_1^j, v_1^j) = d_p(u_1, v_1)$. Moreover, due to Lemma 4.6.2, $\lim_{j\to\infty} d_p(u_t^j, v_t^j) = d_p(u_t, v_t)$. By Lemma 5.3.3, we have

$$d_p(u_t^j, v_t^j) \leq (1-t)d_p(u_0^j, v_0^j) + td_p(u_1^j, v_1^j).$$

Taking limit $j \to \infty$, we get that

$$d_p(u_t, v_t) \leq (1-t)d_p(u_0, v_0) + td_p(u_1, v_1).$$

□

5.3.3 Buseman Convexity for big classes when $p > 1$

As in the case of Lidskii type inequality in Section 5.2, we will first show the Buseman convexity for the analytic singularity case, and then for the minimal singularity case for the big classes.

Assume that θ is a smooth closed real $(1,1)$-form representing a big cohomology class. Let ψ be a θ-psh function with analytic singularities. We will first show that $(\mathcal{E}^p(X,\theta,\psi), d_p)$ is Buseman convex.

Lemma 5.3.5. *For $p > 1$, the metric space $(\mathcal{E}^p(X,\theta,\psi), d_p)$ is a Buseman convex metric space. This means that if $u_0, u_1, v_0, v_1 \in \mathcal{E}^p(X,\theta,\psi)$, and u_t and v_t are the geodesics joining u_0,u_1, and v_0, v_1 respectively, then*

$$d_p(u_t, v_t) \leq (1-t)d_p(u_0, v_0) + td_p(u_1, v_1).$$

Proof. Recall from Section 4.4.1 that there exists a modification $\mu : \tilde{X} \to X$, a big and nef cohomology class $\tilde{\theta}$ and a bounded $\tilde{\theta}$-psh function g on $\tilde{X}$ such that the metric d_p on $\mathcal{E}^p(X,\theta,\psi)$ is given by

$$d_p(u_0, u_1) = d_p(\tilde{u}_0, \tilde{u}_1)$$

where $\mathcal{E}^p(X,\theta,\psi) \ni u \mapsto \tilde{u} := (u-\psi)\circ\mu + g \in \mathcal{E}^p(\tilde{X},\tilde{\theta})$ is a bijection.

By Theorem 5.3.4, the metric space $(\mathcal{E}^p(\tilde{X},\tilde{\theta}), d_p)$ is Buseman convex. Therefore,

$$d_p(u_t, v_t) = d_p(\tilde{u}_t, \tilde{v}_t) \leq (1-t)d_p(\tilde{u}_0, \tilde{v}_0) + td_p(\tilde{u}_1, \tilde{v}_1) = (1-t)d_p(u_0, v_0) + td_p(u_1, v_1).$$

□

Now we can show that the metric space $(\mathcal{E}^p(X,\theta), d_p)$ is Buseman convex for $p > 1$.

Theorem 5.3.6. *For $p > 1$, the metric space $(\mathcal{E}^p(X,\theta), d_p)$ is a Buseman convex metric space. This means that if $u_0, u_1, v_0, v_1 \in \mathcal{E}^p(X,\theta)$, and u_t and v_t are the geodesics joining u_0,u_1, and*

v_0, v_1 *respectively, then*

$$d_p(u_t, v_t) \leq (1-t)d_p(u_0, v_0) + td_p(u_1, v_1).$$

Proof. As described in Section 4.5, let ψ_k be an increasing sequence of θ-psh functions with analytic singularities such $\psi_k \nearrow V_\theta$ a.e.

We denote the projections of u_0, u_1, v_0, and v_1 to $\mathcal{E}^p(X, \theta, \psi_k)$ by $u_0^k = P_\theta[\psi_k](u_0)$, $u_1^k = P_\theta[\psi_k](u_1)$, $v_0^k = P_\theta[\psi_k](v_0)$, and $v_1^k = P_\theta[\psi_k](v_1)$ respectively. We denote by u_t^k and v_t^k the geodesics joining u_0^k, u_1^k, and v_0^k, v_1^k respectively.

The projections $P_\theta[\psi_k](u_t)$ and $P_\theta[\psi_k](v_t)$ do not coincide with u_t^k and v_t^k respectively, but for $p > 1$, using uniform convexity we can show that

$$\lim_{k\to\infty} d_p(u_t^k, P_\theta[\psi_k](u_t)) = 0 \qquad \text{and} \qquad \lim_{k\to\infty} d_p(v_t^k, P_\theta[\psi_k](v_t)) = 0. \tag{5.2}$$

See the proof of [19, Theorem 8.3, Equation 26] where the details are spelled out more clearly.

From Equation (4.16) and 5.2, we get

$$d_p(u_t, v_t) = \lim_{k\to\infty} d_p(P_\theta[\psi_k](u_t), P_\theta[\psi_k](v_t)) = \lim_{k\to\infty} d_p(u_t^k, v_t^k).$$

By Lemma 5.3.5, we can write

$$d_p(u_t^k, v_t^k) \leq (1-t)d_p(u_0^k, v_0^k) + td_p(u_1^k, v_1^k).$$

Combining the last two equations we get

$$d_p(u_t, v_t) = \lim_{k\to\infty} d_p(u_t^k, v_t^k) \leq \lim_{k\to\infty} (1-t)d_p(u_0^k, v_0^k) + td_p(u_1^k, v_1^k)$$
$$= (1-t)d_p(u_0, v_0) + td_p(u_1, v_1).$$

□

5.4 Space of geodesic rays

In this section, we will study the space of geodesic rays and prove Theorem 1.4.1. Let $u \in \mathcal{E}^p(X, \theta)$. A geodesic ray $\{u_t\}_t$ is a finite p-energy geodesic ray emanating from u if $u_t \to u$ as $t \to 0$ in $L^1(X)$ and $u_t \in \mathcal{E}^p(X, \theta)$ for all $t > 0$. We denote by $\mathcal{R}_u^p$ the set of all finite p-energy geodesic rays emanating from u.

We construct a metric d_p^c on $\mathcal{R}_u^p$ as follows. If $\{u_t\}_t, \{v_t\}_t \in \mathcal{R}_u^p$ are two finite energy geodesic rays, then we define

$$d_p^c(\{u_t\}_t, \{v_t\}_t) = \lim_{t\to\infty} \frac{d_p(u_t, v_t)}{t}. \tag{5.3}$$

Due to Buseman convexity of d_p from Theorem 5.0.1, we know that if $t_1 \geq t_2$, then

$$d_p(u_{t_2}, v_{t_2}) \leq \left(1 - \frac{t_2}{t_1}\right) d_p(u_0, v_0) + \frac{t_2}{t_1} d_p(u_{t_1}, v_{t_1}).$$

Since $u_0 = v_0 = u$, we have

$$\frac{d_p(u_{t_2}, v_{t_2})}{t_2} \leq \frac{d_p(u_{t_1}, v_{t_1})}{t_1}.$$

Thus the limit in Equation (5.3) is increasing and is bounded from above due to the triangle inequality. This is because $d_p(u_t, v_t) \leq d_p(u_t, u) + d_p(v_t, u) = td_p(u_1, u) + td_p(v_1, u)$. Thus

$$\frac{d_p(u_t, v_t)}{t} \leq d_p(u_1, u) + d_p(v_1, u).$$

Thus the limit in Equation (5.3) is well defined.

Lemma 5.4.1. *Equation* (5.3) *defines a metric on* $\mathcal{R}_u^p$*.*

Proof. Let $\{u_t\}_t, \{v_t\}_t, \{w_t\}_t \in \mathcal{R}_u^p$ be three finite p-energy geodesic rays. $d_p^c(\{u_t\}_t, \{v_t\}_t) \geq 0$ because $d_p(u_t, v_t) \geq 0$.

If $d_p^c(\{u_t\}_t, \{v_t\}_t) = 0$, then the function $f(t) = d_p(u_t, v_t)/t$ is an increasing function that satisfies $f(0) = 0$ and $\lim_{t\to\infty} f(t) = 0$. Therefore $f(t) = 0$ for all $t \geq 0$. Thus $d_p(u_t, v_t) = 0$ for all $t \geq 0$. Therefore $u_t = v_t$ for all $t \geq 0$. So $\{u_t\}_t = \{v_t\}_t$. This proves non-degeneracy of d_p^c.

The triangle inequality for d_p^c follows from the triangle inequality for d_p.

$$\begin{aligned} d_p^c(\{u_t\}_t, \{v_t\}_t) &= \lim_{t\to\infty} \frac{d_p(u_t, v_t)}{t} \\ &\leq \lim_{t\to\infty} \frac{d_p(u_t, w_t)}{t} + \lim_{t\to\infty} \frac{d_p(w_t, v_t)}{t} \\ &= d_p^c(\{u_t\}_t, \{w_t\}_t) + d_p^c(\{w_t\}_t, \{v_t\}_t). \end{aligned}$$

□

The next theorem shows that the metric space $(\mathcal{R}_u^p, d_p^c)$ are all isometric for all $u \in \mathcal{E}^p(X, \theta)$.

Theorem 5.4.2. *Given $u, v \in \mathcal{E}^p(X,\theta)$, there exists a map $P_{uv} : \mathcal{R}_u^p \to \mathcal{R}_v^p$ that induces a bijective isometry $P_{uv} : (\mathcal{R}_u^p, d_p^c) \to (\mathcal{R}_v^p, d_p^c)$.*

Proof. Let $\{u_t\}_t \in \mathcal{R}_u^p$. We will construct a finite energy geodesic ray $\{v_t\}_t =: P_{uv}(\{u_t\}_t)$ starting from v.

First, assume that $u \geq v$. We can construct a finite energy geodesic $[0,t] \ni l \mapsto v_l^t \in \mathcal{E}^p(X,\theta)$ joining $v = v_0^t$ and $u_t = v_t^t$. Consider $0 \leq t \leq t'$. Since $[0,t'] \ni l \mapsto u_l$ is a finite energy geodesic joining u and $u_{t'}$, whereas $[0,t'] \ni l \mapsto v_l^{t'}$ is a finite energy geodesic joining v and $u_{t'}$. Since $u \geq v$, by the comparison principle, $v_t^{t'} \leq u_t = v_t^t$. Again applying the comparison principle, we get that for $l \leq t$, $v_l^{t'} \leq v_l^t$.

Moreover, by Theorem 5.0.1,

$$d_p(v_l^t, u_l) \leq \left(1 - \frac{l}{t}\right) d_p(u,v).$$

By Lemma 5.1.1, the map $[0,\infty) \ni l \mapsto v_l := \lim_{t\to\infty} v_l^t \in \mathcal{E}^p(X,\theta)$ is a finite energy geodesic ray. Note that the limit geodesic v_l still emanates from v. This is because by Lemma 5.1.1, the path $[0,t] \ni l \mapsto v_l \in \mathcal{E}^p(X,\theta)$ is a geodesic joining v and v_t. Because $v, v_t \in E^p(X,\theta)$ the result follows from the following argument. As $d_p(v_l, v_0) = \frac{l}{t}$, we get that $\lim_{l\to 0} d_p(v_l, v_0) = 0$. Thus by [19, Lemma 4.12] and [15, Theorem 1.2], we get that $v_l \to v_0$ in $L^1(X)$. See [59, Proposition 4.2.1] for a more general result. Now, taking the limit $t \to \infty$ in the equation above we get that $d_p(v_l, u_l) \leq d_p(u,v)$.

A similar proof works when $u \leq v$.

More generally, we can consider $h = P_\theta(u,v) \in \mathcal{E}^p(X,\theta)$. Now $h \leq u, v$. Thus from above, we can construct a finite energy geodesic ray $\{h_t\}_t \in \mathcal{R}_h^p$ that satisfies $d_p(h_t, u_t) \leq$

$d_p(h,u)$. We can now also construct a finite energy geodesic ray $\{v_t\}_t \in \mathcal{R}^p_v$ such that $d_p(h_t,v_t) \leq d_p(h,v)$. Thus by the triangle inequality $d_p(u_t,v_t) \leq d_p(u_t,h_t) + d_p(h_t,v_t) \leq d_p(u,h) + d_p(h,v)$. Thus $\{u_t\}_t, \{v_t\}_t$ are parallel geodesic rays.

To see that $P_{uv} : (\mathcal{R}^p_u, d^c_p) \to (\mathcal{R}^p_v, d^c_p)$ is an isometry consider $\{u^0_t\}_t, \{u^1_t\}_t \in \mathcal{R}^p_u$. Let $P_{uv}(\{u^i_t\}_t) = \{v^i_t\}_t$ for $i = 0, 1$. By the triangle inequality we have

$$\begin{aligned} d^c_p(\{v^0_t\}_t, \{v^1_t\}_t) &= \lim_{t\to\infty} \frac{d_p(v^0_t, v^1_t)}{t} \\ &\leq \lim_{t\to\infty} \frac{d_p(v^0_t, u^0_t) + d_p(u^0_t, u^1_t) + d_p(u^1_t, v^1_t)}{t} \\ &= \lim_{t\to\infty} \frac{d_p(u^0_t, u^1_t)}{t} \\ &= d^c_p(\{u^0_t\}_t, \{u^1_t\}_t). \end{aligned}$$

In the second line, we used that fact that $d_p(u^i_t, v^i_t)$ remain bounded as $t \to \infty$ for $i = 0, 1$. The other side inequality is obtained similarly. Thus $d^c_p(\{u^0_t\}_t, \{u^1_t\}_t) = d^c_p(\{v^0_t\}_t, \{v^1_t\}_t)$. $\square$

Since the spaces $(\mathcal{R}^p_u, d^c_p)$ are all isometric to each other, it is good enough to study just one of them. We will fix $u = V_\theta$ and use $\mathcal{R}^p_\theta$ to denote $\mathcal{R}^p_{V_\theta}$. The goal of the rest of the section is to prove that $(\mathcal{R}^p_\theta, d^c_p)$ is a complete geodesic metric space. First, we prove

Lemma 5.4.3. *The metric space $(\mathcal{R}^p_\theta, d^c_p)$ is a complete metric space.*

Proof. Let $\{u^j_t\}_t \in \mathcal{R}^p_\theta$ be a Cauchy sequence of finite energy geodesic rays. This means that $d^c_p(\{u^j_t\}_t, \{u^k_t\}_t) \to 0$ as $j, k \to \infty$. For a fixed $t \geq 0$, we know that $d_p(u^j_t, u^k_t) \leq t d^c_p(\{u^j_t\}_t, \{u^k_t\}_t)$. Thus u^j_t is a Cauchy sequence in $(\mathcal{E}^p(X,\theta), d_p)$ with limit point $u_t \in \mathcal{E}^p(X,\theta)$.

By the endpoint stability Lemma 5.1.1, u_t is a finite energy geodesic ray. Now we observe that

$$\begin{aligned} d_p^c(\{u_t^j\}_t, \{u_t\}_t) &= \lim_{t\to\infty} \frac{d_p(u_t^j, u_t)}{t} \\ &= \lim_{t\to\infty} \lim_{k\to\infty} \frac{d_p(u_t^j, u_t^k)}{t} \\ &\leq d_p^c(\{u_t^j\}_t, \{u_t^k\}_t) \end{aligned}$$

which can be arbitrarily small. Therefore, $d_p^c(\{u_t^j\}_t, \{u_t\}_t) \to 0$ as $j \to \infty$. □

First, we will show that for $p > 1$, $(\mathcal{R}_\theta^p, d_p^c)$ is a complete geodesic metric space. We have already seen that it is a complete metric space. We just need to show that there are geodesics in this space.

Theorem 5.4.4. *For $p > 1$, $(\mathcal{R}_\theta^p, d_p^c)$ is a complete geodesic metric space.*

Proof. Let $\{u_t^0\}_t, \{u_t^1\}_t \in \mathcal{R}_\theta^p$ be two finite energy geodesic rays. We will construct finite energy geodesic rays $\{u_t^\alpha\}_t \in \mathcal{R}_\theta^p$ for $0 \leq \alpha \leq 1$, such that the path $[0,1] \ni \alpha \mapsto \{u_t^\alpha\}_t \in \mathcal{R}_\theta^p$ is a d_p^c-geodesic.

First, consider the finite energy geodesic $[0,1] \ni \alpha \mapsto v_t^\alpha \in \mathcal{E}^p(X,\theta)$ joining u_t^0 and u_t^1. Let $[0,t] \ni l \mapsto w_l^{\alpha,t} \in \mathcal{E}^p(X,\theta)$ be the finite energy geodesic joining $w_0^{\alpha,t} := V_\theta$ and $w_t^{\alpha,t} = v_t^\alpha$. We claim that for a fixed $l \geq 0$, $w_l^{\alpha,t} \in \mathcal{E}^p(X,\theta)$ is d_p-Cauchy as $t \to \infty$. Thus there exist $u_l^\alpha \in \mathcal{E}^p(X,\theta)$ such that $\lim_{t\to\infty} d_p(w_l^{\alpha,t}, u_l^\alpha) = 0$. By the endpoint stability Lemma 5.1.1, $\{u_t^\alpha\}_t$ is a finite energy geodesic ray.

Now we will prove the claim. Let $s \leq t$. By applying Theorem 5.0.1 to geodesics $[0,t] \ni$

$s \mapsto u_s^0$ joining V_θ and u_t^0, and $[0,t] \ni s \mapsto w_s^{\alpha,t}$ joining V_θ and v_t^α, we get

$$\frac{d_p(u_s^0, w_s^{\alpha,t})}{s} \leq \frac{d_p(u_t^0, v_t^\alpha)}{t} = \alpha \frac{d_p(u_t^0, u_t^1)}{t} \leq \alpha d_p^c(\{u_t^0\}_t, \{u_t^1\}_t) \tag{5.4}$$

Similarly, we have

$$\frac{d_p(u_s^1, w_s^{\alpha,t})}{s} \leq \frac{d_p(u_t^1, v_t^\alpha)}{t} = (1-\alpha) \frac{d_p(u_t^0, u_t^1)}{t} \leq (1-\alpha) d_p^c(\{u_t^0\}_t, \{u_t^1\}_t) \tag{5.5}$$

Since $\frac{d_p(u_s^0, u_s^1)}{s} \nearrow d_p^c(\{u_t^0\}_t, \{u_t^1\}_t)$. We can write $d_p^c(\{u_t^0\}_t, \{u_t^1\}_t) \leq (1+\varepsilon(s)) \frac{d_p(u_s^0, u_s^1)}{s}$ where $\varepsilon(s) \to 0$ as $s \to \infty$. Combining with Equations (5.4) and (5.5), we get

$$d_p(u_s^0, w_s^{\alpha,t}) \leq (\alpha + \varepsilon(s)) d_p(u_s^0, u_s^1)$$

and

$$d_p(u_s^1, w_s^{\alpha,t}) \leq (1 - \alpha + \varepsilon(s)) d_p(u_s^0, u_s^1).$$

By Corollary 4.8.9 of the uniform convexity, we can write

$$d_p(v_s^\alpha, w_s^{\alpha,t}) \leq C \varepsilon(s)^{\frac{1}{r}} d_p(u_s^0, u_s^1).$$

This is the only place where we need the condition that $p > 1$. Applying Theorem 5.0.1, to geodesics $[0,s] \ni l \mapsto w_l^{\alpha,s}$ joining V_θ and v_s^α, and $[0,s] \ni l \mapsto w_l^{\alpha,t}$ joining V_θ and $w_s^{\alpha,t}$, we get for $l \leq s$,

$$\frac{d_p(w_l^{\alpha,s}, w_l^{\alpha,t})}{l} \leq \frac{d_p(v_s^\alpha, w_s^{\alpha,t})}{s} \leq C(\varepsilon(s))^{\frac{1}{r}} d_p^c(\{u_t^0\}_t, \{u_t^1\}_t).$$

As $\varepsilon(s)$ can be made arbitrarily small, we get that $w_l^{\alpha,t}$ is a d_p-Cauchy sequence as $t \to \infty$. By [19, Main Theorem], the metric d_p is complete, so there exists $u_l^\alpha \in \mathcal{E}^p(X,\theta)$ such that $\lim_{t\to\infty} d_p(w_l^{\alpha,t}, u_l^\alpha) = 0$.

Now we will prove that the map $[0,1] \ni \alpha \mapsto \{u_t^\alpha\}_t \in \mathcal{R}_\theta^p$ is a d_p^c-geodesic. Taking the limit $t \to \infty$ in Equations (5.4) and (5.5), we get that

$$\frac{d_p(u_s^0, u_s^\alpha)}{s} \leq \alpha d_p^c(\{u_t^0\}_t, \{u_t^1\}_t).$$

and

$$\frac{d_p(u_s^1, u_s^\alpha)}{s} \leq (1-\alpha) d_p^c(\{u_t^0\}_t, \{u_t^1\}_t).$$

Taking the limit $s \to \infty$, we get that

$$d_p^c(\{u_t^0\}_t, \{u_t^\alpha\}_t) \leq \alpha d_p^c(\{u_t^0\}_t, \{u_t^1\}_t) \qquad \text{and} \qquad d_p^c(\{u_t^1\}_t, \{u_t^\alpha\}_t) \leq (1-\alpha) d_p^c(\{u_t^0\}_t, \{u_t^1\}_t)$$

Combining with the triangle inequality, we get

$$d_p^c(\{u_t^0\}_t, \{u_t^\alpha\}_t) = \alpha d_p^c(\{u_t^0\}_t, \{u_t^1\}_t) \qquad \text{and} \qquad d_p^c(\{u_t^1\}_t, \{u_t^\alpha\}_t) = (1-\alpha) d_p^c(\{u_t^0\}_t, \{u_t^1\}_t)$$

For $0 \leq \alpha \leq \beta \leq 1$, the triangle inequality implies

$$d_p^c(\{u_t^\alpha\}_t, \{u_t^\beta\}_t) \geq d_p^c(\{u_t^0\}_t, \{u_t^\beta\}_t) - d_p^c(\{u_t^0\}_t, \{u_t^\alpha\}_t) = (\beta - \alpha) d_p^c(\{u_t^0\}_t, \{u_t^1\}_t).$$

For the other side, we again use Theorem 5.0.1 on geodesics $[0,t] \ni l \mapsto w_l^{\alpha,t}$ and $[0,t] \ni l \mapsto$

$w_l^{\beta,t}$ joining V_θ with v_t^α and v_t^β respectively. We notice that

$$\frac{d_p(w_l^{\alpha,t}, w_l^{\beta,t})}{l} \leq \frac{d_p(v_t^\alpha, v_t^\beta)}{t} = (\beta-\alpha)\frac{d_p(u_t^0, u_t^1)}{t}$$

Taking limit $t \to \infty$, we get

$$\frac{d_p(u_l^\alpha, u_l^\beta)}{l} \leq (\beta-\alpha) d_p^c(\{u_t^0\}_t, \{u_t^1\}_t).$$

Now taking limit $l \to \infty$, we get

$$d_p^c(\{u_t^\alpha\}_t, \{u_t^\beta\}_t) \leq (\beta-\alpha) d_p^c(\{u_t^0\}_t, \{u_t^1\}_t).$$

Thus

$$d_p^c(\{u_t^\alpha\}_t, \{u_t^\beta\}_t) = (\beta-\alpha) d_p^c(\{u_t^0\}_t, \{u_t^1\}_t)$$

proving that the map $[0,1] \ni \alpha \mapsto \{u_t^\alpha\}_t \in \mathcal{R}_\theta^p$ is a d_p^c-geodesic. □

Now we need to extend this proof to $p = 1$ by approximating it from $p > 1$. First, we will show that the d_p^c-geodesics depend only on the end-points, and not the value of p, just like in the metric space $(\mathcal{E}^p(X,\theta), d_p)$, where the metric geodesics are described by the geodesics whose construction does not depend on p.

Lemma 5.4.5. *If $1 \leq p' \leq p$, then $\mathcal{R}_\theta^p \subset \mathcal{R}_\theta^{p'}$. Moreover, if $[0,1] \ni \alpha \mapsto \{u_t^\alpha\}_t \in \mathcal{R}_\theta^p$ is the d_p^c-geodesic joining $\{u_t^0\}_t$ and $\{u_t^1\}_t$, then $[0,1] \ni \alpha \mapsto \{u_t^\alpha\}_t$ is also the $d_{p'}^c$-geodesic.*

Proof. Recall from the proof of Theorem 5.4.4 that the the geodesics $[0,1] \ni \alpha \mapsto v_t^\alpha \in \mathcal{E}^p(X,\theta)$ joining u_t^0 and u_t^1 do not depend on the value of p. Similarly, the geodesics $[0,t] \ni l \mapsto w_l^{\alpha,t} \in$

$\mathcal{E}^p(X,\theta)$ joining V_θ and v_t^α also do not depend on the value of p.

If we can show that the d_p limit u_l^α of $w_l^{\alpha,t}$ as $t \to \infty$ does not depend on p as well, we will be done. Recall from Lemma 5.1.4, that $d_{p'}(\cdot,\cdot) \leq d_p(\cdot,\cdot)\text{Vol}(\theta)^{\frac{1}{p'}-\frac{1}{p}}$ on $\mathcal{E}^p(X,\theta)$. Thus if $\lim_{t\to\infty} d_p(w_l^{\alpha,t}, u_l^\alpha) = 0$, then

$$\lim_{t\to\infty} d_{p'}(w_l^{\alpha,t}, u_l^\alpha) \leq \lim_{t\to\infty} d_p(w_l^{\alpha,t}, u_l^\alpha)\text{Vol}(\theta)^{\frac{1}{p'}-\frac{1}{p}} = 0.$$

Knowing that the $d_{p'}$ limit of $w_l^{\alpha,t}$ as $t \to \infty$ is u_l^α irrespective of the value of p', the proof of the fact that $[0,1] \ni \alpha \mapsto \{u_t^\alpha\}_t \in \mathcal{R}_\theta^{p'}$ is a $d_{p'}^c$-geodesic goes through as in Theorem 5.4.4. $\square$

Now we will establish the version of Theorem 5.4.4 for $p = 1$. Before that, we need the following

Lemma 5.4.6. *For $p \geq 1$, if $\{u_t^j\}_t, \{u_t\} \in \mathcal{R}_\theta^p$ are finite energy geodesic rays such that $u_t^j \searrow u_t$ then*

$$\lim_{j\to\infty} d_p^c(\{u_t^j\}_t, \{u_t\}_t) = 0.$$

Proof. By Lemma 2.6.1, the maps $t \mapsto \sup_X u_t$ and $t \mapsto \sup_X u_t^j$ are linear. Therefore, by replacing u_t, u_t^j by $u_t - Ct, u_t^j - Ct$, we can assume that $0 \geq V_\theta \geq u_t^j \geq u_t$.

Applying the Lidskii type inequality from Theorem 5.2.3, we get that

$$\frac{d_p(u_t^j, u_t)^p}{t^p} \leq \frac{d_p(u_t, V_\theta)^p - d_p(u_t^j, V_\theta)^p}{t^p} = d_p(u_1, V_\theta)^p - d_p(u_1^j, V_\theta)^p.$$

Since $u_1^j \searrow u_1$, by Lemma 4.6.2, $\lim_{j\to\infty} d_p(u_1^j, 0) = d_p(u_1, 0)$. Thus

$$\lim_{j\to\infty} \frac{d_p(u_t^j, u_t)}{t} = 0$$

uniformly in t. Therefore, $\lim_{j\to\infty} d_p^c(\{u_t^j\}_t, \{u_t\}_t) = 0$. $\square$

With the help of Lemma 5.4.6 and Theorem 2.6.2, we can prove the existence of geodesics in $\mathcal{R}_\theta^1$.

Theorem 5.4.7. *For $p \geq 1$, the space $(\mathcal{R}_\theta^p, d_p^c)$ is a complete geodesic metric.*

Proof. We have already proved the theorem for $p > 1$. We just need to prove it for $p = 1$ now. Let $\{u_t^0\}_t, \{u_t^1\}_t \in \mathcal{R}_\theta^1$ be two finite energy geodeisc rays. As in the proof of Theorem 5.4.4, we can construct geodesics $[0,1] \ni \alpha \mapsto v_t^\alpha \in \mathcal{E}^1(X,\theta)$ joining u_t^0 and u_t^1. We can also construct geodesics $[0,t] \ni l \mapsto w_l^{\alpha,t} \in \mathcal{E}^1(X,\theta)$ joining V_θ and v_t^α. We used $p > 1$ to construct u_l^α as the d_p limit of $w_l^{\alpha,t}$ as $t \to \infty$. In the case $p = 1$, we will use approximation to construct u_l^α.

Fix any $p > 1$. From Theorem 2.6.2 we can construct geodesic rays $\{u_t^{0,j}\}_t, \{u_t^{1,j}\}_t \in \mathcal{R}_\theta^p$, such that $u_t^{0,j} \searrow u_t^0$ and $u_t^{1,j} \searrow u_t^1$ as $j \to \infty$ and for all $t \geq 0$.

Let $[0,1] \ni \alpha \mapsto \{u_t^{\alpha,j}\}_t \in \mathcal{R}_\theta^p \subset \mathcal{R}_\theta^1$ be the d_p^c-geodesic constructed in Theorem 5.4.4. By Lemma 5.4.5 it is also the d_1^c-geodesic. Recall that we constructed $u_l^{\alpha,j}$ as d_p-limit (hence

d_1-limit) of $w_l^{\alpha,t,j}$ as $t \to \infty$. Since $w_l^{\alpha,t,j} \geq w_l^{\alpha,t}$, Theorem 4.6.7 and Lemma 2.7.1 show

$$\begin{aligned}
d_1(w_l^{\alpha,t,j}, w_l^{\alpha,t}) &= I(w_l^{\alpha,t,j}) - I(w_l^{\alpha,t}) \\
&= \frac{l}{t}(I(v_t^{\alpha,j}) - I(v_t^{\alpha})) \\
&= \frac{l}{t}((1-\alpha)I(u_t^{0,j}) + \alpha I(u_t^{1,j}) - (1-\alpha)I(u_t^0) - \alpha I(u_t^1)) \\
&= l(1-\alpha)(I(u_1^{0,j}) - I(u_1^0)) + l\alpha(I(u_1^{1,j}) - I(u_1^1)).
\end{aligned}$$

As $j \to \infty$, $u_1^{0,j} \searrow u_1^0$ and $u_1^{1,j} \searrow u_1^1$. As I is continuous along decreasing sequences (see [29, Proposition 2.10]) we get $I(u_1^{0,j}) \to I(u_1^0)$ and $I(u_1^{1,j}) \to I(u_1^1)$. Hence $d_1(w_l^{\alpha,t,j}, w_l^{\alpha,t}) \to 0$ as $j \to \infty$ uniformly over t.

Now $u_l^{\alpha,j}$ is a decreasing sequence and $u_l^{\alpha,j}$ does not decrease to $-\infty$ as $j \to \infty$ because $I(u_l^{\alpha,j})$ is bounded. Therefore, the limit $\lim_{j\to\infty} u_l^{\alpha,j} =: u_l^\alpha$ has finite I energy due to continuity of I along decreasing sequences. Hence $u_l^\alpha \in \mathcal{E}^1(X,\theta)$. By the proof of Theorem 5.4.4, it remains to show that $\lim_{t\to\infty} d_1(w_l^{\alpha,t}, u_l^\alpha) = 0$. To see this notice that

$$d_1(w_l^{\alpha,t}, u_l^\alpha) \leq d_1(w_l^{\alpha,t}, w_l^{\alpha,t,j}) + d_1(w_l^{\alpha,t,j}, u_l^{\alpha,j}) + d_1(u_l^{\alpha,j}, u_l^\alpha).$$

We have seen above that the first term goes to 0 as $j \to \infty$ uniformly in t. The last term also goes to 0 as $j \to \infty$. For a fixed j, the second term gets arbitrarily small for large t. Therefore, $\lim_{t\to\infty} d_1(w_l^{\alpha,t}, u_l^\alpha) \to 0$. □

5.5 The Ross–Witt Nyström correspondence

The notion of test curves, introduced by Ross–Witt Nyström in [57] in the Kähler setting, and further studied by Darvas–Di Nezza–Lu [17], Darvas–Zhang [37], Darvas–Xia [23] and Darvas–Xia–Zhang [60], is a powerful tool to study geodesic rays in connection with K-stability.

Definition 5.5.1. A map $\mathbb{R} \ni \tau \mapsto \psi_\tau \in \mathrm{PSH}(X, \theta)$ is a test curve, denoted by $\{\psi_\tau\}_\tau$, if

1. The map $\mathbb{R} \ni \tau \mapsto \psi_\tau(x)$ is decreasing, usc, concave for all $x \in X$, and
2. $\psi_\tau = -\infty$ for all τ big enough, and
3. $\psi_\tau \nearrow V_\theta$ a.e. as $\tau \to -\infty$.

From condition (2) above, we can define

$$\tau_\psi^+ = \inf\{\tau \in \mathbb{R} \,:\, \psi_\tau \equiv -\infty\}.$$

Test curves are Legendre dual of sublinear subgeodesic rays starting at V_θ [37]. Several properties of the geodesic rays can be detected by the corresponding test curves. Based on that we define several subclasses of test curves.

Definition 5.5.2. A test curve $\{\psi_\tau\}_\tau$ can have the following properties.

1. It is *maximal* if $P_\theta[\psi_\tau](0) = \psi_\tau$ for all $\tau \in \mathbb{R}$.
2. It has *finite p-energy* for $p \geq 1$, if

$$\int_{-\infty}^{\tau_\psi^+} (-\tau + \tau_\psi^+)^{p-1} \left(\int_X \theta_{V_\theta}^n - \int_X \theta_{\psi_\tau}^n \right) d\tau < \infty$$

3. It is *bounded* if for all τ negative enough, $\psi_\tau = V_\theta$. In this case, we define

$$\tau_\psi^- = \sup\{\tau \in \mathbb{R} \,:\, \psi_\tau = V_\theta\}.$$

When $p = 1$, the notion of finite p-energy test curves was introduced by Darvas–Xia [23] in the Kähler setting, and by Darvas–Zhang [37] in the big setting.

The Ross–Witt Nyström correspondence is the observation of Ross–Witt Nyström [57] that test curves and subgeodesic rays are dual to each other via Legendre transform. If $\{u_t\}_t$ is a sublinear subgeodesic ray starting at V_θ, then its Legendre transform is given by

$$\hat{u}_\tau(x) = \inf_{t>0}(u_t(x) - t\tau).$$

If $\{\psi_\tau\}_\tau$ is a test curve, then its inverse Legendre transform is given by

$$\check{\psi}_t(x) = \sup_{\tau\in\mathbb{R}}(\psi_\tau(x) - t\tau).$$

In particular, we have

Theorem 5.5.3 ([37, Theorem 3.7])**.** *The Legendre transform $\{\psi_\tau\}_\tau \mapsto \{\check{\psi}_t\}_t$ is a bijection with inverse $\{u_t\}_t \mapsto \{\hat{u}_\tau\}_\tau$ between*

1. *test curves and sublinear subgeodesic rays starting at V_θ;*

2. *maximal test curves and geodesic rays starting at V_θ;*

3. *bounded maximal test curves and geodesic rays with minimal singularity type. Moreover, in this case, $V_\theta + \tau_\psi^- t \leq \check{\psi}_t \leq V_\theta + \tau_\psi^+ t$.*

Darvas–Xia [23, Theorem 3.7] in the Kähler setting, and Darvas–Zhang [37, Theorem 3.9] in the big setting proved that the Ross–Witt Nyström correspondence as described above is a bijection between maximal finite 1-energy test curves and finite 1-energy geodesic rays. In this section, we will generalize this statement to finite p-energy test curves and geodesic rays.

For that, first, we need a lemma about the speed of the geodesic segments, which can be seen as a converse of [19, Lemma 5.3].

Lemma 5.5.4. *If $u_0 \in \mathcal{H}_\theta$, $u_1 \in PSH(X,\theta)$, and $[0,1] \ni t \mapsto u_t \in PSH(X,\theta)$ is the geodesic joining u_0 and u_1 such that*

$$\int_X |\dot{u}_0|^p \theta_{u_0}^n < \infty,$$

then $u_1 \in \mathcal{E}^p(X,\theta)$.

Proof. First, assume that $u_0 \geq u_1 + 1$, so that we can find $u_1^j \in \mathcal{H}_\theta$ such that $u_1^j \searrow u_1$ and $u_0 \geq u_1^j$. Let $[0,1] \ni t \mapsto u_t^j \in \mathcal{E}^p(X,\theta)$ be the geodesic joining u_0 and u_1^j. The geodesics are decreasing, therefore $\dot{u}_0, \dot{u}_0^j \leq 0$. By Theorem 4.5.7,

$$d_p(u_0, u_1^j)^p = \int_X |\dot{u}_0^j|^p \theta_{u_0}^n = \int_X (-\dot{u}_0^j)^p \theta_{u_0}^n.$$

Since $u_1^j \geq u_1$, the geodesics satisfy $u_t^j \geq u_t$. Therefore,

$$\dot{u}_0^j = \lim_{t\to 0} \frac{u_t^j - u_0}{t} \geq \lim_{t\to 0} \frac{u_t - u_0}{t} = \dot{u}_0.$$

Hence, $(-\dot{u}_0^j)^p \leq (-\dot{u}_0)^p$. Now,

$$d_p(u_0, u_1^j)^p = \int_X (-\dot{u}_0^j)^p \theta_{u_0}^n \leq \int_X |\dot{u}_0|^p \theta_{u_0}^n < \infty.$$

Therefore, u_1^j is a d_p-bounded decreasing sequence such that $u_1^j \searrow u_1$. Therefore by Lemma 5.1.2 $u_1 \in \mathcal{E}^p(X, \theta)$.

Now we drop the assumption that $u_0 \geq u_1 + 1$. Let $C > 0$ such that $u_1 - C + 1 \leq u_0$. Let $w_1 = u_1 - C$, so that $u_0 \geq w_1 + 1$. If w_t is the geodesic joining u_0 and w_1, then $w_t = u_t - Ct$. Therefore, $\dot{w}_0 = \dot{u}_0 - C$. Hence,

$$\int_X |\dot{w}_0|^p \theta_{u_0}^n = \int_X |\dot{u}_0 - C|^p \theta_{u_0}^n \leq 2^{p-1} \left(\int_X |\dot{u}_0|^p \theta_{u_0}^n + C^p \mathrm{Vol}(\theta) \right) < \infty.$$

By the argument above, we find that $w_1 \in \mathcal{E}^p(X, \theta)$, therefore, $w_1 + C = u_1 \in \mathcal{E}^p(X, \theta)$ as well. $\square$

This Lemma allows us to prove

Theorem 5.5.5. *For $p \geq 1$, the Legendre Transform $\{u_t\}_t \mapsto \{\hat{u}_\tau\}_\tau$ is a bijective map between finite p-energy geodesic rays in $\mathcal{R}_\theta^p$ and maximal finite p-energy test curves.*

Proof. First, we assume that $\sup_X u_t = 0$ for all $t \geq 0$. In this case $\tau_{\hat{u}}^+ = 0$. To see this we observe that u_t decreases as $t \to \infty$. As $\sup_X u_t = 0$, by Hartog's Lemma (see [25, Proposition 8.4]) $\sup_X(\lim_{t\to\infty} u_t) = 0$. Therefore, when $\tau = 0$,

$$\hat{u}_0 = \inf_{t \geq 0} u_t = \lim_{t \to \infty} u_t \not\equiv -\infty$$

Whereas for $\tau > 0$,

$$\hat{u}_\tau = \inf_{t \geq 0}(u_t - t\tau) \equiv -\infty.$$

Thus $\tau_{\hat{u}}^+ = 0$. Again since u_t is decreasing $\dot{u}_0 \leq 0$. We will show that

$$\int_X |\dot{u}_0|^p \theta_{u_0}^n = p \int_{-\infty}^0 (-\tau)^{p-1} \left(\int_X \theta_{V_\theta}^n - \int_X \theta_{\hat{u}_\tau}^n \right) d\tau. \tag{5.6}$$

To see this we start with the left-hand side

$$\begin{aligned} \int_X |\dot{u}_0|^p \theta_{u_0}^n &= p \int_X (-\dot{u}_0)^p \theta_{V_\theta}^n \\ &= p \int_0^\infty \tau^{p-1} \theta_{V_\theta}^n(\{\dot{u}_0 < -\tau\}) d\tau. \end{aligned}$$

Changing the variable from τ to $-\tau$, we can continue

$$= p \int_{-\infty}^0 (-\tau)^{p-1} \theta_{V_\theta}^n(\{\dot{u}_0 < \tau\}) d\tau.$$

Recall that $\hat{u}_\tau = \inf_{t \geq 0}(u_t - t\tau)$. Thus the set $\{\dot{u}_0 \geq \tau\} = \{\hat{u}_\tau = u_0 = V_\theta\}$. Therefore,

$$= p \int_{-\infty}^0 (-\tau)^{p-1} \left(\mathrm{Vol}(\theta) - \theta_{V_\theta}^n(\{\hat{u}_\tau = V_\theta\}) \right) d\tau.$$

Since $\{\hat{u}_\tau\}$ is a maximal test curve, we know that $P_\theta[\hat{u}_\tau] = \hat{u}_\tau$. Therefore, by Theorem 2.4.4, $\theta^n_{P_\theta[\hat{u}_\tau]} = \mathbb{1}_{\{P_\theta[\hat{u}_\tau]=0\}}\theta^n$. Moreover, $\theta^n_{V_\theta} = \mathbb{1}_{\{V_\theta=0\}}\theta^n$. As $P_\theta[\hat{u}_\tau] = \hat{u}_\tau$, and $\{\hat{u}_\tau = 0\} \subset \{V_\theta = 0\}$, we get that $\theta^n_{\hat{u}_\tau} = \mathbb{1}_{\{\hat{u}_\tau=0\}}\theta^n_{V_\theta}$. Thus we can write, $\theta^n_{V_\theta}(\{\hat{u}_\tau = V_\theta\}) = \int_X \theta^n_{\hat{u}_\tau}$. So we get

$$= p\int_{-\infty}^{0} (-\tau)^{p-1}\left(\int_X \theta^n_{V_\theta} - \int_X \theta^n_{\hat{u}_\tau}\right) d\tau.$$

This proves (5.6).

If the test curve $\{\hat{u}_\tau\}_\tau$ has finite p-energy, then by (5.6), $\int_X |\dot{u}_0|^p\theta^n_{u_0} < \infty$. Since $[0,1] \ni s \mapsto w_s := u_{ts}$ is the geodesic joining u_0 and u_t. Thus, $\dot{w}_0 = t\dot{u}_0$ and

$$\int_X |\dot{w}_0|^p\theta^n_{u_0} = t^p\int_X |\dot{u}_0|^p\theta^n_{u_0} < \infty$$

Therefore, by Lemma 5.5.4, $u_t \in \mathcal{E}^p(X,\theta)$. Thus the geodesic ray $\{u_t\}_t$ is a finite p-energy geodesic ray.

Similarly, if $\{u_t\}_t$ is a finite p-energy geodesic ray then $\int_X |\dot{u}_0|^p\theta^n_{u_0} = d_p^p(u_0,u_1) < \infty$, thus by (5.6), $\{\hat{u}_\tau\}_\tau$ is a finite p-energy test curve.

More generally, by Lemma 2.6.1, $\sup_X u_t = Ct$ for some C. Moreover, by the same argument as earlier $C = \tau^+_{\hat{u}}$. If $w_t = u_t - Ct = u_t - \tau^+_{\hat{u}}t$, then $\{w_t\}_t$ is a geodesic whose Legendre dual $\{\hat{w}_\tau\}_\tau$ is given by $\hat{w}_\tau = \hat{u}_{\tau+C} = \hat{u}_{\tau+\tau^+_{\hat{u}}}$.

From (5.6), we can say that

$$
\begin{aligned}
\int_X |\dot{w}_0|^p \theta^n_{V_\theta} &= p\int_{-\infty}^{0} (-\tau)^{p-1}\left(\int_X \theta^n_{V_\theta} - \int_X \theta^n_{\hat{w}_\tau}\right) d\tau \\
&= p\int_{-\infty}^{0} (-\tau)^{p-1}\left(\int_X \theta^n_{V_\theta} - \int_X \theta^n_{\hat{u}_{\tau+\tau^+_{\hat{u}}}}\right) d\tau \\
&= p\int_{-\infty}^{\tau^+_{\hat{u}}} (-\tau+\tau^+_{\hat{u}})^{p-1}\left(\int_X \theta^n_{V_\theta} - \int_X \theta^n_{\hat{u}_\tau}\right) d\tau
\end{aligned}
$$

As $\dot{w}_0 = \dot{u}_0 - \tau^+_{\hat{u}}$, we get that

$$
\int_X |\dot{u}_0|^p \theta^n_{V_\theta} < \infty \iff \int_X |\dot{w}_0|^p \theta^n_{V_\theta} < \infty.
$$

Combining the two equations above, we get that $\{u_t\}_t \in \mathcal{R}^p_\theta$ is the finite p-energy geodesic ray iff the Legendre transform $\{\hat{u}_\tau\}_\tau$ is a maximal finite p-energy test curve. $\square$

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