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A ZERO - DIMENSIONAL SHOCK

by

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1. Introduction

In this note, we add a new wrinkle to the very old problem of determining the motion of a mass point on a spring. We adopt a general model for the spring in which the force needed to compress it to zero length is infinite. (Consequently the motion is governed by a singular nonlinear second-order ordinary differential equation.) In this setting we entertain the possibility, permitted by the governing equations, that such a total compression is actually attained. This total compression corresponds to a kind of shock. We then extract from the governing equations all the illumination they can shed on the physical behavior. Our problem, which presents novel features for ordinary differential equations, captures in microcosm deep and unresolved issues involving shocks and their suppression, which arise in the study of quasilinear hyperbolic and parabolic partial differential equations. We comment briefly on these issues in Section 8.

Notation. t denotes the time. Ordinary derivatives with respect to it are denoted by superposed dots. All other ordinary derivatives are denoted by primes. We denote partial derivatives by subscripts. Certain symbols such x appear in two roles: as a real-valued argument of a function and in an expression $x(t)$ for the value of a real-valued function of t. For the sake of precision we denote the function with values $x(t)$ by $x(\cdot)$.

2. Formulation of the Governing Equations

A spring (e.g., in the form of a steel helix or a rubber

cylinder) is confined to a horizontal groove along the x-axis. The "left" end of the spring is fixed to a wall at $x = -1$. A unit mass point is attached to the "right" end. Without loss of generality we assume that the natural (unstretched) length of the spring is 1. We let $x(t)$ denote the position of the mass point. ($x(t)$ is also the displacement of the mass point from its equilibrium state.) See Fig. 1.

We assume that the force exerted at time t by the spring on the mass point depends only on the position $x(t)$ and the velocity $\dot{x}(t)$ of the mass point. We denote this force by $-n(x(t),\dot{x}(t))$. The function n is assumed to be given. Let us set

$$n(x,0) = \varphi'(x) \quad \text{with} \quad \varphi'(0) = 0,\ \varphi(0) = 0. \tag{1}$$

Then n has the form

$$n(x,y) = \varphi'(x) + \nu g(x,y) \quad \text{with} \quad g(x,0) = 0. \tag{2}$$

Here ν is a non-negative parameter. $\varphi'(x)$ is the <u>tensile force</u> exerted on the mass point when it has position x and is not moving. It effectively describes the elastic properties of the spring. g accounts for forces that come into play when the mass point moves. These are the internal frictional or <u>viscous</u> forces of the spring. When the mass point moves both the groove and the ambient air can exert frictional forces on the mass point. Such forces can be absorbed in g. For our purposes, however, it is conceptually advantageous to regard these external forces as absent (in consequence of a perfect lubrication of the groove and of the placement of the system in a vacuum). We assume that no other forces act on the spring. Then the equation of motion for

the mass point is

$$\ddot{x}(t) + \varphi'(x(t)) + \nu g(x(t),\dot{x}(t)) = 0. \tag{3}$$

For simplicity we assume that n is continuously differentiable. We require that

$$n_x(x,y) > 0 \qquad \forall x>-1 \quad \text{and} \quad \forall y \tag{4}$$

so that an increase in the length of the spring is accompanied by a corresponding increase in the tensile force. We adopt growth conditions compatible with (4): For each fixed y

$$n(x,y) \longrightarrow \begin{Bmatrix} \infty \\ -\infty \end{Bmatrix} \text{ as } x \longrightarrow \begin{Bmatrix} \infty \\ -1 \end{Bmatrix} \tag{5}$$

so that an infinite tensile force is needed to stretch the spring to infinite length and that a negatively infinite tensile force (i.e., an infinite compressive force) is needed to compress the spring to zero length. Conditions (4) and (5) yield corresponding conditions on φ'. We illustrate φ in Fig.2. (φ is called the <u>stored energy</u> of the spring). That $\varphi(x)$ need not approach ∞ as $x \longrightarrow -1$ is crucial in our development.

We assume that

$$n_y(x,y) > 0. \tag{6}$$

Thus an increase in the velocity of the end of the spring is accompanied by a corresponding increase in the tensile force. As we shall see, this requirement ensures that the effect of g is to dissipate energy. When $\nu \neq 0$, we accordingly say that the motion is <u>damped</u>. We adopt a strong version of (6) that is required to hold when the spring is under compression ($x < 0$) and

is being compressed ($\dot{x} < 0$): There is a positive-valued function ψ' such that

(7) $$g(x,y) \le \psi'(x)y \quad \text{for} \quad -1 < x \le 0, \quad y \le 0.$$

(Note that if (7) is an equality, then (4) holds for all x in $(-1,0]$ and for all $y \le 0$ if and only if $\psi''(x) \le 0$.) We take $\psi(0) = 0$. The viscous stress g is allowed to depend upon x because the internal frictional force when the spring is highly compressed may well differ from that when the spring is under tension.

3. Analysis of the Differential Equation for the Undamped Motion

We begin our study of (3) by first studying it for $\nu = 0$. In this case we multiply (3) by $\dot{x}(t)$ and integrate the resulting equation to get the <u>energy equation</u>:

(8) $$\tfrac{1}{2}\dot{x}(t)^2 + \varphi(x(t)) = \tfrac{1}{2}\dot{x}(0)^2 + \varphi(x(0)).$$

This integral is assumed to hold only on time intervals on which $x(\cdot)$ is twice differentiable and satisfies $x(t) > -1$ for all t's in the interval.

We often set $y(t) = \dot{x}(t)$. The phase portrait of (3) consists of the family of curves

(9) $$\mathcal{C}(E): \tfrac{1}{2}\, y^2 + \varphi(x) = E(\text{const})$$

in the (x,y)-plane. Equation (8) says that a phase-plane trajectory $(x(\cdot),\dot{x}(\cdot))$ of the equation is confined to a single curve of the form (9) on any time interval on which $x(\cdot)$ is twice differentiable. (Since $\varphi(x) \longrightarrow \infty$ as $x \longrightarrow \infty$, equation (8) implies that

there is a number C (depending on initial condition) such that $|\dot{x}(t)| < C$, $x(t) < C$. Standard continuation results for ordinary differential equations (cf. [2, Chap.1.]) then ensure that the solution of (3) exists as long as $x(t) > -1$.)

If $\varphi(x) \longrightarrow \infty$ as $x \longrightarrow -1$, the phase portrait is given by Fig. 3, in which no trajectory flirts with the wall $x = -1$ forming the edge of its universe. But if φ has a finite limit as $x \longrightarrow -1$, then the phase portrait has an entirely different character illustrated in Fig.4: For large enough E the trajectories actually touch the wall tangentially. If this happens to a solution, i.e., if there is a τ such that $x(\tau) = -1$, we say that the spring suffers a <u>shock</u>. It is not at all clear what should happen to $x(\cdot)$ after the first time t that $x(t) = -1$. Our goal is to deduce the fate of $x(\cdot)$ i.e., to deduce the <u>shock structure</u>, on the basis of sound mathematical arguments and physical hypotheses.

Let us note that the intersection of the region enclosed by (9) and the half-plane $\{(x,y): x > -1\}$ is convex because φ is a convex function. Since φ has its minimum at $x = 0$, the curve (9) has its greatest separation in the y-direction at $x = 0$.

4. The Equation for Damped Motion

We now examine the equation (3) (with $\nu > 0$) for damped motion, which is equivalent to the system

(10a,b) $$\dot{x}(t) = y(t),\ \dot{y}(t) = -\varphi'(x(t)) - \nu g(x(t),y(t))$$

of first-order equations. Equation (10a) says that trajectories of the phase portrait move right when $y > 0$ and move left when

$y < 0$. The horizontal isocline of (10) is the locus of points at which trajectories are horizontal, i.e., the set of (x,y) such that

$$\varphi'(x) + \nu g(x,y) = 0. \tag{11}$$

Conditions (4), (5) support a global implicit function theorem (based on the intermediate value theorem) ensuring that (11) has a unique solution:

$$x = h(y;\nu) \tag{12}$$

where h is a continuously differentiable function with

$$h(0;\nu) = 0, \quad h_y(y;\nu) \leq 0, \quad h(y;\nu) > -1, \quad h(y;0) = 0. \tag{13}$$

The vertical isocline of (10) is the locus of points at which trajectories are vertical; it is the x-axis as a consequence of (10a).

Let us multiply (3) by $\dot{x}(t)$ obtaining

$$\frac{d}{dt}\left\{\frac{1}{2}\dot{x}(t)^2 + \varphi(x(t))\right\} = -\nu g(x(t),\dot{x}(t))\dot{x}(t). \tag{14}$$

For $\nu > 0$, the properties of g ensure that the right-hand side of (14) is negative when $\dot{x}(t) \neq 0$, i.e., when a phase-plane trajectory is not crossing the x-axis (cf. (10a)). It follows from (14) that trajectories pierce inward through the level curves (9). (This fact supports an existence theorem like that discussed in the remarks following (9).) Equation (14) also says that the energy, which is the term in braces on the left side of (14), is decreasing for $\nu > 0$. It is this fact that justifies characterizing g as dissipative.

5. Shock Absorbers

The dissepativity embodied by (6) ensures that any trajectory (10) originating within or on the distinguished curve $\mathcal{C}(\varphi(-1))$ of (9), which is tangent to the line $x = -1$, must ultimately wind down to the origin. (This fact is a standard result from the theory of ordinary differential equations.) But it is not clear whether a typical trajectory starting outside $\mathcal{C}(\varphi(-1))$ will intersect the line $x = -1$, thereby generating a shock, or will safely negotiate this dangerous edge and ultimately bring the system to rest at $(x,y) = (0,0)$.

Suppose that φ has a finite limit as $x \longrightarrow -1$ (or equivalently, that φ' is integrable on $(0,1]$). Then g is called a <u>shock absorber</u> if for each $\nu > 0$ there is no trajectory starting to the right of the line $x = -1$ that reaches this line. We determine conditions that ensure that g is a shock absorber and then show how shock absorbers can be used to develop a shock structure for the undamped equation.

We integrate (3) from 0 to t obtaining

$$(15) \qquad 0 = \dot{x}(t)-\dot{x}(0) + \nu \int_0^t g(x(\tau),\dot{x}(\tau)d\tau + \int_0^t \varphi'(x(\tau))d\tau.$$

We study (3) and (15) on a time interval $[0,t]$ for which $(x(\cdot),\dot{x}(\cdot))$ lies entirely in the quadrant $Q \equiv \{(x,y):-1<x\leq 0,y<0\}$. Without loss of generality we take initial conditions with $x(0) < 0$, $\dot{x}(0) < 0$. Every trajectory in Q may be regarded as originating from such an initial point. Condition (7) implies that

$$(16) \qquad 0 \le \dot{x}(t)-\dot{x}(0) + \nu \int_0^t \psi'(x(\tau))\dot{x}(\tau)d\tau + \int_0^t \varphi'(x(\tau))d\tau$$

for trajectories in Q. Since $\varphi'(0) = 0$, condition (4) implies that $\varphi'(x) < 0$ for $x < 0$. Thus inequality (16) immediately yields

$$(17) \qquad \nu\psi(x(t)) \ge \dot{x}(0)-\dot{x}(t) + \nu\,\psi(x(0)) - \int_0^t \varphi'(x(\tau))d\tau$$

$$\ge \dot{x}(0) + \nu\psi(x(0))$$

for trajectories in Q.

Now $\dot{x}(0) < 0$ in Q. If $\psi(x) \longrightarrow -\infty$ as $x \searrow -1$, then

$$(18) \qquad x(t) \ge \psi^{-1}(\psi(x(0)) + \dot{x}(0)/\nu) > -1 \qquad \text{for } \nu > 0$$

so that g is a shock absorber. If $\psi(x)$ has a finite limit α as $x \searrow -1$, then there are initial data, including those with $\dot{x}(0) < \nu\alpha$, for which shocks must occur. See Fig. 5. (Note what happens as $\nu \searrow 0$.)

The physical significance of the requirement that $\psi(x) \longrightarrow -\infty$ as $x \searrow -1$ is that internal friction increases without bound as the spring is subjected to increasingly severe compression.

6. Shock Structure

Suppose that the elastic energy $\varphi(x)$ has a finite limit as $x \longrightarrow -1$, in which case the undamped problem has phase portrait Fig.4. Let g be a shock absorber. We adopt the view that the physically natural behavior for solutions of the undamped problem in the presence of shocks is that dictated by the nonuniform limit as $\nu \searrow 0$ for the problem damped with a shock absorber.

To determine this behavior, let us assume that ν is small. Then standard perturbation results of ordinary differential equations (cf. [2, Chap.2], e.g.) tell us that away from the wall $x = -1$ (where the equation ceases to be regular), the trajectories of (3) are close to those of the undamped equation. (I.e., on any compact subset of $\{(x,y): -1 < x\}$ the trajectories of (30) uniformly approach those of the undamped equations as $\nu \searrow 0$.)

We illustrate a typical trajectory in Fig. 6 with large initial data. It stays close to the level curve $\mathcal{C}(E)$ until it approaches the wall. Since g is a shock absorber, the trajectory cannot touch the wall. Its motion in Q satisfies $\dot{x}(\cdot) < 0$, $\dot{y}(\cdot) > 0$ so that it must move nearly parallel to the wall in Q. It is therefore constrained to enter the second quadrant. But this trajectory must pierce inwardly through every level curve (9). Therefore, in the second quadrant it cannot cross the level curve $\mathcal{C}(\varphi(-1))$, which is tangent to $x = -1$. It is therefore subsequently confined to remain within this level curve, slowly winding down to the equilibrium state $(x,y) = (0,0)$.

The behavior of the phase portrait in the limit as $\nu \searrow 0$ is nearly obvious. The limiting trajectory corresponding to that of Fig. 6 is given in Fig. 7. It consecutively passes through points 0, 1^-, 1^+, 2, 3, 4 with the cycle 1^+, 2, 3, 4 repeated periodically. The only question facing us to determine what is happening on the segment $1^-,1^+$. To answer this we revert to the study of (3) when ν is small. Let ε be a small positive number. We study the phase portrait of (3) or (10) in the strip

$\{(x,y): -1 < x < -1+\varepsilon,\ y < 0\}$. Here $\dot{y}(t) > 0$. Thus let $y(\cdot)$, defined on (t_1,t_2), correspond to a trajectory lying in this strip. Let $\hat{t}(\cdot)$ denote its inverse, defined on (y_1,y_2), where $y_1 = y(t_1), y_2 = y(t_2)$. The time lapse in traversing this portion of the trajectory is

$$(19)\quad t_2-t_1 = \int_{y_1}^{y_2} \hat{t}'(y)dy = \int_{y_1}^{y_2} \frac{dy}{\dot{y}(\hat{t}(y))} = -\int_{y_1}^{y_2} \frac{dy}{\varphi'(x(\hat{t}(y)))+\nu g(x(\hat{t}(y)),y)},$$

the last equality coming from (10b). Since $x < -1+\varepsilon$, we find from (19) that

$$(20)\quad t_2 - t_1 \le \frac{y_2-y_1}{\varphi'(-1+\varepsilon)}.$$

As $\varepsilon \searrow 0$, $t_2-t_1 \searrow 0$. We have seen in Fig. 6 that as $\nu \searrow 0$ the trajectory must get closer and closer to $x = -1$ in quadrant Q. In virtue of (20) we therefore conclude that the appropriate time lapse for the trajectory to go from 1^- to 1^+ in Figure 8 is the limit of the time lapse needed to traverse the nearly vertical part of Fig. 6; this limit is 0. Thus we may characterize the shock structure of the undamped problem as being embodied in the phase portrait of Fig. 7 with the jump from 1^- to 1^+ occurring instantaneously. Thus $\dot{x}(\cdot)$ is discontinuous. (A discontinuity in the velocity field is a difining property of <u>shock</u> in the theory of partial differential equations of mechanics. Our notion of <u>shock</u> is consequently consistent with this usage.) We now plot in Figs. 8 the position $x(\cdot)$ and velocity $\dot{x}(\cdot)$ in light of this interpretation of Fig. 7.

This development has some interesting consequences. Suppose that φ is integrable on $(-1,0)$ and that g is a shock absorber. Let us integrate (3) with respect to t over (t_1,t_2) obtaining the Impulse-Momentum Principle

$$(21) \qquad \dot{x}(t_2) - \dot{x}(t_1) = -\int_{t_1}^{t_2} [\varphi'(x(t)) + \nu g(x(t),\dot{x}(t))]dt,$$

which is to hold for (almost) all intervals (t_1,t_2). The left side of (21) is the difference in the linear momentum (since the mass is unity) and the right side, the time integral of the force, is the impulse. The Impulse - Momentum Principle generalizes Newton's Second Law of Motion to handle cases in which the velocity $\dot{x}(\cdot)$ is not a differentiable function of t. More generally, we could regard the impulse over (t_1,t_2), not as an integral of a prescribed force function, but rather as a prescribed functional of $x(\cdot)$.

Suppose that the situation illustrated in Fig.6 prevails. Let $\nu \searrow 0$. Then the trajectory of Figure 7 approaches the line $x = -1$ and in the limit touches it along $1^-,1^+$ of Fig.7 for an instant τ of time. Let $(t_1,t_2) \ni \tau$ and let $t_1 \nearrow \tau$, $t_2 \searrow \tau$. Since $\dot{x}$ suffers a jump at τ in Fig.7, the formal limit of (21) as $\nu \searrow 0$ cannot hold. What happens is that the integrand on the right hand side of (21) converges in the sense of distributions to a Dirac delta. This means that it would be a fruitless exercise to seek a solution of (21) with $\nu = 0$ in the class of functions $x(\cdot)$ for which $\varphi'(x(\cdot))$ is integrable. In this limiting case the right-hand side of (21) should be interpreted as the value of a functional of $x(\cdot)$, depending on

(t_1, t_2).

Let us also note that the energy drops across τ. Indeed, the impulse functional can be expressed in terms of the energies at $\tau-$ and $\tau+$. (cf.(8)). Thus in the limit as ν approaches 0, the effects of dissipation become concentrated at the shock.

7. Example. Quadratic Damping.

We can illuminate the nature of the inequalities (6), (7) by examining a situation in which they are not satisfied, that of quadratic damping:

$$g(x,y) = \tfrac{1}{2}\,\psi'(x)y|y|,\ \psi'(x) \geq 0. \tag{22}$$

On any t-interval for which $\dot{x}(t) < 0$, we can let $\tilde{t}(\cdot)$ be the inverse of $x(\cdot)$ and set $v(x) = \dot{x}(\tilde{t}(x))$. Then $\ddot{x}(\tilde{t}(x)) = v(x)v'(x)$ and (3) reduces to

$$\frac{d}{dx}\left[\tfrac{1}{2}v(x)^2\right] + \varphi'(x) - \tfrac{\nu}{2}\psi'(x)v(x)^2 = 0 \quad \text{where} \quad \dot{x}(t) < 0 \tag{23}$$

(The sign of the last term on the left-hand side of (23) changes where $\dot{x}(t) > 0$.) Let $w = \frac{1}{2}v^2$. Since (23) is a linear equation for $w(\cdot)$, its solution is

$$w(x) = e^{\nu\psi(x)}w(0) - e^{\nu\psi(x)}\int_0^x \varphi'(\xi)e^{-\nu\psi(\xi)}\,d\xi. \tag{24}$$

We assume that $\varphi(x)$ has a finite limit as $x \longrightarrow -1$. Noting that the independent variable in (24) is x, we refer to Figure 6 to deduce that (22) is a shock absorber if for each $w(0) > 0$ and for each ν in $(0,1]$ there is an x in $(-1,0)$ for which $w(x) = 0$. In this case, (24) implies that there is an x in

$(-1,0)$ such that

$$w(0) = \int_0^x \varphi'(\xi)\, e^{-\nu\psi(\xi)} d\xi . \tag{25}$$

If $\psi(x)$ has a finite limit as $x \longrightarrow -1$, then the integrability of φ' ensures that the right-hand side of (25) has an upper bound independent of ν for $\nu \le 1$. Thus (22) cannot be a shock absorber. If $\psi(x) \longrightarrow -\infty$ as $x \longrightarrow -1$, the situation is far more delicate than that for (6), (7), discussed in Section 5. E.g., if $\psi(x) = \ell n(1+x)$, then the right-hand side of (25) reduces to

$$\int_0^x \varphi'(\xi)(1+\xi)^{-\nu} d\xi . \tag{26}$$

If there are numbers $C > 0$, μ in $(0,1)$ such that $\varphi'(x) \ge -C(1+\xi)^{-\mu}$, then (26) is bounded above for $-1 < x \le 0$ when $\nu < 1 - \mu$. Thus even though $\psi(x) \longrightarrow -\infty$ as $x \longrightarrow -1$, equation (22) fails to be a shock absorber. On the other hand, if $\psi(x) = -(1+x)^{-\alpha}$ with $\alpha > 0$ for $x < -1/2$, then for $x < -1/2$ the right-hand side of (25) exceeds

$$[-\varphi'(-1/2)] \int_x^{-1/2} e^{\nu(1-\xi)^{-\alpha}} d\xi$$

$$= [-\varphi'(-1/2)]\alpha^{-1} \int_{(1/2)^{-\alpha}}^{(1+x)^{-\alpha}} e^{\nu\eta}\, \eta^{-(\alpha+1)/\alpha} d\eta \tag{27}$$

$$> [-\varphi'(-1/2)]\alpha^{-1} \int_{(1/2)^{-\alpha}}^{(1+x)^{-\alpha}} [1+\nu\eta+\cdots+(\nu\eta)^k/k!]\eta^{-(\alpha+1)/\alpha} d\eta$$

$$> \nu^k[-\varphi'(-1/2)](\alpha k!)^{-1} \int_{(1/2)^{-\alpha}}^{(1+x)^{-\alpha}} \eta^{k-(\alpha+1)/\alpha} d\eta$$

where k is an integer exceeding $(\alpha+1)/\alpha$. Then as $x \to -1$, the rightmost term of (27) approaches ∞ for each $\nu > 0$. Thus (22) with this class of φ's are shock absorbers.

8. Conclusion

Our treatment in Sections 4-6 produced a doctrine for constructing solutions of the undamped equation near a time τ at which $x(\tau) = -1$. One interpretation of this doctrine is that it yields a criterion for selecting a unique way to continue the solution after the shock. Without such a well-founded doctrine we could have invoked a spurious argument based on the conservation of energy to infer that a trajectory arriving at $(x,\dot{x}) = (-1,-y_0)$ should leave along the tail of the same trajectory at $(x,\dot{x}) = (-1,y_0)$.

One can contemplate problems, analogous to the one we treated, for the longitudinal motion of a viscoelastic rod. This motion is governed by a partial differential equation of the form

$$(28) \qquad \rho(s)u_{tt}(s,t) = n(u_s(s,t),u_{st}(s,t))_s$$
$$= [\varphi'(u_s(s,t)) + \nu g(u_s(s,t),u_{st}(s,t))]_s$$

where ρ is the mass density of the rod per unit reference length, $u(s,t)$ is the displacement of the material point s at time t, and n has the form (2). Pioneering analyses of the existence of solutions for initial-boundary value problems for (28) were carried out by Greenberg, MacCamy, & Mizel [6] for

(29) $$n(x,y) = \varphi'(x) + \nu y, \quad \nu > 0,$$

by Kanel' [8] (and later independently by MacCamy [19]) when there is a number $C > 0$ such that

(30) $$n(x,y) = \varphi'(x) + \psi'(x)y, \quad \psi'(x) \geq C^{-1},$$

and by Dafermos [3] when there is a number $C > 0$ such that

(31) $$n_y \geq C^{-1}, \quad (n_x)^2 \leq Cn_y.$$

It is clear that if φ' has the form show in Fig.2, then (29) satisfies (4)-(7). Obviously (30) satisfies (6),(7). But if $\psi'(x)$ is not constant, then (4) cannot hold for all y. Under mild additional conditions on n it can be shown [1] that the second condition of (31) is incompatible with the second limit of (5).

These observations do not deprive (30) and (31) of their physical significance. Equation (28) subject to (30) is the material (or Lagrangian) form of the one-dimensional Navier-Stokes equations for the flow of a compressible viscous fluid, provided that the first limit of (5) is replaced with the inequality $n(x,y) < 0$. Here u_s+1 is the specific volume and $-n$ is the pressure. Equation (28) subject to (31) furnishes a perfectly good model for one-dimensional shearing motion of a viscoelastic body.

To the best of my knowledge, there is no global existence theory for initial-boundary value problems for (28) for a general class of functions n satisfying (4)-(6). The difficulty lies in the treatment of the possibility that $u(s,t) = -1$ for some s,t and thus reflects a very challenging version of the analogous

obstacle overcome in Sections 4-6. (In technical terms, the uniform parabolicity of (28) inherent in the assumption that $n_y \geq C^{-1}$, is insufficient to overcome the extreme hyperbolicity inherent in the second limit of (5).) I conjecture that an appropriate existence theory can be fashioned on the basis of (7).

The limiting process of Section 6 represents one particularly attractive way to develop a doctrine for handling shocks. Hopf [7] was able to effect a very elegant treatment of the limiting process for a special partial differential equation, but the few extensions of it to more general equations, by DiPerna [4,5], have required formidable exercises in analysis. In all these cases the dissipative terms have the special character of (30).

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Captions

Fig.1 Geometry of the spring-mass system.

Fig.2 Typical stored energy functions φ. The actual graphs shows are for

$$\varphi_1(x) = \frac{1}{3}(1+x)^3 + (1+x)^{-1} - \frac{4}{3},$$

$$\varphi_2(x) = (1+x)^{3/2} - 3(1+x)^{1/2} + 2.$$

Fig.3 Typical phase portrait for the undamped equation when $\varphi(x) \longrightarrow \infty$ as $x \searrow -1$. That plotted here is for φ_1 introduced in the caption of Fig.2.

Fig.4 Typical phase portrait for the undamped equation when $\varphi(-1) < \infty$. That plotted here is for φ_2 introduced in the caption of Fig.2.

Fig.5 A function ψ that generates a shock absorber. That plotted here has the formula

$$\psi(x) = (1+x) - (1+x)^{-1}.$$

The dashed line corresponds to a ψ with $\psi(-1) > -\infty$.

Fig.6 Typical phase portrait of the damped system 10 when g is a shock absorber. The trajectory shown is computed for φ_2 of Fig.2, for $\nu = 0.1$ and for

$$g(x,y) = \begin{cases} \psi'(x)y & \text{for } y \le 0 \\ y & \text{for } y \ge 0 \end{cases}$$

where ψ has the form given in the caption of Fig.5. (That g is not continuously differentiable has no appreciable effect on the phase portrait. This g can

be readily approximated by a function with any prescribed number of derivations for which (4) - (7) hold.)

Fig.7 The limit as $\nu \searrow 0$ of the trajectory of Fig.6.

Fig.8 The displacement $x(\cdot)$ and the discontinuous velocity $v(\cdot)$ for the trajectory of Fig.7.

Preliminary Figures. (Professionally rendered figures are in preparation.)

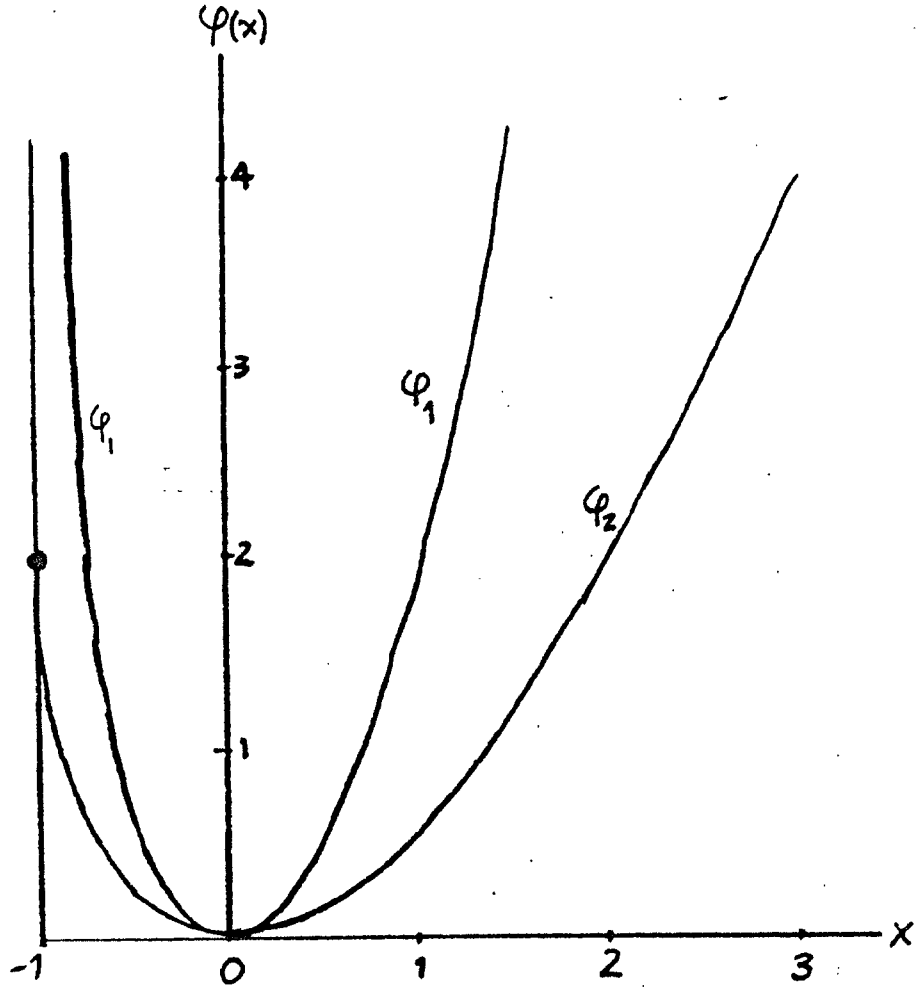


Fig. 2.

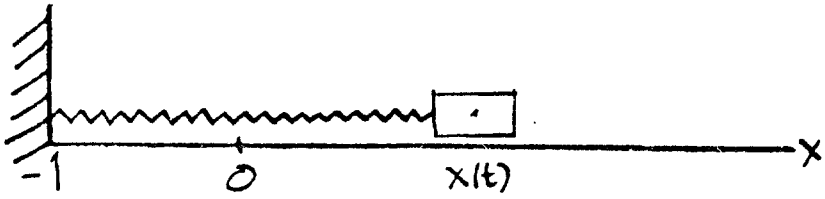


Fig. 1

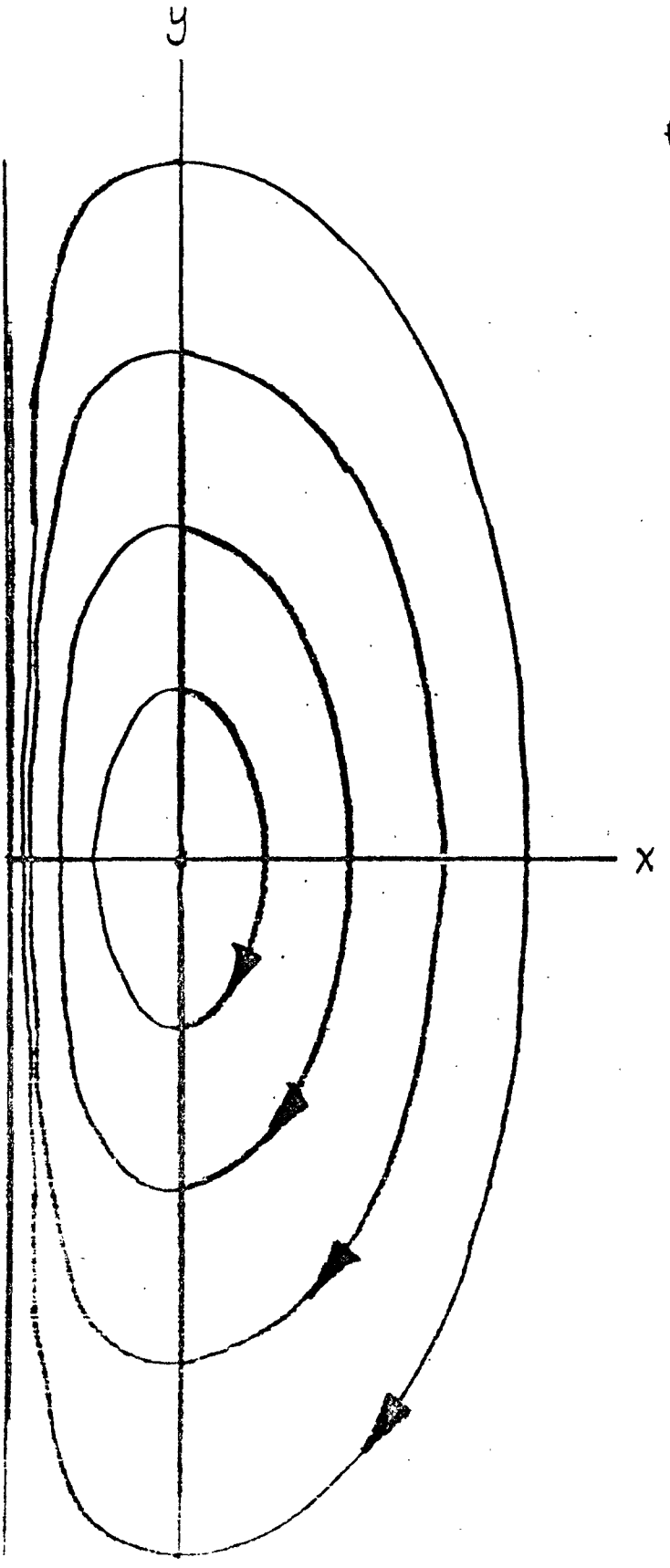
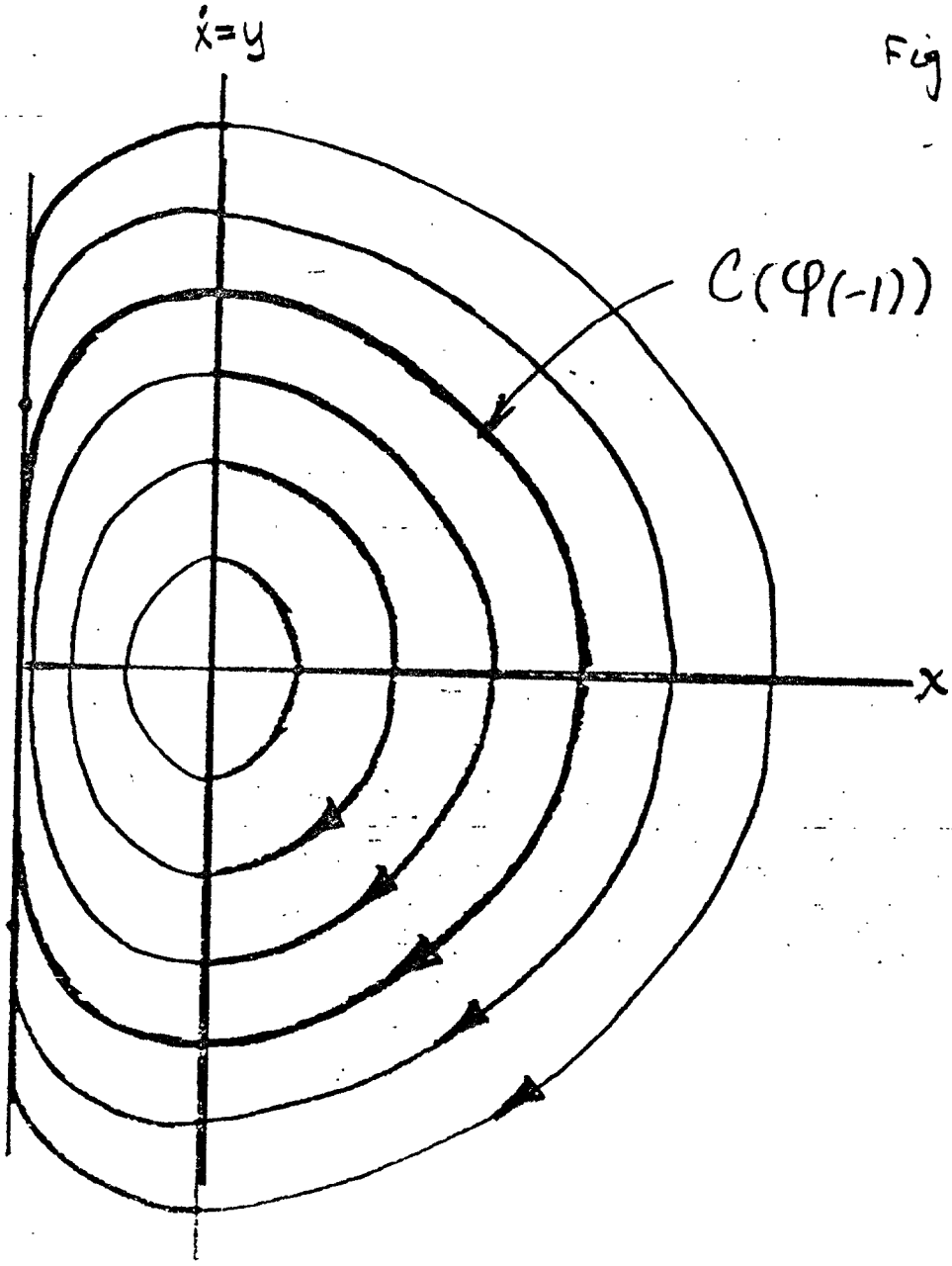


Fig. 3

Fig. 4.



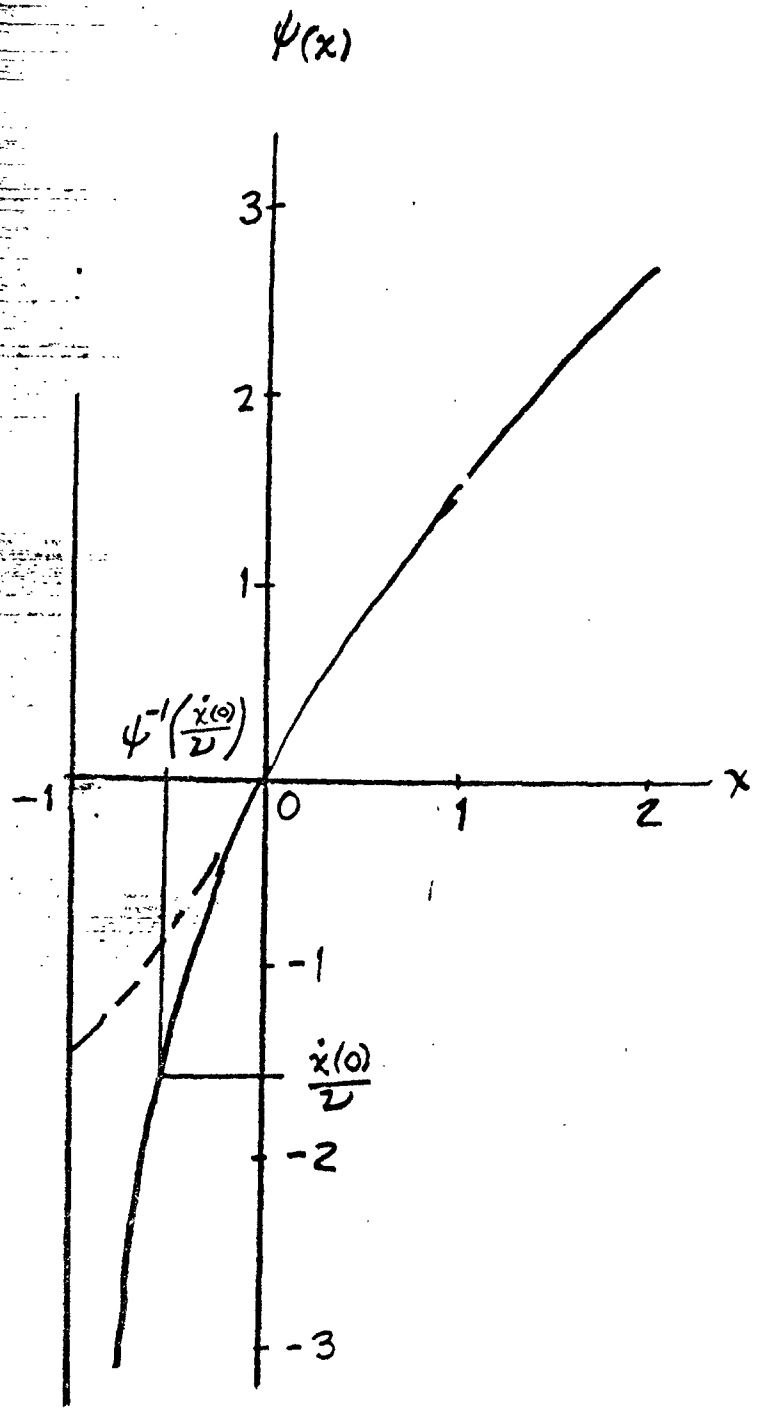


Fig. 5

Fig. 6.

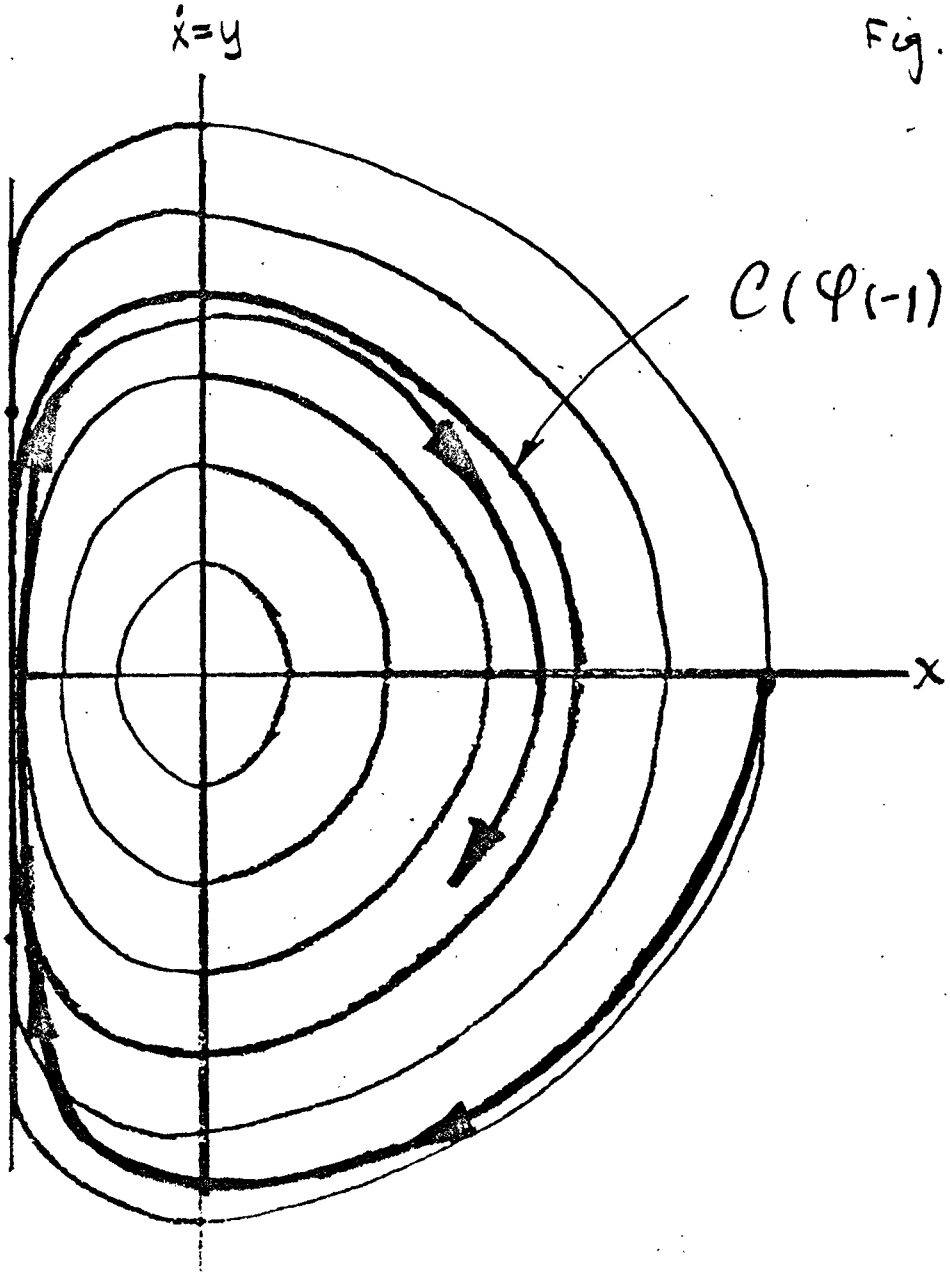
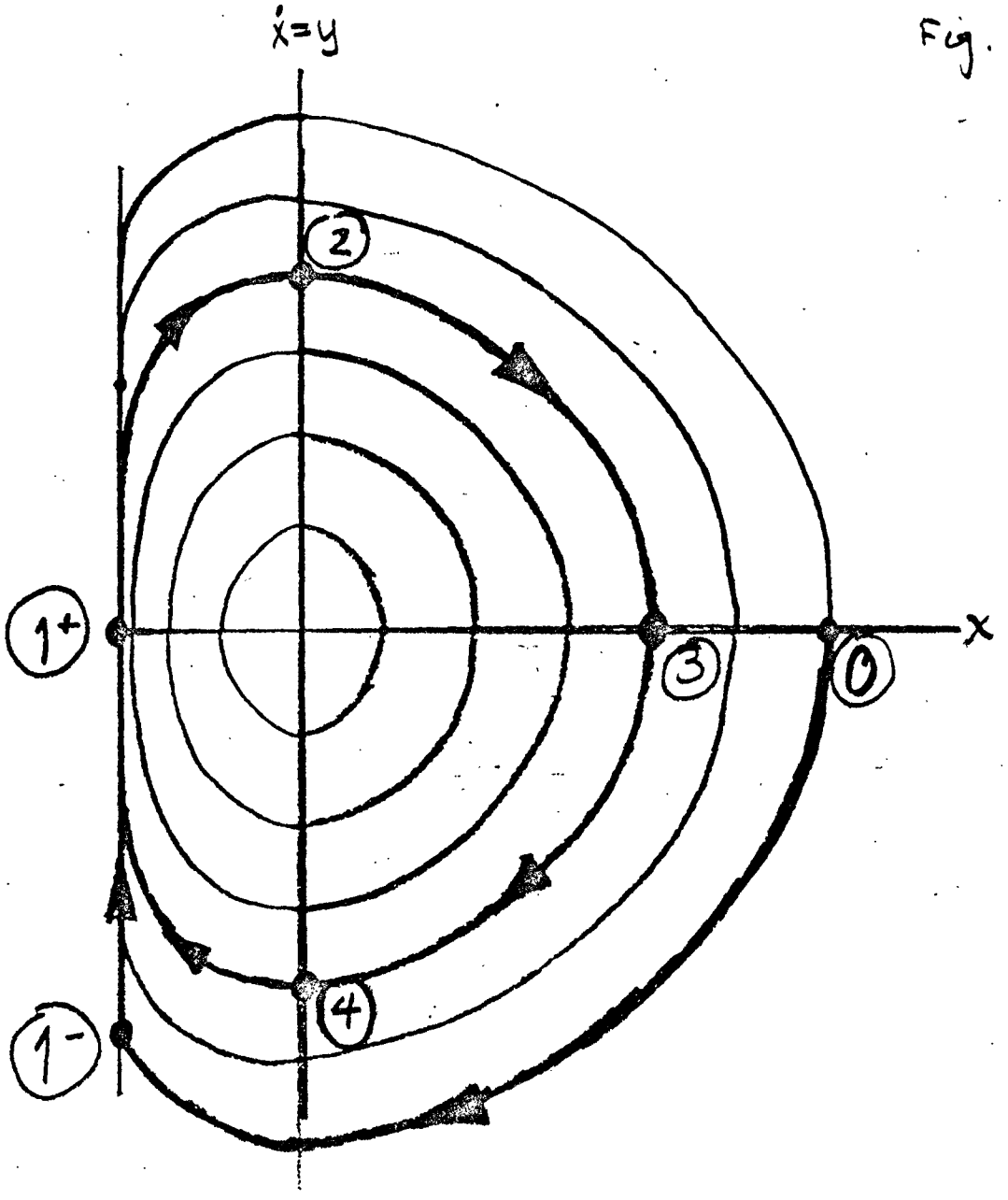


Fig. 7



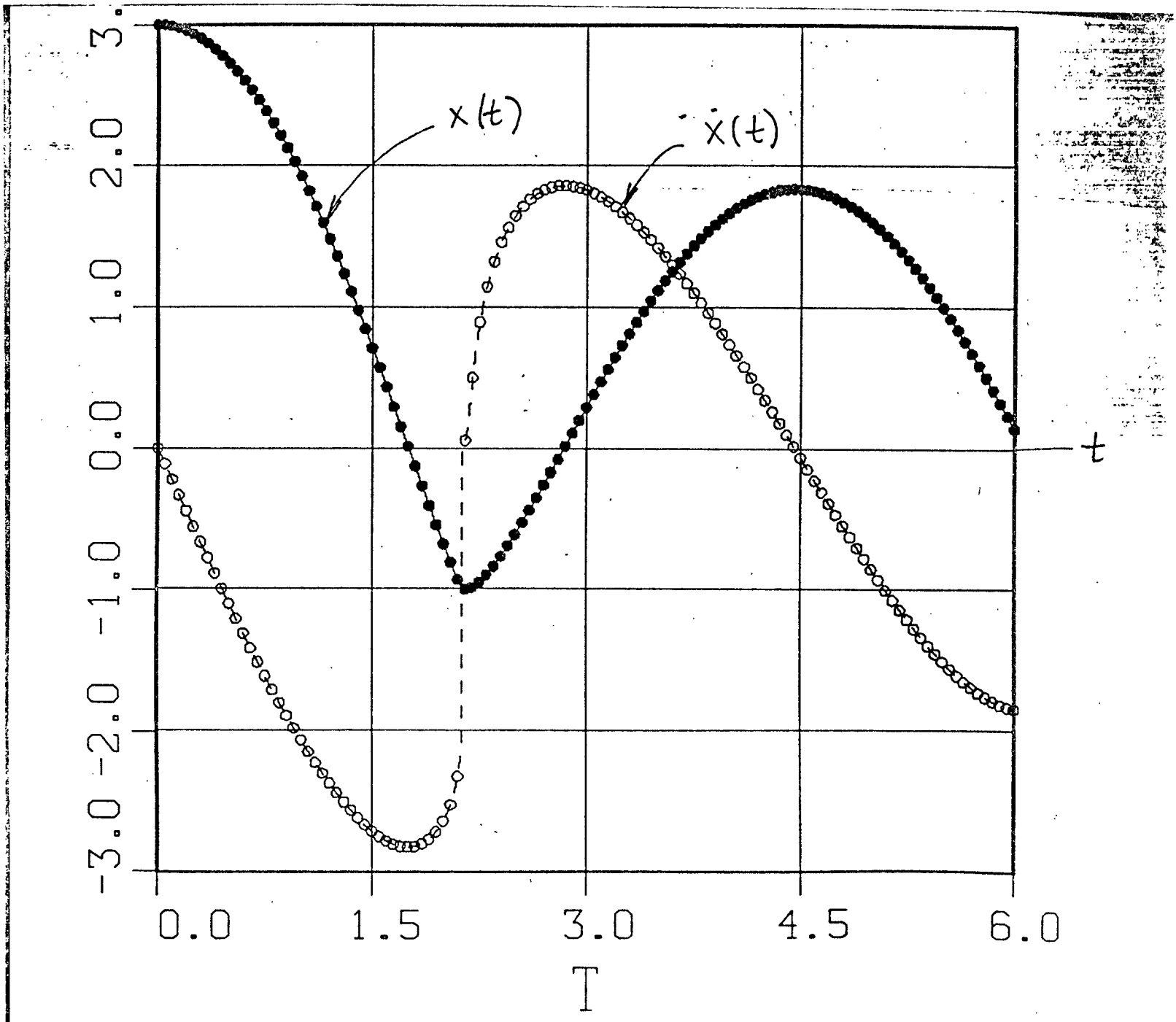


Fig. 8