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"GEOMETRIC PROPERTIES" OF SETS OF LINES

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Abstract

When we regard the plane as a set of points, we can define various geometric properties of subsets of the plane—connectedness, convexity, area, diameter, etc. It is well known that the plane can also be regarded as a set of lines. This note considers methods of defining sets (or fuzzy sets) of lines in the plane, and of defining (analogs of) "geometric properties" for such sets.

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1 Introduction

When we regard the plane as a set of points, we can define various geometric properties of subsets of the plane—connectedness, convexity, area, diameter, etc.; for a review of geometric properties that are of interest in image analysis and computer vision, see [1]. Most of these properties can be generalized to fuzzy subsets of the plane; for reviews of the literature on fuzzy geometry see [2, 3].

The plane can also be regarded as a set of lines; this "dual" viewpoint plays an important role in projective geometry [4]. This note considers methods of defining sets (or fuzzy sets) of lines in the plane, and of defining (analogs of) "geometric properties" for such sets.

A set of lines in the plane can be defined by specifying a set of points in a line parameter space such as Hough space [5]. Section 2 discusses how to define the line parameter space so that the correspondence between points in the parameter space and lines in the plane is one-to-one and continuous. (A parameterization of this type was first introduced in [6].) It also shows that because of the nature of this correspondence, various basic geometric properties are not as "well behaved" for sets of lines as they are for sets of points, (and similarly for fuzzy sets of lines).

In the axiomatic foundations of geometry [7], "incidence axioms" are used to establish relationships between sets of points and sets of lines. Section 3 considers sets of lines that satisfy incidence relations with given sets of points, and defines their geometric properties in terms of properties of these sets of points. It also discusses sets of points that satisfy incidence relations with given sets of lines, and considers conditions under which "duality" holds: If S is a set of points, $L(S)$ is its set of incident lines, and $\overline{S} = S(L(S))$ is the set of incident points of $L(S)$, under what circumstances is $\overline{S} = S$ (or vice versa)? Fuzzy generalizations of incidence relations are also briefly discussed.

2 Sets of lines as subsets of Hough space

2.1 Line parameter space

The set of lines in the plane is a two-parameter family. By choosing the parameters properly, we can define a correspondence between lines in the plane and points in a two-dimensional parameter space. In the computer vision literature, such a parameter space is called a "Hough space" [4]. Sets of lines thus correspond to subsets of Hough space.

It is desirable to choose the parameters in such a way that the correspondence between lines and pairs of parameters is one-to-one, i.e., every line has a well-defined pair of parameter values, and distinct lines have different pairs of values. This requirement strongly constrains the choice of the parameters. For example, a line is determined by its slope (i.e., by the angle that it makes with the x-axis) and by its x- (or y-) intercept (i.e., the distance from the origin to the point where it intersects the x- (or y-) axis); but if the slope is 0 (or $\pi/2$), the point of intersection either does not exist (if the line is parallel to the axis) or is ambiguous (if the line coincides with the axis), so that the parameter values are not well-defined for every line.

A parameterization which avoids this problem (and which is used in the standard "Hough transform" as introduced by Duda and Hart) is based on the so-called "normal form" of the equation of a line; here the parameters are (θ, p), where θ is the slope of the normal to the line, and p is the perpendicular distance from the origin to the line. Every line now has uniquely defined (θ, p) values; but the correspondence between lines and (θ, p) pairs is not one-to-one, because two parallel lines at the same distance from the origin, but on opposite sides of it, have the same (θ, p) values. If we define θ modulo 2π rather than modulo π (so that (θ, p) are the polar coordinates of the foot of the perpendicular), then the two parallel lines have θ's that differ by π; but for a line through the origin ($p = 0$), θ's that differ by π are indistinguishable, so that the correspondence is still not one-to-one.

Another way of making the correspondence one-to-one is to allow p to have both positive and negative signs; for example, if the perpendicular lies in the upper half-plane or on the positive x-axis, we call p positive, and if it lies on the lower half-plane or on the negative x-axis, we call it negative. [Since $0 \leq \theta < \pi$ and $-\infty < p < \infty$, we can regard (θ, p) space as an infinite strip of with π. Since the range of values of θ is cyclically closed (modulo π), we can regard the strip as rolled up into the surface of an infinitely long cylinder.] However, if we do this, the mapping from lines in the plane to points in (θ, p) space is not continuous. For example, consider the set of lines that are tangent to a circle of radius r centered at the origin. All of these lines have $|p| = r$, but by our sign convention, the sign of p is positive if the point of tangency is on the upper half of the circle, and negative if it is on the lower half. Thus the set of tangents maps into the disjoint pair of loci $p = \pm r$ on the cylinder; the mapping has discontinuities where the circle crosses the x-axis.

We can make the mapping continuous by giving the strip a half-twist before joining its opposite edges (so that it becomes an infinitely wide Möbius strip rather than an infinitely long cylinder). On this Möbius strip, the loci $p = \pm r$ are connected to each other at their endpoints, so that the set of tangents to the circle maps into a connected closed curve on the strip. In the rest of this section we shall assume that our Hough space is the Möbius-strip (θ, p) space defined in this way. (For further discussion of the Möbius Hough space, see [6].)

[A one-to-one continuous mapping can also be constructed by defining θ modulo 2π for $p > 0$ and modulo π for $p = 0$, as suggested earlier in this section. The Hough space now looks like a half-infinite cylinder in which we identify diametrically opposite points on the base of the cylinder—i.e., we identify θ with $\theta + \pi$ when $p = 0$. In this Hough space, a family of parallel lines, say with slope ϕ, maps into the two half-lines $\theta = \phi$ and $\theta = \phi + \pi$ (two elements of the cylinder) in Hough space; but this locus is not discontinuous, since we have identified the points where the two half-lines meet the base of the cylinder.]

2.2 "Connected" sets of lines

We call a set of lines "connected" if the corresponding set of points in Hough space is connected. Evidently, by this definition any (single) line is connected, but a finite set of two or more lines cannot be connected.

Proposition 2.2.1 *A pencil of lines is connected.*

Proof: The line through (x_0, y_0) with slope σ has Hough parameters (θ, p) where $\theta = \sigma - \frac{\pi}{2}$ and $p = x_0 \sin\sigma - y_0 \cos\sigma = x_0 \cos\theta + y_0 \sin\theta$ (see Figure 1). Thus the pencil of lines through (x_0, y_0) maps into the sinusoidal curve $p = x_0 \cos\theta + y_0 \sin\theta$ in (θ, p) space. As Figure 2 shows, this is a closed curve; thus it is evidently connected. □

Note that any sector of a pencil of lines is also connected, since it maps into an arc of the sinusoid.

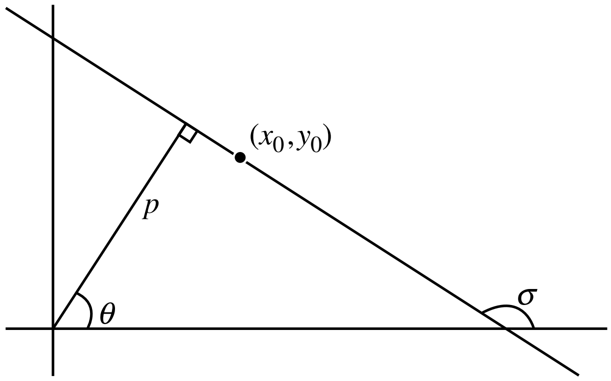

Figure 1:

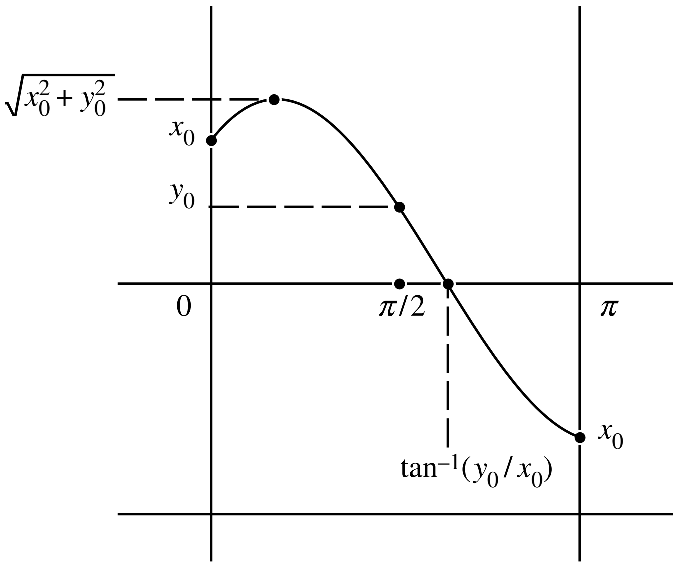

Figure 2:

Proposition 2.2.2 *Let α be a continuously differentiable, rectifiable arc ("arc" for short); then the set of tangents to α is connected.*

Proof: Let the arc have parametric equations $x = x(t), y = y(t)$; then the tangent to the arc at (x, y) has slope $\sigma = \tan^{-1} \dot{y}/\dot{x}$, where the dots denote derivatives with respect to t. As in the proof of Proposition 2.2.1, the Hough parameters of this tangent are $\theta = \sigma - \frac{\pi}{2}$ and $p = x \cos\theta + y \sin\theta = x \sin\sigma - y \cos\sigma = \frac{x(\dot{y}/\dot{x}) - y}{\sqrt{1 + (\dot{y}/\dot{x})^2}} = \frac{x\dot{y} - y\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}$. Since $x, y, \dot{x}$ and $\dot{y}$ are continuous functions of t, so are θ and p; thus the set of tangents to α maps into an arc in Hough space. $\square$

Note that if α is a closed curve, the set of tangents to α maps into a closed curve in Hough space. An analogous argument can be used to prove that if α is an arc in Hough space, so that α defines a family of lines in the plane, then the envelope of these lines is connected; i.e., these lines are the tangents to a connected arc in the plane.

2.3 "Convex" sets of lines

We call a set of lines "convex" if the corresponding set of points in Hough space is convex. Evidently, by this definition any line is convex, but a finite set of two or more lines cannot be convex.

Proposition 2.3.1 *The pencil of lines through the origin is convex.*

Proof: This pencil maps into the locus $p = 0$, which is evidently convex. □

Proposition 2.3.2 *The set of all lines parallel to a given line is convex.*

Proof: This set maps into the locus $\theta =$ constant, which is evidently convex. □

Convexity is defined in terms of collinearity (a set S is convex iff for any two points P, Q in S, any point on the line segment PQ is also in S); but in Hough space, collinearity is not an especially basic relation—for example, the locus $p = a\theta + b$ in Hough space represents the set of tangents to a spiral centered at the origin. On the other hand, the locus $p = c$ represents the set of tangents to a circle centered at the origin, and the locus $\theta = c$ represents a family of parallel lines; thus "orthoconvexity" ($\equiv$ the special case of convexity in which the line segment is parallel to the p- or θ-axis) is perhaps the most interesting type of convexity.

Because the θ coordinate in Hough space is cyclic, θ-convexity (= convexity in the direction parallel to the θ-axis) is a very strong property; if S is θ-convex, and $P, Q \in S$ have the same θ-coordinate (say θ_0), the entire line $\theta = \theta_0$ must be in S (on a cyclically closed dimension we cannot speak about the line "segment" PQ). This proves

Proposition 2.3.3 *If a θ-convex set of lines contains two or more lines at the same distance p from the origin, it contains every line at distance p—i.e., it contains all the tangents to the circle of radius p centered at the origin.* □

If α is an arc (as in Proposition 2.2.2), since $p(t)$ is continuous, the p values of the tangents to α are a subinterval of the p-axis. If $p(t)$ is nonmonotonic, it must take on some nonzero interval I of values twice; by Proposition 2.3.3, this implies that if the set of tangents to α is θ-convex, then every tangent to every circle whose radius lies in I is also a tangent to α, which is evidently impossible (since α would have to take on every slope infinitely often). Note also that for the pencil of lines through a point at distance d from the origin, p takes on the values in a neighborhood of d twice each; hence the pencil cannot be θ-convex unless $d = 0$. These observations imply

Corollary 2.3.4 *The set of tangents to a (non-closed) arc is θ-convex iff no two of the tangents are at the same distance from the origin.* □

Corollary 2.3.5 *The set of tangents to a closed curve is θ-convex iff the curve is a circle centered at the origin.* □

Corollary 2.3.6 *A pencil of lines is θ-convex iff it is the pencil of lines through the origin.* □

Note that the Corollary 2.3.6 and Proposition 2.3.1, θ-convexity implies convexity for pencils of lines; and by Corollary 2.3.5, θ-convexity implies orthoconvexity for the set of tangents to a closed curve. In Corollary 2.3.4, if the set of tangents is also p-convex, there cannot be two parallel tangents that have different distances from the origin, since there would then have to be infinitely many parallel tangents; hence the set of tangents to an arc is orthoconvex iff no two of the tangents have the same distance from the origin, and no two of the tangents are parallel.

2.4 "Metric" properties of sets of lines

We define the "measure" of a set of lines as the measure of the corresponding set of points in Hough space. Evidently, the sets of lines in Propositions 2.2.1–2, 2.3.1–2 and Corollaries 2.3.4–6 all have measure zero. On the other hand, by the remarks following Proposition 2.3.3, the convex hull of a pencil of lines through a point different from the origin, or the convex hull of the set of tangents to a closed curve that is not a circle centered at the origin, has finite, nonzero measure.

We define the θ-extent of a set of lines as the size of the smallest angular interval in which all its θ-values lie. Evidently, a pencil of lines, and the set of tangents to a closed curve, have θ-extent π. Similarly, we define the p-extent of a set of lines as the smallest interval that contains all is (absolute) p-values; evidently, the p-extent of the pencil of lines through a point at distance d from the origin is d.

2.5 Fuzzy sets of lines

A fuzzy subset of Hough space defines a fuzzy set of lines. This allows us to define fuzzy connectedness, fuzzy convexity, etc. for fuzzy sets of lines; we recall [1] that a fuzzy set is fuzzy connected (convex) iff its level sets are all connected (convex), so that the results in Sections 2.2–3 can be used to characterize fuzzy connectedness (convexity) for fuzzy sets of lines. Similarly, it allows us to define "metric" properties of fuzzy sets of lines; for example, the area of a fuzzy set is the integral of its membership function.

3 Sets of lines that meet sets of points

3.1 Sets of lines defined by incidence

Hilbert's incidence axioms [7] for sets of points and lines in the plane require that for any two points in the set of points, the line joining them is in the set of lines. Thus suggests that, for any given set of points S, we can define its set of incident lines $L(S)$ as the set of lines each of which contains at least two points of S. Evidently, $L(S) = \emptyset$ iff $S = \emptyset$ or is a singleton; from now on we will assume that S contains at least two points.

Proposition 3.1.1 *If T surrounds S and is disjoint from S, then $L(S) \subseteq L(T)$.*

Proof: Any ray emanating from a point of S must meet T; hence any line through a point of S must meet T twice. □

Corollary 3.1.2 *If T surrounds S, then $L(S \cup T) = L(T)$.*

We call a set L of line "connected" if $L = L(S)$ for some connected set of points S.

Proposition 3.1.3 *A (singleton) line l is connected.*

Proof: Any segment s of l is a connected set of points, and if s consists of more than a single point, l is the only line that contains (any) two points of s, i.e., $L(s) = \{l\}$. $\square$

Proposition 3.1.4 *A finite set L of (two or more) lines is not connected.*

Proof: Suppose $L = L(S)$ where S is connected. If S is a straight line (segment), L is a singleton; hence S must contain a non-straight connected arc α. This implies that there exists a point P on α such that $L(S)$ contains a nonzero sector of lines emanating from P; thus $L(S)$ is infinite. $\square$

Proposition 3.1.5 *A nonzero sector of a pencil of lines is not connected.*

Proof: In the proof of Proposition 3.1.4, $L(S)$ also contains a line that does not pass through P; thus the lines of $L(S)$ cannot all be concurrent, i.e., $L(S)$ cannot be a sector of a pencil. $\square$

If S has an interior point P (so that a neighborhood of P is contained in S), every line through P meets S in an (open) interval, so $L(S)$ contains the pencil of lines through P.

Corollary: If S is an open set, so that every point of S is an interior point, every line that meets S is in $L(S)$, and $L(S)$ is a union of pencils.

Let S be bounded and "regular", i.e., equal to the closure of its interior. (Note that an arc is not regular, because its interior is empty.) A "line of support" l of S is a line that meets S but has no points of S on one side of it (so that it does not meet the interior of S). It is not hard to see that if $L(S) = L(T)$, they must have the same set of lines of support. Moreover, the lines of support define the halfplanes whose intersection is the convex hull (of S or T). This proves

Proposition 3.1.6 *Let S and T be bounded and regular; then $L(S) = L(T)$ implies $\hat{S} = \hat{T}$ (where $\hat{}$ denotes the convex hull).* $\square$

Corollary 3.1.7 *Let S and T be bounded, regular, and convex; then $L(S) = L(T)$ implies $S = T$.* $\square$

Regularity is essential to these results; in Proposition 3.1.3 we saw that any two collinear line segments S, T have $L(S) = L(T)$, but they obviously do not have the same convex hull. (As a more subtle example, an open disk and its closure (or its boundary) have the same $L(\cdot)$, but have different convex hulls.) The converse of these results is false; for example the S consisting of two touching closed disks is regular, but $L(\hat{S}) \neq L(S)$ because the common tangent of the two disks is in $L(\hat{S})$ but not in $L(S)$.

We call a set L of lines "convex" if $L = L(S)$ for some convex set of points S. By Corollary 3.1.7, in the bounded, regular case, $L(S)$ uniquely determines S. Thus it is meaningful to define metric properties (area, extent,...) of a convex set of lines in terms of the corresponding properties of the (uniquely determined) set of points. For connected sets of lines, such definitions would be ambiguous, since many different S's can yield the same $L(S)$. However, in the bounded, regular case, properties such as extent are uniquely defined since they depend only on the convex hull of S, which is uniquely determined by $L(S)$. Note that for any S, if l is in $L(S)$, it must intersect $\hat{S}$ in an interval, since it intersects S in (at least) two points, and the line segment joining these points must be in $\hat{S}$.

3.2 Other definitions of incidence

If we redefine $L(S)$ as the set of lines that meet S (not necessarily twice), evidently $L(S)$ is the union of the pencils of lines defined by the points of S. This definition is somewhat less satisfactory than the one in Section 3.1; for example, when we use this definition a singleton line is not a "connected" set of lines. However, we will find this definition to be useful when we consider the "duality" between sets of points and sets of lines in Section 3.4.

3.3 Fuzzy incidence

Let μ be a fuzzy set of points, i.e., a function from the set of points of the plane into the interval [0,1]. In terms of μ, we can define fuzzy sets of lines in various ways. For example, we can define $\nu(l) = \sup_{P \in l} \mu(P)$; note that this is a fuzzification of the definition of incidence in Section 3.2 (if μ is crisp, i.e., into $\{0,1\}$, then this definition reduces to $\nu(l) = 1$ iff $\exists P \in l : \mu(P) = 1$).

It is more complicated to fuzzify the definition that we used in Section 3.1. If the value $\sup_{P \in l} \mu(P)$ is not taken on by any P, evidently there are infinitely many P's whose μ values are arbitrarily close to the sup, and we can still use the sup definition; and similarly if the sup is taken on more than once. On the other hand, if the sup is taken on exactly once, say by the point P_0, we must define ν to be the "second highest" μ value on l, i.e., $\sup_{\substack{Q \in l \\ Q \neq P_0}} \mu(Q)$.

Given a definition of ν, we can fuzzify the definitions in Section 3.1; e.g., we can define a fuzzy set of lines ν as being "connected" (or "convex", etc.) if it is defined (using one of the sup definitions) by a connected fuzzy set of points. Since a set of points is fuzzy connected (or convex) iff its level sets $\{P|\mu(P) \geq t \text{ for some } 0 \leq t \leq 1\}$ are connected (or convex), it is straightforward to generalize the results in Section 3.1 to the fuzzy case.

3.4 Duality

The incidence axioms [7] also require that if two lines are in the set of lines, their point of intersection (if it exists) is in the set of points. Thus given a set of lines L, we can define its set of incident points $S(L)$ as the set of intersection points of the lines in L, or equivalently, as the set of points each of which is contained in (lies on) at least two lines of L.

If we start with a set of points S and define its set of incident lines $L(S)$ as in Section 3.1 or 3.2, we can then define the set of incident points $\overline{S} \equiv S(L(S))$. This $\overline{S}$ may or may not be the same as the original S. For example, if we use the definition of $L(S)$ in Section 3.1, and S is a singleton, $L(S)$ is empty, and so is $\overline{S}$; if S consists of a set of collinear points, $L(S)$ is the (singleton) line joining them, and $\overline{S}$ is empty; but if S consists of three noncollinear points, we evidently have $\overline{S} = S$. In general, if S consists of n points, no three of which are collinear, $L(S)$ consists of $n(n-1)/2$ lines, $n-1$ of which meet at each point of S, but for $n > 3$ these lines also have pairwise intersections that do not lie in S, so that $\overline{S}$ strictly contains S. Note that if S contains a neighborhood of any of its points (i.e., S contains an open disk D), then every line that meets D is in $L(S)$, and since every point in the plane is on a sector of such lines, $\overline{S}$ contains every point in the plane. If we use the definition of $L(S)$ in Section 3.2, and $S = \{P\}$ is a singleton, $L(S)$ is a pencil of lines defined by P, and $\overline{S} = \{P\} = S$; while if S contains two points, P and Q, $L(S)$ contains the pencils of lines defined by P and Q, and $\overline{S}$ contains every point in the plane. Thus for either definition of $L(S)$, the above definition of $S(L)$ is "too strong", i.e., $\overline{S}$ is the entire plane unless S has an empty interior (in the first case) or is a singleton (in the second case); and $\overline{S}$ almost always properly contains S (in the first case) if S is finite.

An alternative definition for $S(L)$ is the set of points *every* line through which is in L. Here, if we use the definition of Section 3.1 for $L(S)$, then $\overline{S}$ is empty when S is finite; $\overline{S}$ contains the interior of S (so that $\overline{S}$ contains S when S is open); and if S surrounds T, $\overline{S}$ contains $T - S$. On the other hand, if we use the definition of Section 3.2 for $L(S)$, then $\overline{S} = S$ when S is finite; $\overline{S}$ always contains S; and if S surrounds T, $\overline{S}$ also contains T.

It is of interest to define conditions on S under which "duality" holds, i.e., $\overline{S} = S$, (or analogously, to define conditions on L under which $\overline{L} \equiv L(S(L)) = L$). For the first definition, duality does not hold in many simple cases. [As we have just seen, $\overline{S} \neq S$ when S is finite; and when S is a closed disk, $\overline{S}$ consists of the interior of S, since if P is a border point of S, the tangent to S at P contains only one point of S, so that not every line through P is in $L(S)$.] The situation is somewhat more satisfactory when we use the second definition.

Proposition 3.4.1 *$\overline{S}$ is contained in $\hat{S}$ (the convex hull of S).*

Proof: If $P \notin \hat{S}$, it lies in some halfplane that does not contain S;thus the line through P parallel to this halfplane does not meet S. It follows that not every line through P is in $L(S)$, so that $P \notin \overline{S}$. □

Corollary 3.4.2 *If S is convex, $\overline{S}$ is contained in S (and hence $\overline{S} = S$).* □

Proposition 3.4.3 *If $\overline{S} = S$, every connected component of S is convex.*

Proof: Let C be a connected component of S. If C is not convex, there exists a point P in its convex hull that does not lie in S (if every point in the hull were in S, these points would be connected to C and hence in C, contradiction). Since P is in $\hat{C}$, every line l through P must have points of C on it or on both sides of it; but in the latter case, since C is connected, there is a path in C joining these points, and this path must cross C. Hence any line through P meets $C \subseteq S$, so that $P \in \overline{C} \subseteq \overline{S} = S$, contradiction. □

Corollary 3.4.4 *If S is connected and $\overline{S}$ is contained in S, S is convex.* □

Thus duality holds if S is convex, and conversely if S is connected. If S is not connected, duality need not hold.

The definitions of $S(L)$ suggested in this section have straightforward fuzzy generalizations. Given a fuzzy set ν of lines, we fuzzify the first definition by defining the membership μ of a point P as the sup of the membership of the lines that contain P, except that if the sup is taken on by a unique line l, we define $\mu(P)$ as $\sup\limits_{\substack{l'\neq l \\ P\in l'}} \mu(l')$. To fuzzify the second definition, we define $\mu(P)$ as $\inf\limits_{P\in l} \nu(l)$. The duality results presented above have straightforward generalizations where we use these fuzzy definitions; note that in the fuzzy case, duality means that if we define ν in terms of μ as in Section 3.3, and then define $\overline{\mu}$ in terms of ν as above, then $\overline{\mu} = \mu$.

4 Concluding remarks

"Geometric properties" can be defined for a set of lines L in the plane by associating a set of points S with L and then computing geometric properties of S. This paper has explored two types of methods of associating a set of points with a set of lines. In the first approach, S is the set of Hough-space parameters of the lines in L (so that S is a set of points in Hough space); in the second approach, S is the set of points in the plane that are "incident" with the lines in L. Both approaches can also be used for fuzzy sets of lines, by associating with them fuzzy sets of points. Using both approaches, we have (partially) characterized sets of lines whose corresponding point sets have properties such as (fuzzy) connectedness and convexity.

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