

ABSTRACT

Title of Dissertation: **CLASSIFICATION AND CHARACTERIZATION OF
CRYSTALLINE TOPOLOGICAL INVARIANTS
IN QUANTUM MANY-BODY STATES**

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The theory of topological phases of matter, now a major direction in condensed matter physics, is framed around two complementary problems. The first is to mathematically classify topological states of matter given various symmetries, in terms of suitable topological invariants. The second is to characterize a given state by numerically or experimentally extracting the topological invariants associated with it. Although remarkable progress has been made for topological states with internal symmetries, major open questions remain in the case of many-body systems with crystalline symmetries, especially those with a nonzero Chern number and a magnetic field.

The first part of this thesis develops a theory of crystalline topological response in $(2+1)$ dimensions based on the idea of crystalline gauge fields and their effective actions, which we derive using topological quantum field theory. We use this to obtain a complete classification of topological states with $U(1)$ charge conservation, discrete magnetic translation, and point

group rotation symmetries, finding several new invariants. We separately consider symmetry-enriched topological states of bosons, which admit anyonic excitations with fractional statistics, and invertible fermionic states, which do not.

The second part of this thesis focusses on numerically extracting these invariants from many-body invertible states. First we study two quantized invariants, the discrete shift, and a charge polarization which is quantized by rotational symmetries. We show how to extract these invariants in multiple different ways, which include the fractional charge bound to lattice defects, as well as the angular and linear momentum of magnetic flux. Thereafter, we obtain a *complete* characterization of the theoretically predicted invariants, by studying the expectation value of the ground state under partial rotation operators.

An immediate application of these ideas is to fully characterize the celebrated Hofstadter model of spinless free fermions on a square lattice. Although the Chern number and filling were first computed in this model in 1982, our theory predicts seven nontrivial invariants, including four new invariants which depend on the crystalline symmetry. We compute these numerically and obtain several additional colorings of Hofstadter's butterfly.

CLASSIFICATION AND CHARACTERIZATION OF CRYSTALLINE
TOPOLOGICAL INVARIANTS IN QUANTUM MANY-BODY STATES

by

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Dedication

To my grandfather, D. Ramesh, a colourful character who taught me arithmetic using peanuts and rice grains, and set me along this path.

Acknowledgements

I am very fortunate to have many people to thank for helping me get to this point.

I am especially grateful to my advisor, Professor Maissam Barkeshli, for giving me so much of his time and attention over the last six years. I benefited greatly from the conceptual clarity and the organized approach that he brought to our research. As our work took shape, I was continually impressed by how an abstract calculation that he had previously suggested would bear fruit in an unexpected manner, months or even years later.

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List of Abbreviations

BdG	Bogoliubov-de Gennes
BTC	Braided tensor category
CEP	Crystalline equivalence principle
CFT	Conformal field theory
fCEP	Fermionic crystalline equivalence principle
FCI	Fractional Chern insulator
FQH	Fractional quantum Hall
IQH	Integer quantum Hall
MZM	Majorana zero mode
QFT	Quantum field theory
SPT	Symmetry-protected topological
TQFT	Topological quantum field theory
TSC	Topological superconductor

Chapter 1: Introduction

1.1 General overview

One of the main goals of condensed matter physics is to organize and explain phenomena involving systems composed of large numbers of particles that may be interacting with each other and with their environment. This is in general a difficult task because of the immense range of the phenomena encountered. Even if we only consider atomic systems at very low temperatures, there is so much diversity in the observable properties (such as the microscopic patterns of arrangement, and the response of the system to electric and magnetic fields, thermal gradients, or strains, to name just a few) that attempting such an organization is a formidable task.

To make progress, the field has developed some basic guiding principles. The first is to study *phases* of matter rather than individual states. A phase of matter is a class of states characterized by *universal* low-energy properties that remain fixed or vary smoothly as the microscopic parameters of a system change. The parameters at which universal properties change sharply represent a *phase transition*. If the universal properties of two states can be connected by an adiabatic path in parameter space, without going through a phase transition, the states are said to belong to the same phase of matter.

A second guiding principle is to use the symmetries of the system as a basis for organization. For our purposes, we define the (microscopic) symmetry as the group determined by the operators

that commute with the microscopic Hamiltonian. The symmetry is not only an important universal property in its own right (for example, a symmetry-breaking phase transition is defined by the property that the ground state of the system has different symmetries on either side of the transition), but also serves to constrain other universal properties, such as the response to various perturbations.

The problem of organizing our knowledge of different states of matter can then be conveniently framed as the classification and characterization of phases. To *classify* phases of matter is to write down a complete list of the relevant universal properties (such as symmetries, response coefficients, and other observable quantities), along with the different values that they could take in a numerical simulation or an experiment. To *characterize* a particular state of matter is to measure all the relevant universal properties in that state, using either a numerical or physical experiment, and thus assign the state to one of the phases in the above classification.

In general, the problem is still extremely difficult. It is not at all clear whether a finite set of universal properties can give a ‘complete’ classification of phases. Furthermore, it may be difficult to tell two phases apart even if there is some difference in their properties, because there might not be an obvious phase transition based on available data. A common example is the relationship between water and steam: even though there appears to be a sharp difference between their properties at ambient pressures, water can be converted into steam without crossing a phase transition if the pressure is allowed to vary.

In this thesis, we will focus on a special class of gapped quantum systems modeled at zero temperature (meaning that the universal properties are controlled by the states at lowest energy). We will assume that the ground state preserves a certain symmetry, which we fix at the start. Here the problem simplifies tremendously. First, the relevant universal properties correspond to discrete quantized observables that remain almost perfectly constant with respect

to a wide range of parameters, and only jump between two such quantized values at phase transitions. Second, one can study such systems using a variety of mathematical tools. These predict that for a given symmetry group the number of different phases is well-defined, and each can be completely characterized by measuring a finite number of relevant universal properties. Therefore, at least theoretically, it is possible to make precise the notion of a ‘complete’ classification and characterization of such systems. Since many of the mathematical techniques originate from the branch of topology, the universal properties they describe are referred to as ‘topological invariants’.

The main result of this thesis is to develop a complete classification and characterization of the topological invariants in quantum many-body states with $U(1)$ charge conservation and crystalline symmetries in two space dimensions (we conventionally write this as $(2+1)$ dimensions of spacetime). The crystalline symmetries we assume include discrete lattice translations and discrete point group rotations. The study of crystalline topological invariants has grown immensely over the last decade, and the tools we use were only developed fully within the last five years. However, these tools yield new insight into many interesting models that are much older. An example we will focus on is the spinless free fermion Harper-Hofstadter model, which was first proposed in 1955 [1] and whose energy spectrum was plotted by Hofstadter in 1976 [2]. In fact the first known topological invariant in condensed matter physics, namely the Chern number, was calculated theoretically for this model in 1982 [3]. However, there are several crystalline topological invariants in this model that were not recognized for many years thereafter. One of the major applications of the theory presented here is to find a complete set of topological invariants for the phase diagram of the square lattice Hofstadter model (the famous ‘Hofstadter butterfly’), and thus demonstrate that there are many more interesting invariants in this model

besides the Chern number.

The rest of this Chapter is organized as follows. In Sec. 1.2 we define the topological phases of matter that we study more precisely. In Sec. 1.3 we review prior work in the field and highlight some important unresolved problems. Then in Sec. 1.4 we discuss the outline of each of the following Chapters of this thesis, and explain how they address these open questions.

1.2 Definition of invertible and topologically ordered states

This thesis focuses on gapped (2+1) dimensional topological phases modeled at zero temperature, in which the ground state preserves the symmetry G of the Hamiltonian. The elementary objects in the problem may be bosonic or fermionic; we will be explicit about which case we consider in later Chapters. It will be useful to distinguish two possibilities. In the first case, the system has a unique ground state on any closed spatial manifold, and corresponds to an *invertible* state of matter. The term invertible refers to the fact that the ground state $|\Psi\rangle$ possesses an “inverse” state $|\Psi^{-1}\rangle$, such that $|\Psi\rangle \otimes |\Psi^{-1}\rangle$ (which is referred to as a stack of $|\Psi\rangle$ and $|\Psi^{-1}\rangle$) can be adiabatically connected to a trivial product state.

Important examples of invertible states without any symmetry (other than fermion parity, which is a symmetry in all fermionic systems) include the Majorana chain in (1+1) dimension [4], and the $p + ip$ superconductor in (2+1) dimensions [5]. The canonical example of an invertible state with symmetry is the integer quantum Hall (IQH) state, which is characterized by a quantized topological invariant, the Chern number C , that is protected by U(1) charge conservation symmetry. C is an integer which sets the Hall conductance $\sigma_H = C \frac{e^2}{h}$; this has been measured experimentally with incredible precision, up to 10 decimal places.¹ Other notable

¹Indeed, the Hall conductance in the IQH state is now the experimental standard used to *define* the fundamental

examples of invertible topological states with symmetry include the quantum spin Hall (QSH) insulator (and topological insulators more generally) [6], Chern insulators (which are lattice analogs of the IQH states) as well as symmetry-protected topological states (SPTs) [7, 8], which are invertible states with symmetry G that can be connected to a trivial product state via a G symmetry-breaking adiabatic path, but not by a G symmetry-preserving path. It is also possible to formulate topological superconductors as invertible states in a mean-field treatment, in which phase fluctuations are ignored.

The fact that invertible states indeed have a well-defined inverse under the stacking operation means that their classification can be expressed as a group. For example, stacking two $U(1)$ symmetric invertible states with Chern number 1 gives another $U(1)$ symmetric invertible state with Chern number 2. Since the Chern number can take any integer value and is additive under stacking, the classification based on Chern number alone is given by the group $\mathbb{Z}$. On the other hand, note that if we stack two copies of the quantum spin Hall insulator, the gapless edge states which are protected by time-reversal symmetry in a single copy can be gapped out by local backscattering terms which couple the two copies, and thus the full system can be adiabatically connected to a product state without breaking symmetry. Therefore two copies of the QSH insulator belong to the trivial phase, and the classification is captured by a $\mathbb{Z}_2$ topological invariant. In this sense, the classification of invertible phases with a given symmetry amounts to finding and physically interpreting the group that describes the classification under stacking.

The second possibility is that of *topological order* [9], in which the system defined on a closed manifold with genus g (that is, g holes) has a ground state degeneracy N^g , where $N > 1$ is some integer. Such states carry quasiparticle excitations with fractional statistics, known constant e^2/h , the unit of resistance (ohm), and even the kilogram.

as *anyons*, which can also carry fractional charge and other symmetry quantum numbers. A topologically ordered phase is said to be *Abelian* if its anyons all transform as one-dimensional representations of the braid group, and non-Abelian otherwise. For Abelian topological order, N equals the number of topologically distinct anyons; this is not true for non-Abelian states. Canonical examples of topologically ordered states include the fractional quantum Hall (FQH) states and their lattice analogs, the fractional Chern insulators, as well as quantum spin liquids [9]. Apart from exhibiting an array of exotic and counterintuitive phenomena, they are also a tantalizing potential platform for fault-tolerant quantum computation [10].

Note that anyons, along with their fusion and braiding properties, are *intrinsic* topological invariants that can be defined without invoking any conventional symmetries. On the other hand, the fractional charges of anyons and the associated fractionally quantized response properties can only be defined in the presence of symmetry; see below for an example. In our work we will fix the intrinsic topological invariants and then try to classify the possible symmetry-protected invariants.

Topologically ordered states are *not* invertible: it is not possible to stack two topologically ordered states and adiabatically connect the resulting state to a trivial product state. In particular, topologically ordered states with symmetry (technically referred to as *symmetry-enriched topological* (SET) states) do not form a group under stacking. Instead, their classification is captured by two layers of data: the symmetry fractionalization (or equivalently the fractional symmetry quantum numbers of the anyons), and the fractionally quantized response properties [11].

Let us work with the example of fractional quantum Hall states, and assume only $U(1)$ charge conservation symmetry. Here, the fractional charge Q_a of each anyon a is fixed by the

mutual braiding phase between a and a special anyon v :

$$e^{2\pi i Q_a} = M_{a,v}. \quad (1.1)$$

The anyon v is referred to as the vison, fluxon, or as the ‘charge vector’. The number of different mathematically allowed fractional charge assignments is therefore given by the number of choices of v , which is in turn equal to the number of distinct Abelian anyons. Similar considerations apply when classifying more general symmetry quantum numbers.

Next, the fractional charge of the anyons fixes the fractional part of the Hall conductance:

$$e^{2\pi i \sigma_H} = e^{2\pi i Q_v}. \quad (1.2)$$

This relation determines σ_H up to an integer. In general, the braiding properties of the anyons fix the fractional part of the quantized topological response. The integer part of the response can be thought of as a contribution from an invertible state (in this example, an IQH state). For a given numerical or experimental model, it is therefore necessary to determine v and σ_H (or more generally, the full set of symmetry fractionalization anyons and quantized response properties) in order to characterize the system fully. To give a particular example, in the 1/3 Laughlin state [12], v is the quasihole with charge $+e/3$, while $\sigma_H = \frac{1}{3} \frac{e^2}{h}$.

Although the relationship between fractional charge and Hall conductance was long known in the FQH effect, the general understanding of the classification of SET states in terms of two layers of data is much more recent and less well appreciated. Furthermore, although general mathematical theories of invertible as well as SET states are now available, they have not yet

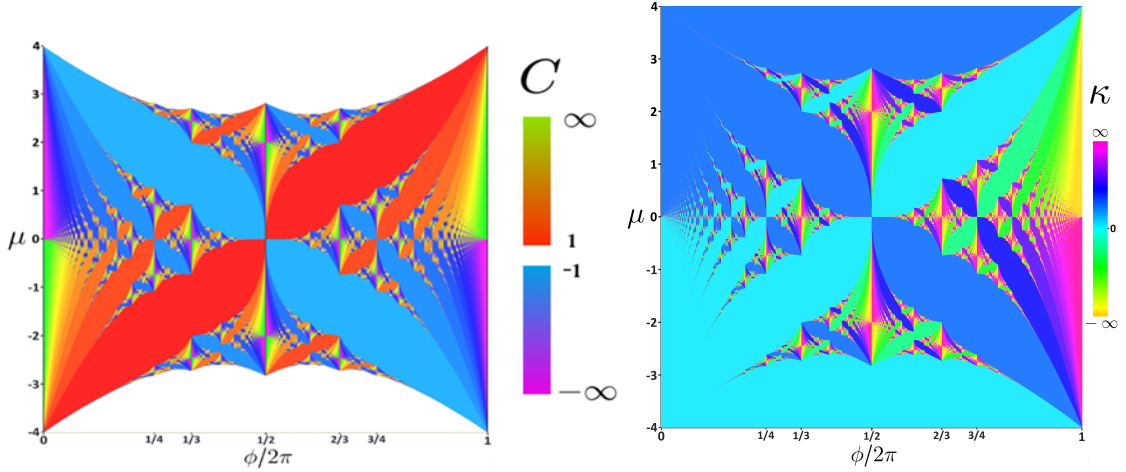


Figure 1.1: Hofstadter butterflies for the Chern number C and the filling invariant κ , plotted using the analytical results in [3]. The figure for the Chern number is adapted from Ref. [13]. The total filling is $\nu = C\phi/2\pi + \kappa$.

been systematically applied to many symmetries of interest (and indeed, doing so is a major goal of our thesis). Below we briefly review some major previous developments in the field that give more context for our work, and then highlight some open problems.

1.3 Review of prior work

1.3.1 Theory of invertible states

As mentioned above, the first experimentally discovered topological state was the integer quantum Hall state in 1980 [14], which is invertible. The Hall conductance in IQH states was understood mathematically as an instance of the Chern number (and was computed analytically in the Hofstadter model) by the group known as TKNN in 1982 [3] (see Fig. 1.1). A model for a Chern insulator without a magnetic field was suggested by Haldane soon after [15]. Further examples of invertible states were discovered in the form of the bosonic Haldane chain [16] and the Kitaev Majorana chain [4], both in 1d, and the $p + ip$ superconductor [5] in 2d. However,

the search for invertible phases exploded with the discovery of topological insulators in 2005 [6, 17, 18], which made it clear that there exist nontrivial invertible states protected by general symmetries, and that charge conservation is not unique in this sense. The next several years saw the development of a range of mathematical techniques to classify invertible states with internal symmetries, assuming free fermions (using K -theory, or equivalently topological band theory) [8, 19, 20], interacting bosons (using group cohomology and cobordism theory) [21, 22] and interacting fermions (using fermionic generalizations of these methods) [7, 11, 23–28]. Currently a complete classification framework is available in each of these cases, but many symmetries of interest have not been fully explored. We will discuss the important case with crystalline symmetries separately below.

1.3.2 Theory of topologically ordered and SET states

Interestingly, states with topological order appeared in the literature in the 1970s in the context of lattice gauge theories [29], much before the concept of topological order existed. The possibility of fractional statistics in (2+1)D systems was appreciated soon after [30, 31]. However, the first experimental system to display topological order was the FQH system (in the form of a fractional Hall conductance) [32]. The term ‘topological order’ was later proposed [33] to describe the properties of FQH states and certain quantum spin liquids, which displayed fractionalized excitations. Important later theoretical advances included schemes to construct exactly solvable models of topologically ordered states [34, 35], as well as proposals to extract topological data from ground state wave functions, using the entanglement entropy [36, 37] and the entanglement Hamiltonian [38–46].

A theory to classify Abelian topological orders using Abelian (equivalently K matrix) Chern-Simons theory was developed in Ref. [47]. It was applied to partially classify SET phases in Ref. [48]. The full theory of SET phases covering non-Abelian anyons and various subtleties pertaining to fermions was only achieved quite recently using the framework of G -crossed braided tensor categories (BTCs) [11, 28, 49–51]. Following this, FQH states and gapped quantum spin liquids have come to be incorporated into the general understanding of SET phases [52, 53]. Currently we have a well-developed theory of SET phases with unitary symmetries, but only a partial theory for antiunitary symmetries [49, 54]

1.3.3 Crystalline symmetries

Given the centrality of crystalline symmetries to our work, we discuss them separately. Even before crystalline topological phases became an active research area, several topological invariants of crystalline (or more generally spatial) symmetries were recognised. These include the filling per unit cell in many lattice models such as the Hofstadter model [3] (see Fig. 1.1), and the shift and spin vector in quantum Hall states [55], which are associated to continuous rotational symmetry in the plane and physically manifest through a quantized Hall viscosity [56–60]. Lattice translation symmetry furthermore plays an important role in the Lieb-Shultz-Mattis constraint and its various generalizations [61–65], in which a system is forbidden from having a trivial product ground state given certain symmetry conditions (such as a fermionic system with translation symmetry and spin-1/2 per unit cell).

The study of topological phases with crystalline symmetries has grown massively in the last decade, with the advent of general techniques to classify invertible and SET phases. Progress

has been especially rapid in the context of free fermions, where the main tool is topological band theory. The classification of topological crystalline insulators and superconductors [66], as well as their higher-order versions (in which fractional charges or unpaired Majorana zero modes are predicted at hinges, corners, and lower-dimensional surfaces of the spatial manifold) [67, 68] has been accomplished to a large degree [69–75]; connections to realistic materials are also being explored in these works, which have initiated the field of topological quantum chemistry.

The classification of bosonic and fermionic SPT phases with interactions and with different crystalline symmetries has been taken forward using a number of ideal wave function or ideal Hamiltonian constructions [76–81]. These methods are applicable quite generally as long as the system does not have gapless chiral edge states. The main idea is to adiabatically connect a given system to an idealized limit in which charges can be placed exclusively at special high-symmetry points within the unit cell (this can be in real space or in momentum space, depending on the problem). In this limit, it is often straightforward to construct ground state wavefunctions and their associated Hamiltonians, and also to determine the abstract classification of topological phases.

An alternative approach, which we in fact develop further and use in this thesis, is known as the *crystalline equivalence principle* [82–85]. This principle (which is a conjecture that has many explicit checks, but no complete proof as yet) states that the classification of topological phases with spatial symmetry G is identical to that of topological phases with some internal symmetry G^{eff} , and moreover that G^{eff} can be determined from G using a concrete algebraic formula [85]. It allows us to use the powerful techniques available for internal symmetries, but in the context of spatial symmetries.

The characterization of lattice models and ground state wave functions through topological

response properties is also an active direction, albeit less well developed. Particular crystalline invariants that have been studied previously include a discrete analog of the Wen-Zee shift [86–88], and a charge polarization that can be quantized by point group symmetries [88–94]; most results on these invariants are however limited to the case with zero Chern number (with notable exceptions [95]). Substantial progress has also been made to extract many-body crystalline invariants using partial symmetry operations [40–42, 96, 97].

1.3.4 Open theoretical questions

Despite the remarkable advances made over the past decade and more, there are a number of basic theoretical questions that need addressing.

First, there are important classes of systems that have not been fully explored using the existing classification techniques. Let us focus on systems with charge conservation and crystalline symmetries. Note that the ideal wavefunction or ideal Hamiltonian constructions mentioned above cannot be applied in the case where the system has chiral edge states (that is, a nonzero chiral central charge). Therefore one needs to generalize the available classification tools to properly include this case. Furthermore, systems with magnetic translation symmetry have a complicated group structure whose implications for topological invariants have not yet been systematically studied. A consequence of this is that many results about crystalline topological invariants have only been studied in detail when $C = 0$ and $\phi = 0$, with limited results in the general case. Crystalline topological invariants have not been systematically studied in classic models such as the Hofstadter model, which have $C, \phi \neq 0$.

A more general problem is that until recently, technical tools to fully capture the topological

response in crystalline topological phases were lacking. Note that it is standard to study quantum Hall systems using effective response theories in terms of background gauge fields (such as Chern-Simons field theory) [47]. This idea existed even earlier for bosonic systems with internal symmetries, in the context of Dijkgraaf-Witten theories [98]. However, the concept of gauge fields and associated topological actions for crystalline symmetries did not exist until recently. This represented a major gap because such topological actions are a particularly convenient way to fully encode topological response properties. Moreover, the standard techniques to study crystalline topological phases, such as ideal wavefunction constructions and topological band theory, are very useful for classification but are not best suited to identifying topological responses and extracting them from lattice models. A number of recent works have sought to address these problems [53, 82, 83, 85, 99–101], and some of them form the core of this thesis.

1.3.5 Experimental progress

We close this section by highlighting recent experimental advances that make the problem of fully characterizing crystalline topological states even more timely. We focus on ordinary and fractional Chern insulators, which form the main subject of this thesis.

1.3.5.1 Chern insulators

As a paradigmatic model for a Chern insulator, the Hofstadter model has attracted immense experimental interest in the last decade. Various states in the Hofstadter model have now been effectively realized in moiré superlattice systems [102–107], ultracold atoms [108–113], and photonic systems [114–116]. In twisted bilayer graphene, many recent experiments have also

demonstrated Chern insulators that exist down to zero magnetic field [117–120].

Photonic crystals (see Ref. [121] for a review) have also been used to study zero modes and fractional photon numbers bound to lattice defects such as disclinations and dislocations [122, 123]. These are especially interesting because defect charges directly encode various crystalline topological invariants.

1.3.5.2 Fractional quantum Hall states

The most precise and well-controlled quantization results on topologically ordered systems have been obtained in FQH systems. Progress has, however, been incremental: the fractional Hall conductance was seen in 1982 [32], the fractional charge of anyons was first convincingly measured in the $1/3$ Laughlin state in 1997 [124, 125]; and the first direct demonstrations of fractional statistics only appeared in 2020 [126, 127]. A direct experimental signature of non-Abelian anyons (such as non-Abelian braiding) has not been observed till date.

1.3.5.3 Fractional Chern insulators (FCIs)

In the last decade, much attention has focussed on FCIs (reviewed in Refs. [128–130]), for several reasons. From a theoretical perspective, they exhibit several phenomena not seen in continuum quantum Hall systems (including the nontrivial crystalline topological invariants studied in this thesis). Moreover, they offer the potential of being realized at low or even zero magnetic fields, in contrast to FQH states, which require strong magnetic fields of ~ 10 T.

The first reported observation of FCI states was in a bilayer graphene heterostructure aligned with a hexagonal boron nitride substrate [131]. This setup however required a strong

magnetic field of 15-30T. FCIs were also reported in magic angle twisted bilayer graphene (TBG) at lower magnetic fields of $\sim 5\text{T}$ [132]. Related 2d systems such as twisted transition metal dichalcogenides (e.g. $\text{WSe}_2, \text{MoTe}_2$) also appear to have Chern bands as well as strong interactions in their moiré lattices [133], and could also potentially host FCIs. Recently, FCI states were claimed to exist even down to zero field in twisted MoTe_2 bilayers [134, 135].

Although FCIs have not been realized in photonic systems, there has been substantial effort towards simulating bosonic fractional Chern insulators, in particular the $1/2$ Laughlin state, in photonic lattices [136–139]. In a promising development, Laughlin states have been realized in a gas of polaritons (bound states of photons with a Rydberg excitation) [140].

1.3.5.4 NISQ processors

In the last five or so years, noisy intermediate-scale quantum (NISQ) devices, built from superconducting qubits, trapped ions, or cold atoms, have attracted significant interest. Although they are prone to errors and have finite-size limitations, these synthetic platforms may be able to simulate a number of phenomena that are extremely hard to realize in conventional solid-state systems.

For example, various experimental signatures of $\mathbb{Z}_2$ quantum spin liquids were studied in quantum simulators built from superconducting qubits [141] and a lattice of Rydberg atoms [142]; the latter experiment also showed signatures of anyonic statistics. Although these platforms could potentially be used to simulate fractional Chern insulators, this has not yet been demonstrated.

More recent experiments have claimed further progress, such as the observation of a non-Abelian zero mode in a $\mathbb{Z}_2$ spin liquid simulated with a superconducting qubit processor [143].

Even more excitingly, a recent work has claimed to observe intrinsic non-Abelian topological order and non-Abelian braiding in a trapped ion quantum simulator [144].

Although the theory of topological phases has run well in advance of experimental capabilities for several years, the pace of recent progress offers the hope that this gap will eventually close. Therefore it is imperative that we develop a more complete understanding of the possible phenomena that could theoretically be simulated on these various platforms.

1.4 Main results and outline of later chapters

1.4.1 Statement of the problem

First we define the symmetries that we study in this thesis. In Chapter 2 we consider bosonic SET phases with the symmetry $G = \text{U}(1) \times G_{\text{space}}$, where G_{space} is an orientation-preserving wallpaper group in two space dimensions. We write $G_{\text{space}} = \mathbb{Z}^2 \rtimes \mathbb{Z}_M$ where $\mathbb{Z}^2$ refers to discrete translation symmetry, and $\mathbb{Z}_M$ for $M = 1, 2, 3, 4, 6$ refers to discrete point group symmetry (often written as C_M in the physics literature). In particular, $M = 2$ defines a parallelogram lattice, $M = 4$ a square lattice, and $M = 3, 6$ either a triangular or a honeycomb lattice. In Chapter 2 we assume that elementary translations commute, so $\phi = 0$.

In all subsequent chapters we consider invertible fermionic phases with the symmetry $G_f = \text{U}(1)^f \times_{\phi} G_{\text{space}}$. G_{space} is as above. The symbol f in G_f and $\text{U}(1)^f$ implies that these groups contain a subgroup of order 2 generated by $(-1)^F$, the fermion parity. $\phi \bmod 2\pi$ denotes the magnetic flux per unit cell, and we assume $\phi = 2\pi p/q$ where p, q are coprime integers. The symbol $\times_{\phi}$ means that the many-body magnetic translation operators generated by $\tilde{T}_{\mathbf{x}}, \tilde{T}_{\mathbf{y}}$ do not

commute; rather, they satisfy

$$\tilde{T}_y^{-1} \tilde{T}_x^{-1} \tilde{T}_y \tilde{T}_x = e^{i\phi \hat{N}} \quad (1.3)$$

where $\hat{N}$ is the number operator.

The main problem can be formulated in two parts. Consider bosonic SET phases with symmetry G (Chapter 2), or invertible fermionic topological phases with symmetry G_f (Chapters 3, 4, 5). Then, we have the

Classification problem: Find the full classification of topological phases with the above symmetries. Note that for SET phases we require the complete data corresponding to the symmetry fractionalization and the topological response. For invertible fermionic phases we require the group which gives the classification under stacking, and an interpretation of this group in terms of topological response properties.

Characterization problem: Given a lattice Hamiltonian or a ground state wave function of an invertible fermionic state with symmetry G_f , develop a numerical scheme to extract all the many-body topological invariants predicted by the above classification. (One can ask the same question for SET phases, but we do not consider this in our thesis.)

For concreteness, we state the same problem but in the specific context of the Hofstadter model, which will be our main numerical example:

Classification and characterization problem for Hofstadter model: Consider the spinless free fermion Hofstadter model with flux ϕ per unit cell on a square lattice, with Hamiltonian

$$H = - \sum_{\langle ij \rangle} e^{-iA_{ij}} c_i^\dagger c_j + \text{h.c.}$$

(We can also consider other lattices, but numerically we focus on the square lattice in this thesis.) For this model, extract a complete set of crystalline topological invariants, and for each invariant obtain a coloring of Hofstadter's butterfly analogous to the coloring with Chern number. Furthermore, if two disconnected regions in the Hofstadter butterfly have the same Chern number, examine if they can be distinguished using one or more crystalline invariants.

We should clarify that although the Hofstadter model is a noninteracting model, we want to study its many-body topological invariants, that is, the ones that remain well-defined even in the presence of interactions that may destroy the band structure, as long as they do not break symmetry or close the gap. Therefore our methods all need to be applicable in the presence of interactions.

Below we will outline the subsequent Chapters of this thesis and explain how these problems are addressed.

1.4.2 Outline of Chapter 2

In this Chapter we study bosonic SET phases described by an Abelian topological order. For a concrete example of what we are seeking, note that there is a well-known topological response theory for Abelian fractional quantum Hall states in terms of K matrix Chern-Simons theory coupled to a background $U(1)$ vector potential A :

$$\mathcal{L} = \frac{1}{4\pi} \epsilon^{\mu\nu\lambda} a_\mu^I K_{IJ} \partial_\nu a_\lambda^J + \frac{1}{2\pi} q_I \epsilon^{\mu\nu\lambda} a_\mu^I \partial_\nu A_\lambda + \frac{k_1}{2\pi} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu A_\lambda. \quad (1.4)$$

The non-degenerate, symmetric integer matrix K characterizes the intrinsic topological order, including the Abelian anyons, their fusion and braiding. The charge vector q_I determines the

fractional U(1) charge of each anyon $\vec{a}$, through the relation $Q_{\vec{a}} = \vec{q}^T K^{-1} \vec{a} \bmod 1$. The coefficient k_1 represents a bosonic integer quantum Hall state that can be added to $\mathcal{L}$ so as to change the Hall conductance without changing the fractional charge of the anyons.

Upon integrating out the dynamical gauge fields a^I , this topological action gives an effective response theory

$$\mathcal{L}_{\text{resp}} = \frac{\vec{q}^T K^{-1} \vec{q} + 2k_1}{4\pi} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda. \quad (1.5)$$

From this one can directly read off the Hall conductance $\sigma_H = (\vec{q}^T K^{-1} \vec{q} + 2k_1) \frac{e^2}{h}$. The different choices of $\vec{q}, k_1$ determine the full classification of bosonic SET phases, assuming a fixed topological order with only U(1) symmetry.

Here the only background symmetry gauge field is the U(1) gauge field A . We wish to generalize this action to include gauge fields for the crystalline symmetries. The main result of Chapter 2 is to develop gauge fields for the full symmetry G defined above, in terms of a triple $B = (A, \vec{R}, \omega)$. A is the usual U(1) gauge field; flux of A corresponds to magnetic flux. The new ingredients are a $\mathbb{Z}^2$ translation gauge field $\vec{R}$ and a discrete $\mathbb{Z}_M$ rotation gauge field ω . Fluxes of $\vec{R}$ and ω correspond to lattice dislocations and disclinations respectively, which may be familiar from elasticity theory. These gauge fields can be defined either discretely on a simplicial manifold, or as differential 1-forms in the continuum.

A major question is how to write down quantized topological terms that couple these gauge fields to the dynamical gauge fields a^I . We show how to do this using the framework of group cohomology and topological quantum field theory (TQFT). The resulting effective action is given in Eq. (2.30), and the response theory derived from it is given in Eq. (2.44). We find that the symmetry fractionalization is specified by four parameters $q_I, s_I, \vec{t}_I, m_I$ which generalize the

charge vector, while the response is fixed by seven additional parameters $k_i, 1 \leq i \leq 7$, which generalize the integer Hall conductance k_1 (see Tables 2.1 and 2.2).

We discuss each topological term in detail. Although many quantitative predictions of the response theory are specific to bosonic SET phases, the qualitative predictions are general and also apply to the invertible fermionic phases we study in later Chapters. In particular, at each high symmetry point $\mathbf{o}$ of the unit cell, we can define two response coefficients $\mathcal{S}_{\mathbf{o}}$ and $\vec{\mathcal{P}}_{\mathbf{o}}$. The theory predicts that they can be measured in (at least) two complementary ways:

Defect charge response: A disclination of angle Ω centered at $\mathbf{o}$ binds fractional charge $\frac{\mathcal{S}_{\mathbf{o}}\Omega}{2\pi}$, while a dislocation with Burgers vector $\vec{b}_{\mathbf{o}}$ binds fractional charge $\vec{\mathcal{P}}_{\mathbf{o}} \cdot \vec{b}_{\mathbf{o}}$.

Dual response (angular and linear momentum of flux): A state $|\Psi\rangle$ with m total flux quanta has angular momentum $l_{\mathbf{o}} = \mathcal{S}_{\mathbf{o}}m + \dots \pmod{M}$, where $e^{\frac{2\pi i}{M}l_{\mathbf{o}}}$ is the eigenvalue under the operator that generates $\mathbb{Z}_M$ point group rotations, and $\dots$ refers to contributions from other terms in the action. The same state also has *linear* momentum in the $i = x, y$ direction $p_{\mathbf{o},i} = \mathcal{P}_{\mathbf{o},i}m + \dots$, where $e^{2\pi i p_{\mathbf{o},i}}$ is the phase of the ground state expectation value of the operator that generates $\mathbb{Z}^2$ translations along the i direction.

The existence of two complementary dual responses for the same invariant is a powerful prediction of TQFT that is difficult to arrive at using any alternative method. Note that these predictions do not account for many subtleties that are only apparent when we consider actual lattice models. Making these predictions more precise and testing them in concrete lattice models is the goal of Chapter 4 (although there we consider invertible fermionic states rather than bosonic SET states).

An interesting application of this theory is to count the number of distinct SET states in specific cases of interest. We find, for example, that there are 2304 distinct SET states with the

intrinsic topological order of the $1/2$ Laughlin state on the square lattice, once the integer part of the filling and Hall conductivity are fixed. Thus, at least for simple topological orders, the classification problem can be solved exactly with our approach.

1.4.3 Outline of Chapter 3

One specific motivation for the theory developed in Chapter 2 was to fully understand the topological response properties of fractional Chern insulators. However, it is much more practical numerically and experimentally to first study the same response properties in ordinary Chern insulators, which are invertible states with $U(1)$ charge conservation and lattice symmetries, together with a nonzero Chern number.

We therefore consider invertible fermionic states with symmetry $G_f = U(1)^f \times_\phi [\mathbb{Z}^2 \rtimes \mathbb{Z}_M]$, which was defined above and consists of $U(1)^f$ charge conservation, discrete $\mathbb{Z}^2$ magnetic translations, and $\mathbb{Z}_M$ point group rotations. Although we set $\phi = 0$ in Chapter 2, many standard models of Chern insulators, such as the Hofstadter model, have $\phi \neq 0$. Thus, we need to generalize the computations in Chapter 2 in two ways, by including invertible fermionic states, and also by allowing a rational flux $\phi = 2\pi \frac{p}{q}$ per unit cell.

It is mathematically most convenient to first develop a general classification framework for invertible fermionic states, and then specialize to the above symmetry. Note that free fermion invertible states have been classified in detail using the mathematical formalism of K-theory, which classifies all inequivalent topological band structures that can exist for a given symmetry group [19, 20, 69, 70]. However, we require a theory which is also valid for interacting systems, in which free fermion classifications are known to break down [145–147]; moreover, certain

invertible states can only be realized in the presence of interactions, and cannot be classified by free fermion methods [27, 80, 85, 148].

Invertible fermionic phases with interactions were first classified using cobordism theory [24]. Although very powerful, this formalism is mathematically very abstract and computationally challenging; moreover, it is difficult to relate cobordism data to observable physical properties. Therefore we use a different mathematical framework developed in Ref. [27] (see also [28, 149]), based on group cohomology and G -crossed braided tensor category theory. This framework very naturally relates the desired topological action to certain group cohomology data that can in principle be computed for any given symmetry by solving a set of algebraic equations. This gives a more physically transparent and computationally simpler classification method than cobordism theory.

This framework strictly applies to internal symmetries, and so we need a systematic argument to extend it to spatial symmetries. In Chapter 2 we argue that the classification of bosonic topological phases with spatial symmetry G is identical to that of bosonic topological phases with internal symmetry G , as long as G is orientation preserving. This is a special case of the *crystalline equivalence principle* (CEP) [82–85]. For fermionic states with symmetry G_f , it is possible to formulate a fermionic crystalline equivalence principle (fCEP), which is much more subtle. According to the fCEP, the classification of fermionic topological phases with spatial symmetry G_f is in one-to-one correspondence with that of topological phases with internal symmetry G_f^{eff} ; but in general $G_f \neq G_f^{\text{eff}}$, even if G_f is orientation preserving. An important achievement of Ref. [85] was to provide a computationally useful algebraic formula of the fCEP relating G_f to G_f^{eff} , which is also valid for reflection and antiunitary symmetries, and which we reproduce here.

This Chapter therefore synthesizes three new ingredients: the crystalline gauge fields used in Chapter 2 with $\phi \neq 0$, the theory of invertible fermionic phases developed in Ref. [27], and an algebraic formulation of the fCEP [85]. Starting with the desired spatial symmetry G_f , we use the fCEP to find G_f^{eff} , and then derive a crystalline gauge theory for invertible fermionic states with the symmetry G_f^{eff} .

The main result of this Chapter is to provide an explicit solution to the classification problem. For invertible fermionic states with $G_f = \text{U}(1)^f \times_{\phi} [\mathbb{Z}^2 \rtimes \mathbb{Z}_4]$ (that is, on a square lattice), the classification is given by the group

$$\mathbb{Z}^3 \times \mathbb{Z}_8 \times \mathbb{Z}_4^2 \times \mathbb{Z}_2.$$

Similar classifications are obtained for $M = 1, 2, 3$ and 6 (see Table 3.1). This classification is contained in a topological effective action that we derive, see Eqs. (3.20) and (3.23).

The three integer invariants are fixed by the chiral central charge c_- , the Chern number C , and the filling ν . The remaining four invariants are described by various quantized coefficients in the effective action, for which we obtain several concrete predictions (see Sec. 3.3.3.3). Extracting these invariants from lattice models forms the focus of Chapters 4 and 5. Interestingly, the overall group structure of the classification does not depend on the value of ϕ , but response coefficients such as the filling ν acquire a ϕ -dependence.

1.4.4 Outline of Chapter 4

The success of TQFT in predicting a classification and a response theory for crystalline topological phases, as outlined in Chapters 2, 3, raises some new problems. Given a set of

topological invariants predicted by field theory, how do we match the field theory to numerics and robustly extract the predicted invariants from lattice models? In this Chapter, we present detailed numerical studies of two invariants, the ‘discrete shift’ $\mathcal{S}_o$ and a quantized charge polarization $\vec{\mathcal{P}}_o$, and verify the predictions regarding them. We compute these invariants in the Hofstadter model, and obtain several new colorings of Hofstadter’s butterfly (see Fig. 4.1). It is easy to verify that regions with the same Chern number can have different values of $\mathcal{S}_o$ and $\vec{\mathcal{P}}_o$; therefore these quantities represent new invariants that extend the phase diagram of the model.

1.4.4.1 Discrete shift $\mathcal{S}_o$

The discrete shift $\mathcal{S}_o$ is a lattice analog of the Wen-Zee shift [55] in clean isotropic continuum quantum Hall systems, which characterizes the response to geometric curvature [56–60, 150–154]. $\mathcal{S}_o$ is an invariant that depends on a discrete $\mathbb{Z}_M$ rotational symmetry and $U(1)^f$ charge conservation; it is defined with respect to a high symmetry point o of the unit cell which is symmetric under M_o -fold rotations. For invertible fermionic topological states, we find that $2\mathcal{S}_o$ is an integer defined mod $2M_o$. We show that for fixed Chern number C , $\mathcal{S}_o$ can in principle take one of M_o distinct values. Thus, in the square lattice unit cell (see Fig. 4.2), there are three invariants $\mathcal{S}_\alpha, \mathcal{S}_\beta, \mathcal{S}_\gamma$, which can take 4, 4 and 2 values respectively for fixed C . For each o , we find that $\mathcal{S}_o \bmod 1 = \frac{C}{2} \bmod 1$. These results agree exactly with the predictions derived in Chapter 3.

We moreover explicitly verify the prediction in Chapter 2 that $\mathcal{S}_o$ can be found in two ways, in terms of a quantized contribution to the fractional charge bound to lattice disclinations (see Eq. (4.35)) and, dually, the fractional angular momentum bound to magnetic flux (see Eq. (4.40)).

Remarkably, $\mathcal{S}_o$ refines the known phase diagram of the Hofstadter model, leading to a new set of square lattice Hofstadter butterflies (Fig. 4.1), which we numerically compute. We theoretically justify several properties of $\mathcal{S}_o$ that are evident from Fig. 4.1, and also propose empirical formulas for $\mathcal{S}_o$ (Eq. (4.41), (4.42)).

1.4.4.2 Quantized charge polarization $\vec{\mathcal{P}}_o$

The calculations in Chapter 3 predict the existence of a quantized many-body polarization in the presence of 2, 3, 4 or 6-fold rotational symmetry. For invertible topological phases, the polarization acquires a $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_2$, or $\mathbb{Z}_1$ classification, respectively. The TQFT not only predicts the presence of the invariant, but also its bulk physical manifestation. As explained in Chapter 2, this is in terms of a fractional quantized contribution of the charge bound to lattice defects (Eq. (4.44)) and a dual response, the momentum of the ground state in the presence of additionally inserted magnetic flux (Eq. (4.21)).

We show how $\vec{\mathcal{P}}_o$ can be extracted from bulk response properties of microscopic models in four different ways. In addition to the above two methods, we extract $\vec{\mathcal{P}}_o$ from the 1d polarization along the $\hat{i}$ direction by treating the system as an effectively 1d system along $\hat{i}$ and using Resta's formula (see Eq. (4.73)). Finally, assuming we know the filling ν , we can measure $\vec{\mathcal{P}}_o$ by taking suitable linear combinations of $\mathcal{S}_o$ for different o (see Table 4.1).

As an application, we extract the quantized charge polarization for the Hofstadter model, providing yet another way to color Hofstadter's butterfly (see Fig. 4.1). We also find empirical formulas for $\vec{\mathcal{P}}_o$ on the square lattice: see Eqs. (4.52), (4.53).

1.4.4.3 Significance of our definition of $\vec{\mathcal{P}}_0$ when $C \neq 0$

Remarkably, the TQFT prediction of a quantized charge polarization applies also in the presence of a non-zero Chern number and a non-trivial magnetic field. In contrast, the modern theory of polarization in insulators is based on the Berry-Zak phase of single-particle wave functions in momentum space [95, 155–157]. This Berry phase theory of polarization assumes the phase of the single-particle states is globally well-defined throughout the Brillouin zone, which applies only in the case of zero Chern number. For the case of non-zero Chern number, while there has been work showing how one may define a notion of polarization in the single-particle context by fixing an origin in the Brillouin zone [95], its quantization from crystalline symmetries, the effects of non-zero magnetic field, and its implication for bulk response properties have not been studied. In fact the TQFT prediction appears to be in tension with several statements made previously in the literature about whether the polarization is well-defined in the presence of a non-zero Chern number [89, 94].

In many-body systems with interactions, the single-particle Berry phase formulation breaks down. It can be replaced with a Berry phase theory based on twisted boundary conditions or with an expectation of Resta’s exponentiated polarization operator [158]. However these apply only to the effective 1d polarization, meaning the system is viewed as an effective one-dimensional system; such a 1d polarization is no longer an intensive quantity in a higher dimensional system.

The implication of this Chapter is that one can indeed define a quantized charge polarization in an intrinsically many-body fashion and in the presence of both non-zero Chern number and non-zero magnetic field. This is not an effective 1d polarization obtained by viewing the system as a 1d system – rather, this is an intrinsic bulk 2d polarization, which has non-trivial bulk

responses mentioned above.

1.4.5 Outline of Chapter 5

Although Chapter 4 makes significant progress towards characterizing crystalline topological invariants for the symmetries we assume, it only studies four invariants in the response theory $(C, \nu, \mathcal{S}_o, \vec{\mathcal{P}}_o)$ out of a total of seven; moreover, one can show that computing $\mathcal{S}_o, \vec{\mathcal{P}}_o$ for each o is insufficient to obtain a full characterization of topological invariants. In Chapter 5, we use a different method, based on partial rotations, to obtain a complete characterization for systems on a square lattice.

Our main result is to numerically extract a set of many-body invariants $\{\Theta_o^\pm\}$ associated to the crystalline symmetries by computing expectation values of partial rotation operators centered at high symmetry points o of the unit cell (see Eq. (5.1)). For example, if o is preserved by rotations of order M_o where M_o is even, then for a fixed Chern number, Θ_o^+ and Θ_o^- can take $M_o/2$ and $2M_o$ different values respectively, for each o .

Recall that the expected theoretical classification for the symmetry G_f on the square lattice is given by the group $\mathbb{Z}^3 \times \mathbb{Z}_8 \times \mathbb{Z}_2 \times \mathbb{Z}_4^2$. We find that the invariants $\{\Theta_o^\pm\}$, together with the Chern number C , the chiral central charge c_- and the filling ν , give a *complete* characterization of these invariants. In particular, the discrete shift $\mathcal{S}_o$ and quantized charge polarization $\vec{\mathcal{P}}_o$ studied in Chapter 4 can be obtained from $\{\Theta_o^\pm, C, c_-, \nu\}$.

Our numerical results are in remarkable agreement with analytical calculations from conformal field theory and TQFT. Specifically, the invariants $\{\Theta_o^\pm\}$ encode the G -crossed modular S, T matrices of symmetry defects [11, 53]. Furthermore, our methods to extract $\{\Theta_o^\pm\}$ require only a

single bulk ground state on a disk, without needing to insert additional defects, adding to a line of work that extracts topological invariants from single bulk ground states [36, 37, 42–46, 159]. With this result, we arrive at a solution to the main problems posed in this thesis.

1.4.6 Outline of Chapter 6

Finally, in Chapter 6, we conclude and discuss some interesting future directions.

Chapter 2: Crystalline gauge theory and quantized discrete geometric response for Abelian topological phases with lattice symmetry

The material in this Chapter is based on the paper *Crystalline gauge fields and quantized discrete geometric response for abelian topological phases with lattice symmetry* by Naren Manjunath and Maissam Barkeshli, Phys. Rev. Research **3**, 013040, Jan 2021.

Notational remarks:

The crystalline gauge fields $\vec{R}, \omega$ which we define in this Chapter all implicitly have a dependence on some origin $\mathbf{o}$ within the lattice unit cell. Therefore all the coefficients in the effective action, Eq. (2.30), and the response theory, Eq. (2.44), may also depend on the choice of $\mathbf{o}$. We will suppress this dependence here, but make it explicit in Chapters 3, 4, 5. In later Chapters we also use $\pm$ superscripts for the coefficients in the response theory; see Chapter 3 for details. Since the superscripts are only necessary in fermionic systems, we will not use them here.

Here we use ν_c to denote the U(1) charge per unit cell and $\vec{\mathcal{P}}_c$ to denote the charge polarization. In later chapters we instead use ν and $\vec{\mathcal{P}}$ respectively.

2.1 Introduction

One of the most striking discoveries in physics is the quantized Hall conductivity of integer and fractional quantum Hall (FQH) systems [160, 161]. The quantized Hall conductivity [?], which requires $U(1)$ charge conservation to define, is however only one of many symmetry-protected topological invariants of FQH systems. In the continuum, clean isotropic quantum Hall systems possess additional symmetry-protected invariants, such as a quantized Hall viscosity [56–58, 151, 162, 163], the shift, and fractional orbital spin of quasiparticles [55]. These invariants define quantized responses to deformations of the spatial geometry [55, 150–152, 164].

The problem of interacting particles in the continuum is in many cases an approximation to interacting particles on a lattice. This approximation is typically only valid in a dilute limit where the lattice effects can be ignored. However topologically ordered phases can also occur when lattice effects are strong, such as in fractional Chern insulators or quantum spin liquids [128, 165]. The crystalline symmetry can in principle allow for new topological invariants that are not possible in continuum systems, while also modifying the known invariants of continuum systems. It is therefore important to understand the possible topological invariants that are protected by the crystalline symmetry of the lattice, together with the on-site symmetry.

In this paper we develop such an understanding in the case of (2+1)D Abelian bosonic topological phases with symmetry group $G = U(1) \times G_{\text{space}}$, where $G_{\text{space}} = \mathbb{Z}^2 \rtimes \mathbb{Z}_M$, for $M = 1, 2, 3, 4, 6$, is a discrete orientation-preserving space group symmetry of a lattice. To do this, we develop a theory of discrete “crystalline gauge fields” coupled to the emergent dynamical $U(1)$ gauge fields that describe the topological order. The crystalline gauge fields include gauge fields associated with the discrete translation and rotation symmetries, which keep track of certain

Parameter	Characterizing symmetry fractionalization				SPT terms: integer contributions to response theory						
	$\vec{q}$	$\vec{s}$	$\vec{t}$	$\vec{m}$	k_1	k_2	k_3	$\vec{k}_4$	$\vec{k}_5$	k_6	k_7
Allowed values	$\mathbb{Z}^D$	$\mathbb{Z}^D$	$\mathbb{Z}^D \times \mathbb{Z}^D$	$\mathbb{Z}^D$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}$	$\mathbb{Z}$
Trivial values	$K\vec{\Lambda}$	$K\vec{\Lambda}_1 + M\vec{\Lambda}_2$	$\begin{pmatrix} K\vec{\Lambda}_1 \\ K\vec{\Lambda}_2 \end{pmatrix} + (1 - U(\frac{2\pi}{M}))\begin{pmatrix} \vec{\Gamma}_1 \\ \vec{\Gamma}_2 \end{pmatrix}$	$K\vec{\Lambda}$	0	$M\mathbb{Z}$	$M\mathbb{Z}$	$(1 - U(\frac{2\pi}{M}))\mathbb{Z}^2$	$(1 - U(\frac{2\pi}{M}))\mathbb{Z}^2$	0	$M\mathbb{Z}$
Classification	$\mathcal{A}$	$\mathcal{A}/M\mathcal{A}$	$K_M \otimes \mathcal{A}$	$\mathcal{A}$	$\mathbb{Z}$	$\mathbb{Z}_M$	$\mathbb{Z}_M$	K_M	K_M	$\mathbb{Z}$	$\mathbb{Z}_M$

Table 2.1: Summary of the parameters defining topologically non-trivial terms in the effective action (Eq. 2.30) for $G = \text{U}(1) \times G_{\text{space}}$, and their classification. $\vec{q}, \vec{s}, \vec{t}, \vec{m}$ characterize symmetry fractionalization, while k_i parameterize additional SPT (Dijkgraaf-Witten) terms in effective action. The topological order is characterized by a $D \times D$ K matrix, and the vectors $\vec{\Gamma}_i, \vec{\Lambda}_i$ are arbitrary $D \times 1$ integer vectors. $\mathcal{A}$ is the Abelian group arising from fusion of the anyons. $K_M = \mathbb{Z}_1, \mathbb{Z}_2^2, \mathbb{Z}_3, \mathbb{Z}_2, \mathbb{Z}_1$, for $M = 1, 2, 3, 4, 6$ respectively. Relabelling the dynamical gauge fields can give redundancies among different choices of the above parameters.

geometric properties of the lattice, such as the presence of dislocations and disclinations, and areas and lengths of closed cycles in lattice units. As such, they form a discrete analog of the coframe field and spin connection used in continuum geometry. While crystalline gauge fields have been discussed before in the theory of elasticity [166], previous treatments in elasticity theory have not fully taken into account the non-Abelian nature of the space groups involved. We note that recently crystalline gauge fields have also been used in the study of quantum phases of matter, see e.g. Ref. [82, 167], although effective actions involving both translation and rotation gauge fields have not to our knowledge been discussed previously.

Recently a powerful algebraic theory using G-crossed braided tensor categories has been developed to comprehensively characterize and classify (2+1)D topologically ordered phases of matter with symmetry [11]. In the case of Abelian topological orders with symmetries whose action does not permute distinct quasiparticle types, an alternate approach using topological effective actions, which we develop here, is significantly simpler and yields insight into the physical response.

Our results may be of particular relevance in a number of physical systems. These include the experimentally realized fractional Chern insulators in van der Waals heterostructures [132,

Fractional symmetry quantum numbers	
Generalized charge vector	Associated quantum number
$\frac{q_I}{2\pi} a^I \cup dA$	$Q_{\vec{l}} = \vec{l}^T K^{-1} \vec{q}$, fractional charge of $\vec{l}$
$\frac{s_I}{2\pi} a^I \cup d\omega$	$L_{\vec{l}} = \vec{l}^T K^{-1} \vec{s}$, fractional angular momentum of $\vec{l}$
$\frac{\vec{t}_I}{2\pi} a^I \cup d\vec{R}$	$\vec{P}_{\vec{l}} = (1 - U^T(\frac{2\pi}{M}))^{-1} (\vec{l}^T K^{-1} \vec{t}_x, \vec{l}^T K^{-1} \vec{t}_y)^T$, fractional linear momentum of $\vec{l}$
$\frac{m_I}{2\pi} a^I \cup A_{XY}$	$\tau_{\vec{l}} = \vec{l}^T K^{-1} \vec{m}$, fractionalization of translation algebra: $T_{x,\vec{l}} T_{y,\vec{l}} = T_{y,\vec{l}} T_{x,\vec{l}} e^{i\tau_{\vec{l}}}$
Quantized fractional response terms	
Response theory term	Associated response property
$\frac{\sigma_H}{2} A \cup dA$	σ_H = Hall conductivity
$\frac{\mathcal{S}}{2\pi} A \cup d\omega$	Defines discrete analog of shift. Charge of $2\pi/M$ disclination is $\mathcal{S}/M$, angular momentum of ϕ flux given by $\mathcal{S}\phi/2\pi$.
$\frac{\ell_s}{4\pi} \omega \cup d\omega$	Angular momentum of elementary disclination equals ℓ_s/M (up to framing anomaly)
$\frac{\vec{\mathcal{P}}_c}{2\pi} \cdot A \cup d\vec{R}$	Fractional quantized charge polarization: (i) Charge of dislocation with Burgers vector $\vec{b}$ equals $\vec{\mathcal{P}}_c \cdot \vec{b}$; (ii) Charge per unit length on a boundary along $\hat{e}$ equals $\vec{\mathcal{P}}_c \cdot \hat{e}$; (iii) A U(1) flux of ϕ has linear momentum equal to $\vec{\mathcal{P}}_c \phi/2\pi$
$\frac{\vec{\mathcal{P}}_s}{2\pi} \cdot \omega \cup d\vec{R}$	Fractional quantized angular momentum polarization: Angular momentum of dislocation with Burgers vector $\vec{b}$ equals $\vec{\mathcal{P}}_s \cdot \vec{b}$
$\frac{\Pi_{ij}}{4\pi} R_i \cup dR_j$	Fractional quantized torsional response: Momentum of dislocation with Burgers vector $\vec{b}$ is $\vec{P}_{\text{disloc},\vec{b}} = \Pi \vec{b}$
$\frac{\nu_c}{2\pi} A \cup A_{XY}$	ν_c = charge per unit cell (filling)
$\frac{\nu_s}{2\pi} \omega \cup A_{XY}$	ν_s = angular momentum per unit cell
$\frac{\vec{\nu}_p}{2\pi} \cdot \vec{R} \cup A_{XY}$	$\vec{\nu}_p = \vec{P}_{\vec{m}}$ linear momentum per unit cell

Table 2.2: Summary of the quantized topological terms that arise in the effective action for a topological order coupled to a background crystalline gauge field $B = (A, \vec{R}, \omega)$ for the symmetry $G = \text{U}(1) \times G_{\text{space}}$. $A, \vec{R}$ and ω refer to the U(1), translation ($\mathbb{Z}^2$) and point group rotation ($\mathbb{Z}_M$) components of the gauge field, while A_{XY} denotes the area element and can be written in terms of $(\vec{R}, \omega)$. $d\vec{R}$ is defined in Eq. 2.14. Symmetry quantum numbers are associated to the coupling terms between B and the Abelian topological order, specified by a K matrix of internal gauge fields. $U(\frac{2\pi}{M})$ is the elementary point group rotation matrix. The response coefficients are obtained by integrating out the internal gauge fields. The classification of the parameters in the effective action is summarized in Table 2.1.

168] and synthetic quantum Hall systems in photonics [115, 123] or ultracold atoms [138, 169]. These platforms may in particular be able to directly measure the (fractionally) quantized charges bound to lattice dislocations and disclinations. Our results are also of relevance for the study of quantum Hall systems with crystalline symmetries on orbifolds [150, 170, 171], polygons, and two-dimensional surfaces of polyhedra.

Our results are summarized in Tables 2.1 and 2.2. We find that in general symmetry fractionalization for $G = \text{U}(1) \times G_{\text{space}}$ is determined by four invariants, which are specified by a charge vector $\vec{q}$, a discrete spin vector $\vec{s}$, a discrete torsion vector $(\vec{t}_x, \vec{t}_y)$, and an area vector $\vec{m}$. The discrete spin vector $\vec{s}$ is a discrete version of the well-known spin vector used in continuum FQH states [55], which specifies a fractional orbital angular momentum for the anyons [170]. The discrete torsion vector $\vec{t}$ has no analog in the continuum and can only be non-trivial for $M = 2, 3, 4$ -fold lattice rotational symmetry; it specifies a fractional linear momentum for the anyons that does not appear to have been discussed in previous studies of topological phases of matter. Finally the area vector $\vec{m}$, which also has no analog in the continuum, specifies the anyon per unit cell [172] and determines how the anyons effectively fractionalize the translation algebra [172–177]. The discrete spin and torsion vectors $\vec{s}$ and $\vec{t}$ furthermore can only be non-trivial when there is some appropriate commensuration between M , the order of the point group symmetry, and the group structure of the fusion rules of the anyons.

The quantized response theory, obtained by integrating out the dynamical $\text{U}(1)$ gauge fields, provides the response of the system to background gauge fields describing background electromagnetic fields and geometrical defects of the lattice (see Eq. 2.44). We find, for example,

1. A discrete analog of the shift of FQH states. This binds a quantized fractional charge

(modulo the charge of the anyons) to disclinations and angular momentum to magnetic flux.

2. Fractional quantized angular momentum for disclinations.
3. Fractional quantized charge polarization for $M = 2, 3, 4$ -fold rotational symmetry. This implies a fractional charge bound to lattice dislocations (modulo the charge of the anyons), fractional charge per unit length along boundaries (modulo the charge of the anyons), and associates a quantized momentum to $U(1)$ flux.
4. An angular momentum analog of the fractional charge polarization, which associates a fractional angular momentum to dislocations (modulo the angular momentum of the anyons).
5. Fractional quantized charge ν_c , angular momentum ν_s , and linear momentum $\vec{\nu}_p$ per unit cell. The charge filling ν_c gives a generalized Lieb-Schulz-Mattis constraint that imposes constraints on the topological order, $\vec{m}$, and $\vec{q}$ given the charge per unit cell [172].
6. Fractional quantized torsional response which associates momentum to dislocations. In particular, this addresses a long-standing issue raised by Ref. [178, 179], where the coupling to continuum geometry gave an unquantized torsional Hall response; our work predicts that properly taking into account the discrete crystalline space group symmetry gives rise to a fractional quantized torsional response, but only for $M = 2, 3, 4$.

Our effective field theory allows us to explicitly classify all distinct symmetry-enriched topological phases for a given Abelian topological order (for the case where symmetries do not permute the anyons). We find, for example, that there are 2304 distinct symmetry-enriched

topological states with the intrinsic topological order of the $1/2$ Laughlin state on the square lattice, once the integer part of the filling and Hall conductivity are fixed.

The outline of this Chapter is as follows. In Section 2.2 we define the background crystalline gauge field for $G = \text{U}(1) \times G_{\text{space}}$ on a manifold $\mathcal{M}$ with a triangulation, and in Section 2.3 we study their gauge transformations and the symmetry fluxes associated to them. The effective action for SET phases with G symmetry is discussed in Section 2.4 by coupling the crystalline gauge field and background $\text{U}(1)$ gauge field to the dynamical gauge fields that specify the intrinsic topological order. In this Section we also study the effective response theory obtained by integrating out the internal gauge fields. Specific examples involving the $1/2$ Laughlin topological order and the $\mathbb{Z}_2$ gauge theory are discussed in Section 2.5. In Section 2.6, we compare our formulation of crystalline gauge theory on a discrete triangulation to the more standard continuum field theory approach and compare the crystalline gauge fields to the coframe fields and spin connection used in continuum geometry. We conclude with a discussion in Section 2.7.

The main purpose of this chapter is to give a physical interpretation of the response coefficients in the topological action. We therefore do not show how to compute the full classification of SET phases in the main text. However, in Appendix A.2, we give several examples showing the full classification, and also discuss how it is reduced when we account for relabellings of the gauge fields.

2.2 Crystalline gauge fields

At a formal mathematical level, our theory of crystalline gauge fields is equivalent to treating the discrete space group symmetry G_{space} as an internal symmetry of the topological

effective field theory. The main difference with usual internal symmetries, which arise from on-site symmetries of a microscopic lattice model, is the physical interpretation of the crystalline gauge fields, which in turn requires certain gauge-invariant quantities to be determined by geometric properties of the underlying lattice, as we describe below.

Ultimately, the topological field theory that we develop in terms of the quantum Chern-Simons theory possesses an implicit dependence on a space-time metric, which is the framing anomaly associated with the chiral central charge c_- [152, 180]. To be physically meaningful, this space-time metric must be determined by the crystalline gauge fields (see Section 2.4.2.7). Further discussion regarding the relation between the space group symmetry in bosonic lattice systems and internal symmetries of the topological effective field theory is presented in Sec. 2.7. A more complete discussion which applies to fermionic systems and to more general symmetries can be found in Chapter 3.

We consider a $(2+1)$ D space-time manifold $\mathcal{M} = \Sigma^2 \times \mathbb{R}$, where Σ^2 is the space on which the clean lattice system is defined. We fix an arbitrary triangulation of $\mathcal{M}$ and we define on the links a gauge field valued in the symmetry group $G = \text{U}(1) \times G_{\text{space}}$. $G_{\text{space}} = \mathbb{Z}^2 \rtimes \mathbb{Z}_M$ contains translation symmetry and a discrete M -fold rotation symmetry for $M = 1, 2, 3, 4, 6$. Physical results will be independent of triangulation. We define a $\text{U}(1)$ gauge field A_{ij} on the link ij of the triangulation, with the link directed towards j (with $A_{ij} = -A_{ji}$ and $A_{ij} \sim A_{ij} + 2\pi$). Next, we define the crystalline gauge field

$$B_{ij} = (\vec{R}_{ij}, \omega_{ij}). \quad (2.1)$$

Here,

$$\vec{R}_{ij}^T = (X_{ij}, Y_{ij}) = ((R_{ij})_x, (R_{ij})_y) \in 2\pi\mathbb{Z}^2 \quad (2.2)$$

is an integer gauge field corresponding to $\mathbb{Z}^2$ translations. The field ω corresponds to point group rotations, where we take $\omega_{ij} \in \frac{2\pi}{M}\mathbb{Z}$, with $\omega_{ij} \sim \omega_{ij} + 2\pi$. Group multiplication is given by $(\vec{R}_1, \omega_1)(\vec{R}_2, \omega_2) = (\vec{R}_1 + U(\omega_1)\vec{R}_2, \omega_1 + \omega_2)$, where we use addition in place of multiplication when the group is abelian. $U(\omega_1)$ is the 2×2 rotation matrix corresponding to ω_1 . Formally $\frac{1}{2\pi}B$ is a lift of an element of G_{space} to $\mathbb{Z}^2 \rtimes \frac{1}{M}\mathbb{Z}$, while A is a lift from $U(1)$ to $\mathbb{R}$.

The gauge freedom in $\vec{R}$ corresponds to the freedom to relabel lattice coordinates. It arises from the well-known ambiguity in elasticity theory that the displacement vector is only meaningful up to an integer lattice vector [166]. The gauge freedom in ω corresponds to the freedom in locally orienting the x and y axes at every point in space and time. For example, if for $M = 4$ we have $\omega_{ij} = \pi/2$ on some link ij , this means the local coordinate axes at i and j will be rotated relative to each other by an angle $\pi/2$.

Under a gauge transformation which places the gauge variable $(\vec{r}_i, h_i)$ at the vertex i , we have:

$$\begin{aligned} B_{ij} &\rightarrow (\vec{r}_i, h_i)^{-1} B_{ij} (\vec{r}_j, h_j) \\ &= (U(-h_i)(\vec{R}_{ij} + U(\omega_{ij})\vec{r}_j - \vec{r}_i), -h_i + \omega_{ij} + h_j). \end{aligned} \quad (2.3)$$

The underlying lattice of the physical system specifies the gauge invariant quantities of the crystalline gauge field. Flux of ω corresponds to disclinations: $\oint_\gamma \omega$ gives the total angle of

disclinations within the cycle γ . If ω vanishes everywhere, then $\oint_{\gamma} \vec{R}$ gives the total Burgers vector of dislocations contained in γ . If space is a torus and ω vanishes everywhere, then $\oint_x X$, $\oint_y Y$ give the lengths of the torus in the x and y directions, while $\oint_y X$ gives the shear in the x direction upon traversing the y cycle, and similarly for $\oint_x Y$.

When ω is non-zero, one needs to take into account the local change of coordinate frame along γ . Let o be the origin of γ and define a path $o12 \dots n$, with $o = 0$. Consider the product $B_{o1}B_{12} \dots B_{n-1,n}$, where $B_{ij} \in G_{\text{space}}$. The translation component of this product is given by

$$\int \vec{\mathcal{R}}^{(0)} := \sum_{k=0}^{n-1} U(\omega_{o1} + \omega_{12} + \dots + \omega_{k-1,k}) R_{k,k+1} \quad (2.4)$$

Motivated by this, we define a Burgers vector $\oint_{\gamma} \vec{\mathcal{R}}^{(0)}$, where

$$\vec{\mathcal{R}}_{k,k+1}^{(0)} = U(\omega_{o1} + \omega_{12} + \dots + \omega_{k-1,k}) \vec{R}_{k,k+1} \quad (2.5)$$

for some arbitrary choice of origin o and path from o to k . The extra ω factors play a role analogous to the covariant derivative allowing parallel transport of $\vec{R}$ on the lattice. Under a gauge transformation,

$$\oint_{\gamma} \vec{\mathcal{R}}^{(0)} \rightarrow U(-h_0) \oint_{\gamma} \vec{\mathcal{R}}^{(0)}, \quad (2.6)$$

corresponding to the fact that the Burgers vector rotates under rotation of the local coordinate system at the origin o . The value of this Burgers vector is invariant under the $\vec{r}$ -dependent part of the gauge transformation (i.e. the translation gauge transformations), but is only well-defined up to an overall rotation. In general the value of this integral around a closed loop γ defines the total

Burgers vector for any dislocations located inside γ . In the special case of a closed loop in a flat configuration, $\oint_{\gamma} \vec{\mathcal{R}}^{(0)} = 0$. To compare Burgers vectors in different regions, it is important that a common origin 0 is chosen.

$(\vec{R}, \omega)$ thus play a role similar to the coframe field and spin connection used in continuum geometry (see Section 2.6.2 for further discussion); it is useful to distinguish them because $(\vec{R}, \omega)$ have discrete gauge transformations, which play a crucial role in the classification of topological terms. Note that we do not consider the continuous elastic response of the crystal due to stresses and strains, which does not receive any topological, quantized contributions [181, 182].

2.3 Symmetry fluxes

In order to construct the effective topological field theory, we need to understand how to construct symmetry fluxes that can be used in the effective action. While symmetry fluxes for A and ω are relatively straightforward, the symmetry fluxes for the translation gauge field are more complicated, particularly in the presence of the rotation gauge field ω . Mathematically, when the gauge fields are flat the symmetry fluxes correspond to representative 2-cocycles associated with the second group cohomology $\mathcal{H}^2(G, \mathbb{Z})$.

The U(1) gauge flux

$$dA[012] = A_{01} + A_{12} - A_{02} \quad (2.7)$$

defined on a 2-simplex [012] of the triangulation is gauge-invariant, with $dA \sim dA + 2\pi$. Note that mathematically d corresponds to the coboundary operation on the triangulation.

ω behaves mathematically like a discrete version of A ; the flux $\int_D d\omega$ for any region D is

gauge-invariant and gives the total angle of disclinations within D . Below we will discuss the fluxes associated to translation symmetry, which are less familiar.

2.3.1 The flux $d\vec{\mathcal{R}}$ and its relation to dislocation density

Naively one may think that $d\vec{\mathcal{R}}^{(0)}$ should be the gauge-invariant physical quantity corresponding to the dislocation density. However $d\vec{\mathcal{R}}^{(0)}$ depends on a choice of origin together with a choice of local coordinate frame at that origin. Therefore $d\vec{\mathcal{R}}^{(0)}$ is both non-local in general and also not gauge invariant. Moreover, in the presence of a disclination, the value of $\vec{\mathcal{R}}_{ij}^{(0)}$ depends on the precise path chosen between the origin and i , and is therefore ambiguous up to a rotation by the disclination angle.

The solution is to instead use the $\vec{R}$ fields themselves, which are local. But there is considerable ambiguity in $\vec{R}$ under gauge transformations. In particular, we now show that gauge transformations preserve the value of $d\vec{R}$ only up to terms of the form $(1 - U(\frac{2\pi}{M}))d\vec{\Gamma}$ where $\frac{1}{2\pi}\vec{\Gamma} \in \mathbb{Z}^2$.

We argue as follows. From the definition of $\vec{\mathcal{R}}^{(0)}$ we have

$$\vec{R}_{ij} = \vec{\mathcal{R}}_{ij}^{(0)} + (1 - U(\omega_{0 \rightarrow i}))\vec{R}_{ij}. \quad (2.8)$$

Here we have defined $\omega_{0 \rightarrow i} = \int_{\gamma} \omega$ for some given path γ from the origin 0 to the point i . The last term is of the form $(1 - U(\frac{2\pi k}{M}))\vec{R}_{ij}$, for some integer k . Let $\omega_{0 \rightarrow i} = \frac{2\pi k_{0,i}}{M}$. Using the fact

that $1 - U^k = (1 - U)(1 + U + \dots + U^{k-1})$, we conclude that

$$\vec{R}_{ij} = \vec{\mathcal{R}}_{ij}^{(0)} + (1 - U(2\pi/M))(1 + U(2\pi/M) + \dots + U^{(k_{0,i}-1)}(2\pi/M))\vec{R}_{ij} \quad (2.9)$$

$$:= \vec{\mathcal{R}}_{ij}^{(0)} + (1 - U(2\pi/M))\vec{\Gamma}_{ij} \quad (2.10)$$

The last line defines the vector field $\vec{\Gamma}$ in terms of $\vec{R}$, with $\frac{1}{2\pi}\vec{\Gamma} \in \mathbb{Z}^2$.

A general gauge transformation sends $\vec{R}_{ij} \rightarrow U(-h_i)(\vec{R}_{ij} + U(\omega_{ij})\vec{r}_j - \vec{r}_i)$. But the above relation will still hold with $\vec{\Gamma}$ replaced by some $\vec{\Gamma}'$ where $\frac{1}{2\pi}\vec{\Gamma}' \in \mathbb{Z}^2$. Now under gauge transformations, assume that the coordinate axes at the origin are rotated by the angle $2\pi m/M$. Then $d\vec{R}$ transforms as

$$d\vec{R} = d\vec{\mathcal{R}}^{(0)} + (1 - U(2\pi/M))d\vec{\Gamma} \quad (2.11)$$

$$\rightarrow U(2\pi m/M)d\vec{\mathcal{R}}^{(0)} + (1 - U(2\pi/M))d\vec{\Gamma}' \quad (2.12)$$

$$= d\vec{\mathcal{R}}^{(0)} + (1 - U(2\pi/M))(d\vec{\Lambda} + d\vec{\Gamma}') \quad (2.13)$$

where $\vec{\Lambda} = (1 + U(2\pi/M) + \dots + U^{m-1}(2\pi/M))\vec{\mathcal{R}}^{(0)}$, and also satisfies $\frac{1}{2\pi}\vec{\Lambda} \in \mathbb{Z}^2$.

Therefore gauge transformations preserve the value of $d\vec{R}$ only up to terms of the form $(1 - U(\frac{2\pi}{M}))d\vec{\Gamma}$.

To summarize, the correct definition of a Burgers vector, given in terms of $\vec{\mathcal{R}}^{(0)}$, is nonlocal due to the choice of origin 0, and so we are forced to use the field $\vec{R}$ instead in the effective action. $d\vec{R}$ is not gauge-invariant: it is determined only up to terms of the form $(1 - U(\frac{2\pi}{M}))d\vec{\Gamma}$. However, $d\vec{R}$ and $d\vec{\mathcal{R}}^{(0)}$ are gauge-equivalent up to such terms. Therefore the fractional part of

M	$(1 - U(\frac{2\pi}{M}))^{-1}(a, b)^T$	Gauge invariants mod 1	K_M
2	$\frac{1}{2}(a, b)^T$	$\frac{1}{2}\{(0, 0), (1, 0), (0, 1), (1, 1)\}$	$\mathbb{Z}_2^2$
3	$\frac{1}{3}(2a + b, b - a)^T$	$\{(0, 0), (1/3, 1/3), (2/3, 2/3)\}$	$\mathbb{Z}_3$
4	$\frac{1}{2}(a + b, b - a)^T$	$\{(0, 0), (1/2, 1/2)\}$	$\mathbb{Z}_2$
6	$(a - b, a)^T$	$(0, 0)$	$\mathbb{Z}_1$

Table 2.3: Gauge invariant, locally well-defined part of the Burgers vectors for different rotation point groups, with $\frac{1}{2\pi} \oint \vec{\mathcal{R}}^{(0)} = (a, b)^T \in \mathbb{Z}^2$. We use a lattice basis where $U(2\pi/M)$ takes $\hat{x} \rightarrow \hat{y}$ for $M \neq 2$ (see Appendix A.1).

$(1 - U(\frac{2\pi}{M}))^{-1}d\vec{R}$ is (i) local, (ii) gauge-invariant, and (iii) equal to the physically meaningful quantity $\frac{1}{2\pi}(1 - U(\frac{2\pi}{M}))^{-1}d\vec{\mathcal{R}}^{(0)} \bmod 1$.¹ This motivates us to define the local quantity

$$d\vec{\mathcal{R}} = \left(1 - U\left(\frac{2\pi}{M}\right)\right)^{-1} d\vec{R}, \quad (2.14)$$

which captures the local, gauge-invariant part of a Burgers vector.

The possible holonomies thus fall into different classes based on the distinct values taken by

$$\frac{1}{2\pi} \oint_{\partial D} \vec{\mathcal{R}} \bmod 1. \quad (2.15)$$

Eq. 2.15 defines a finite group grading on Burgers vectors, where we denote the finite group as K_M , and which is formally defined as

$$K_M = \mathbb{Z}^2 / (1 - U(2\pi/M)\mathbb{Z}^2). \quad (2.16)$$

To understand this physically, note that to each region D we can assign a local Burgers vector

¹Note that this quantity is invariant if the origin changes by a lattice vector, but is *not* invariant if the origin changes by a fraction of a lattice vector. See Chapter 4 for a detailed discussion.

with the choice of origin $0 \in D$. Without picking a common origin, the Burgers vector for a region containing two subregions D and D' is thus ambiguous up to separate local rotations of the coordinate axes for the origins $0 \in D$ and $0' \in D'$. This is explained below in more detail. The part of the Burgers vector that is gauge invariant and can be defined locally defines a finite group grading on Burgers vectors, where we denote K_M as the finite group. The results for various M are given in Table 2.3.

Note that a nontrivial Burgers vector is associated to dislocation defects as well as disclination defects, which additionally have a nonzero holonomy of ω . A disclination dipole is a composite of two defects in which the individual ω holonomies are equal and opposite; however, the net $\vec{R}$ holonomy may still be nonzero. This is the gauge-theoretic formulation of the well-known fact that a disclination dipole is physically equivalent to a dislocation.

2.3.1.1 Understanding the group K_M

There are a number of ways to understand K_M more intuitively and physically. Let us consider the most direct way following the mathematical derivation above. A second derivation based on rotationally symmetric configurations of boundary charge is discussed in Section 2.4.2.

Let us first consider the case $M = 2$, and start by considering a small region with a locally defined Burgers vector (a, b) (see Fig. 2.1). Under a local rotation of the space, this Burgers vector transforms to $(a, b) \rightarrow (-a, -b)$. Thus the Burgers vector $(1, 0) \sim (-1, 0)$ and $(0, 1) \sim (0, -1)$. Now consider two regions, each with a locally defined Burgers vector (a, b) and (a', b') . The combined Burgers vector thus would be $(a + a', b + b')$. Upon a π rotation of the second region however, $(a', b') \rightarrow (-a', -b')$, so $(a + a', b + b') \rightarrow (a - a', b - b')$. Therefore, when

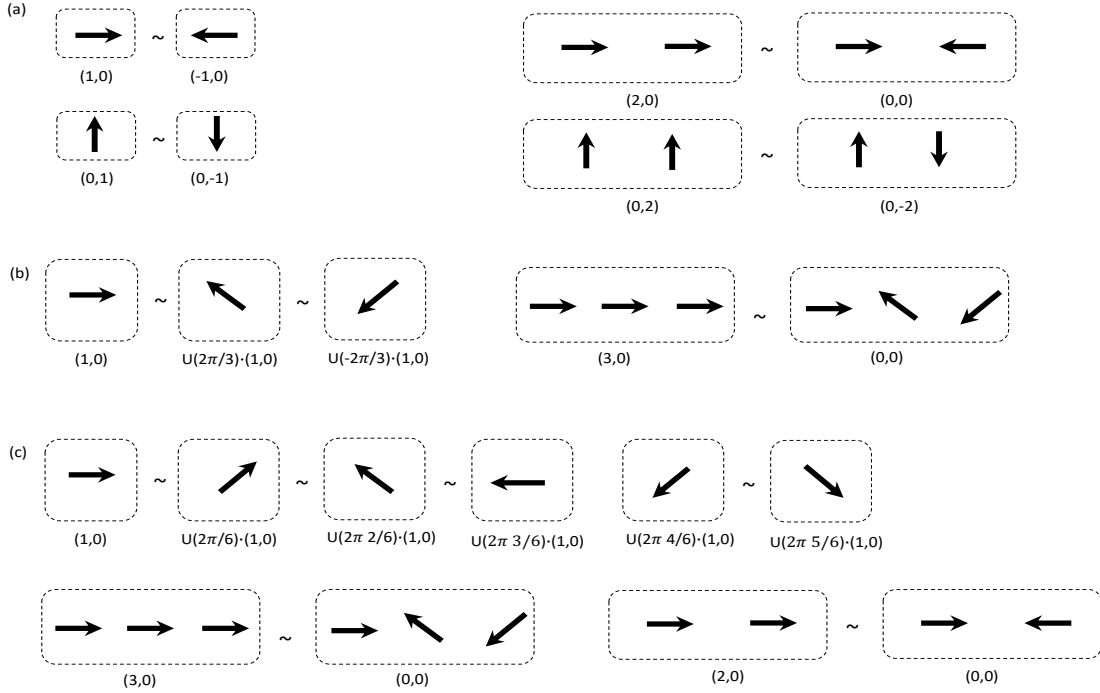


Figure 2.1: Visual representation of how the groups K_M classify dislocation Burgers vectors. (a) For $M = 2$, the vectors (a, b) and $(-a, -b)$ are in the same equivalence class. Moreover, the sum of two neighbouring Burgers vectors can be viewed as either $(a, b) + (a', b')$ or $(a, b) - (a', b')$; this gives the relations $(0, 0) \sim (2, 0) \sim (0, 2)$, which reduce the classification to a group $\mathbb{Z}_2 \times \mathbb{Z}_2$. (b) For $M = 3$, we see that $(3, 0) \sim (1 + U(2\pi/3) + U(4\pi/3))(1, 0)^T = (0, 0)$; in general $(2a + b, b - a) \sim (0, 0)$, so the classification is given by $K_3 \cong \mathbb{Z}_3$. (c) For $M = 6$, we can combine the $M = 2$ and $M = 3$ results to show that $(0, 0) \sim (2, 0) \sim (3, 0)$; thus $(0, 0) \sim (1, 0)$, and similarly $(0, 0) \sim (0, 1)$. Therefore every Burgers vector can be trivialized. Similar reasoning applied to the $M = 4$ case gives $K_4 \cong \mathbb{Z}_2$.

considering the Burgers vector of a large region containing Burgers vectors in smaller regions, $(2, 0) \sim (0, 0)$ and similarly $(0, 2) \sim (0, 0)$. We see that the Burgers vectors form the group $\mathbb{Z}_2 \times \mathbb{Z}_2$, due to the fact that the Burgers vector of a region, when including these local rotations, is only partially well-defined. An equivalent analysis for $M = 3, 4, 6$ gives the groups $\mathbb{Z}_3$, $\mathbb{Z}_2$, and the trivial group (see Fig. 2.1).

In general, dislocations whose Burgers vectors are of the form $(1 - U(2\pi/M))\vec{b}$ are equivalent to zero. If we have two neighbouring dislocations with $\vec{b}$ and $-\vec{b}$, the total Burgers vector associated to a loop containing the dislocations is zero. However a local rotation of $-\vec{b}$ by the angle $2\pi/M$ will give a net holonomy equal to $\vec{b} - U(2\pi/M)\vec{b}$ around the same loop. These values of Burgers vectors are therefore considered to be in the trivial equivalence class. This is what we mean by the statement that rotation gauge symmetry induces a finite group grading on Burgers vectors. They are thus classified by elements of $\mathbb{Z}^2$ modulo $(1 - U(2\pi/M))\mathbb{Z}^2$, which can be taken as the mathematical definition of K_M .

Mathematically, if we consider a generic group element in G_{space} , we can define the K_M grading of the translation component of the group element. One can show that this K_M grading is invariant under conjugation; therefore this K_M grading can be viewed as an invariant of conjugacy classes of G_{space} . The same idea can be expressed intuitively as follows. Suppose we have two well-separated symmetry defects p and q which are defined by the holonomies of B as follows: $\oint_p B = B_p = (\vec{R}_p, \omega_p)$ and $\oint_q B = B_q = (\vec{R}_q, \omega_q)$. Now the holonomy of B around a loop encircling both p and q can be measured equally by the group element $B_p B_q = (\vec{R}_p + U(\omega_p)\vec{R}_q, \omega_p + \omega_q)$ or by the group element $B_q B_p = (\vec{R}_q + U(\omega_q)\vec{R}_p, \omega_q + \omega_p)$. These values of the holonomy should therefore be treated as physically equivalent. They are in

fact gauge-equivalent: the difference in the translation component of the two holonomies equals

$$\begin{aligned} & \vec{R}_p + U(\omega_p)\vec{R}_q - (\vec{R}_q + U(\omega_q)\vec{R}_p) \\ &= (1 - U(\omega_q))\vec{R}_p - (1 - U(\omega_p))\vec{R}_q. \end{aligned} \quad (2.17)$$

In the most general case, the rhs is a multiple of the matrix $(1 - U(\frac{2\pi}{M}))$ by an integer vector. Therefore, a dislocation Burgers vector which takes such values should be regarded as trivial. Indeed, we can always find a gauge transformation which sets these values of Burgers vectors to zero.

In the same way, we can consider three well-separated defects p, q, r , whose holonomies are given by the group elements B_p, B_q and $B_r = B_q^{-1}$. Now, the holonomy of the gauge field can be written either as $B_p B_q B_q^{-1} = B_p$ or as $B_q B_p B_q^{-1}$. Therefore two defects in the same conjugacy class must be regarded as physically equivalent; the corresponding translation components will be gauge equivalent and thus have the same K_M grading.

The groups K_M arise naturally while classifying the allowed fractional $U(1)$ charges associated to the $\vec{R}$ holonomy of a disclination. This was shown for topological crystalline insulators in free fermion systems in Ref. [88]; we will carry out a similar analysis for bosonic SET phases in Section 2.4.2.

2.3.2 Area flux

In terms of the translation gauge fields we can also construct a flux A_{XY} , which is quadratic in $\vec{R}$ and corresponds to an area element. We will see that A_{XY} is not by itself gauge-invariant, which is analogous to the fact that area elements are not invariant under general diffeomorphisms

in continuum geometry. Nevertheless, we will see that under a gauge transformation, when $B = (\vec{R}, \omega)$ is flat, A_{XY} changes by an integer-valued coboundary, so that it gives a well-defined area on closed manifolds. Physically this area corresponds to the number of unit cells of the clean (defect free) lattice. With some minor modifications, we will see that A_{XY} can also provide a well-defined area for spaces with boundary.

We define

$$A_{XY}[ijk] = \frac{1}{4\pi} \vec{R}_{ij} \times (U(\omega_{ij}) \vec{R}_{jk}), \quad (2.18)$$

where $\times$ is the cross product of vectors. When $\omega = 0$ everywhere, this gives the usual area element as expected, and it is easy to verify that on a torus T^2 whose side lengths are L_x and L_y ,

$$\frac{1}{2\pi} \int_{T^2} A_{XY} = L_x L_y. \quad (2.19)$$

The factor $U(\omega_{ij})$ keeps track of the relative orientation of the coordinate axes at i and j when $\omega \neq 0$. In the absence of dislocations, A_{XY} is gauge-invariant up to a boundary term, so that A_{XY} integrated over a closed surface is gauge invariant. To obtain a well-defined area on spaces with boundary, we require the translation gauge transformations to reduce from $\mathbb{Z}^2$ to the subgroup of translations preserved by the boundary. Here we study the behavior of the area flux A_{XY} under a gauge transformation and discuss its properties in the presence of dislocations and boundaries.

The area flux on a 2-simplex $[ijk]$ can be written as

$$A_{XY}[ijk] = \frac{1}{4\pi} \vec{R}_{ij} \times U(\omega_{ij}) \vec{R}_{jk} \quad (2.20)$$

$$= \frac{1}{4\pi} U(\omega_{0 \rightarrow i}) \vec{R}_{ij} \times U(\omega_{0 \rightarrow i}) U(\omega_{ij}) \vec{R}_{jk} \quad (2.21)$$

$$= \frac{1}{4\pi} \vec{\mathcal{R}}_{ij}^{(0)} \times \vec{\mathcal{R}}_{jk}^{(0)} \quad (2.22)$$

where $\times$ refers to the cross product: $\vec{v} \times \vec{u} = v_x u_y - v_y u_x$. The second line uses the fact that the cross product is invariant under an equal rotation of both arguments; the symbol $\omega_{0 \rightarrow i}$ refers to the sum of ω 's on any given path from the origin 0 to the point i . The last line uses the definition of $\vec{\mathcal{R}}^{(0)}$. Note that the cross product of two $\vec{\mathcal{R}}^{(0)}$ fields is thus local even though a single such field is not. Since A_{XY} is independent of the choice of origin 0, we drop this superscript and simply write

$$A_{XY} = \frac{1}{4\pi} \vec{\mathcal{R}}_{ij} \times \vec{\mathcal{R}}_{jk}, \quad (2.23)$$

with the understanding that $\vec{\mathcal{R}}_{ij}$ is defined with respect to an arbitrary choice of origin 0.

Under a gauge transformation, this equality implies that

$$4\pi A_{XY}[ijk] = \vec{\mathcal{R}}_{ij} \times \vec{\mathcal{R}}_{jk} \quad (2.24)$$

$$\rightarrow (\vec{\mathcal{R}}_{ij} + d\tilde{r}_{ij}) \times (\vec{\mathcal{R}}_{jk} + d\tilde{r}_{jk}) \quad (2.25)$$

Here we have defined $\tilde{r}_i = U(\omega_{0 \rightarrow i}) \vec{r}_i$ (for the same arbitrary choice of origin 0 used to define

$\mathcal{R}$). The difference δA_{XY} can be written as

$$4\pi\delta A_{XY}[ijk] = (\vec{\mathcal{R}}_{ij}) \times d\vec{r}_{jk} + d\vec{r}_{ij} \times (\vec{\mathcal{R}}_{jk} + d\vec{r}_{jk}) \quad (2.26)$$

Defining

$$f_{ij} = \vec{\mathcal{R}}_{ij} \times \vec{r}_j + \vec{r}_i \times (\vec{\mathcal{R}}_{ij} + d\vec{r}_{ij}), \quad (2.27)$$

we see that δA_{XY} is a coboundary whenever $d\vec{\mathcal{R}} = 0$:

$$4\pi\delta A_{XY}[ijk] = df[ijk] \quad (2.28)$$

Therefore when A_{XY} is integrated over the entire manifold, this property implies that a gauge transformation will only contribute boundary terms to the integral (assuming $\vec{\mathcal{R}}$ is flat). Therefore $\frac{1}{2\pi} \int_{\Sigma^2} A_{XY}$ over a closed 2-manifold Σ^2 is gauge-invariant (when $\vec{\mathcal{R}}$ is flat), which we physically interpret as the area of the space Σ^2 . Note that since the cross-product gives the area of a parallelogram, the integration over the whole space covers the manifold twice, such that A_{XY} is quantized to be an integer multiple of 2π when integrated over a 2-cycle.

Although we have defined a gauge-invariant area only for closed manifolds, we can also define a gauge-invariant area for manifolds with boundary by restricting the gauge transformations on the boundary. Specifically, we require that the quantity f defined above must vanish for every boundary 1-simplex. For this to occur, it is sufficient that the boundary fields $\vec{\mathcal{R}}_{ij}$ and the boundary gauge transformation variables $\vec{r}_j$ be parallel to each other. This requirement can also be viewed as a consequence of the fact that a boundary can be chosen to break one of the two $\mathbb{Z}$

translation symmetries, so that the $\vec{R}$ field essentially reduces to a $\mathbb{Z}$ gauge field on the boundary.

For example, suppose the space is a square formed by the region $0 \leq x, y \leq a$ with origin $(0,0)$. For simplicity let $\omega = 0$ everywhere on the boundary except on links associated with the corners, which have $\omega = \pi/2$. Let the fields $\vec{R}_{ij}$ on the $y = 0$ line have zero Y -component. Now as we meet the corner $(a, 0)$, we meet a 1-simplex with $\omega = \pi/2$. The above condition on $\vec{R}$ now means that on the $x = a$ line, $\vec{R}$ has zero X -component. In fact, one component of $\vec{R}$ is always constrained to vanish on the boundary.

The discussion above has so far required that $\vec{R}$ be flat. In particular, when $\vec{R}$ is flat, $\vec{R}_{ij} \times \vec{R}_{jk} = \vec{R}_{jk} \times \vec{R}_{ki}$, so the definition of A_{XY} does not depend on the ordering of the vertices. But if we assume that the simplex $[ijk]$ contains a dislocation, this equality no longer holds. we instead have $\vec{R}_{ij}^{(0)} + \vec{R}_{jk}^{(0)} + \vec{R}_{ki}^{(0)} = \vec{b}^{(0)}$ ² (where we have reinstituted the explicit dependence on the origin), so

$$\vec{R}_{ij}^{(0)} \times \vec{R}_{jk}^{(0)} - \vec{R}_{jk}^{(0)} \times \vec{R}_{ki}^{(0)} = \vec{b}^{(0)} \times \vec{R}_{jk}^{(0)} \neq 0. \quad (2.29)$$

This means that the area of a simplex with nonvanishing holonomy of $\vec{R}$ is not well defined. This is physically expected: on a lattice with a dislocation, the number of unit cells within a region containing a dislocation cannot be obtained purely from the dimensions of the boundary. In fact, the number of unit cells in a small region containing a dislocation is not well-defined. Moreover, as the dislocation moves, additional unit cells are added or removed. Therefore extensive observables such as the total charge or angular momentum will no longer be gauge-invariant. However, intensive quantities such as the filling or angular momentum per unit cell

²The same quantity is denoted $\vec{b}_o$ in later Chapters, where $o = 0$.

will still be well-defined, because they are a ratio of two extensive quantities computed with the same triangulation.

A well-defined area can be defined for a given fixed configuration of dislocations by cutting out the regions containing the dislocations. Then the system is viewed as a manifold with boundary, and a gauge-invariant area can be defined as discussed above by restricting the gauge transformations on the boundary. Effectively this approach treats the dislocation as a hole in the simplicial formulation. In principle we can consider alternatively treating it as a puncture (for example, a sphere S^2 with a puncture would correspond to the plane $\mathbb{R}^2$), but then we cannot describe the open set near the puncture in terms of a finite triangulation.

2.4 Effective action and response theory

With this understanding of the local gauge-invariant fluxes of the crystalline gauge fields, we are now ready to study the effective action.

2.4.1 Effective action

To derive the effective action, we rely heavily on group cohomology, which classifies the distinct, inequivalent topological terms that can appear. The derivation of these terms from group cohomology is detailed in Appendix [A.4](#) and their classification is summarized in Table [2.1](#).

The topological effective Lagrangian is

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4\pi} a^I \cup K_{IJ} da^J + \mathcal{L}_{frac} + \mathcal{L}_{SPT} \\
\mathcal{L}_{frac} &= \frac{1}{2\pi} a^I \cup (q_I dA + s_I d\omega + \vec{t}_I \cdot d\vec{\mathcal{R}} + m_I A_{XY}) \\
\mathcal{L}_{SPT} &= \frac{k_1}{2\pi} A \cup dA + \frac{k_2}{2\pi} A \cup d\omega + \frac{k_3}{2\pi} \omega \cup d\omega + \frac{1}{2\pi} A \cup (\vec{k}_4 \cdot d\vec{\mathcal{R}}) \\
&\quad + \frac{1}{2\pi} \omega \cup (\vec{k}_5 \cdot d\vec{\mathcal{R}}) + \left(\frac{k_6}{2\pi} A + \frac{k_7}{2\pi} \omega \right) \cup A_{XY}.
\end{aligned} \tag{2.30}$$

We have used the cup product from cohomology: $(A \cup dA)[ijkl] = A_{ij} dA[jkl]$ for a 3-simplex $[ijkl]$.

The non-degenerate $D \times D$ symmetric integer matrix K [9], which couples the dynamical $U(1)$ gauge field a^I , characterizes the intrinsic topological order³. Topologically distinct quasiparticles correspond to integer vectors $\vec{l} \sim \vec{l} + K\vec{\Lambda}$, where $\vec{l}, \vec{\Lambda} \in \mathbb{Z}^D$. The quasiparticles form an Abelian group $\mathcal{A} = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_D}$ under fusion, where n_i are the diagonal entries in the Smith normal form of K . This simplicial formulation of the Abelian CS theory was recently used in Ref. [183] to develop a local bosonic model for chiral topological phases.

$\mathcal{L}_{frac}$, which contains the coupling between the background gauge fields and a^I , specifies symmetry fractionalization, i.e. how the anyons carry fractional symmetry quantum numbers. Mathematically this is classified by the second group cohomology $\mathcal{H}^2(G, \mathcal{A})$ [11, 176]⁴. The distinct terms in $\mathcal{L}_{frac}$ are consistent with, and in fact can be derived from, the group cohomology

³Formally a^I here are the lifts from $U(1)$ to $\mathbb{R}$.

⁴Note that the symmetry fractionalization anomaly [11, 49, 184] always vanishes here because $\mathcal{H}^4(U(1) \times G_{space}, U(1))$ is trivial.

classification (see Appendix A.4)

$$\mathcal{H}^2(G, \mathcal{A}) = \mathcal{A} \times (\mathcal{A}/M\mathcal{A}) \times (K_M \otimes \mathcal{A}) \times \mathcal{A}, \quad (2.31)$$

for $G = \mathrm{U}(1) \times [\mathbb{Z}^2 \rtimes \mathbb{Z}_M]$. Here $\otimes$ denotes the tensor product of groups, defined in Appendix A.4; for example, $\mathbb{Z}_q \otimes \mathbb{Z}_p = \mathbb{Z}_{\mathrm{gcd}(p,q)}$.

The terms in $\mathcal{L}_{SPT}$ correspond to Dijkgraaf-Witten (DW) terms, classified by $\mathcal{H}^3(G, \mathrm{U}(1)) \cong \mathcal{H}^4(G, \mathbb{Z})$ [98]. In our case, we have

$$\mathcal{H}^3(\mathrm{U}(1) \times [\mathbb{Z}^2 \rtimes \mathbb{Z}_M], \mathrm{U}(1)) = \mathbb{Z}^2 \times \mathbb{Z}_M^3 \times K_M^2 \quad (2.32)$$

The terms in $\mathcal{L}_{SPT}$ correspond explicitly to representative cocycles in $\mathcal{H}^3(G, \mathrm{U}(1))$, as discussed in detail in Appendix A.4. Physically the DW terms can be understood in terms of stacking symmetry-protected topological (SPT) states [11, 21].

While we have defined our topological field theory using the framework of discrete gauge theory, we can equivalently use integral, real-valued differential forms, as discussed in Section 2.6.1.

The terms we have written above are complete for bosonic systems. For invertible fermionic states, in which there are no additional anyons, it is most helpful to use a classification based on G -crossed braided tensor categories. This will be summarized in Chapter 3. In the fermionic case, there may also be symmetry-enriched topological phases beyond group supercohomology [28, 50, 51], which cannot be fully described by the above effective action. We leave a comprehensive understanding of the fermionic case for future work.

Note that the above action is only uniquely defined when the gauge fields are flat: $da^I, dA, d\omega, d\vec{R}_i \in 2\pi\mathbb{Z}$. When the gauge fields are not flat, the action is not invariant under the shift of a , A , or ω by 2π on a single 1-simplex. More generally, for non-flat gauge fields, one can add additional terms to the action which depend on the field strength and which are not uniquely specified [185]. Non-trivial fluxes of a , A , ω , and $\vec{R}$ can be included by treating them as punctures or holes in the spatial manifold around which the gauge fields have non-trivial holonomy, such that the gauge fields remain flat. The above also implies the action is invariant under changes of lift $a_{ij} \rightarrow a_{ij} + 2\pi$ as long as $\vec{q}, \vec{s}, \vec{t}, \vec{m}$ are integer vectors.

In what follows, to read off physical properties, we use the fact that objects charged under A , ω , R correspond to $U(1)$ charge, angular momentum, and linear momentum. The generalized charges can be defined physically through the Berry phase obtained by adiabatically braiding charges around the associated fluxes.

2.4.1.1 Charge vector $\vec{q}$

The charge vector $\vec{q} \in \mathbb{Z}^D$ assigns fractional electric charge

$$Q_{\vec{l}} = \vec{q}^T K^{-1} \vec{l} \quad (2.33)$$

to the anyon $\vec{l}$. Alternatively, this term induces an anyon $\vec{q}$ under insertion of 2π flux. As such, $\vec{q}$ is also sometimes referred to as a vison or fluxon.

Two charge vectors $\vec{q}, \vec{q}'$ describe the same anyon if $\vec{q}' = \vec{q} + K\vec{\Lambda}$ for some $\vec{\Lambda} \in \mathbb{Z}^D$. Therefore the group of inequivalent choices for $\vec{q}$ is $\mathcal{A}$. Note that for a fixed state, this equivalence is realized in the effective action by relabelling $\vec{a} \rightarrow \vec{a} - \vec{\Lambda}A$. Shifting $\vec{q}$ thus also changes the

values of k_1, k_2, k_4 and k_6 , which couple A . The full equivalence relation is

$$(\vec{q}; k_1, k_2, k_{4,i}, k_6) \sim (\vec{q} + K\vec{\Lambda}; k_1 - \vec{q} \cdot \vec{\Lambda} - \vec{\Lambda}^T K \vec{\Lambda} / 2, \\ k_2 - \vec{s} \cdot \vec{\Lambda}, k_{4,i} - \vec{t}_i \cdot \vec{\Lambda}, k_6 - \vec{m} \cdot \vec{\Lambda}). \quad (2.34)$$

2.4.1.2 Discrete spin vector $\vec{s}$

The discrete spin vector $\vec{s} \in \mathbb{Z}^D$ is the analog for discrete rotational symmetry of the spin vector defined previously for continuum FQH systems [55]. However, as we discuss below, this term is only non-trivial when there is a compatibility between the intrinsic topological order and the order M of the point group symmetry.

This term induces an anyon $\vec{s}$ under the insertion (fusion) of M elementary disclinations. In particular, this term contributes a phase $e^{i2\pi\vec{s}^T K^{-1}\vec{l}}$ to the adiabatic transport of an anyon $\vec{l}$ around M elementary disclinations. Alternatively, this term associates a fractional orbital angular momentum

$$L_{\vec{l}} = \vec{s}^T K^{-1} \vec{l} \quad (2.35)$$

to the quasiparticle $\vec{l}$, which contributes a braiding phase $e^{2\pi i L_{\vec{l}}/M}$ to an anyon $\vec{l}$ encircling a $2\pi/M$ disclination.

Consider a continuum FQH state where we adiabatically transport an anyon $\vec{l}$ around a region Σ of a manifold with curvature. The resulting Aharonov-Bohm phase $\gamma_{AB} = \gamma_{AB,1} + \gamma_{AB,2}$ receives two contributions [170]. The first contribution $\gamma_{AB,1}$ is associated to the fractional U(1)

charge of $\vec{l}$ and equals

$$\gamma_{AB,1} = Q_a \Phi(\Sigma), \quad (2.36)$$

where $\Phi(\Sigma)$ is the total magnetic flux through Σ . The second contribution is due to coupling to the spatial curvature:

$$\gamma_{AB,2} = \left(\frac{\vec{l}^T K^{-1} \vec{l}}{2} + \vec{l}^T K^{-1} \vec{s} \right) N_R(\Sigma) \quad (2.37)$$

Here $N_R(\Sigma)$ is the integrated curvature flux through Σ . The quantity in parantheses defines the total spin of $\vec{l}$,

$$S_{\vec{l}} = L_{\vec{l}} + \frac{\vec{l}^T K^{-1} \vec{l}}{2}. \quad (2.38)$$

The first contribution is the orbital angular momentum, which comes from the symmetry fractionalization, and can be understood as the braiding of $\vec{l}$ with the anyon $\vec{s}$ associated to a 2π curvature flux. The second contribution arises because of self-interaction effects that result in the anyon $\vec{l}$ braiding around itself as it is transported around a closed loop. For an explicit calculation of the full A-B phase in a continuum geometry the reader is referred to Refs. [170, 186].

In the discrete case that we are considering in this paper, the same equations are expected to hold, with the modification that the curvature $N(\Sigma)$ arises only due to point sources of $2\pi/M$ curvature flux arising from disclinations and corners.

Note that taking $\vec{s} = M\vec{\Lambda}$ for $\vec{\Lambda} \in \mathbb{Z}^D$ is trivial, since it can be completely accounted for by binding an anyon $\vec{\Lambda}$ to an elementary disclination, which can in turn always be done by

adjusting the local energetics at disclinations. The non-trivial case cannot be captured simply by associating an anyon to an elementary disclination. Therefore we have two equivalence relations:

$$(\vec{s}, \{k_i\}) \sim (\vec{s} + K\vec{\Lambda}, \{k'_i\}) \quad (2.39)$$

(by relabelling $a \rightarrow a - \vec{\Lambda}\omega$), and

$$(\vec{s}, k_3) \sim (\vec{s} + M\vec{\Lambda}', k_3). \quad (2.40)$$

The choices of $\vec{s}$ inequivalent under both relations constitute the group $\mathcal{A}/M\mathcal{A}$. For $\mathcal{A} = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_D}$, $\mathcal{A}/M\mathcal{A} = \mathbb{Z}_{(n_1, M)} \times \cdots \times \mathbb{Z}_{(n_D, M)}$, where $(n, M) = \gcd(n, M)$. We see, therefore, that the order of the group $\mathcal{A}$ must be compatible with M to obtain a non-trivial fractionalization class.

The equivalence on $\vec{s}$ implies that the theory predicts the angular momentum of an anyon $\vec{l}$ modulo $M(\vec{\Lambda}^T K^{-1} \vec{l})$.

2.4.1.3 Discrete torsion vector $(\vec{t}_x, \vec{t}_y)$

The integer vector $(\vec{t}_x, \vec{t}_y)$, with $\vec{t}_i \in \mathbb{Z}^D$, which we refer to as the discrete torsion vector, does not have an analog in the continuum because torsion (i.e. the gauge-invariant part of the dislocation density) is not quantized in continuum geometry. Furthermore this term is non-trivial only in the presence of rotational symmetry, with $M = 2, 3, 4$, because, as summarized in Table 2.3, the gauge-invariant part of the dislocation density (defined by the group grading K_M) is nontrivial only when $M = 2, 3, 4$.

$\vec{t}$ associates an anyon $(\vec{t}_x, \vec{t}_y) \cdot (a, b)$ to a region with Burgers vector $(1 - U(2\pi/M)) \cdot (a, b)$.

Note that an anyon is attached only for Burgers vectors in the trivial class in K_M . Values of $\vec{t}$ which can be accounted for by attaching an anyon to an elementary dislocation are topologically trivial, as they can be accounted for by adjusting the local energetics of a dislocation. It follows that the topologically distinct values of $(\vec{t}_x, \vec{t}_y)$ are classified by the group $K_M \otimes \mathcal{A}$, which for $M = 2, 3, 4$ equals $\mathcal{A}/2\mathcal{A} \times \mathcal{A}/2\mathcal{A}, \mathcal{A}/3\mathcal{A}$ and $\mathcal{A}/2\mathcal{A}$ respectively (see Appendix A.4 for a definition of the symbol $\otimes$).

The term defining the torsion vector can be written in full as $\frac{1}{2\pi} a^I \cup (t_{I,i} (1 - U(2\pi/M))^{-1} dR_i)$.

From this we obtain that the discrete torsion vector furthermore associates a fractional (linear) momentum

$$\begin{aligned} \vec{P}_{\vec{t}} &= (1 - U(2\pi/M)^T)^{-1} \vec{p}_{\vec{t}} \\ (\vec{p}_{\vec{t}})_i &= \vec{t}^T K^{-1} \vec{t}_i \end{aligned} \quad (2.41)$$

to the anyon $\vec{t}$, which is well-defined (i.e. topologically robust) modulo the equivalence on $\vec{t}$:

$$\begin{aligned} \vec{t}_I &\sim \vec{t}_I + (1 - U(2\pi/M))\mathbb{Z}^2, \quad \text{for } I = 1, \dots, D \\ \vec{t}_i &\sim \vec{t}_i + K\mathbb{Z}^D, \quad \text{for } i = x, y). \end{aligned} \quad (2.42)$$

The momentum $\vec{P}_{\vec{t}}$ of an anyon can be defined by the Berry phase $e^{2\pi i \vec{P}_{\vec{t}} \cdot \vec{b}}$ obtained upon adiabatically braiding the anyon $\vec{t}$ around a dislocation with Burgers vector $\vec{b}$.

Under a $2\pi/M$ rotation, the momentum transforms as $\vec{P}_{\vec{t}}^T \rightarrow P_{\vec{t}}^T U(2\pi/M)$; in other words, under a $2\pi/M$ rotation, the change in momentum is $(U^T(2\pi/M) - 1)\vec{P}_{\vec{t}} = -\vec{p}_{\vec{t}}$. However this

is precisely the first ambiguity in $\vec{t}_I$: shifting $\vec{t}_I \rightarrow \vec{t}_I + (1 - U(2\pi/M))(-\vec{t}_I)$ changes the momentum $\vec{P}_{\vec{l}}$ by $-\vec{p}_{\vec{l}}$. Therefore, the topologically robust part of the fractional momentum $\vec{P}_{\vec{l}}$ is consistent with rotational invariance.

We emphasize that this “crystal momentum fractionalization,” which is only non-trivial for $M = 2, 3, 4$, is fundamentally distinct from the more familiar notion usually discussed in the context of quantum spin liquids (see e.g. [176, 177]). The latter case is associated with non-commutativity of the translation operator restricted to a given anyon and arises from the existence of an anyon per unit cell (discussed below), which can be non-trivial even in the case $M = 1$.

2.4.1.4 Area vector $\vec{m}$

Finally, $\vec{m} \in \mathbb{Z}^D$, which we refer to as the area vector, also has no analog in the continuum. This associates an anyon $\vec{m}$ to each unit cell, as has been discussed algebraically in previous work [65, 172] and gives rise to certain notions of “crystal momentum fractionalization” discussed previously [173–177]. This means that if a quasiparticle $\vec{l}$ is taken around a region S containing $\text{Num}(S)$ unit cells, the wave function acquires a braiding phase $e^{2\pi i \vec{l}^T K^{-1} \vec{m} \text{Num}(S)}$. Algebraically, this means that the translation operators satisfy a magnetic translation algebra when its action is restricted to the anyon $\vec{l}$:

$$T_{x,\vec{l}} T_{y,\vec{l}} = e^{2\pi i \vec{l}^T K^{-1} \vec{m}} T_{y,\vec{l}} T_{x,\vec{l}}, \quad (2.43)$$

where $T_{x,\vec{l}}$ and $T_{y,\vec{l}}$ are the translation operators in the x and y direction, restricted to the anyon $\vec{l}$. See Ref. [11] for a precise formulation of symmetry operations restricted to anyons.

2.4.2 Response theory

Given the topological effective action, we can integrate out the dynamical a gauge fields to obtain an effective response theory:

$$\begin{aligned}\mathcal{L}_{\text{eff}} = & \frac{\sigma_H}{2} A \cup dA + \frac{\mathcal{S}}{2\pi} A \cup d\omega + \frac{\ell_s}{4\pi} \omega \cup d\omega + \frac{\vec{\mathcal{P}}_c}{2\pi} \cdot (A \cup d\vec{R}) + \frac{\vec{\mathcal{P}}_s}{2\pi} \cdot (\omega \cup d\vec{R}) \\ & + \frac{1}{2\pi} (\nu_c A + \nu_s \omega) \cup A_{XY} + \frac{\vec{\nu}_p}{2\pi} \cdot \vec{R} \cup A_{XY} + \frac{\Pi_{ij}}{4\pi} R_i \cup dR_j + \frac{\alpha}{4\pi} A_{XY} \cup d^{-1} A_{XY} + \mathcal{L}_{\text{anom}},\end{aligned}\tag{2.44}$$

$$\mathcal{L}_{\text{anom}} = -\frac{\text{sgn}(K)}{48\pi} \omega \cup d\omega.\tag{2.45}$$

Note that as usual, the effective response theory is not well-defined on compact manifolds due to the fractional values of the coefficients; nevertheless, the response theory can be used to read off the fractionally quantized responses of the system on an open patch of space.

The first term is the well-known Hall conductivity, which is given by

$$\sigma_H = (2k_1 + \vec{q}^T K^{-1} \vec{q})/2\pi.\tag{2.46}$$

The second and third terms are discrete analogs of the known continuum geometric response of FQH states [55, 56, 150–152, 162–164]. The remaining terms in $\mathcal{L}_{\text{eff}}$ are intrinsic to the lattice and have no analog in continuum FQH states. In what follows, we discuss them individually in detail.

We note that the term formally written as $A_{XY} \cup d^{-1} A_{XY}$, with $\alpha = \vec{m}^T K^{-1} \vec{m}$, corresponds to $A_{XY} \cup c$, where $dc = A_{XY}$. This term arises from the fact that an anyon is associated with

each unit cell. However it is not clear how or whether this term can be physically measured as a quantized geometric response. We thus do not discuss this term further below.

2.4.2.1 Discrete shift $\mathcal{S}$ and fractional charge of disclinations

The second term gives a discrete analog of the shift [55, 86, 87, 92] $\mathcal{S}$, where

$$\mathcal{S} = k_2 + \vec{q}^T K^{-1} \vec{s}. \quad (2.47)$$

In particular, this term implies that lattice corners and disclinations carry fractional U(1) charge. Both an elementary $2\pi/M$ disclination and a corner of angle $2\pi/M$ carry a fractional U(1) charge of

$$Q_{\text{disclin}, 2\pi/M} = \mathcal{S}/M = \frac{k_2 + \vec{q}^T K^{-1} \vec{s}}{M}. \quad (2.48)$$

For example, if $M = 4$ and the system is defined at the surface of a 3D cube, there are effectively 8 disclinations, each one carrying a fractional charge $\mathcal{S}/4$. If the system is defined on a square, each corner also has a fractional charge $\mathcal{S}/4$. This term therefore implies the system is a fractional “higher order” topological state [148, 187, 188]. Note that when the edge of the system is gapped, the corner charge is clearly well-defined; however when the corner lies along a chiral gapless boundary of the system, it is not clear whether any remnant of the corner charge persists.

Since the $A \cup d\omega$ term defining the shift can also be written as $\omega \cup dA$, this term also

associates an angular momentum to a ϕ flux given by

$$L_{A,\phi} = \frac{\phi}{2\pi} \mathcal{S} = \frac{\phi}{2\pi} (k_2 + \vec{q}^T K^{-1} \vec{s}). \quad (2.49)$$

Note that the fractional part of the angular momentum of a 2π flux equals $L_{\vec{q}} = \vec{q}^T K^{-1} \vec{s} \pmod{1}$, which is the angular momentum of the anyon $\vec{q}$ associated to a 2π flux.

Note that the response theory only predicts the fractional charge, angular momentum, and linear momentum of the dislocations and disclinations up to those of the elementary anyons, as anyons can always be bound to these defects by adjusting the local energetics. Therefore in this case, the fractional charge $\mathcal{S}/M$ is determined only modulo the charges $Q_{\vec{l}} = \vec{q}^T K^{-1} \vec{l}$, for any integer vector $\vec{l}$.

2.4.2.2 Disclination angular momentum

The third term gives a fractionally quantized contribution $L_{2\pi/M}$ to the angular momentum of an elementary $2\pi/M$ disclination,

$$L_{2\pi/M} = \frac{\ell_s}{M} - \frac{1}{M} \frac{c_-}{12}$$

$$\ell_s = (2k_3 + \vec{s}^T K^{-1} \vec{s}) \quad (2.50)$$

The contribution proportional to the chiral central charge $c_- = \text{sgn}(K)$, where $\text{sgn}(K)$ is the signature of K , arises from the framing anomaly, $\mathcal{L}_{\text{anom}}$, which we discuss further in Sec. 2.4.2.7.

2.4.2.3 Fractional quantized charge polarization $\vec{\mathcal{P}}_c$ and fractional charge of dislocations

The term with

$$\begin{aligned}\vec{\mathcal{P}}_c &= (1 - U^T(2\pi/M))^{-1}(\vec{k}_4 + \vec{p}_{\vec{q}}) \\ &= \vec{P}_{\vec{q}} + (1 - U^T(2\pi/M))^{-1}\vec{k}_4\end{aligned}\tag{2.51}$$

is referred to as a fractionally quantized charge polarization. As we now discuss, this leads to three basic properties that are predicted by the topological response theory:

1. Fractionally quantized charge of dislocations (modulo charge of anyons)
2. Fractionally quantized momentum of U(1) flux
3. Fractionally quantized charge per unit length along boundaries (modulo charge of anyons)

The quantization arises due to the rotational symmetry of the lattice. Without rotational symmetry ($M = 1$), the polarization is a non-quantized topological response [167]. Furthermore, $\mathcal{P}_{c,j}$ is only well-defined modulo $\mathbb{Z}$.

This term associates a fractional charge

$$Q_{\text{disloc};\vec{b}} = \vec{\mathcal{P}}_c \cdot \vec{b}\tag{2.52}$$

to a dislocation with Burgers vector $\vec{b}$. Note that, as in the case of the disclination charge, the topological response theory only predicts the dislocation charge modulo the charges of the

anyons.

Whether the fractional charge of a dislocation actually signifies a non-trivial value of the discrete torsion vector is a somewhat subtle issue. In principle, the dislocation charge can be fractions of the minimal anyon charge even when the discrete torsion vector is trivial, due to the interplay between the SPT term $\vec{k}_4$ and the minimal anyon charge. Observe that the fractional charge receives two contributions: one from the intrinsic topological order and symmetry fractionalization, which arises from $\vec{P}_{\vec{q}}$, and one from the SPT term k_4 . The SPT term can contribute a fractional charge in multiples of $1/2$ (for $M = 2, 4$) or $1/3$ (for $M = 3$). Together with the charge of the anyons $Q_{\vec{t}}$ which can be trivially bound to dislocations due to local energetics, this implies that in principle one can obtain fractional charges at dislocations that may be fractions of the anyon charge, but which arise from a trivial value of $\vec{t}$. For example, consider the case of the $1/2$ Laughlin topological order on a honeycomb lattice ($M = 3$) and $k_4 = 1$. There, all choices of discrete torsion vector $\vec{t}$ are trivial, because $\mathbb{Z}_3 \otimes \mathbb{Z}_2 = \mathbb{Z}_1$; nevertheless, even a trivial value of $\vec{t}$ can give rise to a dislocation charge $1/3 - 1/2 = 1/6$. On the other hand, on the square lattice ($M = 4$), a dislocation charge of $1/4$ can only occur for the non-trivial choice of discrete torsion vector $\vec{t} \in \mathbb{Z}_2 \otimes \mathbb{Z}_2 = \mathbb{Z}_2$, while the trivial choice can only give multiples of $1/2$.

Let us compare the charge of a dislocation with Burgers vector $\vec{b}$ and its rotated counterpart

$U(2\pi/M)\vec{b}$. The difference is given by

$$\begin{aligned}
\Delta Q &= Q_{\text{disloc};\vec{b}} - Q_{\text{disloc};U(2\pi/M)\vec{b}} \\
&= \vec{\mathcal{P}}_c^T (1 - U(2\pi/M)) \vec{b} \\
&= (k_{4;i} + \vec{q}^T K^{-1} \vec{t}_i) b_i \\
&= (k_{4;i} + Q_{\vec{t}_i}) b_i.
\end{aligned} \tag{2.53}$$

In other words, the difference is some integer multiple of the fractional charge of $\vec{t}_x$ and $\vec{t}_y$. Thus the contribution to the dislocation charge from the topological response theory, which is only well-defined modulo the charges of the anyons, is rotationally invariant.

The charge polarization term also contributes to the charge of a disclination, if it has nontrivial $\vec{R}$ holonomy. Interestingly, the classification of free fermion SPT phases based on their disclination charges is shown to equal $\mathbb{Z}_M \times K_M$ in Ref. [88]. This agrees with the bosonic crystalline gauge theory, which predicts that the disclination charge is classified by the terms $\frac{1}{2\pi} A \cup (k_2 d\omega + \vec{k}_4 \cdot d\vec{R})$, where $k_2 \in \mathbb{Z}_M$ and $\vec{k}_4 \in K_M$. (The coefficients in a crystalline gauge theory of fermions can in principle have different quantization conditions than in the bosonic case, but we will not discuss the fermionic case in detail here.) The classification approach in Ref. [88] based on Wannier orbitals centred at high-symmetry points is an example of a defect network construction. The problem of establishing a correspondence between the topological responses in the defect network picture and the group cohomology picture is briefly alluded to in Section 2.7.

If the dislocation described by $\vec{b}$ is connected to an edge of the system, the holonomy at the edge is changed by the amount $-\vec{b}$. Hence there must be a compensating fractional charge at the

edge. However since the dislocation line is not by itself well-defined, this boundary fractional charge can be delocalized along the boundary.

This term also associates a momentum

$$\vec{P}_{A,\phi} = \vec{\mathcal{P}}_c \phi / 2\pi \quad (2.54)$$

to a U(1) flux of ϕ spread uniformly throughout the system. The momentum of 2π flux has been discussed previously in the context of Dirac spin liquids in Refs. [189, 190]; our results are consistent with these works for systems with orientation-preserving symmetries. Note that the contribution to $\vec{\mathcal{P}}_c$ from the intrinsic topological order is equal to the momentum $\vec{P}_{\vec{q}}$ of the anyon $\vec{q}$, which is the anyon associated to a 2π flux.

Finally, this term associates a fractional charge per unit length $\vec{\mathcal{P}}_c \cdot \hat{e}$ to a boundary along the direction $\hat{e}$. This corresponds to a fractional charge polarization $\vec{P}_c = \vec{\mathcal{P}}_c \times \hat{z}$ for a system defined on a space with boundary. As above, this fractional charge per unit length is only topologically robust modulo the charge of the anyons. Under a rotation, the charge per unit length along the boundary stays invariant up to the charge of the elementary anyons. Therefore the contribution of the topological response theory to the boundary charge per unit length is rotationally invariant.

We note that because the boundary charge per unit length is only topologically protected modulo the charge of the anyons, the system does not necessarily have a non-zero polarization on a space with boundary; one can arrange the local energetics along the boundary so that the boundary charge per unit length is the same on all boundaries. Nevertheless, the three physical effects described above are all intimately related to the quantum theory of polarization in higher dimensions [167], which is why we refer to this term as the fractional charge polarization.

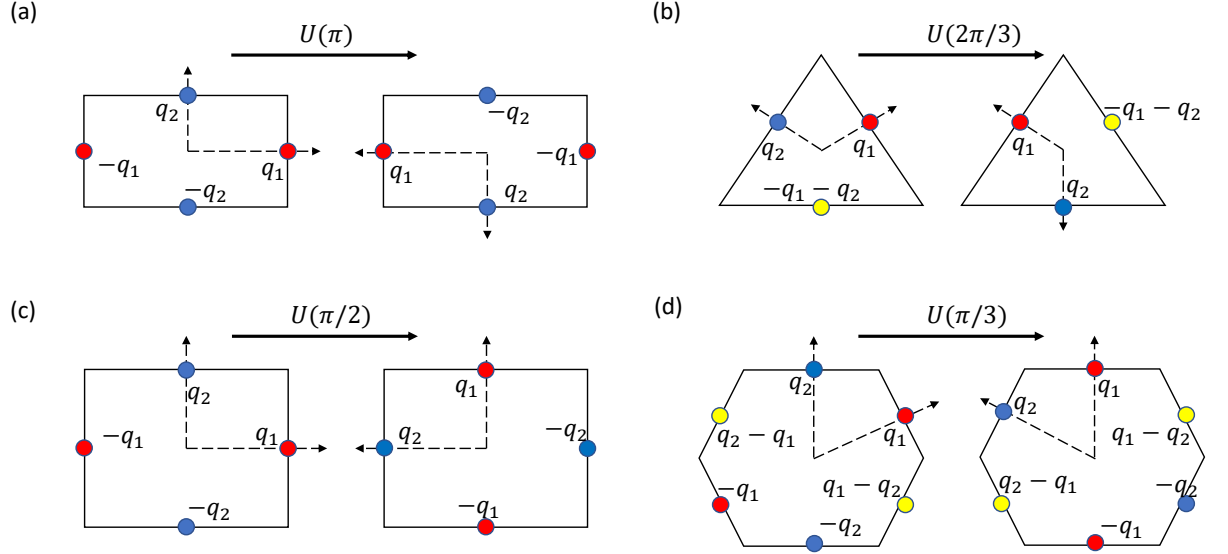


Figure 2.2: The K_M classification of rotationally symmetric configurations of boundary charge for (a) $M = 2$, (b) $M = 3$, (c) $M = 4$, and (d) $M = 6$. We choose our coordinate axes to be normal to the boundaries, and place a charge per unit length equal to $(q_1, q_2)^T \vec{n}$ on the boundary with normal vector $\vec{n}$. Thus in (b), for $M = 3$ we have the arrangement $A = (q_1, q_2, -q_1 - q_2)$ as we proceed anticlockwise around the boundary segments. Now under a $2\pi/3$ rotation of axes, the charge per unit length at the same three segments gets redefined as $A' = (-q_1 - q_2, q_1, q_2)$. Since the fractional charge per unit length on each boundary segment remains the same if we only rotate the coordinate axes, we should have $A = A' \pmod{1}$. This implies that $q_1 = q_2$ and $3q_1 \in \mathbb{Z}$; the three distinct choices of q_1 now determine the group K_3 . We can follow similar reasoning in (a), (c) and (d).

The polarization response can be used to obtain another simple way to understand the group K_M . The group K_M corresponds to the group of allowed fractional charges per unit length along the boundary when the bulk has no intrinsic topological order, as we explain below.

Consider a system with fractional charge per unit length along its boundary given by $\vec{P} \cdot \hat{n}$, where $\vec{P}$ is the polarization vector and $\hat{n}$ is the normal to the boundary. An integer value of $\vec{P}$ corresponds to placing an integer charge per unit length on the boundary, which can always be done locally. This is shown pictorially in Fig. 2.2, where we assign fractional charge per unit length to each boundary segment under one choice of coordinate axes. For example, if we consider a system with $M = 4$, the charge per unit length on the boundaries normal to $\hat{x}, \hat{y}, -\hat{x}, -\hat{y}$ are $(q_1, q_2, -q_1, -q_2)$ respectively. Now we can perform rotations of the axes by $2\pi/M$, which will relabel the charge on each segment since the normal vectors $\vec{n}$ get redefined. In this case, the coordinate axes are rotated by an angle $\pi/2$, and the charges on the same boundary segments will now be labelled as $(-q_2, q_1, q_2, -q_1)$ (see Fig. 2.2). However, the fractional charge on each edge should be the same from either calculation. Therefore we must have $(q_1, q_2, -q_1, -q_2) = (-q_2, q_1, q_2, -q_1) \pmod{1}$. We can see that the only solutions are $(q_1, q_2) = (0, 0)$ or $(1/2, 1/2)$. Therefore the group of distinct assignments of charge at the boundary is $K_4 \cong \mathbb{Z}_2$. One can work out the other cases similarly.

We note that in our initial discussion of Section 2.3.1.1, the K_M classification arose from general properties of the dislocations that do not depend on a particular Lagrangian, while in the second derivation given here, it arose from demanding rotational invariance of a physical response related to the term $\frac{\vec{\mathcal{P}}}{2\pi} A \cup dR$ in the Lagrangian.

Finally, we look at the case with $M = 1$, corresponding to the absence of rotation symmetry. We cannot directly apply the previous reasoning in this case to obtain a useful classification. In

a system without rotation symmetry, the Burgers vector of any dislocation is well-defined: the value of $d\vec{R}$ is gauge-invariant. Since there is no gauge transformation relating them, there is no grading of Burgers vectors. In the example of boundary charge, one can now have any assignment of fractional charges per unit length on the boundary of such a system. In either case, the group classifying inequivalent dislocations or fractional boundary charge configurations is not a finite group. However, if we define K_1 so that it classifies the *quantized* fractional charges per unit length that can be assigned to a boundary, the group is trivial. The quantization was a direct result of discrete rotation symmetry, which is broken when $M = 1$.

2.4.2.4 Quantized angular momentum polarization $\vec{\mathcal{P}}_s$

The term with $\vec{\mathcal{P}}_s$ is the rotational analog of $\vec{\mathcal{P}}_c$, where

$$\begin{aligned}\vec{\mathcal{P}}_s &= (1 - U^T(2\pi/M))^{-1}(\vec{k}_5 + \vec{p}_s) \\ &= \vec{P}_s + (1 - U^T(2\pi/M))^{-1}\vec{k}_5\end{aligned}\tag{2.55}$$

It associates a fractionally quantized angular momentum

$$L_{\text{disloc};\vec{b}} = \vec{\mathcal{P}}_s \cdot \vec{b}\tag{2.56}$$

to a dislocation with Burgers vector $\vec{b}$. Note that the contribution to $\vec{\mathcal{P}}_s$ coming from the symmetry fractionalization is simply $\vec{P}_s$, the linear momentum of $\vec{s}$.

However it is not clear whether dual response, which is the analog of attaching momentum to a U(1) flux, which here would formally correspond to a momentum of a disclination, is well-

defined. It is also unclear whether the analog of the boundary charge per unit length has any meaning in this context, because the boundary is not fixed by a rotation.

2.4.2.5 Charge, linear momentum, and angular momentum filling: $\nu_c, \nu_s, \vec{\nu}_p$

The term proportional to $A \cup A_{XY}$ corresponds to a charge of

$$\nu_c = k_6 + \vec{q}^T K^{-1} \vec{m} \quad (2.57)$$

per unit area. This gives a generalized Lieb-Schulz-Mattis constraint [172] which imposes constraints on $\vec{q}$, K , and $\vec{m}$ in terms of the filling ν_c . Likewise, the term proportional to $\omega \cup A_{XY}$ associates a fractional angular momentum of

$$\nu_s = (k_7 + \vec{s}^T K^{-1} \vec{m}) \quad (2.58)$$

to each unit area.

The term $\vec{R} \cup A_{XY}$ associates a momentum of

$$(\vec{\nu}_p)_j = \sum_i \vec{t}_i^T K^{-1} \vec{m} (1 - U(2\pi/M)_{ij}^{-1}) \quad (2.59)$$

per unit area of the system. It arises from the fact that there is an anyon $\vec{m}$ per unit cell, which in turn carries a momentum as specified by the coupling $\vec{t}$. Indeed, observe that

$$\vec{\nu}_p = \vec{P}_{\vec{m}}. \quad (2.60)$$

Remarkably, this implies that the ground state may carry momentum, depending on the area of the system; only for certain commensurate areas is the ground state momentum trivial. Further, this term is also only non-trivial for $M = 2, 3, 4$ -fold rotational symmetry.

2.4.2.6 Fractionally quantized torsional response

The term with Π_{ij} associates a fractionally quantized momentum of

$$\vec{P}_{\text{disloc}, \vec{b}} = \sum_j \Pi_{ij} b_j \quad (2.61)$$

to a dislocation with Burgers vector $\vec{b}$. Here

$$\begin{aligned} \Pi_{ij} &= e_i^T K^{-1} e_j, \\ e_i &= \sum_{i'} (1 - U(2\pi/M))_{ii'}^{-1} \vec{t}_{i'} \end{aligned} \quad (2.62)$$

This is closely related to the torsional Hall response that has been discussed for continuum Dirac theories [178, 179], although there the corresponding term is not quantized and is sensitive to the ultraviolet cutoff. The non-trivial quantization only occurs for lattice systems with $M = 2, 3, 4$ -fold rotational symmetry.

We note that here we read off momentum as being defined by the charge of the translation gauge field. It is not clear how to define the momentum of a dislocation microscopically. For example, naively one would define the charge in terms of the Aharonov-Bohm phase obtained by braiding with a flux; in this case this naively corresponds to the phase obtained by braiding dislocations around each other. However to define this microscopically, the restricted mobility of

the dislocations on a lattice with a conserved number of atoms must be taken into account.

2.4.2.7 Framing anomaly

We note that the topological field theory itself does possess a continuous space-time symmetry corresponding to diffeomorphism invariance, which corresponds to the retriangulation invariance of the path integral for a given fixed configuration of flat gauge fields. For chiral topological phases, a gravitational CS term, proportional to the chiral central charge c_- , for the full $\text{SO}(2, 1)$ spin connection Ω also arises upon evaluating the path integral. This arises from the implicit metric dependence in the path integral measure required for gauge fixing and quantizing the CS theory, and is referred to as the framing anomaly [152, 180]. In a continuum formulation, this is written as:

$$\mathcal{L}_{\text{anom}} = -\frac{c_-}{96\pi} \text{Tr} \left(\Omega d\Omega + \frac{2}{3} \Omega^3 \right). \quad (2.63)$$

This term may also be viewed as the gravitational anomaly of the (1+1)D boundary of the system, which hosts a chiral CFT with central charge c_- .

We note that the quantization of the CS theory also gives rise to another contribution to the effective theory, given by the Ray-Singer analytic torsion [180]. This term is a topological invariant of the underlying space-time manifold, and is unimportant for our discussion.

Mathematically we may consider Ω to be a separate quantity depending on an underlying space-time metric, and to be distinct from ω and $\vec{R}$. However to be physically meaningful, the space-time manifold $\mathcal{M}$ should split into space and time separately as assumed in this work, with

the time-components of Ω vanishing:

$$\begin{aligned}\Omega_{0,\mu}^a &= \Omega_{b,\mu}^0 = 0 \\ \omega_{\text{SC},\mu} &\equiv \Omega_{2,\mu}^1.\end{aligned}\tag{2.64}$$

Furthermore, the spatial component of Ω , which we have denoted $\omega_{\text{SC},\mu}$, is an $SO(2)$ gauge field whose field strength corresponds to the curvature of the system. The physical origin of this curvature in a lattice system arises from lattice disclinations, so we require that ω_{SC} should be determined by the lattice rotation gauge field ω . We can relate the continuum definition of ω_{SC} to the definition of ω on the triangulation by integrating over a 1-simplex $[ij]$ of the triangulation:

$$\int_{ij} \omega_{\text{SC},\mu} = \omega_{ij}.\tag{2.65}$$

We see therefore that the framing anomaly contributes the following term to the effective response theory:

$$\mathcal{L}_{\text{anom}} = -\frac{c_-}{48\pi} \omega \cup d\omega.\tag{2.66}$$

This term will then contribute an additional angular momentum to disclinations proportional to c (see Eq. [2.50](#)).

2.4.2.8 Additional coboundary terms in response theory

When we consider a system with only $U(1)$ charge conservation and $\mathbb{Z}^2$ translation symmetries (i.e. $M = 1$), the charge polarization corresponds to a non-quantized topological term [167]. In our notation, this term has the form $\frac{\vec{\mathcal{P}}_c}{2\pi} \cdot \vec{R} \cup dA$ in $2 + 1$ dimensions, where $\vec{\mathcal{P}}_c$ is a pair of real numbers defined modulo 1. This non-quantized term is not associated to symmetry fractionalization or to SPTs; rather than corresponding to a non-trivial 3-cocycle, the above term can be understood as a 3-coboundary of the group $U(1) \times \mathbb{Z}^2$. Nevertheless, such a term can be physically meaningful. This means that for a complete understanding of the topological terms, we should also study response terms that are not associated to SPT responses but which correspond to group 3-coboundaries. In this section we consider these possibilities when $G = U(1) \times G_{\text{space}}$, and the rotation symmetry is nontrivial.

We first note that in the presence of rotation symmetry, we do not find any non-quantized topological terms (i.e. terms that are retriangulation invariant in our simplicial formulation). For example, the nonquantized polarization term mentioned above becomes quantized as a result of the rotation symmetry. However, we do find that we can add certain additional quantized topological terms beyond the SPT terms in the effective action, Eq. (2.30). Although we have not explicitly found a coboundary representation for these cocycles, these terms correspond to coboundaries because the SPT terms already present in the effective action form a complete set of cocycle representatives of $\mathcal{H}^3(G, U(1))$ (see Appendix A.4).

First consider the response term $\frac{\Pi_{ij}}{4\pi} R_i \cup dR_j$, where $\Pi_{ij} = (1 - U^T(2\pi/M))^{-1} (t_i^T K^{-1} t_j) (1 - U^T(2\pi/M))^{-1}$. This coefficient can be modified in a manner that preserves gauge-invariance, as

follows: we can define

$$\tilde{\Pi}_{ij} \equiv (1 - U^T(2\pi/M))^{-1}((\vec{t}_i)^T K^{-1} \vec{t}_j + k_{ij})(1 - U^T(2\pi/M))^{-1} \quad (2.67)$$

where $k_{ij} \in \mathbb{Z}$. In some cases, this shift in the momentum of a dislocation due to k_{ij} can be considered to be trivial, and part of the equivalence in the definition of $\vec{t}_i$. However, in general this contribution may not be completely accounted for by the equivalences on $\vec{t}_i$.

Similarly, consider the response term $\frac{\vec{\nu}_p}{2\pi} \cdot \vec{R} \cup A_{XY}$. We can modify the coefficient of this term as follows:

$$(\tilde{\nu}_p)_j = \sum_i (\vec{t}_i^T K^{-1} \vec{m} + k_i)(1 - U(2\pi/M))_{ij}^{-1} \quad (2.68)$$

where $k_i \in \mathbb{Z}$.

Finally, in principle we can have terms which are not related to the response terms already present in Eq. (2.44). For example, we can consider terms proportional to $R \cup R \cup R$, or terms composed of various powers of A, R and ω . Most terms of this kind will not be topological, i.e. will not satisfy the requirement of retriangulation invariance. Those terms that are retriangulation invariant will be coboundaries or equivalent to one of the existing SPT terms, since we already have a complete set of SPT cocycles. To our knowledge, none of these terms are associated to non-quantized topological responses. However, we have not checked all the possibilities systematically.

2.5 Examples

2.5.1 1/2 Laughlin topological order

Consider the 1/2 Laughlin topological order on a square lattice ($M = 4$), with symmetry $U(1) \times [\mathbb{Z}^2 \rtimes \mathbb{Z}_4]$. We have $K = 2$ and $\mathcal{A} = \mathbb{Z}_2$, with the anyons given by $I = 0 \pmod{2}$ and $S = 1 \pmod{2}$. The symmetry fractionalization classification is $\mathcal{H}^2(G, \mathcal{A}) \cong \mathbb{Z}_2^4$, with $\mathcal{A}/4\mathcal{A} \cong K_4 \otimes \mathcal{A} \cong \mathbb{Z}_2$. Thus there are two inequivalent symmetry fractionalization classes associated to each of $q, s, \vec{t}$ and m . Throughout this discussion we will define the elementary rotation matrix as $U(\pi/2) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. At various points, we will comment on the differences in the analysis when we consider different values of M .

The charge, spin and area vectors are each determined by choosing $q, s, m \in \{I, S\}$. The fractional charge and angular momentum of the anyon a are thus given by

$$Q_a = \frac{qa}{2} \pmod{1}, \quad L_a = \frac{sa}{2} \pmod{1} \quad (2.69)$$

respectively. The charge filling gives an LSM constraint on m and q :

$$\nu_c = \frac{qm}{2} \pmod{1}. \quad (2.70)$$

Therefore half-filling (i.e. half charge per unit cell) necessarily fixes $q = m = S$, while integer filling requires at least one of q or m to be trivial.

There are two inequivalent choices of discrete torsion vector, corresponding to $(t_x, t_y) =$

(I, I) and (S, I) , with $(I, I) \sim (S, S)$ and $(S, I) \sim (I, S)$. To see this, note that naively the possible discrete torsion vectors are $(t_x, t_y) \in \{(I, I), (I, S), (S, I), (S, S)\}$. The equivalence condition satisfied by them is

$$\begin{pmatrix} t_x \\ t_y \end{pmatrix} \sim \begin{pmatrix} t_x \\ t_y \end{pmatrix} + (1 - U(\pi/2)) \begin{pmatrix} t'_x \\ t'_y \end{pmatrix} = \begin{pmatrix} t_x \\ t_y \end{pmatrix} + \begin{pmatrix} t'_x - t'_y \\ t'_x + t'_y \end{pmatrix}. \quad (2.71)$$

This condition implies that the symmetry fractionalization class is completely determined by the value of $(t_x + t_y) \bmod 2$. Therefore the assignments $(I, S), (S, I)$ are equivalent and correspond to nontrivial symmetry fractionalization, while the assignment (S, S) is in fact trivial. The latter assignment is seen to be trivial because we have $\begin{pmatrix} 1 \\ 1 \end{pmatrix} = (1 - U(\pi/2)) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and thus (S, S) corresponds to attaching the anyon $S^{b_x} I^{b_y}$ to a dislocation with Burgers vector (b_x, b_y) .

The momentum associated to each anyon a is

$$\vec{P}_a = (1 - U(\pi/2))^{-1} \vec{t}^T \frac{1}{2} a = \frac{a}{4} \begin{pmatrix} t_x + t_y \\ t_y - t_x \end{pmatrix}. \quad (2.72)$$

Recall this momentum is only well-defined (i.e. topologically robust) up to the equivalences on $\vec{t}$ (and the representative a of the anyon). Therefore for the trivial choice $(t_x, t_y) = (I, I) \sim (S, S)$, we have $2P_{a,i} \in \mathbb{Z}$; that is, we have $\vec{P}_I, \vec{P}_S = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} \sim \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}$. Therefore half-integer momentum components should be regarded as trivial. Physically this can be understood from the fact that the change in the braiding phase between a and an elementary dislocation can be compensated for by attaching a semion to the elementary dislocation. For the non-trivial choice of $(t_x, t_y) = (S, I) \sim (I, S)$, we have $\vec{P}_S = \begin{pmatrix} 1/4 \\ -1/4 \end{pmatrix} \sim \begin{pmatrix} 1/4 \\ 1/4 \end{pmatrix}$. Observe that under a rotation, $2\vec{P}_S$ is invariant modulo 1.

The above analysis shows that for the $1/2$ Laughlin state, a momentum whose components are integer or half-integer is an indication of trivial symmetry fractionalization: it corresponds to attaching the anyon S to an elementary dislocation in some fixed direction. Thus, while considering some other M , we will continue to associate the momentum $1/2$ with trivial symmetry fractionalization. For the $1/N$ Laughlin state with N even, an analogous argument would imply that a momentum of $1/N$ units corresponds to trivial symmetry fractionalization, and therefore it is enough to check whether $NP_{a,x}$ and $NP_{a,y}$ are nontrivial.

Let us consider the classification of spin vectors in more generality. For the $1/2$ Laughlin state with M even, the spin vector is always nontrivial if it equals $1 \pmod 2$. However, for $M = 3$, we have $s = 3s \pmod 2$. This means that every spin vector can be thought of as associating the anyon s to an elementary $2\pi/3$ disclination. Therefore in this case, the choice $s = 1$ is in fact trivial. If we generalize to $1/N$ Laughlin states with $\mathcal{A} = \mathbb{Z}_N$, the number of distinct spin vectors equals $\mathbb{Z}_N/M\mathbb{Z}_N = \mathbb{Z}_{(M,N)}$.

Next we discuss the fractionally quantized responses. The Hall conductivity is given by $\sigma_H = \frac{1}{2\pi}(q^2/2 + 2k_1)$, as usual.

The discrete shift is defined as $\mathcal{S} = \frac{qs}{2} + k_2$. Therefore the fractional charge associated to a $\pi/2$ disclination is

$$Q_{\text{disclin}, \pi/2} = \frac{\mathcal{S}}{4} = \frac{qs}{8} + \frac{k_2}{4}. \quad (2.73)$$

Thus we see that shifting k_2 by an integer changes the fractional charge by $1/4$; shifting $k_2 \rightarrow k_2 + 4$ adds a trivial integer charge to the elementary disclination. Furthermore, when $q = s = S$, we obtain a $1/8$ charge at the elementary disclination.

The angular momentum of a $\pi/2$ disclination is

$$L_{\pi/2} = \ell_s/4 - c_-/48 = \frac{s^2}{8} + \frac{k_3}{2} - \frac{1}{48} \quad (2.74)$$

where we have included the contribution $c_-/48$ from the central charge $c_- = 1$ which arises due to the framing anomaly. Note that the fractional part of the angular momentum remains the same when we shift $k_3 \rightarrow k_3 + 2$, even though k_3 has a $\mathbb{Z}_4$ classification.

The charge of a dislocation with Burgers vector $\vec{b}$ is $Q_{disloc, \vec{b}} = \vec{\mathcal{P}}_c \cdot \vec{b}$, where

$$\begin{pmatrix} \mathcal{P}_{c,x} \\ \mathcal{P}_{c,y} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} k_{4,x} + k_{4,y} \\ k_{4,x} - k_{4,y} \end{pmatrix} + \frac{q}{4} \begin{pmatrix} t_x + t_y \\ t_x - t_y \end{pmatrix}. \quad (2.75)$$

Observe that the SPT contribution from $\vec{k}_4$ can only take two inequivalent values: $(0,0)^T$ or $(1/2, 1/2)^T$. This follows from demanding rotational invariance of the polarization up to integers, i.e. of $\vec{\mathcal{P}}_c$ modulo integers. The non-trivial symmetry fractionalization ($q = S$ and $(\vec{t}_x, \vec{t}_y) = (S, I) \sim (I, S)$), then contributes $(1/4, 1/4)^T \sim (1/4, -1/4)^T$. Therefore dislocations can carry charge of $\pm 1/4$, even though the minimal anyon charge is $1/2$. A similar calculation can be performed for the angular momentum polarization.

On a space with boundary, the non-trivial symmetry fractionalization class ($q = S$ and $(\vec{t}_x, \vec{t}_y) = (S, I) \sim (I, S)$) therefore contributes a charge of $1/4 \pmod{1/2}$ per unit length along the boundary. The other symmetry fractionalization classes contribute a 0 charge $\pmod{1/2}$ per unit length along the boundary.

The momentum per unit cell is given by the momentum of the anyon per unit cell, $\vec{v}_p =$

$\vec{P}_{\vec{m}} = \frac{m}{4} \begin{pmatrix} t_x + t_y \\ t_x - t_y \end{pmatrix}$. For trivial fractionalization (either m or $\vec{t}$ trivial), $2\vec{\nu}_p$ is an integer vector. The non-trivial fractionalization gives rise to $\vec{\nu}_p = (1/4, 1/4) \sim (1/4, -1/4)$.

Finally, we compute the momentum of a dislocation with Burgers vector $\vec{b}$. The i component of the momentum equals $\Pi_{ij}b_j$, where $\Pi_{ij} = (1 - U^T(\pi/2))^{-1}t_i^T K^{-1}t_j(1 - U(\pi/2))^{-1}$. Thus we obtain

$$\Pi_{ij} = \frac{1}{8} \begin{pmatrix} (t_x + t_y)^2 & t_x^2 - t_y^2 \\ t_y^2 - t_x^2 & (t_x - t_y)^2 \end{pmatrix}. \quad (2.76)$$

For nontrivial symmetry fractionalization, where $t_x \pm t_y$ is odd, we see that Π_{ij} has diagonal components equal to $1/8$. On the other hand, if we have trivial symmetry fractionalization, the only possible values of the components are 0 and $1/2$, which correspond to trivial values of crystal momentum, as discussed above.

Finally, we note that different choices of the parameters k_i do not necessarily give different SET phases. This is because of redundancies that arise when we consider gauge field relabellings, as we discuss for the $1/2$ Laughlin state in App. A.2.

2.5.2 $\mathbb{Z}_2$ gauge theory

In this example we consider the case where the intrinsic topological order is given by $\mathbb{Z}_2$ gauge theory (i.e. that of the $\mathbb{Z}_2$ toric code), which has $\mathcal{A} = \mathbb{Z}_2 \times \mathbb{Z}_2$. This is the case relevant for gapped $\mathbb{Z}_2$ quantum spin liquids.

The system is described by a K -matrix $K = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$. The anyons are given by $I = (0, 0)^T$, $e = (1, 0)^T$, $m = (0, 1)^T$ and $\psi = (1, 1)^T$, and we have $\vec{a}^T K^{-1} \vec{b} = \frac{a_1 b_2 + a_2 b_1}{2}$.

The symmetry fractionalization classes are specified by inequivalent choices of $\{\vec{q}, \vec{s}, \vec{t}_i, \vec{m}\} \in \mathcal{A} \times (\mathcal{A}/M\mathcal{A}) \times (K_M \otimes \mathcal{A}) \times \mathcal{A}$. Thus the classification of $\vec{q}$ and $\vec{m}$ is individually $\mathbb{Z}_2 \times \mathbb{Z}_2$, irrespective of the value of M . Suppose we wish to determine $\vec{q}$. To do so, we first compute the fractional charge $Q_{\vec{l}} = \vec{q}^T K^{-1} \vec{l} = \frac{q_1 l_2 + q_2 l_1}{2}$, for each anyon $\vec{l}$. From this data, we can uniquely determine the integers $q_I \bmod 2$, which fix the charge fractionalization. A similar method allows us to determine the anyon $\vec{m}$. Together these determine the charge filling, which places a LSM-type constraint on the fractionalization data:

$$\nu_c \bmod 1 = \vec{q}^T K^{-1} \vec{m} = \frac{q_1 m_2 + q_2 m_1}{2}. \quad (2.77)$$

Next we turn to the spin vector. We have $\mathcal{A}/M\mathcal{A} = \mathbb{Z}_2 \times \mathbb{Z}_2$ if M is even, and is $\mathbb{Z}_1$ otherwise. As in the previous example, we see that the anyons $\vec{s}$ and $M\vec{s}$ are equivalent when $M = 3$, so any choice of $\vec{s}$ can be understood in terms of attaching anyons to elementary disclinations in this case. Now suppose M is even. When $\vec{s} = e$, the fractional angular momentum of each anyon (modulo 1), given by $\vec{s}^T K^{-1} \vec{l}$, equals

$$L_I = 0, L_e = 0, L_m = 1/2, L_\psi = 1/2. \quad (2.78)$$

If we choose $\vec{s} = m$, a similar calculation yields

$$L_I = 0, L_e = 1/2, L_m = 0, L_\psi = 1/2; \quad (2.79)$$

and choosing $\vec{s} = \psi$ gives

$$L_I = 0, L_e = 1/2, L_m = 1/2, L_\psi = 0. \quad (2.80)$$

Note that measuring the angular momentum for any one anyon does not uniquely fix the value of $\vec{s}$. This result emphasizes that in general we need to know the angular momentum of *every* anyon in order to fix the symmetry fractionalization class. All these calculations could formally be done in the same manner for M odd; however, each set of angular momentum values thus calculated would correspond to trivial symmetry fractionalization.

For $M = 2, 3, 4$, the distinct torsion vectors are classified by

$$K_2 \otimes (\mathbb{Z}_2 \times \mathbb{Z}_2) = \mathbb{Z}_2^4 \quad (2.81)$$

$$K_3 \otimes (\mathbb{Z}_2 \times \mathbb{Z}_2) = \mathbb{Z}_1 \quad (2.82)$$

$$K_4 \otimes (\mathbb{Z}_2 \times \mathbb{Z}_2) = \mathbb{Z}_2 \times \mathbb{Z}_2 \quad (2.83)$$

We can understand the $\mathbb{Z}_2^4$ classification as follows: when $M = 2$, the anyons $\vec{t}_x$ and $\vec{t}_y$ are fixed independently, and each can be equal to I, e, m or ψ . The equivalence relation on $\vec{t}_x, \vec{t}_y$ does not provide any additional constraint.

For $M = 3$, the equivalence relation is

$$\begin{pmatrix} \vec{t}_x \\ \vec{t}_y \end{pmatrix} \sim \begin{pmatrix} \vec{t}_x \\ \vec{t}_y \end{pmatrix} + (1 - U(2\pi/3)) \begin{pmatrix} t'_x \\ t'_y \end{pmatrix} = \begin{pmatrix} \vec{t}_x \\ \vec{t}_y \end{pmatrix} + \begin{pmatrix} t'_x - t'_y \\ t'_x + 2t'_y \end{pmatrix} \quad (2.84)$$

(see Appendix A.1 for the explicit forms of the rotation point group matrices). Notice that every

anyon can be written in the form $\begin{pmatrix} t'_x - t'_y \\ t'_x + 2t'_y \end{pmatrix} \mod 2$. Therefore every assignment is trivial. In this case, although we can certainly adjust $\vec{t}_x, \vec{t}_y$ so as to obtain nontrivial values of momentum for the anyons, the symmetry fractionalization class is still trivial.

For $M = 4$, we find, as in the previous example, that $\vec{t}_x$ and $\vec{t}_y$ are not independent: we can only fix $\vec{t}_x + \vec{t}_y \in \{I, e, m, \psi\}$. This leads to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ classification. Finally, for $M = 6$, the torsion vectors are always trivial, irrespective of the structure of $\mathcal{A}$.

Let us consider below the fractional U(1) charges of the defects, for the special case of the standard gapped $\mathbb{Z}_2$ spin liquid at half-filling and on the square lattice, where $M = 4$, $\vec{q} = m = (0, 1)^T$, $\vec{m} = e = (1, 0)^T$ and all $k_i = 0$. In this case, the Hall conductivity vanishes, $\sigma_H = 0$, and depending on the value of $\vec{s}$ the U(1) charge of a pure $\pi/2$ disclination is calculated as $\mathcal{S}/4 = \vec{q}^T K^{-1} \vec{s}/4$. To find the U(1) charge of a dislocation, we compute

$$\begin{pmatrix} \mathcal{P}_{c,1} \\ \mathcal{P}_{c,2} \end{pmatrix} = (1 - U(\pi/2))^{-1} \begin{pmatrix} \vec{q}^T K^{-1} \vec{t}_x \\ \vec{q}^T K^{-1} \vec{t}_y \end{pmatrix} \quad (2.85)$$

$$= \frac{1}{4} \begin{pmatrix} \vec{t}_{x,1} + \vec{t}_{y,1} \\ \vec{t}_{x,1} - \vec{t}_{y,1} \end{pmatrix} \quad (2.86)$$

The four fractionalization classes related to the torsion vector are specified by choosing $\vec{t}_x + \vec{t}_y \sim I, e, m$ or ψ . Note that $\vec{t}_{i,1} = 1$ if $\vec{t}_i = e, \psi$, while $\vec{t}_{i,1} = 0$ if $\vec{t}_i = I, m$. Thus, if the momentum fractionalization class is specified by I or m (i.e. it is trivial), the polarization will take values of the form $(0, 0)^T, (0, 1/2)^T, (1/2, 0)^T$. The charge of a dislocation computed using these values will be a multiple of $1/2$ and can be understood as the charge of some anyon associated to that dislocation. If the momentum fractionalization is specified by e or ψ , the polarization will take

values of the form $(1/4, \pm 1/4)$. Then the charge of a dislocation can take the values $1/4$ or $3/4$, which cannot be understood through the attachment of anyons to each dislocation. This is a feature of nontrivial momentum fractionalization. Note that the seeming asymmetry between e and m in this example is due to our choice of $\vec{q} = m$.

The rest of the responses are straightforward to compute in this example given our general theory and we leave them for more detailed studies of $\mathbb{Z}_2$ spin liquids.

2.6 Crystalline gauge theory: continuum approach

In this section we discuss two aspects of crystalline gauge theory. The first is that our method of defining discrete crystalline gauge fields on simplices and using simplicial calculus to evaluate the action was a practical choice to make direct the relation with the group cohomology classifications of symmetry-enriched topological states (SETs) [11]. However we expect that the same results can also be obtained by working with real-valued differential forms. The second aspect is that the discrete translation and rotation gauge fields defined in this work are directly related to the coframe field and the spin connection that arise in continuum geometry and are known to be closely related to elasticity theory.

2.6.1 Crystalline gauge fields as differential forms

In order to construct actions from discrete gauge fields, it was convenient to work in terms of simplicial cohomology and simplicial calculus (see Appendix A of Ref [185] for a review). There, our translation gauge fields could be viewed as $\mathbb{Z}^2$ -valued 1-cochains defined on the triangulated space-time manifold $\mathcal{M}$; that is, $X, Y \in C^1(\mathcal{M}, \mathbb{Z})$. Similarly, the rotation gauge

field can be viewed as a $\mathbb{Z}_M$ valued 1-cochain, $\omega \in C^1(\mathcal{M}, \mathbb{Z}_M)$ (strictly speaking, in the main text ω corresponded to a lift of the $\mathbb{Z}_M$ gauge field to $\frac{2\pi}{M}\mathbb{Z}$. The action is then invariant under changes of lift, e.g. shifting $\omega_{ij} \rightarrow \omega_{ij} + 2\pi$ for a single 1-simplex ij).

We can consider instead a formulation where we take the gauge fields to be real-valued differential 1-forms. We thus can define

$$a^I, A, X, Y, \omega \in \Omega^1(\mathcal{M}, \mathbb{R}), \quad (2.87)$$

where $\Omega^k(\mathcal{M}, \mathbb{R})$ denotes the space of real-valued differential k -forms. $a^I, A, \vec{R} = (X, Y)$, and ω are the internal, U(1), translation, and rotation gauge fields, respectively, now defined as differential 1-forms.

The discreteness of the gauge fields enters through constraints on the holonomies of these gauge fields. Given a cycle γ , we require

$$\oint_{\gamma} \vec{R} \in 2\pi\mathbb{Z}^2, \quad \oint_{\gamma} \omega \in \frac{2\pi}{M}\mathbb{Z}, \quad (2.88)$$

with the equivalence

$$\begin{aligned} \oint_{\gamma} \omega &\sim \oint_{\gamma} \omega + 2\pi, & \oint_{\gamma} A &\sim \oint_{\gamma} A + 2\pi \\ \oint_{\gamma} a^I &\sim \oint_{\gamma} a^I + 2\pi \end{aligned} \quad (2.89)$$

Dislocations and disclinations must therefore correspond to singular sources of flux for X, Y, ω .

Differential forms which are required to integrate to discrete values along cycles are referred to

as integral differential forms.

The gauge transformations are also real-valued. In particular large gauge transformations for a^I , A and ω must be quantized in units of 2π .

We then write the effective action using the wedge product:

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4\pi} K_{IJ} a^I \wedge da^J + \mathcal{L}_{frac} + \mathcal{L}_{SPT} \\
\mathcal{L}_{frac} &= \frac{1}{2\pi} a^I \wedge (q_I dA + s_I d\omega + \vec{t}_I \cdot d\vec{\mathcal{R}} + m_I A_{XY}) \\
\mathcal{L}_{SPT} &= \frac{k_1}{2\pi} A \wedge dA + \frac{k_2}{2\pi} A \wedge d\omega + \frac{k_3}{2\pi} \omega \wedge d\omega + \frac{1}{2\pi} A \wedge (\vec{k}_4 \cdot d\vec{\mathcal{R}}) + \frac{1}{2\pi} \omega \wedge (\vec{k}_5 \cdot d\vec{\mathcal{R}}) \\
&\quad + \left(\frac{k_6}{2\pi} A + \frac{k_7}{2\pi} \omega \right) \wedge A_{XY}
\end{aligned} \tag{2.90}$$

Here A_{XY} is the continuum analog of the area element we defined in the simplicial formulation. For example, when $\omega = 0$, $A_{XY} = \frac{1}{4\pi}(X \wedge Y - Y \wedge X)$. Note that the terms in $\mathcal{L}$ aside from those involving A_{XY} and $d\vec{\mathcal{R}}$ are standard. To ensure that the terms involving $d\vec{\mathcal{R}}$ are invariant under large gauge transformations of a^I , A , and ω , we require $\frac{1}{2\pi} \int_W d\vec{\mathcal{R}} \in \mathbb{Z}^2$ over any closed 2-cycle W . When W is the space, for example, this physically corresponds to the fact that the total Burgers vector of the whole closed space is trivial.

While we do not pursue a formal proof here, we expect that the effective action defined using this continuum formulation yields identical physical results as compared with the lattice gauge theory formulation used in previous sections.

Given a triangulation of the space-time manifold $\mathcal{M}$, we can understand the relation between the discrete formulation and the continuum formulation as follows. Given a link (1-simplex) ij with vertices i and j , the discrete gauge fields, A_{ij} , ω_{ij} , X_{ij} and Y_{ij} are taken to be the integral from i to j along the 1-simplex ij of their continuum counterparts. Note that only those

continuum gauge field configurations can be used that give rise to the appropriate discrete values of ω , X , and Y . Since the only gauge invariant quantities for $\vec{R}$ and ω are associated with disclinations and dislocations, we expect that such gauge configurations can always be found.

We can see how to specify the action of ω on $\vec{R}$ by noting that in the continuum setting, $\vec{R}$ and ω correspond exactly to the continuum coframe fields e and spin connection ω_{SC} . In the following section we discuss this correspondence in more detail.

2.6.2 Gauge fields for continuous spacetime symmetries: coframe field and spin connection

The Euclidean group $\mathbb{E}^2 = \mathbb{R}^2 \rtimes SO(2)$ is a semidirect product of the group of continuous rotations in 2D, $SO(2) = U(1)$ and the group of continuous translations, $\mathbb{R}^2$. In this case we can consider background gauge fields associated with the continuous translation and rotation symmetries.

The translation gauge fields in the continuum setting now correspond to the 1-form coframe fields e_μ^a , $a = 1, 2$ associated with the space Σ^2 . For physically realistic space-time manifolds of the form $\mathcal{M} = \Sigma^2 \times \mathbb{R}$, where Σ^2 is space, we choose e^1, e^2 to be of the form $e_i^a dx^i = e_x^a dx + e_y^a dy$. There is also a fixed time-component of the coframe field, $e^0 = dt$. Below we will assume the space Σ^2 can be curved, but time is separate, as is appropriate for directly describing a condensed matter system. That is, the metric tensor $g = g_{ij} dx^i dx^j + g_{tt} dt^2$.

The coframe fields diagonalize the metric tensor

$$g_{ij} = e_i^a e_j^b \delta_{ab}, \quad (2.91)$$

where δ_{ij} (the Kronecker delta) is the flat space metric. In the linearized approximation where $e_i^a = \delta_i^a + \tilde{e}_i^a$, we have

$$g_{ij} = \delta_{ij} + \tilde{e}_j^i + \tilde{e}_i^j, \quad (2.92)$$

where $\delta_i^a = \delta_{ai}$ is the Kronecker delta. A translation gauge transformation can be identified as an infinitesimal diffeomorphism:

$$x^i \rightarrow f^i(x) = x^i + \epsilon^i(x), \quad (2.93)$$

under which

$$\begin{aligned} e_i^a &\rightarrow \partial_i f^j e_j^a = (\delta_i^j + \partial_i \epsilon^j) e_j^a \\ &= (\delta_i^j + \partial_i \epsilon^j)(\delta_j^a + \tilde{e}_j^a) = \delta_i^a + \tilde{e}_i^a + \partial_i \epsilon^a + \dots \\ &= e_i^a + \partial_i \epsilon^a + \dots, \end{aligned} \quad (2.94)$$

where the $\dots$ indicate the subleading term which we ignore in the linearized approximation. We see therefore that in the linearized approximation, the gauge transformations of e_i^a are the continuous analog of the discrete translation gauge transformations $\vec{R}_{ij} \rightarrow \vec{R}_{ij} + \vec{r}_j - \vec{r}_i$ on the lattice. Note that as in the discrete case, the continuous translation gauge transformations should preserve the gauge-invariant holonomies associated with e^a . In particular, the gauge transformations therefore correspond to diffeomorphisms that preserve the lengths along non-contractible cycles.

Physically, the continuous translation gauge fields e_i^a correspond to the plastic distortion tensor discussed in Ref. [166]. The full strain tensor u is the sum of the elastic strain tensor u^{el} and the plastic strain tensor u^p : $u = u^{el} + u^p$. The gauge-invariant combination is $u^{el} = u - u^p$.

In addition to the translation gauge transformations, there are also rotation gauge transformations. These correspond to locally rotating the coordinate axes by an element of $SO(2)$, at every point. The gauge field associated with these gauge transformations is the spin connection, which is a 1-form gauge field ω_{SC} that corresponds to the continuous spatial rotation symmetry. The spin connection specifies how the frame fields at nearby points are rotated relative to each other. In terms of the full 3D space-time spin connection $\Omega_{b,\mu}^a$, the spin connection associated with spatial rotations corresponds to $\omega_{SC,\mu} = \Omega_{2,\mu}^1$. In this language, we can explicitly write the correspondence between the continuum and discrete gauge fields as $\vec{R} \sim (e^1, e^2)$ and $\omega \sim \omega_{SC}$. We emphasize that when the continuous $\mathbb{E}^2$ symmetry is broken down to a discrete space group symmetry, there is no distinction between $(\vec{R}, \omega)$ and $(\vec{e}, \omega_{SC})$. The gauge-invariant properties associated to $(\vec{R}, \omega)$ can equally be calculated using (e^a, ω_{SC}) .

To further clarify the correspondence, we calculate the contribution of ω_{SC} to the covariant derivative of e^a using our discrete formulation with certain limiting arguments. At a point $\vec{r} + \delta\vec{r}$, the coframe field (written here using the translation gauge field notation of the main text) is

$$R_j(\vec{r} + \delta\vec{r}) \approx R_j(\vec{r}) + \partial_i R_j \cdot \delta r_i. \quad (2.95)$$

As stated in the main text, the vector $\vec{R}(\vec{r})$ parallel transported to $\vec{r} + \delta\vec{r}$ is $U(\omega_{\vec{r}, \vec{r} + \delta\vec{r}}) \vec{R}(\vec{r})$, where we have chosen $\vec{r}$ as the origin. In the continuum, we can write $U(\omega_{\vec{r}, \vec{r} + \delta\vec{r}} = \theta)$ as a rotation matrix $e^{i\theta\sigma_y}$ (this would not be appropriate on a lattice, where we need to use $GL(2, \mathbb{Z})$ matrices

in a lattice basis, but it is not a problem in the continuum). The total rotation applied between $\vec{r}'$ and $\vec{r}' + d\vec{r}'$ is written in terms of ω_{SC} as $e^{-i \int_{\vec{r}}^{\vec{r}+\delta\vec{r}} \vec{\omega}_{\text{SC}}(\vec{r}') \cdot d\vec{r}' \sigma_y}$. Here we have written the spin connection as a vector with components $\omega_{\text{SC},\mu}$. This representation of ω shows that it directly corresponds to ω_{SC} in the continuum.

To first order in $\delta\vec{r}$, we can approximate

$$U(\omega_{\vec{r},\vec{r}+\delta\vec{r}}) = e^{-i \int_{\vec{r}}^{\vec{r}+\delta\vec{r}} \vec{\omega}_{\text{SC}}(\vec{r}') \cdot d\vec{r}' \sigma_y} \quad (2.96)$$

$$\approx 1 - i\sigma_y \vec{\omega}_{\text{SC}}(\vec{r}) \cdot \delta\vec{r} \quad (2.97)$$

The covariant derivative of $\vec{R}$ in the direction x^i can then be written as

$$D_i R_j(\vec{r}) = \lim_{x^i \rightarrow 0} \frac{1}{x^i} (R_j(\vec{r} + x^i) - (U(\omega_{\vec{r},\vec{r}+x^i}) \vec{R}(\vec{r}))_j) \quad (2.98)$$

$$= \lim_{x^i \rightarrow 0} \frac{1}{x^i} (\partial_i R_j x^i + \omega_{\text{SC},i}(\vec{r}) x^i \times (i\sigma_y R)_j) \quad (2.99)$$

$$= \partial_i R_j + \omega_{\text{SC},i}(\vec{r}) \epsilon_{jk} R_k. \quad (2.100)$$

This is precisely the formula for the covariant derivative D of e^a in terms of ω_{SC} , which is written in the usual notation as

$$T^a \equiv D e^a = d e^a + \epsilon_b^a \omega_{\text{SC}}^b = d e^a + \Omega_b^a \wedge e^b. \quad (2.101)$$

Here $\Omega_{b,\mu}^a$ is the full spin connection. We have proved this formula using the fact that $\Omega_b^a = -\Omega_a^b$ is anti-symmetric, so that $\Omega_1^1 = \Omega_2^2 = 0$ and $\Omega_2^1 = -\Omega_1^2 = \omega_{\text{SC}}$.

T^a is the torsion 2-form, which characterizes how the frame field is rotated along the

path traced by a curve in spacetime. The torsion as defined above can be directly related to the dislocation density, i.e. to the holonomy of translation gauge fields after accounting for parallel transport, similar to the quantity $d\vec{\mathcal{R}}$ used in our work. Furthermore, the flux associated to rotational symmetry alone ($d\omega$ in the lattice formulation, or $d\omega_{\text{SC}}$ in the continuum) gives the curvature of the manifold, which is directly related to the disclination density. Therefore couplings involving $d\omega$ or $d\omega_{\text{SC}}$ are essentially coupling the system to curvature. Given that torsion is not quantized in the continuum, there cannot be any quantized topological terms formed by coupling anyons or symmetry charges to the torsion (although nonquantized terms which are topological in the sense of being independent of changes in the underlying metric are well-known).

The classification of SET phases with $U(1) \times \mathbb{E}^2$ symmetry is identical to the classification for $U(1) \times U(1)$ symmetry (this can be proved, for example, by computing the relevant cohomology groups) [53]. So while the translation group $\mathbb{R}^2$ has associated gauge fields X and Y , the Lagrangian does not have any contribution from X and Y ; the only relevant terms for Euclidean group symmetry fractionalization and for the associated SPT states are given by $\frac{s_I}{2\pi} a^I \wedge d\omega_{\text{SC}}$ and $\frac{k}{2\pi} \omega_{\text{SC}} \wedge d\omega_{\text{SC}}$ respectively.

2.7 Discussion

2.7.1 Spatial vs. internal symmetries

As we have discussed, at a formal level our mathematical treatment of crystalline gauge fields is equivalent to treating the symmetry as an internal symmetry of the low energy quantum field theory. The main difference is (1) the physical interpretation of the fluxes in terms of

geometrical properties of the lattice, with certain holonomies being restricted by the lattice area and lengths, and (2) the fact that we ultimately tie the space-time metric of the low energy topological quantum field theory, which arises from the framing anomaly, to the crystalline gauge fields. Here we will begin by discussing this issue in some more detail.

We have two levels of description of the system. The first is the microscopic lattice model, which has a global symmetry $G = U(1) \times G_{\text{space}}$, for some spatial symmetry group G_{space} . The second is the effective field theory description, which in our case is a topological field theory. The symmetry of the topological field theory is $G_{\text{IR}} \times \text{Diff}(M)$, where G_{IR} is the internal symmetry of the field theory and $\text{Diff}(M)$ is the group of diffeomorphisms of the space-time manifold M . Here the internal symmetry G_{IR} allows us to couple the field theory to background principal G_{IR} bundles. The action of the microscopic G symmetry in the low energy field theory is described by a group homomorphism:

$$\alpha : G \rightarrow G_{\text{IR}} \times \text{Diff}(M). \quad (2.102)$$

When $\mathbf{g} \in G$ is a purely on-site symmetry of the microscopic lattice model, then $\alpha(\mathbf{g}) = (\alpha(\mathbf{g})|_{G_{\text{IR}}}, \mathbf{1})$, where $\alpha(\mathbf{g})|_{G_{\text{IR}}}$ denotes the restriction of α to the first factor and $\mathbf{1}$ refers to the identity element of $\text{Diff}(M)$. That is, an on-site symmetry $\mathbf{g}$ in the microscopic lattice model is mapped to an internal symmetry in the field theory. On the other hand, if $\mathbf{g} \in G$ is a purely spatial symmetry of the microscopic lattice model, then $\alpha(\mathbf{g})$ maps $\mathbf{g}$ to a combination of an internal symmetry and an element of $\text{Diff}(M)$. For example, a $\mathbb{Z}_M$ spatial rotation in the microscopic lattice model will be mapped in general to a $\mathbb{Z}_M$ internal symmetry combined with a $\mathbb{Z}_M$ rotation of space in the field theory. The distinct ways that a microscopic lattice symmetry can

act in the field theory is taken into account by the different ways of coupling the effective field theory to background G_{IR} gauge fields.

Another way to state the above is that given any spatial symmetry $\mathfrak{g}$, one can always consider the combination $\alpha(\mathfrak{g})$ followed by an appropriate element of $\text{Diff}(M)$, to obtain a symmetry action in the field theory that has trivial component in $\text{Diff}(M)$. Therefore, given any spatial symmetry in the microscopic lattice model, the effective field theory description can also in general contain a corresponding internal symmetry. To fully describe all possible SETs, we thus take $G_{\text{IR}} = G$, and we classify all the ways that the effective field theory can be coupled to G bundles.

Observe that in the above description, the symmetry defects associated with spatial symmetries in the microscopic lattice model, such as dislocations and disclinations, should therefore be described in the field theory by symmetry fluxes of the internal symmetry of the field theory and, simultaneously, torsion and curvature defects in the space-time metric of the effective field theory. This reflects the fact that α maps to both G_{IR} and $\text{Diff}(M)$. This explains why we equate the spin connection of the space-time metric to the rotation gauge field in Eq. 2.65.

The above explanation is not a proof that spatial symmetries in lattice models can always be treated as internal symmetries in the effective field theory description. Nevertheless, all known examples of effective field theories of quantum many-body systems can be understood via the above paradigm. As a simple example, consider the action of translation symmetries in spin chains and their description in the low energy Luttinger liquid theory [191].

The conjecture that spatial symmetries can always be treated as internal symmetries in the field theory (with some technical caveats) has recently been formalized in Ref. [82] as the “crystalline equivalence principle,” where additional arguments have also been given in support

of it. This principle has also received significant support from the theory of crystalline SPTs, where the SPT classifications obtained by treating spatial symmetries as internal symmetries can be compared with other more direct methods [76,77,79,83], and the results agree with each other. We will present a careful statement of this principle in Sec. 3.2.

2.7.2 Relation to defect network constructions

Ref. [83] gives a general construction of crystalline SET phases in terms of defect networks; a similar approach has been studied for invertible phases in Ref. [79]. Here the manifold $\mathcal{M}$ is decomposed by means of a cellulation, and the defects in the theory, which include anyons as well as symmetry defects, are assumed to live on the 0-cells (vertices) of the cellulation.

The authors of Ref. [83] show that the defect network picture is equivalent to the crystalline equivalence principle. Our formalism is equivalent to assuming the crystalline equivalence principle and proceeding with the G -crossed braided tensor category [11] and associated group cohomology classifications of SET phases. In this sense, we expect that our approach formally yields the same classification results as the defect network picture.

However the two approaches differ in details of physical interpretation. Let us restrict to the SPT case for concreteness. In this special case the defect network picture is mathematically related to an equivariant cohomology theory, in which one considers the high-symmetry points of a space group unit cell and places symmetry charges on these high symmetry points. Two configurations of symmetry charge are in different SPT phases if they cannot be deformed into one another by local, symmetry-preserving unitaries. (This procedure is essentially the "block state" construction of SPT phases developed in Refs [76, 77].) It is not fully clear how this

approach is equivalent to the topological response theory that we have described in our work. We can also express this distinction as follows: the equivariant cohomology approach has symmetry charges, but in this picture it is not apparent how these arrangements of charge give rise to different responses upon introducing symmetry fluxes. Reconciling the two pictures properly is an interesting direction, but beyond the scope of the present work.

2.7.3 Outlook

We have predicted a type of momentum fractionalization, characterized by the discrete torsion vector, which can only be non-trivial for $M = 2, 3, 4$ -fold rotation symmetry together with translation symmetry. This term leads to a number of fractionally quantized response properties with no analog in the continuum. Perhaps most notably this includes a fractionally quantized charge polarization, which can assign non-trivial fractional charges to dislocations and fractional charges per unit length to boundaries (modulo the anyon charge). In addition to this, the theory predicts fractionally quantized linear and angular momenta for disclinations, dislocations, and units of area. It is important to verify the predictions of this crystalline gauge theory through microscopic studies of model Hamiltonians and wave functions; we will do this in the setting of invertible fermionic states in Chapters 4 and 5. While the fractional charges of dislocations and disclinations can in principle also be probed by experiments on fractional Chern insulators with sufficiently weak disorder, it is an interesting theoretical question to understand the extent to which the fractionally quantized linear and angular momenta of anyons, lattice defects, and units of area can be experimentally measured.

Our theory is expected to be complete for topological phases of bosons, where symmetries

do not permute anyon types. For fermionic states, which are most relevant for experimental studies of fractional Chern insulators in solid state systems, we need to use a modified theory which is discussed in Chapter 3. When the space group symmetries do permute anyon types [192, 193], lattice defects can be non-Abelian and the classification of SETs is different. Furthermore, certain values of the coefficients of the response theory may be constrained by the symmetry permutation. A detailed study of this is left for future work.

The crystalline gauge theory we have developed treats the lattice defects as a fixed background configuration that is described in terms of a fixed background gauge field. Such a gauge theory apparently does not take into account the restricted mobility of dislocations and disclinations in a crystalline environment. The restricted mobility of these lattice defects can be described using higher rank tensor gauge fields, which are known to be dual to fracton theories (see e.g. Refs [194–196]). It would be interesting to understand the relation between the topological field theory developed here and a formulation including higher rank tensor gauge fields which explicitly takes into account the restricted mobility of the lattice defects.

Finally, we note that in general, given a symmetry G of a condensed matter system, the effective field theory must include coupling to background gauge fields of the symmetry in order to be fully specified. It would be interesting to revisit the large family of effective field theories used throughout condensed matter physics, including gapless theories, and to properly understand the coupling to background crystalline gauge fields.

2.8 Acknowledgements

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Chapter 3: Crystalline gauge theory for invertible fermionic topological phases

This chapter is organized as follows. In Sec. 3.1, we present a brief review of the classification of invertible fermionic phases in (2+1) dimensions with internal symmetry G_f . In Sec. 3.2 we explain how to extend the theory to the case with spatial symmetries, and state the general version of the fermionic crystalline equivalence principle. Finally, in Sec. 3.3 we state the classification of invertible fermionic phases with symmetry $G_f = \text{U}(1)^f \times_\phi [\mathbb{Z}^2 \rtimes \mathbb{Z}_M]$ in terms of a topological action (Eq. (3.20)), and point out several of its predictions that we will test numerically in Chapters 4 and 5. The complete derivation of this action is given in App. B.3.

This Chapter contains material based on the following papers:

Classification of (2+1)D invertible fermionic topological phases with symmetry, by Maissam Barkeshli, Yu-An Chen, Po-Shen Hsin, and Naren Manjunath, Phys. Rev. B 105, 235143 (2022)
Non-perturbative constraints from symmetry and chirality on Majorana zero modes and defect quantum numbers in (2+1)D by Naren Manjunath, Vladimir Calvera, and Maissam Barkeshli, Phys. Rev. B 107, 165126 (2023).

The calculations in Appendix B.3 have not previously appeared in full, although the results in it have partially appeared in Refs. [100, 101].

Notational remarks: Compared to the notation in Chapter 2, there are two differences. Here we explicitly show how each topological invariant depends on the choice of origin $\mathbf{o}$ (by

writing $\mathcal{S}_0^+, \vec{\mathcal{P}}_0$ and so on). This dependence was present but not made explicit in Chapter 2. Moreover, in fermionic systems there are two natural choices of rotation symmetry operators about $\mathbf{o}$, which we denote as $\tilde{C}_{M_0}^+$ and $\tilde{C}_{M_0}^-$. Invariants are given $+$ and $-$ superscripts when they are defined with respect to $\tilde{C}_{M_0}^+$ and $\tilde{C}_{M_0}^-$ respectively. This distinction did not arise in Chapter 2, because we restricted to bosonic systems.

3.1 Review of classification of invertible fermionic phases

3.1.1 Overview

Let G_f be the symmetry group of the invertible fermionic phase (assumed to be internal in this section). Define $G_b = G_f / \mathbb{Z}_2^f$ where $\mathbb{Z}_2^f$ denotes the fermion parity symmetry. Let $(\mathbf{g}, a)$ denote an element of G_f , where $\mathbf{g} \in G_b, a \in \mathbb{Z}_2^f$. Then, the group law in G_f can be written as

$$(\mathbf{g}_1, a_1)(\mathbf{g}_2, a_2) = (\mathbf{g}_1\mathbf{g}_2, a_1 + a_2 + \omega_2(\mathbf{g}_1, \mathbf{g}_2)) \quad (3.1)$$

where ω_2 is a 2-cocycle representative of the group $\mathcal{H}^2(G_b, \mathbb{Z}_2)$. The homomorphism $s_1 : G_b \rightarrow \mathbb{Z}_2$ specifies if $\mathbf{g} \in G_b$ is unitary ($s_1(\mathbf{g}) = 0$) or antiunitary ($s_1(\mathbf{g}) = 1$).

If the symmetry is of the form $G_f = G_b \times \mathbb{Z}_2^f$, we can set $\omega_2 = 0$. If $[\omega_2]$ belongs to a nontrivial class, there is some combination of operations $\{\mathbf{g}_i\} \in G_b$ with $\prod_i \mathbf{g}_i = \mathbf{0}$, such that

$$\prod_i (\mathbf{g}_i, 0) = (\mathbf{0}, 1). \quad (3.2)$$

Thus a sequence of symmetry operations which acts as the identity on bosonic operators instead

acts as $(-1)^F$ on fermionic operators. The symmetry acts through a projective representation of G_b (which is still however a linear representation of G_f). In this situation the fermion is said to have fractional G_b quantum numbers.

For example, consider a system with $G_b = \text{SO}(3)$ in which a 2π rotation within G_b equals 1 or $(-1)^F$. Such a system has integer and half-integer spin respectively. As another example, consider a system with $G_b = \mathbb{Z}_2^T$ time-reversal symmetry implemented by $\mathbf{T}$, satisfying $\mathbf{T}^2 = (-1)^F$; we can also express this as $\omega_2(\mathbf{T}, \mathbf{T}) = 1 \pmod{2}$. In this case the relevant fractional G_b quantum number is the Kramers degeneracy carried by a fermion.

The main result of Ref. [27] is that each invertible fermionic phase with symmetry group G_f is classified by the data (c_-, n_1, n_2, ν_3) , where c_- is either an integer or a half-integer, and $n_1 : G_b \rightarrow \mathbb{Z}_2, n_2 : G_b \times G_b \rightarrow \mathbb{Z}_2, \nu_3 : G_b \times G_b \times G_b \rightarrow \text{U}(1)$ are functions (‘cochains’) in 1, 2 and 3 variables respectively. They satisfy the following consistency conditions:

$$dn_1 = 0 \tag{3.3}$$

$$dn_2 = \mathcal{O}_3[c_-, n_1] \tag{3.4}$$

$$d\nu_3 = \mathcal{O}_4[c_-, n_1, n_2]. \tag{3.5}$$

The operator d is a derivative defined on cochains, see Appendix B.1. The quantities $\mathcal{O}_3, \mathcal{O}_4$ have complicated expressions and will be discussed further below (see Eqs. (3.13) and (3.15) respectively). Different sets of data (c_-, n_1, n_2, ν_3) can describe the same physical system, and are thus equivalent. The classification of invertible phases for a given G_f is obtained by finding all possible solutions to the above equations, and then modding out by such equivalences.

In the rest of this section, we will explain the general meaning of the above data and

equations. While the mathematical content in this section is the same as in Ref. [27], we include several examples and additional physical arguments to aid readers who are new to the formalism. We refer the reader to [11, 27, 49] for background on the general formalism that we use here.

3.1.2 Definition of (c_-, n_1, n_2, ν_3)

3.1.2.1 c_-

Let us fix G_f (and hence ω_2). The quantity c_- denotes the chiral central charge of the invertible fermionic phase. It can be measured from the thermal Hall conductance κ_{xy} at a temperature T through the relation

$$\kappa_{xy} = c_- \frac{\pi^2 k_B^2}{3h} T. \quad (3.6)$$

c_- is an integer or a half-integer; in a topological superconductor, $2c_-$ is the spectral Chern number of the associated Bogoliubov-de Gennes (BdG) Hamiltonian.

The basic data fully specify the quantized topological properties of symmetry defects in the system, such as their fusion rules, their quantum numbers under G_f , and so on. Symmetry defects are objects labelled by a group element in G_f . In a Hamiltonian picture, a g defect is a static modification of the Hamiltonian along some spatial cut, so that a particle which crosses the cut gets acted upon by g . Topologically distinct defects corresponding to the same group element are denoted a_g, b_g , and so on.

c_- determines the nature of the fermion parity $(0, 1)$ fluxes in the system. These are also sometimes referred to as ‘fermion parity defects’ or ‘fermion parity vortices.’ If c_- is a half-

integer, there is a single fermion parity flux, denoted $\sigma_{(0,1)}$, which is nonabelian and hosts an unpaired Majorana zero mode (MZM). (A notational remark: for fermion parity fluxes, we will usually drop the subscript and instead just write σ .) It satisfies the fusion rules $\sigma \times \psi = \psi \times \sigma = \sigma$; $\sigma \times \sigma = 1 + \psi$, where ψ denotes a fermion. This is the situation in the weak-pairing spinless $p + ip$ superconductor, where $c_- = 1/2$.

If c_- is an integer, there are two fermion parity fluxes, which do not carry an unpaired MZM. When c_- is even we denote them by m and $e = m \times \psi$. They satisfy $e \times e = m \times m = 1$. If c_- is odd, we instead denote the fermion parity fluxes as $v, \bar{v} = v \times \psi$. In this case, $v \times v = \bar{v} \times \bar{v} = \psi$. When c_- is an unspecified integer, by convention we will use e, m .

Note that if we gauge the fermion parity symmetry of an invertible fermion phase, the fermion parity fluxes are promoted to anyons with the same fusion rules as given above. The topological twist θ of these anyons satisfies $\theta = e^{i2\pi \frac{c_-}{8}}$ and thus encodes $c_- \bmod 8$.

3.1.2.2 n_1 and MZMs

Next we have a parameter $n_1 : G_b \rightarrow \mathbb{Z}_2$. The following interpretation of n_1 is valid for unitary as well as antiunitary symmetries. Suppose we gauge the fermion parity. Then, if $n_1(\mathbf{g}) = 1$, any fermion parity flux (say we denote it as m) is converted to $m \times \psi$ when acted upon by $\mathbf{g}$. Thus the action of $\mathbf{g}$ changes the fermion parity of m . We write this as $\mathbf{g}m = m \times \psi$.

When $\mathbf{g}$ is a unitary operation, there is a second equivalent interpretation which is useful: c_- and n_1 together completely specify which symmetry defects carry unpaired MZMs. First suppose $c_- \in \mathbb{Z}$. Now, if $n_1(\mathbf{g}) = 0$, then any $(\mathbf{g}, a)$ defect in the invertible fermionic phase is abelian, i.e. does not carry an unpaired MZM, while if $n_1(\mathbf{g}) = 1$, every $(\mathbf{g}, a)$ defect carries an

unpaired MZM.

When c_- is a half-integer, we saw that the $(0, 1)$ defect always hosts an unpaired MZM. If $n_1(\mathbf{g}) = 0$, the situation is the same as when $\mathbf{g} = 0$: the $(\mathbf{g}, 1)$ defect hosts an unpaired MZM while the $(\mathbf{g}, 0)$ defect does not. On the other hand, $n_1(\mathbf{g}) = 1$ implies that a $(\mathbf{g}, 0)$ defect hosts an unpaired MZM, but a $(\mathbf{g}, 1)$ defect does not.

When $\mathbf{g}$ is the time-reversal operation, a $\mathbf{g}$ defect is not in general a well-defined concept in a Hamiltonian framework.¹ However, there is still an interpretation of $n_1(\mathbf{g}) = 1$: it means that a fermion parity flux carries two degenerate states with different fermion parities which are related by $\mathbf{g}$. These states can be identified with m and $e = m \times \psi$. When $\mathbf{g}$ is a general antiunitary operation, this latter interpretation extends to all cases in which $n_1 = s_1$ [85]. However, since we are interested in unitary symmetries, we will not discuss this further here.

By considering the fusion of a $(\mathbf{g}_1, a)$ -defect with a $(\mathbf{g}_2, b)$ -defect into a $(\mathbf{g}_1, a)(\mathbf{g}_2, b)$ -defect, we can prove that n_1 should be a homomorphism:

$$\begin{aligned} n_1(\mathbf{g}_1) + n_1(\mathbf{g}_2) &= n_1(\mathbf{g}_1 \mathbf{g}_2) \pmod{2} \\ \implies dn_1 &= 0 \pmod{2}. \end{aligned} \tag{3.7}$$

3.1.2.3 ω_2 and fractional quantum numbers of ψ

The content of Eq. (3.1) is summarized by the definition

$$\eta_\psi(\mathbf{g}_1, \mathbf{g}_2) := (-1)^{\omega_2(\mathbf{g}_1, \mathbf{g}_2)}, \tag{3.8}$$

¹In a Euclidean space-time path integral, one can think of a time-reversal defect as an orientation-reversing wall in space-time. See e.g. [54].

where $\mathbf{g}_1, \mathbf{g}_2 \in G_b$. Physically, $\eta_\psi(\mathbf{g}_1, \mathbf{g}_2)$ measures the phase difference between acting on the fermion ψ by $\mathbf{g}_1$ and $\mathbf{g}_2$ separately as opposed to $\mathbf{g}_1\mathbf{g}_2$. A detailed discussion of the meaning of the η symbols in general can be found in Refs. [11, 27, 49, 50].

The symbols η_ψ also determine the fractional G_b quantum numbers carried by ψ . Given some G_b , the choices of $[\omega_2]$ are classified by $\mathcal{H}^2(G_b, \mathbb{Z}_2) \cong (\mathbb{Z}_2)^r$ for some $r \geq 0$. Thus there are r $\mathbb{Z}_2$ invariants which together specify $[\omega_2]$. We denote these as $Q_\psi^i \bmod 1, i = 1, 2, \dots, r$. Each $Q_\psi^i \in \{0, 1/2\} \bmod 1$ is a G_b quantum number of ψ , and is defined by some ‘gauge-invariant’ combination of η_ψ symbols. The precise definition of Q_ψ^i is symmetry-dependent. Whenever $Q_\psi^i = \frac{1}{2} \bmod 1$ (i.e. Q_ψ^i is a *fractional* G_b quantum number), there exists a sequence of group operations which act trivially on any bosonic operator but transform any fermionic operator by a minus sign.

Example. Consider a system with $G_f = \text{U}(1)^f$ charge conservation symmetry. Here G_f and $G_b = \text{U}(1)^f / \mathbb{Z}_2^f$ are both isomorphic to $\text{U}(1)$, but the operator generating a rotation by θ in G_b only generates a rotation by $\theta/2$ in G_f . Note that two $\theta = \pi$ rotations in G_b together act trivially on any bosonic operator, but transform a fermionic operator by a sign. Thus $\omega_2(\pi, \pi) = 1$. We can also define

$$\eta_\psi(\pi, \pi) = -1 =: e^{2\pi i Q_\psi}. \quad (3.9)$$

This means that ψ has G_b charge $Q_\psi = \frac{1}{2}$. Since $\mathcal{H}^2(\text{U}(1), \mathbb{Z}_2) \cong \mathbb{Z}_2$, we only define one invariant Q_ψ .

3.1.2.4 n_2 and fractional quantum numbers of fermion parity fluxes

We can similarly define η symbols for the fermion parity fluxes, say e and m . $\eta_a(\mathbf{g}, \mathbf{h}) \in \text{U}(1)$ for $a = e, m, \psi$ essentially describes the phase difference between applying $\mathbf{g}$ and $\mathbf{h}$ sequentially as compared to $\mathbf{gh}$ (See [11] for a detailed definition). The η symbols in general encode the fractional G_b quantum numbers of excitations.

Just as η_ψ encodes ω_2 , η_e and η_m together encode a function $n_2 : G_b \times G_b \rightarrow \mathbb{Z}_2$. For example, when $c_- = 0$, we can define

$$\begin{aligned}\eta_m(\mathbf{g}, \mathbf{h}) &= (-1)^{n_2(\mathbf{g}, \mathbf{h})} \\ \eta_e(\mathbf{g}, \mathbf{h}) &= (-1)^{(n_2 + \omega_2)(\mathbf{g}, \mathbf{h})}.\end{aligned}\tag{3.10}$$

The η symbols for general c_- in terms of n_2 are given in Ref. [27].

Example. Consider a crystalline topological insulator with charge conservation and translation symmetries, with $c_- = 0$ and a fermion per unit cell (implying a filling $\nu = 1$). Here $G_f = \text{U}(1)^f \times \mathbb{Z}^2$. Note that transporting an e (or m) particle around a single unit cell results in an overall phase of -1, because the fermion within the unit cell is seen as a π -flux for the e and m particles. Thus the elementary translations do not commute on a state with a fermion parity flux ($T_{\mathbf{x}}T_{\mathbf{y}} = -T_{\mathbf{y}}T_{\mathbf{x}}$ in such a state). This is the ‘nontrivial G_b quantum number’ encoded by the η_e, η_m symbols. In particular, we can choose a gauge where $\eta_e(\mathbf{g}, \mathbf{h}) = \eta_m(\mathbf{g}, \mathbf{h}) = (-1)^{X_1(\mathbf{g})X_2(\mathbf{h})}$ where X_i is the projection to the i -th component of $\mathbb{Z}^2$. Then, on a state with an e or m particle, we have $T_{\mathbf{x}}T_{\mathbf{y}}(T_{\mathbf{y}}T_{\mathbf{x}})^{-1} = \eta_e(T_{\mathbf{x}}, T_{\mathbf{y}})/\eta_e(T_{\mathbf{y}}, T_{\mathbf{x}}) = -1$.

Although we will not require them for this paper, we note for completeness that the theory also contains a set of U symbols $U_{\mathbf{g}}(a, b; a \times b)$ for $a, b \in \{e, m, \psi\}$, which can all be set to 1 for unitary symmetries. Through appropriate ‘gauge-invariant’ combinations of the η and U symbols, we can define fractional G_b quantum numbers Q_e^j, Q_m^j for the fermion parity fluxes, where $j = 1, \dots, k$ is an index and k is the number of independent fractional quantum numbers. For a discussion of gauge transformations on the basic data, see Appendix A of Ref. [27].

3.1.2.5 $\mathcal{O}_3$ obstruction to defining n_2

By requiring that the symmetry action on the fermions and the fermion parity fluxes respect their fusion rules, we can obtain various relations among the η symbols. For example, when the symmetry is unitary, we can show that

$$\eta_a \eta_b = \eta_c \tag{3.11}$$

whenever c is a fusion product of a and b , and a, b, c correspond to ψ or to a fermion parity flux. (For antiunitary symmetries, see Eq. 25 in Ref. [85].) This places constraints on the fractional G_b quantum numbers (or equivalently, on n_2). For example, if $G_f = \text{U}(1)^f$ as above and $c_- = 0$, we can define $e^{2\pi i Q_a} := \eta_a(\pi, \pi)$ for $a = e, m, \psi$. Then the relation $e \times m = \psi$ along with Eq. (3.11) implies that $Q_e + Q_m = Q_\psi \pmod{1}$.

Another important relation (again written for unitary symmetries) takes the form

$$\eta_a(\mathbf{g}, \mathbf{h}) \eta_a(\mathbf{gh}, \mathbf{k}) = \eta_{\bar{\mathbf{g}}a}(\mathbf{h}, \mathbf{k}) \eta_a(\mathbf{g}, \mathbf{hk}). \tag{3.12}$$

Here $\bar{\mathbf{g}}a$ is the result of permuting a by $\bar{\mathbf{g}} = \mathbf{g}^{-1}$. Note that if $n_1 \neq 0$, there is a group element

which permutes e and m . In that case the above condition may introduce further constraints on Q_e and Q_m . These relations serve to constrain n_2 in terms of n_1, c_- and ω_2 .

The constraint on n_2 is summarized by a cocycle $\mathcal{O}_3 \in Z^3(G_b, \mathbb{Z}_2)$. The general form of $\mathcal{O}_3$ is

$$\mathcal{O}_3 := n_1 \cup (\omega_2 + s_1 \cup n_1) + c_- \omega_2 \cup_1 \omega_2 \mod 2. \quad (3.13)$$

For reference, the same equation is written after expanding the $\cup$ -cup products in Appendix B.1. $\mathcal{O}_3$ is viewed as an obstruction, because if $\mathcal{O}_3$ is in a non-trivial cohomology class of $\mathcal{H}^3(G_b, \mathbb{Z}_2)$, then the equation $dn_2 = \mathcal{O}_3$ cannot be solved, as the lhs is a 3-coboundary and thus trivial in $\mathcal{H}^3(G_b, \mathbb{Z}_2)$. In particular, for a unitary symmetry, $\mathcal{O}_3$ encodes how ω_2 and c_- fix whether or not symmetry defects can host unpaired MZMs.

3.1.2.6 $\mathcal{O}_4$ obstruction and ν_3

Above we discussed constraints at the level of n_2 . Even if n_2 is well-defined, however, the invertible phase may not be well-defined if the data $(\omega_2, c_-, n_1, n_2)$ have a nontrivial obstruction (often referred to as 't Hooft anomaly), given by a class $[\mathcal{O}_4] \in \mathcal{H}^4(G_b, \text{U}(1))$. The obstruction is trivial if there exists some $\nu_3 : C^3(G_b, \text{U}(1))$ such that

$$d\nu_3 = \mathcal{O}_4[c_-, n_1, n_2]. \quad (3.14)$$

When $n_1 = 0$, the expression for $\mathcal{O}_4$ (both unitary and antiunitary symmetries) is

$$\mathcal{O}_4 = \frac{c_- \mathcal{P}(\omega_2)}{8} + \frac{1}{2} n_2 \cup (\omega_2 + n_2) \mod 1 \quad (3.15)$$

where $\mathcal{P} : \mathcal{H}^2(G_b, \mathbb{Z}_2) \rightarrow \mathcal{H}^4(G_b, \mathbb{Z}_4)$ is the Pontryagin square which is a cohomology operation that refines the square - in the sense that $2\mathcal{P}(\omega_2) = 2\omega_2 \cup \omega_2 \pmod{4}$ (see Appendix B.1). Expressions for $\mathcal{O}_4$ when $n_1 \neq 0$ can be found in Ref. [27].

If $[\mathcal{O}_4]$ vanishes, we can define a consistent set of fusion and braiding data for all the defects; this requires an additional parameter $\nu_3(\mathbf{g}, \mathbf{h}, \mathbf{k})$, which satisfies Eq. (3.14). As described below, ν_3 specifies the G_b quantum numbers of G_b symmetry defects. However, if $[\mathcal{O}_4]$ is nonvanishing, the system with data $(\omega_2, c_-, n_1, n_2)$ can only be defined in conjunction with another system that cancels the obstruction. For example, such a system may exist on the boundary of a bosonic SPT in (3+1) dimensions.

Examples. First consider an integer quantum Hall state with $G_f = \text{U}(1)^f$. There are no obstructions, so ν_3 can be defined consistently. In this case Ref. [27] showed that ν_3 directly encodes the Hall conductance. This can be generalized: just as the Hall conductance measures the $\text{U}(1)$ charge bound to magnetic flux, in general ν_3 encodes the G_b quantum numbers associated to G_b symmetry defects.

An example of a system with nontrivial $\mathcal{O}_4$ (and hence no solution for ν_3) is a translationally symmetric system in (2+1)D with spin-1/2 per unit cell. Here $G_f = \mathbb{Z}^2 \times \text{SU}(2)^f$ and $G_b = \mathbb{Z}^2 \times \text{SO}(3)$. The statement that fermions carry spin-1/2 under $\text{SO}(3)$ implies that ω_2 is nontrivial, while the statement that there is a fermion in each unit cell implies that n_2 is nontrivial. In this case, the data $(\omega_2, c_- = 0, n_1 = 0, n_2)$ give a nontrivial $\mathcal{O}_4$. This recovers a version of the well-known Lieb-Shultz-Mattis theorem, which states that there cannot be an invertible state in 2+1 dimensions satisfying the above criteria.

While solving the obstruction equations, it may happen that certain choices of n_2 that

satisfy Eq. (3.13) do not satisfy Eq. (3.14). In this sense the $\mathcal{O}_4$ obstruction is a more restrictive constraint on c_- and n_1 , so it is not enough to solve only the $\mathcal{O}_3$ equation and ignore $\mathcal{O}_4$.

Remark: In this chapter we identify $U(1) \cong \mathbb{R}/\mathbb{Z}$ and we will interchangeably use multiplicative and additive notation for ν_3 , which are related as follows:

$$\nu_3^{\text{multiplicative}} = \exp(2\pi i \nu_3^{\text{additive}})$$

where ν_3^{additive} is to be understood as a real number modulo 1.

3.2 Extending the formalism to crystalline symmetries

The formalism above is strictly valid only when G_f is an internal symmetry. However, with the aid of the fermionic crystalline equivalence principle (fCEP) [80, 82–84], it can be applied in situations where G_f acts on space. We note that the fCEP has only been discussed for completely general symmetries in [82]; furthermore [82] used the more abstract language of vector bundles. Below we present a concrete formula for the fCEP that is more useful in our context.

3.2.1 Fermionic crystalline equivalence principle

We consider a bosonic symmetry group G_b that may include spatial symmetries. We assume that the clean 2d system without any defects is defined on the infinite plane, so that the spatial elements in G_b are specified by a map $(\vec{R}, \rho_s) : G_b \rightarrow \mathbb{R}^2 \rtimes O(2)$ where $\mathbb{R}^2 \rtimes O(2)$ is the group of continuous translations, rotations and reflections in 2 dimensions. Here $\vec{R} : G_b \rightarrow \mathbb{R}^2$ and $\rho_s : G_b \rightarrow O(2)$. When $(\vec{R}, \rho_s)$ is non-trivial, we include it as part of the symmetry data of the system. Note that previous works checking the CEP for bosonic and fermionic systems [79–81]

considered $G_b = H \times G_{\text{space}}$ where H is internal and G_{space} is a point-group or wallpaper group symmetry. But our result below is expected to hold for arbitrary G_b .

The fCEP states that the classification of invertible fermionic topological phases with *spatial* symmetry G_f defined by the data $(G_b, s_1, \omega_2, (\vec{R}, \rho_s))$ is in 1-1 correspondence with the classification of invertible fermionic topological phases with an effective internal symmetry G_f^{eff} that has data $(G_b, s_1^{\text{eff}}, \omega_2^{\text{eff}})$ where G_b acts trivially on space. $s_1^{\text{eff}}, \omega_2^{\text{eff}}$ are completely determined by s_1, ω_2, ρ_s .

We conjecture that the internal symmetry data are given by

$$\begin{aligned} s_1^{\text{eff}} &= s_1 + w_1 \\ \omega_2^{\text{eff}} &= \omega_2 + w_2 + w_1(s_1 + w_1) \end{aligned} \tag{3.16}$$

where $w_1 = \rho_s^* w_{1,r}$, $w_2 = \rho_s^* w_2$ are obtained by pulling back the Stiefel-Whitney classes $w_{1,r}, w_2$ that generate $\mathcal{H}^1(\text{O}(2), \mathbb{Z}_2), \mathcal{H}^2(\text{SO}(2), \mathbb{Z}_2)$ respectively. If there are no reflections in G_f , $w_1 = 0$; if there are no rotations, $w_2 = 0$. In App. B.2 we explain the notation further, check each term in the above formula with examples and give some heuristic justification.

Note that this formula treats unitary translations exactly as if they were internal symmetries: they do not appear in Eq. (3.16). The formula can also be straightforwardly generalized to d spatial dimensions, as we discuss in App. B.2.

Each term above can be understood as follows. The term w_1 in the equation for s_1^{eff} implies that spatial orientation-reversing symmetries are mapped to internal antiunitary symmetries.

The term w_2 in the equation for ω_2^{eff} implies that if a 2π spatial rotation acts as the identity in G_f , it should act as $(-1)^F$ in G_f^{eff} , and vice versa. The term w_1^2 implies that if two successive

reflections in G_b act as the identity in G_f , they should act as $(-1)^F$ in G_f^{eff} , and vice versa. The term $w_1 s_1$ implies that if G_b has time-reversal and reflection symmetries generated by $\mathbf{T}, \mathbf{R}$ respectively that act on fermions as $\mathbf{R}\mathbf{T} = \mathbf{T}\mathbf{R}$, the fCEP maps them to a pair of anti-unitary internal symmetries $\mathbf{T}, \mathbf{R}'$ that satisfy $\mathbf{R}'\mathbf{T} = (-1)^F \mathbf{T}\mathbf{R}'$.

This principle has not been proven in complete generality for all invertible phases, but its predictions have been tested in several examples using different techniques. For example, Ref. [80] provides evidence through a real space construction that if $G_b = H \times G_{\text{space}}$ with $s_1 = 0$, where H is an internal symmetry and G_{space} is a 2d wallpaper group, then s_1^{eff} and ω_2^{eff} should be defined as in Eq. 3.16.

Our general strategy to study crystalline invertible phases is thus to identify s_1, ω_2 , say from the system Hamiltonian, and then compute $s_1^{\text{eff}}, \omega_2^{\text{eff}}$. Thereafter we study solutions to the $\mathcal{O}_3$ and $\mathcal{O}_4$ obstructions using $s_1^{\text{eff}}, \omega_2^{\text{eff}}$ instead of the original spatial symmetry data. We refer the reader to Ref. [85] for several examples that check the fCEP in models of topological superconductors which have been proposed in the recent literature.

3.3 Classification of invertible fermionic phases with $G_f = \text{U}(1) \times_{\phi} [\mathbb{Z}^2 \rtimes \mathbb{Z}_M]$

3.3.1 Simplifications for our choice of G_f

For the symmetries we consider, the above formalism simplifies considerably:

1. The symmetry is unitary, therefore $s_1 = 0$. It is also spatial orientation-preserving, therefore from Eq. (3.16), $s_1^{\text{eff}} = 0$.
2. Since the symmetry contains $\text{U}(1)^f$ charge conservation as a subgroup, the chiral central

charge c_- must be an integer.

3. Eq. (3.13) turns out to have a solution only when $n_1 = 0$ (see Ref. [85] for a mathematical derivation of this result). Physically, the above two conditions both stem from the well-known fact that a system with $U(1)^f$ charge conservation symmetry cannot host unpaired Majorana zero modes at symmetry defects. Therefore we can set $n_1 = 0$ throughout.

The main simplification when $n_1 = 0$ is that the topological action can be obtained directly from ν_3 via the pullback operation. Let B be a G_b gauge field. When $\phi = 0$, B was discussed in detail in Chapter 2; the definition of B for $\phi \neq 0$ is similar and will be given below. Then, when $n_1 = 0$ we simply have

$$2\pi\mathcal{L} = B^*\nu_3. \quad (3.17)$$

Note that Eq. (3.17) is valid only when $n_1 = 0$; we do not have a complete understanding of how to relate $\mathcal{L}$ to (c_-, n_1, n_2, ν_3) when $n_1 \neq 0$. Furthermore, for our choice of G_b we can further simplify Eqs. (3.13), (3.15) as follows:

$$\begin{aligned} dn_2 &= 0 \\ d\nu_3 &= \frac{1}{2}n_2 \cup (n_2 + \omega_2^{\text{eff}}) + \frac{c_-}{8}\omega_2^{\text{eff}} \cup \omega_2^{\text{eff}} \mod 1. \end{aligned} \quad (3.18)$$

Here ω_2^{eff} determines the effective internal symmetry G_f^{eff} as a group extension of G_b by $\mathbb{Z}_2^f$. An expression for ω_2^{eff} and n_2 in terms of group cocycles is given in App. B.3, see Eqs. (B.18), (B.27).

3.3.2 Definitions

3.3.2.1 Symmetry operators

Note that all point group operation operators must be defined with respect to a specified origin within the elementary unit cell (see Fig. 4.2). Consider any high symmetry point $\mathbf{o}$ of the unit cell. Let M_o be the order of the rotational symmetry group which preserves $\mathbf{o}$. There is a canonical choice of magnetic point group rotation operators $\tilde{C}_{M_o}^+$ which are centered at $\mathbf{o}$ and satisfy $(\tilde{C}_{M_o}^+)^{M_o} = +1$. In a system with flux ϕ per unit cell, the operator $\tilde{C}_{M_o}^+$ inserts zero excess flux at a disclination of angle $2\pi/M_o$ centered at $\mathbf{o}$. We also define another set of operators $\tilde{C}_{M_o}^- := e^{i\frac{\pi}{M_o}\hat{N}}\tilde{C}_{M_o}^+$, and $\tilde{C}_{M_o,\chi}^\pm := e^{i\chi\frac{2\pi}{M_o}\hat{N}}\tilde{C}_{M_o}^\pm$. Note that $\tilde{C}_{M_o}^- = \tilde{C}_{M_o,1/2}^+$ and $(\tilde{C}_{M_o}^-)^{M_o} = (-1)^F$ where $(-1)^F$ is fermion parity.

3.3.2.2 Definition of G_b gauge field

Note that $G_b = G_f/\mathbb{Z}_2^f = \text{U}(1) \times_{2\phi} [\mathbb{Z}^2 \rtimes \mathbb{Z}_M]$ (corresponding to 2ϕ flux per unit cell). Intuitively, we can see this by noting that a boson is composed of an even number of fermions, and therefore always sees a multiple of 2ϕ flux per unit cell. On a manifold $\mathcal{M}^3$ with a triangulation, we define a flat G_b gauge field $B = (\delta A_b, \vec{R}, \omega)$, whose gauge transformations compose according to the group law of G_b . We have $\delta A_b \in \mathbb{R}$ with $\delta A_b \sim \delta A_b + 2\pi$. We also have $\frac{1}{2\pi}\vec{R} \in \mathbb{Z}^2$, and $\frac{1}{2\pi}\omega \in \mathbb{Z}/M\mathbb{Z}$ with $\omega \sim \omega + 2\pi$.

An important point that was not made fully explicit in Chapter 2 is that these gauge fields implicitly depend on a choice of origin $\mathbf{o}$ within the unit cell; therefore all gauge fluxes and quantized topological terms will in general carry an origin-dependence, which we make explicit

below.

In Chapter 2 we denoted the $U(1)_b$ gauge field by A ; here we use the notation δA_b . We write the $U(1)$ component of B as δA_b because it is the *deviation* of the full bosonic $U(1)$ gauge field A_b from some reference that assigns uniform flux 2ϕ per unit cell. When $\phi = 0$, as in Chapter 2, the reference can be set to zero and therefore we have $\delta A_b = A_b$. We use the subscript b in this case to distinguish A_b from the usual vector potential A , which is a $U(1)^f$ gauge field seen by the fermions. This distinction did not arise in Chapter 2.

Next we discuss the fluxes of B . The quantities $d\omega, d\vec{R}, A_{XY}$ are defined exactly as in Chapter 2 (if we take $G \equiv G_b$), and are interpreted as disclination flux, dislocation flux, and the area element respectively. Note that in a system with a nonzero disclination angle, the dislocation Burgers vector $\vec{b}_o = \frac{1}{2\pi} \int d\vec{R}$ depends on the origin o of the loop used to define $\int d\vec{R}$. This is discussed in more detail in Chapter 4 (see in particular Eq. (4.28)).

Importantly, in a system with $\phi \neq 0$, the $U(1)$ component of the field strength of B is not $d\delta A_b$, but $d\delta A_b + \frac{2\phi}{2\pi} A_{XY}$. Mathematically, this is a consequence of the group multiplication law, which specifies that magnetic translations only commute up to a $U(1)$ rotation. The second term measures the background flux per unit cell seen by the bosons, while the first term measures the excess flux above this background. But if A is the full vector potential ($U(1)^f$ gauge field) in the system, and dA represents the total flux seen by the *fermions*, we must have

$$dA = \frac{1}{2} \left(d\delta A_b + \frac{2\phi}{2\pi} A_{XY} \right), \quad (3.19)$$

because the elementary fermions see half the total flux as the elementary bosons. It will be much more convenient to write the field theory in terms of A instead of δA_b , because A appears directly

M	Classification
1	$\mathbb{Z}^3$
2	$\mathbb{Z}^3 \times \mathbb{Z}_4^3 \times \mathbb{Z}_2$
3	$\mathbb{Z}^3 \times \mathbb{Z}_3^5$
4	$\mathbb{Z}^3 \times \mathbb{Z}_8 \times \mathbb{Z}_4^2 \times \mathbb{Z}_2$
6	$\mathbb{Z}^3 \times \mathbb{Z}_{12} \times \mathbb{Z}_6 \times \mathbb{Z}_3$

Table 3.1: Classification of invertible fermionic states with symmetry $G_f = \text{U}(1)^f \times_\phi [\mathbb{Z}^2 \rtimes \mathbb{Z}_M]$ for $M = 1, 2, 3, 4, 6$. The result is independent of ϕ .

in a fermionic lattice Hamiltonian as opposed to δA_b . Therefore in App. B.3, we will first derive the response theory in terms of δA_b , but then replace it with A . The final result is stated below.

3.3.3 Classification result

3.3.3.1 Topological action

For general $\mathfrak{o}$, the response theory defined with respect to $\tilde{C}_{M_o}^+$ reads

$$\begin{aligned} \mathcal{L} = & \frac{C}{4\pi} A \wedge dA + \frac{\mathcal{S}_o^+}{2\pi} A \wedge d\omega + \frac{\ell_{s,o}^+ - c_-/12}{4\pi} \omega \wedge d\omega + \frac{\vec{\mathcal{P}}_o}{2\pi} \cdot \vec{R} \wedge dA \\ & + \frac{\vec{\mathcal{P}}_{s,o}^+}{2\pi} \cdot \vec{R} \wedge d\omega + \frac{\kappa}{4} A \wedge A_{XY} + \frac{\nu_s^+}{4} \omega \wedge A_{XY}. \end{aligned} \quad (3.20)$$

The response theory defined with respect to $\tilde{C}_{M_o}^-$ can be obtained from Eq. (3.20) after relabelling $A \rightarrow A + \omega/2$. We have used $+$ superscripts only for the coefficients that change under this transformation. Note the general similarity between this action and the one written for bosonic SET phases in Eq. (2.44).

Eq. (3.20) is actually derived using a discrete simplicial formulation of the crystalline gauge theory. However, as argued in Chapter 2, the same effective action can be derived using a continuum formulation, in which the gauge fields are defined as differential 1-forms with

quantized holonomies, and simplicial cup products are replaced by wedge products. It is this form of the topological field theory that we present in Eq. (3.20), and use in subsequent Chapters.

This action has a term $-\frac{c_-}{48\pi}\omega \wedge d\omega$ that arises from the framing anomaly [152, 180]. This term cannot be derived directly from the calculations presented in App. B.3, but must be present in any invertible topological state with $c_- \neq 0$.

3.3.3.2 Quantization of coefficients

Below we fix $\mathfrak{o}$ such that $M_{\mathfrak{o}} = M = 4$ (that is, we consider a square lattice). The calculation outlined in App. B.3 gives the quantization conditions in two main steps. The first step is to determine a partial quantization of the coefficients, without including certain equivalences that arise in the computation. The second step is to include these equivalences.

After the first step, we find that

$$\begin{aligned}
C &= c_- + 8k_1, & k_1 &\in \mathbb{Z} \\
\mathcal{S}_{\mathfrak{o}}^+ &= \frac{c_-}{2} + [s_{\mathfrak{o}}]_2 + 2k_{2,\mathfrak{o}}^+ \pmod{8}, & s_{\mathfrak{o}} &\in \mathbb{Z}_2, k_{2,\mathfrak{o}}^+ \in \mathbb{Z}_4 \\
\ell_{s,\mathfrak{o}}^+ &= \frac{c_-}{4} + 2k_{3,\mathfrak{o}}^+ \pmod{8}, & k_{3,\mathfrak{o}}^+ &\in \mathbb{Z}_4 \\
\vec{\mathcal{P}}_{\mathfrak{o}} &= \frac{([t_{\mathfrak{o}}]_2 + 2k_{4,\mathfrak{o}})}{2}(1, 1) \pmod{\mathbb{Z}^2}, & t_{\mathfrak{o}}, k_{4,\mathfrak{o}} &\in \mathbb{Z}_2 \\
\vec{\mathcal{P}}_{s,\mathfrak{o}}^+ &= \frac{k_{5,\mathfrak{o}}^+}{2}(1, 1) \pmod{\mathbb{Z}^2}, & k_{5,\mathfrak{o}}^+ &\in \mathbb{Z}_2 \\
\kappa &\in \mathbb{Z} \\
\nu_s^+ &\in \mathbb{Z}_4.
\end{aligned} \tag{3.21}$$

We have used $+$ superscripts only for the coefficients that change when we take $A \rightarrow A + \omega/2$.

In the second step, we find two equivalences that reduce the classification. These correspond to relabelling disclination and dislocation defects by fermions; such relabellings can modify the coefficients but should be considered trivial, since fermions can be created by local operators. For $a, b \in \mathbb{Z}_2$, these relabellings imply that

$$\begin{aligned}(k_{2,o}^+, k_{3,o}^+) &\simeq (k_{2,o}^+ + 2a, k_{3,o}^+ + 2a) \\ (k_{4,o}, k_{5,o}^+) &\simeq (k_{4,o} + b, k_{5,o}^+ + b)\end{aligned}\tag{3.22}$$

where we assume all other coefficients remain unchanged. After modding out by the equivalences, we can verify that a complete and independent set of (integer quantized) invariants is given by

$$\begin{aligned}c_-, \kappa, k_1 &\in \mathbb{Z} \\ \mathcal{S}_o^+ - \ell_{s,o}^+ - \frac{c_-}{4} &\in \mathbb{Z}_8 \\ \frac{1}{2} \left(\ell_{s,o}^+ - \frac{c_-}{4} \right) &\in \mathbb{Z}_2 \\ 2(\vec{\mathcal{P}}_o - 2\vec{\mathcal{P}}_{s,o}) \cdot (1, 0) &\in \mathbb{Z}_4 \\ \nu_s^+ &\in \mathbb{Z}_4.\end{aligned}\tag{3.23}$$

3.3.3.3 Main theoretical predictions

With this, we obtain the following important predictions:

1. The full classification of invertible fermionic states with the spatial symmetry $G_f = \text{U}(1)^f \times_\phi [\mathbb{Z}^2 \rtimes \mathbb{Z}_M]$ is given by a group $\mathbb{Z}^3 \times (\mathbb{Z}_8 \times \mathbb{Z}_2) \times \mathbb{Z}_4 \times \mathbb{Z}_4$ and is independent of the value of ϕ . The three integer factors can be obtained from the chiral central charge c_- , the Chern

number C , and the integer κ . These are well-defined even without point group rotational symmetry. Moreover, $C = c_- + 8k_1$ where k_1 is a bosonic SPT invariant. In a free fermion state, we have $k_1 = 0$, and therefore $c_- = C$.

2. We have the following interpretation of κ . Suppose the filling per unit cell is ν . On a spatial torus T^2 with N unit cells, the total charge predicted by the field theory is

$$\nu N = \int_{T^2} \frac{\delta \mathcal{L}}{\delta A} = \frac{C}{2\pi} \int_{T^2} dA + \frac{\kappa}{2\pi} \int_{T^2} A_{XY} \quad (3.24)$$

$$= \left(\frac{C\phi}{2\pi} + \kappa \right) N. \quad (3.25)$$

Here we used $\int_{T^2} dA = \phi N$ and $\int_{T^2} A_{XY} = 2\pi N$. Since N is arbitrary, κ must be related to the filling ν as follows:

$$\nu = \frac{C\phi}{2\pi} + \kappa. \quad (3.26)$$

This relation was previously derived in special cases such as in the Hofstadter model [3], and more generally for Chern insulators using flux-threading arguments [65], in which the role of symmetry was not fully clear. The above derivation shows that it is a direct consequence of the magnetic translation symmetry in the problem.

3. The coefficients $\mathcal{S}_0^+, \ell_{s,0}^+$ both depend on c_- :

$$\mathcal{S}_0^+ = \frac{c_-}{2} \mod 1; \quad \ell_{s,0}^+ = \frac{c_-}{4} \mod 1.$$

In the free fermion models that we study numerically, we can set $c_- = C$.

4. For a fixed value of c_- , there are four distinct values of $\mathcal{S}_0^+$ (corresponding to the choices

of $s_0, k_{2,0}^+$), two distinct values of $\vec{\mathcal{P}}_0$ (corresponding to the choice of t_0), and two distinct values of $\ell_{s,0}^+$ (corresponding to the choice of $k_{3,0}^+$). These statements hold whenever $M_0 = 4$ but are modified for general M ; see App. B.3.

5. Interestingly, the classification of the *pair* $(\mathcal{S}_0^+ - \ell_{s,0}^+ - c_-/4, \ell_{s,0}^+/2 - c_-/8)$ is $\mathbb{Z}_8 \times \mathbb{Z}_2$, that is, there are in total 16 different choices. This seems to contradict our statement that $\mathcal{S}_0^+$ and $\ell_{s,0}^+$ can take 4 and 2 different values respectively. The explanation is that before including any equivalences, $\mathcal{S}_0$ and $\ell_{s,0}^+$ take 8 and 4 different values respectively, for fixed c_- ; the equivalences in Eq. (3.22) then reduce the total classification by a factor of 2, giving $8 \times 4/2 = 16$. The seeming inconsistency arises because $\mathcal{S}_0^+, \ell_{s,0}^+$ do not form an independent set of invariants spanning the $\mathbb{Z}_8 \times \mathbb{Z}_2$ subgroup of the classification; however, the pair $(\mathcal{S}_0^+ - \ell_{s,0}^+ - c_-/4, \ell_{s,0}^+ - c_-/4)$ gives more information. In Chapter 5, we show how to obtain precisely this $\mathbb{Z}_8 \times \mathbb{Z}_2$ invariant from partial rotation operations.

When $c_- = 0$ and $k_1 = 0$, we recover the classification derived in Ref. [80] using a real-space construction. The classification result for $G_f = \text{U}(1) \times_\phi [\mathbb{Z}^2 \rtimes \mathbb{Z}_M]$ with $M = 1, 2, 3, 4, 6$ is given in Table 3.1. The field theory is also stated and derived for these cases in App. B.3.

3.4 Conclusion

Using the general theory of invertible fermionic states, combined with the fermionic crystalline equivalence principle, we have obtained a number of powerful predictions that can be tested numerically. In Chapter 4 we discuss the invariants $\mathcal{S}_0$ and $\vec{\mathcal{P}}_0$ in detail, understand how to measure them in several complementary ways, and obtain precise agreement between theoretical and numerical calculations. Then, in Chapter 5, we show that the *entire* set of invariants predicted

above can be extracted numerically in a given lattice model; indeed, this can be done from a single wave function.

Chapter 4: Discrete shift and quantized charge polarization as many-body invariants in (2+1)D crystalline topological states, and Hofstadter butterflies

In this chapter we present a thorough numerical study of two invariants predicted by the theory in Chapter 3. This Chapter is based on the paper *Quantized charge polarization as a many-body invariant in (2+1)D crystalline topological states and Hofstadter butterflies*, by Yuxuan Zhang, Naren Manjunath, Gautam Nambiar, and Maissam Barkeshli, arXiv: 2211.09127 (2022).

Notational comments: As in Chapter 3, we explicitly show the dependence of each topological invariant on the chosen origin $\mathbf{o}$ in the unit cell. We use the same field theory as the one in Chapters 3 (Eq. (3.20)) with $+$ superscripts on the field theory coefficients, but here we drop the superscript for ease of notation. In particular, the quantity $\mathcal{S}_{\mathbf{o}}$ studied here is the same as the quantity $\mathcal{S}_{\mathbf{o}}^+$ studied in Chapters 3 and 5. We use the same operator referred to as $\tilde{C}_{M_{\mathbf{o}}}^+$ in Chapters 3 and 5, but call it $\tilde{C}_{M_{\mathbf{o}}}$ here.

4.1 Overview of main results

We consider a gapped phase of matter with the symmetry group

$$G = \mathrm{U}(1) \times_{\phi} [\mathbb{Z}^2 \rtimes \mathbb{Z}_M], \quad (4.1)$$

M	$\mathcal{S}_{\mathbf{o}+\vec{v}}$	$\vec{\mathcal{P}}_{\mathbf{o}+\vec{v}}$
2	$\mathcal{S}_{\mathbf{o}} - 4\vec{v} \cdot \vec{\mathcal{P}}_{\mathbf{o}} + 4\kappa(v_x^2 + v_y^2 + v_x v_y)$	$\vec{\mathcal{P}}_{\mathbf{o}} + (-v_y \kappa, v_x \kappa)$
4	$\mathcal{S}_{\mathbf{o}} + 2\overline{\mathcal{P}}_{\mathbf{o}} - \kappa$	$\vec{\mathcal{P}}_{\mathbf{o}} + (-\frac{1}{2}\kappa, \frac{1}{2}\kappa)$
3	$\mathcal{S}_{\mathbf{o}} - 3v_y \overline{\mathcal{P}}_{\mathbf{o}} - 3\kappa(v_x^2 + v_y^2 + v_x v_y)$	$\vec{\mathcal{P}}_{\mathbf{o}} + (-v_y \kappa, v_x \kappa)$
6	$\mathcal{S}_{\mathbf{o}}$	0

Table 4.1: Transformation of $\mathcal{S}_{\mathbf{o}}$ and $\vec{\mathcal{P}}_{\mathbf{o}}$ under $\mathbf{o} \rightarrow \mathbf{o} + (v_x, v_y)$, which shifts the origin from one C_M symmetric point to another C_M symmetric point. Note that $\mathcal{S}_{\mathbf{o}}$ and $\vec{\mathcal{P}}_{\mathbf{o}}$ are only defined up to equivalences as described in the main text. For $M = 4$ we have taken the unique non-trivial choice $\vec{v} = (\frac{1}{2}, \frac{1}{2})$.

where $\mathbb{Z}^2$ denotes magnetic lattice translations and $\mathbb{Z}_M$ for $M = 2, 3, 4, 6$ denotes point group rotations.¹ The symbol $\times_\phi$ implies that the magnetic translation operators, generated by $\tilde{T}_{\mathbf{x}}, \tilde{T}_{\mathbf{y}}$, obey the algebra $\tilde{T}_{\mathbf{y}}^{-1} \tilde{T}_{\mathbf{x}}^{-1} \tilde{T}_{\mathbf{y}} \tilde{T}_{\mathbf{x}} = e^{i\phi \hat{N}}$ where $\hat{N}$ is the total fermion number. The tilde superscript indicates that the definition of the operator involves a U(1) gauge transformation.

The charge conservation and translation symmetries allow us to define a charge per unit cell ν . Each unit cell can be divided into M subcells with equal flux ϕ_{sub} . The total flux per unit cell is then $\phi = M\phi_{sub}$. Note that for our purposes, depending on the microscopic model we may need to specify the flux within even smaller subregions of the unit cell. Therefore we assume that the 2d system is embedded in a continuum, and that the magnetic field B is specified at each continuum point. This allows us to specify ϕ, ϕ_{sub} exactly as real numbers, even though the symmetry only requires us to define $\phi \bmod 2\pi$. We comment further on this in Sec. 4.3.

¹More specifically G is the fermionic symmetry group, which acts non-trivially on fermionic operators. It is sometimes written as $U(1)^f \times_\phi [\mathbb{Z}^2 \rtimes \mathbb{Z}_M]$. The symbol $U(1)^f$ means that the order 2 element of this group is identified with the fermion parity operation.

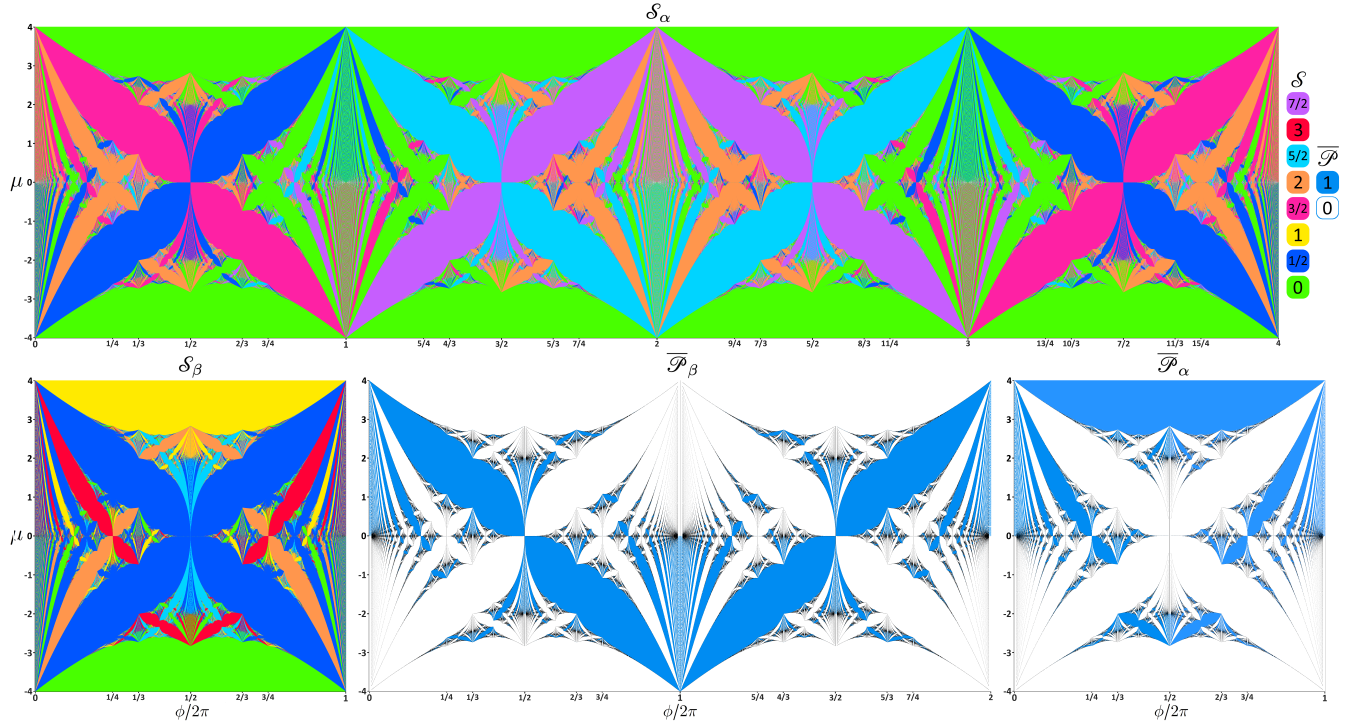


Figure 4.1: A menagerie of butterflies in the spinless square lattice Hofstadter model. The black solid line represents hoppings. α and β represent a plaquette center and vertex respectively. For any C_4 symmetric origin $\mathbf{o}$, $\mathcal{S}_{\mathbf{o}}$ has a $\mathbb{Z}_4$ classification, while $\mathcal{P}_{\mathbf{o}}$ has a $\mathbb{Z}_2$ classification. We empirically find that $\{\mathcal{S}_{\beta}, \mathcal{S}_{\alpha}, \overline{\mathcal{P}}_{\beta}, \overline{\mathcal{P}}_{\alpha}\}$ follow Eqs. $\{(4.41), (4.42), (4.52), (4.53)\}$ respectively. Note that $\mathcal{S}_{\alpha}$ has period 8π in ϕ , $\overline{\mathcal{P}}_{\beta}$ has period 4π in ϕ , while $\mathcal{S}_{\beta}$ and $\overline{\mathcal{P}}_{\alpha}$ has period 2π in ϕ . See Sec. 4.3 and 4.4 for discussions.

Let C be the Chern number of the system. We then define the integer

$$\kappa \equiv \nu - C \frac{\phi}{2\pi}. \quad (4.2)$$

κ is a $\mathbb{Z}$ topological invariant for the system if ϕ is known exactly (and not just modulo 2π).

Fixing C and ϕ , if the charge per unit cell increases by an integer l , then $\kappa \rightarrow \kappa + l$. For further intuition about κ and a heuristic derivation of Eq. (4.2), see App. C.1.1.4.

For a given high symmetry point $\mathbf{o}$ of the lattice unit cell, we will see that one can define a set of topological invariants $\{\mathcal{S}_{\mathbf{o}}, \vec{\mathcal{P}}_{\mathbf{o}}\}$. The transformation of $\{\mathcal{S}_{\mathbf{o}}, \vec{\mathcal{P}}_{\mathbf{o}}\}$ under a change of $\mathbf{o}$ is

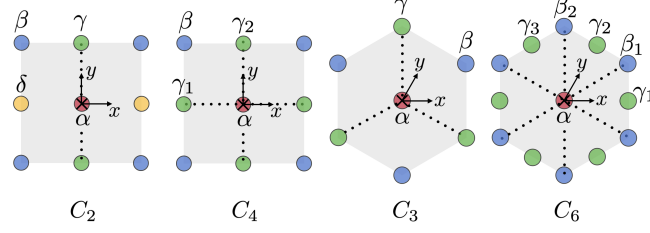


Figure 4.2: Maximal Wyckoff positions for C_2 , C_3 , C_4 and C_6 symmetries (colored circles). $\times$ marks the unit cell center, which we denote as α . The high symmetry points β_i and γ_i each belong to a single maximal Wyckoff position (β or γ), but are all inequivalent under lattice translations. The dotted lines show a possible division of the unit cell into M subcells.

fully determined if $\mathfrak{o}$ is preserved by a $\mathbb{Z}_M$ rotation symmetry group. Therefore it is sufficient to specify $\{\mathcal{S}_{\mathfrak{o}}, \vec{\mathcal{P}}_{\mathfrak{o}}\}$ for a single such $\mathfrak{o}$.

A subtle point is that the definition of the invariants requires a choice of operators that represent the symmetry group elements. In this work, we take $\tilde{C}_{M_0}$ to be a ‘magnetic’ rotation operator about $\mathfrak{o}$ (a spatial rotation combined with a gauge transformation), such that $(\tilde{C}_{M_0})^{M_0} = 1$ and M_0 is the smallest integer for which this holds. (This is the same operator referred to as $\tilde{C}_{M_0}^+$ in Chapters 3 and 5.) If we change this choice by $\tilde{C}_{M_0} \rightarrow \tilde{C}_{M_0} e^{i \frac{2\pi}{M_0} \chi \hat{N}}$ for some real number χ , then the invariant $\mathcal{S}_{\mathfrak{o}}$ also transforms, $\mathcal{S}_{\mathfrak{o}} \rightarrow \mathcal{S}_{\mathfrak{o}} + \chi C$. Nevertheless, we will see that there are canonical choices for $\tilde{C}_{M_0}$ that can be made.

We caution that the vector $\vec{\mathcal{P}}_{\mathfrak{o}}$ we refer to throughout the text is different from the standard charge polarization vector $\vec{P}_{\mathfrak{o}}$, which for zero Chern number satisfies $\mathbf{j} = \partial_t \vec{P}_{\mathfrak{o}}$ where $\mathbf{j}$ is the induced current. We show in App. C.2 that

$$(\mathcal{P}_{\mathfrak{o},x}, \mathcal{P}_{\mathfrak{o},y}) = (P_{\mathfrak{o},y}, -P_{\mathfrak{o},x}) = \vec{P}_{\mathfrak{o}} \times \hat{z}. \quad (4.3)$$

However, since $\vec{\mathcal{P}}_{\mathfrak{o}}$ appears most naturally in our theory, we will work in terms of $\vec{\mathcal{P}}_{\mathfrak{o}}$ throughout and refer to it as the polarization, recognizing this as a slight abuse of terminology.

4.1.1 Warmup: C_4 symmetric lattice

We first illustrate our main results for the square lattice. A representative unit cell with high symmetry points (HSPs) α (unit cell center), β (unit cell vertices), γ_i (edge centers) is shown in Fig. 4.2. The points γ_1, γ_2 are not translation-equivalent but are related by rotations about α . α, β, γ refer to maximal Wyckoff position (MWP), which are collections of points related by lattice symmetries; the precise definition of a MWP is given in App. C.1. α and β have an order $M_\alpha = M_\beta = 4$ site symmetry group generated by the “magnetic” rotation operators $\tilde{C}_{M_\alpha}, \tilde{C}_{M_\beta}$ (which also include a gauge transformation), with $\tilde{C}_{M_\alpha}^4 = \tilde{C}_{M_\beta}^4 = 1$. The point γ has an order $M_\gamma = 2$ site symmetry group generated by the operator $\tilde{C}_{M_\gamma}$, with $\tilde{C}_{M_\gamma}^2 = 1$. We can pick any of these points as our origin $\mathbf{o}$.

First we define the invariants $\mathcal{S}_\mathbf{o}, \vec{\mathcal{P}}_\mathbf{o}$ for each possible HSP $\mathbf{o}$ and list their properties; thereafter, we explain how to use them to characterize the topological phase of the given system.

Suppose $\mathbf{o} = \alpha$ or β . Then, $\mathcal{S}_\mathbf{o}$ is defined mod 4 and can take one of 4 possible values for a fixed Chern number C . Next, suppose $\mathbf{o} = \gamma$, which is a C_2 symmetric point. Then $\mathcal{S}_\gamma$ is only defined mod 2. In all cases, we have the constraint [100]

$$\mathcal{S}_\mathbf{o} \mod 1 = \frac{C}{2} \mod 1. \quad (4.4)$$

We next turn to $\vec{\mathcal{P}}_\mathbf{o}$. For $\mathbf{o} = \alpha, \beta$,

$$\vec{\mathcal{P}}_\mathbf{o} \in \{(0, 0), (\frac{1}{2}, \frac{1}{2})\} \quad (4.5)$$

up to integer vectors. We write

$$\vec{\mathcal{P}}_o = \frac{\overline{\mathcal{P}}_o}{2}(1, 1) \quad (4.6)$$

in this case; $\overline{\mathcal{P}}_o$ is an integer defined mod 2. For $o = \gamma$, there are 4 possible choices:

$$\vec{\mathcal{P}}_\gamma \in \{(0, 0), (\frac{1}{2}, 0), (0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2})\} \quad (4.7)$$

up to integer vectors.

$\mathcal{S}_o, \vec{\mathcal{P}}_o$ for a C_4 symmetric point, together with κ , determines $\mathcal{S}_{o'}, \vec{\mathcal{P}}_{o'}$ for all other o' . For example,

$$\{\mathcal{S}_\beta, \overline{\mathcal{P}}_\beta, \kappa\} = \{\mathcal{S}_\alpha + 2\overline{\mathcal{P}}_\alpha - \kappa, \overline{\mathcal{P}}_\alpha + \kappa, \kappa\}. \quad (4.8)$$

If we only know $\mathcal{S}_\gamma, \vec{\mathcal{P}}_\gamma, \kappa$, we can determine $\vec{\mathcal{P}}_\alpha$ and $\vec{\mathcal{P}}_\beta$ fully, but can only determine $\mathcal{S}_\alpha$ and $\mathcal{S}_\beta$ mod 2 and not mod 4. The relevant formulas are given in Table 4.1. Thus, to fully specify $\mathcal{S}_o, \vec{\mathcal{P}}_o$ for each high symmetry point in the unit cell, we need to determine them for some o with $M_o = M$.

In Fig. 4.1, we show colored Hofstadter butterflies for two different origins α, β extracted for the square lattice Hofstadter model of spinless fermions. We find that $\{\mathcal{S}_\beta, \mathcal{S}_\alpha, \overline{\mathcal{P}}_\beta, \overline{\mathcal{P}}_\alpha\}$ follow the empirical equations Eqs. {(4.41), (4.42), (4.52), (4.53)} respectively. In this figure β corresponds to a site and α to a plaquette center, where there is no site.

To distinguish two phases of matter based on $\mathcal{S}_o, \vec{\mathcal{P}}_o$ (assuming C, κ and all other invariants are equal), we first fix a common origin o (which must be a C_4 symmetric point) and find

M	2	3	4	6
$U(2\pi/M)$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$

Table 4.2: Elementary rotation matrices $U(2\pi/M)$, for the coordinate basis shown in Fig. 4.2.

$\mathcal{S}_o, \vec{\mathcal{P}}_o$ for the two systems. If their values are not equal, the two systems cannot be adiabatically connected to each other in a symmetry-preserving manner. It is important to note that comparing the crystalline topological invariants of two phases is only meaningful after fixing a common origin.

4.1.2 Basic properties and classification for $\{\mathcal{S}_o, \vec{\mathcal{P}}_o\}$

We now generalize the above discussion to the case with $M = 2, 3, 4, 6$.² In Table 4.2 we define the 2×2 matrices $U(2\pi/M)$, corresponding to elementary $2\pi/M$ rotations around an origin o with M -fold rotational symmetry. These describe the action of the rotation operator $\tilde{C}_{M_o}$ on space. As above we fix $(\tilde{C}_{M_o})^{M_o} = +1$.

Then for any given o which is fixed under an order- M rotation, $\{\mathcal{S}_o, \vec{\mathcal{P}}_o\}$ have a $\mathbb{Z}_M \times K_M$ classification, where $K_M = \{\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_2, \mathbb{Z}_1\}$ for $M = 2, 3, 4, 6$. A derivation is given in App. C.1. More specifically, $\mathcal{S}_o$ is an integer or half-integer defined modulo M , and it satisfies Eq. (4.4). $\vec{\mathcal{P}}_o$ is a 2-component vector with the following quantization condition and equivalence relation:

$$(1 - U(2\pi/M))\vec{\mathcal{P}}_o \in \mathbb{Z}^2,$$

$$\vec{\mathcal{P}}_o \sim \vec{\mathcal{P}}_o + \vec{\Lambda}, \quad \vec{\Lambda} \in \mathbb{Z}^2. \quad (4.9)$$

²For the case $M = 1$, there are no quantized discrete shift and polarization invariants [52]

For 4-fold and 6-fold point groups, it is possible for the HSP $\mathbf{o}$ to only be invariant under a smaller M_0 -fold rotation. For example we can have $M_0 = 2$ when $M = 4$, or $M_0 = 2, 3$ when $M = 6$. In these cases, the possible values of $\{\mathcal{S}_\mathbf{o}, \vec{\mathcal{P}}_\mathbf{o}\}$ have a $\mathbb{Z}_{M_0} \times K_{M_0}$ classification. The relations defining them will be as above, with M replaced by M_0 .

It will be convenient to parameterize $\vec{\mathcal{P}}_\mathbf{o}$ in the following way:

$$\vec{\mathcal{P}}_\mathbf{o} = \begin{cases} \frac{1}{2}(\overline{\mathcal{P}}_{\mathbf{o},x}, \overline{\mathcal{P}}_{\mathbf{o},y}) & M_0 = 2 \\ \overline{\mathcal{P}}_\mathbf{o} \frac{1}{2}(1, 1) & M_0 = 4 \\ \overline{\mathcal{P}}_\mathbf{o} \frac{1}{3}(1, 2) & M_0 = 3 \\ 0 & M_0 = 6. \end{cases} \quad (4.10)$$

Here M_0 is the maximal integer such that $\mathbf{o}$ is a fixed point under rotations of order M_0 . $\overline{\mathcal{P}}_\mathbf{o}$ can take any integer value according to the K_{M_0} classification. For example, when $M_0 = 2$, there are 4 inequivalent choices for $\vec{\mathcal{P}}_\mathbf{o}$: $\vec{\mathcal{P}}_\mathbf{o} \in \{(0, 0), (1/2, 0), (0, 1/2), (1/2, 1/2)\}$. When $M_0 = 3$, there are 3 inequivalent choices: $\vec{\mathcal{P}}_\mathbf{o} \in \{(0, 0), (1/3, 2/3), (2/3, 1/3)\}$. This is derived in App. C.1.3.

Next we discuss the origin dependence of $\mathcal{S}_\mathbf{o}, \vec{\mathcal{P}}_\mathbf{o}$. If we shift $\mathbf{o} \rightarrow \mathbf{o}' = \mathbf{o} + (v_x, v_y)$, then we can determine $\{\mathcal{S}_{\mathbf{o}'}, \vec{\mathcal{P}}_{\mathbf{o}'}\}$ from $\{\mathcal{S}_\mathbf{o}, \vec{\mathcal{P}}_\mathbf{o}\}$ and κ , as specified in Table 4.1. Note that (v_x, v_y) can be fractions of a lattice unit. For $\{\mathcal{S}_{\mathbf{o}'}, \vec{\mathcal{P}}_{\mathbf{o}'}\}$ to be fully specified in terms of $\{\mathcal{S}_\mathbf{o}, \vec{\mathcal{P}}_\mathbf{o}\}$, the minimal rotation angle which preserves $\mathbf{o}'$ must be a multiple of the minimal rotation angle which preserves $\mathbf{o}$. In other words, the site symmetry group of $\mathbf{o}'$ is isomorphic to a subgroup of the site symmetry group of $\mathbf{o}$. For example, if $\mathbf{o}$ and $\mathbf{o}'$ have site symmetry groups $\mathbb{Z}_4$ and $\mathbb{Z}_2$ respectively, then $\{\mathcal{S}_{\mathbf{o}'}, \vec{\mathcal{P}}_{\mathbf{o}'}\}$ can be determined using the $M = 2$ entry of Table 4.1.

Thus, in order to obtain a complete specification for each high symmetry point, we need to

know $\{\mathcal{S}_o, \vec{\mathcal{P}}_o\}$ for at least one o which is invariant under an M -fold rotation. Otherwise we will not be able to fully recover $\{\mathcal{S}_{o'}, \vec{\mathcal{P}}_{o'}\}$ for each o' . Note, in particular, that the dependence of $\vec{\mathcal{P}}_o$ on o is completely determined by κ , and the dependence of $\mathcal{S}_o$ on o is completely determined by $\vec{\mathcal{P}}_o$ and κ .

Nevertheless, differences

$$\begin{aligned}\Delta\mathcal{S} &= \mathcal{S}_o(\lambda_1) - \mathcal{S}_o(\lambda_2) \\ \Delta\vec{\mathcal{P}} &= \vec{\mathcal{P}}_o(\lambda_1) - \vec{\mathcal{P}}_o(\lambda_2)\end{aligned}\tag{4.11}$$

are independent of o . Here λ is some tuning parameter in the Hamiltonian, $H[\lambda]$, which keeps the invariant κ fixed and preserves the crystalline symmetry. This can be done for example by fixing $\{\nu, \phi_{\text{sub}}\}$. The reason we need to fix κ is discussed in Sec. 4.5.1.

As another example, if we take $M = 3$, we have 3 maximal Wyckoff positions invariant under 3-fold rotation symmetry: α , β , and γ , with $\beta = \alpha + (1/3, 1/3)$ and $\gamma = \alpha + (-1/3, 2/3)$ (see Fig. 4.2). Then

$$\begin{aligned}\{\mathcal{S}_\beta, \overline{\mathcal{P}}_\beta, \kappa\} &= \{\mathcal{S}_\alpha - \overline{\mathcal{P}}_\alpha - \kappa, \overline{\mathcal{P}}_\alpha - \kappa, \kappa\} \\ \{\mathcal{S}_\gamma, \overline{\mathcal{P}}_\gamma, \kappa\} &= \{\mathcal{S}_\alpha + \overline{\mathcal{P}}_\alpha - \kappa, \overline{\mathcal{P}}_\alpha + \kappa, \kappa\}.\end{aligned}\tag{4.12}$$

4.1.3 Extracting $\{\mathcal{S}_o, \vec{\mathcal{P}}_o\}$ from microscopic models

For a given microscopic model, we can extract $\{\mathcal{S}_o, \vec{\mathcal{P}}_o\}$ in several distinct ways, as summarized in Table 4.3. To set up the calculations, we first need to fix a rotation operator $\tilde{C}_{M_o^*}$, where the

high symmetry point $\mathbf{o}^*$ is invariant under $2\pi/M_{\mathbf{o}^*}$ rotations and $M_{\mathbf{o}^*} \leq M$. In our examples we will always choose this operator so that $(\tilde{C}_{M_{\mathbf{o}^*}})^{M_{\mathbf{o}^*}} = +1$. We also define translation operators $\tilde{T}_{\mathbf{x}}, \tilde{T}_{\mathbf{y}}$ corresponding to the elementary lattice vectors $\mathbf{x}, \mathbf{y}$, which obey the magnetic translation algebra.

Our numerical procedure is guided by a topological response theory derived using TQFT ideas [27, 52, 53]. This gives a Lagrangian density in terms of background gauge fields:

$$\begin{aligned} \mathcal{L} = & \frac{C}{4\pi} A \wedge dA + \frac{\mathcal{S}_{\mathbf{o}}}{2\pi} A \wedge d\omega + \frac{\vec{\mathcal{P}}_{\mathbf{o}}}{2\pi} \cdot A \wedge \vec{T} + \frac{\kappa}{2\pi} A \wedge A_{XY} \\ & + \frac{\tilde{\ell}_s}{4\pi} \omega \wedge d\omega + \frac{\vec{\mathcal{P}}_s}{2\pi} \cdot \omega \wedge \vec{T} + \frac{\nu_s}{2\pi} \omega \wedge A_{XY} + \dots \end{aligned} \quad (4.13)$$

Here, A is a background $U(1)$ gauge field, and is defined so that $\int dA$ represents the full magnetic field (and not just its deviation from some background value) ω is a background $\mathbb{Z}_M$ ‘rotation’ gauge field, which is treated as a real field with quantized holonomies. A_{XY} and $\vec{T}$ are the area element and torsion 2-form, respectively, which are constructed using translation gauge fields (see Chapters 2 and 3 for a complete discussion). Note that this action is the same as Eq. (3.20), but we will drop the $+$ superscripts in this Chapter for ease of notation.

Importantly, the coefficients of these terms are all quantized in specific patterns and defined modulo certain equivalence relations, which were systematically derived using group cohomology in Chapter 3. We find that the quantization conditions on $\vec{\mathcal{P}}_{\mathbf{o}}$ in invertible fermionic systems (i.e. without fractionalized excitations) are the same as for invertible bosonic systems, in contrast to the Chern number and discrete shift [100].

4.1.3.1 $\{\mathcal{S}_o, \vec{\mathcal{P}}_o\}$ from fractional charge of lattice disclinations and dislocations

Given the magnetic rotation operator $\tilde{C}_{M_o^*}$ about a high symmetry point o^* and translation operators $\tilde{T}_x, \tilde{T}_y$, and the Hamiltonian for the clean system with the full crystalline symmetry, H_{clean} , one can define a Hamiltonian in the presence of a lattice disclination or dislocation H_{defect} . This is done through a cut-and-glue procedure described in App. C.3. H_{defect} is uniquely defined up to local operators at the core of the defect. In our numerics we will take H_{clean} to be a free fermion Hofstadter model, usually with nearest neighbor hopping terms, but our methods conceptually apply more generally, as we discuss in Secs. 4.3. In App. C.7, we demonstrate that the dislocation charge calculation generalizes naturally to Hamiltonians with next nearest neighbour hopping.

A lattice disclination has a non-zero disclination angle Ω (Frank angle $-\Omega$), and $\vec{b}_o$ is the Burgers vector. Here, the subscript o means that the Burgers vector is measured by the holonomy of a loop encircling the defect, which starts at the point o . Note that o and o^* need not be equal in general. As we explain in Sec. 4.2, for a disclination with $\Omega \neq 0$, the value of $\vec{b}_o$ as defined above depends on o . However, for a lattice dislocation, which has $\Omega = 0$, the value of $\vec{b}_o$ is independent of o .

We can compute the fractional charge in the ground state, $Q_W \bmod 1$, in a large region W surrounding a lattice disclination or dislocation. We require that the boundaries of the region W align with the boundaries of the unit cell Θ . The linear size of W must be much larger than the

correlation length. We first define the charge in a region W :

$$Q_W = \sum_{i \in W} \text{wt}(i) Q_i, \quad (4.14)$$

where $\text{wt}(i) = 1$ if i is in the interior of W , while if i lies on the boundary ∂W , $2\pi \text{wt}(i)$ is the angle subtended by ∂W in the interior of W at i . Q_i is the charge on site i in the ground state of H_{defect} . An example is shown in Fig. 4.6.

Using Eq. (4.13) along with Eq. (4.2), we then find

$$Q_W = C \frac{\Phi_W}{2\pi} + \mathcal{S}_o \frac{\Omega}{2\pi} + \vec{\mathcal{P}}_o \cdot \vec{b}_o + \kappa n_{W,o} \mod 1 \quad (4.15)$$

$$= C \frac{\delta\Phi_{W,o}}{2\pi} + \mathcal{S}_o \frac{\Omega}{2\pi} + \vec{\mathcal{P}}_o \cdot \vec{b}_o + \nu n_{W,o} \mod 1. \quad (4.16)$$

Φ_W is the total flux through the region W . $n_{W,o}$ is the number of unit cells in W , and may be fractional. Φ_W has two contributions, $\Phi_W = \phi n_{W,o} + \delta\Phi_{W,o}$. Here, $\phi n_{W,o}$ is the reference background flux within W . $\delta\Phi_{W,o}$ is the excess magnetic flux in the region W relative to this reference. The precise microscopic definitions of $\delta\Phi_{W,o}$ and $n_{W,o}$ are quite subtle and non-trivial and explained in detail in Secs 4.3, 4.4.

Importantly, $\delta\Phi_{W,o}$ and $n_{W,o}$ in general depend on the position of o relative to the chosen unit cell Θ . Nevertheless, the final results for $\mathcal{S}_o$ and $\vec{\mathcal{P}}_o$ are independent of Θ . This is explained using the *trimming* method developed in App. C.4.

Naively it may seem that the coefficients in Eq. (4.15) should also depend on o^* . One reason for our notation is that Eq. (4.15) comes from a TQFT which is only sensitive to o . But

even in microscopic calculations, we find that neither $\mathcal{S}_o$ nor $\vec{\mathcal{P}}_o$ actually depends on o^* . This is easily seen for $\vec{\mathcal{P}}_o$, which can be defined using pure dislocations, for which $\Omega = 0$ and o^* does not appear. To show that $\mathcal{S}_o$ is independent of o^* , we give a theoretical argument when $C = 0$, in App. C.2.3. We also have extensive numerical evidence for this when $C \neq 0$.

The above discussion implies that we can consider any defect Hamiltonian, and extract $\mathcal{S}_o$ and $\vec{\mathcal{P}}_o$ (which only depend on o) by suitably defining $\delta\Phi_{W,o}$ and $n_{W,o}$ along with the appropriate Burgers vector. To simplify the disclination charge calculation of $\mathcal{S}_o$ we will often choose $o = o^*$, but this is not a requirement of the theory.

4.1.3.2 $\mathcal{S}_o$ from angular momentum

Alternatively, we can examine the action of rotations and translations on the ground state in order to extract an angular momentum or linear momentum. These dual responses are a valuable consistency check on the value of $\mathcal{S}_o$ obtained from the above charge response. Importantly, to compare the values of discrete shift from the disclination charge and angular momentum calculations, we need to set $o = o^*$ in both cases.

Let $|\Psi(m)\rangle$ be the ground state of the clean translationally invariant system on a torus in the presence of m flux quanta, m being an integer. Then

$$\tilde{C}_{M_o}|\Psi\rangle = e^{2\pi i l_o/M_o}|\Psi\rangle, \quad (4.17)$$

where recall that M_o is the largest integer such that o is invariant under $2\pi/M_o$ rotations centered

Charge response	$Q_W = C \frac{\delta\Phi_{W,o}}{2\pi} + \frac{\Omega_W \mathcal{S}_o}{2\pi} + \vec{\mathcal{P}}_o \cdot \vec{b}_o + \nu(k + n_{\text{irreg},o,\Omega,\vec{b}}) \mod 1$	$\mathbf{o}$ = origin of loop used to measure Burgers vector $\vec{b}_o$
Angular momentum	$l_o(m) = \frac{Cm^2}{2} + \mathcal{S}_o m + K(C, L) \mod M$	$\mathbf{o}$ = rotation center
Linear momentum	$p_{\lambda,y}(m) = -\mathcal{P}_{o,y} m + K_y \mod 1$	$\mathbf{o}$ is determined by the ‘gauge origin’ $\bar{\mathbf{o}}$ through Eq. (4.61)
1d polarization	$-\mathcal{P}_{\mathcal{O},x} = \frac{Ck\phi}{2\pi} L_y + L_y \mathcal{P}_{o,y} + K \mod 1$	$\mathcal{O}$ = origin used in Resta’s formula, Eq. (4.26); $\mathbf{o}, \mathcal{O}$ must satisfy Eq. (4.75).

Table 4.3: General equations for charge response (including dislocation and disclination charge), angular momentum and linear momentum. The last column explains the different special points that arise in each calculation.

at $\mathbf{o}$. We find that the angular momentum l_o obeys the formula

$$l_o(m) = \mathcal{S}_o m + C \frac{m^2}{2} + K(C, L) \mod M_o \quad (4.18)$$

where $K(C, L)$ is a constant independent of m , depending on the system size L and the Chern number C . The bare numerical calculation for l_o is shown in Fig. C.6. For $|\Psi\rangle$ to be an eigenstate of $\tilde{C}_{M_o}$, appropriate global holonomies of the background gauge field and certain commensurate system sizes must be chosen, as discussed in Ref. [100]. We note that one can also recover $\mathcal{S}_o$ by locally inserting flux and performing partial rotations; see Chapter 5.

4.1.3.3 $\vec{\mathcal{P}}_o$ from linear momentum

The topological field theory, Eq. (4.13), predicts that the polarization $\vec{\mathcal{P}}_o$ also specifies the linear momentum of U(1) flux [52]. We have found empirically that $\vec{\mathcal{P}}_o$ can be extracted by studying expectation values of an approximate translation operator, as we briefly summarize

below. See Sec. 4.4 for additional details.

Suppose we wish to measure $\mathcal{P}_{0,y}$ on the square lattice. We consider a state on a clean torus with $m = \frac{\phi}{2\pi} L_x L_y$ total flux quanta, where L_x, L_y are the number of unit cells in the x and y directions. While the infinite plane possesses an infinite magnetic translation symmetry along the two directions, on the torus with magnetic flux it is not possible to fully preserve translation symmetry along y unless $m/L_y = \frac{\phi}{2\pi} L_x$ is an integer. For general m , on the torus we can insert the flux using a Landau-like gauge that is almost translation symmetric along y , except for a small strip which forms a cycle along y .

We then define an approximate translation operator

$$\tilde{T}_{\mathbf{y}} := \hat{T}_{\mathbf{y}} e^{i \sum_j \lambda_j c_j^\dagger c_j}. \quad (4.19)$$

The expectation value of $\tilde{T}_{\mathbf{y}}$ defines the linear momentum $p_{\lambda,y}$ in the y direction:

$$\langle \Psi(m) | \tilde{T}_{\mathbf{y}} | \Psi(m) \rangle = e^{-\gamma + i 2\pi p_{\lambda,y}}. \quad (4.20)$$

In our numerics we define λ using Eq. (4.59). In particular, we find empirically that there exist special choices of λ for which $p_{\lambda,y}$ determines the quantized polarization $\mathcal{P}_{0,y}$ throughout the Hofstadter butterfly, as follows.

We find that for the Hofstadter model, for appropriately chosen λ , the amplitude $e^{-\gamma(m)}$ in general oscillates as a function of m and it vanishes for certain special values of m . Whenever

the amplitude is nonzero, the linear momentum is found to obey the following relation:

$$p_{\lambda,y}(m) = -\mathcal{P}_{o,y}m + K_y \mod 1 \quad (4.21)$$

where K_y is piecewise constant in m (it can jump at the values of m where the amplitude vanishes). The bare numerical calculation for $p_{\lambda,y}$ is shown in Fig. C.6.

The origin o is determined as follows. We first define a point $\bar{o}$, referred to as the ‘gauge origin’ for the vector potential, which has the property that the holonomy of the vector potential is trivial along the x and y cycles of the torus that meet at $\bar{o}$ (see Eq. (4.56)). Then the origin o used to obtain $\vec{\mathcal{P}}_o$ is expressed in terms of $\bar{o}$; see Eq. (4.61).

In defining $\tilde{T}_y$ we in principle have the freedom to combine it with an arbitrary global $U(1)$ rotation:

$$\tilde{T}_y \rightarrow \tilde{T}_y e^{i\chi\hat{N}}, \quad (4.22)$$

which corresponds to a shift $\lambda_j \rightarrow \lambda_j + \chi$ for each j . Once o is fixed, then χ is fixed to be an integer multiple of ϕ by fixing the flux through a dislocation created using $\tilde{T}_y$, as explained in Sec. 4.4 and Appendix C.3. Thus we only need to consider the case where

$$\chi = \phi v_x = \frac{2\pi m}{L_x L_y} v_x, \quad (4.23)$$

for some integer v_x . Under such a shift in $\tilde{T}_y$,

$$\begin{aligned} p_{\lambda,y} &\rightarrow p_{\lambda,y} + mv_x \nu \\ &= p_{\lambda,y} + mv_x \left(\frac{Cm}{L_x L_y} + \kappa \right) \end{aligned} \quad (4.24)$$

If we consider only the term linear in m , this implies that

$$\mathcal{P}_{o,y} \rightarrow \mathcal{P}_{o,y} - \kappa v_x = \mathcal{P}_{o,y} \mod 1. \quad (4.25)$$

Note that one could consider the case where v_x is fractional but quantized, and this would effectively correspond to a shift of the HSP $\mathbf{o}$ by $(v_x, 0)$ to a different HSP, as explained in Sec. 4.4.

Analogous equations hold for $\mathcal{P}_{o,x}$, if we instead start with a Landau-like gauge along x . Furthermore, our procedure straightforwardly generalizes to rotational symmetries of order $M = 2, 3, 4, 6$ [101].

We have also measured $\vec{\mathcal{P}}_o$ by studying the expectation values of a partial translation operator $\tilde{T}_y|_D$, which is $\tilde{T}_y$ restricted to some suitably chosen region D . This method also allows to extract a quantized $\vec{\mathcal{P}}_o$ consistent with dislocation charge, for a suitable choice of $\bar{\mathbf{o}}$ and of the region D , when M is even. We discuss this further in Secs. 4.4.

4.1.3.4 $\vec{\mathcal{P}}_o$ from dimensional reduction and 1d polarization

One can also define a 1d polarization along the $\hat{i}$ direction by treating the system as an effectively 1d system along $\hat{i}$. Let us first consider $\hat{i} = \hat{x}$. Then we can calculate the 1d

polarization using Resta's formula [158]:

$$\mathcal{P}_{\mathcal{O},x} = \frac{1}{2\pi} \arg \langle \Psi | e^{i \frac{2\pi}{L_x} \sum_j j_x \hat{n}_j} | \Psi \rangle. \quad (4.26)$$

The above expression depends on a choice of origin $\mathcal{O}_x$ for j_x , which we have made explicit, i.e.

$j_x \in \{-\mathcal{O}_x, 1 - \mathcal{O}_x, 2 - \mathcal{O}_x, \dots, L_x - 1 - \mathcal{O}_x\}$. Empirically, we find that

$$-\mathcal{P}_{\mathcal{O},x} = \frac{Ck\phi}{2\pi} L_y + L_y \mathcal{P}_{\mathbf{o},y} + K' \pmod{1}, \quad (4.27)$$

where $k := \frac{L_x}{2} - \bar{\mathbf{o}}_x + \mathcal{O}_x$. Knowing the value of k is not crucial in extracting $\mathcal{P}_{\mathbf{o},y}$. This result agrees with a field theory prediction which is derived in Sec. 4.4.4.

Similar to the linear momentum calculation, we extract a value of $\vec{\mathcal{P}}_{\mathbf{o}}$ that is consistent with other approaches only when $\mathbf{o}, \mathcal{O}$ satisfy a certain relation, see Eq. (4.75). This calculation is independent of the specific details of the gauge, as long as Φ_y is linear in ϕ . The full discussion is contained in Sec. 4.4.4.

4.2 Basic properties of lattice defects

Before discussing the numerical calculations in detail, we review some useful background material on lattice defects and their properties. A more extensive background review can be found in App. C.1. The quantum mechanical details of constructing a defect Hamiltonian H_{defect} through a cut-and-clue procedure are described in detail for dislocation defects in App. C.3.

We illustrate the procedure for constructing a dislocation defect in Fig. 4.4. First we make a cut on an infinite clean lattice and define the left (L) and right (R) sides of the cut. We replace

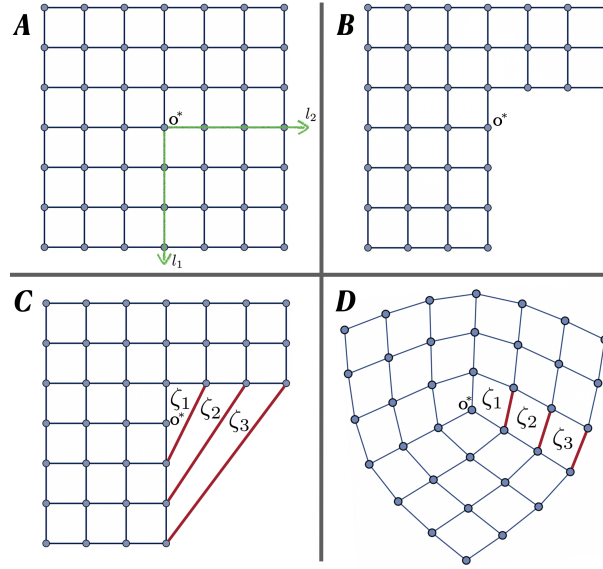


Figure 4.3: Cut and glue procedure of constructing a disclination. **A.** Original lattice on an open plane; **B.** Cutting; **C.** Gluing, this creates new plaquettes ζ_i ; **D.** Reorganizing.

all bonds that cross the cut so that a point L_i , originally connected to some point R_i , is now connected to $R_{i+\vec{b}^*}$. Here $\vec{b}^*$ is an integer vector related to the dislocation Burgers vector, which we define below.

We illustrate the construction of a disclination defect in Fig. 4.3. We draw two rays l_1, l_2 which meet at the point o^* , such that l_2 is obtained from l_1 by rotating about o^* through the angle $\Omega > 0$. Now we delete all points within the wedge enclosed by l_1 and l_2 , except those that lie exactly on l_1 . We then reconnect the bonds so that a link $i_1 j$ (where i_1 lies on l_1) is replaced by a link $i_1 j'$, where j' is obtained from j upon rotating by Ω about o^* .

The disclination angle Ω can be directly measured from the defect lattice alone. It is the angle by which a unit vector is rotated upon being parallel transported around the defect. ‘Pure’ dislocation defects are those with zero disclination angle. They are characterized by a dislocation Burgers vector $\vec{b}$, which is defined as follows. Starting from a point o , consider a sequence of lattice translations which encircles the defect (and no other defects) in counterclockwise fashion,

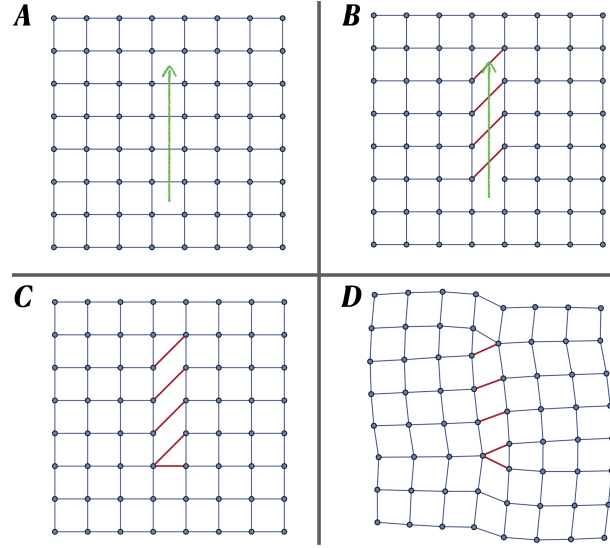


Figure 4.4: Dislocation construction on a square lattice. **A.** Drawing a cut in the $\hat{y}$ direction. **B.** Conjugate hoppings. **C.** Doing local moves. **D.** Reorganizing.

and ends at $\mathbf{o}$. For a dislocation, the sum of these translations will not equal zero, but instead some integer vector, which we define as $\vec{b}$. The above dislocation construction gives two dislocations at the two ends of the cut, with $\vec{b} = \pm \vec{b}^*$. For a pure dislocation, the shape of the loop and the choice of $\mathbf{o}$ do not affect the value of $\vec{b}$.

The Burgers vector of a disclination with $\Omega \neq 0$ can be measured similarly; note that $\mathbf{o}^*$ and $\mathbf{o}$ need not be related. Importantly, the Burgers vector when $\Omega \neq 0$ sensitively depends on the choice of $\mathbf{o}$, assuming $\mathbf{o}^*$ is fixed. If we shift $\mathbf{o} \rightarrow \mathbf{o} + \vec{v}$, then

$$\vec{b}_{\mathbf{o}+\vec{v}} = \vec{b}_{\mathbf{o}} + (1 - U(\Omega))\vec{v}. \quad (4.28)$$

This is derived in App. C.1. As an illustration, Fig. 4.5 shows the same $\Omega = \frac{2\pi}{3}$ disclination but with different Burgers vectors depending on the choice of $\mathbf{o}$. Since choosing $\vec{v} = \vec{\Lambda}$ to be an

integer vector should give us an equivalent characterization of the defect, we have the equivalence

$$\vec{b}_o \simeq \vec{b}_o + (1 - U(\Omega))\vec{\Lambda}. \quad (4.29)$$

Thus, if $\vec{v}$ is fractional, $\vec{b}_{o+\vec{v}}$ will not be equivalent to $\vec{b}_o$. If $\Omega = \frac{2\pi k}{M_o}$ where k, M_o are coprime, the distinct classes of Burgers vectors form a group K_{M_o} as defined in Sec. 4.1.

Note that our construction ensures the following relation:

$$\vec{b}_{o^*} \simeq (0, 0), \quad (4.30)$$

which can be verified by constructing the various classes of disclinations for $M = 2, 3, 4, 6$. Setting $o = o^*$ thus ensures that the defect has trivial Burgers vector. This will be a convenient choice to make in the following sections. Note that we sometimes also write $\vec{b}_o \in [(0, 0)]$ to indicate that $\vec{b}_o$ is in the same equivalence class as $(0, 0)$.

4.3 Calculation of $\mathcal{S}_o$ on the square lattice

This section and the next are devoted to numerically checking the predictions of the field theory for the square lattice. This section reviews and generalizes the main results from Ref. [100]. Analogous calculations for $M = 2, 3, 6$ can be performed, but for reasons of space we do not show them here.

We fix our origin at a HSP o which has fourfold rotational symmetry. There are two choices, α and β , as shown in Fig. 4.2. α denotes the center of the unit cell, while β denotes a corner. For calculations on the simplest square lattice, we pick the unit cell shown in Fig. 4.6 C, where

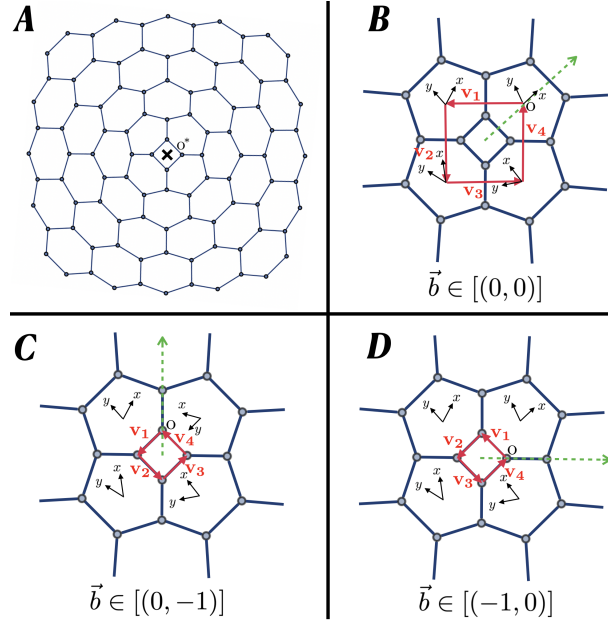


Figure 4.5: **A.** A pure disclination with $\Omega = \frac{2\pi}{3}$ created using $\tilde{C}_{M_o^*}$ via a cut-and-glue construction. The disclination core o^* is marked as ‘ $\times$ ’. We can define a ‘frame’ i.e. an x, y basis at every point on the defect lattice (they are plotted at the four plaquette centers near the disclination). The frame rotates by $-2\pi/3$ upon crossing the green dotted branch cut which passes through o^* and o . We calculate $\vec{b}_o = \vec{v}_1 + \vec{v}_2 + \vec{v}_3 + \vec{v}_4$ for different o **B.** o at a plaquette center. $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\} = \{(-1, 1), (-1, 0), (0, -1), (1, -1)\}$. **C.** o at one of the sites. $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\} = \{(-2/3, 1/3), (-1/3, -1/3), (1/3, -2/3), (2/3, -1/3)\}$. **D.** o at the other site. $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\} = \{(-1/3, 2/3), (-2/3, 1/3), (-1/3, -1/3), (1/3, -2/3)\}$

α corresponds to a plaquette center and β corresponds to a site. Note that the formulas for $o = \gamma$ are contained in Table 4.3, if we take $M = 2$.

For either choice of o , we have two topological invariants, $\mathcal{S}_o$ and $\vec{\mathcal{P}}_o = \frac{1}{2}\overline{\mathcal{P}}_o(1, 1)$. $\mathcal{S}_o$ is defined mod 4 and satisfies $\mathcal{S}_o = \frac{C}{2} \pmod{1}$. $\overline{\mathcal{P}}_o$ is an integer defined mod 2.

We extract $\mathcal{S}_o$ in two physically different ways, through the disclination charge and the angular momentum of flux. The exact choice of unit cell does not affect the final result; we show this in App. C.4.

The gauge transformations used to define the translation operators will be discussed in Sec. 4.4.

4.3.2 Construction of clean Hamiltonian H_{clean}

In our numerical work we consider the Hofstadter model, which has a spinless free fermion Hamiltonian of the form

$$H_{\text{clean}} = - \sum_{\langle ij \rangle} t_{ij} c_i^\dagger c_j + \text{h.c.} \quad (4.33)$$

where the nearest neighbor hopping terms $t_{ij} = t e^{i A_{\text{clean}, ij}}$ depend on a background vector potential A_{clean} , which assigns flux ϕ per unit cell. The parameters in H_{clean} are discussed in detail in App. C.1.

Although we mainly consider nearest-neighbor hopping in our numerics, our theoretical predictions as well as our numerical scheme apply much more generally. For example, we can consider arbitrary next neighbor hoppings. To illustrate this, in App. C.7 we give evidence showing that invariants extracted numerically in the square lattice Hofstadter model remain well-defined upon adding next neighbor hopping terms. Below we will also argue that our procedure works if H_{clean} has N -body interaction terms for $N > 2$.

An important point is that we require the magnetic field B to be defined everywhere within the unit cell. This means that the total magnetic flux within any subregion of the unit cell is specified as a real number. This requirement goes beyond what is directly specified by the crystalline symmetry (which only demands flux $\phi \bmod 2\pi$ per unit cell). But it is a physically natural requirement, since the most general lattice models with the given symmetry have some small amount of further neighbor hopping, between different points within a single unit cell.

In fact, specifying a nearest-neighbor Hamiltonian with just $\phi \bmod 2\pi$ is ill-defined in a sense, because it does not specify how to consistently perturb the model with further neighbor hopping terms. Such a specification requires a choice of ϕ as a real number, not just modulo 2π .

4.3.3 $\mathcal{S}_0$ from disclination charge

4.3.3.1 Construction of defect Hamiltonian H_{defect}

Start with a clean lattice Hamiltonian H_{clean} which depends on a background $U(1)$ vector potential A_{clean} through the variables $e^{iA_{ij,\text{clean}}}$. Suppose we create a lattice disclination (a detailed construction can be found in Ref. [100]), and arrive at a defect Hamiltonian $H_{\text{defect}}[e^{iA_{ij,\text{defect}}}]$ through a cut-and-glue procedure. Here A_{defect} is the vector potential on the defect lattice. Note that irregular unit cells may exist at the center of each defect. For example, there could be a triangular unit cell at the center of an $\Omega = \frac{\pi}{2}$ disclination (see Fig. 4.6B), or a triangular unit cell at the center of a square lattice dislocation (see Fig. 4.7). Irregular unit cells can have different shapes depending on the value of M .

Away from the defect, the flux in any region is fully determined by A_{clean} , and we can ensure that the system has flux ϕ per unit cell. However, the flux in the immediate vicinity of the defect depends on A_{clean} as well as on the definition of the symmetry operator which creates the defect. In particular, if we require a specific value of flux in the unit cells immediately adjacent to the defect, there will be a constraint on the definition of the symmetry operators we use.

4.3.3.2 Charge prediction from field theory

Having fixed H_{defect} , we compute the charge Q_W in a region W containing the defect. We always require that the boundary of W coincides with the boundary of a unit cell. This ensures that the only irregular unit cells in W are near the center of the defect. Q_W is defined by the formula

$$Q_W = \sum_{i \in W} \text{wt}(i) Q_i. \quad (4.34)$$

The weights $\text{wt}(i)$ were defined in Sec. 4.1; this definition ensures that $Q_W + Q_{W'} = Q_{W \sqcup W'}$ when two regions W, W' overlap only on their boundaries, as required by the response theory. Note that the value of Q_W depends on the definition of the unit cell.

This method of measuring Q_W applies to any local gapped Hamiltonian, even if it has interaction terms. As long as the system has a correlation length much smaller than the linear size of W , the ambiguity in H_{defect} near the defect core will not affect the value of Q_W for sufficiently large W .

The next issue is how to assign this charge to the different terms in the response theory, which predicts that for large enough W ,

$$Q_W = C \frac{\delta \Phi_{W,o}}{2\pi} + \mathcal{S}_o \frac{\Omega}{2\pi} + \vec{\mathcal{P}}_o \cdot \vec{b}_o + \nu n_{W,o} \pmod{1}. \quad (4.35)$$

Here Ω and $\vec{b}_o$ are the disclination angle and Burgers vector of the defect, respectively. We fix $\vec{b}_o = (0, 0)$ throughout this section, so the term with $\vec{\mathcal{P}}_o$ does not contribute (this is equivalent to saying that $o = o^*$).

The parts of this equation which require special care are $\delta\Phi_{W,o}$ and $n_{W,o}$ ³. We discuss how to define these below.

4.3.3.3 Definition of $\delta\Phi_{W,o}$ and $n_{W,o}$

$\delta\Phi_{W,o}$ is, intuitively, the excess flux in the region W . To define the excess flux, we must compute the flux in W , and compare it to some background reference flux. There are two possibilities. If $o = \beta$, then all unit cells in W are regular (see Fig. 4.6A), and $n_{W,\beta} = 0 \bmod 1$. Here $\delta\Phi_{W,o} = 0$ if all unit cells have flux ϕ . On the other hand, if $o = \alpha$, then there is an irregular unit cell at the disclination core (see Fig. 4.6B).

In general, we can arrange to have at most one irregular unit cell in the core of the defect, with some flux ϕ_{irreg} which is determined by our choice of rotation operators. Its area is denoted $n_{\text{irreg},o}$, which can be fractional. Note that

$$n_{W,o} = n_{\text{irreg},o} \bmod 1. \quad (4.36)$$

Then, if we choose our symmetry operators so that the flux through each regular unit cell in H_{defect} is ϕ , we have

$$\delta\Phi_{W,o} = \phi_{\text{irreg}} - n_{\text{irreg},o}\phi. \quad (4.37)$$

Thus the value of $\delta\Phi_{W,o}$ depends on the value we assign to $n_{\text{irreg},o}$.

³This was referred to as $n_{u.c.,W}$ in Ref. [100].

4.3.3.4 Computation of $n_{\text{irreg},o}$

If $o = \beta$, there are no irregular unit cells, as discussed above, so we simply define $n_{\text{irreg},\beta} = 0 \pmod{1}$. If $o = \alpha$, we find $n_{\text{irreg},\alpha} = \frac{3}{4} \pmod{1}$. A heuristic argument is that the cut-and-glue procedure for a $\pi/2$ disclination removes $1/4$ of a unit cell at the disclination center, i.e. the irregular unit cell at the disclination core consists of 3 subcells. Each subcell contributes $1/4$ th of a full unit cell, therefore $n_{\text{irreg},\alpha} = \frac{3}{4} \pmod{1}$. (A consistency check on this result is to demand that the value of $\mathcal{S}_o$ corresponding to some physical point o be the same whether we choose the unit cell to satisfy $o = \alpha$ or $o = \beta$. If $n_{\text{irreg},\beta} = 0 \pmod{1}$, then for $\mathcal{S}_o$ to be invariant we find it is necessary to have $n_{\text{irreg},\alpha} = \frac{3}{4} \pmod{1}$.)

We note that the previous step involved another subtlety. Even if we know $n_{\text{irreg},o}$, there can be a further ambiguity in $\delta\Phi_{W,o}$. If $n_{\text{irreg},o}$ is an integer, then Eq. (4.37) is perfectly well-defined modulo 2π . However, if $n_{\text{irreg},o} = a/b$ is a fraction with a, b coprime, $\delta\Phi_{W,o}$ is not invariant under the transformation $\phi \rightarrow \phi + 2\pi$. To keep $\delta\Phi_{W,o}$ invariant, we need to pick a lift of ϕ from $[0, 2\pi)$ to $[0, 2\pi b)$. But this is why we insist on specifying the actual magnetic field everywhere in the unit cell. This fixes ϕ as a real number, not just mod 2π , and so fractions of ϕ can be defined unambiguously.

4.3.3.5 Computing $\mathcal{S}_o$

For a $\pi/2$ disclination with $\vec{b}_o = (0, 0)$, Eq. (4.35) predicts that

$$\frac{\mathcal{S}_o}{4} = Q_W - \nu n_{W,o} - C \frac{\delta\Phi_{W,o}}{2\pi} \pmod{1}. \quad (4.38)$$

This determines $\mathcal{S}_0/4 \bmod 1$ in terms of well-defined quantities. Note that our procedure to define $n_{\text{irreg},0}$ and $\delta\Phi_W$ is independent of the details of the Hamiltonian, in particular if there are further neighbor hopping terms and interaction terms. Therefore we expect that $\mathcal{S}_0$ can be robustly extracted for any H_{clean} with a symmetric, gapped ground state.

4.3.4 Angular momentum of flux

We can also compute $\mathcal{S}_0$ from the angular momentum eigenvalues of $\tilde{C}_{M_0^*}$ after inserting additional flux, if we set $\mathbf{o} = \mathbf{o}^*$. On the torus, there are two positions on the torus that are left invariant under a C_4 rotation, $\mathbf{o}^*$ and $\mathbf{o}^* + (L/2, L/2)$. If L is odd, then the two positions are not the same point in the unit cell. This is deemed unnatural since we only want a single origin. Thus, we consider an even length system on a torus, insert m total flux quanta uniformly, and define

$$\tilde{C}_{M_0} |\Psi(m)\rangle \equiv e^{i\frac{\pi}{2}l_0} |\Psi(m)\rangle. \quad (4.39)$$

The field theory predicts that there will be a contribution to l_0 which equals $\mathcal{S}m$. Indeed, we numerically find that

$$l_0(m) = \frac{Cm^2}{2} + m\mathcal{S}_0 + K(C, L) \bmod 4. \quad (4.40)$$

(see Fig.C.6 for the details). Note that a thorough discussion of the related method of partial rotations will be given in Chapter 5.

4.3.5 Application to Hofstadter model

Below we obtain $\mathcal{S}_o$ for the Hofstadter model, taking $o = \beta$ (at a site), and also $o = \alpha$ (at a plaquette center). The values of $\mathcal{S}_o$ obtained using both disclination charge and angular momentum are consistent. In the limit of small ϕ and ν , our procedure also agrees with known results for continuum Landau levels, i.e. $\mathcal{S}_o = \frac{C^2}{2} \bmod 4$ for either $o = \beta$ or $o = \alpha$. The Hofstadter butterflies for $\mathcal{S}_o$ are plotted in Fig. 4.1.

We find that they obey the following empirical formulas (recall that β is chosen to be a site). For $C > 0$,

$$\mathcal{S}_\beta(\phi) = \frac{C^2}{2} - (C + 1) \left\lfloor \frac{C\phi}{2\pi} \right\rfloor + 2 \sum_{\substack{\frac{p}{q} < \frac{\phi}{2\pi} \\ \text{odd } q}} \left\lfloor \frac{C + q}{2q} \right\rfloor \bmod 4, \quad (4.41)$$

where the third term of Eq.(4.41) we sum over all $\frac{p}{q}$ in the Farey sequence of order C that satisfy $\frac{p}{q} < \frac{\phi}{2\pi}$ and q odd. The value of $\mathcal{S}_\beta$ for $C < 0$ can be obtained from the transformation $\mathcal{S}_\beta(\mu, \phi) = 1 - \mathcal{S}_\beta(-\mu, \phi)$ which flips the sign of C . $\mathcal{S}_\beta = 1$ for the fully filled state with $\nu = 1, C = 0$.

Similarly, we can set $o = \alpha$ which is the plaquette center. For $C > 0$,

$$\mathcal{S}_\alpha(\phi) = \frac{C^2}{2} + C \left\lfloor \frac{C\phi}{2\pi} \right\rfloor + 2 \sum_{\substack{\frac{p}{q} < \frac{\phi}{2\pi} \\ \text{odd } q}} \left\lfloor \frac{C + q}{2q} \right\rfloor \bmod 4. \quad (4.42)$$

The value of $\mathcal{S}_\alpha$ for $C < 0$ lobes can be obtained from the transformation $\mathcal{S}_\alpha(\mu, \phi) = 2C - \mathcal{S}_\alpha(-\mu, \phi)$ which flips the sign of C . $\mathcal{S}_\alpha = 0$ when $C = 0$.

One can verify that $\mathcal{S}_\beta$ has period 2π in ϕ , but $\mathcal{S}_\alpha$ has period 8π in ϕ . The physical reason is that the defect lattice with $o = \alpha$ has a triangular plaquette at the disclination core. Under $\phi \rightarrow$

$\phi + 2\pi$, the background flux through this triangular plaquette transforms as $3\phi/4 \rightarrow 3\phi/4 + \frac{3\pi}{4}$.

Therefore H_{defect} is invariant under $\phi \rightarrow \phi + 8\pi$ up to gauge transformations.

4.4 Calculation of $\vec{\mathcal{P}}_o$ on the square lattice

We next discuss three different ways to calculate $\vec{\mathcal{P}}_o$ for $o = \alpha, \beta$ directly on the square lattice, using dislocation charge, linear momentum, and a 1d polarization response. (The case $o = \gamma$ is similar and is summarized in Table 4.3.) In fact, we show that $\vec{\mathcal{P}}_\alpha, \vec{\mathcal{P}}_\beta$ can also be computed if we know $\mathcal{S}_\alpha, \mathcal{S}_\beta$ alone: see the results in Table 4.1, in particular Eq. (4.8). These results are derived in Sec. 4.5 and App. C.6. So in principle we can also measure the polarization indirectly from the disclination charge and angular momentum responses. All the calculations yield the same numerical values of $\vec{\mathcal{P}}_{\alpha(\beta)}$, and are consistent with field theory predictions.

In this section we will not require the rotation operator $\tilde{C}_{M_o^*}$, but will still use the translation operators satisfying Eq. (4.32). A notational remark: in Secs 4.4 and the related Apps. C.3, C.5 we will use A to denote the entire vector potential on a lattice without dislocation or disclination defects, so A has the same meaning as A_{clean} from the previous section. The vector potential in a system with lattice defects will be denoted as A_{defect} .

4.4.1 $\vec{\mathcal{P}}_o$ from dislocation charge

First we consider the charge bound to a single dislocation (for an explicit construction of such a dislocation defect, see App. C.3). We follow the procedure outlined in Sec. 4.3.3; similar arguments can be applied in this case.

For a defect with zero disclination angle and dislocation Burgers vector $\vec{b} = (0, \pm 1)$ or

$(\pm 1, 0)$, there is a triangular plaquette at the dislocation, and all other plaquettes within a large radius are ensured to be regular through our construction. Suppose the triangular plaquette has flux ϕ_{irreg} . If the translation operator used to construct the dislocation is

$$\tilde{T}_{\mathbf{y}} = \hat{T}_{\mathbf{y}} e^{i \sum_j \lambda_j \hat{n}_j},$$

we find that when $\vec{b} = (0, 1)$,

$$\phi_{\text{irreg}} = A_{j_0, j_0 + \hat{y}} - \lambda_{j_0} \quad (4.43)$$

where j_0 is a specific point in the irregular unit cell at the dislocation (see App. C.3.2 for a proof). Thus, the value of ϕ_{irreg} is set by the combined choice of A , λ and j_0 . In particular, we show in App. C.3.2 that for $v_x \in \mathbb{Z}$, the following transformations all take $\phi_{\text{irreg}} \rightarrow \phi_{\text{irreg}} + \phi v_x$: (1) taking $j_0 \rightarrow j_0 + (v_x, 0)$ keeping A , λ fixed; (2) taking $\lambda_j \rightarrow \lambda_j - \phi v_x$ for each j , keeping A and j_0 fixed. (Here we can even take $v_x \in \mathbb{R}$.) In this section it will turn out that fixing a precise value of ϕ_{irreg} is not essential, but the above discussion will also be useful in Sec. 4.4.3.

The field theory predicts that the total charge in a region W surrounding the dislocation is

$$Q_W = C \frac{\delta \Phi_{W, \mathbf{o}}}{2\pi} + \frac{\overline{\mathcal{P}}_{\mathbf{o}}}{2} + (k + n_{\text{irreg}, \mathbf{o}}) \nu \mod 1, \quad (4.44)$$

where we have written $n_{W, \mathbf{o}} = k + n_{\text{irreg}, \mathbf{o}}$ for some integer k . Here $n_{\text{irreg}, \mathbf{o}}$ is the effective number of unit cells we assign to the irregular unit cell at the dislocation. We use the subscript $\mathbf{o}$ in $n_{\text{irreg}, \mathbf{o}}$ because the value assigned to it will turn out to be different for different high symmetry points $\mathbf{o}$.

Recall from the general discussion in Sec. 4.3.3 that if all regular unit cells have flux ϕ ,

$$\delta\Phi_{W,o} = \phi_{\text{irreg}} - \phi n_{\text{irreg},o}$$

where $\phi n_{\text{irreg},o}$ is the background flux assigned to the dislocation plaquette.

Next we determine $n_{\text{irreg},o}$. First we consider $o = \beta$ (see Fig. 4.7). The choice of $n_{\text{irreg},\beta}$ should satisfy a few sanity checks. First, it should lead to an integer-valued $\overline{\mathcal{P}}_\beta$. Numerically, only $n_{\text{irreg},\beta} = 0$ or $1/2$ give integer values of $\overline{\mathcal{P}}_\beta$; this can also be seen directly from Eq. (4.44) by looking at the limit of zero Chern number and $\nu \in \mathbb{Z}$.

Second, the system with the same Hamiltonian but with all orbitals filled has zero Chern number. It can be adiabatically connected to one in which the points in the α, β, γ maximal Wyckoff positions have integer charge $N_\alpha, N_\beta, N_\gamma$ respectively. In this situation, the polarization can also be defined using the dipole moment within each subcell, and it is possible to show, independent of any dislocation charge calculation, that

$$\overline{\mathcal{P}}_\alpha = N_\beta + N_\gamma \pmod{2}; \quad (4.45)$$

$$\overline{\mathcal{P}}_\beta = N_\alpha + N_\gamma \pmod{2}. \quad (4.46)$$

See App. C.2 for the details. Now we show that Eq. (4.44) only agrees with this result if we set

$$n_{\text{irreg},\beta} = \frac{1}{2}.$$

To derive this explicitly, consider Fig. 4.7. We choose W so that ∂W overlaps with β sites.

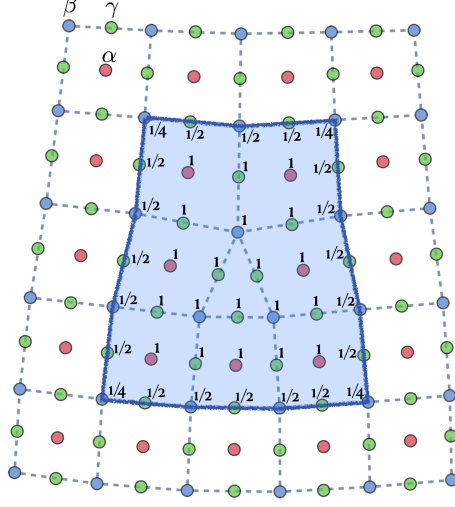


Figure 4.7: Region W for a square lattice dislocation with $\vec{b} = (1, 0)$. The weightings $\text{wt}(i)$ are labeled on each relevant MWPs. Dashed lines represent unit cell boundaries, and colored circles represent the MWPs α (red), β (blue) and γ (green).

Using our weighting procedure, we get

$$Q_W = 7N_\alpha + 7.5N_\beta + 15.5N_\gamma, \quad (4.47)$$

while from the field theory, if we set $\mathbf{o} = \beta$,

$$Q_W = \kappa(7 + n_{\text{irreg},\beta}) + \frac{\overline{\mathcal{P}}_\beta}{2} \mod 1, \quad (4.48)$$

where $\kappa = N_\alpha + N_\beta + 2N_\gamma$. Using Eq. (4.46) in the above equations, and simplifying, we obtain

$$n_{\text{irreg},\beta} = 1/2 \mod 1.$$

The above calculation was done in a limit with Chern number zero. We now assume that $n_{\text{irreg},\beta} = 1/2$ even when $C \neq 0$. This is reasonable because in the actual model under study, changing the Chern number should not change $n_{\text{irreg},\beta}$. After this step, we get the following

prediction:

$$Q_W = \frac{\overline{\mathcal{P}}_\beta}{2} + (k + \frac{1}{2})\nu + C \frac{\delta\Phi_{W,\beta}}{2\pi} \mod 1 \quad (4.49)$$

where W encloses only one dislocation, with one triangular plaquette. Note that

$$\delta\Phi_{W,\beta} = \phi_{\text{irreg}} - \frac{\phi}{2}$$

is completely well-defined mod 2π since ϕ is specified as a real number (see Sec. 4.3.3.4). This allows us to find $\overline{\mathcal{P}}_\beta \mod 2$ unambiguously.

Now let us discuss $\overline{\mathcal{P}}_\alpha$. Again consider Fig. 4.7. Let us use the exact same W (i.e. the same unit cell choice), so

$$Q_W = 7N_\alpha + 7.5N_\beta + 15.5N_\gamma. \quad (4.50)$$

However, if we choose $o = \alpha$, the field theory now predicts that

$$Q_W = \kappa(7 + n_{\text{irreg},\alpha}) + \frac{\overline{\mathcal{P}}_\alpha}{2} \mod 1. \quad (4.51)$$

In the Chern number 0 limit, demanding consistency with Eq. (4.45) implies that $n_{\text{irreg},\alpha} = 0 \mod 1$.

We have thus found that $n_{\text{irreg},o}$ depends on whether o lies at the corner or the center of a unit cell. Even when we consider other values of M , we find that $n_{\text{irreg},o}$ must be fixed for each o by requiring consistency with analytical results in the limit of full filling.

4.4.1.1 Comments on definition of $n_{\text{irreg},0}$

An important general point is that our definitions of $n_{\text{irreg},0}$ are only sensitive to the value of M and not to the actual fine structure of the microscopic lattice. Conceptually, we can imagine tiling the plane with unit cells (square for $M = 4$, hexagonal for $M = 3, 6$, and so on), and the only constraint is that the centers and corners of these unit cells need to be at appropriate high symmetry points of the microscopic lattice. In particular, the tiling does not have to match the structure of hopping or interaction terms in the microscopic Hamiltonian. Then, we note that the number of unit cells in any region W is a property of the tiling alone. Similarly, we can apply a cut and glue procedure on the infinite plane tiling to get a tiling for a surface with a dislocation or disclination defect. Then the quantities $n_{\text{irreg},0}$ for disclinations and dislocations are properties of this defect tiling alone. As such, these are independent of microscopic details of the Hamiltonian such as the distribution of sites and the hoppings between them. The fact that our prescription depends on the tiling rather than on microscopic details such as hopping and interaction terms makes it readily generalizable beyond the nearest neighbor Hofstadter models that we have mainly studied in this work.

4.4.2 Application to Hofstadter model

We apply Eq. (4.49) to the usual square lattice Hofstadter model, choosing α at a plaquette center and β at a site as before. The Hofstadter butterfly for $\overline{\mathcal{P}}_\beta$ is plotted in Fig. 4.1. $\overline{\mathcal{P}}_\beta$ follows the empirical formula

$$\overline{\mathcal{P}}_\beta(\phi) = C\kappa \pmod{2}. \quad (4.52)$$

Note that $\overline{\mathcal{P}}_\beta$ has period 4π in ϕ and not 2π , because its definition involves the quantity $\phi/2$. (Also, if $\phi \rightarrow \phi + 2\pi$ for fixed ν , Eq. (4.52) changes by $C \bmod 2$ so a shift of 4π is needed to leave it invariant.) We use Eq. (4.52), along with an eigenvalue database [13], to generate the Hofstadter butterfly for $\overline{\mathcal{P}}_\beta$ in Fig. 4.1. We have only plotted it for $0 < \phi \leq 2\pi$. The values for $2\pi < \phi \leq 4\pi$ can be obtained either using Eq. 4.52 or by reflecting the butterfly about $\phi = 2\pi$.

Similarly, with α at a plaquette center, and $n_{\text{irreg},\alpha} = 0$, we find that

$$\overline{\mathcal{P}}_\alpha = (C + 1)\kappa \bmod 2. \quad (4.53)$$

$\overline{\mathcal{P}}_\alpha$ has period 2π in ϕ , and is also plotted in Fig. 4.1. Shifting $\phi \rightarrow \phi + 2\pi$ for fixed ν changes Eq. (4.53) by $C(C + 1) \bmod 2$, but as $C(C + 1)$ is even, this change is trivial.

Finally, note that there is a general relation between $\overline{\mathcal{P}}_\alpha$ and $\overline{\mathcal{P}}_\beta$ which goes beyond just the Hofstadter model:

$$\overline{\mathcal{P}}_\alpha - \overline{\mathcal{P}}_\beta = \kappa \bmod 2. \quad (4.54)$$

Eq. (4.54) can be proven from field theory; we do this in Sec. 4.5.

4.4.3 $\vec{\mathcal{P}}_0$ from linear momentum

On a closed manifold, the field theory term $\frac{\vec{\mathcal{P}}_0}{2\pi} \cdot A \wedge \vec{T}$ can be rewritten as $\frac{\vec{\mathcal{P}}_0}{2\pi} \cdot \vec{R} \wedge F$, plus some additional terms, where F is the total magnetic flux, and $\vec{R} = (X, Y)$ is the translation gauge field mentioned below Eq. (4.13). Consider the linear momentum in, say, the y direction associated to a state with total flux $2\pi m = \phi L_x L_y$. The linear momentum of the inserted flux,

which is the charge under the translation gauge field, is predicted to receive a contribution

$$\frac{\delta \mathcal{L}}{\delta Y_0} = \frac{\mathcal{P}_{\mathbf{o},y}}{2\pi} \int F = \mathcal{P}_{\mathbf{o},y} m = \frac{1}{2} \overline{\mathcal{P}}_{\mathbf{o}} m \mod 1 \quad (4.55)$$

from this term. The mod 1 normalization is due to the convention chosen to define Y . We now verify this prediction from numerical calculations on the Hofstadter model for $\mathbf{o} = \alpha, \beta$ by studying the system on a torus.

On a torus with magnetic flux $2\pi m = \phi L_x L_y$, a translation by $\mathbf{y}$ cannot be an exact symmetry of the Hamiltonian in general, even after applying a gauge transformation. This is because, for a fixed x , the holonomy $\oint A_x dx$ along a noncontractible cycle of the torus changes by $\frac{2\pi m}{L_y}$ under $\tilde{T}_{\mathbf{y}}$. If $\frac{m}{L_y} \in \mathbb{Z}$, $\tilde{T}_{\mathbf{y}}$ is an exact symmetry because the change in $\oint A_x dx$ can be undone by a large gauge transformation. For other values of m , $\tilde{T}_{\mathbf{y}}$ needs to be accompanied by an operator $\mathcal{F}_{\frac{2\pi m}{L_y}}$ which adiabatically inserts flux $\frac{2\pi m}{L_y}$ through the cycle running along x , to make it an exact symmetry. In our work we do not use the exact translation operator because it is difficult to numerically implement $\mathcal{F}_{\frac{2\pi m}{L_y}}$.

If the value of $|m| \mod L_y$ is of order 1, the inserted flux from $\mathcal{F}_{\frac{2\pi m}{L_y}}$ is $O(1/L_y)$, which vanishes in the thermodynamic limit; therefore $\tilde{T}_{\mathbf{y}} \simeq \tilde{T}_{\mathbf{y}} \mathcal{F}_{\frac{2\pi m}{L_y}}$ is a good approximation [94]. For general values of m , we find that the expectation value of $\tilde{T}_{\mathbf{y}}$ oscillates between 0 and 1 in amplitude as a function of m (indicating the closeness of the approximation).

Apart from the flux in each unit cell, the vector potential has another gauge invariant quantity called the ‘gauge origin’ $\bar{\mathbf{o}}$, which we define below. The operator $\tilde{T}_{\mathbf{y}}$ also requires us to fix a gauge transformation $\{\lambda_j\}$. The main result of this section is that for a fixed system size and a fixed choice of $\mathbf{o}$, we find only one choice of $\bar{\mathbf{o}}$ and λ for which $\vec{\mathcal{P}}_{\mathbf{o}}$ is quantized throughout

the Hofstadter model. And remarkably, this value of $\vec{\mathcal{P}}_0$ agrees with results from the dislocation charge calculation.

We find that we can similarly measure $\vec{\mathcal{P}}_0$ using the expectation value of a partial translation operator $\tilde{T}_y|_D$ (again not an exact symmetry), which is $\tilde{T}_y$ restricted to an appropriate region D . Further details of the linear momentum calculations can be found in App. C.5.

4.4.3.1 Definition of vector potential

We assume, as in the charge calculation, that the holonomies of the vector potential A can be specified along any loop in the continuum in which the lattice is embedded. We define a gauge-invariant point, the ‘gauge origin’ $\bar{o} = (\bar{o}_x, \bar{o}_y)$, such that the holonomy of A is zero on the x and y cycles of the torus that intersect $\bar{o}$. Note that $\bar{o}$ need not correspond to a lattice site in general. Vector potentials with different values of $\bar{o}$ are not gauge equivalent, and will be distinguished in the treatment below.

To measure $\mathcal{P}_{0,y}$ in our numerics, we insert a total of m flux quanta on the torus using a Landau-like gauge along y . One form of this gauge, shown in Fig. 4.8, is

$$\begin{aligned} A_{j,j+\hat{x}} &= -\frac{2\pi(y - \bar{o}_y)m}{L_y} \delta_{j_x, L_x-1} \\ A_{j,j+\hat{y}} &= \frac{2\pi(x - \bar{o}_x)m}{L_x L_y} \end{aligned} \quad (4.56)$$

$$j = (j_x, j_y), j_x \in \{0, \dots, L_x - 1\}, j_y \in \{0, \dots, L_y - 1\}.$$

Here, $m \equiv \frac{\phi}{2\pi} L_x L_y$ is the total number of flux quanta through the torus. $\bar{o}$ can be determined from $\bar{o}$, as we explain in the subsequent section.

To measure $\mathcal{P}_{0,x}$, we define a similar gauge, but instead along x .

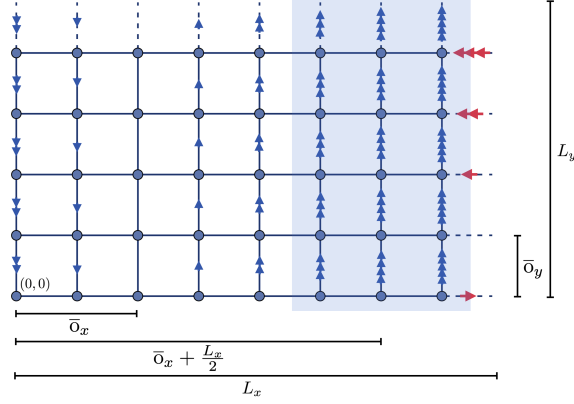


Figure 4.8: Gauge choices defined in Eq. (4.56) on an $L_x \times L_y$ torus. Each blue arrow represents a vector potential $A_{ij} = \frac{2\pi}{L_x L_y}$. Each red arrow represents a vector potential $A_{ij} = \frac{2\pi}{L_y}$. $\bar{o}_x$ and $\bar{o}_y$ mark the distance between $(0, 0)$ and the cycle with trivial holonomy in the x and y directions respectively. The blue region is the partial translation region D which is centered around $\bar{o}_x + \frac{L_x}{2}$.

4.4.3.2 Approximate translation operator

Next we define the operator

$$\tilde{T}_{\mathbf{y}} := \hat{T}_{\mathbf{y}} e^{i \sum_j \lambda_j c_j^\dagger c_j}. \quad (4.57)$$

λ_j is a function of m . If $\tilde{T}_{\mathbf{y}}$ were an exact symmetry, there would be a set $\{\lambda_j\}$ satisfying

$$A_{i+\hat{y}, j+\hat{y}} = A_{ij} - \lambda_i + \lambda_j \quad (4.58)$$

everywhere on the torus. But as mentioned above it is not possible to have exact symmetry when $m/L_y \notin \mathbb{Z}$. Thus we can at best ensure that this relation holds everywhere except for a small strip that we require to be centered at a fixed x . Then the freedom in choosing λ is completely fixed up to (i) an overall shift $\lambda_j \rightarrow \lambda_j + \chi$ for each j , (ii) the geometry of the strip, including the position of its center and its thickness, and (iii) the choice of λ for points that lie within the strip

(if any). We will discuss these further below.

In our numerics we use the gauge

$$\tilde{T}_{\mathbf{y}} = \begin{cases} \hat{T}_{\mathbf{y}} e^{i \sum_j - \frac{\pi m}{L_y} c_j^\dagger c_j} & j_x < \bar{o}_x \\ \hat{T}_{\mathbf{y}} e^{i \sum_j \frac{\pi m}{L_y} c_j^\dagger c_j} & j_x > \bar{o}_x \\ \hat{T}_{\mathbf{y}} & j_x = \bar{o}_x. \end{cases} \quad (4.59)$$

Here j_x is the x coordinate of the site j . For this $\tilde{T}_{\mathbf{y}}$, Eq. (4.58) is violated only on the horizontal links that touch the line $j_x = \bar{o}_x$.

To measure $\mathcal{P}_{o,y}$, we need to express o in terms of gauge-invariant properties of A or λ . If we fix A, λ as above, we can define o as follows. First we find the flux ϕ_{irreg} in a dislocation plaquette created using $\tilde{T}_{\mathbf{y}}$. Recall from Sec. 4.4.1 that ϕ_{irreg} depends on A and λ as well as the position of the dislocation, which is fixed by a point j_0 ; and if we shift j_0 by an integer vector, ϕ_{irreg} can change by multiples of ϕ . For the above choices of A and λ , we thus define o so as to satisfy the following relation, with $j_{0,x} \in \mathbb{Z}$:

$$\phi_{\text{irreg}} = \phi n_{\text{irreg},o} \mod \phi, \quad (4.60)$$

where $n_{\text{irreg},o}$ was computed in Sec. 4.4.1. In the language of that section, our choice of o ensures that the excess flux $\delta\Phi_{W,o}$ in the dislocation plaquette constructed using $\tilde{T}_{\mathbf{y}}$ is zero (mod ϕ). Note that for an arbitrary choice of ϕ_{irreg} , there may be no solution corresponding to any high symmetry

point o. But in App. C.3 we prove that for our choice of A, λ , Eq. 4.60 can be solved and implies

$$o_x = \bar{o}_x + \frac{L_x}{2} + \frac{1}{2} \mod 1. \quad (4.61)$$

Having fixed λ_j , we define the linear momentum $p_{\lambda,y}$ as the expectation value of $\tilde{T}_y$ in the ground state $|\Psi\rangle$ with m total flux quanta:

$$\langle \Psi | \tilde{T}_y | \Psi \rangle := e^{-\gamma + i2\pi p_{\lambda,y}}. \quad (4.62)$$

$e^{-\gamma}$ is the amplitude of the expectation value. Empirically we find that

$$p_{\lambda,y} = -\mathcal{P}_{o,y}m + K_y \mod 1 \quad (4.63)$$

The term linear in m is predicted by the response theory; K_y is piecewise constant in m in our numerics, and can change only when the amplitude vanishes. For the bare numerical calculation, see Fig.C.6.

Using the above choice of λ in the Hofstadter model, we find that the amplitude $e^{-\gamma}$ oscillates with m whenever $C \neq 0$, and for a discrete set of m values it vanishes.⁴ For other values of m , the magnitude is non-vanishing and the linear momentum is well-defined. As a result, for a fixed L_x , we can obtain the expected value of $\mathcal{P}_{o,y}$ throughout the Hofstadter butterfly except for a finite set of ϕ values where $\mathcal{P}_{o,y}$ is not determined. But for these ϕ values, we can pick some other L_x and then extract $\mathcal{P}_{o,y}$. The values of $\mathcal{P}_{o,y}$ are quantized and agree with results

⁴For example, when L_y is even, these are given by $m = \frac{L_y}{2} + kL_y$, where $k \in \mathbb{Z}$. This verifies our earlier comment that $\tilde{T}_y$ is a worse approximate symmetry when $|m| \mod L_y$ is large.

from the dislocation charge measurement.

Now we discuss alternative choices of λ . First there is the freedom in shifting $\lambda_j \rightarrow \lambda_j + \chi$ for each j . As stated in Sec. 4.4.1, taking $\chi = \phi v_x = \frac{2\pi m v_x}{L_x L_y}$ changes the flux ϕ_{irreg} assigned to an irregular unit cell at a dislocation by $-\phi v_x$ (where $v_x \in \mathbb{R}$). If we continue to define $\mathbf{o}$ using Eq. (4.60), then when v_x is an arbitrary real number there is no solution for $\mathbf{o}$. But if v_x is quantized to 0 or $1/2 \bmod 1$, we see that $\mathbf{o}$ must change (in fact, we show in App. C.3 that $\mathbf{o}_x \rightarrow \mathbf{o}_x - v_x$). Hence, only discrete sets of global $U(1)$ transformations are allowed, and the effect of such transformations is simply to change the value of $\mathbf{o}$.

There is additional freedom in choosing the strip that violates translation symmetry. The location of the center of the strip can in principle be shifted by $\delta \in \mathbb{R}$ if we define λ_j as in Eq. (C.39) (see Appendix C.5). In the Hofstadter model, we find empirically that in order to get any quantized result for $\mathcal{P}_{\mathbf{o},y}$ throughout the butterfly, δ must be fixed so that the center of the strip coincides with $\bar{\mathbf{o}}_x$, as in Eq. (4.59), i.e. $\delta = 0$. Remarkably, the quantized $\vec{\mathcal{P}}_{\mathbf{o}}$ extracted from the above linear momentum calculation agrees fully with the result from dislocation charge calculations.

When $C = 0$ in this model, taking $\delta \neq 0$ is equivalent to taking $\lambda_j \rightarrow \lambda_j + \chi$ for some χ .⁵ In this case any choice of δ will give a quantized result $\mathcal{P}_{\mathbf{o}',y}$, but for a high symmetry point $\mathbf{o}'$ that depends on δ . In App. C.5 we discuss an alternative way to understand the choice of δ using gauge-invariant quantities.

Empirically we find that the thickness of the strip does not matter, as long as λ_j varies linearly in j_x within the strip (otherwise we do not obtain quantized results throughout the

⁵A quick argument is that in this limit, $|\Psi(m)\rangle$ is an almost exact eigenstate of $\hat{n}_j$ with eigenvalue κ , for each j . Thus $\hat{T}_{\mathbf{y}} = \hat{T}_{\mathbf{y}} e^{i\kappa \sum_j \lambda_j}$. Therefore any change in λ_j (e.g. by redefining δ) is equivalent to shifting $\lambda_j \rightarrow \lambda_j + \chi$ for each j with some suitable χ .

butterfly for $\mathcal{P}_{o,y}$). We can pick the parameters $\{L_y, \bar{o}_y\}$ arbitrarily, and this does not affect the result for $\mathcal{P}_{o,y}$. In the limit of zero strip thickness, Eq. (4.59) is the only choice of λ that we have found that gives quantized results throughout the butterfly.

4.4.3.3 Partial translations

We can also use a partial translation operator $\tilde{T}_y|_D := \hat{T}_y|_D e^{i\sum_{j \in D} \lambda_j c_j^\dagger c_j}$ to extract the same linear momentum. To get quantized results for $\vec{\mathcal{P}}_o$, we empirically find that D needs to be mirror symmetric about some cycle ℓ along which the holonomy $e^{i\sum_{j \in \ell} A_{j,j+\hat{y}}}$ is real valued.

Here we define the linear momentum of a state with m flux quanta as

$$\langle \Psi | \tilde{T}_y|_D | \Psi \rangle := e^{-\gamma_D + i2\pi p_{\lambda,y}}. \quad (4.64)$$

Numerically, we obtain $\overline{\mathcal{P}}_o$ through the equation

$$p_{\lambda,y} = -\mathcal{P}_{o,y}m + K_y \pmod{2}. \quad (4.65)$$

Here K_y is some constant which only depends on the Chern number and the system size.

We find that there are only two choices of the parameters $\{D, \bar{o}, \lambda\}$ which give a quantized $\mathcal{P}_{o,y}$ throughout the butterfly. In one case,

1. D is centered at the cycle with $x = \bar{o}_x$ (trivial holonomy)
2. $\bar{o}$ is arbitrary
3. $\tilde{T}_y|_D := \hat{T}_y|_D$, i.e. $\lambda_j \equiv 0$ for $j \in D$.

However, empirically these choices give $\mathcal{P}_{0,y} = 0$ everywhere in the butterfly, which does not agree with the results obtained from other methods.

In the second case, we choose

1. D centered at the cycle with $x = \bar{o}_x - \frac{L_x}{2}$ (holonomy $(-1)^m$), for each m
2. $\bar{o} = o + \frac{L_x}{2} + (\frac{1}{2}, \frac{1}{2})$
3. $\tilde{T}_y|_D$ is defined as

$$\tilde{T}_y|_D := \begin{cases} \hat{T}_y|_D e^{i \sum_j - \frac{\pi m}{L_y} c_j^\dagger c_j} & j_x \leq \bar{o}_x \\ \hat{T}_y|_D e^{i \sum_j \frac{\pi m}{L_y} c_j^\dagger c_j} & j_x > \bar{o}_x. \end{cases} \quad (4.66)$$

This choice of $\lambda, \bar{o}, D$ is the only one we have found that gives a quantized but non-trivial result for $\mathcal{P}_{0,y}$ throughout the butterfly. Furthermore, remarkably this agrees with the other methods used in this section (dislocation charge, linear momentum from full translations, and the 1d polarization discussed below). We motivate this choice further in App. C.5.

4.4.4 $\vec{\mathcal{P}}_0$ from 1d polarization

It is natural to ask what the invariant $\vec{\mathcal{P}}_0$ as defined above has to do with other traditional many-body definitions of the polarization. In this section we provide one concrete answer: we consider a torus and compute the 1d polarization along x as defined by Resta [156], treating L_y as a parameter. We show that in addition to the usual term proportional to C , there is a term $L_y \mathcal{P}_{0,y}$ which fixes the dependence of the 1d polarization on the dimensionally reduced coordinate.

Consider the 1d polarization in the x direction, denoted $\mathcal{P}_x$. It is defined by the following

action:

$$\mathcal{L}_{1d} = -\mathcal{P}_x \int dx dt E_x, \quad (4.67)$$

where $E_x = \partial_t A_x - \partial_x A_t$ is the x component of the electric field. In this section we assume that A denotes the entire vector potential and not just its deviation from some background. The sign in Eq. (4.67) is chosen so that the 1d current j satisfies $j = \frac{\delta \mathcal{L}_{1d}}{\delta A_t} = \partial_t \mathcal{P}_x$.

We now use the (2+1)D field theory to make a prediction for $\mathcal{P}_x$ if the original 2d system is dimensionally reduced to an effective 1d system in the x direction. We set the rotation gauge field ω to zero for this calculation. First consider the term with C :

$$\begin{aligned} & \frac{C}{4\pi} \int dx dy dt A \wedge dA \\ &= \frac{C}{4\pi} \int dx dy dt (A_t B - A_x E_y + A_y E_x). \end{aligned} \quad (4.68)$$

Here E_x, E_y, B are the full electric and magnetic fields respectively. Note that the Chern-Simons term is usually written in terms of the deviation of the vector potential from some background, but we have instead used the full vector potential. This rewriting is motivated by the topological field theory, Eq. (3.20), and will be justified by the empirical results we show below.

If $A_y \neq 0$ and E_x is independent of y , then we can rewrite this term as

$$\frac{C}{2\pi} \int dx dt \left(\oint A_y(x) dy \right) E_x + \dots \quad (4.69)$$

This step involves an integration by parts, which contributes a factor of 2. Since the system on the torus is not exactly translationally symmetric, the holonomy $\Phi_y(x) = \oint A_y(x) dy$ has an x -dependence. In order to extract a spatially averaged polarization, which is what we calculate

microscopically using Resta's formula below, we assume E_x is a constant. Then the above term becomes

$$\frac{C}{2\pi} \Phi_{y,\text{avg}} \int dx dt E_x + \dots \quad (4.70)$$

Here, $\Phi_{y,\text{avg}} := \frac{1}{L_x} \int dx dy A_y$ is the average of the global holonomies in the y direction. Thus the Chern number contributes to the 1d polarization. The value of $\Phi_{y,\text{avg}}$ depends on the range of integration for x . If we assume $\mathcal{O}_x \leq x \leq L_x + \mathcal{O}_x$ for some $\mathcal{O}_x$ and $\Phi_y(x)$ is continuous in this range, we can show that

$$\Phi_{y,\text{avg}} = \phi L_y \left(\frac{L_x}{2} - \bar{\phi}_x + \mathcal{O}_x \right) \mod 2\pi. \quad (4.71)$$

Now notice that there is also a contribution from the field theory term $Y \wedge dA$:

$$\begin{aligned} & \frac{\mathcal{P}_{\phi,y}}{2\pi} \int dx dy dt Y \wedge dA \\ &= \frac{\mathcal{P}_{\phi,y}}{2\pi} \int dx dy dt (Y_t B - Y_x E_y + Y_y E_x) \\ &= \frac{\mathcal{P}_{\phi,y}}{2\pi} \oint dy Y_y \int dx dt E_x + \dots \\ &= \mathcal{P}_{\phi,y} L_y \int dx dt E_x + \dots \end{aligned} \quad (4.72)$$

Note that due to a normalization convention, $\oint dy Y_y = 2\pi L_y$. The remaining terms in the field theory do not contribute. Thus we naively expect that the 1d polarization should be equal to

$$-\mathcal{P}_{\mathcal{O},x} = \frac{C\phi}{2\pi} L_y \left(\frac{L_x}{2} - \bar{\phi}_x + \mathcal{O}_x \right) + L_y \mathcal{P}_{\phi,y} + K' \mod 1, \quad (4.73)$$

where we assume K is a constant that is independent of L_x and L_y . We have used the subscript $\mathcal{O}$ in $\mathcal{P}_{\mathcal{O},x}$ to specify the dependence of this quantity on $\mathcal{O}_x$. As we now discuss, to match Eq. (4.73) to a numerical calculation, we demand that $\mathcal{O}_x$ be the origin of coordinates in that calculation.

Numerically we compute the quantity $\mathcal{P}_{\mathcal{O},x}$ using Resta's formula:

$$\mathcal{P}_{\mathcal{O},x} = \frac{1}{2\pi} \arg \langle \Psi | e^{i \frac{2\pi}{L_x} \sum_j j_x c_j^\dagger c_j} | \Psi \rangle, \quad (4.74)$$

where j_x is defined w.r.t. the origin $\mathcal{O} \equiv (\mathcal{O}_x, \mathcal{O}_y)$, i.e. $j_x \in \{-\mathcal{O}_x, 1 - \mathcal{O}_x, 2 - \mathcal{O}_x, \dots, L_x - 1 - \mathcal{O}_x\}$. Here $\mathcal{O}$ refers to the point on the torus where the coordinate is chosen to be $(0, 0)$.

Empirically, we find that $\mathcal{P}_{\mathcal{O},x}$ exactly obeys Eq. (4.73). Let us take a square lattice with $L_x = 20$, $L_y = 12, 13$ and $\mathcal{O}_x = 0, \frac{1}{2}$ as an example. The constant in ϕ contribution $L_y \mathcal{P}_{\mathcal{O},y} + K'$ for the two different L_y is plotted in Fig. 4.9. Empirically we find that the difference of the two gives the desired answer $\mathcal{P}_{\mathcal{O},y}$ when

$$\mathcal{O}_x = \mathcal{O}_x + \frac{L_x}{2} + \frac{1}{2} \mod 1. \quad (4.75)$$

The presence of the $\frac{L_x}{2} + \frac{1}{2}$ term may seem mysterious, but we show why it should be present in the $C = 0$ case below. Consider $\mathcal{P}_{\mathcal{O},x}$ for a square lattice with one orbital per site and at full filling $\nu = 1$. We analytically calculate $\mathcal{P}_{\mathcal{O},x}|_{\nu=1}$ using Resta's formula:

$$\begin{aligned} \mathcal{P}_{\mathcal{O},x}|_{\nu=1} &= \sum_r \frac{j_x}{L_x} \mod 1 \\ &= \frac{L_x(L_x - 1 - 2\mathcal{O}_x)}{2} \frac{L_y}{L_x} \mod 1 \\ &= L_y \left(\frac{L_x}{2} - \frac{1}{2} - \mathcal{O}_x \right) \mod 1. \end{aligned} \quad (4.76)$$

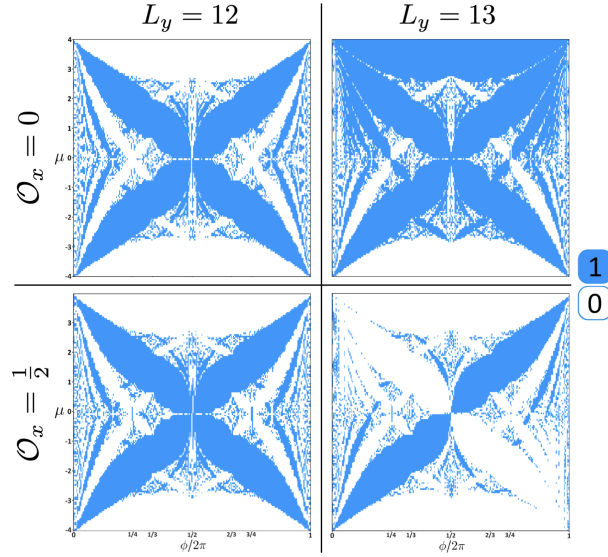


Figure 4.9: Butterflies showing the constant in ϕ contribution $L_y \mathcal{P}_{o,y} + K'$ of the 1d polarization $\mathcal{P}_{O,x}$, for different O_x and L_y . Fixing $L_x = 20$, the result depends only on the parity of L_y . For a fixed O , $\mathcal{P}_{o,y}$ is the difference between the $\phi = 0$ intercept for $L_y = 13$ and $L_y = 12$. O and o satisfy the relation Eq. (4.75).

The coefficient of L_y equals $-\mathcal{P}_{o,y}$. In this special case $\mathcal{P}_{o,y}$ just equals $o_x \pmod 1$ (zero at a site, and $1/2$ at a plaquette center). By equating these two expressions for $\mathcal{P}_{o,y}$, we get Eq. (4.75). We expect that when $C \neq 0$ Eq. (4.75) will continue to hold, and we have verified this expectation numerically.

The relationship between O and o is reminiscent of the linear momentum calculation in which one has to shift the gauge origin $\bar{o}$ by $(\frac{L_x}{2}, \frac{L_y}{2}) + (\frac{1}{2}, \frac{1}{2})$ relative to o in order to obtain $\vec{\mathcal{P}}_{\bar{o}}$. We have numerically checked that Eq. (4.73) gives consistent answers for different gauge choices and for different system sizes.

4.5 Origin dependence of $\mathcal{S}_o, \vec{\mathcal{P}}_o$

In this section we sketch how to derive the formulas in Table 4.1 that express $\mathcal{S}_{o+\vec{v}}, \vec{\mathcal{P}}_{o+\vec{v}}$ in terms of $\mathcal{S}_o, \vec{\mathcal{P}}_o$, and κ . $\vec{v}$ is a quantized vector (perhaps fractional) such that $o, o + \vec{v}$ are

both symmetric under M -fold rotations. These relationships were commented on previously in Secs. 4.3 and 4.4. We leave the technical computations to App. C.6. Background details and definitions regarding the response theory which we use in this section can be found in Chapter 3.

Let A be the full vector potential for the system (meaning that $dA = F$ measures the total magnetic field). The full Lagrangian which involves A is

$$\mathcal{L} = \frac{C}{4\pi} A \wedge dA + \mathcal{L}_{A,o}, \quad (4.77)$$

$$\mathcal{L}_{A,o} := \frac{1}{2\pi} A \wedge \left(\mathcal{S}_o d\omega + \vec{\mathcal{P}}_o \cdot \vec{T} + \kappa A_{XY} \right). \quad (4.78)$$

Here dA is the total U(1) flux. The total U(1) charge and flux in any region should physically be invariant under a shift of origin, and therefore the contribution from the term with C to any response property remains invariant upon shifting the origin. Hence we only consider the transformation of the remaining terms as given by $\mathcal{L}_{A,o}$. Crucially, in $\mathcal{L}_{A,o}$ the coefficient of the term with $\frac{1}{2\pi} A \wedge A_{XY}$ is κ instead of ν .

To understand the transformation of $\mathcal{L}_{A,o}$, it will be most convenient to use the discrete simplicial formulation of the response theory, as developed in Chapter 2; there, the above wedge product should be interpreted as a cup product, but we will stick to wedge product notation below. First note that a space group operation which performs a rotation about o and then a translation takes the form $g = (\mathbf{r}, h)$, where $\mathbf{r} \in \mathbb{Z}^2, h \in \mathbb{Z}_M$. The same group element with respect to the shifted origin $o' = o + \vec{v}$ takes the form

$$g' = (\vec{v}, 0)(\mathbf{r}, h)(-\vec{v}, 0).$$

Since the crystalline gauge fields encode some configuration of group elements on a manifold, a shift of origin by a fractional lattice vector $\vec{v}$ can be viewed as a ‘fractional’ gauge transformation of the crystalline gauge fields. In particular, because this is not a true gauge transformation, the crystalline gauge fluxes also get redefined, leading to a transformation of the response coefficients.

In the simplicial formulation, if the gauge field is flat, the crystalline gauge fluxes are defined by the terms $d\omega, A_{XY}, (1 - U(\frac{2\pi}{M}))^{-1} d\vec{R}$. (Note that in the discrete case we can equivalently write $\vec{T}$ as $d\vec{R}$; we will do this for the rest of the section.)⁶ Under a shift of origin, these fluxes transform into new quantities $d\omega', (1 - U(\frac{2\pi}{M}))^{-1} d\vec{R}', A'_{XY}$. In App. C.6 we show that

$$d\omega' = d\omega \quad (4.79)$$

$$d\vec{R}' = d\vec{R} + \vec{\tau}_M d\omega \quad (4.80)$$

$$A'_{XY} = A_{XY} + \rho_M d\omega + \vec{\mu}_M \cdot d\vec{R}, \quad (4.81)$$

where $\vec{\tau}_M, \rho_M, \vec{\mu}_M$ depend on $\vec{v}$. This transforms the Lagrangian as follows:

$$\mathcal{L}_{A, \mathfrak{o}'} = \frac{1}{2\pi} A \wedge \left(\mathcal{S}_{\mathfrak{o}'} d\omega' + \vec{\mathcal{P}}_{\mathfrak{o}'} \cdot d\vec{R}' + \kappa' A'_{XY} \right). \quad (4.82)$$

Assuming a fixed choice of unit cell, the total charge measured in any given region W should be

⁶These fluxes correspond to a set of 2-cocycle representatives which generate the group $\mathcal{H}^2(G_{\text{space}}, \mathbb{Z})$ when $G_{\text{space}} = \mathbb{Z}^2 \rtimes \mathbb{Z}_M$ [52]. We have $\mathcal{H}^2(G_{\text{space}}, \mathbb{Z}) \cong \mathbb{Z}_M \times \mathbb{Z} \times K_M$. The 2-cocycles corresponding to $d\omega, A_{XY}, (1 - U(\frac{2\pi}{M}))^{-1} d\vec{R}$ generate these three factors respectively.

the same for either choice of origin. Thus

$$\begin{aligned} & \int_W (\mathcal{S}_o d\omega + \vec{\mathcal{P}}_o \cdot d\vec{R} + \kappa A_{XY}) \\ &= \int_W (\mathcal{S}_{o'} d\omega' + \vec{\mathcal{P}}_{o'} \cdot d\vec{R}' + \kappa' A'_{XY}). \end{aligned} \quad (4.83)$$

We now use (4.79),(4.80),(4.81), and compare coefficients. This gives us

$$\kappa = \kappa' \quad (4.84)$$

$$\vec{\mathcal{P}}_o = \vec{\mathcal{P}}_{o' - \vec{v}} = \vec{\mathcal{P}}_{o'} + \kappa' \vec{\mu}_M \quad (4.85)$$

$$\mathcal{S}_o = \mathcal{S}_{o' - \vec{v}} = \mathcal{S}_{o'} + \vec{\mathcal{P}}_{o'} \cdot \vec{\tau}_M + \kappa' \rho_M. \quad (4.86)$$

These equations finally give the results in Table 4.1: the calculation reduces to showing (4.79),(4.80),(4.81), and finding $\rho_M, \vec{\mu}_M, \vec{\tau}_M$ in terms of $\vec{v}$. We have performed these calculations in Appendix C.6; to obtain values for $\rho_{M=2,4}$ which are consistent with results for Chern number 0, we need to make an assumption on the functional form of A_{XY} , which we explain there. The results are contained in Eqs. (C.54) ($\vec{\tau}_M$), (C.68) (ρ_M) and in Sec. C.6.0.3 ($\vec{\mu}_M$).

Note that Eq. (C.54) implies

$$d\vec{R}' = d\vec{R} + M(1 - U(2\pi/M))d\omega. \quad (4.87)$$

This is in fact equivalent to Eq. (4.28), which reads

$$\vec{b}_{o+\vec{v}} = \vec{b}_o + (1 - U(\Omega))\vec{v}. \quad (4.88)$$

The equivalence can be shown as follows. First let us identify $d\vec{R}' = 2\pi\vec{b}_{o+\vec{v}}$, $d\vec{R} = 2\pi\vec{b}_o$, $d\omega = \Omega = \frac{2\pi k}{M}$. Then Eq. (4.87) becomes

$$\vec{b}_{o+\vec{v}} = \vec{b}_o + k(1 - U(2\pi/M))\vec{v}. \quad (4.89)$$

To prove the equivalence, we need to show that the two expressions for $\vec{b}_{o+\vec{v}}$ are in the same equivalence class:

$$(1 - U(2\pi k/M))\vec{v} = k(1 - U(2\pi/M))\vec{v} + (1 - U(2\pi/M))\vec{\Lambda} \quad (4.90)$$

for some $\vec{\Lambda} \in \mathbb{Z}^2$. But this follows from the fact that $(1 - U(2\pi/M))\vec{v}$ is always an integer vector.

As a check on Table 4.1, we can see that the values of $\mathcal{S}_o$ and $\vec{\mathcal{P}}_o$ for zero Chern number given in App. C.2 follow the equations in the Table.

In this derivation we made the crucial assumption that F is invariant under a shift of origin. Note that $F = d\delta A + \frac{\phi}{2\pi}A_{XY}$ where δA is the deviation of the U(1) gauge field from its reference value. F can indeed be made invariant under the transformation of Eq. (4.81), if we take

$$\delta A \rightarrow \delta A - \frac{\phi}{2\pi}(\rho_M\omega + \vec{\mu}_M \cdot \vec{R}). \quad (4.91)$$

Note that $d\delta A$ only changes around a defect, where $d\omega$ and $d\vec{R}$ can be nonzero. The transformations of $d\delta A$, A_{XY} indicate that our conventions for ‘background flux’ and ‘excess flux’ in each plaquette (measured by ϕA_{XY} and $d\delta A$ respectively) change by equal and opposite amounts. But the total flux in each plaquette is invariant.

4.5.1 Transformation of $\vec{\mathcal{P}}_0$

Although $\rho_M, \vec{\tau}_M$ depend sensitively on M , our calculations show that

$$\vec{\mu}_M = (v_y, -v_x),$$

irrespective of the choice of M . This implies that

$$\vec{\mathcal{P}}_{0+\vec{v}} = \vec{\mathcal{P}}_0 + \kappa(-v_y, v_x) \quad (4.92)$$

for $M \in 2, 3, 4, 6$.

The fact that $\vec{\mathcal{P}}_0$ transforms proportionally to κ has an important consequence. If we consider two Hamiltonians 1 and 2, which can for example be the end points of some path in parameter space, we may naively imagine differences of the form $\vec{\mathcal{P}}_0^{(2)} - \vec{\mathcal{P}}_0^{(1)}$ to be completely independent of κ . But for $\vec{\mathcal{P}}_0$ as defined in this work, this is true only if $\kappa_2 = \kappa_1$. Indeed, Eq. (4.92) implies that

$$\vec{\mathcal{P}}_{0+\vec{v}}^{(2)} - \vec{\mathcal{P}}_{0+\vec{v}}^{(1)} = \vec{\mathcal{P}}_0^{(2)} - \vec{\mathcal{P}}_0^{(1)} + (\kappa_2 - \kappa_1)(-v_y, v_x). \quad (4.93)$$

Thus, in order to measure an origin-independent quantity through differences of $\vec{\mathcal{P}}_0$, we must ensure that the initial and final values of κ are equal.

4.5.2 Formula for $n_{\text{irreg}, \mathbf{o} + \vec{v}, \vec{b}}$

An interesting corollary of these results is that they allow us to determine the correct assignments of $n_{\text{irreg}, \mathbf{o}, \vec{b}}$ for a fixed $\vec{b}$, as we vary $\mathbf{o}$. On an infinite plane lattice with only dislocations, we can set $\omega = 0$, and so Eq. (4.81) becomes

$$A'_{XY} = A_{XY} + \vec{\mu}_M \cdot d\vec{R}. \quad (4.94)$$

With $\vec{\mu}_M$ as above, we integrate over a region W surrounding the dislocation, and reduce mod 1 to obtain

$$n_{\text{irreg}, \mathbf{o} + \vec{v}, \vec{b}} = n_{\text{irreg}, \mathbf{o}, \vec{b}} - v_x b_y + v_y b_x \mod 1. \quad (4.95)$$

Once $n_{\text{irreg}, \mathbf{o}, \vec{b}}$ is known for a single origin $\mathbf{o}$, the values of $n_{\text{irreg}, \mathbf{o} + \vec{v}, \vec{b}}$ can be determined using Eq. (4.95). This non-trivial transformation rule is confirmed by our numerical results and by the analytical results at $C = 0$.

4.6 Discussion

In this paper we have described several complementary many-body approaches to measure the quantized charge polarization $\vec{\mathcal{P}}_0$ and the discrete shift $\mathcal{S}_0$ in gapped topological phases, including those with a nonzero Chern number and magnetic field. We extract $\vec{\mathcal{P}}_0$ by studying the fractional charge bound to dislocations, the linear momentum bound to flux, and from an effective 1d polarization response. We have obtained explicit numerical results for the spinless Hofstadter model with C_M rotational symmetry, for $M = 2, 3, 4, 6$, by matching our microscopic

calculations to field theory predictions. We have also obtained a theoretical understanding of the origin dependence of $\vec{\mathcal{P}}_o$ and the discrete shift $\mathcal{S}_o$. Together with the Chern number C and $\kappa \equiv \nu - C\phi/2\pi$, the quadruple $\{C, \mathcal{S}_o, \vec{\mathcal{P}}_o, \kappa\}$ completely specifies the quantized charge response in systems with charge conservation, magnetic translation, and point group rotation symmetry.

An important issue we wish to emphasize is how to understand $\mathcal{S}_o, \vec{\mathcal{P}}_o$ as invariants describing a topological phase. In the case of an origin-independent invariant such as C or κ , knowing that the invariant differs between two systems is enough distinguish them as topological phases. However, in order to distinguish two systems based on their respective values of $\mathcal{S}_o$ or $\vec{\mathcal{P}}_o$, it is essential to first fix a common origin o for both systems, and then compare the various numbers. This is because two systems are in the same phase only if they can be adiabatically connected to each other without closing the gap or breaking the symmetry, *and while keeping their common origin o fixed*. Without fixing this common origin, we cannot meaningfully define the notion of adiabatic equivalence between two systems.

Another important issue is whether the definitions of charge polarization we have proposed in terms of dislocation charge, linear momentum etc. continue to hold when $C \neq 0$, in the absence of rotational symmetry. Note that the discrete rotational symmetry allows us to define $\vec{\mathcal{P}}_o$ in terms of differences in $\mathcal{S}_o$ for different o (see Table 4.1); this is clearly not possible without rotational symmetry. Moreover, in the dislocation charge as well as the linear momentum calculations, the rotational symmetry served as a crucial guide in fixing the free parameters in each case. Therefore when $C \neq 0$ and in the absence of rotational symmetry, it is not clear to what extent the charge polarization is well-defined. In fact it has already been noted that there can be in general multiple inequivalent definitions of the charge polarization [157, 167], although these ambiguities may not be related to the subtleties that arise when $C \neq 0$.

Since $\mathcal{S}_o$ and $\vec{\mathcal{P}}_o$ are topological invariants that depend on crystalline symmetry, their values will cease to be quantized if we break the crystalline symmetry by adding disorder. In particular, if we introduce onsite or bond disorder which breaks the rotational symmetry, $\mathcal{S}_o$ is no longer quantized. If the disorder breaks either the translational or the rotational symmetry, $\vec{\mathcal{P}}_o$ is no longer quantized. Nevertheless, the disorder averages of $\mathcal{S}_o, \vec{\mathcal{P}}_o$ remain quantized to their respective clean values, although their standard deviations increase with an increase in disorder strength. This was shown numerically for $\mathcal{S}_o$ in Ref. [100] and is expected to hold for $\vec{\mathcal{P}}_o$ as well.

Next we comment on some unresolved issues in this work. In both the linear momentum and 1d polarization calculations, we need to pick a distinguished point and relate it to o . For the linear momentum calculation $\bar{o}$ is the gauge origin, and in the 1d polarization calculation $\mathcal{O}$ determines the coordinates for each site in Resta's formula, Eq. (4.26). In both calculations, we extract $\vec{\mathcal{P}}_o$ correctly only if o satisfies a certain relation with $\bar{o}$ or $\mathcal{O}$. In the linear momentum calculation, we also found that if $\tilde{T}_y = \hat{T}_y e^{i \sum_j \lambda_j c_j^\dagger c_j}$, for a fixed $\bar{o}$ there is only one choice of λ that gives a quantized $\mathcal{P}_{o,y}$ throughout the Hofstadter butterfly. These observations are completely empirical; we leave a full explanation for future work.

For our linear momentum calculations we used an approximate translation operator and thus had to work with its expectation values rather than exact eigenvalues. It would be useful to compute the linear momentum exactly by incorporating a flux insertion operator that makes the translation symmetry exact.

Additionally, under a shift of origin, the field theory does not fix a unique transformation rule for $\mathcal{S}_o, \vec{\mathcal{P}}_o$ when $M = 2, 4$; instead it gives a few different possibilities as explained in detail in App. C.6. To fix the transformation consistent with numerics and physical expectations, we need to make some additional choices in the field theory that are allowed but do not have an

obvious physical interpretation.

We close by pointing out some related open questions. Previously, Ref. [95] proposed a way to define the charge polarization as a single-particle Berry phase in momentum space when $C \neq 0$, by picking a suitable origin for the Brillouin zone. It would be useful to understand whether this choice can be related to the ones we have made in defining $\vec{\mathcal{P}}_0$ using the dislocation charge and the linear momentum.

We have not commented on how the invariants $\mathcal{S}_0, \vec{\mathcal{P}}_0$ manifest at corners and edges of the system. The relation between disclination charge and fractional corner charge when $C = 0$ has been discussed in several places, see e.g. Ref. [188]. Refs. [94, 95] specifically discussed an edge charge interpretation of the polarization; Ref. [95] did so in the context of Chern insulators, and Ref. [94] did so from the perspective of a boundary Luttinger theorem when $C = 0$. It would be interesting to understand the corner and edge charges in the context of our results, which apply for general C and in the presence of a magnetic field.

Another interesting direction is to study the charge polarization in *fractional* Chern insulators. In fact, Chapter 2 used field theory to predict that this can indeed be defined systematically and that its quantized fractional values are sensitive to M as well as to the anyon content of the theory. In such topologically ordered phases, the charge polarization can encode a novel, nontrivial form of symmetry fractionalization which was called the ‘discrete torsion vector’ in Chapter 2.

Chapter 5: Complete crystalline topological invariants from partial rotations in (2+1)D invertible fermionic states and Hofstadter's butterfly

This Chapter is based on the paper *Complete crystalline topological invariants from partial rotations in (2+1)D invertible fermionic states and Hofstadter's butterfly* by Yuxuan Zhang, Naren Manjunath, Ryohei Kobayashi, and Maissam Barkeshli, arXiv: 2203.16919.

Notational comments: The notation in this Chapter is the same as in Chapter 3. We use the subscript o to denote the dependence on the origin o in the unit cell. We define the operators $\tilde{C}_{M_o}^\pm$ and use the superscript $\pm$ to denote the dependence of an invariant on these operators. (In Chapter 4, we only considered $\tilde{C}_{M_o}^+$ and its associated invariants, and dropped the superscript.)

5.1 Introduction

There has been intense work over the past decade in developing a comprehensive characterization and classification of topological phases of matter with internal [7, 11, 19–22, 24, 26–28, 50] and crystalline symmetries [8, 52, 53, 64, 70, 71, 73, 77, 78, 80, 82, 197]. Despite much progress, the problem of how to fully characterize such systems, in particular by numerically extracting a complete set of invariants, remains only partially solved [42, 100, 101, 198, 199].

In this paper, we study (2+1)D topological states of fermions with symmetry group $G = \text{U}(1) \times_\phi [\mathbb{Z}^2 \rtimes \mathbb{Z}_M]$, which consists of $\text{U}(1)$ charge conservation, discrete magnetic translation

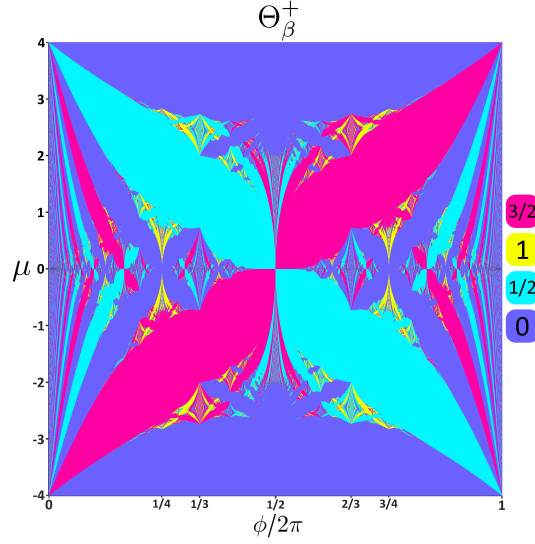


Figure 5.1: $\Theta_{\beta}^{+} \bmod 2$ for Hofstadter model, plotted using an empirical formula Eq. (5.7) and Eq. (D.2) in App. D.1.

and point group rotation symmetries. We focus on invertible fermionic phases, which have a unique ground state on all manifolds [27, 28]. We show how to numerically extract a set of many-body invariants $\{\Theta_{\mathbf{o}}^{\pm}\}$ associated to the crystalline symmetries by computing expectation values of partial rotations centered at high symmetry points $\mathbf{o}$ of the unit cell. For example, if $\mathbf{o}$ is preserved by rotations of order $M_{\mathbf{o}}$ where $M_{\mathbf{o}}$ is even, then for a fixed Chern number, $\Theta_{\mathbf{o}}^{+}$ and $\Theta_{\mathbf{o}}^{-}$ define $\mathbb{Z}_{M_{\mathbf{o}}/2}$ and $\mathbb{Z}_{2M_{\mathbf{o}}}$ invariants respectively, for each $\mathbf{o}$. Our numerical results are in remarkable agreement with analytical calculations from conformal and topological field theory; specifically, the invariants $\{\Theta_{\mathbf{o}}^{\pm}\}$ encode the G -crossed modular S, T matrices of symmetry defects [11, 53].

For square lattice and $U(1)$ charge conservation symmetries, the expected theoretical classification of topological invariants is given by the group $\mathbb{Z}^3 \times \mathbb{Z}_8 \times \mathbb{Z}_2 \times \mathbb{Z}_4^2$ (see Chapter 3). We find that the invariants $\{\Theta_{\mathbf{o}}^{\pm}\}$, together with the Chern number C , the chiral central charge c_{-} and the filling ν , give a *complete* characterization of these invariants. In particular, the discrete shift $\mathcal{S}_{\mathbf{o}}$ and quantized charge polarization $\vec{\mathcal{P}}_{\mathbf{o}}$ studied in Refs. [100, 101] can be obtained from

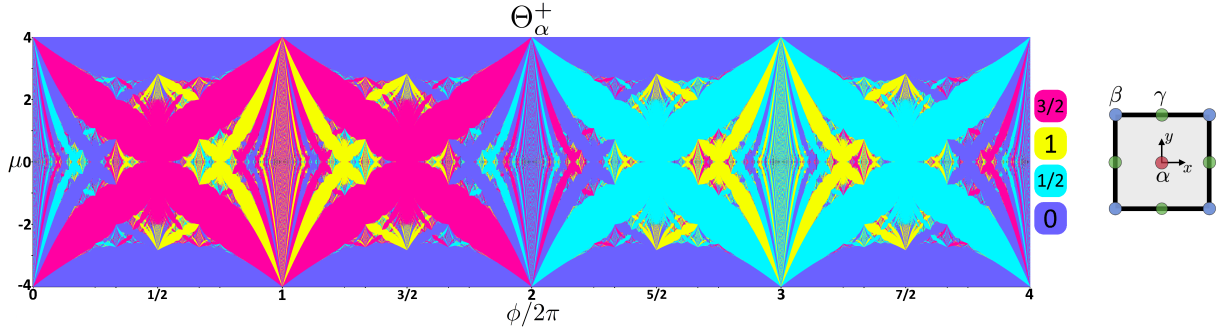


Figure 5.2: $\Theta_{\alpha}^{+} \bmod 2$ for Hofstadter model, where α is a plaquette center, plotted using an empirical formula Eq. (5.7) and Eq. (D.1) in App. D.1. The choice of square lattice unit cell is shown on the right.

$\{\Theta_{\circ}^{\pm}, C, c_{-}, \nu\}$. Our methods to extract $\{\Theta_{\circ}^{\pm}\}$ require only a single bulk ground state on a disk, without needing to insert additional defects, adding to a line of work that extracts topological invariants from single bulk ground states [36, 37, 42–46, 159].

As an application, we study the square lattice Hofstadter model and obtain a number of additional colorings of Hofstadter’s butterfly, extending the colorings by $\mathcal{S}_{\circ}$ and $\vec{\mathcal{P}}_{\circ}$ discovered in [100, 101]. Analogous classification results hold for point group rotations of order 2, 3 and 6, and will be mentioned at the end of the paper.

Partial rotations and partial translations on a cylinder have been studied in prior works and have been shown to yield invariants such as topological spins of anyons, central charges [40, 41, 200], and invariants of symmetry-protected topological (SPT) states [42, 97, 201–203]. In contrast our results apply both in the presence of arbitrary rational magnetic flux, $C \neq 0$, demonstrate a complete set of invariants for crystalline symmetries, and also incorporate several subtleties missed in prior work. These include the proper use of G -crossed modular matrices, the quantization of $\{\Theta_{\circ}^{\pm}\}$, and their dependence on $\mathfrak{o}$.

5.2 Symmetries and model

We study gapped phases of matter with the symmetry group $G = \text{U}(1) \times_{\phi} [\mathbb{Z}^2 \rtimes \mathbb{Z}_M]$, where $\mathbb{Z}^2$ denotes magnetic lattice translations, $\mathbb{Z}_M$ for $M = 2, 3, 4, 6$ denotes point group rotations. We assume $M = 4$ throughout, and briefly comment on $M = 2, 3, 6$ at the end of the chapter.

The symbol $\times_{\phi}$ implies that the magnetic translation operators, generated by $\tilde{T}_{\mathbf{x}}, \tilde{T}_{\mathbf{y}}$, obey the algebra $\tilde{T}_{\mathbf{y}}^{-1} \tilde{T}_{\mathbf{x}}^{-1} \tilde{T}_{\mathbf{y}} \tilde{T}_{\mathbf{x}} = e^{i\phi \hat{N}}$ where $\hat{N}$ is the total fermion number. The tilde indicates that the definition of the operator involves a $\text{U}(1)$ gauge transformation. Note $e^{i\pi \hat{N}} = (-1)^F$, which is fermion parity.

Let $2\pi/M_0$ be the smallest angle of rotation which preserves the rotation center $\mathbf{o}$. The possible high symmetry points $\mathbf{o}$ of the square lattice are shown in Fig. 5.2; they are α, β (plaquette center and vertex respectively, with $M_{\alpha} = M_{\beta} = 4$) and γ (edge center, with $M_{\gamma} = 2$)¹. As reviewed in App. D.1, there is a canonical choice of magnetic point group rotation operators $\tilde{C}_{M_0}^+$ which are centered at $\mathbf{o}$ and satisfy $(\tilde{C}_{M_0}^+)^{M_0} = +1$. We also define another set of operators $\tilde{C}_{M_0}^- := e^{i\frac{\pi}{M_0} \hat{N}} \tilde{C}_{M_0}^+$, and $\tilde{C}_{M_0, \chi}^{\pm} := e^{i\chi \frac{2\pi}{M_0} \hat{N}} \tilde{C}_{M_0}^{\pm}$. Note that $\tilde{C}_{M_0}^- = \tilde{C}_{M_0, 1/2}^+$ and $(\tilde{C}_{M_0}^-)^{M_0} = (-1)^F$.

We study the spinless free fermion Hofstadter model with flux ϕ per unit cell, with Hamiltonian $H = - \sum_{\langle ij \rangle} e^{-iA_{ij}} c_i^{\dagger} c_j + \text{h.c.}$. Our numerical methods and theoretical analysis hold even if we consider further neighbor hopping and/or interaction terms.

5.3 Partial rotation calculation

Consider a state $|\Psi\rangle$, which can be defined on a system with either open or periodic boundary conditions. Let $\tilde{C}_{M_0}^{\pm}|_D$ be the restriction of $\tilde{C}_{M_0}^{\pm}$ to an invariant subregion D centered

¹Note we will abuse language somewhat and use $\mathbf{o}$ both as the rotation center and as a maximal Wyckoff position.

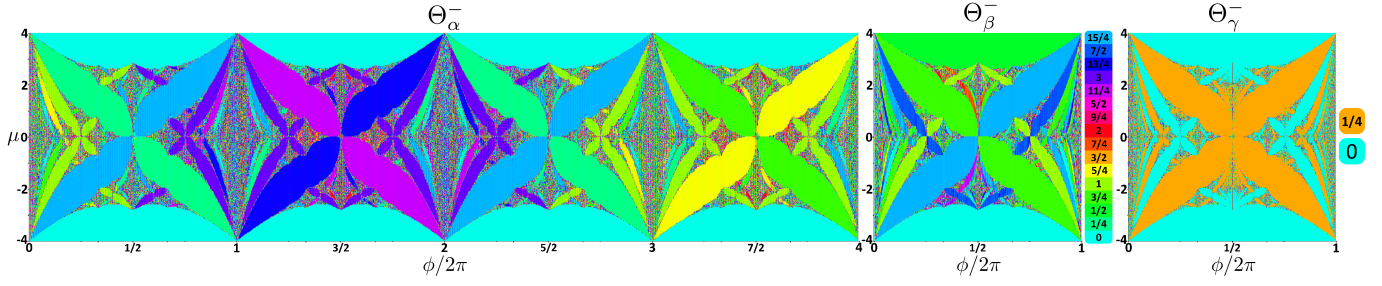


Figure 5.3: Raw data for Θ_o^- . Θ_α^- and Θ_γ^- are calculated on a 30×30 open disc; Θ_β^- is calculated on a 30×30 torus. The diameter of the partial rotation region D is taken to be roughly half the system size. In the $C = \{1, 2, 3\}$ main Landau levels, Θ_β^- quantizes to $\{15/4 \pm 0.003, 1 \pm 0.022, 3/4 \pm 0.051\}$.

at o.

We find that $\langle \Psi | \tilde{C}_{M_o, \chi}^\pm | \Psi \rangle = e^{\frac{2\pi i}{M_o} l_{D, o, \chi}^\pm - \gamma_{D, o, \chi}^\pm}$. $\gamma_{D, o, \chi}^\pm \propto |\partial D|$ sets the real-valued amplitude.

When χ is an integer, we find

$$l_{D, o, \chi}^\pm = \frac{C\chi^2}{2} + \mathcal{S}_o^\pm \chi + K_o^\pm \mod M_o. \quad (5.1)$$

$\mathcal{S}_o^+$ is the discrete shift, which was studied in Chapter 4 and was referred to there as $\mathcal{S}_o$. Here we define $\mathcal{S}_o^-$ analogously, as the term linear in χ in Eq. 5.1. Empirically we observe that $K_o^\pm \mod M_o$ may not be invariant throughout a given lobe in the Hofstadter model: K_o^+ can jump by multiples of $M_o/2$ within a single lobe, but there are no jumps in K_o^- . For an intuitive explanation of these jumps, which are system size dependent, see App. D.1. We thus define

$$\Theta_o^+ := K_o^+ \mod \frac{M_o}{2}, \quad \Theta_o^- := K_o^- \mod M_o. \quad (5.2)$$

The main result of this paper is that $\{\Theta_o^\pm\}$ are topological invariants protected by point group rotational symmetry about o. Θ_o^+ is in general an integer or half-integer, while Θ_o^- is a multiple of $1/4$. For a fixed Chern number, they can take $M_o/2$ and $2M_o$ different values respectively and

thus define $\mathbb{Z}_{M_0/2}$ and $\mathbb{Z}_{2M_0}$ invariants, respectively. We find both empirically and analytically using Eq. 5.10 that $\Theta_\gamma^+ = 0 \pmod{1}$, and so it will henceforth be ignored. When $C = 0$, $\{\Theta_o^\pm\}$ are closely related to the many-body real-space indicators defined in [199].

The values of $\Theta_o^\pm$ are plotted for the square lattice Hofstadter model in Figs. 5.1, 5.2, and 5.3. The values of Θ_o^+ satisfy the empirical relation

$$\Theta_o^+ = k_{3,o}^+ - \frac{C}{2} \pmod{2}, \quad o = \alpha, \beta \quad (5.3)$$

where $k_{3,o}^+$ is an integer. The subscript 3 is used to match the convention in the field theory of Chapter 3. Θ_o^- satisfies

$$\Theta_o^- = \begin{cases} k_{3,o}^- + \frac{k_{s,o}}{2} - \frac{3C}{4} \pmod{4}, & o = \alpha, \beta \\ k_{3,o}^- + \frac{k_{s,o}}{2} - \frac{C}{4} \pmod{2}, & o = \gamma, \end{cases} \quad (5.4)$$

where $k_{s,o} \in \{0, 1\}$ and $k_{3,o}^-$ is an integer. Eqs. (5.3), (5.4) are in excellent agreement with a conformal field theory calculation, which predicts the basic quantization and C dependence of $\Theta_o^\pm$ exactly, but does not give the actual values of $k_{3,o}^\pm$ and $k_{s,o}$ for a particular model. The CFT prediction is summarized by Eq. (5.10).

If we view $\Theta_o^\pm$ in the square lattice Hofstadter model as a function of ϕ , $\Theta_\beta^\pm$ has period 2π while $\Theta_\alpha^\pm$ has period 8π . A similar periodicity depending on origin was observed for the discrete shift in Chapter 4. Mathematically, the lack of periodicity in 2π arises because $\tilde{C}_{M_0}$ is not a gauge-invariant operator, and H is only periodic (without a gauge transformation) under $\phi \rightarrow \phi + 8\pi$ (For details see App. D.1). The lack of 2π periodicity is physically meaningful and

arises because ϕ must be defined as a real number to consistently specify generic perturbations of H .

The continuum limit of the Hofstadter model is obtained by taking $\phi \rightarrow 0^+, \nu \rightarrow 0^+$. In this limit states with Chern number C are equivalent to C filled Landau levels (LLs), and we have

$$\Theta_{\text{o,LL}}^+ = \frac{C^3}{6} - \frac{2C}{3} \mod 2, \quad \text{o} = \alpha, \beta \quad (5.5)$$

$$\Theta_{\text{o,LL}}^- = \begin{cases} \frac{C^3}{6} + \frac{C^2}{4} - \frac{2C}{3} \mod 4 & \text{o} = \alpha, \beta \\ \frac{C^3}{6} + \frac{C^2}{4} - \frac{C}{6} \mod 2 & \text{o} = \gamma \end{cases}. \quad (5.6)$$

As explained below, we can derive Eqs. 5.5, 5.6 analytically using results from CFT and TQFT, and also verify them empirically in our numerical calculations. $\Theta_{\text{o}}^\pm$ are closely related to the topological term $\frac{\ell_{s,\text{o}}^\pm}{4\pi} \omega \wedge d\omega$ which appears in the effective response theory of continuum LLs coupled to an $\text{SO}(2)$ spin connection ω [55].

The empirical data for Θ_{o}^+ can be fit to precise formulas throughout the square lattice Hofstadter model, which we use to fully color the butterflies in Fig. 5.1, 5.2. These equations all have the general form

$$\Theta_{\text{o}}^\pm = \Theta_{\text{o,LL}}^\pm + \Theta_{\text{o,diff}}^\pm. \quad (5.7)$$

Eq. (5.7) indicates that we can express any given state with $C \neq 0$ as a stack of C filled LLs (which contribute $\Theta_{\text{o,LL}}^\pm$) and a state with zero Chern number (which contributes $\Theta_{\text{o,diff}}^\pm$). Thus, $\Theta_{\text{o,diff}}^\pm$ specifies how the given state differs from the limit of C filled LLs. The empirical formulas for $\Theta_{\text{o,diff}}^+$ throughout the butterfly are summarized in App. D.1. We have not obtained empirical formulas for $\Theta_{\text{o,diff}}^-$ as these are substantially more complicated.

5.4 Verification using conformal field theory

Eqs. (5.3), (5.4) can be analytically derived by using the cut-and-glue approach established in [39], which describes the entanglement spectrum of the disk subregion in the long wavelength limit by that of the (1+1)D CFT on its edge. That is, the reduced density matrix for the disk subregion D is effectively given by $\rho_D = \rho_{\text{CFT}}$, where ρ_{CFT} denotes the CFT on the edge of the disk. The edge of the disk entangled with the complement subsystem is described by a thermal density matrix of a perturbed edge CFT within a fixed topological sector [38]. The form of the perturbation in the entanglement Hamiltonian is not universal. In the following, we assume that the entanglement Hamiltonian is that of the unperturbed CFT: $\rho_{\text{CFT}} = e^{-\beta H}$, and check the validity of this assumption with our numerics.

In accordance with the crystalline equivalence principle [82,85], the $\tilde{C}_{M_0}^\pm$ rotation symmetry acts as a translation symmetry combined with an internal $\mathbb{Z}_{M_0}$ symmetry of the CFT on the boundary of D . The expectation value of $\tilde{C}_{M_0}^\pm$ for the disk D without any additional flux insertion is hence evaluated in terms of the symmetry generators in the CFT as follows:

$$\begin{aligned} \langle \Psi | \tilde{C}_{M_0}^\pm | \Psi \rangle &= \frac{\text{Tr}[e^{iQ_{M_0} \frac{\pi}{M_0}} e^{i\tilde{P} \frac{L}{M_0}} e^{-\frac{\xi}{v} H}]}{\text{Tr}[e^{-\frac{\xi}{v} H}]} \\ &= e^{-\frac{2\pi i}{24M_0} c_-} \frac{\sum_{a=1,\psi} \chi_a(\frac{i\xi}{L} - \frac{1}{M_0}; [\text{AP}, 0], [\text{AP}, 1])}{\sum_{a=1,\psi} \chi_a(\frac{i\xi}{L}; [\text{AP}, 0], [\text{AP}, 0])} \end{aligned} \quad (5.8)$$

where we introduced the velocity v of the CFT, finite temperature correlation length of the edge theory $\xi := \beta v$, the length of the boundary $L = |\partial D|$, and $\{\text{AP}, \text{P}\}$ denotes the boundary condition with respect to $\mathbb{Z}_2^f$ fermion parity symmetry. The action of the M_0 -fold rotation on the CFT is expressed as the combination $e^{iQ_{M_0} \frac{\pi}{M_0}} e^{i\tilde{P} \frac{L}{M_0}}$, where Q_{M_0} generates an internal $\mathbb{Z}_{2M_0}^f$ (resp. $\mathbb{Z}_{M_0}$)

symmetry when we take the rotation symmetry to be $\tilde{C}_{M_0}^+$ (resp. $\tilde{C}_{M_0}^-$). $\tilde{P}$ is the normalized translation operator

$$\tilde{P} := \frac{1}{v}(H - E_0) = \frac{2\pi}{L} \left[L_0 - \frac{c_-}{24} - \langle L_0 - \frac{c_-}{24} \rangle \right] \quad (5.9)$$

so that $\tilde{P}|\text{vac}\rangle = 0$ on the vacuum state $|\text{vac}\rangle$ of the CFT. $\chi_a(\tau; [s, j], [s', j'])$ with $s, s' \in \{\text{AP}, \text{P}\}$, $j, j' \in \mathbb{Z}_{M_0}$ is the CFT character that corresponds to the partition function on a torus equipped with spin structure and $\mathbb{Z}_{M_0}$ gauge field. As shown in App. D.3, the above CFT characters at high temperature $\xi \ll L$ can be evaluated using the modular S, T matrices of the G -crossed braided fusion category [11, 53] describing the invertible phase. Our calculation uses the precise defect sectors of the G -crossed modular S, T matrices that correspond to the boundary conditions which define the CFT character; the use of G -crossed modularity allows us to readily treat any symmetry action on the state encoded in the classification of fermionic invertible phases in (2+1)D [27, 28], some of whose details are missed in Ref. [42]. The results for even M_0 are to leading order given by

$$\begin{aligned} \langle \Psi | \tilde{C}_{M_0}^+ |_D | \Psi \rangle_{\text{CFT}} &\propto e^{-\frac{2\pi i}{24}(M_0 - \frac{1}{M_0})c_-} \mathcal{I}_{M_0}^+ = e^{\frac{2\pi i}{M_0}\Theta_0^+} \\ \langle \Psi | \tilde{C}_{M_0}^- |_D | \Psi \rangle_{\text{CFT}} &\propto e^{-\frac{2\pi i}{24}(M_0 + \frac{2}{M_0})c_-} \mathcal{I}_{M_0}^- = e^{\frac{2\pi i}{M_0}\Theta_0^-} \end{aligned} \quad (5.10)$$

where

$$\mathcal{I}_{M_0}^\pm := e^{\frac{2\pi i}{M_0} \frac{\ell_{s,0}^\pm}{2}} = e^{\frac{2\pi i}{M_0} \left((1 \mp 1) \frac{k_{s,0}}{4} + (1 \pm 1) \frac{c_-}{16} + k_{3,0}^\pm \right)} \quad (5.11)$$

for integers $k_{3,o}^\pm \in \mathbb{Z}_{M_o}$ and $k_{s,o} \in \mathbb{Z}_2$. Eqs. (5.10) and (5.11) directly give Eqs. (5.3), (5.4). Moreover, we have an equivalence $k_{3,o}^+ \simeq k_{3,o}^+ + M_o/2$ which comes from relabelling symmetry defects in the CFT by fermions, but no such equivalence for $k_{3,o}^-$. Combining Eqs. (5.10) and (5.11) gives

$$\ell_{s,o}^\pm = \begin{cases} \frac{11 \mp 1}{8} C + 2\Theta_o^\pm \pmod{4} & \text{o} = \alpha, \beta \\ \frac{1 \mp 1}{4} C + 2\Theta_o^\pm \pmod{2} & \text{o} = \gamma. \end{cases} \quad (5.12)$$

5.5 Relation to topological action

As we have seen, $\ell_{s,o}^\pm$ appear in the topological action derived previously in Chapters 2, 3, in terms of a U(1) gauge field A , and crystalline gauge fields $(\vec{R}, \omega)$. ω is a background ‘rotation’ gauge field, which is treated as a real field with quantized holonomies. We can define ω as a $\mathbb{Z}_M$ gauge field or a $\mathbb{Z}_{2M}$ gauge field corresponding to the subgroups of G generated by $\tilde{C}_{M_o}^+$ and $\tilde{C}_{M_o}^-$ respectively. We use $+$ and $-$ superscripts for the coefficients that appear in the action in each case. The coefficients with superscript $-$ can be obtained by replacing $A \rightarrow A + \omega/2$ in the action with only $+$ superscripts.

The full action includes a term $\frac{\ell_{s,o}^\pm - c_-/12}{4\pi} \omega \wedge d\omega$, where $\ell_{s,o}^\pm$ is quantized mod M_o , while $-\frac{c_-}{12}$ is a contribution from the framing anomaly [152, 180]. The quantities $\ell_{s,o}^\pm$ determine the invariant of invertible phases protected purely by rotations about o [78], and arise in the CFT computation above through the G -crossed modular S, T matrices. $\ell_{s,o}^\pm$ are directly related to $\Theta_o^\pm$, by Eq. (5.12).

For continuum Landau levels, $\ell_{s,\text{LL}}^\pm$ is an origin- independent $\mathbb{Z}$ invariant associated to

continuous $\text{SO}(2)$ rotational symmetry. Now $\ell_{s,\text{LL}}^+$ is the coefficient of the term $\frac{1}{4\pi}\omega \wedge d\omega$ that arises in the effective action upon integrating out the fermion fields [55], $\mathcal{L} = \frac{1}{4\pi} \sum_{n=1}^C (A + s_n \omega) \wedge d(A + s_n \omega)$, where $s_n = \frac{2n-1}{2}$ is the spin of the fermion in the n th LL; and $\ell_{s,\text{LL}}^-$ is similarly obtained after replacing $A \rightarrow A + \omega/2$. Moreover, in the continuum limit of the Hofstadter model, we have $\ell_{s,0,\text{LL}}^\pm = \ell_{s,\text{LL}}^\pm \bmod M_0$. After simplification (see App. D.4), we obtain

$$\begin{aligned}\ell_{s,0,\text{LL}}^+ &= \frac{C^3}{3} - \frac{C}{12} \bmod M_0 \\ \ell_{s,0,\text{LL}}^- &= \frac{C^3}{3} + \frac{C^2}{2} + \frac{C}{3} \bmod M_0.\end{aligned}\tag{5.13}$$

Combining Eq. (5.13) and Eq. (5.12) provides the analytical derivation of Eqs. (5.5) and (5.6).

5.6 Classification

There are various relationships involving $\{\Theta_0^\pm\}$:

$$2(\Theta_0^- - \Theta_0^+) = \mathcal{S}_0^+ \bmod M_0,\tag{5.14}$$

$$\Theta_\alpha^+ + \Theta_\beta^+ = \Theta_\alpha^- + \Theta_\beta^- + 2\Theta_\gamma^- - \frac{\kappa}{2} + C \bmod 2.\tag{5.15}$$

Eqs. (5.14), (5.15) can be derived in the LL limit (where $\kappa = 0$ but $C \neq 0$) using Eq. (5.5), (5.6) and in the case of $C = 0$ using a real-space Wannier function argument [204]; combining the two results using linearity under stacking gives (5.14), (5.15). Eq. (5.15) implies that the 5 invariants $\{\Theta_\alpha^+, \Theta_\beta^+, \Theta_\alpha^-, \Theta_\beta^-, \Theta_\gamma^-\}$ define 4 independent invariants.

Eqs. (5.3), (5.4) imply that $I_1 := 2\Theta_\alpha^- + C/2$, $I_2 := \Theta_\alpha^+ + C/2$, and $I_3 := 2\Theta_\gamma^- + C/2$

are integers modulo 8, 2, and 4, respectively. Additionally, $I_4 := \Theta_\alpha^- + \Theta_\beta^- + 2\Theta_\gamma^- - \kappa/2$ is an integer modulo 4. This can also be established in the LL limit using (5.5), (5.6) and in the case of $C = 0$ using a real-space Wannier function argument, and finally by combining the two using linearity under stacking.

The above results imply that $\{I_1, I_2, I_3, I_4\} \in \mathbb{Z}_8 \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_4$. But note that the overall classification of invertible fermionic phases with $G = \text{U}(1) \times_\phi [\mathbb{Z}^2 \rtimes \mathbb{Z}_4]$ is a group $\mathbb{Z}^3 \times \mathbb{Z}_8 \times \mathbb{Z}_2 \times \mathbb{Z}_4^2$ (see Chapter 3). The three $\mathbb{Z}$ invariants are c_- , C , $\kappa = \nu - \frac{C\phi}{2\pi}$. A subset of this classification which assumes $c_- = 0, C = 0$ was derived in Ref. [80] through a real space construction. The derivation in Chapter 3 also shows that I_1, I_2, I_3, I_4 are independent. In fact, they are related to the manifestly independent invariants in Eq. (3.23) as follows:

$$\begin{aligned} \mathcal{S}_\alpha^+ - \ell_{s,\alpha}^+ - \frac{c_-}{4} &= I_1 - 4I_2 \pmod{8} \\ \frac{1}{2} \left(\ell_{s,0} - \frac{c_-}{4} \right) &= I_2 \pmod{2} \\ 2(\vec{\mathcal{P}}_\alpha - 2\vec{\mathcal{P}}_{s,\alpha}) \cdot (1, 0) &= 2I_4 + \kappa - C - I_1 - I_3 \pmod{4} \\ \nu_s^+ &= I_4 \pmod{4}. \end{aligned} \tag{5.16}$$

From this, we can conclude that $\{c_-, C, \kappa, I_1, I_2, I_3, I_4\}$, or equivalently $\{c_-, C, \kappa, \Theta_0^\pm\}$ fully characterize invertible fermionic states with symmetry group G .

5.7 Discussion

We have obtained the remarkable result that we can extract a *complete* set of invariants (apart from C, c_-, ν) from partial rotations, without inserting any extra magnetic flux or lattice

defects. As a corollary, we can extract the previously studied invariants $\mathcal{S}_0^+, \vec{\mathcal{P}}_0^+$ in terms of $\{\Theta_0^\pm, C, c_-, \nu\}$ as well, using Eq. (5.14) and the relationships between $\mathcal{S}_0^+$ and $\vec{\mathcal{P}}_0^+$ derived in [101]. We also showed how to understand the topological response theory coefficients $\ell_{s,0}^\pm$ in terms of $\Theta_0^\pm$.

We can instead study the eigenvalues of a *global* rotation operator on a torus; this was used in [100, 101] to find $\mathcal{S}_0^+, \vec{\mathcal{P}}_0^+$, and also discussed in Ref. [199]. Some preliminary results for global rotations are shown in App. D.2; it is unclear how much of the classification can be obtained in this way.

It is worth highlighting that we have extracted $\{\Theta_0^\pm\}$ from a single wave function without requiring the insertion of additional magnetic flux or lattice defects. Note that one can also extract from a single wave function the quantities c_- [38, 40, 41, 45], C [42–44, 46], and ν . The above discussion then implies that one can in principle extract all the many-body invariants characterizing an invertible state with $G = \text{U}(1) \times_\phi [\mathbb{Z}^2 \rtimes \mathbb{Z}_M]$ from a single wavefunction.

Above, we focussed on the square lattice, with $M = 4$ and $M_0 = 2$ or 4 . For any orientation preserving space group symmetry, we have the following results [204]. If $M_0 = 2, 4, 6$, Θ_0^+ defines a $\mathbb{Z}_{M_0/2}$ invariant, while Θ_0^- defines a $\mathbb{Z}_{2M_0}$ invariant. (This means that if we know C , Θ_0^+ and Θ_0^- can take $M_0/2$ and M values respectively, although their precise quantization will depend on C .) On the other hand, if $M_0 = 3$, both Θ_0^+ and Θ_0^- define $\mathbb{Z}_{M_0}$ invariants. In either case, $\{\Theta_0^\pm\}$ along with C, ν and c_- , fully characterize the many-body crystalline invariants of systems with symmetry $\text{U}(1) \times_\phi [\mathbb{Z}^2 \rtimes \mathbb{Z}_M]$, and these can be extracted using a single wave function.

Chapter 6: Conclusion and outlook

In this thesis we posed a classification and characterization problem for topological phases with charge conservation and crystalline symmetries in (2+1) dimensions. We addressed the classification problem for bosonic SET phases in Chapter 2 and for invertible fermionic phases in Chapter 3, using methods from topological quantum field theory (TQFT). We partially addressed the characterization problem in Chapter 4, studying the discrete shift and quantized charge polarization, which had not been explored previously in systems with a nonzero Chern number and magnetic field. Finally in Chapter 5, we used the method of partial rotations to obtain a complete characterization of the invariants predicted by TQFT, from a single ground state wave function on a disk.

Throughout this thesis we have advocated a many-body approach to the classification and characterization problem, best illustrated by the fact that we study the Hofstadter model using many-body techniques even though the model itself is noninteracting. We do so in order to have results that are applicable to a wide class of numerical and experimental systems, which may have strong interactions. This approach also ensures that our methods can be generalized very naturally to SET phases such as fractional Chern insulators.

The many-body response properties we study fall into two main categories, as evidenced by the results in Chapters 4 and 5. The first is to insert symmetry defects, such as magnetic fluxes or lattice defects, and measure different symmetry quantum numbers of the resulting ground state.

The second approach is to apply *partial* symmetry operations to the ground state without inserting any defects, and measure the argument of the overlap between the original and the transformed state.

This thesis provides a firm theoretical justification for both approaches and explains their complementary nature in terms of TQFT. Some of the new features of this work include a general algebraic statement of the crystalline equivalence principle, a number of TQFT computations for specific crystalline symmetries of interest, and a detailed discussion of how to match TQFT predictions to numerics. The techniques developed here highlight the fact that TQFT methods are applicable to crystalline systems to a much greater degree than previously thought.

To conclude, we discuss several interesting questions that extend the line of work presented here.

In Chapter 2 we derived a general classification for SET phases with crystalline symmetries, but did not formulate a concrete set of numerical predictions along the lines of Chapters 4, 5. One natural generalization is to apply this classification to specific lattice models of fractional Chern insulators and quantum spin liquids, and characterize them numerically.

In this work, we did not consider many other symmetries of interest, such as reflections/glides, spin rotation, or time reversal. These symmetries can change the quantization of the invariants presented here, and also protect new invariants that we have not studied. Therefore, another major direction is to develop a similar theory for each of the above symmetries, and make progress towards a full theory of crystalline topological states.

We can also consider *gapless* systems (such as massless Dirac fermions) and develop a theory of crystalline topological response in this case. This is a particularly interesting direction because we do not as yet have a firm mathematical basis to understand gapless states, analogous

to the theory of invertible or topologically ordered states.

Finally, in Chapter 1 we discussed how various invertible and topologically ordered states have been realized in moiré, photonic and NISQ systems. This thesis has pointed out many new topological invariants that can provide a deeper characterization of these states. A major direction is to develop protocols by which these invariants can be accessed using available experimental technology. Until very recently, the theory of topological phases of matter ran far in advance of experimental capability; however, the gap is being closed, and we can now reasonably hope that the crystalline topological invariants studied in this thesis will be measured experimentally at some point in the future.

Appendix A: Appendix for Chapter 2

A.1 2d Point group rotation matrices

An important role in the main text was played by the 2×2 rotation matrix $U\left(\frac{2\pi}{M}\right)$, associated with the generator of point group rotations. Due to the presence of a lattice, there is a natural basis in which point group rotation matrices $U\left(\frac{2\pi}{M}\right)$ can be defined. We define the x and y axes to be the lattice vectors, such that for M -fold point group rotations, the x and y axes subtend an angle $\frac{2\pi}{M}$. For $M = 2$, $U\left(\frac{2\pi}{M}\right) = -1$, where 1 here denotes the 2×2 identity matrix. For $M = 3, 4, 6$ an elementary $\frac{2\pi}{M}$ rotation can always be defined to take $x \rightarrow y$. In turn, the existence of a lattice ensures the rotated position of y can be expressed as a linear combination of the original x and y . The result for $U\left(\frac{2\pi}{M}\right)$ is given in Table A.1, along with the matrices $(1 - U\left(\frac{2\pi}{M}\right))^{-1}$ that also arise frequently.

In our calculations we have assumed that the lengths L_x, L_y are defined along these possibly nonorthogonal axes. Moreover, integrals $\int f(x, y) dx dy$ should be carried out with x and y defined by this lattice-specific coordinate system. The advantage of using these coordinates is that we always work with integer vectors and matrices, so the coefficients of the theory are always integers or fractions of integers.

M	2	3	4	6
$U\left(\frac{2\pi}{M}\right)$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$
$(1 - U\left(\frac{2\pi}{M}\right))^{-1}$	$\frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\frac{1}{3} \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$	$\frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$

Table A.1: Elementary rotation matrices $U\left(\frac{2\pi}{M}\right)$ for different M .

A.2 Classification of SETs and reduction of $\mathcal{H}^3(G, \text{U}(1))$

A.2.1 Recovering the $\mathcal{H}^2(G, \mathcal{A})$ and $\mathcal{H}^3(G, \text{U}(1))$ classification

The four generalized charge vectors $\vec{q}, \vec{s}, \vec{t}, \vec{m}$ described in Chapter 2 can all be included independently in the effective action for $G = \text{U}(1) \times G_{\text{space}}$. Therefore the group classification of the generalized charge vectors is $\mathcal{A} \times (\mathcal{A}/M\mathcal{A}) \times (K_M \otimes \mathcal{A}) \times \mathcal{A}$, which equals $\mathcal{H}^2(G, \mathcal{A})$ as expected. For $M = 1$ the correct $\mathcal{A} \times \mathcal{A}$ classification is produced by taking K_1 to be trivial. When the magnetic flux per unit cell is not an integer, the group structure becomes a non-trivial central extension of G_{space} by $\text{U}(1)$ due to the magnetic translation algebra. However, it is found that the group cohomology classification remains the same. See App. B.3.3 for an argument.

The full classification of the allowed SPT terms is given by $\mathbb{Z}^2 \times \mathbb{Z}_M^3 \times K_M^2$, which indeed equals $\mathcal{H}^3(G, \text{U}(1))$, as we derive in Appendix A.4. The classification based on $\mathcal{H}^2(G, \mathcal{A})$ and $\mathcal{H}^3(G, \text{U}(1))$ is summarized in Table 2.1. However not all of these choices give topologically distinct phases of matter [11, 48] as some of them can be trivialized by field redefinitions. The particular redundancies that appear depend sensitively on the choice of K -matrix and the generalized charge vectors [48] as we will describe below. For example, we find that the 1/2 Laughlin topological order on a square lattice ($M = 4$) possesses 2304 distinct symmetry-enriched topologi-

N	Generalized charge vectors	Relabelled SPT parameters $(k'_2, k'_3, k'_{5,i}, k'_7)$
$4N'$	$(q \bmod N, s \bmod 4, \vec{t}, m \bmod N)$	$(k_2 - q, k_3 - \frac{N}{2} - s, k_{5,i} - t_i, k_7 - m)$
$4N' + 2$	$(q \bmod N, s \bmod 2, \vec{t}, m \bmod N)$	$(k_2 - 2q, k_3 - 2s, k_{5,i}, k_7 - 2m)$

Table A.2: The effect of relabellings on the SET classification for the $1/N$ Laughlin state (for N even) with $U(1)$ charge conservation and $p4 = \mathbb{Z}^2 \rtimes \mathbb{Z}_4$ space group symmetry. In the left column, N' is an integer. The charge, spin and area vectors are now integers, while the torsion vector $\vec{t}_i$ is valued in $\mathbb{Z}^2$, and $t_x + t_y \bmod 2$ specifies the symmetry fractionalization. The internal gauge fields can be relabelled so as to leave the partition function invariant but transform the SPT parameters (k_2, k_3, k_5, k_7) to $(k'_2, k'_3, k'_{5,i}, k'_7)$ (shown in the last column). Note that the precise transformation of these coefficients is different for different symmetry fractionalization classes and for different values of N .

cal states when the integer part of the charge filling per unit area (k_6) and the Hall conductivity (k_1) are fixed.

A.2.2 Reduction of $\mathcal{H}^3(G, U(1))$ due to relabellings

Recall that $\mathcal{L}_{frac}$ specifies the symmetry fractionalization class through the choice of the generalized charge vectors, which correspond to the classification $\mathcal{H}^2(G, \mathcal{A})$. $\mathcal{L}_{SPT}$ contains additional terms depending only on the background gauge fields, and is classified by $\mathcal{H}^3(G, U(1))$. The choice of $\mathcal{H}^3(G, U(1))$ representative, which corresponds to changing the coefficients k_i in $\mathcal{L}_{SPT}$, can be understood as stacking (2+1)D SPT states. Physically, the effect of changing the action by a choice of $\mathcal{H}^3(G, U(1))$ is to change the braiding and fusion properties of the symmetry defects [11].

Depending on the choice of symmetry fractionalization class and the precise topological order involved, it is possible that changing the action by a non-trivial choice of $\mathcal{H}^3(G, U(1))$ does not yield a distinct phase of matter. Therefore, keeping the symmetry fractionalization choice fixed, the true classification of distinct symmetry-enriched topological states (SETs) is reduced

Count of SETs for 1/2 Laughlin topological order ($\mathcal{A} = \mathbb{Z}_2$) with $G = \text{U}(1) \times [\mathbb{Z}^2 \rtimes \mathbb{Z}_M]$				
M	$\mathcal{H}^2(G, \mathbb{Z}_2)$	$\mathcal{H}^3(G, \text{U}(1))$	Naive SET count (k_1, k_6 fixed)	Reduced SET count (k_1, k_6 fixed)
2	$\mathbb{Z}_2^5$	$\mathbb{Z}^2 \times \mathbb{Z}_2^7$	4096	800
3	$\mathbb{Z}_2^2$	$\mathbb{Z}^2 \times \mathbb{Z}_3^5$	972	972
4	$\mathbb{Z}_2^4$	$\mathbb{Z}^2 \times \mathbb{Z}_2^2 \times \mathbb{Z}_4^3$	4096	2304
6	$\mathbb{Z}_2^3$	$\mathbb{Z}^2 \times \mathbb{Z}_6^3$	1728	972

Table A.3: Count of SETs for the 1/2 Laughlin topological order with $G = \text{U}(1) \times [\mathbb{Z}^2 \rtimes \mathbb{Z}_M]$. We have fixed k_1 and k_6 , which are the integer parts of the Hall conductivity and the charge filling. See Appendix A.3 for the derivation.

from $\mathcal{H}^3(G, \text{U}(1))$ to a subgroup. In the G-crossed braided tensor category formulation [11], this reduction corresponds to cases where changing the algebraic theory of defects by an element of $\mathcal{H}^3(G, \text{U}(1))$ can be completely accounted for by a relabeling of the symmetry defects.

We can also see this reduction from $\mathcal{H}^3(G, \text{U}(1))$ in the context of our topological effective action. In this context, we see that field redefinitions can be made to absorb the effect of changing the couplings in $\mathcal{L}_{SPT}$ by certain amounts. Since this analysis is heavily dependent on the precise topological order (precise choice of K matrix) involved, here we will focus on some simple examples.

To illustrate the main idea, let us begin by taking $G = \mathbb{Z}_M$, with the symmetry fractionalization class specified by the spin vector $\vec{s}$, and the associated defect class given by $k \in \mathbb{Z}_M$. The $\mathbb{Z}_M$ gauge field ω couples to a as follows:

$$\mathcal{L} = -\frac{1}{4\pi} K_{IJ} a^I \cup da^J + \frac{s_I}{2\pi} a^I \cup d\omega + \frac{k}{2\pi} \omega \cup d\omega \quad (\text{A.1})$$

In this case, there are naively M distinct choices of k , $k = 0, \dots, M-1$, corresponding to $\mathcal{H}^3(\mathbb{Z}_M, \text{U}(1)) = \mathbb{Z}_M$. First, we note that the choice of couplings $(\vec{s}, k)$ has the following

redundancies:

$$(\vec{s} + M\vec{\Lambda}, k) \sim (\vec{s}, k) \sim (\vec{s}, k + M) \quad (\text{A.2})$$

The first equivalence is because $M \frac{\vec{\Lambda}}{2\pi} \cdot \vec{a} \cup d\omega$ is trivial, as explained in the main text. The second equivalence follows from $\mathcal{H}^3(\mathbb{Z}_M, \text{U}(1)) = \mathbb{Z}_M$.

Next, observe that we can rewrite the Lagrangian as

$$\mathcal{L} = -\frac{1}{4\pi} K_{IJ} (a^I + u^I \omega) \cup d(a^J + u^J \omega) + \frac{s_I + K_{IJ} u^J}{2\pi} (a^I + u^I \omega) \cup d\omega + \frac{2k - \vec{u}^T K \vec{u} - 2\vec{s} \cdot \vec{u}}{4\pi} \omega \cup d\omega, \quad (\text{A.3})$$

where $\vec{u} \in \mathbb{Z}^D$. Since a^I is dynamical, the shift $a^I \rightarrow a^I + u^I \omega$ can be trivially absorbed by redefining the integration variables. Note that $a^I + u^I \omega$ still obeys the flux quantization condition since $d\omega$ integrates to $2\pi\mathbb{Z}$ over any 2-cycle.

Therefore, we have the additional equivalence

$$(\vec{s}, k) \sim (\vec{s} + K\vec{u}, k - \frac{\vec{u}^T K \vec{u}}{2} - \vec{s}^T \vec{u}). \quad (\text{A.4})$$

Combining the equivalences in (A.4) and (A.2), we see that whenever $K\vec{u} = M\vec{\Lambda}$, we get

$$(\vec{s}, k) \sim (\vec{s}, k - \frac{\vec{u}^T K u}{2} - \vec{s}^T \vec{u}). \quad (\text{A.5})$$

For a fixed choice of $\vec{s}$, this corresponds in general to a reduction of $\mathcal{H}^3(\mathbb{Z}_M, \text{U}(1))$.

Now we can work out some specific examples. Consider the $1/N$ Laughlin state with N

even, for which $K = N$, and take $M = 2$. Since $s \sim s + M$, there are two possible spin vector classes, given by s odd or s even. Suppose we choose $u = 1$ and $\Lambda = N/2$. Then we have

$$(s, k) \sim (s, (k - N/2 - s) \bmod 2) \quad (\text{A.6})$$

For $s = 1$, then this relabelling will take $k \rightarrow (k - 1 - N/2) \bmod 2$. Hence, if N is a multiple of 4, the SET classes corresponding to $(s, k) = (1, 0)$ and $(1, 1)$ are the same, while the two classes $(s, k) = (0, 0)$ and $(1, 1)$ are distinct. The above result was previously also obtained using the edge physics of Chern-Simons theories in Ref [48]; here we have reproduced their result with the field theory in the bulk.

One can use similar reasoning to obtain the SET classification for general K -matrix states and for general symmetries $G = \text{U}(1) \times G_{\text{space}}$, as explained in Appendix A.3. The summary of the relabeling analysis for $1/N$ bosonic Laughlin topological orders and for the case where $G_{\text{space}} = \mathbb{Z}^2 \rtimes \mathbb{Z}_4$ is summarized in Table A.2. In Appendix A.3 we further do an explicit counting of SET states for the $1/2$ Laughlin topological order, with $G = \text{U}(1) \times G_{\text{space}}$ and considering all 5 orientation-preserving 2d space group symmetries, $G_{\text{space}} = \mathbb{Z}^2 \rtimes \mathbb{Z}_M$ for $M = 1, 2, 3, 4, 6$; these results are shown in Table A.3.

A.3 Count of SETs for the Laughlin state with $G = \text{U}(1) \times G_{\text{space}}$

In Section A.2, we discussed a general procedure to account for redundancies in the $\mathcal{H}^3(G, \text{U}(1))$ classification of SET phases using relabellings of the gauge field, when $G = \mathbb{Z}_M$. Here we will generalize that procedure to $G = \text{U}(1) \times G_{\text{space}}$.

We first recall the effective action written in Eq. (2.30):

$$\begin{aligned}
\mathcal{L} &= -\frac{1}{4\pi} a^I \cup K_{IJ} da^J + \mathcal{L}_{frac} + \mathcal{L}_{SPT} \\
\mathcal{L}_{frac} &= \frac{1}{2\pi} a^I \cup (q_I dA + s_I d\omega + \vec{t}_I \cdot d\vec{\mathcal{R}} + m_I A_{XY}) \\
\mathcal{L}_{SPT} &= \frac{k_1}{2\pi} A \cup dA + \frac{k_2}{2\pi} A \cup d\omega + \frac{k_3}{2\pi} \omega \cup d\omega + \frac{1}{2\pi} A \cup (\vec{k}_4 \cdot d\vec{\mathcal{R}}) + \frac{1}{2\pi} \omega \cup (\vec{k}_5 \cdot d\vec{\mathcal{R}}) \\
&\quad + \left(\frac{k_6}{2\pi} A + \frac{k_7}{2\pi} \omega \right) \cup A_{XY}.
\end{aligned} \tag{A.7}$$

The integer coefficients k_1 through k_7 have the following independent redundancies arising from the group structure of $\mathcal{H}^3(G, \text{U}(1))$ (k_1 and k_6 have no redundancy):

$$\begin{aligned}
k_2 &\sim k_2 + M\lambda_2 \\
k_3 &\sim k_3 + M\lambda_3 \\
\vec{k}_4 &\sim \vec{k}_4 + (1 - U\left(\frac{2\pi}{M}\right))\vec{\lambda}_4 \\
\vec{k}_5 &\sim \vec{k}_5 + (1 - U\left(\frac{2\pi}{M}\right))\vec{\lambda}_5 \\
k_7 &\sim k_7 + M\lambda_7
\end{aligned} \tag{A.8}$$

where $\lambda_i \in \mathbb{Z}$. We also have the following independent equivalence for the anyon $\vec{s}$, introduced in Eq. (A.2):

$$s_I \sim s_I + M\Lambda_{s,I}, \tag{A.9}$$

where $\Lambda_{s,I}$ are integers.

Now there is an additional independent equivalence involving the anyons $\vec{t}_i$. Note that the following term is trivial, and can therefore be added to the effective action with no change to the partition function:

$$\frac{(1 - U(\frac{2\pi}{M}))}{2\pi} \vec{\Lambda}_{t,I} da^I \cup \vec{\mathcal{R}} = \frac{1}{2\pi} \vec{\Lambda}_{t,I} da^I \cup \vec{R}, \quad (\text{A.10})$$

where $\vec{\Lambda}_{t,I}$ are integer vectors. The second expression is always a multiple of 2π for a flat gauge field configuration of a^I , and is therefore trivial. This implies the following equivalence relation, which we have discussed previously:

$$\vec{t}_I \sim \vec{t}_I + (1 - U(\frac{2\pi}{M})) \vec{\Lambda}_{t,I}. \quad (\text{A.11})$$

Now, the most general relabelling of the gauge fields a^I which preserves the flux quantization condition $\int da^I \in 2\pi\mathbb{Z}$, includes the gauge field $\vec{\mathcal{R}}$ as well as the gauge field ω :

$$a^I \rightarrow a^I + u^I \omega + \vec{v}^I \cdot \vec{\mathcal{R}}, \quad (\text{A.12})$$

where $u^I, \vec{v}^I$ are all integers. For a flat background gauge field configuration, both $d\omega$ and $d\vec{\mathcal{R}}$ are multiples of 2π , and therefore this relabelling does not affect the flux quantization condition.

We can now repeat the procedure adopted in Section A.2. First we relabel the fields a^I as indicated above. Then we find the constraints on $u^I, \vec{v}^I$ such that s_I and $\vec{t}_I$ can be shifted back to their original values by the addition of trivial terms. Finally, we compute the change in the

coefficients k_1 through k_7 that is required in order to leave the effective action invariant after this relabelling.

The result is the following: s_I and $\vec{t}_I$ can be shifted back to their original values when

$$\begin{aligned} Ku^I &= M\Lambda_{s,I} \\ K\vec{v}^I &= (1 - U\left(\frac{2\pi}{M}\right))\vec{\Lambda}_{t,I}. \end{aligned} \tag{A.13}$$

For such relabellings, the SPT coefficients change simultaneously, in the following way:

$$\begin{aligned} k_2 &\rightarrow k_2 - q^I u_I \\ k_3 &\rightarrow k_3 - \frac{u^I K_{IJ} u^J}{2} - s^I u_I \\ \vec{k}_4 &\rightarrow \vec{k}_4 - \vec{v}^I q_I \\ \vec{k}_5 &\rightarrow \vec{k}_5 - \vec{v}^I s_I - u^I \vec{t}_I - u^I K_{IJ} \vec{v}^J \\ k_7 &\rightarrow k_7 - m^I u_I \end{aligned} \tag{A.14}$$

Note that all coefficients except k_1 and k_6 can be transformed in principle by these relabellings.

We will now use this result to perform some specific computations. Consider an example with the $1/N$ Laughlin state (with N even) and $U(1) \times p4 = U(1) \times [\mathbb{Z}^2 \rtimes \mathbb{Z}_4]$ symmetry. In this case we have $M = 4$ and $K = N$. The integer s is defined modulo $\gcd(4, N)$, which is either 2 or 4, since we consider N to be even; the integers q, m are defined modulo N . The equivalence class of the torsion vector $\vec{t}$ is given by the value of $t_x + t_y \pmod{2}$. The condition on the relabelling

indices $u, \vec{v}$, Eq. (A.13), now becomes

$$Nu = 4\Lambda_s \quad (\text{A.15})$$

$$N\vec{v} = (1 - U(\frac{\pi}{2}))\vec{\Lambda}_t \quad (\text{A.16})$$

The transformation of SPT coefficients, Eq. (A.14), now reads

$$\begin{aligned} k_2 &\rightarrow k_2 - qu \\ k_3 &\rightarrow k_3 - \frac{Nu^2}{2} - su \\ \vec{k}_4 &\rightarrow \vec{k}_4 - q\vec{v} \\ \vec{k}_5 &\rightarrow \vec{k}_5 - s\vec{v} - u\vec{t} \\ k_7 &\rightarrow k_7 - mu. \end{aligned} \quad (\text{A.17})$$

Note that the SPTs parametrized by $\vec{k}_4, \vec{k}_5$ are nontrivial only if $k_{4,x} + k_{4,y}$ (respectively $k_{5,x} + k_{5,y}$) is odd. Since N must be even, the term $-Nu\vec{v}$ in the transformation of $\vec{k}_5$ is trivial, and has been ignored. With N a multiple of 4, we can without loss of generality take $u = 1$ to satisfy the constraint; however, we must choose v_i to be even. Hence $\vec{v}$ will not be responsible for any nontrivial relabellings.

When N is of the form $4N' + 2$ we must choose $u = 2$, while $\vec{v}$ must be chosen so that $v_x + v_y$ is even. However, this means that $\vec{v}$ still gives a trivial contribution to the relabellings of $\vec{k}_4, \vec{k}_5$, and hence we only need to consider transformations due to u .

The SET equivalences for this example are summarized in Table A.2. Note that in our examples, it is crucial that $\omega, \vec{\mathcal{R}}$ be discrete, so that we can add trivial terms such as $\frac{Ms'_L}{2\pi}a^I \cup d\omega$.

This is not possible for continuous symmetry gauge fields: a term $\frac{q_I}{2\pi}a^I \cup dA$ cannot be trivial on its own for any nonzero integer value of q_I . This means that there is no chain of equivalences relating different elements of $\mathcal{H}^3(\text{U}(1), \text{U}(1))$ while keeping the charge vector $\vec{q}$ fixed. This is consistent with the fact that the different $\text{U}(1)$ SETs with the same charge vector all have different Hall conductivities, and are thus physically distinct states of matter.

In our final example below, we will count the number of distinct SETs associated to the $1/2$ Laughlin state with $\text{U}(1) \times G_{\text{space}}$ symmetry. We will only present the results, which can be derived using the arguments above. In this case, the parameters q, s, t_x, t_y, m can correspond to the identity particle I or to the semion S . The parameters k_1 and k_6 will not be affected by relabellings and will always contribute a factor of $\mathbb{Z} \times \mathbb{Z}$ to the overall SET classification; we assume they are fixed. The remaining SPT parameters $k_2, k_3, \vec{k}_4, \vec{k}_5, k_7$ are classified by the group $\mathbb{Z}_M \times \mathbb{Z}_M \times K_M \times K_M \times \mathbb{Z}_M$. The relabelling equation is now

$$\begin{aligned}
k_2 &\rightarrow k_2 - qu \\
k_3 &\rightarrow k_3 - u^2 - su \\
\vec{k}_4 &\rightarrow \vec{k}_4 - q\vec{v} \\
\vec{k}_5 &\rightarrow \vec{k}_5 - s\vec{v} - u\vec{t} - 2u\vec{v} \\
k_7 &\rightarrow k_7 - mu
\end{aligned} \tag{A.18}$$

where

$$\begin{aligned} 2u &= M\Lambda_s \\ 2\vec{v} &= (1 - U\left(\frac{2\pi}{M}\right))\vec{\Lambda}_t. \end{aligned} \tag{A.19}$$

We analyze $M = 2, 3, 4, 6$ separately below:

1. For $M = 2$, there are 2 ways to choose each of q, s, t_x, t_y, m . We also have $|\mathbb{Z}_2^3 \times K_2^2| = 2^7$.

We can choose u and v arbitrarily. There are now different cases. If $q = s = I$ there is a factor 2 reduction due to u . If exactly one of q, s equals S , there is a factor $2^3 = 8$ reduction due to both u and $\vec{v}$. By varying $\vec{v}$, we see that if $q = S$, all values of $\vec{k}_4$ are trivial, while if $s = S$, all values of $\vec{k}_5$ are trivial. Finally, if $q = s = S$, we have a factor $2^5 = 32$ reduction, and both $\vec{k}_4$ and $\vec{k}_5$ will be trivial.

Therefore we obtain

$$2^7 \left(\frac{8}{2} + \frac{16}{8} + \frac{8}{32} \right) = 800 \tag{A.20}$$

SETs.

2. For $M = 3$ there are 2 choices each for q and m , but all possible choices for s and $\vec{t}$ are trivial. Thus there are 4 symmetry fractionalization classes. We also have $|\mathbb{Z}_3^3 \times K_3^2| = 3^5$. There are no relabellings involving either u or $\vec{v}$, since we must choose both u and $v_x - v_y$ to be multiples of 3. Hence we get $3^5 = 243$ distinct SETs for each symmetry fractionalization class, and $4 \times 243 = 972$ SETs in total.

3. For $M = 4$, there are 2 choices each for q, s, m and $t_x + t_y$, giving 16 choices of charge vectors in total. We also have $|\mathbb{Z}_4^3 \times K_4^2| = 2^8$. First we note that we have to choose $v_x + v_y$ even, implying that $\vec{v}$ is not responsible for any nontrivial equivalences. Therefore we only consider equivalences due to $u = 2$.

We have $2^2 \times 4^3 = 256$ SETs whenever $q = s = m = I$ (there are no relabellings); otherwise we have $2 \times 4^3 = 128$ SETs. This gives $2 \times 256 + 14 \times 128 = 2304$ SETs in total.

4. For $M = 6$, there are 2 choices each for q, s, m , while $\vec{t}$ is anyway trivial, and so we do not have to consider relabellings involving $\vec{v}$. We also have $|\mathbb{Z}_6^3| = 6^3$. The only relabellings come from setting $u = 1$. Now we have $6^3 = 216$ SETs whenever $q = M = I$ and $s = S$ (there are no relabellings in these cases); otherwise we have $6^3/2 = 108$ SETs. This gives $1 \times 216 + 7 \times 108 = 972$ SETs in total.

If we specialize to the $M = 4$ case, we find that there are 2304 distinct SETs, in contrast to the naive estimate of $|\mathcal{H}^2(G, \mathcal{A}) \times \mathcal{H}^3(G, \text{U}(1))| = 4096$. Note that much of the analysis was simplified by our choice of the 1/2 Laughlin topological order. If we consider more complicated topological orders, the analysis will become much more involved.

A.4 Topological terms and group cohomology

The correspondence between the topological effective action and the group cohomology formulation runs deeper than giving the same overall classification. There is a one-to-one correspondence between topological terms in the action involving flat background G gauge fields and cocycles in

group cohomology. In this section we will explain this relationship through concrete calculations.

Let us first summarize the relationship between $\mathcal{H}^3(G, \text{U}(1))$ and the topological terms in $\mathcal{L}_{SPT}$, which correspond to topological effective actions for (2+1)D SPT states. See Ref. [21, 98] for a more detailed discussion. For an overview of simplicial calculus, see Ref. [185].

1. A topological Lagrangian for an SPT involving flat G gauge fields (defined on 1-simplices) can be integrated over a 3-simplex of a triangulation, which gives an action S associated to a single 3-simplex. The resulting e^{iS} , which depends on the values of the flat gauge field defined on the 1-simplices, is thus a 3-cochain of G valued in $\text{U}(1)$, i.e. an element of $C^3(G, \text{U}(1))$.
2. In fact e^{iS} is a 3-cocycle of G valued in $\text{U}(1)$, i.e. an element of $Z^3(G, \text{U}(1))$. The 3-cocycle condition arises by demanding that the theory be independent of the triangulation.
3. Gauge transformations applied to the G gauge fields on a triangulation change the value of e^{iS} by an amount $e^{i \int df}$, which corresponds to a 3-coboundary of G valued in $\text{U}(1)$, or an element of $B^3(G, \text{U}(1))$.

Therefore we see that gauge-inequivalent topological actions for flat G gauge fields fall into equivalence classes determined by the quotient $\mathcal{H}^3(G, \text{U}(1)) := \frac{Z^3(G, \text{U}(1))}{B^3(G, \text{U}(1))}$. It has been shown that this fully characterizes topological gauge theories for gauge group G [98], and also believed to fully characterize (2+1)D SPTs [7, 21]. It is also known to classify the fusion and braiding properties of symmetry defects in (2+1)D SETs once the symmetry fractionalization class has been fixed [11].

Let us now summarize the relationship between $\mathcal{H}^2(G, \mathcal{A})$ and $\mathcal{L}_{\text{frac}}$. Consider the coupling of flat G gauge fields to flat dynamical gauge fields describing the Abelian topological order (we assume that the symmetry does not permute anyons):

1. Consider a single internal $U(1)$ gauge field a . Consider a topological term which is an integer multiple of $\frac{1}{2\pi}a \cup B$, where $B \in 2\pi\mathbb{Z}$ is obtained in terms of the G gauge field and is defined on 2-simplices. Note that $B \in 2\pi\mathbb{Z}$ in order for this term to be invariant under large gauge transformations of a .
2. This action can be thought of as an action for $U(1) \times G$ symmetry. Demanding retriangulation invariance implies that $\frac{1}{2\pi}a \cup B$ must be a 3-cocycle: $\frac{1}{2\pi}d(a \cup B) = \frac{1}{2\pi}(da \cup B + a \cup dB) \in 2\pi\mathbb{Z}$. Since a is flat, $da \in 2\pi\mathbb{Z}$, so we find $dB = 0$.

A G gauge transformation which takes $B \rightarrow B + d\Gamma$, where $\frac{1}{2\pi}\Gamma \in \mathbb{Z}$, changes the Lagrangian by a 2-coboundary of G with $\mathbb{Z}$ coefficients. Therefore the gauge inequivalent actions fall into equivalence classes determined by the quotient $\mathcal{H}^2(G, \mathbb{Z}) := \frac{Z^2(G, \mathbb{Z})}{B^2(G, \mathbb{Z})}$.

When there are D independent internal gauge fields, the coefficient changes from $\mathbb{Z}$ to $\mathbb{Z}^D$.

3. The K -matrix coupling ensures that if B is of the form $K\vec{\Lambda}$ where $\frac{1}{2\pi}\vec{\Lambda}$ is an integer vector, the theory is trivial. This is because the anyon associated to the “symmetry flux” B is trivial. Therefore the correct coefficients which classify physically distinct couplings of the K -matrix theory to the background G gauge field are $\mathbb{Z}^D / K\mathbb{Z}^D \cong \mathcal{A}$. This is in fact the definition of $\mathcal{A}$, the group of anyons. Therefore the classification of such actions is given by $\mathcal{H}^2(G, \mathcal{A})$.

In what follows we describe in more detail the precise relation between the topological terms in the effective action and the group cohomology cocycles for the symmetry group discussed in this paper, $G = \text{U}(1) \times G_{\text{space}}$. Note that the symbol $\mathcal{H}$ refers to cohomology with measurable cochains, also known as Borel cohomology. For $n > 0$, the groups $\mathcal{H}^n(G, \mathbb{Z})$ and $H^n(BG, \mathbb{Z})$ coincide. Here $H(BG, \mathbb{Z})$ refers to the cohomology of the classifying space BG of G .

A.4.1 Cocycle representatives for $G = \text{U}(1)$ and $G = \mathbb{Z}_M$

$G = \text{U}(1)$: In this case we have

$$\mathcal{L}_{\text{frac}} = \frac{1}{2\pi} q_I a^I \cup dA \quad (\text{A.21})$$

$$\mathcal{L}_{\text{SPT}} = \frac{k}{2\pi} A \cup dA \quad (\text{A.22})$$

where $q_I, k \in \mathbb{Z}$. Here $\vec{q}$ is the charge vector. Define a flat, real-valued gauge field A such that $A_{12} = a$ and $A_{23} = b$. Formally A is the lift of a $\text{U}(1)$ gauge field to $\mathbb{R}$. A corresponding element of $\text{U}(1) = \mathbb{R}/2\pi\mathbb{Z}$ is written as $[a] = a \bmod 2\pi$; therefore $a = [a] + 2\pi n_a$ for some $n_a \in \mathbb{Z}$. Now for the 3-simplex $[0123]$, $\mathcal{L}_{\text{frac}}$ becomes

$$\frac{q_I}{2\pi} a_{01}^I dA_{123} = \frac{q_I}{2\pi} a_{01}^I ([a] + [b] - [a+b] + dn(a, b)) \quad (\text{A.23})$$

where $dn(a, b) = n_a + n_b - n_{a+b}$. The quantity $\frac{q_I}{2\pi} ([a] + [b] - [a+b] + dn(a, b))$ defines an anyon, i.e. an element in $\mathcal{A}$, and thus can be viewed as an $\mathcal{A}$ -valued 2-cocycle, i.e. an element of $Z^2(\text{U}(1), \mathcal{A})$. The quantity $\frac{q_I}{2\pi} dn(a, b)$ is an $\mathcal{A}$ -valued 2-coboundary, i.e. an element

of $B^2(\mathrm{U}(1), \mathcal{A})$. In general, coboundaries correspond to changes of lift. Inequivalent choices of $\vec{q}$ determine inequivalent classes in the cohomology group $\mathcal{H}^2(\mathrm{U}(1), \mathcal{A}) \cong \mathcal{A}$.

A similar analysis can be made for the SPT term $\frac{k}{2\pi} A \cup dA$. In this case, choose a 3-simplex $[0123]$ with $A_{01} = a$, $A_{12} = b$, $A_{23} = c$ (the other values are fixed by the flatness of A). Again, A is formally a lift from $\mathrm{U}(1)$ to $\mathbb{R}$. Then we have

$$\begin{aligned} \frac{k}{2\pi} A \cup dA[0123] &= \frac{k}{2\pi} A_{01} \times dA_{123} \\ &= \frac{k}{2\pi} ([a] + n_a)([b] + [c] - [b + c] + dn(b, c)). \end{aligned} \quad (\text{A.24})$$

When evaluated modulo 2π , the rhs is a 3-cocycle which represents a cohomology class in $\mathcal{H}^3(\mathrm{U}(1), \mathrm{U}(1)) \cong \mathbb{Z}$ identified by k ; the terms which explicitly depend on n arise by choosing alternative lifts. For each choice of charge vector $\vec{q}$, it is possible to add a $\mathbb{Z}$ worth of SPT states. This means that for each symmetry fractionalization class, one can obtain a set of topological phases related to each other by stacking G -SPT states, given by elements of $\mathcal{H}^3(G, \mathrm{U}(1))$.

$G = \mathbb{Z}_M$: Effective SPT actions for $G = \mathbb{Z}_M$ have been related to $\mathbb{Z}_M$ group cocycles in previous work [205]. The action for $G = \mathbb{Z}_M$ is

$$\mathcal{L}_{\text{frac}} = \frac{s^I}{2\pi} a^I \cup d\omega \quad (\text{A.25})$$

$$\mathcal{L}_{\text{SPT}} = \frac{k}{2\pi} \omega \cup d\omega \quad (\text{A.26})$$

Define a flat gauge field $\omega \in \frac{2\pi}{M}\mathbb{Z}$ such that $\omega_{12} = 2\pi a/M$ and $\omega_{23} = 2\pi b/M$ where a, b are integers. Formally ω is a lift from $\mathbb{Z}_M$ to $\frac{2\pi}{M}\mathbb{Z}$. A corresponding element of $\mathbb{Z}_M$ is written as $\frac{2\pi[a]_M}{M} = \frac{2\pi a}{M} \bmod 2\pi$, where we define $[a]_M = a \bmod M$; therefore $a = [a]_M + Mn_a$ for

some $n_a \in \mathbb{Z}$. Now $\mathcal{L}_{frac}$ becomes

$$\begin{aligned} & \frac{s_I}{2\pi} a_{01}^I d\omega_{123} \\ &= \frac{s_I}{M} a_{01}^I ([a]_M + [b]_M - [a+b]_M + Mdn(a, b)). \end{aligned} \quad (\text{A.27})$$

The quantity $\frac{s_I}{M}([a]_M + [b]_M - [a+b]_M + Mdn(a, b))$ is a 2-cocycle in the group $Z^2(\mathbb{Z}_M, \mathcal{A})$. The quantity $s_I dn(a, b)$, which is the difference between two different choices of lifts, is a 2-coboundary in the group $B^2(\mathbb{Z}_M, \mathcal{A})$. Note that the most general coboundary relation implies that shifting s_I by a multiple of M corresponds to changing the lift; therefore $s_I + M\Lambda_I$ for $\Lambda_I \in \mathbb{Z}$ is equivalent to s_I . With these conditions we see that the equivalence classes of $\vec{s}$ are in bijection with cohomology classes $[\mathfrak{w}] \in \mathcal{H}^2(\mathbb{Z}_M, \mathcal{A})$. When $\mathcal{A} = \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_r}$, we simply have $\mathcal{H}^2(\mathbb{Z}_M, \mathcal{A}) = \mathbb{Z}_{(M, n_1)} \times \cdots \times \mathbb{Z}_{(M, n_r)} = \mathcal{A}/M\mathcal{A}$ ($M\mathcal{A}$ is defined as $\{Ma | a \in \mathcal{A}\}$).

Next we analyze $\mathcal{L}_{SPT}$. The Lagrangian integrated on a 3-simplex with $\omega_{01} = a, \omega_{12} = b, \omega_{23} = c$ gives

$$\begin{aligned} & \frac{k}{2\pi} \omega \cup d\omega[0123] \\ &= \frac{2\pi k}{M^2} ([a]_M + Mn_a)([b]_M + [c]_M - [b+c]_M + Mdn(b, c)). \end{aligned} \quad (\text{A.28})$$

Taken modulo 2π , this function is a 3-cocycle in $Z^3(\mathbb{Z}_M, \text{U}(1))$. Choosing k to be a multiple of M results in a 3-coboundary; therefore the classification is $\mathcal{H}^3(\mathbb{Z}_M, \text{U}(1)) \cong \mathbb{Z}_M$. Since the 3-cocycles of $\mathbb{Z}_M$ and $\text{U}(1)$ have a similar form, the resulting SPT terms, which are of the form $A \cup dA$ and $\omega \cup d\omega$, also have the same Chern-Simons structure.

A.4.2 Calculation of $\mathcal{H}^2(G_{\text{space}}, \mathbb{Z})$ and $\mathcal{H}^3(G_{\text{space}}, \text{U}(1))$

The part of the action with terms from the group G_{space} is

$$\mathcal{L}_{\text{frac}} = \frac{s_I}{2\pi} a^I \cup d\omega + \frac{\vec{t}_I}{2\pi} a^I \cup d\vec{R} + \frac{m_I}{2\pi} a^I \cup A_{XY} \quad (\text{A.29})$$

$$\mathcal{L}_{\text{SPT}} = \frac{k_3}{2\pi} \omega \cup d\omega + \frac{\vec{k}_5}{2\pi} \omega \cup d\vec{R} + \frac{k_7}{2\pi} \omega \cup A_{XY} \quad (\text{A.30})$$

Since the group cocycles for G_{space} are less common than those of $\text{U}(1)$ or $\mathbb{Z}_M$, we will first derive them abstractly and then discuss their relationship to the gauge fields $\vec{R}$ and ω . A space group G_{space} can always be written as a group extension of a point group H by the group of translations $\mathbb{Z}^2$, with some action $\theta : H \rightarrow \text{Aut}(\mathbb{Z}^2)$, as summarized by the short exact sequence

$$1 \rightarrow \mathbb{Z}^2 \rightarrow G_{\text{space}} \rightarrow H \rightarrow 1. \quad (\text{A.31})$$

When a group G can be expressed in terms of a direct product extension, we can use the Künneth formula and its associated decomposition to determine its cohomology groups. When H is a rotation point group, the above extension is on the other hand always a semidirect product. If it were possible to apply the Künneth decomposition to the above semidirect product extension for the cohomology of G_{space} with $\mathbb{Z}$ coefficients, we would obtain

$$\mathcal{H}^n(G_{\text{space}}, \mathbb{Z}) = \prod_{k=0}^n \mathcal{H}_{\theta_k}^k(H, \mathcal{H}^{n-k}(\mathbb{Z}^2, \mathbb{Z})). \quad (\text{A.32})$$

This equation will be further explained below; however we first note the following caveats. For a general semidirect product extension, it is not possible to use the Künneth decomposition. The

more general technique that is applicable in this case involves what are referred to as *spectral sequences* (see eg. Refs. [206, 207] for an introduction). When $G = G_{\text{space}}$, however, the cohomology groups $\mathcal{H}^n(G_{\text{space}}, \mathbb{Z})$ can be numerically computed using a program such as GAP, as was done in Ref. [82]. Although we do not show the calculations here, we can apply spectral sequence techniques (specifically, we use the Lyndon-Hochschild-Serre spectral sequence) and compare them to the known numerical results. From this, we can infer that the correct expansion for $\mathcal{H}^n(G_{\text{space}}, \mathbb{Z})$ is indeed given by the Künneth decomposition. Knowing this result, we can finally use the Künneth decomposition again to obtain the cohomology of the group $U(1) \times G_{\text{space}}$.

In (A.32), the action θ_k is not on $\mathbb{Z}^2$ itself, but on the cohomology group $\mathcal{H}^{n-k}(\mathbb{Z}^2, A)$; it is induced by the action θ of H on $\mathbb{Z}^2$, and will be discussed further below.

Let us first study the classification of symmetry fractionalization. It is easiest to first compute $\mathcal{H}^2(G_{\text{space}}, \mathbb{Z})$ and then shift to $\mathcal{A}$ coefficients. Eq. (A.32) gives

$$\begin{aligned} \mathcal{H}^2(G_{\text{space}}, \mathbb{Z}) &\cong \mathcal{H}_{\theta_2}^2(\mathbb{Z}_M, \mathcal{H}^0(\mathbb{Z}^2, \mathbb{Z})) \\ &\times \mathcal{H}_{\theta_1}^1(\mathbb{Z}_M, \mathcal{H}^1(\mathbb{Z}^2, \mathbb{Z})) \times \mathcal{H}_{\theta_0}^0(\mathbb{Z}_M, \mathcal{H}^2(\mathbb{Z}^2, \mathbb{Z})) \end{aligned} \quad (\text{A.33})$$

$$= \mathcal{H}_{\theta_2}^2(\mathbb{Z}_M, \mathbb{Z}) \times \mathcal{H}_{\theta_1}^1(\mathbb{Z}_M, \mathbb{Z} \times \mathbb{Z}) \times \mathcal{H}_{\theta_0}^0(\mathbb{Z}_M, \mathbb{Z}) \quad (\text{A.34})$$

$$= \mathbb{Z}_M \times K_M \times \mathbb{Z}. \quad (\text{A.35})$$

The first line is the Künneth decomposition. In the second line, we substituted the known cohomology groups $\mathcal{H}^k(\mathbb{Z}^2, \mathbb{Z}) = \mathbb{Z}^{\binom{2}{k}}$. The result of evaluating these cohomology groups is shown on the last line; these calculations will be discussed further below.

Using Eq.(A.32), we can next compute

$$\begin{aligned} \mathcal{H}^4(G_{\text{space}}, \mathbb{Z}) &\cong \mathcal{H}_{\theta_4}^4(\mathbb{Z}_M, \mathcal{H}^0(\mathbb{Z}^2, \mathbb{Z})) \\ &\times \mathcal{H}_{\theta_3}^3(\mathbb{Z}_M, \mathcal{H}^1(\mathbb{Z}^2, \mathbb{Z})) \times \mathcal{H}_{\theta_2}^2(\mathbb{Z}_M, \mathcal{H}^2(\mathbb{Z}^2, \mathbb{Z})) \end{aligned} \quad (\text{A.36})$$

$$= \mathcal{H}_{\theta_4}^4(\mathbb{Z}_M, \mathbb{Z}) \times \mathcal{H}_{\theta_3}^3(\mathbb{Z}_M, \mathbb{Z} \times \mathbb{Z}) \times \mathcal{H}_{\theta_2}^2(\mathbb{Z}_M, \mathbb{Z}) \quad (\text{A.37})$$

$$= \mathbb{Z}_M \times K_M \times \mathbb{Z}_M. \quad (\text{A.38})$$

For a finite group or a compact Lie group G , it is a general result that $\mathcal{H}^n(G, \text{U}(1)) \cong \mathcal{H}^{n+1}(G, \mathbb{Z})$ when $n > 0$. Thus we can compute the cohomology groups of $\text{U}(1)$ with both $\mathbb{Z}$ and $\text{U}(1)$ coefficients. However, this statement does not hold for arbitrary G . Verifying this requires additional spectral sequence computations, which we will not show here. However, these computations indeed reveal that

$$\mathcal{H}^3(G_{\text{space}}, \text{U}(1)) \cong \mathcal{H}^4(G_{\text{space}}, \mathbb{Z}) \quad (\text{A.39})$$

for $G_{\text{space}} = \mathbb{Z}^2 \rtimes \mathbb{Z}_M$ (we in fact expect this relation to hold for a general 2D space group, although we have not done the more general computation). Finally, we use the Künneth decomposition

for the direct product $U(1) \times G_{\text{space}}$ and use the above results to obtain ¹

$$\begin{aligned} \mathcal{H}^3(U(1) \times G_{\text{space}}, U(1)) &\cong \mathcal{H}^3(U(1), U(1)) \\ &\times \mathcal{H}^2(G_{\text{space}}, \mathbb{Z}) \times \mathcal{H}^3(G_{\text{space}}, U(1)) \end{aligned} \quad (\text{A.40})$$

$$= \mathbb{Z}^2 \times Z_M^3 \times K_M^2. \quad (\text{A.41})$$

Note that since $U(1)$ is a continuous group, and we are working with measurable (Borel) cohomology, it is difficult to compute its cohomology groups directly, and in doing so we must rely on technical mathematical results. To provide some additional intuition about the cohomology of $U(1)$ using results on finite groups, Ref. [21] computed the cohomology groups of $\mathbb{Z}_n$ and showed how they were related to those of $U(1)$ upon taking an appropriate limit where $n \rightarrow \infty$. In a similar spirit, we can compute the cohomology of $G = \mathbb{Z}_n \times G_{\text{space}}$ for an arbitrary integer n using the Künneth decomposition, and obtain

$$\begin{aligned} \mathcal{H}^3(\mathbb{Z}_n \times G_{\text{space}}, U(1)) &\cong \mathcal{H}^3(\mathbb{Z}_n, U(1)) \\ &\times \mathcal{H}^2(G_{\text{space}}, \mathbb{Z}_n) \times \mathcal{H}^3(G_{\text{space}}, U(1)) \end{aligned} \quad (\text{A.42})$$

$$= \mathbb{Z}_n^2 \times Z_M^2 \times K_M \times \mathbb{Z}_{(M,n)} \times (K_M \otimes \mathbb{Z}_n). \quad (\text{A.43})$$

Thus we obtain a result which bears a significant resemblance to the claimed result for $U(1) \times G_{\text{space}}$: the difference is that some groups in the above classification depend on the commensuration between n and M . If we choose n to be a multiple of M , and take $n \rightarrow \infty$, so that the initial factors of $\mathbb{Z}_n$ are replaced by $\mathbb{Z}$, we recover the result for the group $U(1) \times G_{\text{space}}$.

¹This version of the Kunneth decomposition assumes that the coefficients have the discrete topology; we have assumed here that we can also use $U(1)$ coefficients in the formula.

A.4.3 Cocycle representatives for $\mathcal{H}^2(G_{\text{space}}, \mathbb{Z})$

Note that all 2-cocycles must satisfy the following condition:

$$f_2(g_1, g_2) + f_2(g_1 g_2, g_3) = f_2(g_2, g_3) + f_2(g_1, g_2 g_3) \quad (\text{A.44})$$

where, if $g_i = (\vec{r}_i, h_i)$, then $g_1 g_2 = (\vec{r}_1 + U(h_1)\vec{r}_2, h_1 + h_2)$. In what follows, we assume that the translation gauge field $\vec{R}$ is valued in $2\pi\mathbb{Z}^2$, while the $\mathbb{Z}^2$ group elements $\vec{r}_i$ are assumed to be integer-valued. Cocycles in $Z^d(G_{\text{space}}, \mathbb{Z})$ are denoted as f_d .

A.4.3.1 $\mathcal{H}_{\theta_2}^2(\mathbb{Z}_M, \mathcal{H}^0(\mathbb{Z}^2, \mathbb{Z}))$

The cocycles in the coefficient group $\mathcal{H}^0(\mathbb{Z}^2, \mathbb{Z})$ in the first term of Eq. (A.33) are constant functions valued in $\mathbb{Z}$. The $\mathbb{Z}_M$ rotations, which act on $\mathbb{Z}^2$, therefore do not change the value of these functions, so that θ_2 is the trivial action. The first term is thus isomorphic to $\mathcal{H}^2(\mathbb{Z}_M, \mathbb{Z})$, and the associated cocycle representatives of G_{space} are $f_2(g_1, g_2) = \frac{s}{M}([h_1]_M + [h_2]_M - [h_1 + h_2]_M)$ with $s \in \mathbb{Z}_M$ as discussed previously. The corresponding field-theoretic element is $\frac{s}{2\pi}d\omega$.

A.4.3.2 $\mathcal{H}_{\theta_1}^1(\mathbb{Z}_M, \mathcal{H}^1(\mathbb{Z}^2, \mathbb{Z}))$

Now we consider the second term of (A.33). The coefficient module $\mathcal{H}^1(\mathbb{Z}^2, \mathbb{Z})$ has cocycle representatives of the form $p_{\vec{t}}$, where $p_{\vec{t}}(\vec{r}) = \vec{t} \cdot \vec{r}$ for some $\vec{t} \in \mathbb{Z}^2$. Under a rotation $U(h)$, $p_{\vec{t}}$ gets transformed as $\vec{t} \cdot U(h)\vec{r} = (U^T(h)\vec{t}) \cdot \vec{r} = p_{U^T(h)\vec{t}}(\vec{r})$. This means that the induced action on the coefficients is equivalent to the rotation action $\theta_1 = \theta$ on vectors in $\mathbb{Z}^2$.

The first observation is that the group $\mathcal{H}_{\theta_1}^1(\mathbb{Z}_M, \mathbb{Z}^2)$ classifies functions f_1 taking elements h of $\mathbb{Z}_M$ to vectors in $\mathbb{Z}^2$. We have $\mathcal{H}_{\theta}^1(\mathbb{Z}_M, \mathbb{Z}^2) \cong \frac{\mathbb{Z}^2}{(I-U(\frac{2\pi}{M}))\mathbb{Z}^2} \cong K_M$, using standard results on the cohomology of cyclic groups (see for eg. Ref [21]). A representative cocycle f_1 of this group has the form

$$f_1(h) = \frac{1 - U(h)}{1 - U(\frac{2\pi}{M})} f_1(2\pi/M), \quad (\text{A.45})$$

where $f_1(2\pi/M) = \vec{t}$ for some $\vec{t} \in \mathbb{Z}^2$, and the U matrices act on $\vec{t}$ by rotation.

Next we consider the more detailed decomposition $\mathcal{H}_{\theta_1}^1(\mathbb{Z}_M, \mathcal{H}^1(\mathbb{Z}^2, \mathbb{Z}))$. A cocycle of this group maps an element $h \in \mathbb{Z}_M$ to a *cohomology class* $[p_{f_1(h)}] \in \mathcal{H}^1(\mathbb{Z}^2, \mathbb{Z})$ whose representatives are functions $p_{f_1(h)}$.

The desired 2-cocycle of G_{space} is completely determined in terms of $p_{f_1(h)}$ as follows:

$$f_2(g_1, g_2) = p_{f_1(h_1)}(\vec{r}_2) \quad (\text{A.46})$$

$$= \left(\frac{1 - U(h_1)}{1 - U(\frac{2\pi}{M})} \vec{t} \right) \cdot \vec{r}_2 \quad (\text{A.47})$$

This function, with parameter $\vec{t}$, satisfies the 2-cocycle condition for G_{space} . Values of $\vec{t}$ which are of the form $\vec{t} = (1 - U(2\pi/M))\vec{t}'$ are trivial, as the resulting cocycles are actually 2-coboundaries db , where $b(g) = \vec{t}' \cdot \vec{r}$. It is easy to motivate this function by looking at a 2-simplex [012]. If $\frac{1}{2\pi}\vec{R}_{01} = \vec{r}_1$, $\omega_{01} = h_1$ and $\frac{1}{2\pi}\vec{R}_{12} = \vec{r}_2$, then from flatness of $(\vec{R}, \omega)$ we have $\frac{1}{2\pi}\vec{R}_{02} = \vec{r}_1 + U(h_1)\vec{r}_2$. Therefore $\frac{1}{2\pi}d\vec{R}[012] = \vec{r}_1 + \vec{r}_2 - (\vec{r}_1 + U(h_1)\vec{r}_2) = (1 - U(h_1))\vec{r}_2$. Since $U(h_1)$ is a power of $U(\frac{2\pi}{M})$, this function is always a multiple of $(1 - U(\frac{2\pi}{M}))$. Therefore, $f_2(g_1, g_2) = \vec{t} \cdot \frac{1 - U(h_1)}{1 - U(\frac{2\pi}{M})} \vec{r}_2$ is integer valued for all $\vec{t} \in \mathbb{Z}^2$. However, it cannot be generated on a 2-simplex by

a 2-coboundary $df(g_1, g_2)$ (the only function that would give df is $f(g) = \vec{t} \cdot (1 - U(\frac{2\pi}{M}))^{-1} \vec{r}$, which is not integer-valued, unless $\vec{t}$ has the trivial form). The field theory element giving this value is $\frac{\vec{t}}{2\pi} \cdot d\vec{R} = \frac{\vec{t}}{2\pi} \cdot (1 - U(\frac{2\pi}{M}))^{-1} d\vec{R}$.

A.4.3.3 $\mathcal{H}_{\theta_0}^0(\mathbb{Z}_M, \mathcal{H}^2(\mathbb{Z}^2, \mathbb{Z}))$

Finally, we study the third term of (A.33). The coefficient module $\mathcal{H}^2(\mathbb{Z}^2, \mathbb{Z})$ has representatives w_m for $m \in \mathbb{Z}$, satisfying $w_m(\vec{r}_1, \vec{r}_2) - w_m(\vec{r}_2, \vec{r}_1) = m\vec{r}_1 \times \vec{r}_2$. Although the rotation action changes the form of w_m , the above cross product (and hence the value of m) is rotationally invariant, and in this sense the action θ_0 is trivial. Now the group $\mathcal{H}_{\theta_0}^0(\mathbb{Z}_M, \mathbb{Z})$ classifies functions f_0 taking each $h \in \mathbb{Z}_M$ to some fixed integer $f_0(h) = m \in \mathbb{Z}$. Therefore a cocycle in the group $\mathcal{H}_{\theta_0}^0(\mathbb{Z}_M, \mathcal{H}^2(\mathbb{Z}^2, \mathbb{Z}))$ should take h to the cohomology class $[w_{f_0(h)}]$ whose representatives $w_{f_0(h)}$ are such that $w_{f_0(h)}(\vec{r}_1, \vec{r}_2) - w_{f_0(h)}(\vec{r}_2, \vec{r}_1)$ is rotationally invariant.

It can be verified that the following function is a 2-cocycle of G_{space} with these properties:

$$f_2(g_1, g_2) = w_{f_0(h_1)}(\vec{r}_1, U(h_1)\vec{r}_2) = mr_{1,x}(U(h_1)\vec{r}_2)_y \quad (\text{A.48})$$

In this case we have $w_{f_0(h_1)}(\vec{r}_1, U(h_1)\vec{r}_2) - w_{f_0(h_1)}(U(h_1)\vec{r}_2, \vec{r}_1) = m\vec{r}_1 \times U(h_1)\vec{r}_2$. The cross product is invariant under rotations and is a measure of area. If operations 2 and 1 are performed successively, the rotation h_1 changes the relative orientation of axes used to measure the two translations. The vector $\vec{r}_2$ is therefore rotated by $U(h_1)$ so as to meaningfully take a cross product with $\vec{r}_1$.

Consider the quantity $w'_{f_0(h_1)}(\vec{r}_1, U(h_1)\vec{r}_2) = \frac{m}{2}\vec{r}_1 \times U(h_1)\vec{r}_2$. Although it is not an integer-valued cocycle, it satisfies the 2-cocycle condition with $\frac{1}{2}\mathbb{Z}$ coefficients (hence it can be used to

obtain a topologically invariant action on 3-simplices). This function satisfies $w'_{f_0(h_1)}(\vec{r}_1, U(h_1)\vec{r}_2) - w'_{f_0(h_1)}(U(h_1)\vec{r}_2, \vec{r}_1) = m\vec{r}_1 \times U(h_1)\vec{r}_2$, i.e. it has the same gauge-invariant property as $f_2(g_1, g_2)$; moreover, it is already rotationally invariant. We use this $\frac{1}{2}\mathbb{Z}$ -valued cocycle in the field theory because it is closely related to the integer-valued space group cocycles, and furthermore the cross product is an intuitive measure of area. The corresponding field theory object is $\frac{m}{2\pi}A_{XY}$, where $A_{XY}[012] = \frac{1}{4\pi}(\vec{R}_{01}) \times U(\omega_{01})\vec{R}_{12}$. The gauge transformation behaviour of A_{XY} and its physical relationship to the area element were discussed in Section 2.3.2.

A.4.3.4 Classification

The classification $\mathcal{H}^2(G_{\text{space}}, \mathbb{Z})$ is seen from the discussion above to be $\mathbb{Z}_M \times K_M \times \mathbb{Z}$. To obtain the classification of symmetry fractionalization, we use the Universal Coefficient Theorem [208] to write

$$\begin{aligned} \mathcal{H}^2(G_{\text{space}}, \mathcal{A}) \\ &= \mathcal{H}^2(G_{\text{space}}, \mathbb{Z}) \otimes \mathcal{A} \times \text{Tor}(\mathcal{H}^3(G_{\text{space}}, \mathbb{Z}), \mathcal{A}) \end{aligned} \quad (\text{A.49})$$

$$= \mathcal{H}^2(G_{\text{space}}, \mathbb{Z}) \otimes \mathcal{A} \quad (\text{A.50})$$

(we can check that the group $\mathcal{H}^3(G_{\text{space}}, \mathbb{Z})$ vanishes when G_{space} is orientation-preserving). The $\otimes$ (tensor product) symbol defines the tensor product $G \otimes H$ of abelian groups G and H . The group $G \otimes H$ is defined as the set of pairs $g \otimes h$ where $g \in G, h \in H$, where $\otimes$ is a bilinear operation such that $g \otimes h$ is trivial if either g or h is trivial. For example if $ng = 1_G$ (the identity element of G), $n(g \otimes h) = (ng) \otimes h = 1_{G \otimes H}$; and this argument runs similarly for h . The group

$G \otimes H$ is completely defined by the following properties:

$$G \otimes H \cong H \otimes G \quad (\text{A.51})$$

$$\left(\prod_i G_i\right) \otimes \left(\prod_j H_j\right) \cong \prod_{i,j} (G_i \otimes H_j) \quad (\text{A.52})$$

$$G \otimes \mathbb{Z} \cong G \quad (\text{A.53})$$

$$\mathbb{Z}_m \otimes \mathbb{Z}_n \cong \mathbb{Z}_d, d = \gcd(m, n) \quad (\text{A.54})$$

The topological terms classified by $\mathcal{H}^2(G_{\text{space}}, \mathbb{Z}) \otimes \mathcal{A}$ are thus consistent with the group structure of symmetry fluxes (classified by $\mathcal{H}^2(G_{\text{space}}, \mathbb{Z})$) as well as that of anyons (classified by $\mathcal{A}$). The formal effect of the $\otimes$ symbol is to replace the $\mathbb{Z}$ coefficients by $\mathcal{A}$ coefficients. This means that the coefficients $s, \vec{t}, m$ are replaced by vectors $s_I, \vec{t}_I, m_I$ in $\mathbb{Z}^D$. Moreover, if these parameters take the form $K\vec{\Lambda}$, they are trivial. With this change, the above cocycles all become cocycle representatives for $\mathcal{H}^2(G_{\text{space}}, \mathcal{A}) \cong (\mathcal{A}/M\mathcal{A}) \times (K_M \otimes \mathcal{A}) \times \mathcal{A}$. (Note that $\mathbb{Z}_M \otimes \mathcal{A} \cong \mathcal{A}/M\mathcal{A}$.) Effective actions corresponding to these cocycles are recovered by taking a cup product of the vector a_I of internal gauge fields with the field theory term corresponding to a representative of $\mathcal{H}^2(G, \mathcal{A})$.

A.4.4 Cocycle representatives for $\mathcal{H}^3(G_{\text{space}}, \text{U}(1))$

With our knowledge of $\mathcal{H}^2(G_{\text{space}}, \mathbb{Z})$, it is easy to understand the group $\mathcal{H}^3(G_{\text{space}}, \text{U}(1))$. We can derive its cocycle representatives in the following direct way. The G_{space} charges are classified by $\mathcal{H}^1(G_{\text{space}}, \text{U}(1)) \cong \mathbb{Z}_M$ (corresponding to the charges of ω), whose generator is represented by the cocycle $f_1(h) = 2\pi[h]_M/M \mod 2\pi$. The associated field theory element

is just ω . The fluxes are classified by the group $\mathcal{H}^2(G_{\text{space}}, \mathbb{Z})$. Therefore SPT cocycles, which associate symmetry flux to an elementary symmetry charge, are all of the form $\nu(g_1, g_2, g_3) = \frac{2\pi[h_1]}{M}\beta(g_2, g_3) \bmod 2\pi$, where $[\beta] \in \mathcal{H}^2(G_{\text{space}}, \mathbb{Z})$. These functions satisfy the 3-cocycle condition for $\mathcal{H}^3(G_{\text{space}}, \text{U}(1))$, and correspond to taking the cup product of a cocycle in $\mathcal{H}^1(\mathbb{Z}_M, \text{U}(1))$ with another from $\mathcal{H}^2(G_{\text{space}}, \mathbb{Z})$.

To obtain the relevant SPT cocycle representatives, consider the three subgroups S_k , $k = 0, 1, 2$, of $\mathcal{H}^2(G_{\text{space}}, \mathbb{Z})$, defined as

$$S_k := \mathcal{H}^k(\mathbb{Z}_M, \mathcal{H}^{2-k}(\mathbb{Z}^2, \mathbb{Z})). \quad (\text{A.55})$$

From the definition of the tensor product, the classification of SPT terms obtained by associating an elementary $\mathbb{Z}_M$ charge to a flux represented by a cocycle of S_i is $\mathbb{Z}_M \otimes S_i$. The full SPT classification is therefore $\prod_{i=0}^2 \mathbb{Z}_M \otimes S_i = (\mathbb{Z}_M \otimes \mathbb{Z}_M) \times (\mathbb{Z}_M \otimes K_M) \times (\mathbb{Z}_M \otimes \mathbb{Z}) = \mathbb{Z}_M \times K_M \times \mathbb{Z}_M$. This is the same as the Künneth decomposition result: $\mathcal{H}^3(G_{\text{space}}, \text{U}(1)) \cong \mathcal{H}^3(\mathbb{Z}_M, \text{U}(1)) \times \mathcal{H}_\theta^2(\mathbb{Z}_M, \text{U}(1) \times \text{U}(1)) \times \mathcal{H}^1(\mathbb{Z}_M, \mathcal{H}^2(\mathbb{Z}^2, \text{U}(1))) \cong \mathbb{Z}_M^2 \times K_M$. Therefore the flux-charge construction accounts for all the group cohomology SPTs. The cocycles so obtained are moreover in one-to-one correspondence with cocycle representatives of $\mathcal{H}^4(G_{\text{space}}, \mathbb{Z})$.

The cocycles for mixed SPTs of $\text{U}(1)$ and G_{space} symmetry are obtained by a cup product of a 1-cocycle representative of $\mathcal{H}^1(\text{U}(1), \text{U}(1))$ (generated by $f_1(a) = [a] \bmod 2\pi$) and a 2-cocycle representative of $\mathcal{H}^2(G_{\text{space}}, \mathbb{Z})$. Finally, the full $\mathcal{H}^3(G_{\text{space}} \times \text{U}(1), \text{U}(1))$ classification can also be obtained from the Künneth decomposition: it equals $\mathbb{Z}^2 \times \mathbb{Z}_M^3 \times K_M^2$. In this case, the possible charges are classified by the group $\mathbb{Z} \times \mathbb{Z}_M$, corresponding to charge of A and ω respectively. These charges couple to fluxes, i.e. representatives of the group $\mathcal{H}^2(G_{\text{space}}, \mathbb{Z})$, to

give the full SPT action for the group $G_{\text{space}} \times \text{U}(1)$.

Appendix B: Appendix for Chapter 3

B.1 Results from group cohomology

B.1.1 Cup product definitions used in the main text

Here we state some basic definitions in group cohomology. A more detailed discussion of the quantities defined here can be found in Appendix B of Ref. [27]. Let $C^n(G_b, M)$ be the set of functions in n variables from G_b to the abelian group M . Each element of $C^n(G_b, M)$ is called an n -cochain. We commonly take $M = \mathbb{Z}, \mathbb{Z}_2, \text{U}(1)$: for example, in defining $n_1, n_2, \mathcal{O}_3$ we take $M = \mathbb{Z}_2$, and in defining $\nu_3, \mathcal{O}_4$ we take $M = \text{U}(1)$.

The differential d takes an element in $C^m(G_b, M)$ to an element in $C^{m+1}(G_b, M)$, and satisfies $d \circ d = 0$. Examples are shown below.

When M is a ring, there is a binary operation on cochains called the cup product, denoted $\cup : C^n(G_b, M) \times C^m(G_b, M) \rightarrow C^{n+m}(G_b, M)$, defined as

$$(f_n \cup f_m)(\mathbf{g}_1, \dots, \mathbf{g}_{n+m}) := f_n(\mathbf{g}_1, \dots, \mathbf{g}_n) f_m(\mathbf{g}_{n+1}, \dots, \mathbf{g}_{n+m}). \quad (\text{B.1})$$

Similarly we can define cup- i products, denoted $\cup_i$, for $i \geq 0$. These are binary operations

$$\cup_i : C^n(G_b, M) \times C^m(G_b, M) \rightarrow C^{n+m-i}(G_b, M).$$

Here we give the expanded form of some of the equations used regularly in the main text that used cup products or other cohomology operations. If $\mathbf{g}, \mathbf{h}, \mathbf{k}, \mathbf{l} \in G_b$, and $f_i \in C^i(G_b, M)$,

$$df_1(\mathbf{g}, \mathbf{h}) := f_1(\mathbf{g}) + {}^{\mathfrak{g}}f_1(\mathbf{h}) - f_1(\mathbf{gh}) \quad (\text{B.2})$$

$$df_2(\mathbf{g}, \mathbf{h}, \mathbf{k}) := -(f_2(\mathbf{g}, \mathbf{h}) + f_2(\mathbf{gh}, \mathbf{k})) + (f_2(\mathbf{g}, \mathbf{hk}) + {}^{\mathfrak{g}}f_2(\mathbf{h}, \mathbf{k})) \quad (\text{B.3})$$

$$f_1 \cup f_2(\mathbf{g}, \mathbf{h}, \mathbf{k}) := f_1(\mathbf{g})f_2(\mathbf{h}, \mathbf{k}) \quad (\text{B.4})$$

$$f_2 \cup_1 f_2(\mathbf{g}, \mathbf{h}, \mathbf{k}) := f_2(\mathbf{g}, \mathbf{hk})f_2(\mathbf{gh}, \mathbf{k}) + f_2(\mathbf{g}, \mathbf{hk})f_2(\mathbf{h}, \mathbf{k}) \mod 2 \quad (\text{B.5})$$

$$\begin{aligned} df_3(\mathbf{g}, \mathbf{h}, \mathbf{k}, \mathbf{l}) &:= f_3(\mathbf{g}, \mathbf{h}, \mathbf{k}) - f_3(\mathbf{gh}, \mathbf{k}, \mathbf{l}) + f_3(\mathbf{g}, \mathbf{hk}, \mathbf{l}) \\ &\quad - f_3(\mathbf{g}, \mathbf{h}, \mathbf{kl}) + {}^{\mathfrak{g}}f_3(\mathbf{h}, \mathbf{k}, \mathbf{l}). \end{aligned} \quad (\text{B.6})$$

The superscript ${}^{\mathfrak{g}}f_n$ refers to the action of G_b on M . In Eq. (3.4), which defines $\mathcal{O}_3$, the action is trivial. In Eq. (3.5), while defining $d\nu_3$, we assume that antiunitary operations act on the $\text{U}(1)$ coefficients by conjugation. Therefore ${}^{\mathfrak{g}}\nu_3(\mathbf{h}, \mathbf{k}, \mathbf{l}) = (1 - 2s_1(\mathbf{g}))\nu_3(\mathbf{h}, \mathbf{k}, \mathbf{l})$. We will interchangeably use multiplicative and additive notation for $\text{U}(1)$ by identifying $x \mod 1$ (in additive notation) with $e^{2\pi i x}$ (in multiplicative notation).

B.1.2 Pontryagin square

The Pontryagin square is a cohomology operation that refines the square coming from the cup product. In particular, consider $\omega \in \mathcal{H}^{2d}(G, \mathbb{Z}_{2L})$, then $\mathcal{P}(\omega) \in \mathcal{H}^{4d}(G, \mathbb{Z}_{4L})$ such that $2\mathcal{P}(\omega) = 2\omega \cup \omega \mod 4L$ in $\mathcal{H}^{4d}(G, \mathbb{Z}_{4L})$. An explicit expression for this operation is as follows.

Take W an integral lift of ω , i.e. $W \in C^{2d}(G, \mathbb{Z})$ such that $W = \omega \mod 2L$. Then

$$\mathcal{P}(\omega) = W \cup W - W \cup_1 dW \mod 4L$$

is independent of the lift W as a cohomology class in $\mathcal{H}^{4d}(G, \mathbb{Z}_{4L})$.

B.1.3 Comment about continuous groups

When G is not a discrete group, $\mathcal{H}^*(G, M)$ is taken to mean group cohomology with measurable cochains. (M is an arbitrary G -module.)

On the other hand, a useful cohomology related to a group G is the singular cohomology of its classifying space (BG) denoted by $H^*(BG, M)$. There are several results about these cohomology groups and they come with a geometrical intuition that is sometimes useful.

As reviewed in **Theorem J.1** of Ref. [53], if G is finite-dimensional, locally compact, σ -compact group and M is discrete, there is an isomorphism

$$\mathcal{H}^n(G, M) \cong H^n(BG, M). \quad (\text{B.7})$$

The above assumptions are in particular true for finite dimensional compact Lie groups and finite groups.

We use the isomorphism in Eq. B.7 to identify $\mathcal{H}^n(G, M)$ and $H^n(BG, M)$ in the remaining appendices. In later appendices, the spectral sequence calculations involving Lie groups are done using the singular cohomology of the classifying space $H^n(BG, M)$.

B.2 Algebraic formula for the fCEP

B.2.1 Notation

First we discuss the notation used for Eq. (3.16) in detail. Consider the group

$$G_E = (\mathbb{R}^d \rtimes \mathrm{O}(d)) \times \mathbb{Z}_2^{\mathbf{T}}$$

where $d = D - 1$ is the space dimension, which is the group of translations, rotations, reflections, and time reversal. The symmetry of the TQFT describing topological phases of matter contains G_E as a subgroup. A symmetry which acts on both space and time is implemented via the map

$$\bar{\rho} : G_b \rightarrow G_E.$$

We can write $\bar{\rho} = (\vec{R}, \rho_s, \rho_t)$ where the three terms project to $\mathbb{R}^d, \mathrm{O}(d), \mathbb{Z}_2^{\mathbf{T}}$ respectively. Note that the restriction $(\vec{R}, \rho_s)$ of $\bar{\rho}$ fully specifies the action of G_b on space.

We first define the following classes, which generate their respective groups:

$$\begin{aligned} [w_{1,r}] &\in \mathcal{H}^1(\mathrm{O}(d), \mathbb{Z}_2) \\ [w_{1,t}] &\in \mathcal{H}^1(\mathbb{Z}_2^{\mathbf{T}}, \mathbb{Z}_2) \\ [w_2] &\in \mathcal{H}^2(\mathcal{S}_0(d), \mathbb{Z}_2). \end{aligned} \tag{B.8}$$

Correspondingly we can define

$$\begin{aligned}
[w_1] &= [\rho_s^* w_{1,r}] \in \mathcal{H}^1(G_b, \mathbb{Z}_2) \\
[s_1] &= [\rho_t^* w_{1,t}] \in \mathcal{H}^1(G_b, \mathbb{Z}_2) \\
[w_2] &= [\rho_s^* w_2] \in \mathcal{H}^2(G_b, \mathbb{Z}_2).
\end{aligned} \tag{B.9}$$

Here the pullback ρ^* is defined so that $\rho^* f(\mathbf{g}_1, \dots, \mathbf{g}_i) = f(\rho(\mathbf{g}_1), \dots, \rho(\mathbf{g}_i))$ where $\mathbf{g}_1, \dots, \mathbf{g}_i \in G_b$. Note that s_1 has the same information as ρ_t but differs in that it takes an antiunitary element in G_b to the number $1 \in \mathbb{Z}_2$ while ρ_t takes an antiunitary element to $\mathbf{T} \in \mathbb{Z}_2^{\mathbf{T}}$. This is only a formal distinction but will be useful in the next section.

The data for the spatial symmetry are thus given by $(G_b, s_1, \omega_2, (\vec{R}, \rho_s))$. Note that the data for the effective internal symmetry are fully specified by $(G_b, s_1^{\text{eff}}, \omega_2^{\text{eff}})$ because $(\vec{R}, \rho_s)$ are both trivial and can be dropped. The two sets of data are related by the fCEP formula, Eq. (3.16).

B.2.2 Checks

Below we give examples showing why each term in the above formula should be present.

B.2.2.1 w_2

To see why the term $w_2 = \rho_s^* w_2$ should be present, note that a single layer of a spinless $p+ip$ superconductor with $c_- = 1/2$ has a unitary C_2 rotational symmetry that acts as $\hat{C}_2^2 = (-1)^F$ on fermions. Conversely, a single layer of a spinless $p+ip$ SC with $c_- = 1/2$ can only have a $\mathbb{Z}_2$ internal unitary symmetry if the generator $\mathbf{h}$ satisfies $\mathbf{h}^2 = +1$. For these two states to be related by the fCEP, ω_2^{eff} and ω_2 should differ by a nontrivial 2-cocycle in $\mathcal{H}^2(G_b, \mathbb{Z}_2)$. This nontrivial

cocycle can be constructed using ρ_s as $\rho_s^* w_2 = w_2$ because w_2 is the cocycle that corresponds to lifting 2π rotations to $(-1)^F$. Therefore

$$\omega_2^{\text{eff}} = \omega_2 + w_2. \quad (\text{B.10})$$

Note that in this example, $s_1 = 0, w_1 = 0$, so the above result is consistent with the general formula.

B.2.2.2 w_1^2

Next, note that we can only have a nontrivial topological superconductor (TSC) with $\mathbb{Z}_2^{\mathbf{R}}$ reflection symmetry if it acts as $\mathbf{R}^2 = +1$ on fermions. In this case, we can place a 1d Kitaev chain on the reflection axis. We cannot do so if $\mathbf{R}^2 = (-1)^F$, for in that case the onsite symmetry on the reflection axis would be $\mathbb{Z}_4^f$, and there are no 1d invertible phases with unpaired MZMs in this case.

On the other hand, consider the equivalent internal symmetry, and assume that reflections map to time-reversal. Here we know that the only nontrivial TSC has $\mathbf{T}^2 = (-1)^F$. This means that in this example, ω_2 and ω_2^{eff} should differ by a nontrivial 2-cocycle in $\mathcal{H}^2(G_b, \mathbb{Z}_2)$. This nontrivial cocycle can be constructed using ρ_s as $\rho_s^* w_{1,r}^2 = w_1^2$ because $w_{1,r}^2$ is the cocycle that corresponds to lifting $\mathbf{R}^2$ to $(-1)^F$. Thus we should have

$$\omega_2^{\text{eff}} = \omega_2 + w_1^2. \quad (\text{B.11})$$

Moreover, note that in this example $w_2 = 0$ and $s_1 = 0$, so the above is consistent with the

general formula.

B.2.2.3 $s_1 w_1$

In the previous literature, the formula in Eq. 3.16 has appeared in the case where there are no anti-unitary symmetries. We now argue for the new term $s_1 \cup w_1$ as follows.

Consider a fermionic system in (2+1)D with a reflection $\mathbf{R}$ and time-reversal symmetry $\mathbf{T}$ such that

$$\mathbf{T}^2 = \mathbf{R}^2 = (-1)^F; \quad (\mathbf{RT})^2 = 1. \quad (\text{B.12})$$

This can be written as $G_f = \mathbb{Z}_2^{\mathbf{RT}} \times \mathbb{Z}_4^{\mathbf{T}f}$. This corresponds to $G_b = \mathbb{Z}_2^{\mathbf{RT}} \times \mathbb{Z}_2^{\mathbf{T}}$. Let $\tau, \sigma \in \mathcal{H}^1(G_b, \mathbb{Z}_2)$ be defined as follows. If an element in G_b is written as $[a, b]$ with $a \in \mathbb{Z}_2^{\mathbf{RT}}, b \in \mathbb{Z}_2^{\mathbf{T}}$, then $\tau([a, b]) = a + b$ and $\sigma([a, b]) = a$. The symmetry data is $s_1 = \tau$, $\omega_2 = \tau^2 + \sigma^2$ and $\rho_s([a, b]) = ((-1)^a, 1)$.

Note that $w_1 = \sigma$ and $w_2 = 0$. Applying the fCEP using Eq. (3.16) gives $s_1^{\text{eff}} = \tau + \sigma$ and $\omega_2^{\text{eff}} = \sigma(\sigma + \tau)$. This corresponds to $G_f^{\text{eff}} = \mathbb{Z}_4^{\mathbf{R}'f} \rtimes \mathbb{Z}_2^{\mathbf{R}'\mathbf{T}}$ - $\mathbf{R}'$ is an anti-unitary internal symmetry that squares to $(-1)^F$ and anticommutes with the unitary $\mathbb{Z}_2$ symmetry $\mathbf{R}'\mathbf{T}$. This example has a $\mathbb{Z}_8$ classification and it is generated by free fermions. This is consistent with the result found in Ref. [209].

If we did not include the term $s_1 \cup w_1$ in Eq. 3.16, we would find that $\omega_2^{\text{eff}} = \sigma^2$ and $G_f^{\text{eff}} = \mathbb{Z}_4^{\mathbf{R}'f} \times \mathbb{Z}_2^{\mathbf{R}'\mathbf{T}}$. Which is equivalent to $G_f = \mathbb{Z}_4^{\mathbf{T}f} \times \mathbb{Z}_2$ so that $\omega_2 = s_1^2$. Let x be projection to the second factor. We find that $n_1 \in \{0, s_1\}$ and $n_2 = 0$. There is only one non-trivial solution with $n_1 = n_2 = 0$ for ν_3 and it is equal to $\frac{1}{2}x^3$. Note that even if we do not fully apply the stacking rules, these results would imply a classification of $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4$, which cannot be equal

to the $\mathbb{Z}_8$ proposed in Ref. [209].

B.2.3 Heuristic argument

Here we assume the abstract statement of the fCEP without further justification, i.e. that the classification of fermionic topological phases with the spatial symmetry data $(G_b, s_1, \omega_2, (\vec{R}, \rho_s))$ is isomorphic to that of fermionic topological phases with some internal symmetry given by the data $(G_b, s_1^{\text{eff}}, \omega_2^{\text{eff}})$. We now discuss some heuristic arguments to connect this abstract statement to the concrete formula in Eq. (3.16). We note that the abstract statement of the fCEP is argued for in Refs. [82–84]. One can directly derive the formula, with several assumptions, by using the mathematical statements given in these references, but we will not present this here.

A result that we use below is that $\mathcal{H}^i(G_E, \mathbb{Z}_2)$ can be identified with $\mathcal{H}^i(\text{O}(d) \times \mathbb{Z}_2^{\text{T}}, \mathbb{Z}_2)$, i.e. the $\mathbb{R}^d$ translations do not contribute to the cohomology (see e.g. Theorem J7 of Ref. [53]).

Now if we believe that the fCEP exists, then for fermionic phases it must relate the data $s_1, \omega_2, \bar{\rho}$ to $s_1^{\text{eff}}, \omega_2^{\text{eff}}$. Clearly it should take the form

$$\begin{aligned} s_1^{\text{eff}} &= s_1 + f_1 \\ \omega_2^{\text{eff}} &= \omega_2 + f_2 \end{aligned} \tag{B.13}$$

where $f_i \in \mathcal{H}^i(G_b, \mathbb{Z}_2)$. To include $\bar{\rho}$ in the above formula, it is natural to construct f_i by considering cohomology classes $\kappa_i \in \mathcal{H}^i(G_E, \mathbb{Z}_2)$ and then setting $f_i = \bar{\rho}^* \kappa_i$, i.e. $f_i(\mathbf{g}) = \kappa_i(\bar{\rho}(\mathbf{g}))$ for $\mathbf{g} \in G_b$; the pullback $\bar{\rho}^*$ was defined previously.

Now $\mathcal{H}^1(G_E, \mathbb{Z}_2) \cong \mathcal{H}^1(\text{O}(d) \times \mathbb{Z}_2^{\text{T}}, \mathbb{Z}_2)$ is generated by the cocycles $w_{1,r}, w_{1,t}$. Thus f_1 must be a linear combination of $\rho_s^* w_{1,r} = w_1$ and $\rho_t^* w_{1,t} = s_1$. We note that the space-time

orientation-reversing elements are specified by $s_1 + w_1$ for the spatial symmetry, and by s_1^{eff} for the internal symmetry. Since both the spatial symmetry and its effective internal symmetry should have the same orientation-reversing action on space-time, we should set

$$s_1^{\text{eff}} = s_1 + w_1. \quad (\text{B.14})$$

Next we discuss f_2 . Note that $\mathcal{H}^2(G_E, \mathbb{Z}_2) \cong \mathcal{H}^2(\text{O}(d) \times \mathbb{Z}_2^{\mathbf{T}}, \mathbb{Z}_2)$ is generated by the cocycles $w_{1,t}^2, w_{1,t}w_{1,r}, w_{1,r}^2, w_2$. Thus f_2 should be some linear combination of $\rho_t^* w_{1,t}^2 = (s_1)^2, \rho_t^* w_{1,t} \rho_s^* w_{1,r} = s_1 w_1, \rho_s^* w_{1,r}^2 = w_1^2, \rho_s^* w_2 = w_2$. The previously discussed examples show that the last three terms should indeed appear in f_2 .

If we also included the remaining term in f_2 , we would have $\omega_2^{\text{eff}} = \omega_2 + (s_1)^2 + w_1 s_1 + w_1^2 + w_2$. But when ρ_s is trivial (i.e. G_b acts trivially on space), this would give us $\omega_2^{\text{eff}} = \omega_2 + (s_1)^2$, which would give an equivalence between any system that has $\mathbf{T}^2 = +1$ with a system that has $\mathbf{T}^2 = (-1)^F$. This is a contradiction. Therefore we take $f_2 = w_1 s_1 + w_1^2 + w_2$. This implies Eq. (3.16).

This heuristic argument also indicates why unitary translations do not appear in the formula. This is because the cohomology of $\mathbb{R}^d$ vanishes, so $[\alpha_i] \in \mathcal{H}^i(\mathbb{R}^d, \mathbb{Z}_2)$ is always trivial and therefore so is $[\vec{R}^* \alpha_i]$.

B.3 Derivation of topological action

B.3.1 Details for general M

Here we summarize the steps of the derivation of Eq. (3.20), keeping M general. We fix an origin $\mathbf{o}$ which is symmetric under M -fold rotations, that is, $M_{\mathbf{o}} = M$. The calculations specific to $M = 2, 3, 4, 6$ will be given in the following subsections. Note that Apps B.3.1, B.3.2 assume $\phi = 0$. The extension to $\phi \neq 0$ will be discussed in App. B.3.3.

B.3.1.1 Overview and definition of G_f

We have $G_f = \text{U}(1)^f \times [\mathbb{Z}^2 \rtimes \mathbb{Z}_M]$ (that is, we assume $\phi = 0$), and $G_b = G_f / \mathbb{Z}_2^f = \text{U}(1) \times [\mathbb{Z}^2 \rtimes \mathbb{Z}_M]$. Let $[\omega_2] \in \mathcal{H}^2(G_b, \mathbb{Z}_2)$ determine G_f as a group extension of G_b by $\mathbb{Z}_2^f$. We assume that the group $\mathbb{Z}_M$ is generated by the operator $\tilde{C}_{M_{\mathbf{o}}}^+$ defined in the main text. The general topological action $\mathcal{L}$ for G_f symmetric invertible fermion phases will be derived in two main steps. First we use the fermionic crystalline equivalence principle (fCEP), which states that the desired classification of G_f invertible phases is in one-to-one correspondence with that of invertible phases with an *effective internal* symmetry G_f^{eff} . (In our case it will turn out that $G_f^{\text{eff}} \cong G_f$.) Next, we use the general theory of invertible fermionic phases with internal symmetries developed in Ref. [27] to convert the problem of finding $\mathcal{L}$ into a calculation in group cohomology.

B.3.1.2 Group cohomology definitions

The following calculations make liberal use of results from group cohomology. An introduction aimed at applications to topological phases of matter can be found, for example, in the appendices of Refs. [21] (general introduction and many examples), [52] (computation of several cohomology groups relevant to this work), [53] (several general results, especially a development of spectral sequences), [27] (cup products) and [85] (further computational tricks).

We begin by stating some results about the group cohomology of $G_b = \text{U}(1) \times [\mathbb{Z}^2 \rtimes \mathbb{Z}_M]$. These statements were derived in terms of explicit cocycle representatives in App. A.4, but here we will use a cleaner notation which is independent of the specific choice of cocycle representative. First we have the result

$$\mathcal{H}^2(G_b, \mathbb{Z}) \cong \mathcal{H}^2(\text{U}(1), \mathbb{Z}) \times \mathcal{H}^2(\mathbb{Z}_M, \mathbb{Z}) \times \mathcal{H}^1(\mathbb{Z}_M, \mathcal{H}^1(\mathbb{Z}^2, \mathbb{Z})) \times \mathcal{H}^0(\mathbb{Z}_M, \mathcal{H}^2(\mathbb{Z}^2, \mathbb{Z})) \quad (\text{B.15})$$

$$\cong \mathbb{Z} \times \mathbb{Z}_M \times K_M \times \mathbb{Z} \quad (\text{B.16})$$

where $K_M := \frac{\mathbb{Z}^2}{(1-U(\mathbf{h}))\mathbb{Z}^2}$ and $\mathbf{h}$ is the generator of $\mathbb{Z}_M$, which rotates vectors in $\mathbb{Z}^2$ via the matrix $U(\mathbf{h})$. Thus, elements of K_M are given by integer vectors modulo an equivalence relation. We find that $K_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, $K_3 \cong \mathbb{Z}_3$, $K_4 \cong \mathbb{Z}_2$, $K_6 \cong \mathbb{Z}_1$. This group cohomology result can be proven using a spectral sequence decomposition, as argued in App. A.4.

We denote the generators of this group as

$$[z_2] \in \mathcal{H}^2(\mathrm{U}(1), \mathbb{Z}), \quad [h_2] \in \mathcal{H}^2(\mathbb{Z}_M, \mathbb{Z}),$$

$$[\vec{t}_2] = ([t_{2,x}], [t_{2,y}]) \in \mathcal{H}^1(\mathbb{Z}_M, \mathcal{H}^1(\mathbb{Z}^2, \mathbb{Z})), [a_2] \in \mathcal{H}^0(\mathbb{Z}_M, \mathcal{H}^2(\mathbb{Z}^2, \mathbb{Z})). \quad (\text{B.17})$$

(Here $[x]$ is the cohomology class corresponding to the cocycle representative x .) Note that in this case, $\mathcal{H}^2(G_b, \mathbb{Z}_2) \cong \mathcal{H}^2(G_b, \mathbb{Z}) \otimes \mathbb{Z}_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_{(M,2)} \times (K_M \otimes \mathbb{Z}_2) \times \mathbb{Z}_2$. This means that any $\mathbb{Z}_n$ or $\mathbb{Z}$ factor of $\mathcal{H}^2(G_b, \mathbb{Z})$ reduces to a $\mathbb{Z}_{(n,2)}$ or $\mathbb{Z}_2$ factor of $\mathcal{H}^2(G_b, \mathbb{Z}_2)$, respectively. The symbol (m, n) denotes the greatest common divisor of m and n .

For our choice of G_f , we moreover have

$$\omega_2 = z_2 \pmod{2}, \quad (\text{B.18})$$

indicating that the $\mathrm{U}(1)$ subgroup of G_b nontrivially extends fermion parity, but the spatial part of G_b does not.

Next, we discuss cohomology in degree 4 with integer coefficients. Note that

$$\mathcal{H}^4(G_b, \mathbb{Z}) \cong \mathcal{H}^4(\mathrm{U}(1), \mathbb{Z}) \times \mathcal{H}^2(\mathbb{Z}^2 \rtimes \mathbb{Z}_M, \mathbb{Z}) \times \mathcal{H}^4(\mathbb{Z}^2 \rtimes \mathbb{Z}_M, \mathbb{Z}) \quad (\text{B.19})$$

$$\cong \mathbb{Z} \times [\mathbb{Z} \times \mathbb{Z}_M \times K_M] \times [\mathbb{Z}_M^2 \times K_M]. \quad (\text{B.20})$$

We consider the three factors in the decomposition separately. The generators of this group can be expressed in terms of cup products of the generators of $\mathcal{H}^2(G_b, \mathbb{Z}_2)$ discussed above (see App. B of Ref. [27] for a thorough introduction to cup products and their properties). The first

factor is generated by the cocycle representative $z_2 \cup z_2 \equiv z_2^2$. The second factor is generated by the cocycles $z_2 a_2 (\mathbb{Z})$, $z_2 h_2 (\mathbb{Z}_M)$, $\{z_2 \mathbf{t}_{2,x}, z_2 \mathbf{t}_{2,y}\} (K_M)$. The third factor is generated by the cocycles $h_2^2 (\mathbb{Z}_M)$, $a_2 h_2 (\mathbb{Z}_M)$, $\{h_2 \mathbf{t}_{2,x}, h_2 \mathbf{t}_{2,y}\} (K_M)$.

Note that $\mathcal{H}^3(G_b, U(1)) \cong \mathcal{H}^4(G_b, \mathbb{Z})$. In Ref. [52], a concrete parametrization was obtained for representative cocycles $[\lambda_3] \in \mathcal{H}^3(G_b, U(1))$, and we rewrite it here for convenience. First define z_1, h_1 as the generators of $\mathcal{H}^1(U(1), U(1))$, $\mathcal{H}^1(\mathbb{Z}_M, U(1))$ respectively. Assume that the $U(1)$ coefficient module is defined as $\mathbb{R}/\mathbb{Z}$. Then, we have [52]

$$\lambda_3 = k_1 z_1 z_2 + k_2 h_1 z_2 + k_3 h_1 h_2 + \vec{k}_4 \cdot z_1 \vec{\mathbf{t}}_2 + \vec{k}_5 \cdot h_1 \vec{\mathbf{t}}_2 + k_6 z_1 a_2 + k_7 h_1 a_2 \pmod{1}. \quad (\text{B.21})$$

Here we have suppressed the dependence of the coefficients on the origin $\mathbf{o}$, although this dependence will be made explicit in the final result. The fact that $d\lambda_3 = 0 \pmod{1}$ can be seen by observing that $dz_1 = z_2$ and $dh_1 = h_2$; these hold because the coboundary d is the Bockstein map for the short exact sequence

$$1 \rightarrow U(1) \rightarrow \mathbb{R} \rightarrow \mathbb{Z} \rightarrow 1.$$

Finally, it will be useful to define cocycle representatives for the group $\mathcal{H}^1(G_b, \mathbb{Z}_2)$. This group is isomorphic to $K_M \times \mathbb{Z}_2$ for even M , and is trivial for $M = 3$. When M is even, the first factor is generated by $2\mathbf{t}_{1,x} \pmod{2}$, $2\mathbf{t}_{1,y} \pmod{2}$, where $d\vec{\mathbf{t}}_1 = \vec{\mathbf{t}}_2 \pmod{2}$ by definition. The second factor is generated by $Mh_1 \pmod{2}$.

B.3.1.3 Fermionic crystalline equivalence principle

Let us discuss the first step in finding $\mathcal{L}$, using the formulation of the fCEP in Ref. [85]. Applied to our case, that result simply states that G_f^{eff} is also a group extension of G_b by $\mathbb{Z}_2^f$ corresponding to a cocycle $[\omega_2^{\text{eff}}] \in \mathcal{H}^2(G_b, \mathbb{Z}_2)$, and in particular

$$\omega_2^{\text{eff}} = \omega_2 + h_2 \mod 2 = z_2 + h_2 \mod 2. \quad (\text{B.22})$$

A heuristic justification of the principle can be found in Ref. [85]. Although the fCEP has been discussed in several prior works and verified in many specific examples, we are not aware of a general proof.

B.3.1.4 Definition of G_b gauge field

In the main text, the G_b gauge field was defined as $B = (A_b, \vec{R}, \omega)$, where $A_b \sim A_b + 2\pi, \omega$. In this appendix, we will instead define $\tilde{X} := X/2\pi$ for $X = B, A_b, \vec{R}, \omega$. We use tilded gauge fields in this appendix so that we have a more convenient normalization for ν_3 and $\mathcal{L}$. But we will switch back to the usual untilded notation at the end of the calculation.

Since our system has U(1) charge conservation symmetry, it cannot host unpaired Majorana zero modes [85]. In this case we can obtain $\mathcal{L}$ by setting

$$\mathcal{L} = 2\pi \tilde{B}^* \nu_3$$

where $\nu_3 \in C^3(G_b, \mathbb{R}/\mathbb{Z})$ is a 3-cochain which will be determined in terms of group cohomology

data. Note that this relation does not necessarily hold when the system has unpaired Majorana zero modes, or more formally when $n_1 \neq 0$ in the formalism of Ref. [27].

Let us parameterize elements of G_b as $\mathbf{g} = (z, \vec{r}, h)$ where $z \in \mathbb{R}/\mathbb{Z} \cong \text{U}(1)$, $\vec{r} \in \mathbb{Z}^2$, $h \in \mathbb{Z}/M\mathbb{Z} \cong \mathbb{Z}_M$. On a 3-manifold $\mathcal{M}^3$ with a triangulation, we then define the flat G_b gauge field

$$\tilde{B} = (\tilde{A}_b, \vec{\tilde{R}}, \tilde{\omega}) \in H^1(\mathcal{M}^3, G_b). \quad (\text{B.23})$$

Thus $\tilde{A}_b$ is real valued, with $\tilde{A}_b \sim \tilde{A}_b + 1$. It is a gauge field for the bosonic $\text{U}(1)$ symmetry, which is defined as $\text{U}(1)^f / \mathbb{Z}_2^f$ (note that the fermion has charge 1/2 under this group). $\vec{\tilde{R}} = (\tilde{X}, \tilde{Y})$ is integer valued. Finally $\tilde{\omega}$ is valued in multiples of $1/M$, with $\tilde{\omega} \sim \tilde{\omega} + 1$.

For a flat gauge field $\tilde{B}$, the fluxes of $\tilde{B}$ are related to the cohomology classes previously defined:

$$d\tilde{A}_b = \tilde{B}^* z_2; \quad d\tilde{\omega} = \tilde{B}^* z_2; \quad \tilde{A}_{XY} = \tilde{B}^* a_2. \quad (\text{B.24})$$

Here $2\pi d\tilde{A}_b$ is interpreted as magnetic flux. $2\pi d\tilde{\omega}$ is interpreted as disclination flux: the integral of $2\pi d\tilde{\omega}$ over a region W measures the total disclination angle of defects within W . $d\vec{\tilde{R}}$ is interpreted as dislocation flux: the integral of $d\vec{\tilde{R}}$ within W measures the total Burgers vector of defects within W , up to conjugation by elements in G_b (see Chapter 2). $\tilde{A}_{XY}$ has the physical interpretation of an area element, whose integral over a region W counts the number of unit cells in W .

B.3.1.5 Condition on $\mathcal{L}$

The condition of topological invariance written below comes from the general theory of invertible fermionic phases developed in Ref. [27]. We wish to write down a topological action $\mathcal{L}$ for invertible fermionic phases with (unitary) symmetry G_f^{eff} and chiral central charge c_- . Since the symmetries we consider contain $U(1)^f$ charge conservation as a subgroup, a number of simplifications occur:

1. The chiral central charge c_- must be an integer.
2. A system with $U(1)^f$ charge conservation symmetry cannot host unpaired Majorana zero modes at symmetry defects.

If we work out the theory with these simplifications, we find that

$$\mathcal{L} = 2\pi \tilde{B}^* \nu_3, \quad (\text{B.25})$$

$$d\nu_3 = \mathcal{O}_4 = \frac{1}{2}n_2 \cup (n_2 + \omega_2^{\text{eff}}) + \frac{c_-}{8}\omega_2^{\text{eff}} \cup \omega_2^{\text{eff}} \mod 1 \quad (\text{B.26})$$

where n_2 is a general element of $\mathcal{H}^2(G_b, \mathbb{Z}_2)$, that is, $dn_2 = 0 \mod 2$.¹ In terms of the generators of $\mathcal{H}^2(G_b, \mathbb{Z}_2)$, the most general form of n_2 is

$$n_2 = qz_2 + sh_2 + \vec{t} \cdot \vec{\mathfrak{t}}_2 + ma_2. \quad (\text{B.27})$$

¹In the general form of Eq. (B.26), there is an extra term $\frac{c_-}{8}\omega_2^{\text{eff}} \cup_1 d\omega_2^{\text{eff}}$. Moreover, the equation for n_2 is more generally $dn_2 = c_- \omega_2^{\text{eff}} \cup_1 \omega_2^{\text{eff}} \mod 2$. However, for the symmetries we consider, the terms with $\cup_1$ products turn out to vanish. Moreover, when we write $\frac{c_-}{8}\omega_2^{\text{eff}} \cup \omega_2^{\text{eff}}$, what we mean by ω_2^{eff} is strictly speaking a class in $\mathcal{H}^2(G_b, \mathbb{Z})$ whose mod 2 reduction equals ω_2^{eff} .

Note that in the general theory, n_2 and $n_2 + \omega_2^{\text{eff}}$ are equivalent [27]. Using this fact, we can set $q = 0$ without loss of generality. By substituting Eqs. (B.22), (B.27) into Eq. (B.26), and simplifying, we find

$$d\nu_3 = \frac{1}{2}(t_x \mathfrak{t}_{2,x}^2 + t_y \mathfrak{t}_{2,y}^2 + ma_2^2 + h_2(t_x \mathfrak{t}_{2,x} + t_y \mathfrak{t}_{2,y} + ma_2)) + \frac{1}{2}z_2(sh_2 + t_x \mathfrak{t}_{2,x} + t_y \mathfrak{t}_{2,y} + ma_2) + \frac{c_-}{8}(z_2^2 + 2z_2 h_2 + h_2^2). \quad (\text{B.28})$$

Here we have suppressed the cup product notation for conciseness. We also used the fact that for 2-cocycles α_2, β_2 , $\frac{1}{2}\alpha_2\beta_2 = \frac{1}{2}\beta_2\alpha_2 \pmod{1}$ up to coboundaries; this simplifies many of the mixed terms.

Eq. (B.28) is the main equation we will consider separately for $M = 2, 3, 4, 6$. It can be further simplified using the properties of the cohomology ring of G_b ; this essentially shows that the various summands in Eq. (B.28) are not independent, but rather there exist various relations between them which depend on the value of M .

We can integrate this equation to find ν_3 . Note that there is a constant of integration λ_3 satisfying $d\lambda_3 = 0 \pmod{1}$; this corresponds precisely to a G_b bosonic SPT phase classified by the group $\mathcal{H}^3(G_b, \text{U}(1))$. Therefore the expression for ν_3 should include a generic contribution from a G_b bosonic SPT, represented by the cocycle in Eq. (B.21). This will be verified below.

Using the definition $\mathcal{L} = 2\pi\tilde{B}^*\nu_3$, we obtain an expression in terms of $\tilde{A}_b$ and the other gauge field components. However, this is unsatisfactory because the physical system is described more naturally by the $\text{U}(1)^f$ gauge field $\tilde{A} = \tilde{A}_b/2$, under which the fermion has charge 1. (Note that the correctly normalized vector potential for the system is $A = 2\pi\tilde{A}$.) The final step is thus to replace $\tilde{A}_b \rightarrow 2\tilde{A}$, and then switch to the original, untilded gauge fields.

B.3.2 Calculations for specific M

B.3.2.1 $M = 4$

In the main text we took $M = 4$. Here we find that $[\mathfrak{t}_{2,x}] = [\mathfrak{t}_{2,y}] = [\mathfrak{t}_2]$. Moreover, using the ring structure associated to the group cohomology of G_b , one can show the following relations:

$$[\mathfrak{a}_2^2] = [\mathfrak{h}_2 \mathfrak{a}_2] \pmod{2}, \quad [\mathfrak{t}_2^2] = [\mathfrak{t}_2 \mathfrak{h}_2] \pmod{2}. \quad (\text{B.29})$$

Defining $t = t_x + t_y \pmod{2}$, we can simplify Eq. (B.28) as follows:

$$d\nu_3 = \frac{1}{2} z_2 (s \mathfrak{h}_2 + t \mathfrak{t}_2 + m \mathfrak{a}_2) + \frac{c_-}{8} (z_2^2 + z_2 \mathfrak{h}_2 + \mathfrak{h}_2^2) \pmod{1}. \quad (\text{B.30})$$

The full expression obtained after integrating $d\nu_3$ and simplifying is

$$\begin{aligned} \nu_3 = & \left(\frac{c_-}{8} + k_1 \right) z_1 z_2 + \left(\frac{c_-}{4} + \frac{[s]_2}{2} + k_2^+ \right) \mathfrak{h}_1 z_2 + \left(\frac{c_-}{8} + k_3^+ \right) \mathfrak{h}_1 \mathfrak{h}_2 \\ & + \left(\frac{[t]_2}{2} + k_4 \right) z_1 \mathfrak{t}_2 + k_5^+ \mathfrak{h}_1 \mathfrak{t}_2 + \left(\frac{[m]_2}{2} + k_6 \right) z_1 \mathfrak{a}_2 + k_7^+ \mathfrak{h}_1 \mathfrak{a}_2 \pmod{1}, \end{aligned} \quad (\text{B.31})$$

where

$$k_1, k_6 \in \mathbb{Z}; \quad k_2^+, k_3^+, k_7^+ \in \mathbb{Z}_4; \quad k_4, k_5^+ \in \mathbb{Z}_2. \quad (\text{B.32})$$

The coefficients s, t, m, k_i all in principle have a dependence on the origin $\mathfrak{o}$, which we have suppressed here but will restore at the end of the calculation. Note that $\tilde{B}^*_{z_1} = \tilde{A}_b, \tilde{B}^*_{\mathfrak{h}_1} =$

$\tilde{\omega}, \tilde{B}^* \mathbf{t}_2 = \frac{1}{2}(1, 1) \cdot d\tilde{R}$. Therefore

$$\begin{aligned} \tilde{B}^* \nu_3 &= \left(\frac{c_-}{8} + k_1 \right) \tilde{A}_b \cup d\tilde{A}_b + \left(\frac{c_-}{4} + \frac{[s]_2}{2} + k_2^+ \right) \tilde{\omega} \cup d\tilde{A}_b + \left(\frac{c_-}{8} + k_3^+ \right) \tilde{\omega} \cup d\tilde{\omega} \\ &+ \frac{(1, 1)}{2} \cdot \left(\frac{[t]_2}{2} + k_4 \right) \tilde{A}_b \cup d\tilde{R} + \frac{(1, 1)}{2} \cdot k_5^+ \tilde{\omega} \cup d\tilde{R} + \left(\frac{[m]_2}{2} + k_6 \right) \tilde{A}_b \cup \tilde{A}_{XY} \\ &+ k_7^+ \tilde{\omega} \cup \tilde{A}_{XY} \mod 1. \end{aligned} \quad (\text{B.33})$$

Next, as explained above, we replace $\tilde{A}_b = 2\tilde{A}$, where $\tilde{A}$ is a $U(1)^f$ gauge field, to obtain

$$\begin{aligned} \tilde{B}^* \nu_3 &= \frac{c_- + 8k_1}{2} \tilde{A} \cup d\tilde{A} + \left(\frac{c_-}{2} + [s]_2 + 2k_2^+ \right) \tilde{\omega} \cup d\tilde{A}_b + \left(\frac{c_-}{8} + k_3^+ \right) \tilde{\omega} \cup d\tilde{\omega} \\ &+ \frac{(1, 1)}{2} \cdot ([t]_2 + 2k_4) \tilde{A} \cup d\tilde{R} + \frac{(1, 1)}{2} \cdot k_5^+ \tilde{\omega} \cup d\tilde{R} + ([m]_2 + 2k_6) \tilde{A} \cup \tilde{A}_{XY} + k_7^+ \tilde{\omega} \cup \tilde{A}_{XY} \mod 1. \end{aligned} \quad (\text{B.34})$$

Finally, we multiply by 2π to obtain $\mathcal{L} = 2\pi \tilde{B}^* \nu_3$ in terms of the tilded gauge fields, and then replace each $\tilde{X}$ with $X/2\pi$. The resulting action (with o subscripts restored) is

$$\begin{aligned} \mathcal{L} &= \frac{c_- + 8k_1}{4\pi} A \cup dA + \left(\frac{c_-}{2} + [s_o]_2 + 2k_{2,o}^+ \right) \frac{\omega \cup dA}{2\pi} + \left(\frac{c_-}{4} + 2k_{3,o}^+ \right) \frac{\omega \cup d\omega}{4\pi} \\ &+ \frac{(1, 1)}{2} \cdot ([t_o]_2 + 2k_{4,o}) \frac{A \cup d\vec{R}}{2\pi} + \frac{(1, 1)}{2} \cdot k_{5,o}^+ \frac{\omega \cup d\vec{R}}{2\pi} + ([m]_2 + 2k_6) \frac{A \cup A_{XY}}{2\pi} \\ &+ k_7^+ \frac{\omega \cup A_{XY}}{2\pi} \mod 2\pi. \end{aligned} \quad (\text{B.35})$$

m, k_6, k_7^+ set the charge and angular momentum per unit cell, and are independent of o.

Using the quantization of k_i obtained in Eq. (B.32), we can define the coefficients of the action as given in Eq. (3.21). This completes the derivation of the effective action prior to modding out by equivalences.

From the general theory outlined in Ref. [27], once n_2 is gauge fixed (by setting $q = 0$), the only remaining equivalences are on ν_3 and are given by

$$\nu_3 \simeq \nu_3 + \frac{1}{2}\chi_1 \cup \omega_2^{\text{eff}}, \quad (\text{B.36})$$

where χ_1 runs over the group $\mathcal{H}^1(G_b, \mathbb{Z}_2)$. These equivalences correspond to relabelling G_b defects (in this case, disclination and dislocation defects as determined by the choice of χ_1) by fermions. A general form for χ_1 is

$$\chi_1 = a(4h_1) + b(2t_1) \quad (\text{B.37})$$

where we define $t_1 \in C^1(G_b, \mathbb{R}/\mathbb{Z})$ so that $dt_1 = t_2 \pmod{2}$. Therefore,

$$\begin{aligned} \nu_3 &\simeq \nu_3 + \frac{1}{2}(4ah_1 + 2bt_1)(z_2 + h_2) \\ &\simeq \nu_3 + 2ah_1(z_2 + h_2) + b(z_1 + h_1)t_2. \end{aligned} \quad (\text{B.38})$$

Here we used the fact that $z_1 t_2 = t_1 z_2$ and $h_1 t_2 = t_1 h_2$ up to coboundaries. If we now attribute the change in ν_3 to the coefficients k_i (which is reasonable because the relabelling can be thought of as stacking a bosonic SPT), we obtain Eq. (3.22).

B.3.2.2 $M = 2$

When $M = 2$, we find using the ring structure in group cohomology that

$$[a_2^2] = [h_2 a_2], \quad [t_{2,i}^2] = [t_{2,i} h_2], \quad i = x, y. \quad (\text{B.39})$$

We can simplify Eq. (B.28) as follows:

$$d\nu_3 = \frac{1}{2}z_2(sh_2 + \vec{t} \cdot \vec{t}_2 + ma_2) + \frac{c_-}{8}(z_2^2 + z_2h_2 + h_2^2) \mod 1. \quad (\text{B.40})$$

The full expression for ν_3 is

$$\begin{aligned} \nu_3 = & \left(\frac{c_-}{8} + k_1\right) z_1 z_2 + \left(\frac{c_-}{4} + \frac{[s]_2}{2} + k_2^+\right) h_1 z_2 + \left(\frac{c_-}{8} + k_3^+\right) h_1 h_2 \\ & + \left(\frac{[\vec{t}]_2}{2} + \vec{k}_4\right) \cdot z_1 \vec{t}_2 + \vec{k}_5^+ \cdot h_1 \vec{t}_2 + \left(\frac{[m]_2}{2} + k_6\right) z_1 a_2 + k_7^+ h_1 a_2 \mod 1, \end{aligned} \quad (\text{B.41})$$

where

$$k_1, k_6 \in \mathbb{Z}; \quad k_2^+, k_3^+, k_{4,i}, k_{5,i}^+, k_7^+ \in \mathbb{Z}_2. \quad (\text{B.42})$$

The coefficients s, t, m, k_i all in principle have a dependence on the origin $\mathbf{o}$, which we have suppressed here but will restore at the end of the calculation. Note that $\tilde{B}^* z_1 = \tilde{A}_b \cdot \tilde{B}^* h_1 = \tilde{\omega}$, $\tilde{B}^* \vec{t}_2 = \frac{1}{2} \cdot d\vec{R}$. Therefore

$$\begin{aligned} \tilde{B}^* \nu_3 = & \left(\frac{c_-}{8} + k_1\right) \tilde{A}_b \cup d\tilde{A}_b + \left(\frac{c_-}{4} + \frac{[s]_2}{2} + k_2^+\right) \tilde{\omega} \cup d\tilde{A}_b + \left(\frac{c_-}{8} + k_3^+\right) \tilde{\omega} \cup d\tilde{\omega} \\ & + \frac{1}{2} \left(\frac{[\vec{t}]_2}{2} + \vec{k}_4\right) \cdot \tilde{A}_b \cup d\vec{R} + \frac{1}{2} \vec{k}_5^+ \cdot \tilde{\omega} \cup d\vec{R} + \left(\frac{[m]_2}{2} + k_6\right) \tilde{A}_b \cup \tilde{A}_{XY} + k_7^+ \tilde{\omega} \cup \tilde{A}_{XY} \mod 1. \end{aligned} \quad (\text{B.43})$$

Replacing $\tilde{A}_b = 2\tilde{A}$ gives

$$\begin{aligned}\tilde{B}^* \nu_3 &= \frac{c_- + 8k_1}{2} \tilde{A} \cup d\tilde{A} + \left(\frac{c_-}{2} + [s]_2 + 2k_2^+ \right) \tilde{\omega} \cup d\tilde{A}_b + \left(\frac{c_-}{8} + k_3^+ \right) \tilde{\omega} \cup d\tilde{\omega} \\ &+ \frac{1}{2} ([\vec{t}]_2 + 2\vec{k}_4) \cdot \tilde{A} \cup d\vec{R} + \frac{1}{2} \vec{k}_5^+ \cdot \tilde{\omega} \cup d\vec{R} + ([m]_2 + 2k_6) \tilde{A} \cup \tilde{A}_{XY} + k_7^+ \tilde{\omega} \cup \tilde{A}_{XY} \pmod{1}.\end{aligned}\tag{B.44}$$

Finally, we replace each $\tilde{X}$ with $X/2\pi$ and write the resulting action (with o subscripts restored):

$$\begin{aligned}\mathcal{L} &= \frac{c_- + 8k_1}{4\pi} A \cup dA + \left(\frac{c_-}{2} + [s_o]_2 + 2k_{2,o}^+ \right) \frac{\omega \cup dA}{2\pi} + \left(\frac{c_-}{4} + 2k_{3,o}^+ \right) \frac{\omega \cup d\omega}{4\pi} \\ &+ \frac{1}{2} (\vec{t}_o]_2 + 2\vec{k}_{4,o}) \cdot \frac{A \cup d\vec{R}}{2\pi} + \frac{1}{2} \vec{k}_{5,o}^+ \cdot \frac{\omega \cup d\vec{R}}{2\pi} + ([m]_2 + 2k_6) \frac{A \cup A_{XY}}{2\pi} + k_7^+ \frac{\omega \cup A_{XY}}{2\pi} \pmod{2\pi}.\end{aligned}\tag{B.45}$$

Here m, k_6, k_7^+ are independent of o. Using Eq. (B.42), we can define the coefficients of the action as

$$\begin{aligned}C &= c_- + 8k_1, & k_1 &\in \mathbb{Z} \\ \mathcal{S}_o^+ &= \frac{c_-}{2} + [s_o]_2 + 2k_{2,o}^+ \pmod{4}, & s_o &\in \mathbb{Z}_2, k_{2,o}^+ \in \mathbb{Z}_2 \\ \ell_{s,o}^+ &= \frac{c_-}{4} + 2k_{3,o}^+ \pmod{4}, & k_{3,o}^+ &\in \mathbb{Z}_2 \\ \vec{\mathcal{P}}_o &= \frac{([\vec{t}_o]_2 + 2\vec{k}_{4,o})}{2}, & \vec{t}_o, \vec{k}_{4,o} &\in \mathbb{Z}_2 \times \mathbb{Z}_2 \\ \vec{\mathcal{P}}_{s,o}^+ &= \frac{\vec{k}_{5,o}^+}{2}, & \vec{k}_{5,o}^+ &\in \mathbb{Z}_2 \times \mathbb{Z}_2 \\ \kappa &\in \mathbb{Z} \\ \nu_s^+ &\in \mathbb{Z}_2.\end{aligned}\tag{B.46}$$

This completes the derivation of the effective action prior to modding out by equivalences.

The remaining equivalences on ν_3 are given by

$$\nu_3 \simeq \nu_3 + \frac{1}{2}\chi_1 \cup \omega_2^{\text{eff}}, \quad (\text{B.47})$$

where χ_1 runs over $\mathcal{H}^1(G_b, \mathbb{Z}_2)$. A general form for χ_1 is

$$\chi_1 = a(2h_1) + \vec{b} \cdot (2t\vec{t}_1) \pmod{2} \quad (\text{B.48})$$

where we define $\mathfrak{t}_{1,i} \in C^1(G_b, \mathbb{R}/\mathbb{Z})$ for $i = x, y$ so that $d\vec{\mathfrak{t}}_1 = \vec{\mathfrak{t}}_2 \pmod{2}$. Therefore,

$$\begin{aligned} \nu_3 &\simeq \nu_3 + \frac{1}{2}(2ah_1 + 2b\mathfrak{t}_1)(z_2 + h_2) \\ &\simeq \nu_3 + ah_1(z_2 + h_2) + \vec{b} \cdot (z_1 + h_1)\vec{\mathfrak{t}}_2 \pmod{1}. \end{aligned} \quad (\text{B.49})$$

Attributing this change in ν_3 to the coefficients k_i as before, we obtain

$$\begin{aligned} (k_{2,0}^+, k_{3,0}^+) &\simeq (k_{2,0}^+ + a, k_{3,0}^+ + a) \\ (\vec{k}_{4,0}, \vec{k}_{5,0}^+) &\simeq (\vec{k}_{4,0} + \vec{b}, \vec{k}_{5,0}^+ + \vec{b}) \end{aligned} \quad (\text{B.50})$$

where we assume all other coefficients remain unchanged. After modding out by the equivalences,

a complete and independent set of invariants for a fixed $\mathfrak{o}$ is given by

$$\begin{aligned}
c_-, \kappa, k_1 &\in \mathbb{Z} \\
\mathcal{S}_\mathfrak{o}^+ - \ell_{s,\mathfrak{o}}^+ - \frac{c_-}{4} &\in \mathbb{Z}_4 \\
2(\vec{\mathcal{P}}_\mathfrak{o} - 2\vec{\mathcal{P}}_{s,\mathfrak{o}}^+) &\in \mathbb{Z}_4 \times \mathbb{Z}_4 \\
\nu_s^+ &\in \mathbb{Z}_2.
\end{aligned} \tag{B.51}$$

Thus the full classification is $\mathbb{Z}^3 \times \mathbb{Z}_4^3 \times \mathbb{Z}_2$.

B.3.2.3 $M = 3$

When $M = 3$, we need to set $\vec{\mathfrak{t}}_2 \bmod 2$ to be trivial in $\mathcal{H}^2(G_b, \mathbb{Z}_2)$, although $\vec{\mathfrak{t}}_2$ generates a $\mathbb{Z}_3$ factor in $\mathcal{H}^2(G_b, \mathbb{Z})$. We also need to set $s = 0$, because $\mathfrak{h}_2 \bmod 2$ is trivial in $\mathcal{H}^2(G_b, \mathbb{Z}_2)$.

Furthermore, we have

$$[\mathfrak{a}_2^2] = [\mathfrak{h}_2 \mathfrak{a}_2]. \tag{B.52}$$

Therefore we can simplify Eq. (B.28) as follows:

$$d\nu_3 = \frac{1}{2}z_2(m\mathfrak{a}_2) + \frac{c_-}{8}(z_2^2 + z_2\mathfrak{h}_2 + \mathfrak{h}_2^2) \bmod 1. \tag{B.53}$$

The full expression for ν_3 is

$$\begin{aligned}
\nu_3 = & \left(\frac{c_-}{8} + k_1\right) z_1 z_2 + \left(\frac{c_-}{4} + k_2^+\right) \mathfrak{h}_1 z_2 + \left(\frac{c_-}{8} + k_3^+\right) \mathfrak{h}_1 \mathfrak{h}_2 \\
& + \vec{k}_4 \cdot z_1 \vec{\mathfrak{t}}_2 + \vec{k}_5^+ \cdot \mathfrak{h}_1 \vec{\mathfrak{t}}_2 + \left(\frac{[m]_2}{2} + k_6\right) z_1 \mathfrak{a}_2 + k_7^+ \mathfrak{h}_1 \mathfrak{a}_2 \bmod 1,
\end{aligned} \tag{B.54}$$

where

$$k_1, k_6 \in \mathbb{Z}; \quad \vec{k}_4 := k_4(1, 2); \vec{k}_5^+ := k_5^+(1, 2); \quad k_2^+, k_3^+, k_4, k_5^+, k_7^+ \in \mathbb{Z}_3. \quad (\text{B.55})$$

The coefficients k_i all in principle have a dependence on the origin $\mathbf{o}$, which we have suppressed here but will restore at the end of the calculation. Now using $\tilde{B}^*_{\mathbf{z}_1} = \tilde{A}_b, \tilde{B}^*_{\mathbf{h}_1} = \tilde{\omega}, \tilde{B}^*_{\mathbf{t}_2} = \frac{1}{3}d\vec{R}$. Therefore

$$\begin{aligned} \tilde{B}^*_{\nu_3} = & \left(\frac{c_-}{8} + k_1 \right) \tilde{A}_b \cup d\tilde{A}_b + \left(\frac{c_-}{4} + k_2^+ \right) \tilde{\omega} \cup d\tilde{A}_b + \left(\frac{c_-}{8} + k_3^+ \right) \tilde{\omega} \cup d\tilde{\omega} \\ & + \frac{1}{3}(1, 2) \cdot k_4 \tilde{A}_b \cup d\vec{R} + \frac{1}{3}(1, 2) \cdot k_5^+ \tilde{\omega} \cup d\vec{R} + \left(\frac{[m]_2}{2} + k_6 \right) \tilde{A}_b \cup \tilde{A}_{XY} + k_7^+ \tilde{\omega} \cup \tilde{A}_{XY} \mod 1. \end{aligned} \quad (\text{B.56})$$

Replacing $\tilde{A}_b = 2\tilde{A}$ gives

$$\begin{aligned} \tilde{B}^*_{\nu_3} = & \frac{c_- + 8k_1}{2} \tilde{A} \cup d\tilde{A} + \left(\frac{c_-}{2} + 2k_2^+ \right) \tilde{\omega} \cup d\tilde{A}_b + \left(\frac{c_-}{8} + k_3^+ \right) \tilde{\omega} \cup d\tilde{\omega} \\ & + \frac{k_4}{3}(1, 2) \cdot \tilde{A} \cup d\vec{R} + \frac{k_5^+}{3}(1, 2) \cdot \tilde{\omega} \cup d\vec{R} + ([m]_2 + 2k_6) \tilde{A} \cup \tilde{A}_{XY} + k_7^+ \tilde{\omega} \cup \tilde{A}_{XY} \mod 1. \end{aligned} \quad (\text{B.57})$$

Finally, we replace each $\tilde{X}$ with $X/2\pi$ and write the resulting action (with $\mathbf{o}$ subscripts restored):

$$\begin{aligned} \mathcal{L} = & \frac{c_- + 8k_1}{4\pi} A \cup dA + \left(\frac{c_-}{2} + 2k_{2,\mathbf{o}}^+ \right) \frac{\omega \cup dA}{2\pi} + \left(\frac{c_-}{4} + 2k_{3,\mathbf{o}}^+ \right) \frac{\omega \cup d\omega}{4\pi} \\ & + \frac{k_4}{3}(1, 2) \cdot \frac{A \cup d\vec{R}}{2\pi} + \frac{k_5^+}{3}(1, 2) \cdot \frac{\omega \cup d\vec{R}}{2\pi} + ([m]_2 + 2k_6) \frac{A \cup A_{XY}}{2\pi} + k_7^+ \frac{\omega \cup A_{XY}}{2\pi} \mod 2\pi. \end{aligned} \quad (\text{B.58})$$

Here m, k_6, k_7^+ are independent of $\mathbf{o}$. Using Eq. (B.55), we can define the coefficients of the action as

$$\begin{aligned}
C &= c_- + 8k_1, & k_1 &\in \mathbb{Z} \\
\mathcal{S}_\mathbf{o}^+ &= \frac{c_-}{2} + 2k_{2,\mathbf{o}}^+ \pmod{6}, & k_{2,\mathbf{o}}^+ &\in \mathbb{Z}_3 \\
\ell_{s,\mathbf{o}}^+ &= \frac{c_-}{4} + 2k_{3,\mathbf{o}}^+ \pmod{6}, & k_{3,\mathbf{o}}^+ &\in \mathbb{Z}_3 \\
\vec{\mathcal{P}}_\mathbf{o} &= \frac{k_{4,\mathbf{o}}}{3}(1, 2) \pmod{\mathbb{Z}^2}, & k_{4,\mathbf{o}} &\in \mathbb{Z}_3 \\
\vec{\mathcal{P}}_{s,\mathbf{o}}^+ &= \frac{k_{5,\mathbf{o}}^+}{2}(1, 2) \pmod{\mathbb{Z}^2}, & k_{5,\mathbf{o}}^+ &\in \mathbb{Z}_3 \\
\kappa &\in \mathbb{Z} \\
\nu_s^+ &\in \mathbb{Z}_3.
\end{aligned} \tag{B.59}$$

$$\tag{B.60}$$

In this case there are no further equivalences, so a complete set of invariants is given by

$$\begin{aligned}
&c_-, \kappa, k_1 \in \mathbb{Z} \\
&\mathcal{S}_\mathbf{o}^+ - \frac{c_-}{2}, \ell_{s,\mathbf{o}}^+ - \frac{c_-}{4}, 3\vec{\mathcal{P}}_\mathbf{o} \cdot (1, 0), 3\vec{\mathcal{P}}_{s,\mathbf{o}}^+ \cdot (1, 0), \nu_s^+ \in \mathbb{Z}_3.
\end{aligned} \tag{B.61}$$

Thus the full classification is $\mathbb{Z}^3 \times \mathbb{Z}_3^5$.

B.3.2.4 $M = 6$

When $M = 6$, we need to set $\vec{\mathbf{t}}_2$ to be trivial in $\mathcal{H}^2(G_b, \mathbb{Z})$. Therefore torsional terms depending on $d\vec{R}$ do not appear in the result at all. We also have

$$[\mathbf{a}_2^2] = [\mathbf{h}_2 \mathbf{a}_2]. \tag{B.62}$$

Therefore we can simplify Eq. (B.28) as follows:

$$d\nu_3 = \frac{1}{2}z_2(sh_2 + ma_2) + \frac{c_-}{8}(z_2^2 + z_2h_2 + h_2^2) \mod 1. \quad (\text{B.63})$$

The full expression for ν_3 is

$$\begin{aligned} \nu_3 = & \left(\frac{c_-}{8} + k_1\right) z_1 z_2 + \left(\frac{c_-}{4} + \frac{[s]_2}{2} + k_2^+\right) h_1 z_2 + \left(\frac{c_-}{8} + k_3^+\right) h_1 h_2 \\ & + \left(\frac{[m]_2}{2} + k_6\right) z_1 a_2 + k_7^+ h_1 a_2 \mod 1, \end{aligned} \quad (\text{B.64})$$

where

$$k_1, k_6 \in \mathbb{Z}; \quad k_2^+, k_3^+, k_7^+ \in \mathbb{Z}_6. \quad (\text{B.65})$$

The coefficients s, k_i all in principle have a dependence on the origin $\mathfrak{o}$, which we have suppressed here but will restore at the end of the calculation. Now using $\tilde{B}^* z_1 = \tilde{A}_b, \tilde{B}^* h_1 = \tilde{\omega}, \tilde{B}^* \vec{\mathfrak{t}}_2 = \frac{1}{3}d\tilde{R}$. Therefore

$$\begin{aligned} \tilde{B}^* \nu_3 = & \left(\frac{c_-}{8} + k_1\right) \tilde{A}_b \cup d\tilde{A}_b + \left(\frac{c_-}{4} + \frac{[s]_2}{2} + k_2^+\right) \tilde{\omega} \cup d\tilde{A}_b + \left(\frac{c_-}{8} + k_3^+\right) \tilde{\omega} \cup d\tilde{\omega} \\ & + \left(\frac{[m]_2}{2} + k_6\right) \tilde{A}_b \cup \tilde{A}_{XY} + k_7^+ \tilde{\omega} \cup \tilde{A}_{XY} \mod 1. \end{aligned} \quad (\text{B.66})$$

Replacing $\tilde{A}_b = 2\tilde{A}$ gives

$$\begin{aligned} \tilde{B}^* \nu_3 = & \frac{c_- + 8k_1}{2} \tilde{A} \cup d\tilde{A} + \left(\frac{c_-}{2} + [s]_2 + 2k_2^+\right) \tilde{\omega} \cup d\tilde{A}_b + \left(\frac{c_-}{8} + k_3^+\right) \tilde{\omega} \cup d\tilde{\omega} \\ & + ([m]_2 + 2k_6) \tilde{A} \cup \tilde{A}_{XY} + k_7^+ \tilde{\omega} \cup \tilde{A}_{XY} \mod 1. \end{aligned} \quad (\text{B.67})$$

Finally, we replace each $\tilde{X}$ with $X/2\pi$ and write the resulting action (with o subscripts restored):

$$\begin{aligned} \mathcal{L} = & \frac{c_- + 8k_1}{4\pi} A \cup dA + \left(\frac{c_-}{2} + [s_o]_2 + 2k_{2,o}^+ \right) \frac{\omega \cup dA}{2\pi} + \left(\frac{c_-}{4} + 2k_{3,o}^+ \right) \frac{\omega \cup d\omega}{4\pi} \\ & + ([m]_2 + 2k_6) \frac{A \cup A_{XY}}{2\pi} + k_7^+ \frac{\omega \cup A_{XY}}{2\pi} \mod 2\pi. \end{aligned} \quad (\text{B.68})$$

Here m, k_6, k_7^+ are independent of o. Using Eq. (B.65), we can define the coefficients of the action as

$$\begin{aligned} C &= c_- + 8k_1, & k_1 &\in \mathbb{Z} \\ \mathcal{S}_o^+ &= \frac{c_-}{2} + [s_o]_2 + 2k_{2,o}^+ \mod 12, & s_o &\in \mathbb{Z}_2, k_{2,o}^+ \in \mathbb{Z}_6 \\ \ell_{s,o}^+ &= \frac{c_-}{4} + 2k_{3,o}^+ \mod 12, & k_{3,o}^+ &\in \mathbb{Z}_6 \\ \kappa &\in \mathbb{Z} \\ \nu_s^+ &\in \mathbb{Z}_6. \end{aligned} \quad (\text{B.69})$$

$$(\text{B.70})$$

The remaining equivalences on ν_3 are given by

$$\nu_3 \simeq \nu_3 + \frac{1}{2} \chi_1 \cup \omega_2^{\text{eff}}, \quad (\text{B.71})$$

where χ_1 runs over $\mathcal{H}^1(G_b, \mathbb{Z}_2)$. A general form for χ_1 is

$$\chi_1 = a(6h_1) \mod 2. \quad (\text{B.72})$$

Therefore,

$$\nu_3 \simeq \nu_3 + ah_1(z_2 + h_2) \pmod{1}. \quad (\text{B.73})$$

This gives

$$(k_{2,0}^+, k_{3,0}^+) \simeq (k_{2,0}^+ + 3a, k_{3,0}^+ + 3a), \quad a \in \mathbb{Z}_2. \quad (\text{B.74})$$

A complete set of invariants is given by

$$\begin{aligned} c_-, \kappa, k_1 &\in \mathbb{Z} \\ \mathcal{S}_0^+ - \ell_{s,0}^+ - \frac{c_-}{4} &\in \mathbb{Z}_{12} \\ \frac{1}{2} \left(\ell_{s,0}^+ - \frac{c_-}{4} \right) &\in \mathbb{Z}_3 \\ \nu_s^+ &\in \mathbb{Z}_6. \end{aligned} \quad (\text{B.75})$$

Thus the full classification is $\mathbb{Z}^3 \times \mathbb{Z}_{12} \times \mathbb{Z}_3 \times \mathbb{Z}_6$.

B.3.3 The case with $\phi \neq 0$

Extending the derivation to any $\phi \neq 0$ turns out to be straightforward. The final result is that for any M , the complete topological action is still given by Eq. (3.20) with the coefficients still quantized according to Eqs. (3.23), (B.51), (B.61), (B.75). Here A is understood as the full vector potential, so that $\int_W dA = \phi \int_W A_{XY} + \int_W \delta A$ gives the full magnetic flux within the system, including a uniform background contribution proportional to ϕ , and any excess flux, if present. We explain this below.

We have $G_b = \text{U}(1) \times_{2\phi} [\mathbb{Z}^2 \rtimes \mathbb{Z}_M]$. The notation $\times_{2\phi}$ is used because the elementary boson, which is a bound state of two fermions, sees 2ϕ flux per unit cell. Proceeding as in App. B.3.1, we define a G_b gauge field $\tilde{B} = (\delta\tilde{A}_b, \tilde{\vec{R}}, \tilde{\omega})$. Crucially, $\delta\tilde{A}_b$ is not the full vector potential for the bosonic $\text{U}(1)$ symmetry; it is the deviation of $\tilde{A}_b$ from a reference gauge field that assigns uniform flux 2ϕ per unit cell. But if $\phi = 0$, we have $\delta\tilde{A}_b = \tilde{A}_b$.

Since the fermion sees half the total flux seen by an elementary boson, the full vector potential $\tilde{A}$ is related to $\delta\tilde{A}_b$ as follows:

$$d\tilde{A} = \frac{1}{2}d\tilde{A}_b = \frac{1}{2} \left(d\delta\tilde{A}_b + \frac{2\phi}{2\pi}\tilde{A}_{XY} \right). \quad (\text{B.76})$$

The main steps for $\phi \neq 0$ only differ in terms of the definitions of n_2, ω_2 . Note that $z_2 = dz_1$ is not longer an element of $\mathcal{H}^2(G_b, \mathbb{Z})$ when $\phi \neq 0$; we need to replace this with the quantity $dz_1 + \frac{2\phi}{2\pi}a_2$, which is a valid cocycle representative. The other generators of $\mathcal{H}^2(G_b, \mathbb{Z})$ remain unchanged.

If we then go through the calculation as before, we eventually obtain Eq. (B.28) with z_2 (the $\phi = 0$ result) replaced by $dz_1 + \frac{2\phi}{2\pi}a_2$. Therefore in the resulting field theory we should replace $\tilde{B}^*_{z_2} = d\delta\tilde{A}_b$ with

$$\tilde{B}^*(z_2 + \frac{2\phi}{2\pi}a_2) = d\delta\tilde{A}_b + \frac{2\phi}{2\pi}A_{XY}.$$

But from Eq. (B.76), this just equals $\tilde{B}^*(2d\tilde{A})$. Therefore, in symbolic terms, we need to make the same replacement for $\phi = 0$ and $\phi \neq 0$. Thus we have proven our main claim that Eq. (3.20) continues to hold, but A is now understood as the full vector potential for the system.

It is also straightforward to verify that the quantization of the response coefficients does

not depend on ϕ . Note, however, that some important properties which do not directly appear in Eq. (3.20) indeed acquire a ϕ -dependence, such as the filling per unit cell ν ; this was explained in the main text.

Appendix C: Appendix for Chapter 4

C.1 Review

This section has three parts. First we introduce some standard definitions of quantities on a clean lattice. Next we introduce dislocation and disclination defects, and define the disclination angle and dislocation Burgers vector. After that we discuss the basic properties of $\mathcal{S}_0$ and $\vec{\mathcal{P}}_0$ from the perspective of disclination and dislocation charge respectively.

C.1.1 Definitions and background

C.1.1.1 Maximal Wyckoff positions

The definitions in this section are taken from Ref. [210]. Fix an origin o . Suppose we are given a set of points in the infinite plane which form a lattice with spatial symmetry group G_{space} . On this lattice (assumed to be without any defects), we define a Hamiltonian H_{clean} with translation operators $\hat{T}_x, \hat{T}_y$ corresponding to translations by the elementary lattice vectors, and additional point group symmetry operators, defined w.r.t. o . The full symmetry group G of H_{clean} can contain operations in G_{space} as well as internal symmetry operations.

For any point p in the plane, the *site symmetry group* G_p is the subgroup of operations in G_{space} that leaves p invariant. Two points p, p' are said to have site symmetry groups that are

conjugate to each other if $G_{p'} = \mathbf{g}G_p\mathbf{g}^{-1}$ for some $\mathbf{g} \in G_{\text{space}}$. The *Wyckoff position* containing p is the set of all points whose site symmetry groups are conjugate to G_p . For example, every point p' obtained from p by a lattice translation or rotation is in the same Wyckoff position as p . Furthermore, every point with a trivial site symmetry group belongs to a single Wyckoff position.

p is in a *maximal* Wyckoff position (MWP) if G_p is not a proper subgroup of $G_{p'}$ for any other site p' . In Fig. 4.2, we show the MWPs for the wallpaper groups p2, p3, p4, p6. Note that when $G_{\text{space}} = \text{p4}$, the high symmetry point γ_i has site symmetry group $G_\gamma \cong \mathbb{Z}_2$ but is still in a MWP, even though there are other points with site symmetry group $\mathbb{Z}_4$. This is because there is no point with site symmetry group $\mathbb{Z}_4$ that contains G_γ . We will always choose $\mathbf{o}$ to belong to a MWP.

It is important to distinguish a MWP, which is a collection of points, from a single high symmetry point (HSP) of a unit cell. Our notation for HSPs is β_i, γ_i where β, γ denotes the MWP and i runs over the corresponding HSPs which are inequivalent under lattice translations.

C.1.1.2 Unit cells and subcells

A *unit cell* Θ for the given lattice corresponds to a division of lattice points into elementary repeating units. Starting with a clean lattice on the infinite plane, pick a HSP α for which the site symmetry group contains the full point group. (This choice may not be unique.) Then define α and its lattice translates as the centers of each unit cell. This is the convention used in Fig. 4.2. Now consider any other point q . If q is equidistant from $n > 1$ unit cell centers, q is assigned to be on the common boundary of n unit cells centered around α . If q is closest to one particular α point, we say that q lies in the interior of the unit cell. By convention, the corners of the unit cell

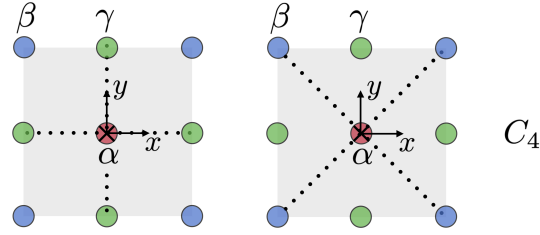


Figure C.1: Two representative ways of dividing a C_4 unit cell into subcells.

are labelled as β (or β_i , if there are multiple HSPs in the same MWP); any other points on the unit cell boundary are denoted $\gamma, \delta, \dots$ as required.

Note that this definition does not fully determine the shape of the unit cell; we only require the sub-cells to rotate into each other, and there is still a lot of freedom in drawing the exact boundaries.

Now let us specialize to the case where H_{clean} has an M fold rotational symmetry. We can subdivide an M -fold rotationally symmetric unit cell into M subcells. Fig. 4.2 illustrates such a division. We only require that subcells rotate into each other under rotations about α . We have no constraint on the shape of the subcells: in Fig. C.1 we have shown two equally valid shapes. Therefore, although the vertices of each subcell are fixed by our definition of the unit cell, the boundaries are otherwise arbitrary.

C.1.1.3 Parameters in H_{clean}

In our numerics, we consider Hofstadter models, i.e. free fermion Hamiltonians of the form

$$H_{\text{clean}} = - \sum_{ij} t_{ij} c_i^\dagger c_j + \text{h.c.} \quad (\text{C.1})$$

where i, j are site indices and the hopping terms $t_{ij} = te^{-iA_{\text{clean},ij}}$ depend on a background vector potential A_{clean} . We mainly take $t_{ij} = 1$ if i, j are nearest neighbors, and $t_{ij} = 0$ otherwise. However, in App. C.7 we also consider the Hofstadter model with next neighbor hopping terms, as an illustration.

For any loop ℓ , the enclosed flux equals $\arg\left(\prod_{\langle ij \rangle \in \ell} t_{ij}\right) = \sum_{\langle ij \rangle \in \ell} A_{\text{clean},ij} \mod 2\pi$, where the sum is taken counterclockwise. We assume that H_{clean} has the symmetry $G_f = \text{U}(1)^f \times_\phi G_{\text{space}}$, where $\text{U}(1)^f$ denotes the group $\text{U}(1)$ whose order 2 element is the fermion parity operation. This means that the total flux in each unit cell equals $\phi \mod 2\pi$.

If H_{clean} is defined on a torus with side lengths L_1, L_2 instead of the infinite plane, the $\mathbb{Z}^2$ translation symmetry is broken down to $\mathbb{Z}_{L_1} \times \mathbb{Z}_{L_2}$ if there is no flux; if there is flux, the symmetry is still broken down to a subgroup but the details are more complicated.

As explained in the main text, we demand that the magnetic field is specified everywhere within a unit cell. This requirement goes beyond the specification of H_{clean} and of the symmetry data. We also note (see in particular App. C.7) that our numerical methods are expected to generalize to arbitrary gapped systems with next neighbor hopping and interaction terms.

C.1.1.4 Definition of ν and κ

A lattice system with $\text{U}(1)$ charge conservation symmetry has a filling per unit cell given by $\frac{N_e \eta_{u.c.}}{N_{\text{orb}}}$ where N_e is the number of electrons, N_{orb} is the number of orbitals, and $\eta_{u.c.}$ is the number of orbitals per unit cell. On a clean torus, this reduces to $\nu := \frac{N_e}{N_{u.c.}}$, where $N_{u.c.}$ is the total number of unit cells. If the system has a unique, gapped ground state and has Chern number C , ν and C are related by Eq. (4.2), which also defines the integer κ . Eq. (4.2) was derived in

Chapter 3. In particular, if $C = 0, \nu = \kappa$.

In the limit of small flux per unit cell, $0 \leq C\phi/2\pi \leq 1$, κ has a simple interpretation. When $\kappa = 0$, a state with Chern number C corresponds to a system of C filled Landau levels. Therefore κ is simply the deviation of the total filling from the Landau level limit. In a system with $\phi = 2\pi p/q$ where p, q are coprime, the filling on a torus takes the form $\nu = r/q$ for some integer r . Then Eq. (4.2) can be written as a Diophantine equation $r = q\kappa + pC$ for the integers C, κ . In the Hofstadter model it can further be shown that $|C| \leq q/2$. This Diophantine equation is often used in the study of the Hofstadter model.

Finally, suppose we place the system on a closed manifold with lattice defects. This could for example be the surface of a cube, which has 8 corners, each being a disclination with disclination angle $\pi/2$. In this case, the actual filling per unit cell differs from ν . In fact the difference is proportional to the discrete shift, as discussed in Ref. [100].

C.1.2 Origin dependence of $\vec{b}_o$

Here we prove Eq. (4.28) in the main text. We define a $\mathbb{Z}^2$ translation gauge field $\vec{R}$ and a $\mathbb{Z}_M$ rotation gauge field ω (see Chapter 3 for a detailed definition). The holonomy of $(\vec{R}, \omega)$ around a defect equals some space group element which we denote as $(\vec{b}_o, \Omega)$, with $\vec{b}_o \in \mathbb{Z}^2, M\Omega/2\pi \in \mathbb{Z}_M$. $\vec{b}_o$ and Ω encode the dislocation Burgers vector and the disclination angle of each defect. We assume that the starting point of the loop is an integer vector away from o . Now suppose we choose a different starting point for the loop $o' = o + \vec{v}$, where $\vec{v}$ may be fractional. We wish to relate $\vec{b}_{o'}$ to $\vec{b}_o$.

The group element which describes the same defect but w.r.t. the new origin o' is obtained

from (b_o, Ω) by conjugating with the translation $(\vec{v}, 0)$:

$$(\vec{v}, 0)(\vec{b}_o, \Omega)(-\vec{v}, 0) = (\vec{b}_o + \vec{v} - U(\Omega)\vec{v}, \Omega). \quad (\text{C.2})$$

Here $U(\Omega)$ is the rotation matrix with angle Ω . The above result follows from the multiplication law for the space group. This group element must give the holonomy of $\vec{R}, \omega$ w.r.t a loop starting at o' . Thus we find that

$$\vec{b}_{o'} = \vec{b}_o + \vec{v} - U(\Omega)\vec{v}, \quad (\text{C.3})$$

as claimed.

C.1.3 Deriving the quantization of $\vec{\mathcal{P}}_o$

In Chapter 3 we derived the quantization of $\mathcal{S}_o, \vec{\mathcal{P}}_o$ from rather abstract TQFT arguments. Here we provide a simpler, more physical way to explain the quantization of $\vec{\mathcal{P}}_o$.

Consider a lattice defect with parameters $(\vec{b}_o, \Omega)$. The response theory predicts that the fractional charge at such a defect receives a contribution

$$\vec{\mathcal{P}}_o \cdot \vec{b}_o \pmod{1}$$

from the term with $\vec{\mathcal{P}}_o$. Let us shift the origin by a lattice vector $\vec{\Lambda}$. This takes $\vec{b}_o \rightarrow \vec{b}_o + \vec{\Lambda} - U(\Omega)\vec{\Lambda}$. The charge at the defect with respect to the new origin is

$$\vec{\mathcal{P}}_o \cdot (\vec{b}_o + \vec{\Lambda} - U(\Omega)\vec{\Lambda}) \pmod{1}.$$

For the theory to be consistent under a shift of origin, the above quantities must agree mod 1, i.e.

$$\vec{\mathcal{P}}_0 \cdot (\vec{\Lambda} - U(\Omega)\vec{\Lambda}) = 0 \pmod{1}. \quad (\text{C.4})$$

We can now take the minimal allowed disclination angle $\Omega = \frac{2\pi}{M_0}$, and obtain the desired quantization:

$$\vec{\mathcal{P}}_0 \cdot (\vec{\Lambda} - U(\frac{2\pi}{M_0})\vec{\Lambda}) = 0 \pmod{1}. \quad (\text{C.5})$$

The distinct choices of $\vec{\mathcal{P}}_0$ compatible with this condition form a group which we denote K_{M_0} . Using the rotation matrices given in Table 4.2, we show below that $\vec{\mathcal{P}}_0$ is a nontrivial topological invariant only when $M_0 = 2, 3, 4$. The general parametrization for $\vec{\mathcal{P}}_0$ is given in Eq. (4.10). The matrix representation of $U(2\pi/M_0)$ for different M_0 (chosen to be consistent with the coordinate axis definitions in Fig. 4.2) is given in Table 4.2.

When $M_0 = 2$, Eq. (C.5) gives

$$2\mathcal{P}_{0,x}\Lambda_x + 2\mathcal{P}_{0,y}\Lambda_y = 0 \pmod{1} \quad (\text{C.6})$$

for any Λ_x, Λ_y . Therefore we set $\mathcal{P}_{0,x} = \frac{\overline{\mathcal{P}}_{0,x}}{2}$, $\mathcal{P}_{0,y} = \frac{\overline{\mathcal{P}}_{0,y}}{2}$ where $\overline{\mathcal{P}}_{0,x}, \overline{\mathcal{P}}_{0,y}$ are integers.

When $M_0 = 3$, Eq. (C.5) gives after simplification

$$\mathcal{P}_{0,x} = \mathcal{P}_{0,y} = -2\mathcal{P}_{0,x} \pmod{1} \quad (\text{C.7})$$

implying that we can set $\vec{\mathcal{P}}_0 = \frac{\overline{\mathcal{P}}_0}{3}(1, 2)$ for some integer $\overline{\mathcal{P}}_0$, with $\overline{\mathcal{P}}_0 \simeq \overline{\mathcal{P}}_0 + 3$.

When $M_o = 4$, Eq. (C.5) gives after simplification

$$\mathcal{P}_{o,x} = \mathcal{P}_{o,y} = -\mathcal{P}_{o,y} \mod 1 \quad (\text{C.8})$$

implying that we can set $\vec{\mathcal{P}} = \frac{\overline{\mathcal{P}}_o}{2}(1, 1)$ for some integer $\overline{\mathcal{P}}_o$, with $\overline{\mathcal{P}}_o \simeq \overline{\mathcal{P}}_o + 2$.

When $M_o = 6$, Eq. (C.5) gives after simplification

$$\mathcal{P}_{o,x} = \mathcal{P}_{o,y} = 0 \mod 1 \quad (\text{C.9})$$

implying that $\vec{\mathcal{P}}_o$ must be an integer vector and is therefore trivial.

C.2 Calculation of $\vec{\mathcal{P}}_o$ and $\mathcal{S}_o$ when $C = 0$

C.2.1 $\vec{\mathcal{P}}_o$

Here we show how to calculate $\vec{\mathcal{P}}_o$ in the limit of zero Chern number, where the system can be adiabatically connected to an insulator in which the charge density is a sum of delta functions at the maximal Wyckoff positions $i = \alpha, \beta, \gamma, \delta$. The charge at these points is always an integer, of the form N_i . In this limit, the polarization $\vec{\mathcal{P}}_o$ is related to the dipole moment $\vec{P}_o := M \sum_{j \in \Theta_{sub}} \text{wt}(j) Q_j \vec{r}_j \mod \mathbb{Z}^2$, where Θ_{sub} is any subcell of a unit cell, and $\vec{r}_j$ is a representative position vector for the point j .

In defining $\vec{P}_o$, the weights $\text{wt}(j)$ are defined as in the main text. If, as for $M = 4, 6$, there are two HSPs in a subcell that are rotation-equivalent but not translation equivalent, we include them separately in the sum.

Next we relate $\vec{P}_0$ to the desired quantity $\vec{\mathcal{P}}_0$: we show that

$$(\mathcal{P}_{0,x}, \mathcal{P}_{0,y}) = (P_{0,y}, -P_{0,x}) \mod \mathbb{Z}^2. \quad (\text{C.10})$$

This is true irrespective of any rotational symmetries. The argument is as follows (it is adapted from statements in Ref. [94], although our result differs from theirs by a sign). We consider a clean lattice in which the vector potential satisfies $A_t = 0$, and the translation gauge fields have components satisfying the ‘natural’ choice

$$X_x = Y_y = 1; X_y = Y_x = 0$$

(implying that $\int_{\vec{r}_i}^{\vec{r}_f} (X_x, Y_y) = \vec{r}_f - \vec{r}_i$). We set $\omega = 0$ for convenience and assume that

$$\mathcal{L} = \frac{C}{4\pi} A \wedge dA + \frac{\vec{\mathcal{P}}_0}{2\pi} \cdot A \wedge d\vec{R} + \nu A \wedge A_{XY},$$

neglecting all other terms. With the above assumptions, $\mathcal{L}$ can be expanded as follows:

$$\begin{aligned} \mathcal{L} = & \frac{C}{4\pi} (-A_x \partial_t A_y + A_y \partial_t A_x) \\ & - \frac{\mathcal{P}_{0,x}}{2\pi} \partial_t A_y + \frac{\mathcal{P}_{0,y}}{2\pi} \partial_t A_x. \end{aligned} \quad (\text{C.11})$$

Therefore the expression for the current $\mathbf{j}$ is

$$\begin{aligned}\mathbf{j} &= \left(\frac{\delta \mathcal{L}}{\delta A_x}, \frac{\delta \mathcal{L}}{\delta A_y} \right) \\ &= \frac{1}{2\pi} \partial_t (-\mathcal{P}_{0,y}, \mathcal{P}_{0,x}) + \frac{C}{2\pi} (-E_x, E_y)\end{aligned}\tag{C.12}$$

where E_x, E_y are the electric field components. But when $C = 0$, we can also write (after picking a suitable normalization)

$$\mathbf{j} = \frac{1}{2\pi} \partial_t \vec{P}_0.\tag{C.13}$$

Comparing the equations for $\mathbf{j}$ then gives Eq. (4.3). The above argument holds on the infinite plane or the torus, with and without rotational symmetry.

Below we consider $M = 2, 3, 4$ separately; for $M = 6$, $\vec{\mathcal{P}}_0 \simeq (0, 0)$. We first determine $\vec{P}_0$ as a dipole moment, and then find the corresponding $\vec{\mathcal{P}}_0$.

M=2: Using Fig. 4.2 as reference, we find that, modulo 1,

$$\begin{aligned}\vec{P}_\alpha &= \frac{1}{2}(N_\beta + N_\gamma, N_\beta + N_\delta) \\ \vec{P}_\beta &= \frac{1}{2}(N_\alpha + N_\delta, N_\alpha + N_\gamma) \\ \vec{P}_\gamma &= \frac{1}{2}(N_\alpha + N_\delta, N_\beta + N_\delta) \\ \vec{P}_\delta &= \frac{1}{2}(N_\beta + N_\gamma, N_\alpha + N_\gamma).\end{aligned}\tag{C.14}$$

This implies that, modulo 1,

$$\begin{aligned}
\vec{\mathcal{P}}_\alpha &= \frac{1}{2}(N_\beta + N_\delta, N_\beta + N_\gamma) \\
\vec{\mathcal{P}}_\beta &= \frac{1}{2}(N_\alpha + N_\gamma, N_\alpha + N_\delta) \\
\vec{\mathcal{P}}_\gamma &= \frac{1}{2}(N_\beta + N_\delta, N_\alpha + N_\delta) \\
\vec{\mathcal{P}}_\delta &= \frac{1}{2}(N_\alpha + N_\gamma, N_\beta + N_\gamma).
\end{aligned} \tag{C.15}$$

M=3: Fig.4.2 shows that, modulo 1,

$$\begin{aligned}
\vec{P}_\alpha &= \frac{1}{3}(N_\beta + 2N_\gamma, N_\beta + 2N_\gamma) \\
\vec{P}_\beta &= \frac{1}{3}(N_\gamma + 2N_\alpha, N_\gamma + 2N_\alpha) \\
\vec{P}_\gamma &= \frac{1}{3}(N_\alpha + 2N_\beta, N_\alpha + 2N_\beta).
\end{aligned} \tag{C.16}$$

This implies that, modulo 1,

$$\begin{aligned}
\vec{\mathcal{P}}_\alpha &= \frac{1}{3}(N_\beta + 2N_\gamma, 2N_\beta + N_\gamma) \\
\vec{\mathcal{P}}_\beta &= \frac{1}{3}(N_\gamma + 2N_\alpha, 2N_\gamma + N_\alpha) \\
\vec{\mathcal{P}}_\gamma &= \frac{1}{3}(N_\alpha + 2N_\beta, 2N_\alpha + N_\beta).
\end{aligned} \tag{C.17}$$

Since by convention $\vec{\mathcal{P}}_0 = \frac{\overline{\mathcal{P}}_0}{3}(1, 2) \pmod{1}$, the value of $\overline{\mathcal{P}}_0$ can be read off from the x component of the above equations.

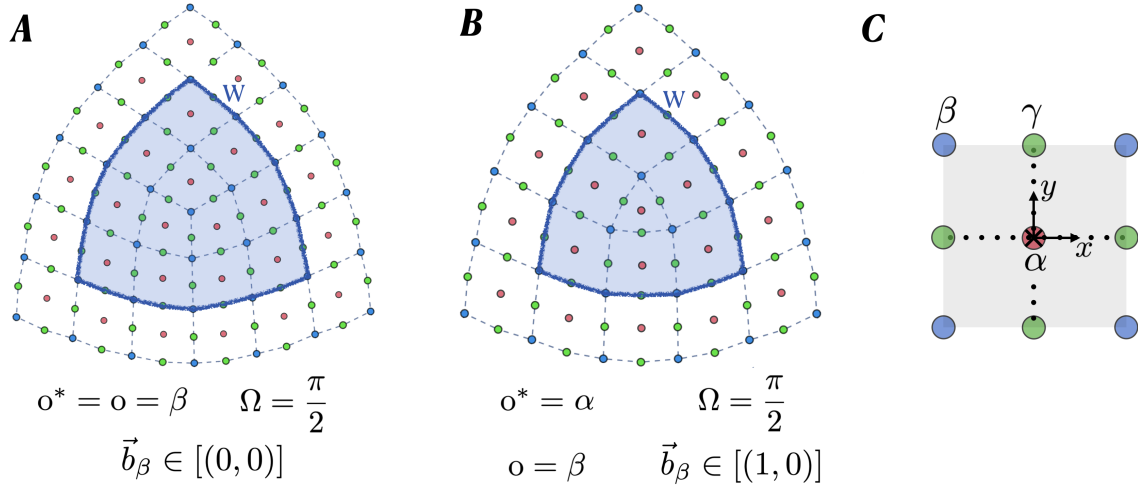


Figure C.2: Pure disclinations with two different choices of o^* : **A.** $o^* = \beta$, **B.** $o^* = \alpha$. Dashed lines represent unitcell boundary **C.** The $M = 4$ unit cell

M=4: Fig. 4.2 shows that, modulo 1,

$$\begin{aligned}\vec{P}_\alpha &= \frac{N_\beta + N_\gamma}{2}(1, 1) \\ \vec{P}_\beta &= \frac{N_\alpha + N_\gamma}{2}(1, 1)\end{aligned}\tag{C.18}$$

In this case, $\vec{P}_o = \vec{\mathcal{P}}_o \mod 1$, and $\vec{\mathcal{P}}_{\alpha(\beta)} = N_{\beta(\alpha)} + N_\gamma \mod 2$. The polarization at the points γ_1, γ_2 with C_2 symmetry can be obtained from the above $M = 2$ results by taking $\gamma_2 = \gamma, \gamma_1 = \delta$, and $N_\gamma = N_\delta$.

C.2.2 $\mathcal{S}_o$

The calculation of $\mathcal{S}_o$ in a system with M -fold rotational symmetry proceeds by constructing a disclination with disclination angle $2\pi/M$ at o and measuring the fractional charge Q_W in a region W around the disclination. When $C = 0$ we can consider the fully filled limit in which the charge at any point in the MWPs $i = \alpha, \beta, \gamma, \delta$ is equal to N_i . Then, by working out the different

cases, we find that

$$\mathcal{S}_o = N_o \mod M \quad (\text{C.19})$$

for $o = \alpha, \beta, \gamma, \delta$.

C.2.3 Argument that $\mathcal{S}_o$ is independent of o^*

In this section we use a square lattice calculation with $C = 0$ as an example to argue that $\mathcal{S}_o$ does not depend on the origin o^* used to construct the disclination, but only on the point o used to measure the dislocation Burgers vector. Say we want to prove that $\mathcal{S}_\beta$ is independent of the origin. We consider $o^* = \beta$ and $o^* = \alpha$ separately and construct two different $\Omega = \frac{\pi}{2}$ disclinations using $\tilde{C}_{M_{o^*}}$ (see Fig. C.2). We fix $o = \beta$ in both cases and extract $\mathcal{S}_\beta$ from the charge response. We have numerically checked that $\mathcal{S}_\beta$ in for these two choices of o^* are the same (see Fig. 4.1). Now we analytically show that $\mathcal{S}_o$ is independent of o^* when $C = 0$.

We pick the region W shown in Fig. C.2. We first consider Fig. C.2A where $o^* = \beta$. In the $C = 0$ limit, the charge response is

$$Q_W = \frac{\mathcal{S}_\beta}{4} + 12\nu \mod 1. \quad (\text{C.20})$$

A direct counting of Wannier orbitals gives

$$Q_W = 12N_\alpha + (12 + \frac{1}{4})N_\beta + 24N_\gamma. \quad (\text{C.21})$$

Since $\nu = N_\alpha + N_\beta + 2N_\gamma$, we obtain $\mathcal{S}_\beta = \frac{1}{4}N_\beta$.

Now we switch to $o = \alpha$ (see Fig. C.2B). The Burgers vector is non-trivial and there is

a irregular unit cell in the defect core contributing $n_{\text{irreg}, \vec{b}, \text{o}} = \frac{1}{2}\nu$. Taking $C = 0$, the impure disclination charge is

$$Q_W = \frac{\mathcal{S}_\beta}{4} + \frac{\overline{\mathcal{P}}_\beta}{2} + (6 + \frac{1}{2})\nu \mod 1. \quad (\text{C.22})$$

A direct counting of Wannier orbitals gives

$$Q_W = 7N_\alpha + (6 + \frac{3}{4})N_\beta + (12 + \frac{3}{2})N_\gamma. \quad (\text{C.23})$$

We now use the $C = 0$ result $\overline{\mathcal{P}}_\beta = N_\alpha + N_\gamma$ which was obtained above. Combined with the above equations, we again obtain $\mathcal{S}_\beta = \frac{1}{4}N_\beta$. This completes our verification that $\mathcal{S}_\beta$ is independent of o^* . We can apply similar arguments to verify that $\mathcal{S}_\text{o}$ is independent of o^* for each o and each $M = 2, 3, 4, 6$.

C.3 Microscopic construction of a dislocation Hamiltonian

In this section, we demonstrate a systematic method to construct a dislocation Hamiltonian H_{defect} in terms of the Hamiltonian H_{clean} on a clean rotationally symmetric lattice with $M = 2, 4$. The procedure is similar in spirit to the construction of disclination defects given in [100] and can be generalized to $M = 3, 6$ [101].

Since the dislocation is a defect of the translation symmetry, the construction necessarily involves the magnetic translation operator $\tilde{T} \equiv e^{i \sum_j \lambda_j c_j^\dagger c_j} \hat{T}$, which is a translation followed by a gauge transformation which ensures that $\tilde{T}$ is a symmetry of H_{clean} . For concreteness, we write a general form of the Landau gauge on the square lattice below (this is analogous to the previously

defined gauge on the torus, Eq.(4.56)):

$$A_{j,j+\hat{y}} = (j_x - \bar{o}_x)\phi \quad (C.24)$$

$$j = (j_x, j_y), \quad j_x \in \mathbb{Z}, j_y \in \mathbb{Z}.$$

This vector potential is defined on an infinite plane. In this appendix we use $A \equiv A_{\text{clean}}$ to denote the entire vector potential on the clean lattice. $\bar{o} = (\bar{o}_x, \bar{o}_y)$ is called the gauge origin.

We now explain how to construct the dislocation Hamiltonian H_{defect} , starting with H_{clean} . In our numerics we use this procedure to construct dislocations with $\vec{b} = (\pm 1, 0), (0, \pm 1)$; all other dislocation Burgers vectors lie in the same equivalence class as one of these choices of $\vec{b}$.

C.3.1 C_2, C_4 dislocation

Let us consider a dislocation on the simplest square lattice as an example. The construction below generalizes naturally to C_2 lattice dislocations since both of them have quadrilateral unit cells. Given an origin o and the desired dislocation Burgers vector $\vec{b}$, the procedure for constructing H_{defect} is summarized in three steps:

1. Draw a cut parallel to $\vec{b}$
2. Define conjugate hopping terms
3. Perform local moves

The first step is to draw a linear cut in the direction of $\vec{b}$ ($\vec{b} = (0, 1)$ in the figure). This cut starts and ends on two plaquette centers. The cut intersects a set of links in the lattice corresponding to hopping terms in H_{clean} . We will only consider 2-fermion hopping terms below.

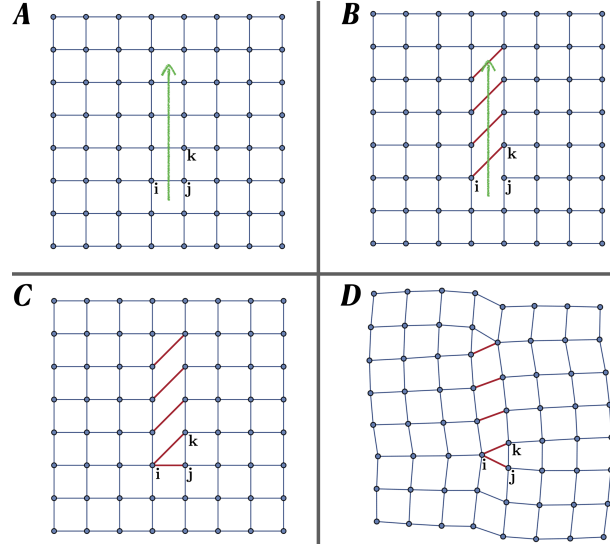


Figure C.3: Dislocation construction procedure on a square lattice **A.** Drawing a cut in the $\hat{y}$ direction. **B.** Conjugate hoppings. **C.** Doing local moves. **D.** Reorganizing.

Consider a particular term which intersects the cut, and is of the form $c_{L_j}^\dagger c_{R_j} e^{-iA_{L_j R_j}}$ in H_{clean} . L_j and R_j are sites on the left and right of the cut respectively. We conjugate *only* the operator c_{R_j} with the translation operator $\tilde{T}_{\mathbf{y}}$:

$$\begin{aligned} e^{-iA_{L_j R_j}} c_{L_j}^\dagger c_{R_j} &\rightarrow e^{-iA_{L_j R_j}} c_{L_j}^\dagger \rightarrow \tilde{T}_{\mathbf{y}} c_{R_j} \tilde{T}_{\mathbf{y}}^\dagger \\ &= e^{-iA_{L_j R_j}} \times e^{-i\lambda_{R_j}} c_{L_j}^\dagger c_{R_{j+1}}. \end{aligned} \quad (\text{C.25})$$

Starting with H_{clean} , we delete the original hoppings $e^{-iA_{L_j R_j}} c_{L_j}^\dagger c_{R_j}$, and add the new hoppings $e^{-i(A_{L_j R_j} + \lambda_{R_j})} c_{L_j}^\dagger c_{R_{j+1}}$ to obtain H_{defect} . In the example shown in Fig. C.3, this procedure creates a dislocation with $\vec{b} = (0, 1)$ on the $-y$ side of the cut and an anti-dislocation with $\vec{b} = (0, -1)$ on $+y$ side of the cut.

We want to consistently ensure that the irregular unit cells in this construction are all triangular, and not pentagonal or in some other irregular shape. Therefore, in the third step, we make one local change of the hopping terms such that both the dislocation and anti-dislocation

have a triangular plaquette (see Fig. C.3 C). This fully determines H_{defect} .

C.3.2 Properties of ϕ_{irreg} on square lattice

Here we derive several statements that were made in Sec. 4.4 regarding the flux ϕ_{irreg} in a dislocation plaquette on a square lattice.

C.3.2.1 Proof of Eq. (4.43)

Consider Fig. C.3. We refer to the point j in this figure as j_0 in the main text. For a $\vec{b} = (0, 1)$ Burgers vector, the flux in the irregular dislocation plaquette can be written in terms of the vector potential A_{defect} on the defect lattice as follows:

$$\phi_{\text{irreg}} = A_{\text{defect},ij} + A_{\text{defect},jk} + A_{\text{defect},ki}. \quad (\text{C.26})$$

From Eq. (C.25), we see that the new hopping created in the defect lattice $tc_i^\dagger c_k e^{-iA_{\text{defect},ik}}$ can be calculated in terms of the vector potential on the clean lattice, i.e.

$$A_{\text{defect},ik} = A_{ij} + \lambda_j.$$

Moreover, $A_{\text{defect},ij} = A_{ij}$ and $A_{\text{defect},jk} = A_{jk}$. Therefore

$$\phi_{\text{irreg}} = A_{jk} - \lambda_j. \quad (\text{C.27})$$

In the main text we explicitly wrote $j = j_0, k = j_0 + \hat{y}$.

C.3.2.2 Change in ϕ_{irreg}

Next we prove the claims in Sec. 4.4.1 about how ϕ_{irreg} changes when either λ or j_0 is changed.

(1): Suppose we change $j_0 \rightarrow j_0 + (v_x, 0)$, $v_x \in \mathbb{Z}$, keeping A, λ fixed. Then the change in ϕ_{irreg} is

$$\begin{aligned} \Delta\phi_{\text{irreg}} &= A_{j_0+(v_x,0),j_0+(v_x,0)+\hat{y}} \\ &\quad - \lambda_{j_0+(v_x,0)} - (A_{j_0,j_0+\hat{y}} - \lambda_{j_0}). \end{aligned} \quad (\text{C.28})$$

Now we use Eq. (4.58). On the infinite plane, Eq. (4.58) holds exactly, while on the torus, where it is not exact everywhere, we ensure that the dislocation is constructed within a region where this equality does hold. Then we get

$$\lambda_{j_0} - \lambda_{j_0+(v_x,0)} = A_{j_0,j_0+(v_x,0)} - A_{j_0+\hat{y},j_0+(v_x,0)+\hat{y}}. \quad (\text{C.29})$$

Therefore

$$\begin{aligned} \Delta\phi_{\text{irreg}} &= A_{j_0+(v_x,0),j_0+(v_x,0)+\hat{y}} - A_{j_0,j_0+\hat{y}} \\ &\quad + A_{j_0,j_0+(v_x,0)} - A_{j_0+\hat{y},j_0+(v_x,0)+\hat{y}} \end{aligned} \quad (\text{C.30})$$

$$= +\phi v_x. \quad (\text{C.31})$$

The last equality is because the sum over A measures the flux in a rectangle of side lengths v_x and 1 in the x and y directions respectively.

(2): Suppose we change $\lambda_j \rightarrow \lambda_j + \chi$ for each j , and $\chi = \phi v_x$. j_0 and A are fixed. Then from Eq. (C.27), $\Delta\phi_{\text{irreg}} = -\phi v_x$.

C.3.2.3 Proof of Eq. (4.61)

With λ_j as defined in Eq. (4.59), we calculate ϕ_{irreg} given that the dislocation is created at $x = j_{0,x}$ on the torus:

$$\phi_{\text{irreg}} = \begin{cases} (j_{0,x} - \bar{o}_x)\phi + \frac{\phi}{2}L_x & j_{0,x} < \bar{o}_x \\ (j_{0,x} - \bar{o}_x)\phi - \frac{\phi}{2}L_x & j_{0,x} > \bar{o}_x \\ 0 & j_{0,x} = \bar{o}_x. \end{cases} \quad (\text{C.32})$$

Since the sites have integer x and y coordinates, we use Eq. (4.60), along with the following relation, which can be found empirically:

$$n_{\text{irreg},o} = -(\mathbf{o}_x + \frac{1}{2}) \mod 1. \quad (\text{C.33})$$

This relation is defined mod 1 because an integer change of $n_{\text{irreg},o}$ does not change $\mathbf{o} \mod \mathbb{Z}^2$.

From these two conditions we find that $\mathbf{o}_x = \bar{o}_x - \frac{1}{2} - \frac{L_x}{2} \mod 1$ whenever $j_{0,x} \neq \bar{o}_x$. This is the same as our numerical observation using Eq. (4.61).

C.4 Independence of $\{\mathcal{S}_o, \vec{\mathcal{P}}_o\}$ under shift of unit cell

In this section we describe the *trimming* method which serves as a tool to prove that $\vec{\mathcal{P}}_o$ and $\mathcal{S}_o$ do not depend on the choice of unit cell Θ . It is worth noting that the linear momentum

calculation and the 1d polarization calculation are completely independent of Θ . Only the charge response calculation depends on Θ . We provide two examples to illustrate the method, for $\mathcal{S}_o$ and $\vec{\mathcal{P}}_o$ respectively.

C.4.1 $\mathcal{S}_o$

Consider a pure disclination on the square lattice with the C_4 symmetric origin o at a site, as shown in Fig.C.4 A. We consider the two choices of unit cell Θ_1, Θ_2 shown in Fig.C.4 C and D. We also define two regions W_1, W_2 whose boundaries are aligned with those of Θ_1, Θ_2 respectively.

Following Eq.(4.13), the charge response for W_1 can be written as

$$Q_{W_1} = \frac{\mathcal{S}_o}{4} + (k_1 + \frac{3}{4})\nu + \frac{C\delta\Phi_{W_1}}{2\pi} \mod 1, \quad (\text{C.34})$$

where k_1 is the integer part of n_{W_1} . Since o is at the center of the unit cell, we have $n_{o,\text{irreg},\Omega} = 1 - \frac{\Omega}{2\pi} = \frac{3}{4}$. The charge response for W_2 can be written as

$$Q_{W_2} = \frac{\mathcal{S}_o}{4} + k_2\nu + \frac{C\delta\Phi_{W_2}}{2\pi} \mod 1. \quad (\text{C.35})$$

The extra flux $\delta\Phi_W$ is only inserted near the defect, and W_1 and W_2 only differ at their boundary.

Therefore $\delta\Phi_{W_1} = \delta\Phi_{W_2}$.

In order to prove that $\mathcal{S}_o$ extracted in the two equations above is the same, we need to show that $Q_{W_1} - Q_{W_2} = (k_3 + \frac{3}{4})\nu$. Since we can arbitrarily add a full unit cell on the boundary of W_1 or W_2 , k_3 can vary freely without changing $\mathcal{S}_o$, but the fractional part in parentheses needs to

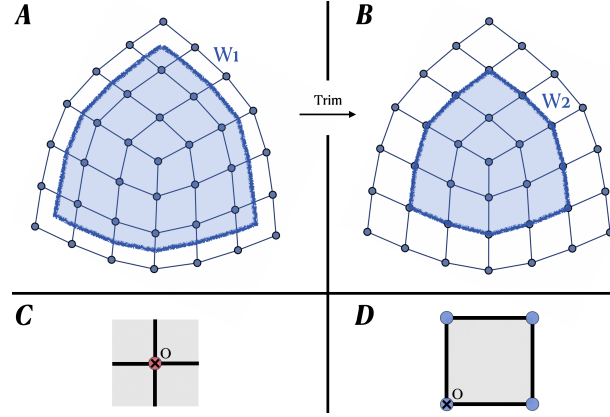


Figure C.4: Trimming method for a pure disclination on the square lattice. We consider two regions **A**. W_1 and **B**. W_2 . W_1, W_2 correspond to choices of the unit cell shown in **C**. and **D**. respectively. The origin o is defined to be at a site. Black solid lines represent hoppings. Here W_1 is trimmed into W_2 by excluding subcells at the boundary.

be $\frac{3}{4}$.

In order to compare Q_{W_1} and Q_{W_2} , we *trim* W_1 into W_2 . This only requires cutting out *subcells* on the boundary, which is assumed to be far away from the defect. Thus, on the boundary we expect the charge in each unit cell to be ν , and the charge in each *subcell* to be $\frac{1}{4}\nu$. With this information, we can explicitly calculate $Q_{W_1} - Q_{W_2}$.

For example, in Fig.C.4 **A** and **B**, $Q_{W_1} - Q_{W_2} = (6 + \frac{3}{4})\nu$, which satisfies the aforementioned condition. The above process is repeated for disclinations with $M = 2, 3, 4, 6$ lattice with arbitrary o and Ω . In all cases, the value of $Q_{W_1} - Q_{W_2}$ obtained from the trimming method is consistent with a fixed value for $\mathcal{S}_o$. This allows us to check that the extracted $\mathcal{S}_o$ does not depend on Θ .

C.4.2 $\vec{\mathcal{P}}_o$

We use a similar procedure to prove that $\vec{\mathcal{P}}_o$ does not depend on the choice of Θ . Consider the honeycomb lattice dislocation shown in Fig.C.5 with Burgers vector $\vec{b} = (0, -1)$. Let us

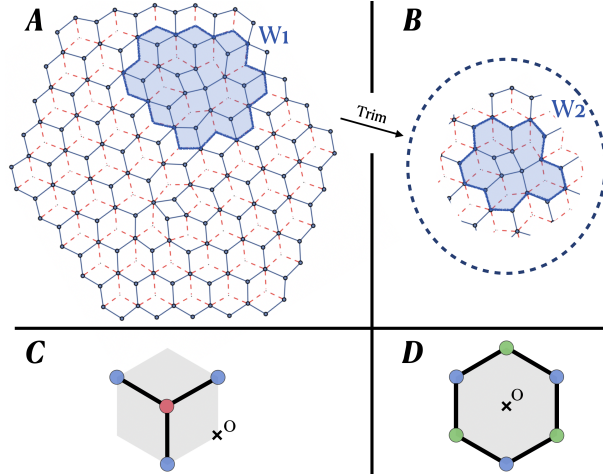


Figure C.5: Trimming method for a dislocation. **A.** Honeycomb dislocation with region W_1 covering the $\vec{b} = (0, -1)$ dislocation. The boundary of W is aligned with unitcell boundary of **C.**, shown as red dashed line. **B.** The region W_2 after trimming. Now the boundary of W_2 is aligned with the hopping, which is also the unit cell boundary of **D.**, shown as black solid line.

consider o to be at a plaquette center. Two choices of Θ are shown in Fig. C.5 C and D.

The charge response with unit cell as in Fig. C.5 C can be written as

$$Q_{W_1} = \vec{\mathcal{P}}_o \cdot \vec{b} + (k_1 + \frac{2}{3})\nu + \frac{C\delta\Phi_{W_1}}{2\pi} \mod 1. \quad (C.36)$$

Again, k_1 is the integer part of the number of unit cells enclosed by W_1 . Since the position of the origin relative to the center of the unit cell is $(\frac{2}{3}, -\frac{1}{3})$, the fractional value of n_{irreg} can be determined to be $1/2$. Similarly, the charge response with Θ_2 as in Fig. C.5 D can be written as

$$Q_{W_2} = \vec{\mathcal{P}}_o \cdot \vec{b} + k_2\nu + \frac{C\delta\Phi_{W_2}}{2\pi} \mod 1. \quad (C.37)$$

Similar to the previous example, $\delta\Phi_{W_1} = \delta\Phi_{W_2}$, and we need to show that $Q_{W_1} - Q_{W_2} = (k_3 + \frac{2}{3})\nu$.

In the example shown in Fig. C.5 A and B, each subcell contributes $\frac{1}{3}\nu$. We have $Q_{W_1} -$

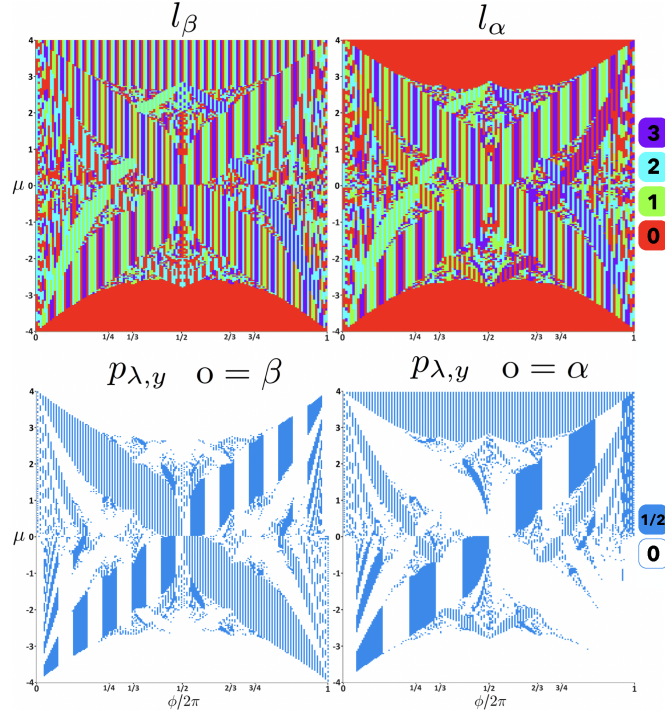


Figure C.6: Bare numerical result of **Top:** angular momentum l_o which follows Eq. (4.40); both calculated on an $L_x \times L_y = 12 \times 12$ torus. **Bottom:** linear momentum $p_{\lambda,y}$ which follows Eq. (4.21); calculated on the $L_x \times L_y = 19 \times 11$ and $L_x \times L_y = 20 \times 11$ torus respectively. All sub-figure are plotted in the $0 < \phi < 2\pi$ range.

$Q_{W2} = \frac{14}{3}\nu = (4 + \frac{2}{3})\nu$, giving the expected answer.

This procedure can be extended to $M = 2, 3, 4, 6$ lattices with arbitrary o and $\vec{b}$. With this we can verify that $\vec{\mathcal{P}}_o$ does not depend on Θ .

C.5 Further details in linear momentum calculation

In this appendix, we state some numerical observations on how the quantization of the linear momentum changes upon starting with the operator $\tilde{T}_y$ as in Eq. (4.59) but then changing the location of the strip that violates translation symmetry. The raw numerical results for linear momentum can be seen in Fig. C.6.

We note that in the main text, the origin o for the charge polarization was defined in terms

of the ‘gauge origin’ $\bar{o}$. This is however not the only way to define o . For a given A, λ where λ is associated to the operator $\tilde{T}_y$, there is a point $\tilde{o}$ which satisfies

$$\lambda_{\tilde{o}} - A_{\tilde{o}, \tilde{o} + \hat{y}} = 0. \quad (C.38)$$

$\tilde{o}$ does not change under gauge transformations, which take $A_{ij} \rightarrow A_{ij} + f_j - f_i$ and $\lambda_j \rightarrow \lambda_j - f_j + f_{j+\hat{y}}$. This transformation of λ_j is obtained by solving Eq. (4.58) after gauge transforming A . Therefore $\tilde{o}$ is a gauge-invariant point, and we can in principle define o in terms of $\tilde{o}$ by constraining the flux at a dislocation defect. This involves arguments similar to the ones in App. C.3. However, we do not use $\tilde{o}$ in the present work.

C.5.1 Freedom in changing location of strip

Consider the choice of λ in Eq. (4.59). Apart from global $U(1)$ transformations, we can consider other choices of λ_j which serve equally well as approximate symmetries. For example, we can consider

$$\tilde{T}'_y = \begin{cases} \hat{T}_y e^{i \sum_j - \frac{\pi m}{L_y} c_j^\dagger c_j} & j_x < \bar{o}_x + \delta \\ \hat{T}_y e^{i \sum_j \frac{\pi m}{L_y} c_j^\dagger c_j} & j_x > \bar{o}_x + \delta \\ \hat{T}_y & j_x = \bar{o}_x + \delta, \end{cases} \quad (C.39)$$

for some real number $\delta \neq 0$. The vector potential is again almost symmetric with respect to $\tilde{T}'_y$; the strip where $\tilde{T}'_y$ is locally not the symmetry of the Hamiltonian is however shifted from $\bar{o}$ by the amount δ in the x direction. However, the expectation value of $\tilde{T}'_y$ does not yield the desired

invariant $\mathcal{P}_{0,y}$. We have numerically checked that for any $\delta \neq 0$, the resulting values of $\mathcal{P}_{0,y}$ are not even quantized throughout each gapped phase of the Hofstadter model when $C \neq 0$. And when $C = 0$, choosing $\delta \neq 0$ continues to give a quantized result, but certain choices of δ will give $\mathcal{P}_{0+(1/2,0),y}$ instead of $\mathcal{P}_{0,y}$.

These observations can be understood as follows. Suppose we calculate $\mathcal{P}_{0,y}$ using $\tilde{T}_{\mathbf{y}}$ for a given choice of strip, and then change the strip position by $\delta\hat{x}$ as in Eq. (C.39). In general, if $\delta \neq 0$, λ_j changes by $k_j\pi m/L_y$ for each j that was crossed by the strip as it was shifted. $k_j = \pm 1$ if j lies exactly on the initial or the final position of the strip, and $k_j = \pm 2$ if j lies in between the initial and final positions of the strip. Let S be the set of points crossed by the strip. Then

$$\tilde{T}_{\mathbf{y}} \rightarrow \tilde{T}_{\mathbf{y}} e^{i \sum_{j \in S} \frac{k_j \pi m}{L_y} \hat{n}_j}. \quad (\text{C.40})$$

When $C = 0$, we verified numerically that the state $|\Psi\rangle$ is an almost exact eigenstate of $\hat{n}_j$, with expectation value κ (the statement being exact for the Hofstadter model). Thus we obtain

$$\tilde{T}_{\mathbf{y}} \rightarrow \tilde{T}_{\mathbf{y}} e^{i \pi m \kappa \sum_{j \in S} \frac{k_j}{L_y}}. \quad (\text{C.41})$$

Since the number of points j corresponding to a fixed k_j is always a multiple of L_y , the extra factor is of the form $(-1)^{k'm}$, where $k' = 0$ or 1 , and is fixed by the actual values of k_j and κ . If $k' = 0$, $\mathcal{P}_{0,y}$ is invariant. But if $k' = 1$, $\mathcal{P}_{0,y} \rightarrow \mathcal{P}_{0,y} + 1/2$.

Note that this reasoning breaks down when $C \neq 0$: $|\Psi(m)\rangle$ is *not* an eigenstate of $\sum_{j \in S} \hat{n}_j$, and $\mathcal{P}_{0,y}$ hence can be expected to depend sensitively on the strip position. As mentioned in the main text, empirically we have found that the strip needs to be aligned with the ‘gauge origin’ $\bar{0}$

in order to obtain consistent results for $\mathcal{P}_{\mathbf{o},y}$.

C.6 Computations for Section 4.5

Here we will use the simplicial formulation of the topological response theory [52] to derive the quantities $\rho_M, \vec{\mu}_M, \vec{\tau}_M$ defined in Eqs. (4.79), (4.80), (4.81).

C.6.0.1 Fractional gauge transformations

The definition of each group element $\mathbf{g}_i = (\mathbf{r}_i, h_i) \in G_{\text{space}}$ implicitly depends on the choice of origin $\mathbf{o}$. For example, $\mathbf{h} = (\mathbf{0}, 1 \bmod M)$ is the element in G_{space} corresponding to a rotation by angle $2\pi/M$ (about $\mathbf{o}$). Now suppose $\mathbf{o}' = \mathbf{o} + \vec{v}$, as before. The group element $\mathbf{g}'_i$ which implements the same transformation but with respect to the new origin $\mathbf{o}'$ can be expressed as follows:

$$\mathbf{g}'_i = (\vec{v}, 0)\mathbf{g}(-\vec{v}, 0), \quad (\text{C.42})$$

which corresponds to translating from $\mathbf{o}'$ to $\mathbf{o}$, applying $\mathbf{g}$, and translating back. Now we use the group multiplication law in G_{space} ,

$$(\mathbf{r}_1, h_1)(\mathbf{r}_2, h_2) = (\mathbf{r}_1 + U(2\pi h_1/M)\mathbf{r}_2, h_1 + h_2), \quad (\text{C.43})$$

to find that

$$\mathbf{g}'_i = (\mathbf{r}_i - U(2\pi h_i/M)\vec{v} + \vec{v}, h_i). \quad (\text{C.44})$$

C.6.0.2 Change in $\mathcal{S}_0$

First we study the transformation of $d\omega$. A cocycle representative for $d\omega$ can be explicitly written on a 2-simplex with the group elements $\mathbf{g}_1, \mathbf{g}_2, (\mathbf{g}_1\mathbf{g}_2)^{-1}$:

$$d\omega(\mathbf{g}_1, \mathbf{g}_2) = \frac{[h_1]_M + [h_2]_M - [h_1 + h_2]_M}{M} \quad (\text{C.45})$$

where $[a]_M := a \bmod M$. Since $d\omega$ does not depend on the translation group variables, it is unaffected by the transformation in Eq. (C.44). Therefore we have

$$d\omega' = d\omega. \quad (\text{C.46})$$

Next we study the quantity $d\vec{R}$. An explicit cocycle representative is

$$\begin{aligned} d\vec{R}(\mathbf{g}_1, \mathbf{g}_2) &= \mathbf{r}_1 + \mathbf{r}_2 - (\mathbf{r}_1 + U(2\pi h_1/M)\mathbf{r}_2) \\ &= (1 - U(2\pi h_1/M))\mathbf{r}_2. \end{aligned} \quad (\text{C.47})$$

This implies that

$$\begin{aligned} d\vec{R}'(\mathbf{g}_1, \mathbf{g}_2) &:= d\vec{R}(\mathbf{g}'_1, \mathbf{g}'_2) \\ &= (1 - U(2\pi h_1/M))(\mathbf{r}_2 + (1 - U(2\pi h_2/M))\vec{v}) \end{aligned} \quad (\text{C.48})$$

$$= d\vec{R} + (1 - U(2\pi h_1/M))(1 - U(2\pi h_2/M))\vec{v}. \quad (\text{C.49})$$

The second term only depends on h_1, h_2 , just like $d\omega$. In fact, it is coboundary equivalent to a

multiple of $d\omega$. This can be seen by computing a cohomology invariant we denote I_Ω , which for a general cocycle $f_2 \in Z^2(G_{\text{space}}, \mathbb{Z})$ is defined as

$$I_\Omega[f_2] := \sum_{j=0}^{M-1} f_2(\mathbf{h}^j, \mathbf{h}) \mod M, \quad (\text{C.50})$$

where we recall $\mathbf{h} = (\mathbf{0}, 1)$. We can show that $I_\Omega(d\omega) = 1$ using Eq. (C.45), and also that $I_\Omega(d\vec{R}) = I_\Omega(A_{XY}) = 0$. Thus I_Ω is an invariant corresponding to the $\mathbb{Z}_M$ subgroup of $\mathcal{H}^2(G_{\text{space}}, \mathbb{Z})$.

Up to coboundaries, we have by assumption

$$d\vec{R}' = d\vec{R} + \vec{\tau}_M d\omega; \quad (\text{C.51})$$

therefore, for any integer vector $\vec{u}$,

$$\begin{aligned} I_\Omega(\vec{u} \cdot d\vec{R}') &= I_\Omega(\vec{u} \cdot d\vec{R}) + I_\Omega(\vec{u} \cdot \vec{\tau}_M d\omega) \\ &= I_\Omega(\vec{u} \cdot d\vec{R}) + \vec{u} \cdot \vec{\tau}_M \mod M. \end{aligned} \quad (\text{C.52})$$

But we can compute

$$\begin{aligned} I_\Omega(\vec{u} \cdot d\vec{R}') &= I_\Omega(\vec{u} \cdot d\vec{R}) \\ &\quad + \vec{u} \cdot \sum_{j=0}^{M-1} (1 - U(2\pi j/M))(1 - U(2\pi/M)) \vec{v} \\ &= I_\Omega(d\vec{R}) + \vec{u} \cdot M(1 - U(2\pi/M)) \vec{v} \mod M. \end{aligned} \quad (\text{C.53})$$

By comparing Eqs. (C.52), (C.53), and using the fact that $\vec{u}$ can be arbitrary, we see that

$$\vec{\tau}_M = M(1 - U(2\pi/M))\vec{v}. \quad (\text{C.54})$$

The components of $\vec{\tau}_M$ are ambiguous up to multiples of M , since I_Ω is a $\mathbb{Z}_M$ invariant. This only means that $\mathcal{S}_{\phi'}$ is determined up to multiples of M , as we already know.

Finally, we study the transformation of the area element A_{XY} . We know that A_{XY} should be quadratic in the translation gauge fields if it corresponds to an area element. Consider the ansatz

$$A_{XY}(\mathbf{g}_1, \mathbf{g}_2) = \mathbf{r}_1^T Q_M U(2\pi h_1/M) \mathbf{r}_2 \quad (\text{C.55})$$

where Q_M is a 2×2 matrix. The 2-cocycle condition for A_{XY} is

$$\begin{aligned} & A_{XY}(\mathbf{g}_1, \mathbf{g}_2) + A_{XY}(\mathbf{g}_1 \mathbf{g}_2, \mathbf{g}_3) \\ &= A_{XY}(\mathbf{g}_2, \mathbf{g}_3) + A_{XY}(\mathbf{g}_1, \mathbf{g}_2 \mathbf{g}_3). \end{aligned} \quad (\text{C.56})$$

Upon substituting in the ansatz for A_{XY} , we find that

$$\mathbf{r}_1^T (U(2\pi/M)^T Q_M U(2\pi/M) - Q_M) \mathbf{r}_2 = 0. \quad (\text{C.57})$$

This implies that

$$U(2\pi/M)^T Q_M U(2\pi/M) = Q_M. \quad (\text{C.58})$$

Thus the particular form we chose for A_{XY} ensures that Q_M is a constant matrix with no dependence

on group variables. We also demand that

$$A_{XY}(\mathbf{x}, \mathbf{y}) - A_{XY}(\mathbf{y}, \mathbf{x}) = 1,$$

which is a normalization condition. Solving these two conditions, we get the following possibilities for Q_M :

$$Q_2 = \begin{pmatrix} Q_{xx} & 1 \\ 0 & Q_{yy} \end{pmatrix}, Q_{xx}, Q_{yy} \in \mathbb{Z} \quad (\text{C.59})$$

$$Q_4 = \frac{1}{2} \begin{pmatrix} 2Q_0 & 1 \\ -1 & 2Q_0 \end{pmatrix} \quad Q_0 \in \mathbb{Z} \quad (\text{C.60})$$

$$Q_{3,6} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}. \quad (\text{C.61})$$

Note that for $M = 2, 4$, Q_M is not uniquely fixed by Eq. (C.58). This will be important in what follows. Upon shifting the origin, we obtain

$$A'_{XY} = (\mathbf{r}_1 - (U(h_1) - 1)\vec{v})^T Q_M U(h_1) (\mathbf{r}_1 - (U(h_2) - 1)\vec{v}). \quad (\text{C.62})$$

We now write

$$A'_{XY} = A_{XY} + \rho_M d\omega + \vec{\mu}_M \cdot d\vec{R}$$

for some $\rho_M, \vec{\mu}_M$, up to coboundaries. Evaluating the invariant I_Ω , we find that

$$I_\Omega(A'_{XY}) = \rho_M = -M((U(2\pi/M) - 1)\vec{v})^T Q_M U(2\pi/M)\vec{v}. \quad (\text{C.63})$$

After expanding out this expression for each M using Table 4.2, we find that

$$\rho_2 = 4(v_x v_y + v_x^2 Q_{xx} + v_y^2 Q_{yy}) \mod 2 \quad (\text{C.64})$$

$$\rho_3 = -3(v_x^2 + v_y^2 + v_x v_y) \mod 3 \quad (\text{C.65})$$

$$\rho_4 = -2(v_x^2 + v_y^2)(1 + 2Q_0) \mod 4 \quad (\text{C.66})$$

$$\rho_6 = 0 \mod 6. \quad (\text{C.67})$$

Note that the transformation rule depends on the choice of Q_{xx}, Q_{yy} for $M = 2$ and on Q_0 for $M = 4$. We find that numerical results are matched only if we set $Q_{xx} = Q_{yy} = 1$ and $Q_0 = 0$.

Upon doing so, we obtain

$$\begin{aligned} \rho_2 &= 4(v_x v_y + v_x^2 + v_y^2) \mod 2 \\ \rho_3 &= -3(v_x^2 + v_y^2 + v_x v_y) \mod 3 \\ \rho_4 &= -2(v_x^2 + v_y^2) \mod 4 \\ \rho_6 &= 0 \mod 6. \end{aligned} \quad (\text{C.68})$$

Knowing $\vec{\tau}_M, \rho_M$ allows us to conclude that

$$\mathcal{S}_{o'} = \mathcal{S}_o + M\vec{\mathcal{P}}_o \cdot (1 - U(2\pi/M))\vec{v} + \rho_M, \quad (\text{C.69})$$

which gives the complete expression for the transformation of $\mathcal{S}_o$. Thus the field theory does not uniquely fix the transformation rules for $M = 2, 4$; we needed to make some additional choices guided by numerics.

C.6.0.3 Change in $\vec{\mathcal{P}}_0$

Finally we find $\vec{\mu}_M$, which fixes the transformation of $\vec{\mathcal{P}}_0$. The calculation is given below.

We discuss the different values of M separately. We assume

$$A'_{XY} = A_{XY} + \rho_M d\omega + \vec{\mu}_M \cdot d\vec{R}. \quad (\text{C.70})$$

M=2: The invariants which characterize the cohomology class of $\vec{\mu}_2 \cdot d\vec{R}$ are $\mathcal{I}_X, \mathcal{I}_Y$, which are defined as

$$\mathcal{I}_X[f_2] := f_2(\mathbf{xh}, \mathbf{xh}) + f_2(\mathbf{h}, \mathbf{h}) \pmod{2} \quad (\text{C.71})$$

$$\mathcal{I}_Y[f_2] := f_2(\mathbf{yh}, \mathbf{yh}) + f_2(\mathbf{h}, \mathbf{h}) \pmod{2}. \quad (\text{C.72})$$

Here $\mathbf{x} = ((1, 0), 0)$, $\mathbf{y} = ((0, 1), 0)$, $\mathbf{h} = ((0, 0), 1)$. We can check that $I_X(d\omega) = I_Y(d\omega) = 0 \pmod{2}$. We can also show that

$$\mathcal{I}_X[\vec{\mu}_2 \cdot d\vec{R}] = 2\mu_{2,x} \pmod{2} \quad (\text{C.73})$$

$$\mathcal{I}_Y[\vec{\mu}_2 \cdot d\vec{R}] = 2\mu_{2,y} \pmod{2}. \quad (\text{C.74})$$

Therefore,

$$I_{X(Y)}(A'_{XY} - A_{XY}) = 2\mu_{2,x(y)}. \quad (\text{C.75})$$

But when $M = 2$, $(1 - U(h_1))\vec{v} = 2h_1\vec{v}$ for any $\vec{v}$ (where h_1 is just an integer mod 2). Using this

in Eq. (C.62), we obtain

$$A'_{XY}(\mathbf{g}_1, \mathbf{g}_2) = (\mathbf{r}_1 + 2\vec{v})^T Q_2(\mathbf{r}_2 + 2h_2\vec{v}).$$

Then, by direct calculation we get

$$\begin{aligned} \mathcal{I}_X[A'_{XY} - A_{XY}] &= (1 + 2v_x, 2v_y)Q_2(1 + 2v_x, 2v_y)^T \\ &\quad - (2v_x, 2v_y)Q_2(2v_x, 2v_y)^T - Q_{xx} \end{aligned} \quad (\text{C.76})$$

$$= 2v_y \pmod{2}; \quad (\text{C.77})$$

$$\begin{aligned} \mathcal{I}_Y[A'_{XY} - A_{XY}] &= (2v_x, 1 + 2v_y)Q_2(2v_x, 1 + 2v_y)^T \\ &\quad - (2v_x, 2v_y)Q_2(2v_x, 2v_y)^T - Q_{yy} \end{aligned} \quad (\text{C.78})$$

$$= 2v_x \pmod{2}. \quad (\text{C.79})$$

This means that $2\vec{\mu}_2 = 2(v_y, v_x) \pmod{\mathbb{Z}^2}$. Putting this back into the transformation rule for $\vec{\mathcal{P}}_o$, we get the following result (mod 1):

$$2\mathcal{P}_{o',x} := 2\mathcal{P}_{o,x} + \kappa\mathcal{I}_X[A'_{XY} - A_{XY}] = 2\mathcal{P}_{o,x} + 2\kappa v_y \quad (\text{C.80})$$

$$2\mathcal{P}_{o',y} := 2\mathcal{P}_{o,y} + \kappa\mathcal{I}_Y[A'_{XY} - A_{XY}] = 2\mathcal{P}_{o,y} + 2\kappa v_x. \quad (\text{C.81})$$

Note that this result is independent of the choice for Q_{xx}, Q_{yy} made above, as these quantities cancel out of the final expression.

M=4: We can directly see the result by setting $\vec{\mathcal{P}}_o = \frac{\vec{\mathcal{P}}_o}{2}(1, 1)$ and $v_x = v_y = \frac{v_0}{2} \pmod{1}$ in

the above calculation for $M = 2$. The result is

$$\overline{\mathcal{P}}_{o'} = \overline{\mathcal{P}}_o + \kappa v_0 \pmod{2}. \quad (\text{C.82})$$

The results for $M = 2, 4$ are in fact all equivalent to Eq. (4.92).

M=3: We wish to prove Eq. (4.92). If we parametrize $\vec{v} = \frac{v_0}{3}(1, 1)$ and $\vec{\mathcal{P}}_o = \frac{\overline{\mathcal{P}}_o}{3}(1, 2)$, this requires us to show that

$$\overline{\mathcal{P}}_{o'} = \overline{\mathcal{P}}_o + 2\kappa v_0 \pmod{3}. \quad (\text{C.83})$$

The invariant giving the cohomology class of $\vec{\mu}_3 \cdot d\vec{R}$ is $\mathcal{I}_X$, defined as

$$\begin{aligned} \mathcal{I}_X[f_2] &:= f_2(\mathbf{xh}, \mathbf{xh}) + f_2((\mathbf{xh})^2, \mathbf{xh}) \\ &\quad - f_2(\mathbf{h}, \mathbf{h}) - f_2(\mathbf{h}^2, \mathbf{h}) \pmod{3}. \end{aligned} \quad (\text{C.84})$$

By direct calculation, we can show that

$$\mathcal{I}_X[\vec{\mu}_3 \cdot d\vec{R}] = 3\mu_{3,x} = \overline{\mathcal{P}}_o \pmod{3}. \quad (\text{C.85})$$

Again we use $A'_{XY}(\mathbf{g}_1, \mathbf{g}_2) = (\mathbf{r}_1 - (1 - U(h_1)\vec{v}))^T Q_3 U(h_1)(\mathbf{r}_2 - (1 - U(h_2)\vec{v}))$. We use the same procedure discussed for $M = 2$. After some (tedious) algebra, we indeed obtain Eq. (C.83), or equivalently Eq. (4.92).

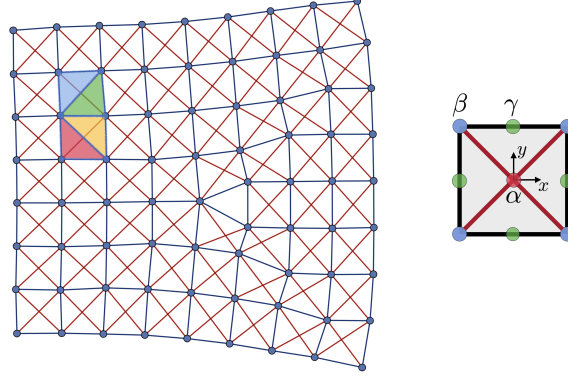


Figure C.7: **Left.** $\vec{b} = (0, 1)$ dislocation of a square lattice with next nearest neighbour hopping (red links). The C_4 symmetry requires all 4 colored triangles with different orientations having same flux $\phi/2$. **Right.** The choice of unit cell. There are *no* sites at the MWP α and γ , and there is only one site at β . The next nearest neighbour hoppings cross each other.

C.7 Generalization to models with next nearest neighbour hopping

In the main text, we argued that our method of extracting $\mathcal{S}_0$ and $\vec{\mathcal{P}}_0$ works for an arbitrarily complicated unit cell with further neighbor hopping and interaction terms, as long as it is C_M symmetric. In order to apply our dislocation or disclination charge calculation to measure $\mathcal{S}_0, \vec{\mathcal{P}}_0$ in these generalized Hamiltonians, we need to construct a defect Hamiltonian H_{defect} , measure the charge Q_W in a region W , and then specify the quantities $n_{\text{irreg},0}, \delta\Phi_{W,0}$. We argued that our numerical procedure allows us to specify these quantities as long as H_{clean} is local and has a gapped, symmetry-preserving ground state, irrespective of its other properties.

We now provide numerical evidence that our procedure gives the expected quantized $\vec{\mathcal{P}}_0$ when we add next neighbor hopping (numerically it is much harder to verify this for interaction terms, and we do not pursue this here). Consider a Hofstadter model with the unit cell shown in Fig.C.7. The Hamiltonian with this unit cell has next neighbour hopping. We set the nearest neighbour hopping amplitude equal to $t = 1$, and tune the next neighbour (diagonal) hopping amplitude t' between 0 and 1. There is a flux ϕ through each unit cell. Since we require the unit

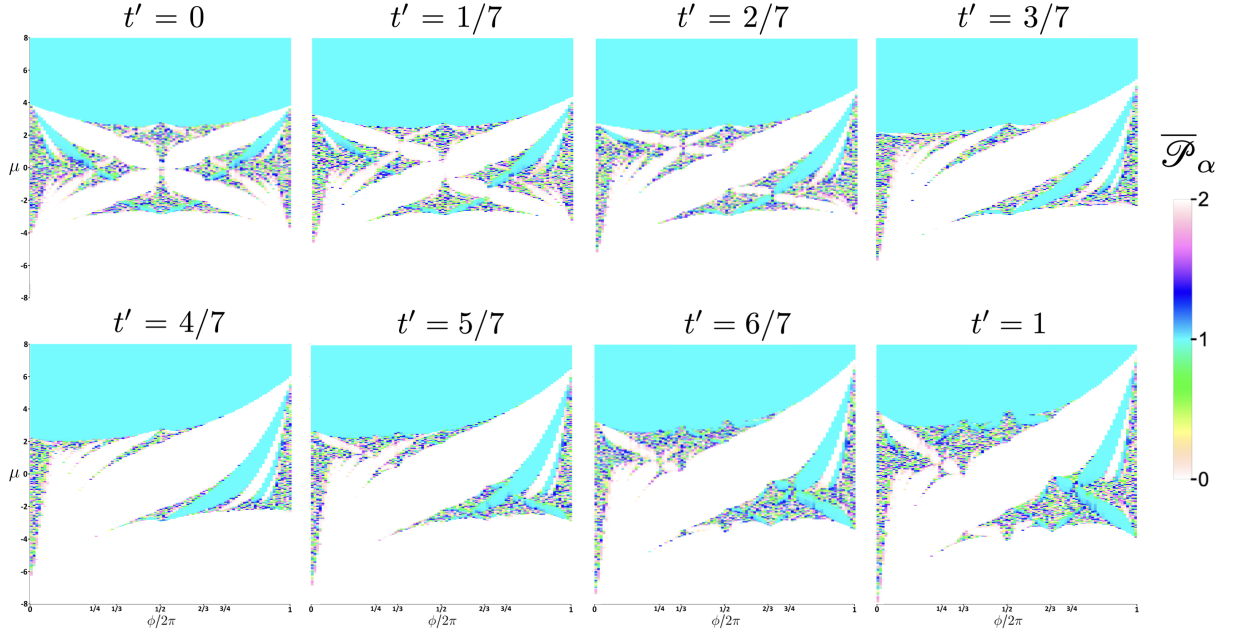


Figure C.8: $\overline{\mathcal{P}}_\alpha$ as a function of the next neighbour hopping amplitude t' ; the nearest neighbor hopping amplitude is $t = 1$. The calculation is done on a $L_x \times L_y = 41 \times 41$ open disk, with a $\vec{b} = (0, 1)$ dislocation placed at the center. The defects are located at the center of the disk. The noisy features appear since the butterfly is numerically calculated on a finite size system rather than analytically derived from an empirical formula as in Fig.4.1.

cell to be C_4 symmetric, we also require that there is a flux $\frac{\phi}{2}$ through each triangle in the unit cell, as shown in Fig.C.7.

We extract the $\mathbb{Z}_2$ invariant $\overline{\mathcal{P}}_\alpha$ from a dislocation charge computation. The respective butterflies as a function of t' are shown in fig. C.8. We see that the invariant is indeed quantized and well-defined mod 2 throughout the main Hofstadter lobes. In the $t' = 0$ limit, the butterfly reduces to the one for $\overline{\mathcal{P}}_\alpha$ in Fig. 4.1.

Appendix D: Appendix for Chapter 5

D.1 Further details on the numerics

D.1.1 Definition of vector potential

In this section we provide a definition of the vector potential A for the square lattice Hofstadter model on a torus. Some of the following details have previously appeared in Ref. [100]. We first define our vector potential on a $L \times L$ torus with L even. Considering even L is sufficient for our calculations on the torus. An example of the gauge choice is shown in Fig. D.1. We can use a similar gauge to define A on an open disc.

D.1.2 Choice of global rotation operator

In systems with $U(1)$ and point group rotation symmetries, we have the ambiguity that any rotation symmetry operator can be multiplied by a phase $e^{i\theta\hat{N}}$ to give another valid symmetry operator. It is necessary to fix this ambiguity in order to uniquely define a rotation operator. Below we illustrate how to do this on the infinite plane, and then on the torus.

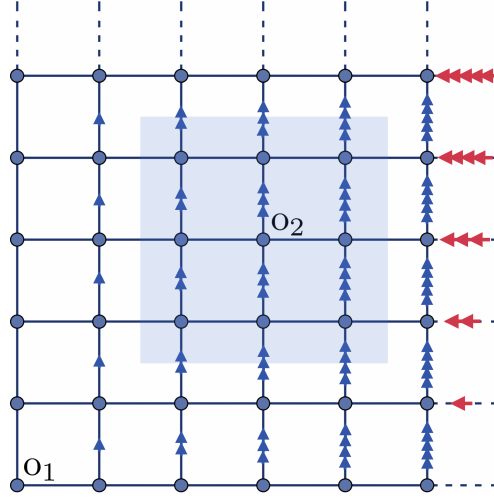


Figure D.1: $L \times L = 6 \times 6$ square lattice with periodic boundary condition indicated by the dotted bonds. We insert $2\pi m$ flux uniformly through the torus, where $m = \frac{\phi}{2\pi} L^2$. Each blue arrow represents a hopping phase $A_{ij} = \frac{2\pi m}{L^2}$; each red arrow represents a hopping phase $A_{ij} = \frac{2\pi m}{L}$. o_1, o_2 are the rotation centers. The 3×3 partial rotation region is defined around o_2 and is colored blue.

D.1.2.1 On the infinite plane

Ref. [100] showed how to canonically define one such operator ($\tilde{C}_{M_0}^+$ in the notation of this paper) on the infinite plane, which assigns zero excess flux around a disclination of angle $2\pi/M_0$ created at o using $\tilde{C}_{M_0}^+$. Since the defect Hamiltonian depends on the specific choice of $U(1)$ gauge transformation defining $\tilde{C}_{M_0}^+$, considering a different operator $e^{i\chi\hat{N}}\tilde{C}_{M_0}^+$ modifies the flux around the disclination by $\chi \bmod 2\pi$, and therefore we can physically distinguish it from $\tilde{C}_{M_0}^+$. See Ref. [100] for the explicit construction of a disclination Hamiltonian with fourfold rotational symmetry.

Let us specialize to the case $M_0 = 4$ and $(\tilde{C}_{M_0}^+)^4 = +1$. Assume each plaquette on the infinite plane has flux $\phi \bmod 2\pi$. If $o = \beta$ (a vertex), the four possible operators assign equal flux $\phi \bmod 2\pi$ to all plaquettes in the lattice with a $\pi/2$ disclination, except one that lies next to the disclination. In this plaquette the assigned flux is $\phi + k\frac{\pi}{2} \bmod 2\pi$, where $k = 0, 1, 2, 3$. Thus

there is a canonical choice of $\tilde{C}_{M_\beta}^+$, corresponding to $k = 0$ (meaning zero excess flux). When $o = \alpha$ (a plaquette center), there is a triangular plaquette at a $\pi/2$ disclination which is assigned a flux $\frac{3\phi}{4} + k\frac{\pi}{2} \pmod{2\pi}$. The operator $\tilde{C}_{M_\alpha}^+$ for which $k = 0$ is also canonical, because it assigns a flux at the disclination that is proportional to ϕ .

To extract the desired topological invariants in this paper, it turns out to be insufficient to use the operators $\tilde{C}_{M_o}^+$ alone, because they only generate groups of order M_o . Thus we also define $\tilde{C}_{M_o}^- := e^{i\frac{\pi}{M_o}\hat{N}}\tilde{C}_{M_o}^+$, which is of order $2M_o$ and satisfies $(\tilde{C}_{M_o}^-)^{M_o} = (-1)^F$. The two sets of operators, together denoted $\tilde{C}_{M_o}^\pm$, are indeed sufficient for our purpose. Arbitrary $U(1)$ rotations can be considered by defining $\tilde{C}_{M_o,\chi}^\pm = e^{i\chi\frac{2\pi}{M_o}\hat{N}}\tilde{C}_{M_o}^\pm$. Note that $\tilde{C}_{M_o,\chi}^+, \tilde{C}_{M_o,\chi}^-$ insert excess flux $2\pi\chi/M_o$ at a $2\pi/M_o$ disclination, compared to $\tilde{C}_{M_o}^+$ and $\tilde{C}_{M_o}^-$ respectively.

D.1.2.2 On the torus

Defining a unique rotation operator on a torus with fourfold rotational symmetry is slightly more subtle than on the infinite plane, because any order 4 rotation preserves 2 points o_1, o_2 . If the side length L of the torus is even, the two fixed points are either both at α or both at β . If L is odd, we have one fixed point each at α and β .

$\tilde{C}_{M_o}^+$ can be defined for $o = o_1$ or o_2 by restricting to a disc D centered around o , and creating a $\pi/2$ disclination within D centered at o , using $\tilde{C}_{M_o}^+|_D$. As in the case of the infinite plane discussed above, we canonically define $\tilde{C}_{M_o}^+$ such that $\tilde{C}_{M_o}^+|_D$ creates a disclination with zero excess flux. We then define $\tilde{C}_{M_o}^- := e^{i\frac{\pi}{M_o}\hat{N}}\tilde{C}_{M_o}^+$ and $\tilde{C}_{M_o,\chi}^\pm = e^{i\chi\frac{2\pi}{M_o}\hat{N}}\tilde{C}_{M_o}^\pm$.

D.1.3 Partial rotation on a torus

Given the global rotation operators $\tilde{C}_{M_0, \chi}^{\pm}$, we wish to restrict them to a region D , hence obtaining $\tilde{C}_{M_0, \chi}^{\pm}|_D$. Note that the chosen operator inserts excess flux $2\pi\chi/M_0$ in a $2\pi/M_0$ disclination centered at o relative to $\tilde{C}_{M_0}^{\pm}|_D$. We find that the result for $l_{D, o, \chi}^{\pm}$ follows Eq. (5.1).

D.1.4 Partial rotation on an open disc

We can also perform the partial rotation calculation on an open disc. This can be done by first defining the vector potential similarly as in Fig. D.1, using a gauge which inserts ϕ flux per plaquette in some region. As shown in Fig. D.2, It is *not* necessary that the vector potential has support on the whole disc: the flux needs to be inserted only in an invariant subregion around o . Besides, D *does not* have to be defined around the center of the subregion with flux, as long as D is enclosed by this subregion. The center of D , which we define as o , can be at any maximal Wyckoff point.

On the torus, we have a global rotation symmetry operator $\tilde{C}_{M_0}^{\pm}$, and we define $\tilde{C}_{M_0}^{\pm}|_D$ to be the restriction of $\tilde{C}_{M_0}^{\pm}$ to region D . On the disc, this cannot be done as there might not be a global symmetry operator to start with. Therefore, we first choose D and then define an operator $\tilde{C}_{M_0}^{\pm}|_D$, which commutes with the Hamiltonian restricted to D . We find that choosing linear size L_D of D to be too large ($L_D \geq L-1$) or too small ($L_D \leq 2$) gives unquantized $K_o^{\pm}$, which is tested this for $L \leq 30$. As before, we need to impose a constraint on $\tilde{C}_{4, o}|_D$ to fix the ambiguity by a global $U(1)$ phase $e^{i\theta\hat{N}}$. This can be done by fixing the excess flux inserted around a disclination created using the $\tilde{C}_{M_0}^{\pm}|_D$ operator as explained in the previous section. We define $\tilde{C}_{M_0, \chi}^+$ (respectively $\tilde{C}_{M_0, \chi}^-$) so that it inserts an excess flux of $2\pi\chi/M_0$ at a $2\pi/M_0$ disclination, relative to $\tilde{C}_{M_0}^+$, $\tilde{C}_{M_0}^-$.

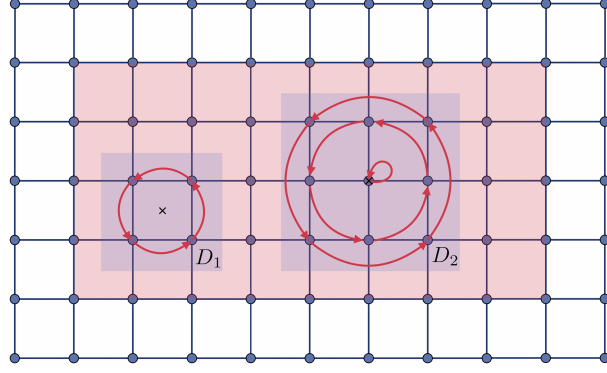


Figure D.2: Calculation on an open disc. Red shading indicates the sub-region where the flux is inserted. Blue shading inside the red region indicates different choices of maximal Wyckoff point as the partial rotation center.

Note that, we obtain the same result, Eq. (5.1), on the torus or the open disc, as long as we choose o at the same maximal Wyckoff position, and choose a partial rotation operator that inserts the same flux around a disclination at o .

D.1.5 Details on the jumps of K_o^+

As stated in the main text, if we use the operators $\tilde{C}_{M_o}^+$, K_o^+ sometimes jumps by $M_o/2$ inside a single Hofstadter lobe, as seen in Fig. D.3. We define $\Theta_o^+ \equiv K_o^+ \bmod M_o/2$ in order to eliminate these jumps. In general, the position of these jumps will depend on the system size and the size of the partial rotation region, but the value of these jumps is always $M_o/2$. We can understand these jumps as follows. The following arguments are equivalent to those given in [199] using the many-body real space invariants defined there.

Suppose $M_o = 4$ in a system where all orbitals are filled ($\nu = 1$). First consider K_o^+ . Let $K_o^+|_{\nu=1}$ be the value of K_o^+ at $\nu = 1$. Now suppose we minimally enlarge the partial rotation region D by enclosing 4 more sites which rotate into each other Under $\tilde{C}_{M_o}^+$. The states at these 4 sites form an orbit of size 4 under $\tilde{C}_{M_o}^+$ and contribute extra phases of $\{1, e^{i2\pi/4}, e^{i4\pi/4}, e^{i6\pi/4}\}$,

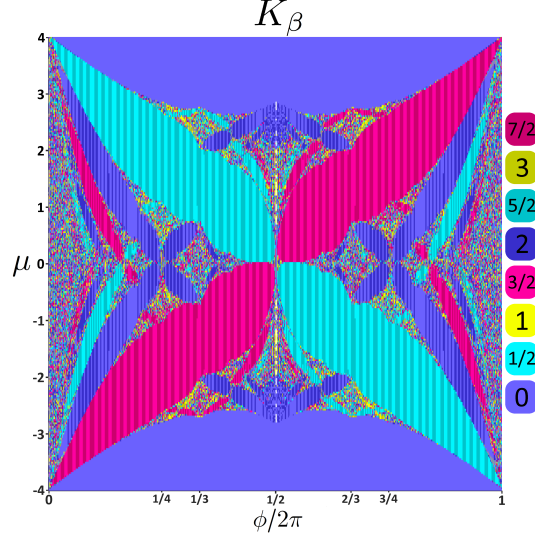


Figure D.3: Raw data for K_β , calculated on a 30×30 open disc. Partial rotation region D is a 15×15 square. K_β jumps by 2 within a single lobe whenever the color switches between bright and dim shades.

which are fourth roots of 1. The expectation value of $\tilde{C}_{M_0}^+$ will gain a total extra phase of $e^{i\pi}$ (which is the product of the 4 phases), and thus $K_0^+|_{\nu=1}$ changes by 2 mod 4. But a change of the size of the partial rotation region should be regarded as trivial. Therefore only $K_0^+ \bmod 2$ is an invariant quantity.

On the other hand, considering $\tilde{C}_{M_0}^-$ in the same argument, the states at the 4 extra sites in the enlarged region contribute phases of $\{e^{i\pi/4}, e^{i3\pi/4}, e^{i5\pi/4}, e^{i7\pi/4}\}$, which are fourth roots of -1. The total phase contribution from these four sites is 1, and thus K_0^- is invariant upon changing D .

Since changing D is trivial, the above example suggests that Θ_0^+ is an invariant mod $M_0/2$, and Θ_0^- is an invariant mod M_0 . In the latter case we indeed verify numerically that there are no jumps of K_0^- inside a given Hofstadter lobe.

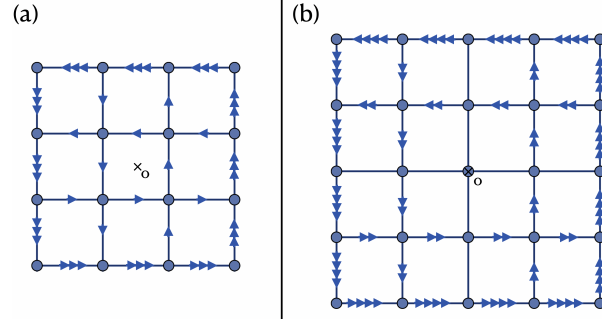


Figure D.4: Symmetric gauge defined with respect to (a) α ; (b) β . Each blue arrow represents the vector potential $\phi/4$.

D.1.6 Periodicities of $\Theta_o^\pm$

We find that as a function of the background flux per unit cell in the Hofstadter model, $\Theta_\beta^\pm$ has periodicity 2π , while $\Theta_\alpha^\pm$ has periodicity 8π . We can straightforwardly see why these invariants need not be 2π periodic in ϕ . Consider a system on the infinite plane, with flux ϕ per unit cell. For a given origin o , it is most convenient to define the vector potential in symmetric gauge around o ; in this case the operator that trivially rotates points without any U(1) gauge transformation is also a symmetry operator that commutes with the Hamiltonian.

As shown in Fig. D.4, if $o = \beta$, the links are assigned a vector potential which is an integer multiple of $\pi/2$, while if $o = \alpha$, the links are assigned a vector potential which is an integer multiple of $\pi/4$. Therefore H is 4π periodic in ϕ when $o = \beta$, and 8π periodic when $o = \alpha$. The actual periodicity of $\Theta_\beta^\pm, \Theta_\alpha^\pm$ in ϕ must therefore be divisible by 4π and 8π respectively. The fact that $\Theta_\beta^\pm$ has a smaller periodicity 2π while $\Theta_\alpha^\pm$ has the maximal allowed periodicity of 8π is a more subtle result that we see empirically, but do not have a simple justification for on a lattice without defects.

However, we can consider the flux inserted at a $\pi/2$ disclination created using $\tilde{C}_{M_\beta}^\pm$ and $\tilde{C}_{M_\alpha}^\pm$ respectively. As seen previously, for $o = \beta$ the flux is always expressed in terms of integer

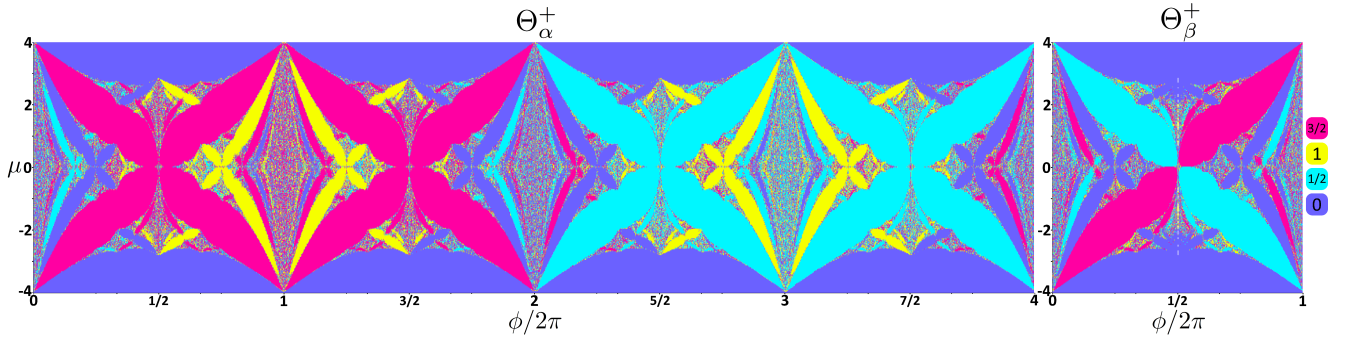


Figure D.5: Raw data for Θ_o . Θ_α^+ , Θ_β^+ , and Θ_γ^+ are calculated on a 30×30 open disc. $\Theta_\gamma^+ = 0 \bmod 1$ everywhere in the Hofstadter butterfly, and is therefore not plotted here. The diameter of the partial rotation region D is taken to be roughly half the system size.

multiples of ϕ (which is 2π periodic), while for $o = \alpha$ the flux is expressed in terms of the quantity $3\phi/4$ (which is 8π periodic). Therefore defects of $\tilde{C}_{M_\beta}^\pm$ and $\tilde{C}_{M_\alpha}^\pm$ do have the same periodicities in ϕ as $\Theta_\beta^\pm, \Theta_\alpha^\pm$. This gives additional justification for the observed periodicities.

D.1.7 Obtaining empirical formulas for Θ_o^+

In this section, we explain how to obtain empirical formula for Θ_α^+ and Θ_β^+ . The result is in Eqs. (D.1), (D.2).

In Fig. 5.3 and D.5 we plot the raw Hofstadter butterflies for $\Theta_o^\pm$. For a fixed Chern number C , the different Hofstadter lobes are separated by the so called Farey sequence of order $2|C|$, which consists of ordered irreducible fractions $\frac{p}{q}$ with $0 < p \leq q \leq 2|C|$. For example, the Farey sequence of order 4 is $\{\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}\}$. If we track the lobes at a fixed C as $\phi/2\pi$ is increased from 0 to 1, $\Theta_o^\pm$ may change its value between lobes which meet at the Farey seq of order $|C|$. (A similar behaviour was observed for the discrete shift $\mathcal{S}_o^+$ in Ref. [100].)

The jumps for Θ_α^+ and Θ_β^+ are tabulated in Figs. D.6, D.7 and can be categorized into two contributions: A jump of 1 at every odd denominator q for $q < |C|$; and a *possible* jump of 1 when q divides C . When $C > 0$ we find that after summing the contribution from each jump

$\begin{smallmatrix} \phi \\ C \end{smallmatrix}$	$\frac{1}{8}$	$\frac{1}{7}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{2}{7}$	$\frac{1}{3}$	$\frac{3}{8}$	$\frac{2}{5}$	$\frac{3}{7}$	$\frac{1}{2}$	$\frac{4}{7}$	$\frac{3}{5}$	$\frac{5}{8}$	$\frac{2}{3}$	$\frac{5}{7}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	$\frac{6}{7}$	$\frac{7}{8}$	1
1																						
2										1												
3							1															
4							1							1								
5							1		1					1				1				
6			1	1					1		1		1					1	1			
7		1		1			1		1	1			1		1	1		1				
8		1		1		1	1		1	1		1	1		1	1		1		1		

Figure D.6: Jumps in Θ_{α}^{+} for fixed C , as $\frac{\phi}{2\pi}$ increases from 0 to $\frac{1}{2}$.

point,

$$\Theta_{\alpha,\text{diff}}^{+} = \left(\sum_{\substack{p < \frac{\phi}{2\pi} \\ q \text{ odd}}} \left\lfloor \frac{C+q}{2q} \right\rfloor \right) + \begin{cases} \left\lfloor \frac{C\phi}{2\pi} \right\rfloor \text{ if } C \bmod 4 = 2 \\ \left\lfloor \frac{C\phi}{4\pi} \right\rfloor \text{ if } C \bmod 4 = 3 \\ 0 \text{ if } C \bmod 4 = 0 \\ \left\lfloor \frac{1}{2} + \frac{C\phi}{4\pi} \right\rfloor \text{ if } C \bmod 4 = 1 \end{cases} \quad (\text{D.1})$$

$$\Theta_{\beta,\text{diff}}^{+} = \left(\sum_{\substack{p < \frac{\phi}{2\pi} \\ q \text{ odd}}} \left\lfloor \frac{C+q}{2q} \right\rfloor \right) + \begin{cases} \left\lfloor \frac{C\phi}{2\pi} \right\rfloor \text{ if } C \bmod 4 = 1 \\ \left\lfloor \frac{C\phi}{4\pi} \right\rfloor \text{ if } C \bmod 4 = 2 \\ 0 \text{ if } C \bmod 4 = 3 \\ \left\lfloor \frac{1}{2} + \frac{C\phi}{4\pi} \right\rfloor \text{ if } C \bmod 4 = 0 \end{cases} \quad (\text{D.2})$$

$$\Theta_{\gamma,\text{diff}}^{+} = 0. \quad (\text{D.3})$$

These equations are all taken mod $M_o/2$. Θ_{α}^{+} in the $C < 0$ lobes can be obtained by the symmetry transformation $\Theta_{\alpha}^{+}(\mu, \phi) = \Theta_{\alpha}^{+}(-\mu, \phi) \bmod 2$; Θ_{β}^{+} in the $C < 0$ lobes can be obtained by the symmetry transformation $\Theta_{\beta}^{+}(\mu, \phi) = -\Theta_{\beta}^{+}(-\mu, \phi) \bmod 2$. We have not found the analogous formulas for Θ_o^{-} , as the jump patterns are different and much more complicated in this case.

$\frac{\phi}{2\pi}$	1	1	1	1	2	3	2	3	1	4	3	5	2	5	3	4	5	6	7	1
C	8	7	6	5	4	3	8	5	7	2	7	5	8	3	7	4	5	6	7	8
1																				
2																				
3						1							1							
4				1		1							1		1					
5						1							1							
6			1					1				1					1			
7	1	1	1		1	1		1	1		1	1		1	1		1		1	
8	1	1	1		1	1	1	1	1		1	1	1	1	1		1		1	1

Figure D.7: Jumps in Θ_β^+ for fixed C , as $\frac{\phi}{2\pi}$ increases from 0 to $\frac{1}{2}$

D.2 Numerical results for global rotations

In this section we present some preliminary numerical results for the invariants of the Hofstadter model obtained from the eigenvalue under global rotations, $\langle \Psi | \tilde{C}_{M_0}^\pm | \Psi \rangle$, on the ground state $|\Psi\rangle$ on an even length torus.

Define $\tilde{C}_{M_0}^\pm$ as in the previous section. We define a set of invariants $\Phi_0^\pm$ as follows:

$$\langle \Psi | \tilde{C}_{M_0}^\pm | \Psi \rangle = e^{il_{0,\text{global}}^\pm}, \quad (\text{D.4})$$

and we empirically find that

$$l_{02,\text{global}}^\pm = \frac{Cm^2}{2} + \mathcal{S}_{02}^\pm m + K_{02,\text{global}}^\pm \pmod{4} \quad (\text{D.5})$$

$$l_{01,\text{global}}^\pm = K_{01,\text{global}}^\pm \pmod{4}. \quad (\text{D.6})$$

If we fix a given Hofstadter lobe, the quantities $K_{0,\text{global}}^+$ jump by multiples of $M_0/2$ as the total length of the torus is changed. (Recall that for partial rotations in a region D , the quantities K_0^+ similarly jump by multiples of $M_0/2$ for the same system, as the size of D is changed.)

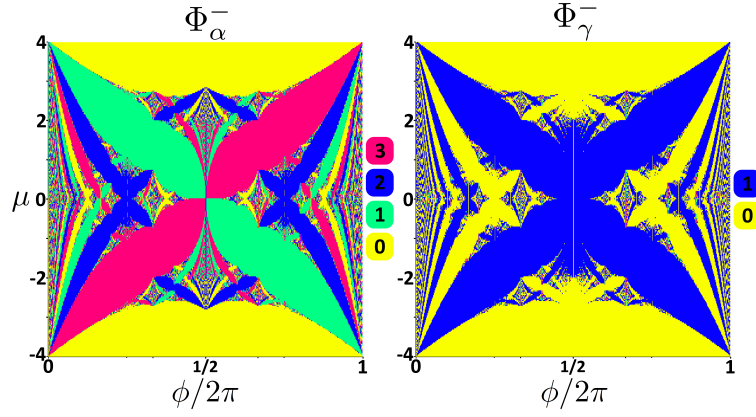


Figure D.8: Raw data for Φ_α^- and Φ_γ^+ , calculated on a 30×30 torus. They follow Eq. D.9 and Eq. D.8 respectively.

However there is no change in the value of $K_{\text{o,global}}^-$. So we define

$$\begin{aligned}\Phi_o^+ &= K_{\text{o,global}}^+ \mod M_o/2 \\ \Phi_o^- &= K_{\text{o,global}}^- \mod M_o.\end{aligned}\tag{D.7}$$

The raw Hofstadter butterflies for $\{\Phi_o^\pm\}$ are shown in Fig. D.8. We see that for either $o = o_1$ or $o = o_2$,

$$\Phi_\alpha^+ = \Phi_\beta^+ = \Phi_\gamma^+ = \Phi_\gamma^- = -C \mod 2\tag{D.8}$$

$$\Phi_\alpha^- = \Phi_\beta^- = -C \mod 4.\tag{D.9}$$

Ref. [199] also studied global rotations on an even length torus in terms of a many-body real space invariant which appears closely related to $\Phi_o^\pm$, and also found that this invariant only depends on C . In the above cases, these invariants do not give any additional information beyond the Chern number. It is not clear how much more information can be obtained by also considering odd length tori.

D.3 CFT calculation

In this appendix, we derive the expression Eq. (5.10) for the partial rotation by evaluating the CFT character,

$$\begin{aligned} \langle \Psi | \tilde{C}_M^\pm |_D | \Psi \rangle &= \frac{\text{Tr}[e^{iQ_M \frac{\pi}{M}} e^{i\tilde{P} \frac{L}{M}} e^{-\frac{\xi}{v} H}]}{\text{Tr}[e^{-\frac{\xi}{v} H}]} \\ &= e^{-\frac{2\pi i}{24M} c_-} \frac{\sum_{a=1, \psi} \chi_a(\frac{i\xi}{L} - \frac{1}{M}; [\text{AP}, 0], [\text{AP}, 1])}{\sum_{a=1, \psi} \chi_a(\frac{i\xi}{L}; [\text{AP}, 0], [\text{AP}, 0])}, \end{aligned} \quad (\text{D.10})$$

where we set M to be even. For ease of notation, we will use M in this appendix and suppress the o subscript, but we assume that some o has been fixed. While comparing with numerics we take $M = M_o = 4$ for $o = \alpha, \beta$ and $M = 2$ for $o = \gamma$. Here, $\chi_a(\tau; [s, j], [s', j'])$ with $s, s' \in \{\text{AP}, \text{P}\}$, $j, j' \in \mathbb{Z}_M$ is the CFT character that corresponds to the partition function on a torus equipped with a spin structure and a background $\mathbb{Z}_M$ gauge field. For example,

$$\chi_a(\tau; [\text{AP}, j], [\text{AP}, j']) = \text{Tr}_{a, [\text{AP}, j]}[e^{iQ_M \frac{j'\pi}{M}} e^{2\pi i \tau (L_0 - \frac{c_-}{24})}] \quad (\text{D.11})$$

where a labels the quasiparticle within the AP (antiperiodic) sector, with the twisted boundary condition corresponding to $j \in \mathbb{Z}_n$ along the spatial cycle.

In order to evaluate the above CFT character, we need to study different cases of the action of the rotation operator $\tilde{C}_M^\pm$, i.e., whether $(\tilde{C}_M^+)^M = +1$ or $(\tilde{C}_M^-)^M = (-1)^F$. This choice of the symmetry action amounts to considering the twisted or untwisted spin structure respectively for the $\mathbb{Z}_M$ gauge field coupled to the CFT. That is, the rotation symmetry $\tilde{C}_M^\pm$ at long wavelengths is expressed as a combination of an internal $\mathbb{Z}_M$ symmetry and the translation symmetry of the

CFT: $\tilde{C}_M^\pm = e^{iQ_M \frac{\pi}{M}} e^{i\tilde{P} \frac{L}{n}}$. Note that the symmetry action of $\tilde{C}_M^+$ (resp. $\tilde{C}_M^-$) is realized by an internal symmetry satisfying $e^{iQ_M \pi} = (-1)^F$ (resp. $e^{iQ_M \pi} = 1$). The former case corresponds to a $\mathbb{Z}_{2M}^f$ symmetry where the $\mathbb{Z}_M$ is nontrivially extended by the fermion parity $(-1)^F$, which corresponds to the twisted spin structure, or equivalently a $\text{spin}^{\mathbb{Z}_{2M}}$ structure. Meanwhile, the latter corresponds to the untwisted spin structure together with a flat $\mathbb{Z}_M$ gauge field.

The above identification of the spatial symmetry $\tilde{C}_M^+(\tilde{C}_M^-)$ with an internal symmetry $\mathbb{Z}_{2M}^f(\mathbb{Z}_M \times \mathbb{Z}_2^f)$ can be thought of a consequence of the crystalline equivalence principle [82, 85]. As we will see below, these two choices of the symmetry action give rise to different modular properties of the CFT characters, and hence distinct values of $\langle \Psi | \tilde{C}_M^\pm |_D | \Psi \rangle$.

D.3.1 The case with $\tilde{C}_M^+$

Let us first take the rotation symmetry $\tilde{C}_M^+$, which has an equivalent internal $\mathbb{Z}_{2M}^f$ symmetry satisfying $e^{iQ_M \pi} = (-1)^F$. The CFT character on the edge can be evaluated by the modular S, T transformations as follows (note that M is even):

$$\begin{aligned} \chi_a \left(\frac{i\xi}{L} - \frac{1}{M}; [\text{AP}, 0], [\text{AP}, 1] \right) &= S_{ab} \chi_b \left(-\frac{1}{\frac{i\xi}{L} - \frac{1}{M}}; [\text{AP}, 1], [\text{AP}, 0] \right) \\ &= (ST^M)_{ab} \chi_b \left(\frac{-iM \frac{\xi}{L}}{\frac{i\xi}{L} + \frac{1}{M}}; [\text{AP}, 1], [\text{P}, 0] \right), \end{aligned} \quad (\text{D.12})$$

where we used the fact that T exchanges $\text{Spin}^{\mathbb{Z}_{2M}}$ structure as follows:

$$T : \begin{cases} ([\text{AP}, j], [\text{AP}, j']) & \rightarrow ([\text{AP}, j], [\text{P} + \frac{[j]_M + [j']_M - [j+j']_M}{M}, [j+j']_M), \\ ([\text{AP}, j], [\text{P}, j']) & \rightarrow ([\text{AP}, j], [\text{AP} + \frac{[j]_M + [j']_M - [j+j']_M}{M}, [j+j']_M). \end{cases} \quad (\text{D.13})$$

Here, $\square_M$ denotes the mod M operation, and the spin structure changes under $\mathbb{Z}_2$ action as $AP + 1 = P, P + 1 = AP$.

To evaluate the action of the S, T modular matrices in the presence of a background $\mathbb{Z}_{2M}^f$ gauge field, we note that the modular transformation in the defect Hilbert space of the CFT can be determined from the modular S, T matrices of the G_b -crossed braided fusion category that describes the bosonic shadow of the invertible phase of interest with bosonic global symmetry G_b in the bulk [11]. In our case, the bosonic symmetry group is $G_b = \mathbb{Z}_M$. Using the data of the $\mathbb{Z}_M$ -crossed braided fusion category, the T -matrix element is expressed as [11]

$$T_{a_{\mathbf{g}}, b_{\mathbf{g}}}^{(\mathbf{g}, \mathbf{h})} = e^{-\frac{2\pi i}{24}c_-} \cdot \theta_{a_{\mathbf{g}}} \cdot \eta_a(\mathbf{g}, \mathbf{h}) \cdot \delta_{a_{\mathbf{g}}, b_{\mathbf{g}}} \quad (\text{D.14})$$

where $\mathbf{g}, \mathbf{h} \in \mathbb{Z}_M$, and $\eta_a(\mathbf{g}, \mathbf{h})$ is a phase that describes the symmetry fractionalization of the bulk topological order [11]. The character is then further rewritten as

$$\begin{aligned} & \chi_a \left(\frac{i\xi}{L} - \frac{1}{M}; [\text{AP}, 0], [\text{AP}, 1] \right) \\ &= \sum_{b \in \mathcal{C}_1} (ST^M)_{ab} \chi_b \left(\frac{-iM \frac{\xi}{L}}{\frac{i\xi}{L} + \frac{1}{M}}; [\text{AP}, 1], [\text{AP}, 0] \right) \\ &= e^{-\frac{2\pi i M}{24}c_-} \sum_{b \in \mathcal{C}_1} S_{ab} \theta_b^n \prod_{j=0}^{M-1} \eta_b(1, j) \times \chi_b \left(\frac{-iM \frac{\xi}{L}}{\frac{i\xi}{L} + \frac{1}{M}}; [\text{AP}, 1], [\text{P}, 0] \right) \\ &= e^{-\frac{2\pi i M}{24}c_-} \sum_{b \in \mathcal{C}_1} \sum_{c \in \mathcal{C}_P} S_{ab} \theta_b^M \prod_{j=0}^{M-1} \eta_b(1, j) \times S_{bc} \chi_c \left(\frac{iL}{M^2 \xi} + \frac{1}{M}; [\text{P}, 0], [\text{AP}, 1] \right), \end{aligned} \quad (\text{D.15})$$

and

$$\chi_a \left(\frac{i\xi}{L}; [\text{AP}, 0], [\text{AP}, 0] \right) = \sum_{b \in \mathcal{C}_0} S_{ab} \chi_b \left(\frac{iL}{\xi}; [\text{AP}, 0], [\text{AP}, 0] \right), \quad (\text{D.16})$$

where $\mathcal{C}_0$ (resp. $\mathcal{C}_1$) is the untwisted (resp. twisted by $1 \in \mathbb{Z}_M$) sector in G_b -crossed extension of super-modular category that consists of defects in the $[\text{AP}, 0]$ sector, namely the identity particle and the fermion (resp. the $[\text{AP}, 1]$ defects, namely the elementary $\mathbb{Z}_M$ fluxes). Meanwhile, $\mathcal{C}_P$ denotes the defects in the $[\text{P}, 0]$ sector that are labelled by fermion parity.

The character in Eq. (D.16) can further be approximated as

$$\chi_b \left(\frac{iL}{\xi}; [\text{AP}, 0], [\text{AP}, 0] \right) \approx e^{-\frac{2\pi L}{\xi} (h_b - \frac{c_-}{24})}. \quad (\text{D.17})$$

The phase of the partial rotation can then be written as

$$\langle \Psi | \tilde{C}_M^+ |_D | \Psi \rangle \propto e^{-2\pi i (M + \frac{1}{M}) \frac{c_-}{24}} \sum_{b \in \mathcal{C}_1} \sum_{c \in \mathcal{C}_P} (S_{1b} + S_{\psi b}) \theta_b^n \prod_{j=0}^{M-1} \eta_b(1, j) \times S_{bc} \chi_c \left(\frac{iL}{M^2 \xi} + \frac{1}{M}; [\text{P}, 0], [\text{AP}, 1] \right). \quad (\text{D.18})$$

While the above expression requires the sum over quasiparticles in the twisted sector $b \in \mathcal{C}_1$, it is more illuminating to express it in terms of the anyons in the untwisted sector. This is done by using the consistency equations and Verlinde formula of the $\mathbb{Z}_M$ -crossed braided fusion category, given by [11]

$$\begin{aligned} \theta_{b_1} &= \theta_{b_0} \theta_{0_1} \cdot (R^{b_0, 0_1} R^{0_1, b_0}), \\ \eta_{b_1}(1, j) &= \eta_{b_0}(1, j) \eta_{0_1}(1, j), \\ S_{a, b_1} &= \frac{1}{d_a} S_{a, b_0} (R^{a, 0_{-1}} R^{0_{-1}, a}), \end{aligned} \quad (\text{D.19})$$

where $0_j \in \mathcal{C}_j$ denotes a $j \in \mathbb{Z}_M$ defect. From the general solution outlined in Ref. [11], one can

work in the gauge where $R^{b,0_1} = 1$, $R^{c,0_1} = 1$ for $b \in \mathcal{C}_0$, $c \in \mathcal{C}_P$. We then obtain

$$\langle \Psi | \tilde{C}_M^+ |_D | \Psi \rangle \propto e^{-\frac{2\pi i}{24}(M+\frac{1}{M})c_-} \theta_{0_1}^M \prod_{j=0}^{M-1} \eta_{0_1}(1, j) \sum_{b \in \mathcal{C}_0} \sum_{c \in \mathcal{C}_P} d_b \theta_b^M \prod_{j=0}^{M-1} \eta_b(1, j) \times S_{bc} \chi_c \left(\frac{iL}{M^2 \xi} + \frac{1}{M}; [P, 0], [AP, 1] \right) \quad (\text{D.20})$$

Here we observe invariants given by the following combination of symmetry fractionalization data:

$$\mathcal{I}_M^\pm := \theta_{0_1}^M \prod_{j=0}^{M-1} \eta_{0_1}(1, j), \quad (\text{D.21})$$

where the $\pm$ superscript is used depending on whether we consider $\tilde{C}_M^+$ or $\tilde{C}_M^-$, and

$$e^{i\pi Q_b} := \prod_{j=0}^{M-1} \eta_b(1, j). \quad (\text{D.22})$$

They can be interpreted as a $\mathbb{Z}_M$ analog of the Hall conductivity, and a fractional $\mathbb{Z}_M$ charge respectively [53]. For $\frac{L}{\xi} \gg 1$, we approximate the CFT character in terms of the highest weight state $|h_b\rangle$:

$$\chi_c \left(\frac{iL}{M^2 \xi} + \frac{1}{M}; [P, 0], [AP, 1] \right) \approx e^{\frac{2\pi i}{M}(h_c - \frac{c_-}{24})} e^{-\frac{2\pi L}{M^2 \xi}(h_c - \frac{c_-}{24})}. \quad (\text{D.23})$$

We then have

$$\langle \Psi | \tilde{C}_M^+ |_D | \Psi \rangle \propto e^{-\frac{2\pi i}{24}(M+\frac{2}{M})c_-} e^{\frac{2\pi i}{M}h_v} \mathcal{I}_M^+ \sum_{b \in \mathcal{C}_0} d_b \theta_b^M e^{i\pi Q_b} S_{bv}, \quad (\text{D.24})$$

where $v \in \mathcal{C}_P$ is the quasiparticle in the periodic sector (fermion parity flux) with the lowest value of spin. For the Chern insulator with chiral central charge c_- , there are two such defects $v, v \times \psi$ with equal topological spin $c_-/8$. The result with $\mathcal{C}_0 = \{1, \psi\}$ is then given by

$$\langle \Psi | \tilde{C}_M^+ |_D | \Psi \rangle \propto e^{-\frac{2\pi i}{24}(M - \frac{1}{M})c_-} \mathcal{I}_M^+. \quad (\text{D.25})$$

D.3.2 The case with $\tilde{C}_M^-$

Here we consider the case where we take $\tilde{C}_M^-$ symmetry, which corresponds to the internal $\mathbb{Z}_M$ symmetry satisfying $e^{iQ_M\pi} = 1$. The CFT character on the edge can be evaluated by the modular S, T transformation as (note that M is even)

$$\begin{aligned} \chi_a \left(\frac{i\xi}{L} - \frac{1}{M}; [\text{AP}, 0], [\text{AP}, 1] \right) &= S_{ab} \chi_b \left(-\frac{1}{\frac{i\xi}{L} - \frac{1}{M}}; [\text{AP}, 1], [\text{AP}, 0] \right) \\ &= (ST^M)_{ab} \chi_b \left(\frac{-iM\frac{\xi}{L}}{\frac{i\xi}{L} + \frac{1}{M}}; [\text{AP}, 1], [\text{AP}, 0] \right), \end{aligned} \quad (\text{D.26})$$

where we used T exchanges $\text{spin} \times \mathbb{Z}_M$ structure as

$$T : \begin{cases} ([\text{AP}, j], [\text{AP}, j']) \rightarrow ([\text{AP}, j], [\text{P}, [j + j']_M]), \\ ([\text{AP}, j], [\text{P}, j']) \rightarrow ([\text{AP}, j], [\text{AP}, [j + j']_M]). \end{cases} \quad (\text{D.27})$$

Using a similar discussion as the previous subsection, one can write the phase of the partial rotation as

$$\langle \Psi | \tilde{C}_M^- |_D | \Psi \rangle \propto e^{-2\pi i \frac{c_-}{24} (M + \frac{1}{M})} \mathcal{I}_M^- \sum_{b, c \in \mathcal{C}_0} d_b \theta_b^M \prod_{j=0}^{M-1} \eta_b(1, j) \times S_{bc} \chi_c \left(\frac{iL}{M^2 \xi} + \frac{1}{M}; [\text{AP}, 0], [\text{AP}, 1] \right), \quad (\text{D.28})$$

where we work on the gauge $R^{b, 0 \frac{\pi}{n}} = 1$ for $b \in \mathcal{C}_0$. The dominant contribution for the sum over anyons c in the trivial sector comes from $c = 1$, where the CFT character is approximated as

$$\chi_1 \left(\frac{iL}{M^2 \xi} + \frac{1}{M}; [\text{AP}, 0], [\text{AP}, 1] \right) \approx e^{-\frac{2\pi i}{M} \frac{c_-}{24}} e^{\frac{2\pi L}{M^2 \xi} \frac{c_-}{24}}. \quad (\text{D.29})$$

We then obtain

$$\langle \Psi | \tilde{C}_M^- |_D | \Psi \rangle \propto e^{-\frac{2\pi i}{24} (M + \frac{2}{M}) c_-} \mathcal{I}_M^- \sum_{b \in \mathcal{C}_0} d_b^2 \theta_b^M e^{i\pi Q_b}. \quad (\text{D.30})$$

For Chern insulators, in which $\mathcal{C}_0 = \{1, \psi\}$, the partial rotation is simply given by

$$\langle \Psi | \tilde{C}_M^- |_D | \Psi \rangle \propto e^{-\frac{2\pi i}{24} (M + \frac{2}{M}) c_-} \mathcal{I}_M^-. \quad (\text{D.31})$$

D.3.3 Calculation of $\mathcal{I}_M^\pm$

In Eq. (D.25), (D.31), we observed that the partial rotation is proportional to the phase $\mathcal{I}_M^\pm$ that should be determined by $\tilde{C}_M^\pm$ symmetry action on the state. Here, we derive the general expression for $\mathcal{I}_M^\pm$ in terms of the data of a given fermionic invertible phase with $G_b = \mathbb{Z}_M$

symmetry and chiral central charge c_- .

The bosonic shadow of this phase is a 16-fold way G_b -enriched topological phase, also with chiral central charge c_- . In our setup, the symmetry does not permute anyons. Let the symmetry fractionalization data be given by

$$\omega_2(\mathbf{g}, \mathbf{h}) = k_{\pm} \frac{[\mathbf{g}]_M + [\mathbf{h}]_M - [\mathbf{gh}]_M}{M} \mod 2 \quad (\text{D.32})$$

$$n_2(\mathbf{g}, \mathbf{h}) = k_s \frac{[\mathbf{g}]_M + [\mathbf{h}]_M - [\mathbf{gh}]_M}{M} \mod 2. \quad (\text{D.33})$$

ω_2 describes the extension of the internal symmetry $\mathbb{Z}_M$ by fermion parity. From the fermionic crystalline equivalence principle, $k_+ = 1 \mod 2$ while $k_- = 0 \mod 2$. For a longer discussion of this point, see for example Ref. [85]. The two cases can be combined by defining $k_{\pm} = (1 \pm 1)_2^{\frac{1}{2}}$.

If $\mathbf{g}_0$ is the generator of G_b , we wish to compute

$$\mathcal{I}_M^{\pm} := \theta_{0_{\mathbf{g}_0}}^M \prod_{j=0}^{M-1} \eta_{0_{\mathbf{g}_0}}(\mathbf{g}_0, \mathbf{g}_0^j). \quad (\text{D.34})$$

We use the general solution outlined in Ref. [11] combined with the general theory of invertible fermionic phases in Ref. [27], which provides explicit expressions for the η and θ symbols in terms of n_2, ω_2, c_- . In particular, for this symmetry we can choose a gauge in which $\theta_{0_{\mathbf{g}_0}} = 1$ and

$$\eta_{0_{\mathbf{k}}}(\mathbf{g}, \mathbf{h}) = \nu_3^{-1}(\mathbf{g}, \mathbf{h}, \mathbf{k}) \quad (\text{D.35})$$

where

$$d\nu_3 = e^{-2\pi i \mathcal{O}_4}, \quad (\text{D.36})$$

$$\mathcal{O}_4 = \frac{1}{2}n_2(n_2 + \omega_2) + \frac{c_-}{8}\omega_2^2 \pmod{1}. \quad (\text{D.37})$$

After plugging in the functional forms of ω_2, n_2 and integrating, we get the following solution for ν_3 :

$$\nu_3(\mathbf{g}, \mathbf{h}, \mathbf{k}) = \exp \left(-2\pi i \left[\frac{k_s(k_\pm + k_s)}{2} + \frac{c_- k_\pm^2}{8} + k_3 \right] \frac{[\mathbf{g}]_M [\mathbf{h}]_M + [\mathbf{k}]_M - [\mathbf{hk}]_M}{M} \right). \quad (\text{D.38})$$

Here k_3 is a bosonic SPT index. Now a computation gives the desired result

$$\mathcal{I}_M^\pm = \exp \left(\frac{2\pi i}{M} \left[\frac{k_s(k_\pm + k_s)}{2} + \frac{c_- k_\pm^2}{8} + k_3 \right] \right). \quad (\text{D.39})$$

Combining Eqs. (D.25), (D.31) with Eq. (D.39), and restoring the o subscripts, we obtain Eq. (5.10) in the main text. Note that there is a phase proportional to c_- coming from $\mathcal{I}_M^+$ but not $\mathcal{I}_M^-$.

D.4 Computation of $\ell_{s,o,LL}$

Here we state the effective response theory for a continuum system of C filled Landau levels with symmetry $U(1) \times SO(2)$ (where $SO(2)$ signifies spatial rotations), and find its relationship to the continuum limit of the Hofstadter model.

In terms of a $U(1)$ gauge field A and an $SO(2)$ gauge field ω (which can be identified with the components of the spin connection on the underlying spatial manifold), the continuum

response theory in the case where a 2π spatial rotation acts trivially on fermions is

$$\mathcal{L}_{eff} = \sum_{n=1}^C \left(\frac{1}{4\pi} (A + s_n \omega) \wedge d(A + s_n \omega) - \frac{1}{48\pi} \omega \wedge d\omega \right) \quad (\text{D.40})$$

where $s_n = \frac{2n-1}{2}$ is the orbital spin per particle in the n -th Landau level. The Chern number of each filled Landau level is 1. The last term represents the gravitational anomaly. Expanding the above Lagrangian, we obtain

$$\mathcal{L}_{eff} = \frac{1}{4\pi} A \wedge dA + \frac{1}{2\pi} \frac{C^2}{2} A \wedge d\omega + \frac{1}{4\pi} \frac{2C^3 - C}{6} \omega \wedge d\omega. \quad (\text{D.41})$$

The coefficient of $\frac{1}{2\pi} A \wedge d\omega$ is identified with the Wen-Zee shift $\mathcal{S}_{LL}^+$. It has no origin dependence, and takes the value $\mathcal{S}_{LL}^+ = C^2/2$. The coefficient $\tilde{\ell}_{s,LL}^+$ of the term $\frac{1}{4\pi} \omega \wedge d\omega$ also has no origin dependence, and takes the values

$$\tilde{\ell}_{s,LL}^+ = \frac{2C^3 - C}{6} = \left\{ \frac{1}{6}, \frac{7}{3}, \frac{17}{2}, \frac{62}{3}, \frac{245}{6}, \dots \right\} \quad (\text{D.42})$$

We define $\ell_{LL} = \tilde{\ell}_s + C/12$, which is the coefficient of $\frac{1}{4\pi} \omega \wedge d\omega$ if we ignore the framing anomaly.

In the continuum limit of the Hofstadter model, assuming that a 2π rotation acts trivially on fermions (that is, the rotation operator is given by $\tilde{C}_{M_0}^+$), the analogous response coefficient is denoted $\ell_{s,o,LL}^+ = \ell_{s,LL}^+ \bmod M_0/2$; it acquires an origin dependence and is quantized mod $M_0/2$ where M_0 is the order of rotations that preserve o . Here we assume M_0 is even. For the square lattice,

$$\ell_{s,o,LL}^+ = \frac{2C^3 - C}{6} + \frac{C}{12} = \left\{ \frac{1}{4}, \frac{5}{2}, \frac{3}{4}, 1, \frac{5}{4}, \dots \right\} \bmod M_0/2. \quad (\text{D.43})$$

To derive the expression for $\ell_{s,0,\text{LL}}^-$ when we instead use the operators $\tilde{C}_{M_0}^-$, we can simply replace $A \rightarrow A + \omega/2$ in Eq. (D.40) (this corresponds to choosing a reference rotation operator $\tilde{C}_{M_0}^-$ that inserts π/M_0 flux at a disclination of angle $2\pi/M_0$) and again compute the $\frac{1}{4\pi}\omega \wedge d\omega$ coefficient, after subtracting off the framing anomaly. The final result is

$$\ell_{s,0,\text{LL}}^- = \frac{2C^3 - C}{6} + \frac{C^2}{2} + \frac{C}{4} + \frac{C}{12} = \{1, 1, 2, 2, 3, 3, 0, 0, 1, \dots\} \mod M_0. \quad (\text{D.44})$$

The above equations reproduce Eqs. (5.5), (5.6) appear in the main text.

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