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The Jacobian Analysis of a Parallel Manipulator Using Reciprocal Screws

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The Jacobian Analysis of a Parallel Manipulator Using the Theory of Reciprocal Screws

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Abstract

In this paper, the theory of reciprocal screws is reviewed. Reciprocal screw systems associated with some frequently used kinematic pairs and chains are developed. Then, the application of reciprocal screw systems for the Jacobian analysis of parallel manipulators is described. The Jacobian and singular conditions of a six-dof parallel manipulator are analyzed.

1 Introduction

The study of the instantaneous motion of kinematic chains was pioneered by Waldron (1966). Since then, various methods of analysis have been proposed (Baker, 1980; and Davies, 1981). The instantaneous kinematics of parallel manipulators is complicated by the existence of numerous number of closed loops. Recently, Mohamed, et al. (1983) developed a procedure for the determination of the instantaneous twists associated with the joints of a limb using the velocity vector-loop equations. Mohamed and Duffy (1985) applied the theory of reciprocal screws, and Sugimoto (1987) used the motor algebra for the Jacobian analysis of parallel manipulators.

In this paper, the theory of reciprocal screws will be reviewed. Although the concept of using reciprocal screws for the instantaneous kinematic analysis of parallel manipulators was first introduced by Mohamed and Duffy (1985), no specific procedure was given for the derivation of reciprocal screws. The purpose of this paper is to provide the readers with a better understanding of the reciprocal screws and their applications for the Jacobain analysis of parallel manipulators. It is shown that the reciprocal screws associated with a kinematic chain can be identified by an inspection of the kinematic structure. To demonstrate the methodology, the Jacobian of a six-dof parallel manipulator is analyzed.

2 Reciprocal Screws

The concept of reciprocal screws was first studied by Ball (1900), followed by Waldron(1969), Hunt (1978), Roth (1984), and others. A unit screw $\hat{\$}$ is defined by a pair of vectors:

$$\hat{\$} = \begin{bmatrix} \mathbf{s} \\ \mathbf{s}_o \times \mathbf{s} + \lambda \mathbf{s} \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{bmatrix}, \tag{1}$$

where $\mathbf{s}$ is a unit vector along the direction of the screw axis, $\mathbf{s}_o$ is the position vector of any point on the screw axis with respect to a reference frame, and λ is called the pitch.

A screw of *intensify* ρ can be written as $\$ = \rho\hat{\$}$. We call the screw a *twist* if it represents an instantaneous motion of a rigid body, and a *wrench* if it represents a system of forces and couples acting on a rigid body. In this regard, the first three components of a twist represent the angular velocity and the last three components represent the linear velocity of a point in the rigid body which is instantaneously coincident with the origin of the reference frame. On the other hand, the first three components of a wrench represent the resultant force and the last three components represent the resultant moment about the origin of the reference frame.

Two screws, $\$$ and $\$_r$, are said to be *reciprocal* if they satisfy the condition:

$$\$_r^T \$ = 0, \tag{2}$$

where the *transpose of a screw* is defined as

$$\$^T = [S_4 \; S_5 \; S_6 \; S_1 \; S_2 \; S_3]$$

such that the *orthogonal product* of the two screws is given by:

$$\$_r^T \$ = S_{r4}S_1 + S_{r5}S_2 + S_{r6}S_3 + S_{r1}S_4 + S_{r2}S_5 + S_{r3}S_6. \tag{3}$$

The reciprocity condition can be stated as the virtual work between a wrench and a twist is equal to zero. From the geometry of the lines associated with the two screws, the reciprocal condition can be derived as:

$$(\lambda + \lambda_r)\cos\alpha - a\sin\alpha = 0, \tag{4}$$

where a is the offset distance along the common perpendicular leading from the screw axis of $\$$ to $\$_r$, and α is the twist angle between the axes of $\$$ and $\$_r$, measured from $\$$ to $\$_r$ about the common perpendicular according to the right-hand screw rule.

Equation (2) imposes one constraint on the two screws. Since a unit screw requires five independent parameters to specify the screw axis direction, location, and the pitch, there is a quadruple infinitude (∞^4) of screws reciprocal to a given screw. Hence, all screws that are reciprocal to a single screw form a five-system. A more detailed description of screw systems can be found in Hunt (1978).

2.1 Reciprocal Screws Of Some Kinematic Pairs

The screws and reciprocal screws associated with some frequently used joints are listed below:

Revolute Joint. The unit screw associated with a revolute joint is a screw of zero pitch pointing along the joint axis. The reciprocal screws form a five-system and, in particular, those zero-pitch reciprocal screws lie on all planes containing the axis of the revolute joint.

Prismatic Joint. The unit screw associated with a prismatic joint is a screw of infinite pitch pointing along the direction of the joint axis. The reciprocal screws form a five-system and, in particular, those zero-pitch reciprocal screws lie on all planes perpendicular to the axis of the prismatic joint.

Spherical Joint. The unit screws associated with a spherical joint form a three-system of zero pitch passing through the center of the joint. The reciprocal screws also form a three-system of zero pitch passing through the center of the sphere.

Universal Joint. The unit screws associated with a universal joint form a two-system of zero pitch. It is a planar pencil radiating from the center of the universal joint and lying on a plane which contains the two axes of revolution. The reciprocal screws form a four-system. In particular, all zero-pitch reciprocal screws either pass through the center of the universal joint or lie on the plane defined by the two joint axes of the universal joint, and there exists a reciprocal screw of infinite pitch which passes through the center of the universal joint and is perpendicular to both joint axes.

2.2 Reciprocal Screws Of Some Kinematic Chains

Using the above information, reciprocal screws associated with a kinematic chain can be obtained by an intersection of the systems of reciprocal screws associated with the joints in the kinematic chain. In what follows, the reciprocal screws associated with several frequently used kinematic chains are developed.

Universal-Spherical Dyad. The joint screws associated with a universal-spherical dyad form a five-system. Hence, the reciprocal screw is a one-system. Because of the presence of a spherical joint and a universal joint, the reciprocal screw is a zero-pitch screw passing through the centers of the two joints.

Revolute-Spherical Dyad. The joint screws associated with a revolute-spherical dyad form a four-system. The reciprocal screws form a two-system. Because of the presence of a spherical joint and a revolute joint, all reciprocal screws are zero-pitch screws forming a planar pencil. Specifically, all reciprocal screws pass through the center of the spherical joint and lie on a plane which contains both the axis of the revolute joint and the center of the spherical joint as shown in Fig. 1(a).

Prismatic-Spherical Dyad. The joint screws associated with a prismatic-spherical dyad form a four-system. The reciprocal screws form a two-system. Because of the presence of a spherical joint and a prismatic joint, all reciprocal screws are zero-pitch screws forming a planar pencil. Specifically, all reciprocal screws pass through the center of the spherical joint and lie on a plane which is perpendicular to the axis of the prismatic joint as shown in Fig. 1(b).

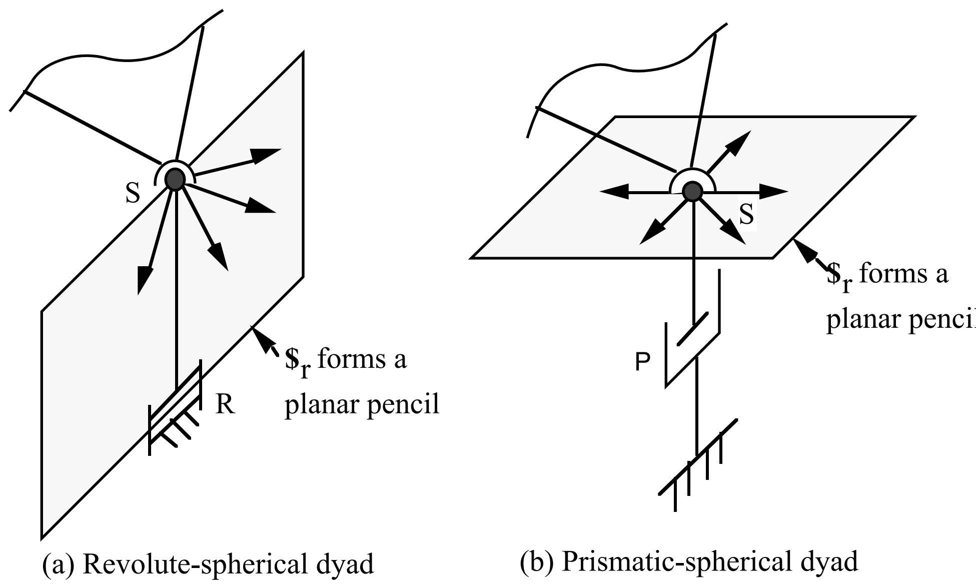

Figure 1: Reciprocal screw systems of two dyads.

3 The Screw Based Jacobian

A parallel manipulator typically consists of several limbs, say m. Each limb is made up of several links connected together by joints. Kinematically, we can replace a cylindrical joint by a revolute joint plus a coaxial prismatic joint, and a spherical joint by three non-coplanar intersecting revolute joints. Hence, we may consider each limb as an open-loop chain connecting the moving platform to the fixed base by ℓ one-dof joints. Figure 2 shows a typical limb, where the first subscript denotes the joint number of a limb and the second subscript represents the limb number. This way, the instantaneous twist, $\$_p$, of the moving platform can be expressed as a linear combination of ℓ instantaneous twists (Mohamed and Duffy, 1985).

$$\$_p = \sum_{j=1}^{\ell} \dot{q}_{j,i} \hat{\$}_{j,i}, \qquad \text{for } i = 1, 2, \cdots, m, \tag{5}$$

where $\dot{q}_{j,i}$ and $\hat{\$}_{j,i}$ denote the intensity and the unit screw associated with the jth joint of the ith limb.

Equation (5) contains many unactuated joint rates which should be eliminated. This can be accomplished by applying the theory of reciprocal screws. Assuming that the actuated joints in each limb appear in the first g terms, we first identify g screws, $\hat{\$}_{rj,i}$, for $j = 1, 2 \cdots, g$, each of them is reciprocal to the screw system associated with all the unactuated joints of the ith limb. Then, we take the orthogonal product of both sides of Eq. (5) with each reciprocal screw. This produces g equations which can be written in a matrix form as follows:

$$J_{x,i} \$_p = J_{q,i} \dot{\mathbf{q}}_i, \tag{6}$$

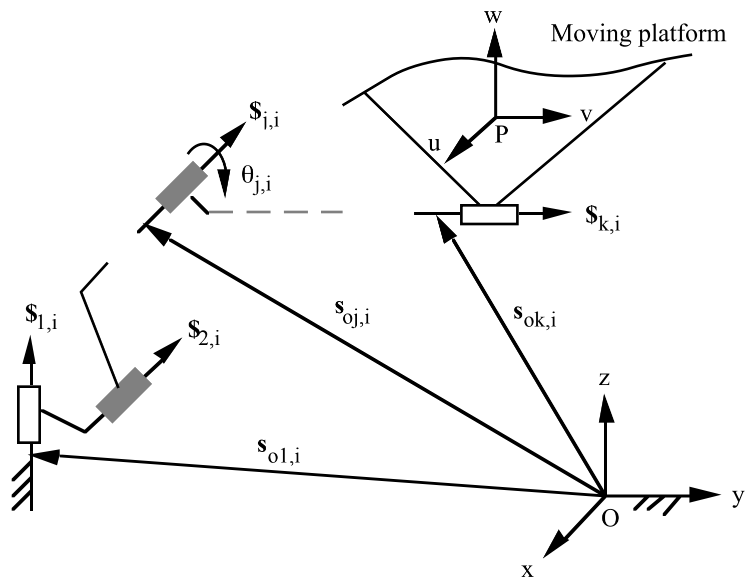

Figure 2: Typical limb of a parallel manipulator.

where $\dot{\mathbf{q}}_i = [\dot{q}_{1,i}, \dot{q}_{2,i}, \cdots, \dot{q}_{g,i}]^T$,

$$J_{x,i} = \begin{bmatrix} \hat{\$}^T_{r1,i} \\ \hat{\$}^T_{r2,i} \\ \vdots \\ \hat{\$}^T_{rg,i} \end{bmatrix}, \quad \text{and} \quad J_{q,i} = \begin{bmatrix} \hat{\$}^T_{r1,i}\hat{\$}_{1,i} & \hat{\$}^T_{r1,i}\hat{\$}_{2,i} & \cdots & \hat{\$}^T_{r1,i}\hat{\$}_{g,i} \\ \hat{\$}^T_{r2,i}\hat{\$}_{1,i} & \hat{\$}^T_{r2,i}\hat{\$}_{2,i} & \cdots & \hat{\$}^T_{r2,i}\hat{\$}_{g,i} \\ \vdots & \vdots & \cdots & \vdots \\ \hat{\$}^T_{rg,i}\hat{\$}_{1,i} & \hat{\$}^T_{rg,i}\hat{\$}_{2,i} & \cdots & \hat{\$}^T_{rg,i}\hat{\$}_{g,i} \end{bmatrix}.$$

Equation (6) written m times, once for each limb, yields $n = m \times g$ linear equations which can be assembled in a matrix form:

$$J_x \$_p = J_q \dot{\mathbf{q}}, \tag{7}$$

where $\dot{\mathbf{q}} = [\dot{q}_{1,i}, \cdots, \dot{q}_{g,i}, \dot{q}_{1,2}, \cdots, \dot{q}_{g,2}, \cdots, \dot{q}_{g,m}]^T$,

$$J_x = \begin{bmatrix} J_{x,1} \\ J_{x,2} \\ \vdots \\ J_{x,m} \end{bmatrix} \quad \text{and} \quad J_q = \begin{bmatrix} [J_{q,1}] & 0 & \cdots & 0 \\ 0 & [J_{q,2}] & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & [J_{q,m}] \end{bmatrix}.$$

The twists associated with all the joints of a limb form an ℓ-system provided that they are linearly independent. Let d be the dimension of the motion space ($d = 6$ for spatial and, $d = 3$ for planar and spherical motions). Clearly, if $\ell = d$, there exists no screw reciprocal to the ℓ-system of twists. If $\ell < d$, there are $(d - \ell)$ linearly independent screws which form a

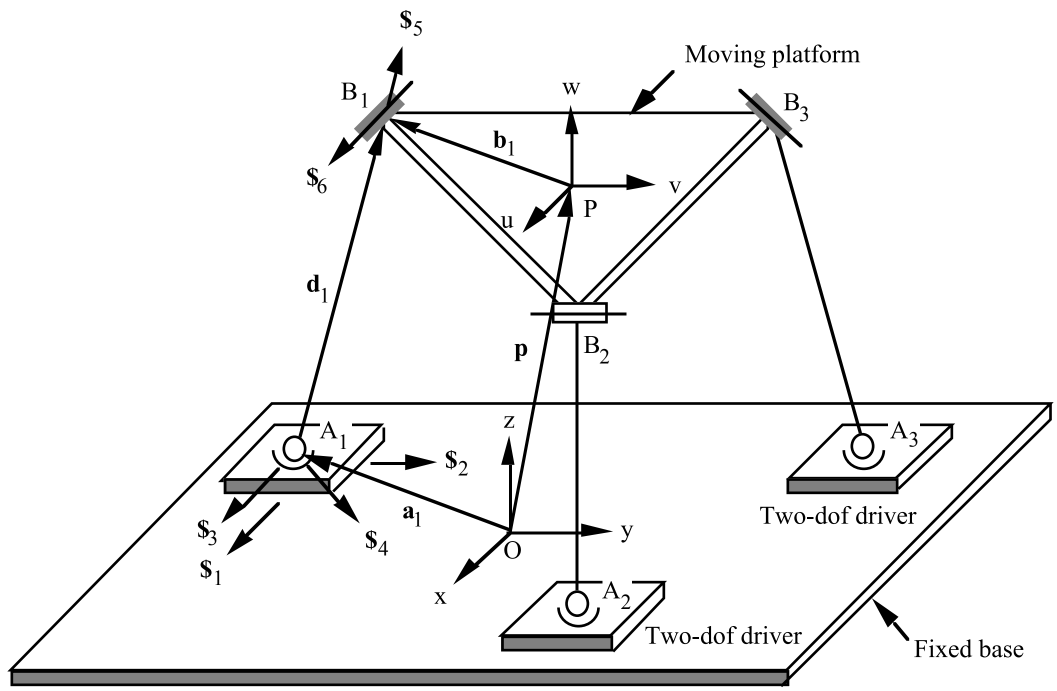

Figure 3: A parallel manipulator driven three planar motors.

$(d-\ell)$-system. Every screw in the $(d-\ell)$-system is reciprocal to the ℓ-system of twists. For a limb with k unactuated joints, there exists $(d-k)$ linearly independent screws that form a $(d-k)$-system. Every screw in the $(d-k)$-system is reciprocal to every unactuated joint screw of the limb. Obviously, the $(d-k)$-system contains the $(d-\ell)$-system. Any screw taken from the $(d-k)$-system, provided that it does not belong to the $(d-\ell)$-system, can be used as $\hat{\$}_{rj,i}$ to formulate of the Jacobian matrices. Hence, there are plenty of reciprocal screws to choose. Furthermore, if each of the reciprocal screws is reciprocal to all the joint screws, except for just one of the actuated joint screws, then J_q reduces to a diagonal matrix. In what follows, the Jacobian of a six-dof parallel manipulator is analyzed to illustrate the theory.

4 The Jacobian Of A 6-DOF Parallel Manipulator

Figure 3 shows a six-dof parallel manipulator developed by (Tsai and Tahmasebi, 1993). This manipulator is made up of a moving platform, a fixed base, and three inextensible limbs, A_iB_i for $i=1$ to 3. The upper end of each limb is connected to the moving platform by a revolute joint while the lower end is connected to a planar two-dof driver by a spherical joint. The three revolute joints lie on the $B_1B_2B_3$ plane. The manipulator motion is obtained by moving the planar motors on the fixed base.

As shown in Fig. 3, two Cartesian coordinate systems, (x,y,z) and (u,v,w), are attached to the fixed base and moving platform, respectively. The origin of the (u, v, w) frame is located at the centroid P of the moving platform with the u and v axes lying on the $B_1B_2B_3$ plane. In the equivalent limb, the spherical joint is replaced by three non-coplanar intersecting revolute joints, $\hat{\$}_{3,i}, \hat{\$}_{4,i}$, and $\hat{\$}_{5,i}$. The axis of $\hat{\$}_{3,i}$ is aligned with the x axis; the axis of $\hat{\$}_{5,i}$ is aligned with the longitudinal axis of the limb; and the axis of $\hat{\$}_{4,i}$ is perpendicular to both $\hat{\$}_{3,i}$ and $\hat{\$}_{5,i}$. Overall, there are six joint screws associated with each limb. The first two are actuated prismatic joints while the remaining four are passive revolute joints.

To facilitate the analysis, we define an instantaneous reference frame, (x', y', z'), with its origin is located at point P and the x', y' and z' axes parallel to the x, y and z axes of the fixed frame. Then, we express all the joint screws with respect to this instantaneous reference frame as follows:

$$\hat{\$}_{1,i} = \begin{bmatrix} \mathbf{0} \\ \mathbf{s}_{1,i} \end{bmatrix}, \quad \hat{\$}_{3,i} = \begin{bmatrix} \mathbf{s}_{3,i} \\ (\mathbf{b}_i - \mathbf{d}_i) \times \mathbf{s}_{3,i} \end{bmatrix}, \quad \hat{\$}_{5,i} = \begin{bmatrix} \mathbf{s}_{5,i} \\ \mathbf{b}_i \times \mathbf{s}_{5,i} \end{bmatrix},$$
$$\hat{\$}_{2,i} = \begin{bmatrix} \mathbf{0} \\ \mathbf{s}_{2,i} \end{bmatrix}, \quad \hat{\$}_{4,i} = \begin{bmatrix} \mathbf{s}_{4,i} \\ (\mathbf{b}_i - \mathbf{d}_i) \times \mathbf{s}_{4,i} \end{bmatrix}, \quad \hat{\$}_{6,i} = \begin{bmatrix} \mathbf{s}_{6,i} \\ \mathbf{b}_i \times \mathbf{s}_{6,i} \end{bmatrix},$$

where $\mathbf{s}_{1,i} = \mathbf{s}_{3,i} = [1, 0, 0]^T, \mathbf{s}_{2,i} = [0, 1, 0]^T$, $\mathbf{b}_i = \overline{PB_i}$, $\mathbf{d}_i = \overline{A_iB_i}$, $\mathbf{s}_{5,i} = \mathbf{d}_i/d_i$, and $\mathbf{s}_{4,i} = \mathbf{s}_{5,i} \times \mathbf{s}_{3,i}$.

We now consider each limb as an open-loop chain and express the instantaneous twist, $\$_p$, of the moving platform in terms of the joint screws of the ith limb.

$$\$_p = v_{x,i}\hat{\$}_{1,i} + v_{y,i}\hat{\$}_{2,i} + \dot{\theta}_{3,i}\hat{\$}_{3,i} + \cdots + \dot{\theta}_{6,i}\hat{\$}_{6,i}, \tag{8}$$

where $v_{x,i}$ and $v_{y,i}$ denote the linear velocities the planar motor along the x and y directions, respectively, and $\dot{\theta}_{3,i}, \cdots, \dot{\theta}_{6,i}$ denote the rotation rates about the passive joint axes.

Since there is a spherical joint at the lower end and a revolute joint at the upper end of each limb, all screws that are reciprocal to the unactuated joint screws form a planar pencil. These reciprocal screws pass through point A_i and lie on a plane H^i which contains point A_i and the axis of $\hat{\$}_{6,i}$.

Let $\mathbf{n}_i = [n_{x,i}, n_{y,i}, n_{z,i}]^T$ be a unit vector defined by the cross product of $\mathbf{s}_{5,i}$ and $\mathbf{s}_{6,i}$:

$$\mathbf{n}_i = \mathbf{s}_{5,i} \times \mathbf{s}_{6,i}. \tag{9}$$

Then, a screw that is reciprocal to all screws except for $\hat{\$}_{1,i}$ is obtained by an intersection of the H^i plane and the plane which contains A_i and is parallel to the $x - z$ plane:

$$\hat{\$}_{r1,i} = \begin{bmatrix} \mathbf{s}_{r1,i} \\ (\mathbf{b}_i - \mathbf{d}_i) \times \mathbf{s}_{r1,i} \end{bmatrix}, \quad \text{where } \mathbf{s}_{r1,i} = \frac{[n_{z,i} \;\; 0 \;\; -n_{x,i}]^T}{\sqrt{1 - n_{y,i}^2}}. \tag{10}$$

Similarly, a screw that is reciprocal to all screws except for $\hat{\$}_{2,i}$ is obtained by an intersection of the H^i plane and the plane which contains A_i and is parallel to the $y - z$ plane:

$$\hat{\$}_{r2,i} = \begin{bmatrix} \mathbf{s}_{r2,i} \\ (\mathbf{b}_i - \mathbf{d}_i) \times \mathbf{s}_{r2,i} \end{bmatrix}, \quad \text{where } \mathbf{s}_{r2,i} = \frac{[0 \;\; n_{z,i} \;\; -n_{y,i}]^T}{\sqrt{1 - n_{x,i}^2}}. \tag{11}$$

Taking the orthogonal products of both sides of Eq. (8) with (10) and (11), we obtain:

$$\begin{bmatrix} ((\mathbf{b}_i-\mathbf{d}_i)\times\mathbf{s}_{r1,i})^T & \mathbf{s}_{r1,i}^T \\ ((\mathbf{b}_i-\mathbf{d}_i)\times\mathbf{s}_{r2,i})^T & \mathbf{s}_{r2,i}^T \end{bmatrix} \begin{bmatrix} \bar{\omega}_p \\ \mathbf{v}_p \end{bmatrix} = \begin{bmatrix} \frac{n_{z,i}}{\sqrt{1-n_{y,i}^2}} & 0 \\ 0 & \frac{n_{z,i}}{\sqrt{1-n_{x,i}^2}} \end{bmatrix} \begin{bmatrix} v_{x,i} \\ v_{y,i} \end{bmatrix}. \tag{12}$$

Writing Eqs. (12) three times, once for each limb, we obtain:

$$J_x\dot{\mathbf{x}} = J_q\dot{\mathbf{q}}, \tag{13}$$

where $\dot{\mathbf{x}} = [\omega_{px}, \omega_{py}, \omega_{pz}, v_{px}, v_{py}, v_{pz}]^T$, $\dot{\mathbf{q}} = [v_{x,1}, v_{y,1}, v_{x,2}, v_{y,2}, v_{x,3}, v_{y,3}]^T$, and

$$J_x = \begin{bmatrix} ((\mathbf{b}_1-\mathbf{d}_1)\times\mathbf{s}_{r1,1})^T & \mathbf{s}_{r1,1}^T \\ ((\mathbf{b}_1-\mathbf{d}_1)\times\mathbf{s}_{r2,1})^T & \mathbf{s}_{r2,1}^T \\ ((\mathbf{b}_2-\mathbf{d}_2)\times\mathbf{s}_{r1,2})^T & \mathbf{s}_{r1,2}^T \\ ((\mathbf{b}_2-\mathbf{d}_2)\times\mathbf{s}_{r2,2})^T & \mathbf{s}_{r2,2}^T \\ ((\mathbf{b}_3-\mathbf{d}_3)\times\mathbf{s}_{r1,3})^T & \mathbf{s}_{r1,3}^T \\ ((\mathbf{b}_3-\mathbf{d}_3)\times\mathbf{s}_{r2,3})^T & \mathbf{s}_{r2,3}^T \end{bmatrix},$$

$$J_q = \begin{bmatrix} \frac{n_{z,1}}{\sqrt{1-n_{y,1}^2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{n_{z,1}}{\sqrt{1-n_{x,1}^2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{n_{z,2}}{\sqrt{1-n_{y,2}^2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{n_{z,2}}{\sqrt{1-n_{x,2}^2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{n_{z,3}}{\sqrt{1-n_{y,3}^2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{n_{z,3}}{\sqrt{1-n_{x,3}^2}} \end{bmatrix}.$$

An inverse kinematic singularity occurs when $n_{z,i} = 0$ for $i = 1$ or 2 or 3. Hence, when one of the limbs points in the z direction, the manipulator loses two degrees of freedom. If two or three limbs simultaneously point in the z direction, four or six degrees of freedom will be lost. A forward kinematic singularity occurs when $n_{x,i} = n_{y,i} = 0$ for all three limbs. Under such condition, the sixth column of J_x becomes zero identically and the manipulator gains one degree of freedom. Physically, this corresponds to the configuration when the moving platform and the three limbs lie on the $x-y$ plane. Another forward kinematic singularity occurs when the following three conditions are simultaneously satisfied: (1) $\triangle B_1B_2B_3$ is an equilateral triangle; (2) the three limbs are of equal length; and (3) A_1, A_2, and A_3 are placed directly under the centroid of the moving platform. When the above conditions are met, the manipulator gains three rotational degrees of freedom. A third direct kinematic singularity occurs when $n_{z,i} = 0$ for $i = 1, 2$ and 3. Under this condition, the fourth and fifth columns of J_x are both equal to zero, and the moving platform can make an infinitesimal translation along any direction which is parallel to the $x-y$ plane. In fact, this is a combined singularity since the manipulator is also under the inverse kinematic singulairy.

5 Summary

First, the definitions of a screw and its reciprocal screws are reviewed. Then, reciprocal screw systems associated with some frequently used kinematic pairs and chains are developed. Using these reciprocal screw systems, it is shown that the Jacobian of a parallel manipulator can be systematically derived. To illustrate the methodology, the Jacobian of a six-dof parallel manipulator is analyzed and its singular conditions are discussed.

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