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Stability Analysis of a Rigid Body with a Flexible Attachment Using the Energy-Casimir Method

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Abstract

We consider a system consisting of a rigid body to which a linear extensible shear beam is attached. For such a system the Energy-Casimir method can be used to investigate the stability of the equilibria. In the case we consider, it can be shown that a test for (formal) stability reduces to checking the positive definiteness of two matrices which depend on the parameters of the system and the particular equilibrium about which the stability is to be ascertained.

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1. Introduction

We consider a rigid body to which a long, flexible appendage is attached. A coordinate reference frame is fixed in the rigid body with the origin at the center of mass of the rigid body. The flexible attachment is assumed to lie along the second coordinate axis when the configuration is at rest. (see figure 1.) The equations of motion for such a configuration, under suitable assumptions and with the appendage modeled as a linear extensible shear beam, are derived by Krishnaprasad and Marsden in [2]. In deriving the equations of motion they use Hamiltonian methods in the context of Poisson manifolds and reduction. (see [2] for the explicit formula for the Poisson brackets involved.)

If we assume that the momentum of the system which arises from the appendage rotating with the rigid body is negligible, then our Hamiltonian is of the form

$$\mathbf{H} = \tfrac{1}{2}\mathbf{J}^{-1}\mathbf{p}\cdot\mathbf{p} + \tfrac{1}{2}\int_0^{\ell} \frac{\|\mathbf{m}(s)\|^2}{\rho_0}\,ds + \tfrac{1}{2}\int_0^{\ell} \mathbf{K}\frac{\partial \mathbf{r}}{\partial s}\cdot\frac{\partial \mathbf{r}}{\partial s}\,ds \tag{1.1}$$

We assume that $\mathbf{J}$ is the inertia matrix of the rigid body and that ρ_0 is the uniform mass per unit length of the attached appendage of length ℓ. The reduced phase space is coordinatized at any time by $\boldsymbol{\omega}$, the convected angular velocity vector of the rigid body; $\mathbf{r}(s)$, the convected displacement of the shear beam at a point s, $0 \le s \le \ell$; and $\mathbf{m}(s)$ the momentum density of shear beam at the point s. The vector $\mathbf{p}$ is the body angular momentum vector of the rigid body, thus $\mathbf{p} = \mathbf{J}\boldsymbol{\omega}$. Finally, $\mathbf{K}$ is the diagonal matrix of elastic coefficients.

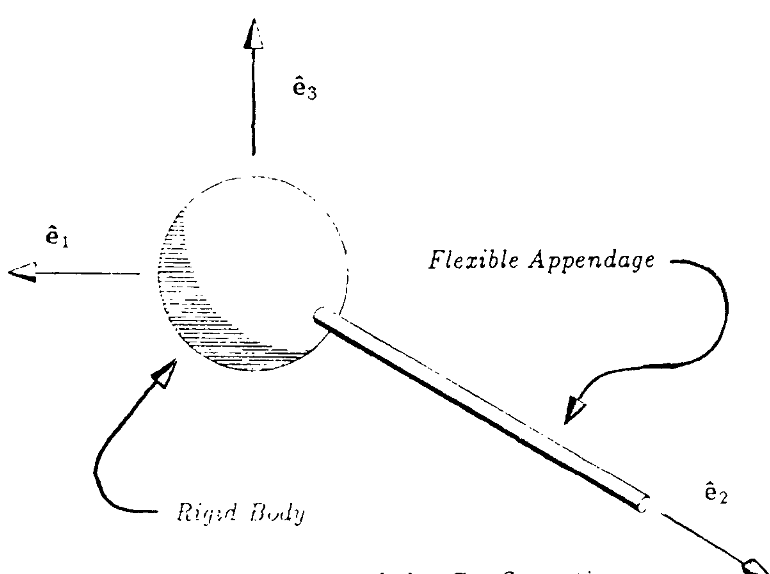

Figure 1. The Geometry of the Configuration

In our investigation we are interested in the stability of the system about equilibria points. These equilibria will satisfy,

$$0 = \mathbf{J}\boldsymbol{\omega} \times \boldsymbol{\omega} + \mathbf{a} \times \mathbf{K}\frac{\partial \mathbf{r}}{\partial s}\bigg|_{s=0} - \mathbf{r}(\ell) \times \mathbf{K}\,\hat{\mathbf{e}}_2 + \int_0^\ell \frac{\partial \mathbf{r}}{\partial s} \times \mathbf{K}\frac{\partial \mathbf{r}}{\partial s}\, ds \tag{1.2}$$

$$0 = \frac{1}{\rho_0}\mathbf{m} + \mathbf{r} \times \boldsymbol{\omega} \tag{1.3}$$

$$0 = \mathbf{K}\frac{\partial^2 \mathbf{r}}{\partial s^2} + \mathbf{m} \times \boldsymbol{\omega} \tag{1.4}$$

Two boundary values are associated with these equations,

$$\frac{\partial \mathbf{r}}{\partial s}\bigg|_{s=\ell} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \mathbf{e}_2, \qquad \text{and} \qquad \mathbf{r}\bigg|_{s=0} = \begin{bmatrix} 0 \\ a \\ 0 \end{bmatrix} = \mathbf{a} \tag{1.4}$$

In [2], a stability algorithm based on the Energy-Casimir method was applied to a specific family of equilibria (see section 4.2 below). The essence of the stability algorithm is to recognize that the relevant Poisson structure $\{\cdot\,,\,\cdot\}$ admits nontrivial Casimirs i.e. functions F that *Poisson–commute* with any function of the phase space. It follows that these are also conserved quantities for the dynamics of (1.1). Specific Casimirs $\mathbf{C}_\phi$ may be found such that the relative equilibria defined by (1.2) – (1.4) are critical points of $(\mathbf{H} + \mathbf{C}_\phi)$ on the reduced phase space. *Formal stability* follows from establishing definiteness conditions for the second variation $D^2(\mathbf{H} + \mathbf{C}_\phi)$ at the relative equilibria. To establish rigorous *nonlinear stability*, one has to carry out certain convexity estimates as in [2].

The purpose of this paper is to establish a *systematic* procedure for carrying out the *formal stability step* for arbitrary equilibria satisfying the equations (1.2) – (1.4). This has useful applications in the engineering context where the model at hand represents the mechanics of a spinning spacecraft with a flexible attachment (such as a boom for carrying instruments or an antenna). See [2] for related remarks and references. The procedure derived here recovers the results of [2] when applied to the specific example considered there. (see section 4.2 below.)

2. Computation of the First and Second Variations

In this section we compute the first and second variations of the Hamiltonian plus the Casimir function, $\mathbf{H} + \mathbf{C}_\phi$. From the previous definitions of these we know

$$\mathbf{H} = \tfrac{1}{2}\mathbf{J}^{-1}\mathbf{p}\cdot\mathbf{p} + \tfrac{1}{2}\int_0^\ell \frac{\|\mathbf{m}(s)\|^2}{\rho_0}\,ds + \tfrac{1}{2}\int_0^\ell \mathbf{K}\frac{\partial \mathbf{r}}{\partial s}\cdot\frac{\partial \mathbf{r}}{\partial s}\,ds, \tag{2.1}$$

and the Casimir function may be taken to be

$$\mathbf{C}_\phi = \tfrac{1}{2}\phi(\|\mathbf{p} + \int_0^\ell \mathbf{r}\times\mathbf{m}\,ds\|^2). \tag{2.2}$$

We will denote the first and second variations by $D(\mathbf{H} + \mathbf{C}_\phi)$, and $D^2(\mathbf{H} + \mathbf{C}_\phi)$. Note that because of the distributed nature of the system we are dealing with we will need to compute variational derivatives instead of ordinary gradients.

2.1. Computation of the First Variation

For the integrals in the Hamiltonian we consider variational differentials defined by

$$D\mathbf{f}(x) = \lim_{\epsilon\to 0}\frac{\mathbf{f}(x+\epsilon h) - \mathbf{f}(x)}{\epsilon} = \int_0^\ell \frac{\delta f}{\delta x}\cdot\delta x\,ds. \tag{2.3}$$

Thus, letting

$$f_2 = \tfrac{1}{2}\int_0^\ell \frac{\|\mathbf{m}(s)\|^2}{\rho_0}\,ds, \tag{2.4}$$

then

$$Df_2(\mathbf{m}) = \int_0^\ell \frac{1}{\rho_0}\mathbf{m}\cdot\delta\mathbf{m}\,ds. \tag{2.5}$$

Similarly, let

$$f_3(\mathbf{r}) = \tfrac{1}{2}\int_0^\ell \mathbf{K}\frac{\partial \mathbf{r}}{\partial s}\cdot\frac{\partial \mathbf{r}}{\partial s}\,ds, \tag{2.6}$$

$$Df_3(\mathbf{r}) = \int_0^\ell \mathbf{K}\frac{\partial \mathbf{r}}{\partial s}\cdot\frac{\partial \delta\mathbf{r}}{\partial s}\,ds. \tag{2.7}$$

If we integrate this equation by parts with the boundary conditions $\delta\mathbf{r}(\ell) = \delta\mathbf{r}(0) = 0$, we get

$$Df_3(\mathbf{r}) = -\int_0^\ell \mathbf{K}\frac{\partial^2 \mathbf{r}}{\partial s^2}\cdot\delta\mathbf{r}\,ds. \tag{2.8}$$

For the integral term in the Casimir function we are taking variational derivatives of a cross product term. If we define

$$||\boldsymbol{\alpha}||^2 = ||\mathbf{p} + \int_0^\ell \mathbf{r} \times \mathbf{m}\, ds||^2, \tag{2.9}$$

then

$$D||\boldsymbol{\alpha}||^2 = 2\boldsymbol{\alpha} \cdot (\delta\mathbf{p} + \int_0^\ell \mathbf{r} \times \delta\mathbf{m}\, ds + \int_0^\ell \delta\mathbf{r} \times \mathbf{m}\, ds). \tag{2.10}$$

If we combine all of the above we get the expression for the first variation

$$\begin{aligned} D(\mathbf{H} + \mathbf{C}_\phi) = \mathbf{J}^{-1}\mathbf{p} \cdot \delta\mathbf{p} + \int_0^\ell \frac{1}{\rho_0}\mathbf{m} \cdot \delta\mathbf{m}\, ds - \int_0^\ell \mathbf{K}\frac{\partial^2 \mathbf{r}}{\partial s^2} \cdot \delta\mathbf{r}\, ds \\ + \phi'(||\boldsymbol{\alpha}||^2)\, \boldsymbol{\alpha} \cdot (\delta\mathbf{p} + \int_0^\ell \mathbf{r} \times \delta\mathbf{m}\, ds + \int_0^\ell \delta\mathbf{r} \times \mathbf{m}\, ds). \end{aligned} \tag{2.11}$$

2.2. Computation of the Second Variation

For the second variation, the starting point is the expression for the first variation. The terms arising from the original Hamiltonian are straight forward to compute, they are

$$D(\mathbf{J}^{-1}\mathbf{p} \cdot \delta\mathbf{p}) = \mathbf{J}^{-1}\delta\mathbf{p} \cdot \delta\mathbf{p}, \tag{2.12}$$

$$D(\int_0^\ell \frac{1}{\rho_0}\mathbf{m} \cdot \delta\mathbf{m}\, ds) = \int_0^\ell \frac{1}{\rho_0}\delta\mathbf{m} \cdot \delta\mathbf{m}\, ds, \tag{2.13}$$

$$D(\int_0^\ell \mathbf{K}\frac{\partial^2 \mathbf{r}}{\partial s^2} \cdot \delta\mathbf{r}\, ds) = \int_0^\ell \mathbf{K}\frac{\partial^2 \delta\mathbf{r}}{\partial s^2} \cdot \delta\mathbf{r}\, ds. \tag{2.14}$$

Note that we can use the boundary conditions on $\delta\mathbf{r}$ to get

$$\int_0^\ell \mathbf{K}\frac{\partial^2 \delta\mathbf{r}}{\partial s^2} \cdot \delta\mathbf{r}\, ds = -\int_0^\ell \mathbf{K}\frac{\partial \delta\mathbf{r}}{\partial s} \cdot \frac{\partial \delta\mathbf{r}}{\partial s}\, ds. \tag{2.15}$$

Next we consider the component which arises from the Casimir function which we added to the Hamiltonian. From the first factor of this term we compute,

$$D\phi'(\boldsymbol{\alpha}) = 2\,\phi''(||\boldsymbol{\alpha}||^2)\, \boldsymbol{\alpha} \cdot (\delta\mathbf{p} + \int_0^\ell \mathbf{r} \times \delta\mathbf{m}\, ds + \int_0^\ell \delta\mathbf{r} \times \mathbf{m}\, ds). \tag{2.16}$$

From the second factor of the Casimir term we compute

$$\begin{aligned} D(\boldsymbol{\alpha} \cdot (\delta\mathbf{p} + \int_0^\ell \mathbf{r} \times \delta\mathbf{m}\, ds + \int_0^\ell \delta\mathbf{r} \times \mathbf{m}\, ds)) = \\ ||\delta\mathbf{p} + \int_0^\ell \mathbf{r} \times \delta\mathbf{m}\, ds + \int_0^\ell \delta\mathbf{r} \times \mathbf{m}\, ds||^2 \\ + 2(\mathbf{p} + \int_0^\ell \mathbf{r} \times \mathbf{m}\, ds) \cdot (\int_0^\ell \delta\mathbf{r} \times \delta\mathbf{m}\, ds). \end{aligned} \tag{2.17}$$

We use the above to get the expression for the second variation

$$
\begin{aligned}
D^2(\mathbf{H}+\mathbf{C}_\phi) = \mathbf{J}^{-1}\delta\mathbf{p}\cdot\delta\mathbf{p} &+ \int_0^\ell \frac{1}{\rho_0}\delta\mathbf{m}\cdot\delta\mathbf{m}\,ds + \int_0^\ell \mathbf{K}\frac{\partial\delta\mathbf{r}}{\partial s}\cdot\frac{\partial\delta\mathbf{r}}{\partial s}\,ds \\
&+ 2\,\phi''(\|\boldsymbol{\alpha}\|^2)\,\left(\boldsymbol{\alpha}\cdot(\delta\mathbf{p}+\int_0^\ell \mathbf{r}\times\delta\mathbf{m}\,ds+\int_0^\ell \delta\mathbf{r}\times\mathbf{m}\,ds)\right)^2 \\
&+ \phi'(\|\boldsymbol{\alpha}\|^2)\{\|\delta\mathbf{p}+\int_0^\ell \mathbf{r}\times\delta\mathbf{m}\,ds+\int_0^\ell \delta\mathbf{r}\times\mathbf{m}\,ds\|^2 \\
&+ 2(\mathbf{p}+\int_0^\ell \mathbf{r}\times\mathbf{m}\,ds)\cdot(\int_0^\ell \delta\mathbf{r}\times\delta\mathbf{m}\,ds)\}. \qquad (2.18)
\end{aligned}
$$

3. Computation of a Stability Criterion

The conditions which assure that the first variation $D(\mathbf{H}+\mathbf{C}_\phi)$ at an equilibrium is zero are

$$\phi'(\|\boldsymbol{\alpha}^e\|^2)\boldsymbol{\alpha}^e = -\boldsymbol{\omega}^e, \qquad (3.1)$$

$$\phi'(\|\boldsymbol{\alpha}^e\|^2)\,\boldsymbol{\alpha}^e\times\mathbf{r}^e = -\frac{1}{\rho_0}\mathbf{m}^e, \qquad (3.2)$$

$$\phi'(\|\boldsymbol{\alpha}^e\|^2)\,\boldsymbol{\alpha}^e\times\mathbf{m}^e = -\mathbf{K}\frac{\partial^2\mathbf{r}^e}{\partial s^2}, \qquad (3.3)$$

where $\boldsymbol{\omega}^e = \mathbf{J}^{-1}\mathbf{p}^e$, and

$$\boldsymbol{\alpha}^e = \mathbf{p}^e + \int_0^\ell \mathbf{r}^e\times\mathbf{m}^e\,ds. \qquad (3.4)$$

We use the superscript e to denote evaluation at an equilibrium. If we dot (3.1) with $\boldsymbol{\alpha}^e$ we have

$$\phi'(\|\boldsymbol{\alpha}^e\|^2) = -\frac{\boldsymbol{\omega}^e\cdot\boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2} \qquad (3.5)$$

If we evaluate the first variation at an equilibrium, incorporating the above, then we can derive conditions which assure the stability of the equilibrium. In the following sequence of steps we demonstrate how this is done.

Step 1 : Evaluate the Second Variation at an Equilibrium

Recall the second variation. If we use the above to substitute for $\phi'(\|\boldsymbol{\alpha}^e\|^2)$ in this expression and rearrange slightly we find that

$$D^2(\mathbf{H}+\mathbf{C}_\phi)_{(\mathbf{p}^e,\mathbf{r}^e,\mathbf{m}^e)} = \mathbf{J}^{-1}\delta\mathbf{p}\cdot\delta\mathbf{p} + \int_0^\ell \frac{1}{\rho_0}\delta\mathbf{m}\cdot\delta\mathbf{m}\,ds + \int_0^\ell \mathbf{K}\frac{\partial\delta\mathbf{r}}{\partial s}\cdot\frac{\partial\delta\mathbf{r}}{\partial s}\,ds$$

$$
\begin{aligned}
&- \frac{\boldsymbol{\omega}^e \cdot \boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2} \, \|\delta \mathbf{p} + \int_0^\ell \mathbf{r}^e \times \delta \mathbf{m} \, ds + \int_0^\ell \delta \mathbf{r} \times \mathbf{m}^e \, ds\|^2 \\
&- 2 \, \frac{\boldsymbol{\omega}^e \cdot \boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2} \, \boldsymbol{\alpha}^e \cdot (\int_0^\ell \delta \mathbf{r} \times \delta \mathbf{m} \, ds) \\
&+ 2 \, \phi''(\|\boldsymbol{\alpha}^e\|^2) \, (\boldsymbol{\alpha}^e \cdot (\delta \mathbf{p} + \int_0^\ell \mathbf{r}^e \times \delta \mathbf{m} \, ds + \int_0^\ell \delta \mathbf{r} \times \mathbf{m}^e \, ds))^2,
\end{aligned}
\tag{3.6}
$$

which corresponds to expression (5.5) in Krishnaprasad and Marsden [2]. In that paper, ϕ is required to satisfy the condition:

$$
\phi''(\|\boldsymbol{\alpha}^e\|^2) = \frac{\boldsymbol{\omega}^e \cdot \boldsymbol{\alpha}^e}{2 \, \|\boldsymbol{\alpha}^e\|^4}
\tag{3.7}
$$

which is consistent with (3.5). In the following development we impose no conditions on $\phi''(\|\boldsymbol{\alpha}^e\|^2)$ at this time.

Step 2: Expand Terms Containing δp

We first note that the fourth and sixth terms in (3.6) can be expanded. For the fourth term we have

$$
\begin{aligned}
- \frac{\boldsymbol{\omega}^e \cdot \boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2} \, &\|\delta \mathbf{p} + \int_0^\ell \mathbf{r}^e \times \delta \mathbf{m} \, ds + \int_0^\ell \delta \mathbf{r} \times \mathbf{m}^e \, ds\|^2 \\
&= - \frac{\boldsymbol{\omega}^e \cdot \boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2} \delta \mathbf{p} \cdot \delta \mathbf{p} \\
&\quad - 2 \, \frac{\boldsymbol{\omega}^e \cdot \boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2} \delta \mathbf{p} \cdot (\int_0^\ell \mathbf{r}^e \times \delta \mathbf{m} \, ds + \int_0^\ell \delta \mathbf{r} \times \mathbf{m}^e \, ds) \\
&\quad - \frac{\boldsymbol{\omega}^e \cdot \boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2} \| \int_0^\ell \mathbf{r}^e \times \delta \mathbf{m} \, ds + \int_0^\ell \delta \mathbf{r} \times \mathbf{m}^e \, ds\|^2
\end{aligned}
\tag{3.8}
$$

while for the sixth term

$$
\begin{aligned}
2 \, \phi''(\|\boldsymbol{\alpha}^e\|^2) \, &(\boldsymbol{\alpha}^e \cdot (\delta \mathbf{p} + \int_0^\ell \mathbf{r}^e \times \delta \mathbf{m} \, ds + \int_0^\ell \delta \mathbf{r} \times \mathbf{m}^e \, ds))^2 \\
&= 2 \phi''(\|\boldsymbol{\alpha}^e\|^2)(\boldsymbol{\alpha}^e \cdot \delta \mathbf{p})^2 \\
&\quad + 4 \phi''(\|\boldsymbol{\alpha}^e\|^2)(\boldsymbol{\alpha}^e \cdot \delta \mathbf{p}) \, (\boldsymbol{\alpha}^e \cdot (\int_0^\ell \mathbf{r}^e \times \delta \mathbf{m} \, ds + \int_0^\ell \delta \mathbf{r} \times \mathbf{m}^e \, ds)) \\
&\quad + 2 \phi''(\|\boldsymbol{\alpha}^e\|^2)(\boldsymbol{\alpha}^e \cdot (\int_0^\ell \mathbf{r}^e \times \delta \mathbf{m} \, ds + \int_0^\ell \delta \mathbf{r} \times \mathbf{m}^e \, ds))^2 .
\end{aligned}
\tag{3.9}
$$

Step 3: Collect Terms Containing $\delta\mathbf{p}$

Now, collect together terms in which the quantity $\delta\mathbf{p}$ appears. Our expression for the second variation at an equilibrium can then be written

$$\begin{aligned} D^2(\mathbf{H}+\mathbf{C}_\phi) = \Big[&\mathbf{J}^{-1}\delta\mathbf{p}\cdot\delta\mathbf{p} - \frac{\boldsymbol{\omega}^e\cdot\boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}(\delta\mathbf{p}\cdot\delta\mathbf{p} + 2\delta\mathbf{p}\cdot(\int_0^\ell \mathbf{r}^e\times\delta\mathbf{m}\,ds + \int_0^\ell \delta\mathbf{r}\times\mathbf{m}^e\,ds)) \\ &+ 2\phi''(\|\boldsymbol{\alpha}^e\|^2)\{(\boldsymbol{\alpha}^e\cdot\delta\mathbf{p})^2 + 2(\boldsymbol{\alpha}^e\cdot\delta\mathbf{p})(\boldsymbol{\alpha}^e\cdot(\int_0^\ell \mathbf{r}^e\times\delta\mathbf{m}\,ds + \int_0^\ell \delta\mathbf{r}\times\mathbf{m}^e\,ds))\}\Big] \\ &- \frac{\boldsymbol{\omega}^e\cdot\boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}\|\int_0^\ell \mathbf{r}^e\times\delta\mathbf{m}\,ds + \int_0^\ell \delta\mathbf{r}\times\mathbf{m}^e\,ds\|^2 \\ &+ 2\phi''(\|\boldsymbol{\alpha}^e\|^2)(\boldsymbol{\alpha}^e\cdot(\int_0^\ell \mathbf{r}^e\times\delta\mathbf{m}\,ds + \int_0^\ell \delta\mathbf{r}\times\mathbf{m}^e\,ds))^2 \\ &- 2\frac{\boldsymbol{\omega}^e\cdot\boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}\boldsymbol{\alpha}^e\cdot(\int_0^\ell \delta\mathbf{r}\times\delta\mathbf{m}\,ds) + \int_0^\ell \frac{1}{\rho_0}\delta\mathbf{m}\cdot\delta\mathbf{m}\,ds + \int_0^\ell \mathbf{K}\frac{\partial\delta\mathbf{r}}{\partial s}\cdot\frac{\partial\delta\mathbf{r}}{\partial s}\,ds \end{aligned} \tag{3.10}$$

Step 4: Complete the Square

The term in square brackets which contains the $\delta\mathbf{p}$ terms can be rewritten

$$\begin{aligned} \Big[\ \cdot\ \Big] = &(\mathbf{J}^{-1} - \frac{\boldsymbol{\omega}^e\cdot\boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}\mathbf{I} + 2\phi''(\|\boldsymbol{\alpha}^e\|^2)\boldsymbol{\alpha}^e\otimes\boldsymbol{\alpha}^e)\delta\mathbf{p}\cdot\delta\mathbf{p} \\ &+ 2(-\frac{\boldsymbol{\omega}^e\cdot\boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}\mathbf{I} + 2\phi''(\|\boldsymbol{\alpha}^e\|^2)\boldsymbol{\alpha}^e\otimes\boldsymbol{\alpha}^e)\delta\mathbf{p}\cdot(\int_0^\ell \mathbf{r}^e\times\delta\mathbf{m}\,ds + \int_0^\ell \delta\mathbf{r}\times\mathbf{m}^e\,ds) \end{aligned} \tag{3.11}$$

In this expression we use $\otimes$ to denote the tensor product and $\mathbf{I}$ the identity. Note that $\boldsymbol{\alpha}^e\otimes\boldsymbol{\alpha}^e$ is a tensor of rank 2. We can complete the square for this expression provided the quantity

$$\mathbf{J}^{-1} - \frac{\boldsymbol{\omega}^e\cdot\boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}\mathbf{I} + 2\phi''(\|\boldsymbol{\alpha}^e\|^2)\boldsymbol{\alpha}^e\otimes\boldsymbol{\alpha}^e \tag{3.12}$$

has an inverse.

We next assume this inverse exists and define the two symmetric matrices $\mathbf{M}$ and $\mathbf{N}$ by,

$$\begin{aligned} \mathbf{M}^T\mathbf{M} &= \mathbf{J}^{-1} - \frac{\boldsymbol{\omega}^e\cdot\boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}\mathbf{I} + 2\phi''(\|\boldsymbol{\alpha}^e\|^2)\boldsymbol{\alpha}^e\otimes\boldsymbol{\alpha}^e \\ &\triangleq \mathbf{J}_e^{-1} \end{aligned} \tag{3.13}$$

$$\begin{aligned} \mathbf{N}^T\mathbf{M} &= -\frac{\boldsymbol{\omega}^e\cdot\boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}\mathbf{I} + 2\phi''(\|\boldsymbol{\alpha}^e\|^2)\boldsymbol{\alpha}^e\otimes\boldsymbol{\alpha}^e \\ &\triangleq \mathbf{Q}_e \end{aligned} \tag{3.14}$$

Completing the square for the term in brackets we now get

$$\left[\ \cdot\ \right] = \| \mathbf{M}\delta\mathbf{p} + \mathbf{N}(\int_0^\ell \mathbf{r}^e \times \delta\mathbf{m}\, ds + \int_0^\ell \delta\mathbf{r} \times \mathbf{m}^e\, ds)\ \|^2$$
$$- \mathbf{N}^T\mathbf{N}(\int_0^\ell \mathbf{r}^e \times \delta\mathbf{m}\, ds + \int_0^\ell \delta\mathbf{r} \times \mathbf{m}^e\, ds) \cdot (\int_0^\ell \mathbf{r}^e \times \delta\mathbf{m}\, ds + \int_0^\ell \delta\mathbf{r} \times \mathbf{m}^e\, ds) \tag{3.15}$$

The term in braces is bounded below by a perfect square when $\mathbf{N}^T\mathbf{N} \geq \mathbf{0}$. For this to be the case we need to assume that the inverted matrix, $\mathbf{J}_e^{-1}$ is positive definite, in general it need not be. Note that this assumption will impose conditions on $\phi''(\|\boldsymbol{\alpha}^e\|^2)$. The requirements on the parameters in this matrix to assure it is strictly positive definite will be expressed in the form of inequalities. These inequalities will be the first conditions that we need to assure stability.

Step 5: The Reformulated Second Variation

The second variation at an equilibrium is thus of the form

$$\begin{aligned}
D^2(\mathbf{H} + \mathbf{C}_\phi) = &\quad (square) \\
& - \mathbf{N}^T\mathbf{N}(\int_0^\ell \mathbf{r}^e \times \delta\mathbf{m}\, ds + \int_0^\ell \delta\mathbf{r} \times \mathbf{m}^e\, ds) \\
&\qquad \cdot (\int_0^\ell \mathbf{r}^e \times \delta\mathbf{m}\, ds + \int_0^\ell \delta\mathbf{r} \times \mathbf{m}^e\, ds) \\
& - \frac{\boldsymbol{\omega}^e \cdot \boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2} \|(\int_0^\ell \mathbf{r}^e \times \delta\mathbf{m}\, ds + \int_0^\ell \delta\mathbf{r} \times \mathbf{m}^e\, ds)\|^2 \\
& + 2\phi''(\|\boldsymbol{\alpha}^e\|^2)\boldsymbol{\alpha}^e \otimes \boldsymbol{\alpha}^e(\int_0^\ell \mathbf{r}^e \times \delta\mathbf{m}\, ds + \int_0^\ell \delta\mathbf{r} \times \mathbf{m}^e\, ds) \\
&\qquad \cdot (\int_0^\ell \mathbf{r}^e \times \delta\mathbf{m}\, ds + \int_0^\ell \delta\mathbf{r} \times \mathbf{m}^e\, ds) \\
& - 2\frac{\boldsymbol{\omega}^e \cdot \boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}\boldsymbol{\alpha}^e \cdot (\int_0^\ell \delta\mathbf{r} \times \delta\mathbf{m}\, ds) + \int_0^\ell \frac{1}{\rho_0}\delta\mathbf{m} \cdot \delta\mathbf{m}\, ds + \int_0^\ell \mathbf{K}\frac{\partial\delta\mathbf{r}}{\partial s} \cdot \frac{\partial\delta\mathbf{r}}{\partial s}\, ds
\end{aligned} \tag{3.16}$$

Where we note that

$$\mathbf{N}^T\mathbf{N} = (-\frac{\boldsymbol{\omega}^e \cdot \boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}\mathbf{I} + 2\phi''(\|\boldsymbol{\alpha}^e\|^2)\boldsymbol{\alpha}^e \otimes \boldsymbol{\alpha}^e)$$

$$(\mathbf{J}^{-1} - \frac{\boldsymbol{\omega}^e \cdot \boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}\mathbf{I} + 2\phi''(\|\boldsymbol{\alpha}^e\|^2)\boldsymbol{\alpha}^e \otimes \boldsymbol{\alpha}^e)^{-1}(-\frac{\boldsymbol{\omega}^e \cdot \boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}\mathbf{I} + 2\phi''(\|\boldsymbol{\alpha}^e\|^2)\boldsymbol{\alpha}^e \otimes \boldsymbol{\alpha}^e)$$

$$= \mathbf{Q}_e\mathbf{J}_e\mathbf{Q}_e \tag{3.17}$$

Step 6: Collect Integrals of Cross Products

Collecting terms containing the integrals of cross products the second variation can be written

$$\begin{aligned} D^2(\mathbf{H} + \mathbf{C}_\phi) = \quad & (square) \\ & - (\mathbf{Q}_e\mathbf{J}_e\mathbf{Q}_e - \mathbf{Q}_e)\Big(\int_0^\ell \mathbf{r}^e \times \delta\mathbf{m}\, ds + \int_0^\ell \delta\mathbf{r} \times \mathbf{m}^e\, ds\Big) \\ & \qquad \cdot \Big(\int_0^\ell \mathbf{r}^e \times \delta\mathbf{m}\, ds + \int_0^\ell \delta\mathbf{r} \times \mathbf{m}^e\, ds\Big) \\ & - 2\frac{\boldsymbol{\omega}^e \cdot \boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}\boldsymbol{\alpha}^e \cdot \Big(\int_0^\ell \delta\mathbf{r} \times \delta\mathbf{m}\, ds\Big) + \int_0^\ell \frac{1}{\rho_0}\delta\mathbf{m} \times \delta\mathbf{m}\, ds + \int_0^\ell \mathbf{K}\frac{\partial \delta\mathbf{r}}{\partial s} \cdot \frac{\partial \delta\mathbf{r}}{\partial s}\, ds \end{aligned} \tag{3.18}$$

Step 7: A Vector Identity

Observe that a simple vector identity enables us to write

$$\begin{aligned} 2\frac{\boldsymbol{\omega}^e \cdot \boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}\boldsymbol{\alpha}^e \cdot \Big(\int_0^\ell \delta\mathbf{r} \times \delta\mathbf{m}\, ds\Big) &= 2\int_0^\ell \Big(\frac{\boldsymbol{\alpha}^e \cdot \boldsymbol{\omega}^e}{\|\boldsymbol{\alpha}^e\|^2}\boldsymbol{\alpha}^e \times \delta\mathbf{r}\Big) \cdot \delta\mathbf{m}\, ds \\ &= 2\int_0^\ell \delta\mathbf{m}^T\mathbf{S}\Big(\frac{(\boldsymbol{\alpha}^e \otimes \boldsymbol{\alpha}^e)\boldsymbol{\omega}^e}{\|\boldsymbol{\alpha}^e\|^2}\Big)\delta\mathbf{r}\, ds \end{aligned} \tag{3.19}$$

where we have used the skew-symmetric matrix $S(\mathbf{x})$ associated with the cross-product

$$\mathbf{S}(\mathbf{x}) = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \tag{3.20}$$

Step 8: A Quadratic Form

Now define the symmetric matrix

$$\mathbf{R} \stackrel{\triangle}{=} \mathbf{Q}_e\mathbf{J}_e\mathbf{Q}_e - \mathbf{Q}_e \tag{3.21}$$

We will see below, that an eigenvalue estimate (3.23) relies on having $\mathbf{R}$ nonnegative definite. We thus require that conditions on the parameters of the problem and $\phi''(\|\boldsymbol{\alpha}^e\|^2)$ hold such that $\mathbf{J}_e^{-1}$ defined in (3.13) is positive definite *and* $\mathbf{R}$ defined in (3.21) is nonnegative definite. The latter will assure that $\mathbf{R}$ has a square root $\mathbf{R}^{1/2}$. We will examine these assumptions again in remark 2 below.

Expanding the second term in (3.18), we can re-express it as a quadratic form,

$$\begin{aligned}
\mathbf{R}(\int_0^\ell \mathbf{r}^e \times \delta\mathbf{m}\, ds + \int_0^\ell \delta\mathbf{r} \times \mathbf{m}^e\, ds) \cdot (\int_0^\ell \mathbf{r}^e \times \delta\mathbf{m}\, ds + \int_0^\ell \delta\mathbf{r} \times \mathbf{m}^e\, ds) \\
= \int_0^\ell \int_0^\ell \mathbf{R}(\mathbf{S}(\mathbf{r}^e(s))\delta\mathbf{m}(s) - \mathbf{S}(\mathbf{m}^e(s))\delta\mathbf{r}(s)) \cdot (\mathbf{S}(\mathbf{r}^e(\sigma))\delta\mathbf{m}(\sigma) - \mathbf{S}(\mathbf{m}^e(\sigma))\delta\mathbf{r}(\sigma))\, ds\, d\sigma \\
= \int_0^\ell \int_0^\ell [\,\delta\mathbf{m}^T(s) \quad \delta\mathbf{r}^T(s)\,] \begin{bmatrix} \mathbf{S}^T(\mathbf{r}^e(s)) \\ -\mathbf{S}^T(\mathbf{m}^e(s)) \end{bmatrix} \mathbf{R}\, [\,\mathbf{S}(\mathbf{r}^e(\sigma)) \quad -\mathbf{S}(\mathbf{m}^e(\sigma))\,] \begin{bmatrix} \delta\mathbf{m}(\sigma) \\ \delta\mathbf{r}(\sigma) \end{bmatrix} ds\, d\sigma \\
= \int_0^\ell \int_0^\ell [\,\delta\mathbf{m}^T(s) \quad \delta\mathbf{r}^T(s)\,] \mathbf{A}^T(s)\mathbf{A}(\sigma) \begin{bmatrix} \delta\mathbf{m}(\sigma) \\ \delta\mathbf{r}(\sigma) \end{bmatrix} ds\, d\sigma \qquad (3.22)
\end{aligned}$$

We now can find a lower bound on the above. The bound we want is obtained from an eigenvalue inequality which we introduce by way of the following lemma

Step 8: An Eigenvalue Inequality

Lemma(3.1): Let $\mathbf{A}(s) \in L_2^{n\times n}(0,\ell)$, and $\mathbf{x}(s) \in L_2^n(0,\ell)$ then

$$\int_0^\ell \int_0^\ell \mathbf{x}^T(s)\mathbf{A}^T(s)\mathbf{A}(\sigma)\mathbf{x}(\sigma)\, d\sigma\, ds \leq \int_0^\ell \mathbf{x}^T(s)\{\int_0^\ell \mathbf{I}\lambda^2(\sigma)\, d\sigma\}\mathbf{x}(s)\, ds, \qquad (3.23)$$

where $\lambda^2(s)$ is the maximum eigenvalue of $\mathbf{A}^T(s)\mathbf{A}(s)$.

Proof: Let $||\cdot||$ denote the standard norm in Euclidean space and also the induced matrix norm associated with it. Then

$$\begin{aligned}
\int_0^\ell \int_0^\ell \mathbf{x}^T(s)\mathbf{A}^T(s)\mathbf{A}(\sigma)\mathbf{x}(\sigma)\, d\sigma\, ds &\leq \int_0^\ell \int_0^\ell |\mathbf{x}^T(s)\mathbf{A}^T(s)\mathbf{A}(\sigma)\mathbf{x}(\sigma)|\, d\sigma\, ds \\
&\leq \int_0^\ell \int_0^\ell ||\mathbf{A}(s)\mathbf{x}(s)||\, ||\mathbf{A}(\sigma)\mathbf{x}(\sigma)||\, d\sigma\, ds \\
&\leq \int_0^\ell ||\mathbf{A}(s)||\, ||\mathbf{x}(s)||\, ds \int_0^\ell ||\mathbf{A}(\sigma)||\, ||\mathbf{x}(\sigma)||\, d\sigma
\end{aligned}$$

where we have used $||\mathbf{A}(s)\mathbf{x}(s)|| \leq ||\mathbf{A}(s)||\, ||\mathbf{x}(s)||$. We can now use the Schwarz inequality

$$(\int_0^\ell ||\mathbf{A}(s)||\, ||\mathbf{x}(s)||\, ds)^2 \leq \int_0^\ell ||\mathbf{A}(s)||^2\, ds \int_0^\ell ||\mathbf{x}(s)||^2\, ds$$

Finally noting that the value of $||\mathbf{A}(s)||$ is simply the square root of the maximum eigenvalue of $\mathbf{A}^T(s)\mathbf{A}(s)$ establishes the result. ▮

If we let $\lambda^2(s)$ be the maximum eigenvalue of

$$\mathbf{A}^T(s)\mathbf{A}(s) = \begin{bmatrix} \mathbf{S}^T(\mathbf{r}^e(s))\mathbf{R}\mathbf{S}(\mathbf{r}^e(s)) & -\mathbf{S}^T(\mathbf{r}^e(s))\mathbf{R}\mathbf{S}(\mathbf{m}^e(s)) \\ -\mathbf{S}^T(\mathbf{m}^e(s))\mathbf{R}\mathbf{S}(\mathbf{r}^e(s)) & \mathbf{S}^T(\mathbf{m}^e(s))\mathbf{R}\mathbf{S}(\mathbf{m}^e(s)) \end{bmatrix} \tag{3.24}$$

and let $\tilde{\lambda}^2 = \int_0^\ell \lambda^2(s)\,ds$ then we have by way of lemma 3.1 a lower bound on the second variation

$$\begin{aligned} D^2(\mathbf{H}+\mathbf{C}_\phi)_{(\mathbf{p}^e,\mathbf{r}^e,\mathbf{m}^e)} \geq\ & (square) \\ & - \tilde{\lambda}^2 \int_0^\ell \delta\mathbf{m}^T\delta\mathbf{m}\,ds - \tilde{\lambda}^2 \int_0^\ell \delta\mathbf{r}^T\delta\mathbf{r}\,ds \\ & - 2\int_0^\ell \delta\mathbf{m}^T\mathbf{S}\left(\frac{(\boldsymbol{\alpha}^e\otimes\boldsymbol{\alpha}^e)\boldsymbol{\omega}^e}{||\boldsymbol{\alpha}^e||^2}\right)\delta\mathbf{r}\,ds \\ & + \int_0^\ell \frac{1}{\rho_0}\delta\mathbf{m}^T\delta\mathbf{m}\,ds + \int_0^\ell \mathbf{K}\frac{\partial\delta\mathbf{r}}{\partial s}\cdot\frac{\partial\delta\mathbf{r}}{\partial s}\,ds \end{aligned} \tag{3.25}$$

Step 9: A Poincaré Type Inequality

If we assume that $\mathbf{K}$ is diagonal and use a Poincaré-type inequality

$$\int_0^\ell \mathbf{K}\frac{\partial\delta\mathbf{r}}{\partial s}\cdot\frac{\partial\delta\mathbf{r}}{\partial s}\,ds \geq c\int_0^\ell \mathbf{K}\delta\mathbf{r}\cdot\delta\mathbf{r}\,ds, \tag{3.26}$$

with $c = (\frac{\pi}{2\ell})^2$, then the second variation can be bounded below as

$$\begin{aligned} D^2(\mathbf{H}+\mathbf{C}_\phi)_{(\mathbf{p}^e,\mathbf{r}^e,\mathbf{m}^e)} \geq\ & (\quad square \quad) \\ & - \tilde{\lambda}^2 \int_0^\ell \delta\mathbf{m}^T\delta\mathbf{m}\,ds - \tilde{\lambda}^2 \int_0^\ell \delta\mathbf{r}^T\delta\mathbf{r}\,ds \\ & - 2\int_0^\ell \delta\mathbf{m}^T\mathbf{S}\left(\frac{(\boldsymbol{\alpha}^e\otimes\boldsymbol{\alpha}^e)\boldsymbol{\omega}^e}{||\boldsymbol{\alpha}^e||^2}\right)\delta\mathbf{r}\,ds \\ & + \int_0^\ell \frac{1}{\rho_0}\delta\mathbf{m}^T\delta\mathbf{m}\,ds + c\int_0^\ell \delta\mathbf{r}^T\mathbf{K}\delta\mathbf{r}\,ds \end{aligned} \tag{3.27}$$

Step 10: Rewrite The Lower Bound

We can reformulate the lower bound in a clearer form as follows

$$\begin{aligned} D^2(\mathbf{H}+\mathbf{C}_\phi)_{(\mathbf{p}^e,\mathbf{r}^e,\mathbf{m}^e)} \geq\ & (square) \\ & + \int_0^\ell \begin{bmatrix} \frac{1}{\rho_0}\mathbf{I} - \mathbf{I}\tilde{\lambda}^2 & -\mathbf{S}(\frac{\boldsymbol{\alpha}^e\otimes\boldsymbol{\alpha}^e}{||\boldsymbol{\alpha}^e||^2}\boldsymbol{\omega}^e) \\ -\mathbf{S}^T(\frac{\boldsymbol{\alpha}^e\otimes\boldsymbol{\alpha}^e}{||\boldsymbol{\alpha}^e||^2}\boldsymbol{\omega}^e) & c\mathbf{K} - \mathbf{I}\tilde{\lambda}^2 \end{bmatrix} \begin{bmatrix} \delta\mathbf{m} \\ \delta\mathbf{r} \end{bmatrix} \cdot \begin{bmatrix} \delta\mathbf{m} \\ \delta\mathbf{r} \end{bmatrix} ds \end{aligned} \tag{3.28}$$

If we define the matrix

$$\mathbf{D}(\mathbf{p}^e, \mathbf{r}^e, \mathbf{m}^e) = \begin{bmatrix} \frac{1}{\rho_0}\mathbf{I} - \mathbf{I}\tilde{\lambda}^2 & -\mathbf{S}(\frac{\boldsymbol{\alpha}^e \otimes \boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}\boldsymbol{\omega}^e) \\ -\mathbf{S}^T(\frac{\boldsymbol{\alpha}^e \otimes \boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}\boldsymbol{\omega}^e) & c\mathbf{K} - \mathbf{I}\tilde{\lambda}^2 \end{bmatrix} \tag{3.29}$$

then we can state the following theorem ;

Theorem(3.2): If the matrix $\mathbf{R} = \mathbf{Q}_e\mathbf{J}_e\mathbf{Q}_e - \mathbf{Q}_e$ defined in (3.21) exists and is nonnegative definite, $\mathbf{J}_e$ defined in (3.13) is positive definite. and the matrix $\mathbf{D}$ defined in equation (3.29) is positive definite, then the system described by equations (1.2)-(1.4) is nonlinearly (formally) stable at the equilibrium point ($\mathbf{p}^e$, $\mathbf{r}^e$, $\mathbf{m}^e$). ▮

Remark 1: This result establishes only formal stability, since it is based on the the definiteness of second variation. To establish rigorous stability of the nonlinear system one generally needs to examine convexity estimates as is done in [2].

Remark 2: Note that if $\mathbf{Q}_e^{-1}$ exists and we use the matrix inversion lemma [4, p.656] we obtain the following

$$\begin{aligned}(\mathbf{Q}_e^{-1} + \mathbf{J})^{-1} &= \mathbf{Q}_e - \mathbf{Q}_e\mathbf{J}_e\mathbf{Q}_e \\ &= -\mathbf{R}\end{aligned} \tag{3.30}$$

Recall that we already have an assumption of nonnegative definiteness on $\mathbf{R}$. Thus we need to specify conditions on the parameters and $\phi''(\|\boldsymbol{\alpha}^e\|^2)$ such that

$$\mathbf{J}^{-1} + \mathbf{Q}_e > 0 \tag{3.31}$$

$$(\mathbf{J} + \mathbf{Q}_e^{-1})^{-1} \leq 0 \tag{3.32}$$

which are the same conditions as $\mathbf{R} \geq 0$ and $\mathbf{J}_e > 0$. In the examples of the next section $\mathbf{Q}_e$ is singular.

Remark 3: A better result can be had by observing that $\mathbf{A}^T(s)\mathbf{A}(s)$ is frequently in the form of a block diagonal matrix

$$\mathbf{A}^T(s)\mathbf{A}(s) = \begin{bmatrix} \mathbf{A}_1^T(s)\mathbf{A}_1(s) & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{A}_k^T(s)\mathbf{A}_k(s) \end{bmatrix} \tag{3.33}$$

where $0 \leq k \leq 6$ and because of the semidefiniteness of $\mathbf{A}^T\mathbf{A}(s)$ some of the diagonal blocks may be zero. If we let $\lambda_i^2(s)$ be the maximum eigenvalue of $\mathbf{A}_i^T(s)\mathbf{A}_i(s)$, $0 \leq i \leq k$ then we can define

$$\mathbf{D}' = \begin{bmatrix} \frac{1}{\rho_0}\mathbf{I} & -\mathbf{S}\left(\frac{\boldsymbol{\alpha}^e \otimes \boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}\boldsymbol{\omega}^e\right) \\ -\mathbf{S}^T\left(\frac{\boldsymbol{\alpha}^e \otimes \boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}\boldsymbol{\omega}^e\right) & c\mathbf{K} \end{bmatrix} - \begin{bmatrix} \mathbf{I}\tilde{\lambda}_1^2 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \mathbf{I}\tilde{\lambda}_k^2 \end{bmatrix} \tag{3.34}$$

Thus, if the conditions of theorem (3.2) are satisfied and also the matrix $\mathbf{D}'$ defined in equation (3.34) is positive definite, then the system described by equations (1.2)-(1.4) is (formally) nonlinearly stable at the equilibrium point ($\mathbf{p}^e$, $\mathbf{r}^e$, $\mathbf{m}^e$). In theorem (3.2) this will mean the special choice $\lambda^2(s) = \max\{\lambda_1^2(s), \ldots, \lambda_k^2(s)\}$.

4. Some Examples

In this section we apply theorem (3.2) to specific equilibria of (1.1) - (1.3). We will assume that that the linear extensible shear beam lies along the same direction as the second principal axis of inertia of the rigid body. From geometric considerations the position of the shear beam will cause the principal axes of the rigid-body-shear-beam configuration to lie in the same directions as those of the rigid body. In this case the addition of the shear beam will have the effect of increasing the moments of inertia about the first and the third principal axes. Because the linear extensible shear beam cannot deflect laterally the principal axes of the of the configuration remain fixed for any longitudinal extension of the shear beam. Thus, for this configuration there are three axes about which the equilibria can exist. These axes will correspond to the three principal axes of the rigid body.

4.1. A Trivial Equilibrium

The simplest case to be considered is when the rotation takes place about the axis along which the linear extensible shear beam lies. In this case the equilibrium will be

$$\boldsymbol{\omega}^e = \omega_2^e\,\hat{\mathbf{e}}_2 \tag{4.1}$$

$$\mathbf{r}^e = (a_2 + s)\,\hat{\mathbf{e}}_2, \qquad 0 \leq s \leq \ell \tag{4.2}$$

$$\mathbf{m}^e = \mathbf{0} \tag{4.3}$$

This describes the linear-extensible-shear-beam being unstretched.

What follows is a special case of the second variation computed in *Step 1* of the previous section. In this and the following example we will assume $\phi''(\|\boldsymbol{\alpha}^e\|^2)$ is the same as in [2], thus recall from (3.7) that if this is the case then

$$\phi''(\|\boldsymbol{\alpha}^e\|^2) = \frac{\boldsymbol{\omega}^e \cdot \boldsymbol{\alpha}^e}{2\|\boldsymbol{\alpha}^e\|^4}$$

And the two quantities, $\mathbf{J}_e^{-1}$, and $\mathbf{Q}_e$, which we define in *Step 4* are

$$\mathbf{J}_e^{-1} = \mathbf{J}^{-1} - \frac{\boldsymbol{\omega}^e \cdot \boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}\left(\mathbf{I} - \frac{\boldsymbol{\alpha}^e \otimes \boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}\right) \tag{4.4}$$

$$\mathbf{Q}_e = -\frac{\boldsymbol{\omega}^e \cdot \boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}\left(\mathbf{I} - \frac{\boldsymbol{\alpha}^e \otimes \boldsymbol{\alpha}^e}{\|\boldsymbol{\alpha}^e\|^2}\right) \tag{4.5}$$

For our example, if we first compute

$$\boldsymbol{\alpha}^e = j_{22}\omega_2\,\hat{\mathbf{e}}_2 \tag{4.6}$$

then

$$\boldsymbol{\alpha}^{eT}\boldsymbol{\omega}^e = j_{22}(\omega_2^e)^2 \qquad \text{and} \qquad \boldsymbol{\alpha}^{eT}\boldsymbol{\alpha}^e = j_{22}^2(\omega_2^e)^2 \tag{4.7}$$

from which we immediately compute

$$\mathbf{I} - \frac{\boldsymbol{\alpha}^e\boldsymbol{\alpha}^{eT}}{\boldsymbol{\alpha}^{eT}\boldsymbol{\alpha}^e} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.8}$$

and finally,

$$\mathbf{J}_e^{-1} = \begin{bmatrix} \frac{j_{11}j_{22}}{j_{22}-j_{11}} & 0 & 0 \\ 0 & j_{22} & 0 \\ 0 & 0 & \frac{j_{33}j_{22}}{j_{22}-j_{33}} \end{bmatrix} \tag{4.9}$$

$$\mathbf{Q}_e = \begin{bmatrix} \frac{1}{j_{22}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{j_{22}} \end{bmatrix} \tag{4.10}$$

For $\mathbf{J}_e$ to be positive definite we require $j_{22} > j_{11}$, and $j_{22} > j_{33}$. This will assure positive elements along the diagonal in the inverse above.

Thus, the quantity $\mathbf{Q}_e\mathbf{J}_e\mathbf{Q}_e$ which appears in the reformulated second variation of *Step 5* will be,

$$\mathbf{Q}_e\mathbf{J}_e\mathbf{Q}_e = \left(\mathbf{J}^{-1} - \frac{\boldsymbol{\alpha}^{eT}\boldsymbol{\omega}^e}{\boldsymbol{\alpha}^{eT}\boldsymbol{\alpha}^e}(\mathbf{I} - \frac{\boldsymbol{\alpha}^e\boldsymbol{\alpha}^{eT}}{\boldsymbol{\alpha}^{eT}\boldsymbol{\alpha}^e})\right)^{-1}\left(\frac{(\boldsymbol{\alpha}^{eT}\boldsymbol{\omega}^e)^2}{(\boldsymbol{\alpha}^{eT}\boldsymbol{\alpha}^e)^2}(\mathbf{I} - \frac{\boldsymbol{\alpha}^e\boldsymbol{\alpha}^{eT}}{\boldsymbol{\alpha}^{eT}\boldsymbol{\alpha}^e})\right)$$
$$= \left(\begin{bmatrix} \frac{1}{j_{11}} & 0 & 0 \\ 0 & \frac{1}{j_{22}} & 0 \\ 0 & 0 & \frac{1}{j_{33}} \end{bmatrix} - \begin{bmatrix} \frac{1}{j_{22}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{j_{22}} \end{bmatrix}\right)^{-1}\left(\begin{bmatrix} \frac{1}{j_{22}^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{j_{22}^2} \end{bmatrix}\right)$$
$$= \begin{bmatrix} \frac{j_{11}j_{22}}{j_{22}-j_{11}} & 0 & 0 \\ 0 & j_{22} & 0 \\ 0 & 0 & \frac{j_{33}j_{22}}{j_{22}-j_{33}} \end{bmatrix}\begin{bmatrix} \frac{1}{j_{22}^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{j_{22}^2} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{j_{11}}{j_{22}(j_{22}-j_{11})} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{j_{33}}{j_{22}(j_{22}-j_{33})} \end{bmatrix} \tag{4.11}$$

where we have used equation (3.17) and the fact that $\mathbf{J}_e$ and $\mathbf{Q}_e$ are diagonal.

We also need the skew symmetric matrix which appears in *Step 7.* Thus, we compute

$$\mathrm{S}(\tfrac{\alpha^e\alpha^{eT}}{\alpha^{eT}\alpha^e}\omega^e) = \begin{bmatrix} 0 & 0 & \omega_2^e \\ 0 & 0 & 0 \\ -\omega_2^e & 0 & 0 \end{bmatrix} \tag{4.12}$$

Now we compute $\mathbf{R}$, which is defined in *Step 8.*

$$\mathbf{R} = \mathbf{Q}_e\mathbf{J}_e\mathbf{Q}_e + \frac{\boldsymbol{\alpha}^{eT}\boldsymbol{\omega}^e}{\boldsymbol{\alpha}^{eT}\boldsymbol{\alpha}^e}(\mathbf{I} - \frac{\boldsymbol{\alpha}^e\boldsymbol{\alpha}^{eT}}{\boldsymbol{\alpha}^{eT}\boldsymbol{\alpha}^e})$$
$$= \begin{bmatrix} \frac{j_{11}}{j_{22}(j_{22}-j_{11})} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{j_{33}}{j_{22}(j_{22}-j_{33})} \end{bmatrix} + \begin{bmatrix} \frac{1}{j_{22}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{j_{22}} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{j_{22}-j_{11}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{j_{22}-j_{33}} \end{bmatrix} \tag{4.13}$$

which, along with the definition of $\mathrm{S}(\cdot)$ in (3.20), we can now use to compute

$$\mathbf{S}^T(\mathbf{r}^e)\mathbf{R}\mathbf{S}(\mathbf{r}^e) = \begin{bmatrix} \frac{1}{j_{22}-j_{33}}r_2^{e\,2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{j_{22}-j_{11}}r_2^{e\,2} \end{bmatrix} \tag{4.14}$$

$$\mathbf{S}^T(\mathbf{r}^e)\mathbf{R}\mathbf{S}(\mathbf{m}^e) = \mathbf{0} \tag{4.15}$$

$$\mathbf{S}^T(\mathbf{m}^e)\mathbf{R}\mathbf{S}(\mathbf{m}^e) = \mathbf{0} \tag{4.16}$$

These matrices are used to form the matrix $\mathbf{A}^T(s)\mathbf{A}(s)$ in (3.34), note that it has only the two nonzero elements (computed in (4.15)). These correspond to the first and second diagonal elements. Hence, $\mathbf{A}^T(s)\mathbf{A}(s)$ is a diagonal matrix and the nonzero eigenvalues are these two elements. As a consequence we will use the modified bound described in *Remark 3*. Thus, the eigenvalue inequality is easily obtained.

After using the Poincaré inequality of *Step 9* we proceed to the final step and construct the $\mathbf{D}'$ matrix in (3.34)

$$\mathbf{D}' = \begin{bmatrix} \frac{1}{\rho_0} - \frac{1}{j_{22}-j_{33}}\int_0^\ell r_2^{e\,2}\,ds & 0 & 0 & 0 & 0 & -\omega_2^e \\ 0 & \frac{1}{\rho_0} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\rho_0} - \frac{1}{j_{22}-j_{11}}\int_0^\ell r_2^{e\,2}\,ds & \omega_2^e & 0 & 0 \\ 0 & 0 & \omega_2^e & (\frac{\pi}{2\ell})^2 k_x & 0 & 0 \\ 0 & 0 & 0 & 0 & (\frac{\pi}{2\ell})^2 k_y & 0 \\ -\,\omega_2^e & 0 & 0 & 0 & 0 & (\frac{\pi}{2\ell})^2 k_z \end{bmatrix} \tag{4.17}$$

To assure that the $\mathbf{D}'$ matrix is positive definite we require

$$j_{22} - j_{11} > \rho_0 \int_0^\ell r_2^{e\,2}\,ds \tag{4.18}$$

$$j_{22} - j_{33} > \rho_0 \int_0^\ell r_2^{e\,2}\,ds \tag{4.19}$$

and also,

$$\left(\frac{1}{\rho_0} - \frac{1}{j_{22}-j_{33}}\int_0^\ell r_2^{e\,2}\,ds\right)\left(\frac{\pi}{2\ell}\right)^2 k_z > (\omega_2^e)^2 \tag{4.20}$$

$$\left(\frac{1}{\rho_0} - \frac{1}{j_{22}-j_{11}}\int_0^\ell r_2^{e\,2}\,ds\right)\left(\frac{\pi}{2\ell}\right)^2 k_x > (\omega_2^e)^2 \tag{4.21}$$

Physically the first two conditions are classical stability conditions on the stable axes of rotation for a rigid body. The term on the right is the additional inertia due to the flexible appendage which adds inertia about both the first and third axes. The second two inequalities are conditions on the admissible rotation rates of the configuration. They have an interesting physical interpretation.

4.2. A Non-Trivial Equilibrium

For the second example we will consider rotations of the rigid-body-shear-beam configuration about the first or third principal axes of inertia. We will examine the case when the rotation is about the first principal axis of inertia, rotations about the third axis are similar. This corresponds to the example in Krishnaprasad and Marsden [2].

$$\boldsymbol{\omega}^e = \omega_1^e\,\hat{\mathbf{e}}_1 \tag{4.22}$$

$$\mathbf{r}^e(s) = \left(\frac{\sin(\sqrt{\frac{\rho_0}{k_v}}\omega_1^e s)}{\sqrt{\frac{\rho_0}{k_v}}\omega_1^e} + a\frac{\cos(\sqrt{\frac{\rho_0}{k_v}}\omega_1^e(s-\ell))}{\cos(\sqrt{\frac{\rho_0}{k_v}}\omega_1^e\ell)} \right)\hat{\mathbf{e}}_2 \tag{4.23}$$

$$\mathbf{m}^e(s) = \rho_0\omega_1 \left(\frac{\sin(\sqrt{\frac{\rho_0}{k_v}}\omega_1^e s)}{\sqrt{\frac{\rho_0}{k_v}}\omega_1^e} + a\frac{\cos(\sqrt{\frac{\rho_0}{k_v}}\omega_1^e(s-\ell))}{\cos(\sqrt{\frac{\rho_0}{k_v}}\omega_1^e\ell)} \right)\hat{\mathbf{e}}_3 \tag{4.24}$$

In these equations we have $0 \le s \le \ell$. For simplicity we will denote the nonzero element of $\mathbf{r}$ as r_2^e, and that of $\mathbf{m}$ as m_3^e.

We first compute

$$\boldsymbol{\alpha}^e = j_{11}\omega_1^e + \int_0^\ell r_2^e m_3^e\,ds\,\hat{\mathbf{e}}_1 \tag{4.25}$$

thus

$$\boldsymbol{\alpha}^{eT}\boldsymbol{\omega}^e = (j_{11}\omega_1^e + \int_0^\ell r_2^e m_3^e\,ds)\omega_1^e \tag{4.26}$$

$$\boldsymbol{\alpha}^{eT}\boldsymbol{\alpha}^e = (j_{11}\omega_1^e + \int_0^\ell r_2^e m_3^e\,ds)^2 \tag{4.27}$$

Subsequently we will denote the first element of $\boldsymbol{\alpha}$ by α_1. We now compute

$$\mathbf{I} - \frac{\boldsymbol{\alpha}^e\boldsymbol{\alpha}^{eT}}{\boldsymbol{\alpha}^{eT}\boldsymbol{\alpha}^e} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.28}$$

and finally, $\mathbf{J}_e^{-1}$ and $\mathbf{Q}_e$ defined in *Step 4* are

$$\mathbf{J}_e^{-1} = \begin{bmatrix} \frac{1}{j_{11}} & 0 & 0 \\ 0 & \frac{\alpha_1 - j_{22}\omega_1^e}{j_{22}\alpha_1} & 0 \\ 0 & 0 & \frac{\alpha_1 - j_{33}\omega_1^e}{j_{33}\alpha_1} \end{bmatrix} \tag{4.29}$$

$$\mathbf{Q}_e = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{\omega_1^e}{\alpha_1} & 0 \\ 0 & 0 & \frac{\omega_1^e}{\alpha_1} \end{bmatrix} \tag{4.30}$$

For $\mathbf{J}_e^{-1}$ to be positive definite we require

$$\alpha_1 > j_{22}\omega_1^e \qquad \text{and} \qquad \alpha_1 > j_{33}\omega_1^e \tag{4.26}$$

These conditions will hold if $j_{11} > j_{22}$, and $j_{11} > j_{33}$ and will assure positive elements along the diagonal in the inverse above. These conditions are the same as (5.10) in [2].

Then from equation (3.17) we have

$$\begin{aligned}
\mathbf{Q}_e\mathbf{J}_e\mathbf{Q}_e &= \left(\mathbf{J}^{-1} - \frac{\boldsymbol{\alpha}^{eT}\boldsymbol{\omega}^e}{\boldsymbol{\alpha}^{eT}\boldsymbol{\alpha}^e}(\mathbf{I} - \frac{\boldsymbol{\alpha}^e\boldsymbol{\alpha}^{eT}}{\boldsymbol{\alpha}^{eT}\boldsymbol{\alpha}^e})\right)^{-1}\left(\frac{(\boldsymbol{\alpha}^{eT}\boldsymbol{\omega}^e)^2}{(\boldsymbol{\alpha}^{eT}\boldsymbol{\alpha}^e)^2}(\mathbf{I} - \frac{\boldsymbol{\alpha}^e\boldsymbol{\alpha}^{eT}}{\boldsymbol{\alpha}^{eT}\boldsymbol{\alpha}^e})\right) \\
&= \left(\begin{bmatrix} \frac{1}{j_{11}} & 0 & 0 \\ 0 & \frac{1}{j_{22}} & 0 \\ 0 & 0 & \frac{1}{j_{33}} \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{\omega_1^e}{\alpha_1} & 0 \\ 0 & 0 & \frac{\omega_1^e}{\alpha_1} \end{bmatrix}\right)^{-1}\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{(\omega_1^e)^2}{\alpha_1^2} & 0 \\ 0 & 0 & \frac{(\omega_1^e)^2}{\alpha_1^2} \end{bmatrix}\right) \\
&= \begin{bmatrix} \frac{1}{j_{11}} & 0 & 0 \\ 0 & \frac{\alpha_1 - j_{22}\omega_1}{j_{22}\alpha_1} & 0 \\ 0 & 0 & \frac{\alpha_1 - j_{33}\omega_1}{j_{33}\alpha_1} \end{bmatrix}^{-1}\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{(\omega_1^e)^2}{\alpha_1^2} & 0 \\ 0 & 0 & \frac{(\omega_1^e)^2}{\alpha_1^2} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{(\omega_1^e)^2 j_{22}}{\alpha_1(\alpha_1 - j_{22}\omega_1^e)} & 0 \\ 0 & 0 & \frac{(\omega_1^e)^2 j_{33}}{\alpha_1(\alpha_1 - j_{33}\omega_1^e)} \end{bmatrix}
\end{aligned} \tag{4.31}$$

The skew symmetric matrix of *Step 7* is

$$S(\tfrac{\alpha^e\alpha^{eT}}{\alpha^{eT}\alpha^e}\omega^e) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\omega_1^e \\ 0 & \omega_1^e & 0 \end{bmatrix} \tag{4.32}$$

Now we compute $\mathbf{R}$ as defined in *Step 8*,

$$\begin{aligned}
\mathbf{R} &= \mathbf{Q}_e\mathbf{J}_e\mathbf{Q}_e + \frac{\boldsymbol{\alpha}^{eT}\boldsymbol{\omega}^e}{\boldsymbol{\alpha}^{eT}\boldsymbol{\alpha}^{eT}}(\mathbf{I} - \frac{\boldsymbol{\alpha}^e\boldsymbol{\alpha}^{eT}}{\boldsymbol{\alpha}^{eT}\boldsymbol{\alpha}^e}) \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & b_{22} & 0 \\ 0 & 0 & b_{33} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{\omega_1^e}{\alpha_1} & 0 \\ 0 & 0 & \frac{\omega_1^e}{\alpha_1} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 \\ 0 & \gamma_{22} & 0 \\ 0 & 0 & \gamma_{33} \end{bmatrix}
\end{aligned} \tag{4.33}$$

where we have

$$\gamma_{22} = \frac{\omega_1^e}{\alpha_1 - j_{22}\omega_1^e} \qquad \text{and} \qquad \gamma_{33} = \frac{\omega_1^e}{\alpha_1 - j_{33}\omega_1^e} \tag{4.34}$$

Note that these are not the same as the γ_1, and γ_2 terms which appear in [2].

We can now compute

$$\mathbf{S}^T(\mathbf{r}^e)\mathbf{R}\mathbf{S}(\mathbf{r}^e) = \begin{bmatrix} \gamma_{33}r_2^{e\,2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4.35}$$

$$\mathbf{S}^T(\mathbf{r}^e)\mathbf{R}\mathbf{S}(\mathbf{m}^e) = \mathbf{0} \tag{4.36}$$

$$\mathbf{S}^T(\mathbf{m}^e)\mathbf{R}\mathbf{S}(\mathbf{m}^e) = \begin{bmatrix} \gamma_{22}m_3^{e\,2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4.37}$$

From this we can compute the matrix $\mathbf{A}^T(s)\mathbf{A}(s)$ in (3.24), note that it has only two nonzero elements. These correspond to second and fourth diagonal elements. Hence, $\mathbf{A}^T(s)\mathbf{A}(s)$ is a diagonal matrix and the nonzero eigenvalues are these two elements. As in the previous example we will use the modified bound described in *Remark 3.*

We can construct the $\mathbf{D}'$ matrix in (3.34)

$$\mathbf{D}' = \begin{bmatrix} \frac{1}{\rho_0} - \gamma_{33}\int_0^\ell r_2^{e\,2}\,ds & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\rho_0} & 0 & 0 & 0 & -\omega_1^e \\ 0 & 0 & \frac{1}{\rho_0} & 0 & \omega_1^e & 0 \\ 0 & 0 & 0 & (\frac{\pi}{2\ell})^2 k_x - \gamma_{22}\int_0^\ell m_2^{e\,2}\,ds & 0 & 0 \\ 0 & 0 & \omega_1^e & 0 & (\frac{\pi}{2\ell})^2 k_y & 0 \\ 0 & -\omega_1^e & 0 & 0 & 0 & (\frac{\pi}{2\ell})^2 k_z \end{bmatrix} \tag{4.38}$$

To assure that the $\mathbf{D}'$ matrix is positive definite we require

$$\frac{1}{\gamma_{33}} > \rho_0 \int_0^\ell r_2^{e\,2}\,ds \tag{4.39}$$

$$\left(\frac{\pi}{2\ell}\right)^2 \frac{k_x}{\gamma_{22}} > \int_0^\ell m_3^{e\,2}\,ds \tag{4.40}$$

and

$$\frac{k_z}{\rho_0}\left(\frac{\pi}{2\ell}\right)^2 > (\omega_2^e)^2 \tag{4.41}$$

$$\frac{k_y}{\rho_0}\left(\frac{\pi}{2\ell}\right)^2 > (\omega_2^e)^2 \tag{4.42}$$

These conditions are exactly those of (5.14) in Krishnaprasad and Marsden and they assure stability about the equilibrium which also satisfies (4.26).

Finally a remark about the difference between [2] and our development. If we integrate the matrix we call $\mathbf{A}^T(s)\mathbf{A}(s)$ then the elements of the integrated matrix would correspond to γ_2, and γ_1 in the paper of Krishnaprasad and Marsden. This suggests modifying the procedure in the previous section to look at the eigenvalues of the integrated matrix rather than integrating the eigenvalues.

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