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Optimal Stationary Behavior In Some Stochastic Nonlinear Filtering Problems-A Bound Approach

by

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OPTIMAL STATIONARY BEHAVIOR IN SOME STOCHASTIC NONLINEAR FILTERING PROBLEMS - A BOUND APPROACH

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ABSTRACT

A lower and upper bound on the a priori optimal mean square error is used to study the stationary behavior of one dimensional nonlinear filters. The long time behavior as $t \rightarrow \infty$ for asymptotically linear systems is investigated. Lower and upper bounds of the Riccati type are derived and it is shown that for nonlinear systems with linear limiting ones, the Kalman filter (KF) formally designed for the limiting systems is asymptotically optimal in some sense. Examples with simulation results are provided.

KEYWORDS : Nonlinear filtering, Kalman filtering, bounds on the optimal mean square error.

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1 INTRODUCTION :

We consider the Ito stochastic model:

$$\begin{aligned} dx_t &= g(t, x_t)\, dt + \sigma(t)\, dw_t \\ dy_t &= h(t, x_t)\, dt + \rho(t)\, dv_t \\ x(0) &= x_0 \end{aligned} \tag{1}$$

where g, h, α and ρ are smooth functions of their arguments, $\{v_t\}$, $\{w_t\}$ are independent Wiener processes, x_0 a random variable independent of $\{v_t\}$, $\{w_t\}$. Given this model one is interested in computing least squares estimates of functions of the signal x_t given $\sigma\{ y_s, 0 \leq s \leq t \}$, the σ-algebra generated by the observations, i.e., quantities of the form $E[\phi(x_t) \mid \sigma\{y_s, 0 \leq s \leq t\}]$. In many applications this computation must be done recursively. This involves the conditional probability density $p^y(t, x)$ which satisfies a nonlinear stochastic partial differential equation, the Kushner-Stratonovich equation [1]. By considering an unnormalized version of p^y, the above problem can be reduced to the study of the Duncan-Mortenson-Zakai (DMZ) equation which is linear ([6]).

The filtering problem was completely solved in the context of finite dimensional linear Gaussian systems by Kalman and Bucy [2], [3] in 1960-61, and the resulting Kalman filter (KF) has been widely applied. Apart from a few special cases [4], [5] the nonlinear case is far more complicated; the evolution of the conditional statistics is, in general, an infinite dimensional system.

Although progress has been made using the DMZ equation, optimal algorithms are not generally available. The performance of suboptimal designs, however derived, may be based on lower and upper bounds on the minimum mean square error (optimal MS-error) $p(t)$. This approach is used here to investigate the asymptotic behavior of a class of nonlinear filtering problems.

Two aspects are treated in detail:

(1) the long time behavior, that is, the asymptotic behavior of the filter as $t \to \infty$ (this paper).

(2) the asymptotic behavior as $\epsilon \to 0$, with ϵ a small parameter in the model (in a companion paper [7]).

To illustrate the ideas, consider the one-dimensional version of the model where g and h have continuous bounded derivatives satisfying

$$\underline{\alpha}(t) \leq g_x(t,x) \leq \overline{\alpha}(t) \tag{2-a}$$

$$\underline{\beta}(t) \leq h_x(t,x) \leq \overline{\beta}(t) \tag{2-b}$$

Let

$$\begin{aligned} p(t) &:= E\,[\,x_t - E(x_t \mid \mathcal{Y}_0^t)\,]^2 \\ p^*(t) &:= E\,(\,x_t - x_t^*\,)^2 \end{aligned} \tag{3}$$

where $\mathcal{Y}_0^t = \sigma\{\,y_s\,, 0 \leq s \leq t\,\}$ and x_t^* is given by :

$$\begin{aligned} dx_t^* &= g(t,x_t^*)dt + \frac{\underline{\beta}(t)}{\rho^2(t)}\,u(t)[dy_t - h(t,x_t^*)dt\,]\ ;\ x^*(0) = 0 \\ \dot{u}(t) &= \sigma^2(t) + 2\overline{\alpha}(t)\,u(t) - \frac{\underline{\beta}^2(t)}{\rho^2(t)}\,u^2(t)\ ;\ u(0) = \sigma_0^2 \\ (\,x_0 &\sim \mathcal{N}(0,\sigma_0^2) \quad \text{assumed}\,) \end{aligned} \tag{BOF}$$

Clearly the BOF (bound optimal filter) is readily implementable, with precomputable gain. It coincides with the Kalman filter if g and h are linear. In section 2 it is shown by applying results from [8], [9] that the BOF is a " best bound" filter in the sense that the associated upper bound $u(t)$ of $p^*(t)$ is the tightest over a class of nonlinear Kalman-like filters and that $p(t)$ is bounded as follow:

$$0 \leq l(t) \leq p(t) \leq p^*(t) \leq u(t)$$

where $l(t)$ satisfies another Riccati equation.

In section 3 these bounds are used to address the long time behavior of asymptotically time invariant systems. In the particular case where

$$g(t,x) = a\,x + \lambda(t)\,f(t,x) \xrightarrow[t \to \infty]{} a\,x$$

and

$$h(t,x) = c\,x + \nu(t)\,k(t,x) \xrightarrow[t \to \infty]{} c\,x$$

it is shown that the BOF is asymptotically optimal in the sense that $\lim_{t \to \infty} (p^*(t) - p(t)) = 0$ and that as far as the long time performance is concerned, the nonlinearities f and k can be ignored in the original model. In other words the " KF" and even the " SSKF" (steady state) formally designed for the underlying linear system are asymptotically optimal.

In section 4 examples with simulation results are given.

2 LOWER AND UPPER BOUNDS ON THE A PRIORI OPTIMAL MS-ERROR

Since the explicit solution of nonlinear filtering problems is impossible in general, one is naturally interested in suboptimal solutions, the performance of which may be avaluated using upper and lower bounds on the (unknown) optimal MS-error.

In fact, the structural complexity which arises is also present at the level of performance testing in the sense that simple and tractable bounds are not generally available for suboptimal estimators unless one puts further restrictions on the type of nonlinearities considered.

Consider the one dimensional *Itô* stochastic differential equation

$$
\begin{aligned}
dx_t &= g(t, x_t)\, dt + \sigma(t)\, dw_t \quad , \quad t \geq 0 \\
dy_t &= h(t, x_t)\, dt + \rho(t)\, dv_t \\
x_0 &\sim p_0(x) \quad , \quad E\, x_0 = 0 \quad , \quad E\, x_0 x_0^T = \sigma_0^2
\end{aligned}
\tag{1}
$$

where $\{w_t\}$ and $\{v_t\}$ are independent standard Wiener processes, x_0 is a random variable (generally taken to be Gaussian) independent of $\{w_t\}$ and $\{v_t\}$; g and h are such that (1) has a unique solution [10], differentiable with continuous partial derivatives. Given this model one is interested in finding bounds on the optimal MS-error :

$$
p(t) = E\, [\, (\, x_t - E(x_t \mid \mathcal{Y}_0^t)\,)^2\,]
\tag{2}
$$

where $\mathcal{Y}_0^t = \sigma\{\, y_s\, , 0 \leq s \leq t\, \}$ is the σ - algebra generated by the observations up to time t, i.e., find functions $l(t)$, $u(t)$ such that :

$$
0 \leq l(t) \leq p(t) \leq u(t)
\tag{3}
$$

In this section, existing results are applied to one dimensional systems for which the nonlinearities have bounded derivatives to obtain lower and upper bounds involving ordinary differential equations of the Riccati type. The upper bound is obtained in section 2-2 by considering a class of nonlinear, Kalman-like suboptimal filters. To each such filter is associated an upper bound on the corresponding mean square error (MSE) and the BOF (bound optimal filter) is defined as the one with the tightest upper bound. The latter is used in inequality (3).

2-1 Lower bound :

The following additional assumptions make it possible to derive a simple, tractable lower bound in the one dimensional case :

H_1 : $\quad | \; g_x(t,x) - \alpha(t) \; | \leq \Delta\alpha(t)$

H_2 : $\quad | \; h_x(t,x) - \beta(t) \; | \leq \Delta\beta(t) \quad , \quad \underline{\beta}(t) := \beta(t) - \Delta\beta(t) \geq 0$

We will denote this by :

$$g \in \; < [\, \alpha(t) \,,\, \Delta\alpha(t) \,]$$
$$h \in \; < [\, \beta(t) \,,\, \Delta\beta(t) \,]$$

Proposition 2-1:

Assume H_1 , H_2 hold and let $p(t) := E\,(\, x_t - E(x_t \mid \overset{t}{y_0}\,)\,)^2$; then $p(t)$ is lower bounded by $l(t)$, i.e., $0 \leq l(t) \leq p(t)$ where $l(t)$ satisfies the following Riccati equation :

$$\begin{aligned} \dot{l}(t) &= \sigma^2(t) + 2\underline{\alpha}(t)l(t) - \frac{1}{\rho^2(t)}\,[\,\overline{\beta}^2(t) + 4\frac{\rho^2(t)}{\sigma^2(t)}\,(\Delta\alpha(t))^2\,]\; l^2(t) \\ l(0) &= \sigma_0^2 \end{aligned} \tag{4}$$

with the notation : $\overline{\alpha} = \alpha + \Delta\alpha \quad , \quad \underline{\alpha} = \alpha - \Delta\alpha$

* * *

Remark:

The above proposition says that the optimal MS-error $p(t)$ corresponding to the nonlinear filtering problem (1) is lower bounded by the optimal MS-error corresponding to the following Kalman filtering problem:

$$\begin{aligned} dz_t &= \underline{\alpha}(t)z_t\,dt + \sigma(t)dw_t \\ dy_t{}' &= \overline{\beta}(t)z_t\,dt + \rho'\,(t)dv_t \\ \rho'\,(t) &= \frac{\rho(t)}{\left(1 + 4\dfrac{\rho^2(t)}{\sigma^2(t)}\,\dfrac{\Delta\alpha^2(t)}{\overline{\beta}^2(t)}\right)^{\frac{1}{2}}} \end{aligned}$$

It is indeed easily seen (e.g. [1]) that:

$$E\,[\, z_t - E\,(\, z_t \mid \sigma\,\{\, y_s{}' \; : 0 \leq s \leq t \,\}\,)\,]^2 = l(t)$$

Proof :

Using the Bobrovsky-Zakai lower bound ([8]) we obtain $\quad L(t) \leq p(t) \quad$ where

$$\dot{L}(t) = \sigma^2(t) + 2a(t)L(t) - \frac{c^2(t)}{\rho^2(t)} L^2(t)$$
$$L(0) = \sigma_0^2$$
$$a(t) = E\ g_x(t,x_t)$$
$$c^2(t) = E\ h_x^2(t,x_t) + \frac{\rho^2(t)}{\sigma^2(t)} var(g_x(t,x_t))$$

Thus, $L(t)$ satisfies a Riccati equations, the coefficients of which are unknown in general.

Clearly H_1 implies : $\underline{\alpha}(t) \leq g_x \leq \overline{\alpha}(t)$ a.s., and hence , $\underline{\alpha}(t) \leq a(t) \leq \overline{\alpha}(t)$.

Thus

$$|\ g_x(t,x_t) - a(t)\ | \leq 2\ \Delta\alpha(t) \qquad a.s$$

and

$$var\ g_x(t,x_t) \leq 4\ (\Delta\alpha(t))^2$$

Similarly H_2 implies: $\quad 0 \leq \underline{\beta}(t) \leq h_x(t,x_t) \leq \overline{\beta}(t)$

hence, $\quad E\ h_x^2(t,x_t) \leq \overline{\beta}^2(t)$.

Therefore:

$$c^2(t) \leq \overline{\beta}^2(t) + 4\ \frac{\rho^2(t)}{\sigma^2(t)}\ (\Delta\alpha(t))^2$$

Since $L(t)$ satisfies a Riccati equation with strictly positive initial condition, then $L(t) > 0$ ([16]) and the right hand side of $\dot{L}(t)$ is hence greater than

$$\sigma^2(t) + 2\underline{\alpha}(t)L(t) - \frac{1}{\rho^2(t)}\ [\ \overline{\beta}^2(t) + 4\frac{\rho^2(t)}{\sigma^2(t)}\ (\Delta\alpha(t))^2\ ]\ L^2$$

By the comparison theorem (see Appendix) we obtain: $\quad l(t) \leq L(t)$

* * *

2.2 Upper bound and bound optimal filter (BOF) :

Let x_t and y_t be as in (1) and assume that

$$H_1 \quad : \quad g_x(t,x) \quad \text{is continuous and} \quad g_x(t,x) \leq \bar{\alpha}(t)$$

$$H_2 \quad : \quad h_x(t,x) \quad \text{is continuous and} \quad h_x(t,x) \geq \underline{\beta}(t) \geq 0$$

Proposition 2-2:

The optimal MS-error $p(t)$ is upper bounded by $u(t)$ where $u(t)$ satisfies the Riccati equation :

$$\begin{aligned} \dot{u}(t) &= \sigma^2(t) + 2\,\bar{\alpha}(t)\,u(t) - \frac{\underline{\beta}^2(t)}{\rho^2(t)}\,u^2(t) \\ u(0) &= \sigma_0^2 \end{aligned} \tag{5}$$

* * *

Note: This says that the optimal MS-error in the nonlinear filtering problem (1) is upper bounded by the optimal MS-error in the following linear one :

$$\begin{aligned} dz_t &= \bar{\alpha}(t)\, z_t \; dt + \sigma(t)\, dw_t \\ dy'_t &= \underline{\beta}(t)\, z_t \; dt + \rho(t)\, dv_t \end{aligned}$$

Proof :

The conditional mean $\hat{x}_t := E\,(x_t \mid \mathcal{Y}_0^t)$ and the conditional optimal MS-error

$$\hat{p}_t := E\,[\,(x_t - \hat{x}_t)^2 \mid \mathcal{Y}_0^t\,]$$

are given by [1]:

$$d\hat{x}_t = \hat{g}(t,x_t)\,dt + \frac{\hat{e}_t}{\rho^2(t)}\,d\bar{w}_t \quad ; \quad \hat{x}_0 = 0$$

$$\begin{aligned} d\hat{p}_t &= [\,\sigma^2(t) + 2(\,(x_t\,g_t)\,\hat{} - \hat{x}_t\,\hat{g}_t\,) - \frac{1}{\rho^2(t)}\,(\hat{e}_t)^2\,]\,dt + \frac{T_t}{\rho^2(t)}\,d\bar{w}_t \\ \hat{p}_0 &= \sigma_0^2 \end{aligned} \tag{6}$$

where $\hat{}$ denotes conditional expectation and

$$g_t = g(t,x_t) \quad ; \quad h_t = h(t,x_t)$$

$$\hat{e}_t = (x_t\,h_t)\,\hat{} - \hat{x}_t\,\hat{h}_t$$

$$T_t = (x_t{}^2 h_t)\,\hat{} - \widehat{x_t^2}\,\hat{h}_t - 2\hat{x}_t\,(x_t\,h_t)\,\hat{} + 2(\hat{x}_t)^2\hat{h}_t$$

and $\overline{dw_t} := dy_t - \hat{h}(t, x_t)dt$ is the innovation process which is a Wiener process on $\mathcal{Y}_0^t$.

Since the expectation of *Itô* integrals is zero and $E\ \hat{p}_t = E\ (x_t - \hat{x}_t)^2 = p(t)$, taking the expectation on both sides of (6), we find

$$\dot{p}(t) = \sigma^2(t) + 2E(\ (x_t g_t)\hat{} - \hat{x}_t \hat{g}_t\ ) - \frac{E(\ \hat{e}_t\ )^2}{\rho^2(t)}$$
$$p(0) = \sigma_0^2$$

The smoothing property of conditional expectations [10] implies

$$E(\ (x_t g_t)\hat{} - \hat{x}_t \hat{g}_t\ ) = E(\ x_t - \hat{x}_t\ )(\ g_t - g(t, \hat{x}_t)\ )$$
$$= E\ \tilde{x}_t\ (\ g_t - g(t, \hat{x}_t)\ )$$

Therefore,

$$\dot{p}(t) = \sigma^2(t) + 2\ E\ \tilde{x}_t\ (\ g_t - g(t, \hat{x}_t)\ ) - \frac{E\ (\ \hat{e}_t\ )^2}{\rho^2(t)} \qquad (7)$$
$$p(0) = \sigma_0^2$$

Jensen's inequality [10] implies that :

$$E\ (\ \hat{e}_t\ )^2 \ge (\ E\ \hat{e}_t\ )^2$$

$$E\ \hat{e}_t = E(\ (x_t h_t)\hat{} - \hat{x}_t \hat{h}_t\ ) = E\ \tilde{x}_t\ (\ h_t - h(t, \hat{x}_t)\ )$$

Now

$$h(t, x_t) - h(t, \hat{x}_t) = \tilde{x}_t \int_0^1 h_x[\ t\ ,\ \hat{x}_t + s\tilde{x}_t\ ]ds := \tilde{x}_t \psi_h$$

Hence

$$E\ \hat{e}_t = E\ \tilde{x}_t^{\ 2}\ \psi_h$$

H_2 implies that $\quad \psi_h \ge \underline{\beta}(t)$ a.s.

$$E\ \hat{e}_t \ge \underline{\beta}(t)\ E\ \tilde{x}_t^{\ 2} = \underline{\beta}(t)\ p(t)$$
$$E\ (\hat{e}_t)^2 \ge (\ E\hat{e}_t\ )^2 \ge \underline{\beta}^2(t)\ p^2(t) \qquad (8)$$

Similarly H_1 implies that

$$E\ \tilde{x}_t\ (\ g_t - g(t, \hat{x}_t)\ ) = E\ \psi_g\ \tilde{x}_t^{\ 2} \le \overline{\alpha}(t)\ E\ \tilde{x}_t^{\ 2} = \overline{\alpha}(t)\ p(t) \qquad (9)$$

Combining (7)-(9) and the comparison theorem (See the appendix) yields: $\quad p(t) \le u(t)$.

* * *

An alternate and more constructive approach to getting the same result, due to A. S. Gilman and I. B. Rhodes ([9]), is outlined below. The upper bound is derived by considering the following family of parametrized nonlinear suboptimal filters, the structure of which is suggested by the kalman filter :

$$\begin{aligned} dx_t^{(k)} &= g(t, x_t^{(k)})\, dt + k(t)\,[\, dy_t - h(t, x_t^{(k)})\, dt \,] \\ x_0^{(k)} &= 0 \end{aligned} \tag{10}$$

where $k(t)$ is a non random, continuous non-negative bounded function.

To each gain $k(t)$ is associated a suboptimal filter given by (10) and denoted $\{x\}_k$. It can be shown ([9], [11]) that:

(1) Corresponding to each $\{x\}_k$ there exists a function $u_k(t)$ satisfying the linear o.d.e:

$$\dot{u}_k(t) = \sigma^2(t) + \rho^2(t)\, k^2(t) + 2[\, \overline{\alpha}(t) - k(t)\beta(t) \,]\, u_k(t) \quad ; \quad u_k(0) = \sigma_0^2 \tag{11}$$

such that

$$p^k(t) := E\,(\, x_t - x_t^k)^2 \leq u_k(t) \tag{12}$$

(2) The suboptimal filter $\{x\}_{k^*}$ obtained for the particular choice $k^*(t) = \dfrac{\beta(t)}{\rho^2(t)}\, u(t)$ i.e.

$$dx_t^* = g(t, x_t^*)dt + \frac{\beta(t)}{\rho^2(t)}\, u(t)\,[dy_t - h(t, x_t^*)dt] \quad ; \quad x_0^* = 0 \tag{13}$$

where $u(t)$ satisfies the Riccati equation :

$$\dot{u}(t) = \alpha^2(t) + 2\overline{\alpha}(t)u(t) - \frac{\beta^2(t)}{\rho^2(t)}\, u^2(t) \quad ; \quad u(0) = \sigma_0^2 \tag{14}$$

is such that $u(t) \leq u^k(t)$ for every continuous nonnegative function $k(t)$. More importantly, we have the following inequalities:

$$p(t) := E\,(\, x_t - E(x_t \mid \mathcal{Y}_0^t)\,)^2 \leq p^*(t) := E\,(\, x_t - x_t^*)^2 \leq u(t) \tag{15}$$

The nonlinear filter given by (13)-(14), subsequently referred to as the bound optimal filter

(BOF), will turn out to be near optimal in many situations of practical importance as it will be seen in the next section and in [7].

In the next subsection we combine results from the previous two subsections in a single statement ready to be used subsequently.

2-3 Summary :

For systems modeled by one dimensional *Itô* SDE's of the form :

$$\begin{aligned} dx_t &= g(t,x_t)\,dt + \sigma(t)\,dw_t \\ dy_t &= h(t,x_t)\,dt + \rho(t)\,dv_t \\ E\,x_0 &= 0 \quad , \quad E\,x_0^2 = \sigma_0^2 \end{aligned} \tag{16}$$

with g and h satisfying

$$H_1 \ : \ | g_x(t,x) - \alpha(t) | < \Delta\alpha(t) \quad \text{denoted by} \quad g \in <[\alpha(t)\,,\,\Delta\alpha(t)] \tag{17}$$

$$H_2 \ : \ | h_x(t,x) - \beta(t) | < \Delta\beta(t) \quad \text{denoted by} \quad h \in <[\beta(t)\,,\,\Delta\beta(t)] \tag{18}$$

define

$$\overline{\alpha}(t) := \alpha(t) + \Delta\alpha(t) \quad ; \quad \underline{\alpha}(t) := \alpha(t) - \Delta\alpha(t) \tag{19}$$

$$\overline{\beta}(t) := \beta(t) + \Delta\beta(t) \quad ; \quad \underline{\beta}(t) := \beta(t) - \Delta\beta(t) \geq 0 \tag{20}$$

$$p(t) := E\,(\,x_t - E(x_t \mid y_0^t)\,)^2 \tag{21}$$

$$p^*(t) := E\,(\,x_t - x_t^*\,)^2 \tag{22}$$

where x_t^* is the BOF and is given by

$$\begin{aligned} dx_t^* &= g(t,x_t^*)\,dt + \frac{\underline{\beta}^2(t)}{\rho^2(t)}\,u(t)\,[\,dy_t - h(t,x_t^*)dt\,] \\ x_0^* &= 0 \end{aligned} \tag{23}$$

$$\dot{u}(t) = \sigma^2(t) + 2\overline{\alpha}(t)u(t) - \frac{\underline{\beta}^2(t)}{\rho^2(t)}\,u^2(t) \quad ; \quad u(0) = \sigma_0^2. \tag{24}$$

Then by combining the results from the previous two sections we readily get the following bounds on the optimal MS-error:

$$0 \leq l(t) \leq p(t) \leq p^{*}(t) \leq u(t) \tag{25}$$

where

$$\begin{aligned} \dot{l}(t) &= \sigma^2(t) + 2\underline{\alpha}(t)l(t) - \frac{1}{\rho^2(t)}\left[\,\overline{\beta}^{\,2}(t) + 4\frac{\rho^2(t)}{\sigma^2(t)}(\Delta\alpha(t))^2\,\right] l^2(t) \\ l(0) &= \sigma_0^2 \end{aligned} \tag{26}$$

and $u(t)$ satisfies (24).

3 ASYMPTOTICALLY LINEAR SYSTEMS

In this section we discuss systems that are asymptotically time invariant, i.e.,

$$\begin{aligned} dx_t &= g(t,x_t)\,dt + \sigma\,dw_t \\ dy_t &= h(t,x_t)\,dt + \rho dv_t \end{aligned} \tag{1}$$

where

$$\begin{aligned} g(t,x) &= g(x) + \lambda(t)\,f(t,x) \\ h(t,x) &= h(x) + \nu(t)\,k(t,x) \end{aligned} \tag{2}$$

$$\begin{aligned} g &\in <[\,a\,,\Delta a\,] \quad ; \quad f \in <[\,\mu(t)\,,\Delta\mu(t)\,] \\ h &\in <[\,c\,,\Delta c\,] \quad ; \quad k \in <[\,\varsigma(t)\,,\Delta\varsigma(t)\,] \end{aligned} \tag{3}$$

and

$$\lim_{t\to\infty}\,[\lambda(t)\,,\nu(t)] = [0,0] \tag{4}$$

In the particular case where $g(x)$ and $h(x)$ are linear (the limiting system is linear), one is interested in knowing whether the Kalman filter (KF) designed formally for the limiting linear system and driven by (the nonlinear observations) y_t in (1) is asymptotically optimal as t becomes large. This situation arises when the nonlinearities are neglected during the modelization process. The nonlinear filter resulting from this scheme will be referred to (wrongly) as the " KF".

More specifically, let

$$
\begin{aligned}
dx_t &= a\ x_t\ dt + \lambda(t)\ f\ (t, x_t)\ dt + \sigma\ dw_t \\
dy_t &= c\ x_t\ dt + \nu(t)\ k\ (t, x_t)\ dt + \rho\ dv_t \\
E\ x_0 &= 0 \quad , \quad E\ x_0^2 = \sigma_0^2 > 0
\end{aligned} \tag{5}
$$

Then the " KF" designed for the limiting system is

$$
dx_t^k = a\ x_t^k\ dt + \frac{c}{\rho^2}\ r(t)\ [\ dy_t - c\ x_t^k\ dt\ ] \quad ; \quad x^k(0) = 0 \tag{6}
$$

$$
\dot{r}(t) = \sigma^2 + 2\ a\ r - \frac{c^2}{\rho^2}\ r^2 \quad ; \quad r(0) = \sigma_0^2 \tag{7}
$$

and the questions of interest are:

- under what conditions is x_t^k (or the BOF x_t^*) asymptotically optimal as $t \to \infty$, i.e. $\lim_{t \to \infty} (\ p^k(t) - p(t)\ ) = 0 \quad (\ \lim_{t \to \infty} (\ p^*(t) - p(t)\ ) = 0)$?

  where

$$
p^k(t) = E\ (\ x_t - x_t^k\ )^2 \tag{8}
$$

$$
p^*(t) = E\ (\ x_t - x_t^*\ )^2 \tag{9}
$$

$$
p(t) = E\ (\ x_t - E(x_t \mid \mathcal{Y}_0^t)\ )^2 \tag{10}
$$

- would the same result hold for the steady state " KF" (" SSKF"), obtained by setting $r(t) = r(\infty)$ in (6) ?

The bounds on the optimal MS-error derived in the previous section are used to answer these questions in the linear limiting case. However, the bounds on the first derivatives do not contain " enough information" to treat similar questions in the general case where $g(x)$ and $h(x)$ are nonlinear.

Consequently, we will only consider the class of nonlinear filtering problems (5) with the assumptions :

H_1 : $f \in <[\ \mu(t)\ ,\ \Delta\mu(t)\ ]$; $k \in <[\ \varsigma(t)\ ,\ \Delta\varsigma(t)\ ]$

H_2 : $\lambda(t)$ and $\nu(t)$ are continuous, vanishing functions on $[0,\infty[$ and nonnegative (for simplicity)

H_3 : $\mu(t)$, $\Delta\mu(t)$, $\varsigma(t)$ and $\Delta\varsigma(t)$ are bounded continuous functions on $[0,\infty[$

$$H_4 \quad : \quad c + \nu(t)\,\underline{\varsigma}(t) \geq \delta_0 > 0 \quad ; \quad c \neq 0 \, .$$

In the next two subsections we show that :

$$\lim_{t \to \infty} \left(p^{*}(t) - p(t) \right) = 0 \;\text{ and }\; \lim_{t \to \infty} \left(p^{k}(t) - p(t) \right) = 0 \tag{11}$$

this is done by bounding $p(t)$ as

$$0 \leq l(t) \leq p(t) \leq p^{*}(t) \leq u(t) \tag{12}$$

$$0 \leq l(t) \leq p(t) \leq p^{k}(t) \leq q(t) \tag{13}$$

and showing that

$$\lim_{t \to \infty} \left(u(t) - l(t) \right) = 0 \quad \text{and} \quad \lim_{t \to \infty} \left(q(t) - l(t) \right) = 0 \tag{14}$$

The result is then generalized to the case

$$g(t,x) = a\,x + \sum_{i=1}^{n} \lambda_i(t)\, f_i(t,x) \tag{15}$$

$$h(t,x) = c\,x + \sum_{i=1}^{m} \nu_i(t)\, k_i(t,x) \tag{16}$$

which in turn can be applied to treat cases where a and c are time varying functions.

3-1 Asymptotic optimality of the BOF:

In the case of (5), we note that H_1 and H_2 imply:

$$g(t,x) = ax + \lambda(t) f(t,x) \in \, < [a + \lambda(t)\mu(t) \, , \, \lambda(t)\Delta\mu(t)] \tag{17}$$

$$h(t,x) = cx + \nu(t) k(t,x) \in \, < [c + \nu(t)\varsigma(t) \, , \, \nu(t)\Delta\varsigma(t)] \tag{18}$$

Thus the results in section 2-3 apply with

$$\overline{\alpha} = a + \lambda(t)\mu(t) + \lambda(t)\Delta\mu(t) = a + \lambda(t)\overline{\mu}(t) \tag{19}$$

$$\underline{\alpha} = a + \lambda(t)\underline{\mu}(t) \tag{20}$$

$$\overline{\beta} = c + \nu(t)\,\overline{\varsigma}(t) \tag{21}$$

$$\beta = c + \nu(t)\, \delta(t) \tag{22}$$

and the BOF is given here by:

$$\begin{aligned} dx_t^* &= ax_t^* dt + \lambda(t) f(t, x_t^*) dt + \frac{\beta(t)}{\rho^2} u(t) \left[dy_t - cx_t^* dt - \nu(t) k(t, x_t^*) dt \right] \\ x_0^* &= 0 \end{aligned} \tag{23}$$

$$\dot{u}(t) = \sigma^2 + 2\overline{\alpha} u(t) - \frac{\beta^2}{\rho^2} u^2(t) \;\; ; \;\; u(0) = \sigma_0^2 \tag{24}$$

The asymptotic optimality of the BOF is a direct consequence of the following lemma:

Lemma 3-1 :

Let θ_1, θ_2, γ_1 and γ_2 be continuous functions on $[\, 0, +\infty\, [$ such that

- $\lim_{t \to \infty} \theta_i(t) = a$

- $\lim_{t \to \infty} \gamma_i^2(t) = c^2 \quad ; \quad t \geq 0 \quad ; \quad i = 1,2$

and consider the Riccati equations :

$$\dot{v}_1 = \sigma^2 + 2\theta_1 v_1 - \frac{\gamma_1^2}{\rho^2} v_1^2 \quad ; \quad v_1(0) = \sigma_0^2 \tag{25}$$

$$\dot{v}_2 = \sigma^2 + 2\theta_2 v_2 - \frac{\gamma_2^2}{\rho^2} v_2^2 \quad ; \quad v_2(0) = \sigma_0^2 \tag{26}$$

If $v_1(t) \geq v_2(t)$ and if one of the assumptions given below holds then:

$$\lim_{t \to \infty} v_1(t) = \lim_{t \to \infty} v_2(t)$$

$A_1 \quad : \quad a < 0$

$A_2 \quad : \quad v_2(t) \geq r(t)\,, \;\; t \geq 0$ and $\gamma_1^2 \geq \delta^2 > 0$ for some δ

Recall that :

$$\dot{r}(t) = \sigma^2 + 2ar - \frac{c^2}{\rho^2} r^2 \quad , \quad r(0) = \sigma_0^2 \tag{27}$$

* * *

Proof:

Let $w(t) = v_1(t) - v_2(t) \geq 0$. Then a straightforward computation yields

$$\begin{aligned} \dot{w} &= 2(\theta_1 - \theta_2)v_2 + \frac{1}{\rho^2}(\gamma_2^2 - \gamma_1^2)v_2^2 + 2(\theta_1 - \frac{\gamma_1^2}{\rho^2} v_2)w - \frac{\gamma_1^2}{\rho^2} w^2 \\ w(0) &= 0 \end{aligned} \tag{28}$$

which we rewrite as

$$\dot{w}(t) = i(t) + 2j(t)w - \frac{\gamma_1^2}{\rho^2}w^2 \quad , \quad w(0) = 0 \tag{29}$$

where

$$i(t) = 2(\theta_1 - \theta_2)v_2 + \frac{1}{\rho^2}(\gamma_2^2 - \gamma_1^2)v_2^2 \tag{30}$$

$$j(t) = \theta_1 - \frac{\gamma_1^2}{\rho^2} v_2 \tag{31}$$

(29) clearly implies:

$$\dot{w} \leq i(t) + 2j(t)w. \tag{32}$$

Depending on the assumption used (A_1 *or* A_2) we will bound $w(t)$ differently using the comparison theorem.

(1) Assumption A_1 :

Since $l(t)$ and $w(t)$ are nonnegative, $w(t)$ can be bounded as

$$\dot{w} \leq i(t) + 2\theta_1 w \tag{33}$$

thus $0 \leq w(t) \leq z(t)$ where

$$\dot{z}(t) = i(t) + 2\theta_1 z \quad ; \quad z(0) = 0 \tag{34}$$

Similarly $v_1(t) \leq V_1(t)$ where

$$\dot{V}_1 = \sigma^2 + 2\theta_1 V_1 \quad , \quad V_1(0) = \sigma_0^2 \tag{35}$$

If $a < 0$ then $\lim_{t \to \infty} \theta_1 = a < 0$ and Perron's theorem (see the Appendix) can be applied to (34) and (35). We get

$$V_1(\infty) \;=\; -\frac{\sigma^2}{2a} \tag{36}$$

Since $v_2(t) \;\leq\; v_1(t) \;\leq\; V_1(t)$ for every $t \geq 0$, (36) implies

$$\lim_{t \to \infty} i(t) \;=\; 0$$

Re-applied to (34) Perron's theorem yields

$$\lim_{t \to \infty} z(t) \;=\; 0 \qquad \textit{that} \quad \textit{is} \qquad \lim_{t \to \infty} w(t) \;=\; 0$$

(2) Assumption A_2 :

Since $v_2(t) \;\geq\; r(t)$, $j(t) \;\leq\; \theta_1 - \frac{\gamma_1^2}{\rho^2}\, r(t)$, (32) then implies that $w(t) \;\leq\; z(t)$, where:

$$\dot{z} \;=\; i(t) + 2\,(\,\theta_1 - \frac{\gamma_1^2}{\rho^2}\, r(t)\,)\, z(t) \quad ; \quad z(0) \;=\; 0 \tag{37}$$

$\lim_{t \to \infty} (\,\theta_1 - \frac{\gamma_1^2}{\rho^2}\, r(t)\,) \;=\; a - \frac{c^2}{\rho^2}\, r(\infty)$; but $r(\infty)$ is the positive root of

$$\sigma^2 + 2ax - \frac{c^2}{\rho^2}\, x^2 \;=\; 0 \tag{38}$$

i.e.

$$r(\infty) \;=\; \frac{\rho^2}{c^2}\,[\, a + (a^2 + \frac{\sigma^2}{\rho^2} c^2)^{1/2}\,]$$

and $a - \frac{c^2}{\rho^2}\, r(\infty) \;=\; -(a^2 + \frac{\sigma^2}{\rho^2}\, c^2)^{1/2}$. Thus $\lim_{t \to \infty} z(t) \;=\; 0$ provided $\lim_{t \to \infty} i(t) \;=\; 0$. For this to happen it suffices that $v_2(t)$ be bounded ($v_1(t)$ be bounded). Using the assumptions and the comparison theorem we immediately get $v_1(t) \;\leq\; V_1(t)$ where

$$\dot{V}_1 \;=\; \sigma^2 + 2\theta_M\, V_1 - \frac{\delta^2}{\rho^2}\, V_1^2 \quad ; \quad V_1(0) \;=\; \sigma_0^2 \tag{39}$$

and θ_M is a nonzero upper bound of $\theta_1(t)$. $V_1(t)$ is clearly bounded. We conclude that $\lim_{t \to \infty} z(t) \;=\; 0$, i.e., $\lim_{t \to \infty} w(t) = 0$.

* * *

Note:

We can conclude in particular that $v_1(\infty) = v_2(\infty) = r(\infty)$ provided one of the following holds:

(1) $v_1(t) \geq v_2(t) \geq r(t)$; $t \geq 0$

(2) $v_1(t) \geq r(t) \geq v_2(t)$ and $a < 0$

(3) $r(t) \geq v_1(t) \geq v_2(t)$ and $a < 0$

This last assertion is obtained by applying the above Lemma to the pair (r, v_2) .

Proposition 3-2 :

If $H_1 - H_4$ and H_5 or H_6 hold, where

H_5 : $a < 0$

H_6 : $l(t) \geq r(t) \qquad t \geq 0$

then the BOF given by (23)-(24) is asymptotically optimal as $t \to \infty$.

* * *

Proof:

We have that : $0 \leq l(t) \leq p(t) \leq p^*(t) \leq u(t)$ where $l(t)$ and $u(t)$ are given by (19), (20), (24) and (26) in section 2-3. Lemma 3-1 can then be applied to $u(t)$ and $l(t)$ by taking :

$$\theta_1(t) = \overline{\alpha}(t) = a + \lambda(t)\overline{\mu}(t) \tag{40}$$

$$\theta_2(t) = \underline{\alpha}(t) = a + \lambda(t)\underline{\mu}(t) \tag{41}$$

$$\gamma_1^2(t) = \underline{\beta}^2(t) = [\, c + \nu(t)\,\underline{s}(t)\,]^2 \tag{42}$$

$$\gamma_2^2(t) = \overline{\beta}^2(t) + 4\,\frac{\rho^2}{\sigma^2}\,(\,\Delta\alpha(t)\,)^2$$

$$\gamma_2^2 = [\, c + \nu(t)\,\overline{s}(t)\,]^2 + 4\,\frac{\rho^2}{\sigma^2}\,\lambda^2(t)\,(\Delta\mu(t))^2 \tag{43}$$

It is readily checked that all hypotheses in Lemma 3-1 are satisfied and the result follows.

* * *

Remark :

(1) It follows directly from lemma 3-1 that if H_5 is replaced by

H'_5 : $a < 0$ and either $u(t) \geq r(t)$ or $u(t) \leq r(t)$

then $p(\infty) = p^*(\infty) = r(\infty) = \frac{\rho^2}{c^2}\left[\, a + (a^2 + \frac{\sigma^2}{\rho^2} c^2)^{1/2} \,\right]$

(2) A sufficient condition for H_6 to hold is $\underline{\mu} \geq 0$ and $(1 + \nu \frac{\overline{\varsigma}}{c})^2 + 4\lambda^2 \frac{\rho^2}{\sigma^2} \frac{(\Delta\mu)^2}{c^2} < 1$ for every $t \geq 0$.

Assuming that $c > 0$ and rewriting the last inequality as $2\nu\frac{\overline{\varsigma}}{c} + \nu^2 \frac{\overline{\varsigma}^2}{c^2} + 4\lambda^2 \frac{\rho^2}{\sigma^2} \frac{(\Delta\mu)^2}{c^2} < 0$, it can be seen that a necessary condition for this last inequality to hold is $\overline{\varsigma} \leq 0$. It turns out that H_6 holds in many cases if f and k lie in the first/third quadrant ($\underline{\mu} \geq 0$) and second/fourth quadrant ($\overline{\varsigma} \leq 0$) respectively (e.g. see example (2)).

In general hypotheses such as H_6 must be checked numerically.

Next we generalize proposition 3-2 to nonlinearities of the following type:

$$
\begin{aligned}
g(t,x) &= a\,x + \sum_{i=1}^{n} \lambda_i(t) f_i(t,x) \\
h(t,x) &= c\,x + \sum_{j=1}^{m} \nu_j(t) k_j(t,x)
\end{aligned}
\tag{44}
$$

with the assumptions H_1 , H_2 and H_3 holding for each $i = 1,....,n$; $j = 1,....,m$.

Using a vector notation, e.g., $\Delta\mu = (\Delta\mu_1 , \cdots , \Delta\mu_n)^T$, and $< \; , \; >_n$ to denote the inner product in R^n , the nonlinearities above can be written in the more condensed form:

$$
\begin{aligned}
g(t,x) &= ax + < \lambda(t) \, , \, f(t,x) >_n \\
h(t,x) &= cx + < \nu(t) \, , \, k(t,x) >_m
\end{aligned}
\tag{45}
$$

and we clearly have

$$
\begin{aligned}
g &\in < [\, a + <\lambda \, , \, \mu>_n \; ; \; <\lambda \, , \, \Delta\mu>_n \,] \\
h &\in < [\, c + <\nu \, , \, \varsigma>_m \; ; \; <\nu \, , \, \Delta\varsigma>_m \,]
\end{aligned}
\tag{46}
$$

Thus if we make the additional hypothesis

H_4' : $\underline{\beta} = c + <\nu \, , \underline{\beta}>_m \geq \delta_0 > 0$ then the same results hold. More precisely the BOF is given by

$$
\begin{aligned}
&dx_t^* = a x_t^* dt + <\lambda(t), f\,(t, x_t^*)> dt + \frac{\underline{\beta}(t)}{\rho^2} u^2(t)[dy_t - c x_t^* dt - <\nu(t), k\,(t, x_t^*)> dt] \\
&x^*(0) = 0
\end{aligned}
\tag{47}
$$

with the corresponding MSE $p^*(t)$ and the optimal MS-error $p(t)$ satisfying

$$
0 \leq l(t) \leq p(t) \leq p^*(t) \leq u(t) \tag{48}
$$

where $l(t)$ and $u(t)$ are given in section 2-3 with

$$
\overline{\alpha}(t) = a + <\lambda(t) \, , \, \overline{\mu}(t)> \tag{49}
$$

$$
\underline{\alpha}(t) = a + <\lambda(t) \, , \, \underline{\mu}(t)> \tag{50}
$$

$$
\Delta\alpha = <\lambda(t) \, , \, \Delta\mu> \tag{51}
$$

$$
\overline{\beta}(t) = c + <\nu(t) \, , \, \overline{\varsigma}(t)> \tag{52}
$$

$$
\underline{\beta}(t) = c + <\nu(t) \, , \, \underline{\varsigma}(t)> \tag{53}
$$

The corollary below is now a direct application of lemma 3-1.

Corollary 3-3 :

If $H_1 - H_4'$ and H_5 or H_6 stated below are satisfied then the BOF (47) is asymptotically optimal as $t \to \infty$.

H_5 : $a < 0$

H_6 : $l(t) \geq r(t)$; $r(t)$ given by (27).

* * *

The corollary above can be used to treat the more general cases where a and c are time varying, i.e.,

$$\begin{aligned} g(t,x) &= a(t)x + \sum_1^n \lambda_i(t) f_i(t,x) \\ h(t,x) &= c(t)x + \sum_1^m \nu_i(t) k_i(t,x) \end{aligned} \tag{54}$$

where $\lim_{t \to \infty} a(t) = a$ and $\lim_{t \to \infty} c(t) = c$

As an illustration, assume that $a(t)$ and $c(t)$ are monotonous and continuous, then (54) may be rewritten as

$$\begin{aligned} g(t,x) &= ax + (a(t) - a)x + <\lambda , f>_n \\ h(t,x) &= cx + (c(t) - c)x + <\nu , k>_m \end{aligned} \tag{55}$$

By letting:

$$\begin{aligned} \lambda_{n+1}(t) &= | a(t) - a | \\ \nu_{m+1}(t) &= | c(t) - c | \\ f_{n+1}(t,x) &= \text{sign}(a(t) - a)x \\ k_{m+1}(t,x) &= \text{sign}(c(t) - c)x \end{aligned}$$

(55) becomes:

$$\begin{aligned} g(t,x) &= ax + <\lambda , f>_{n+1} \\ h(t,x) &= cx + <\nu , k>_{m+1} \end{aligned} \tag{56}$$

and we are in position to apply the above Corollary since λ_{n+1} and ν_{m+1} are continuous vanishing nonnegative functions with f_{n+1} and k_{m+1} belonging to $<[\ \text{sign}(\lambda_{n+1})\ ,\ \delta\ ]$ and $<[\ \text{sign}(\nu_{m+1})\ ,\ \delta\ ]$ respectively, where $\delta \geq 0$ is arbitrary.

3-2 Asymptotic optimality of the KF :

For the nonlinear filtering problem (5), it is clear that (6)-(7) correspond to a regular Kalman filter designed for the underlying linear system obtained when one ignores the nonlinear terms in (5). It should be noted however that (6)-(7) is driven by observations from a nonlinear system. We will, nevertheless, continue to refer to it as the " KF " and " SSKF " (steady state) when $r(t)$ is replaced by $r(\infty)$.

In addition to $H_1 - H_4$, we make the following assumption:

H_0 : $f(t,0)$ and $k(t,0)$ are continuous, bounded on $[0, +\infty[$.

Proposition 3-4 :

If $a < 0$ then both the " KF " and the " SSKF " are asymptotically optimal as $t \to \infty$. Moreover :

$$p(\infty) = p^k(\infty) = r(\infty) = \frac{\rho^2}{c^2}\left[a + (a^2 + \frac{\sigma^2}{\rho^2} c^2)^{1/2} \right] \tag{57}$$

* * *

Proof:

We first derive an upper bound on $p^k(t) := E(x_t - x_t^k)^2$ where x_t^k is given by (6)-(7).

Let $\overline{x}_t = x_t - x_t^k$; then (5) and (6) yield

$$d\overline{x}_t = [\, \overline{g}_t - G(t)\overline{h}_t]dt + \sigma\, dw_t - \rho\, G(t)\, dv_t \tag{58}$$

where

$$G(t) = \frac{c}{\rho^2} r(t) \qquad \left(or \quad \frac{c}{\rho^2} r(\infty) \right) \tag{59}$$

$$\overline{g}_t = a\overline{x}_t + \lambda(t) f(t, x_t) \tag{60}$$

$$\overline{h}_t = c\overline{x}_t + \nu(t) k(t, x_t) \tag{61}$$

Applying *Itô ' s* chain rule ([1]) gives

$$d\overline{x}_t^{\,2} = [\, \sigma^2 + \rho^2\, G^2(t)\,]\, dt + 2\overline{x}_t\, d\overline{x}_t \tag{62}$$

Taking the expectation on both sides yields :

$$\frac{d}{dt}E\ \overline{x}_t^{\,2} = \dot{p}^k(t) = \sigma^2 + \rho^2 G^2(t) + 2E\overline{x}_t [\overline{g}_t - G(t)\overline{h}_t]$$

$$\frac{dp^k}{dt} = \sigma^2 + \rho^2 G^2 + 2\, E\ \overline{x}_t\, \overline{g}_t - 2\, G\ E\ \overline{x}_t\, \overline{h}_t \quad , \quad p^k(0) = \sigma_0^2$$

$$\dot{p}^k = \sigma^2 + \rho^2 G^2 + 2(a - cG)p^k + 2\lambda E\ \overline{x}_t\, f(t, x_t) - 2\nu G E\ \overline{x}_t\, k(t, x) \tag{63}$$

Clearly

$$2E\ \overline{x}_t f(t,x_t) \le E\ \overline{x}_t^{\,2} + E\ f^2(t,x_t) = p^k(t) + E\ f^2(t,x_t) \tag{64}$$

$$-2E\ \overline{x}_t k(t,x_t) \le p^k(t) + E\ k^2(t,x_t) \tag{65}$$

By the comparison theorem : $p^k(t) \le q(t)$; $q(0) = \sigma_0^2$ where

$$\dot{q}(t) = \sigma^2 + \rho^2 G^2 + 2(a - cG)q + \lambda(q + Ef^2) + \nu G(q + Ek^2)$$
$$= \sigma^2 + \rho^2 G^2 + \lambda E f^2 + \nu G E k^2 + [\ 2(a - cG) + \lambda + \nu G\ ]\ q \tag{66}$$

which we rewrite as

$$\dot{q} = i(t) + j(t)q \quad , \quad q(0) = \sigma_0^2 \tag{67}$$

Now, $\lim_{t \to \infty} j(t) = 2(a - \frac{c^2}{\rho^2} r(\infty)) = -2(a^2 + \frac{\sigma^2}{\rho^2} c^2)^{1/2} < 0$. Thus if

$$\lim_{t \to \infty} \lambda(t) E\ f^2(t,x_t) = \lim_{t \to \infty} \nu(t) E\ k^2(t,x_t) = 0 \tag{68}$$

then
$$\lim_{t \to \infty} i(t) = \sigma^2 + \frac{c^2}{\rho^2} r^2(\infty) .$$

Applying Perron's theorem to (67) would give:

$$q(\infty) = -\frac{i(\infty)}{j(\infty)} = -\frac{\sigma^2 + \frac{c^2}{\rho^2} r^2(\infty)}{2\ (a - \frac{c^2}{\rho^2} r(\infty))} \tag{69}$$

But $r(\infty)$ satisfies the algebraic Riccati equation:

$$\sigma^2 + 2a\ r(\infty) - \frac{c^2}{\rho^2} r^2(\infty) = 0$$

It follows that :

$$q(\infty) = -\frac{\sigma^2 + 2a\ r(\infty) - \frac{c^2}{\rho^2} r^2(\infty) - 2(a - \frac{c^2}{\rho^2} r(\infty)) r(\infty)}{2(a - \frac{c^2}{\rho^2} r(\infty))} = r(\infty) \tag{70}$$

If $a < 0$ and $l(t) \le r(t)$ then by letting $v_1(t) = r(t)$ and $v_2(t) = l(t)$ in Lemma 3-1 we conclude that $l(\infty) = r(\infty) = q(\infty)$ and hence $p(\infty) = p^k(\infty) = r(\infty)$.

If $l(t)$ is not less or equal than $r(t)$ for every t then we can always find a lower bound $l'(t)$ which is less or equal than both $l(t)$ and $r(t)$ (see the next remark). Thus we can apply the same lemma with $v_2(t) = l'(t)$ and conclude that

$l'(\infty) = r(\infty) = q(\infty) \quad (= l(\infty))$ and hence $p(\infty) = p^k(\infty) = r(\infty)$.

We now show that (68) holds if $a < 0$ and H_0 hold.

$f \in < [\mu(t), \Delta\mu(t)]$ implies

$$\underline{\mu}(t)x + f(t,0) \leq f(t,x) \leq \overline{\mu}(t)x + f(t,0) \tag{71}$$

where the time functions $\underline{\mu}(t)$, $\overline{\mu}(t)$ and $f(t,0)$ are all bounded continuous for $t \geq 0$. (71) implies in turn that:

$$f^2(t,x) \leq A^2(t)x^2 + B^2(t) \tag{72}$$

for some continuous bounded functions A and B. Therefore, $\lim_{t \to \infty} \lambda(t)Ef^2(t,x_t) = 0$ holds if

$$\lim_{t \to \infty} \lambda(t)Ex_t^2 = 0 \tag{73}$$

Ex_t^2 satisfies the following ode ([1]):

$$\frac{d}{dt} E\, x_t^2 = 1 + 2\lambda(t)E\, x_t f(t,x_t) + 2aE\, x_t^2 \tag{74}$$

$$2E\, x_t f(t,x_t) \leq E\, x_t^2 + E\, f^2(t,x_t) \tag{75}$$

Using (73) and (72) in (74), we conclude by the comparison that $E\, x_t^2$ is bounded by $V(t)$ where :

$$\dot{V} = 1 + \lambda(t)B^2(t) + (2a + \lambda(t) + \lambda(t)A^2(t))V(t)$$

Perron's theorem applies and $V(\infty) = -\dfrac{1}{2a}$. Therefore $\lim_{t \to \infty} \lambda(t)E\, f^2(t,x_t) = 0$.

Clearly the same thing is also true for $\nu(t)\, Ek^2(t,x_t)$.

* * *

Remark :

Let $f \in < [\mu(t), \Delta\mu(t)]$ and $k \in < [\varsigma(t), \Delta\varsigma(t)]$ i.e.

$$f_x \in [\underline{\mu}(t), \overline{\mu}(t)] \quad , \quad k_x \in [\underline{\varsigma}(t), \overline{\varsigma}(t)] \tag{*}$$

The lower bound is then given by :

$$\dot{l}(t) = \sigma^2(t) + 2\underline{\alpha}(t)\, l(t) - \frac{1}{\rho^2}\,[\overline{\beta}^2 + 4\frac{\rho^2}{\sigma^2}\,(\Delta\alpha)^2\,]\; l^2(t) \quad ; \quad l(0) = \sigma_0^2$$

where $\underline{\alpha}(t) = a + \lambda(t)\underline{\mu}(t)$, $\Delta\alpha(t) = \lambda(t)\Delta\mu(t)$ and $\overline{\beta}(t) = c + \nu(t)\overline{\varsigma}(t)$.

Clearly $l(t) \leq r(t)$ if $\underline{\mu}(t) \leq 0$ and $\overline{\varsigma}(t) \geq 0$ where

$$\dot{r}(t) = \sigma^2 + 2a\; r(t) - \frac{c^2}{\rho^2}\, r^2(t) \quad ; \quad r(0) = \sigma_0^2$$

If $\underline{\mu}(t) \leq 0$ and $\overline{\varsigma}(t) \geq 0$ does not hold, we can always choose a worse lower bound $l'(t)$ such that $l'(t) \leq r(t)$. This is possible since (*) implies that $f_x \in [\,\underline{\mu}'(t)\,, \overline{\mu}(t)\,]$; $k_x \in [\,\underline{\varsigma}, \overline{\varsigma}'\,]$ with $\underline{\mu}'(t) \leq 0$ and $\overline{\varsigma}'(t) \geq 0$.

* * *

Let us now turn to the case where g and h are again given by (54). Then under the same assumptions and notations of Corollary 3-3 and the additional obvious additional assumptions introduced by H_0 (namely that it holds for each f_i, k_j), it can be shown ([11]) that the following holds:

Corollary 3-5 :

If $a < 0$ then both the " KF " and the " SSKF " are asymptotically optimal as $t \to \infty$. Moreover

$$p(\infty) = p^k(\infty) = r(\infty) = \frac{c^2}{c^2}\,[\; a + (a^2\, \frac{\sigma^2}{\rho^2}\, c^2)^{1/2}\,]$$

* * *

4 EXAMPLES AND SIMULATION RESULTS :

Example (1) :

Let x_t and y_t be given by:

$$dx_t = ax_t\,dt + e^{-t}\sin^2(\omega t)\,th(x_t)\,dt + \sigma dw_t$$
$$dy_t = cx_t\,dt + \frac{1}{t^2+1}\,x_t\,e^{-x_t^2}\,dt + \rho dv_t$$
$$x_0 \sim N(m_0, \sigma_0^2)$$

Thus

$$\lambda(t) = e^{-t}\sin^2(\omega t) \quad ; \quad f(x) = th(x)$$
$$\nu(t) = \frac{1}{t^2+1} \quad ; \quad k(x) = xe^{-x^2}$$

Simulation resuls were made with the following numerical data:

$$a = -1 \ , \ \omega = 50 \ , \ \sigma = \rho = 0.2$$
$$c = 1 \ , \ m_0 = 0.0 \ , \ \sigma_0^2 = 0.2$$

for which it is readily obtained that $f \in < [\frac{1}{2}, \frac{1}{2}]$ and $k \in < [\frac{1-2e^{-\frac{3}{2}}}{2}, \frac{1+2e^{-\frac{3}{2}}}{2}]$ i.e.

$$\underline{\mu} = 0 \ , \ \overline{\mu} = 1 \ , \ \mu = \frac{1}{2} \ , \ \Delta\mu = \frac{1}{2}$$
$$\underline{\varsigma} = -2e^{-\frac{3}{2}} \ , \ \overline{\varsigma} = 1 \ , \ \varsigma = \frac{1-2e^{-\frac{3}{2}}}{2} \ , \ \Delta\varsigma = \frac{1+2e^{-\frac{3}{2}}}{2}$$

The simulations results, obtained using Monte Carlo simulations, are summarized in the plots of fig. 1 and 2 corresponding to the BOF and " KF" respectively. In fig. 1, the upper and lower bound $(u(t), l(t))$ on the optimal MS-error $p(t) := E\,[\,x_t - E\,(\,x_t \mid y_0^t\,)\,]^2$ together with the MSE corresponding to the BOF are plotted. Fig. 2 contains similar results for the " KF" except that instead of $u(t)$, $r(t)$ is plotted. (Recall that while u was shown to be an upper bound on the BOF MSE $p^*(t)$, neither u nor r are known to be upper bounds for the " KF" MSE $p^k(t)$).

It can be seen that the BOF and the " KF" are indeed both asymptotically optimal in the sense that $\lim_{t \to \infty}(\,p(t) - p^*(t)\,) = 0$ and $\lim_{t \to \infty}(\,p(t) - p^k(t)\,) = 0$ respectively. Moreover:

$$p(\infty) = p^*(\infty) = p^k(\infty) = r(\infty) = \frac{\rho^2}{c^2}\,[\,a + (a^2 + \frac{\sigma^2}{\rho^2}c^2)^{\frac{1}{2}}\,] = 0.017$$

Example (2) :

According to proposition 3-2, the asymptotic optimality of the BOF does not require $a < 0$ provided $l(t) \geq r(t)$ for every $t \geq 0$.

In this example we consider such a case. Here, $\{x_t\}$ and $\{y_t\}$ are given by:

$$\begin{aligned} dx_t &= ax_t\,dt + e^{-t}\,(\,\alpha_1 x_t + sin\,(\alpha_2 x_t)\,)\,dt + \sigma dw_t \\ dy_t &= cx_t + e^{-t}\,(\,\beta_1 x_t + arctg\,(\beta_2 x_t)\,)\,dt + \rho dv_t \\ x_0 &\sim Nn\,(m_0, \sigma_0^2) \end{aligned}$$

Thus:

$$\lambda(t) = \nu(t) = e^{-t}$$

$$f\,(x) = \alpha_1 x + \sin(\alpha_2 x)$$

$$k\,(x) = \beta_1 x + arctg\,(\beta_2 x)$$

The numerical data for this example is

$$\begin{aligned} &a = 0.5 > 0 \;,\; \alpha_1 = 0.4 \;,\; \alpha_2 = 0.2 \;,\; \sigma = 0.2 \\ &c = 1 \;,\; \beta_1 = -0.5 \;,\; \beta_2 = 0.25 \;,\; \rho = 0.2 \\ &m_0 = 0.0 \;,\; \sigma_0 = 0.1 \end{aligned}$$

from which it easily follows that $f \in <[0.4,0.2]$ and $k \in <[-0.375,0.125]$ so that f and k belong to different quadrants. The simulation results are summarized in figures 3 and 4. Conclusions similar to the ones in example (1) can be deduced.

5 CONCLUSION

We investigated the asymptotic behavior question of one dimensional nonlinear filtering problems involving drifts with bounded derivatives using an upper and lower bound approach to show that the a priori mean square error associated with some suboptimal filters approaches the optimal one asymptotically. The upper and lower bounds satisfy ordinary differential equations of the Riccati type.

In particular, it is shown that in the case of asymptotically time invariant systems for which the limiting system is linear, the " KF " and " SSKF " (designed for the limiting linear system) are asymptotically optimal as $t \to \infty$ (section 3). In other words the nonlinearity can be ignored as far as the long time behavior is concerned.

This approach proved that significant information relevant to this type of filtering problems can be infered from the knowledge of the derivative bounds (i.e., of the cone in which the nonlinearities reside), and the main point is that, tractable bounds on the optimal MS-error, when

available, can be used (in addition to performance testing of sub-optimal designs) as a study approach to tackle some questions arising in nonlinear filtering.

APPENDIX :

Theorem (1) : (comparison theorem [12])

Let $F(x,y)$ and $G(x,y)$ be continuous in the rectangle

$$\mathrm{D}: \quad | x - x_0 | < a \quad , \quad | y - y_0 | < b$$

and suppose that $F(x,y) < G(x,y)$ everywhere in D. Let $y(x)$ and $z(x)$ be the solutions of

$$\begin{aligned} \dot{y} &= F(x,y) \quad , \quad y(x_0) = \alpha \\ \dot{z} &= G(x,y) \quad , \quad z(x_0) = \alpha \end{aligned}$$

Let I be the largest subinterval of $(x_0 - a , x_0 + a)$ where both $y(x)$ and $z(x)$ are defined and continuous ; then for $x \in I$

$$\begin{aligned} z(x) &< y(x) \quad , \quad x < x_0 \\ z(x) &> y(x) \quad , \quad x > x_0 \end{aligned}$$

Theorem (2) : (Perron [13])

If $F(t)$, $f_i(t)$, $t_0 \in [\, 0 , \infty \,[$, $i = 1,.....,n$, are real continuous functions of t having finite limits $\lim\limits_{t \to \infty} F(t) = b$, $\lim\limits_{t \to \infty} f_i = a_i$, if the roots

λ_i , $i = 1,\ldots,n$ of the equation

$$\rho^n + a_1 \rho^{n-1} + \ldots\ldots\ldots\ldots + a_n = 0$$

are real, distinct, and different from 0, then the equation

$$\frac{d^n}{dt^n} y(t) + f_1(t) \frac{d^{n-1}}{dt^{n-1}} y(t) + \ldots\ldots + f_n(t)\, y(t) = F(t) \qquad (*)$$

has at least one solution $y(t)$ with

$$\lim_{t \to \infty} y(t) = \frac{b}{a_n} \quad , \quad \lim_{t \to \infty} \frac{d^m}{dt^m} y(t) = 0 \, .$$

If $\lambda_i < 0$, $i = 1,\ldots\ldots,n$, then all solutions of (*) have these properties.

REFERENCES

[1] Jazwinski, A.H. : " Stochastic Processes and Filtering Theory " , Academic Press, New York , 1970

[2] Kalman, R.E. : " A New Approach to Linear Filtering and Prediction Problems " J.Basic Eng., Trans. ASME, Series D, Vol. 82, #1, March 1960, pp34-35

[3] Kalman, R.E. ; Bucy, R.S. : " New Results in Filtering and Prediction theory ", 1961, J. Basic Eng, Trans AMSE, Series D, Vol 83, pp 95-107

[4] Benes, V.E. : " Exact Finite Dimensional Filters for Certain Diffusions with Non-Linear Drifts " , Stochastics, 1981

[5] Wong, W.S. : " New Classes of Finite Dimensional Nonlinear Filters " , Systems and Control Letters, 3, 1983, 155-164

[6] Baras, J. ; Blankenship, G.L. ; Mitter, S.K : " Nonlinear Filtering of Diffusion Processes " , IFAC, Triennial World Congress, Kyoto, Japan, 1981

[7] Saydy, L. ; Blankenship, G.L. : " A Bound Approach To Asymptotic Optimality In Nonlinear Filtering Of Diffusion Processes " , Submitted.

[8] Bobrovsky, B.Z. ; Zakai, M. : " A Lower Bound on the Estimation Error for Certain Diffusion Processes " , IEEE, Trans on Inf. Theory , Vol92, #1, Jan 1976

[9] Gilman, A.S. ; Rhodes, I.B. : " Cone Bounded Nonlinearities and Mean Square Bounds - Estimation Upper Bounds " , IEEE A.C., June 1973

[10] Arnold, L. : " Stochastic Differential Equations " , John Wiley & Sons, New York, 1974

[11] Saydy, L. : " Asymptotic Behavior Of Nonlinear Filters " M.S. Thesis, August 1985, University Of Maryland At College Park.

[12] Hille, E. : " Lecture notes on ODE's " , Addison Wesley, 1965.

[13] Cesari, L. : " Asymptotic Behavior and Stability Problems in Ordinary Differential Equations " , Springer Verlag, 1971

[14] Blankenship, G.L. : " Asymptotic Analysis in Mathematical Physics and Control Theory : Some Problems with Common Features " , Ricerche di Automatica, Vol. 10, #2, Dec. 1979

[15] Blankenship, G.L. : " Some Approximation Methods in Nonlinear Filtering ", IEEE Control And Decision Proc., 1980, 51-56

[16] Bucy, R.S ; Joseph, P.D : " Filtering for Stochastic Processes with Application to Guidance " , N.Y,Interscience Publishers, 1968.

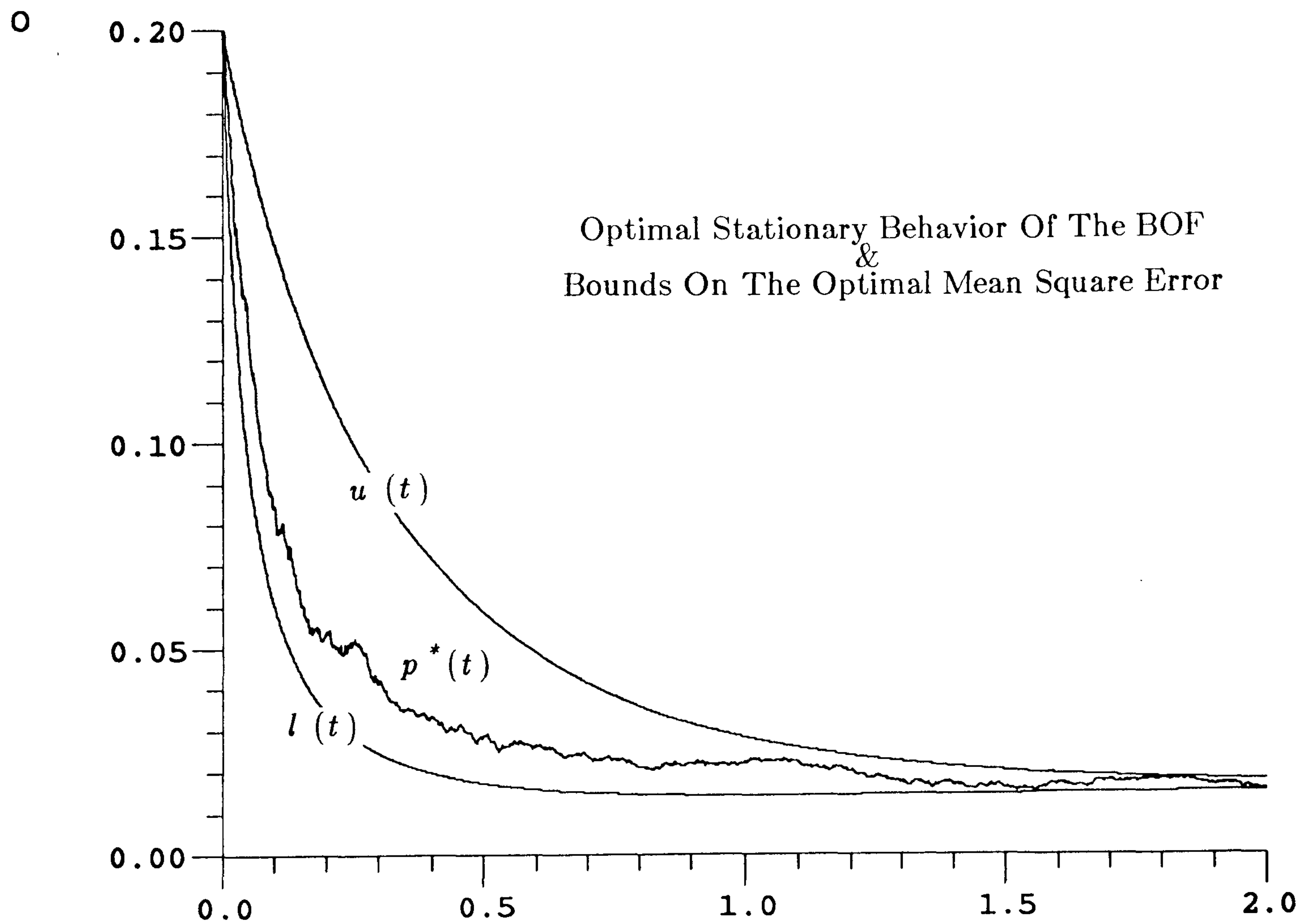

figure 1 : BOF performance

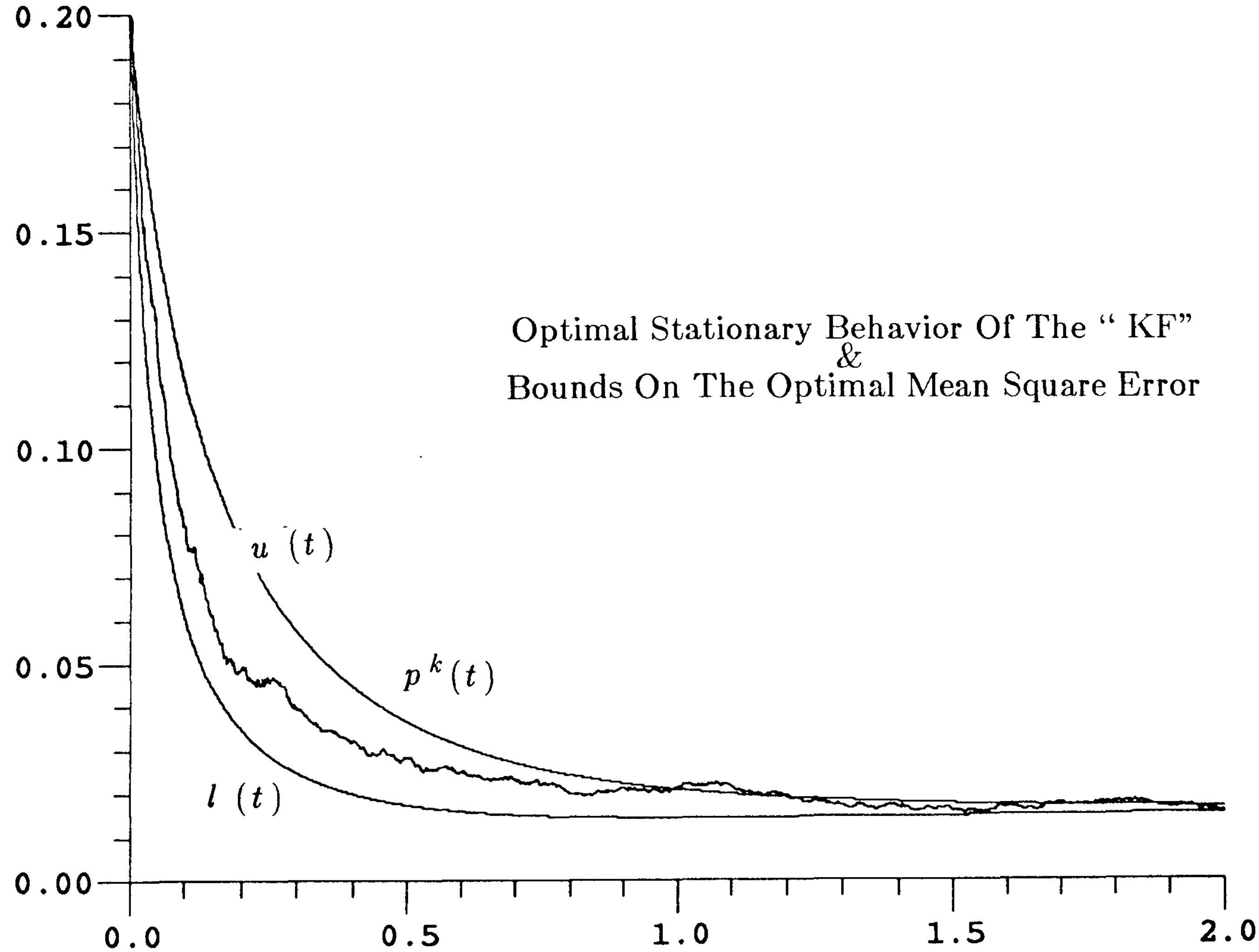

figure 2 : " KF" performance

```
      program tasop
c
c
c     USAGE :
c     =======
c
c     Program "tasop" carries out Monte Carlo simulations
c     for one dimensional filtering problems of the form:
c
c     dxt = a xt dt + lambda(t)  * f(xt)  dt + sig*dwt , 0<=t<=ziT
c     dyt = c xt dt + nu(t)  * k(xt)  dt + rho*dvt
c     x0 ~ N ( xm0 , sig0 ** 2 )
c
c     where the nonlinearities have bounded derivatives :
c
c     xmuu <= fx(x)  <= xmub
c     dzetau <= kx(x)  <= dzetab
c
c     Program tasop generates E [ ( xt - zt )**2 ] together with
c     the upper and lower bounds u(t)  and l(t)  on the optimal
c     MS error.
c     If ioption=1 then zt is the "KF" filtered estimate.
c     If ioption=2 then zt is the BOF filtered estimate.
c     When ioption=0, two sample paths for xt and zt are generated;
c     where zt is either from "KF"  (iflag=1) or BOF  (iflag=2).
c
c     INPUT DATA:
c     ===========
c
c      (i) iT, N:
c          ------
c          iT = time horizon
c          N = number of subdivisions in the time interval [ 0, ziT ]
c               (should be large enough in order for the discretized
c               stochastic differential to yield a good approximation).
c               N<=5000, unless the array dimensions are changed.
c      (ii) xm0,sig0,sig,a,c,rho,dseed0,dseed1,dseed2:
c           ------------------------------------------
c           xm0,sig0,sig,a,c,rho: parameters of the model
c           dseed0,dseed1,dseed2: initializations for the random number
c           generator. These could be any (distinct) numbers between 0
c           and 1.0e20, preferably as large as possible.
c      (iii) M, NS :
c            -------
c            M = number of values to be printed out.
c            NS = number of sample paths used to compute expectations.
c      (iv) ioption, iflag :  already described.
c           ----------------
c      (v) xmuu,xmub,dzetau,dzetab : derivative bounds.
c          -------------------------
c
c     OUTPUT DATA:
c     ============
c
c     ioption = 0 :
c     -------------
c     An array of 2N values is generated. The first set of N numbers
c     corresponds to the (simulated) true state; i.e.:
c     x(i*dT), i=0,1,...,N-1, where dT = ziT/N.
c     The other N values are those of the filtered estimate zt (either
```

```
c       "KF" or BOF, depending on iflag).
c       ioption > 0 :
c       -------------
c       An array of 3N numbers is obtained the first N values of which
c       are those of p(t) = E [(xt - zt)**2] (zt being either "KF" or
c       BOF, depending on ioption), namely:
c       p(i*dT), i=0,1,....,N-1, where dT=ziT/N.
c       Similarily, the second and third set of values are those of u(t)
c       and l(t) respectively.
c       remark: in the case of the "KF", no computable upper bound exists.
c               Instead of u(t), the solution of the riccati equation
c               associated with the limiting linear system r(t) is printed
c
c       TIPS :
c       ======
c
c       (a) Progam tasop uses the IMSLS library for random number
c           generation. E.g. low could be run as follows:
c           % f77 -o runlow low.f -limsls
c           % runlow <inputfile >outputfile
c           where inputfile is a file in which the data is prealably
c           stored.
c       (b) The nonlinearities f and k may be changed by modifying fk and
c           gk accordingly in the subroutines observy, kalfilt and bofilt.
c           To change the time functions lambda(t) and nu(t), additional
c           modifications should be made in the Riccati subroutine (ric)
c           to the quantities qltt, alp and bet.
c           Currently, these functions are f(x)=tanh(x), k(x)=x*exp(-x**2),
c           lambda(t)=exp(-t)*sin(50*t) and nu(t)=1/(t*t+1).
c       (c) The quality of the simulation results depends strongly on
c           how large N and NS are. Typically, N=1000 and NS>=500.
c
c
        dimension er(5000),xx(5000),xxf(5000)
        dimension u(5000),x(4),dx(4)
        double precision dseed0,dseed1,dseed2
        common /const/deltat,sqd,xm0,sig0,sig,c,rho,N,iT
        common /param/a,xmuu,xmub,dzetau,dzetab
c
        read *,iT,N
        read *,xm0,sig0,sig,a,c,rho,dseed0,dseed1,dseed2
        read *,M,NS
        read *,ioption,iflag
        read *,xmuu,xmub,dzetau,dzetab
c
        deltat=1.0*iT/N
        sqd=sqrt(deltat)
c
        if (ioption.eq.0) go to 63
c
        do 50 i=1,N
        er(i)=0.
50      continue
c
        if (ioption.eq.2) go to 29
c
c       ioption=1 ---> N values of u(t)(=r(t) here) are computed
c       and used to compute the mmse for the "kf" applied to
c       the w.n.l filtering pb.
c
```

```
      kswitch=0
      kr=0
      call ric(kswitch,kr,u)
c
      do 60 j=1,NS
      call kfsub(dseed0,dseed1,dseed2,xx,xxf,u)
c
      do 70 k=1,N
      er(k)=er(k)+(xx(k)-xxf(k))**2
70    continue
c
60    continue
      go to 22
c
c     ioption=2 :
c     upper bound (N values:u(0)...u(iT)) are computed
c     and used to compute the BOF mmse error next
c
29    kswitch=0
      kr=1
      call ric(kswitch,kr,u)
      do 37 j=1,NS
      call bofsub(dseed0,dseed1,dseed2,xx,xxf,u)
      do 38 k=1,N
      er(k)=er(k)+(xx(k)-xxf(k))**2
38    continue
37    continue
c
c     M (<=N) values of the mmse error are printed next
c
22    er(1)=sig0**2
      print *,er(1)
      do 80 k=2,M
      er(k)=er(k)/NS
      print *,er(k)
80    continue
      go to 67
c
c     ioption=0 :
c     two sample paths of the true and (*-) filtered state are
c     computed.
c     iflag=1 ----> kf-filtered  ;  iflag=2 ----> bof-filtered
c
63    if (iflag.eq.2) go to 119
      kswitch=0
      kr=0
      call ric(kswitch,kr,u)
      call kfsub(dseed0,dseed1,dseed2,xx,xxf,u)
      go to 121
c
119   kswitch=0
      kr=1
      call ric(kswitch,kr,u)
      call bofsub(dseed0,dseed1,dseed2,xx,xxf,u)
c
121   do 65 k=1,M
      print *,xx(k)
65    continue
c
      do 66 k=1,M
```

```
      print *,xxf(k)
66    continue
c
      do 141 i=1,M
      print *,xxf(i)
 141  continue
      go to 137
c
c     print upper bound u(t)
c     in the case of "KF" , i.e. ioption=1, this is just r(t)
c     which is not an upper bound for neither pk(t) nor p(t).
c
67    do 135 i=1,N
135   print *,u(i)
c
c     compute and print lower bound l(t)
      kswitch=1
      kr=1
      call ric(kswitch,kr,u)
      do 136 i=1,N
 136  print *,u(i)
 137  stop
      end
c
c
c     *****************
c     SOUBROUTINE KFSUB
c     *****************
c
      subroutine kfsub(dseed0,dseed1,dseed2,xx,xxf,u)
c
      real xk,xfk,yk,yyk
      double precision dseed0,dseed1,dseed2
      dimension xx(5000),xxf(5000)
      dimension u(5000)
      common /const/deltat,sqd,xm0,sig0,sig,c,rho,N,iT
      common /param/a,xmuu,xmub,dzetau,dzetab
c
      do 10 k=1,N
      km1=k-1
      call kalfilt(km1,dseed0,dseed1,dseed2,xk,xfk,yyk,u)
      xx(k)=xk
      xxf(k)=xfk
10    continue
c
      return
      end

c     *******************
c     SUBROUTINE BOFSUB :
c     *******************
c
      subroutine bofsub(dseed0,dseed1,dseed2,xx,xxf,u)
c
      real xk,xfk,yk,yyk
      double precision dseed0,dseed1,dseed2
      dimension xx(5000),xxf(5000)
      dimension u(5000)
      common /const/deltat,sqd,xm0,sig0,sig,c,rho,N,iT
      common /param/a,xmuu,xmub,dzetau,dzetab
```

```
c
      do 10 k=1,N
      km1=k-1
      call bofilt(km1,dseed0,dseed1,dseed2,xk,xfk,yyk,u)
      xx(k)=xk
      xxf(k)=xfk
10    continue
c
      return
      end

c     ********************
c     SOUBROUTINE OBSERVY :
c     ********************


      subroutine observy(km1,dseed0,dseed1,dseed2,xk,yk)
c
c     ****************************************************
c     observy generates the observation yk=y(k*deltat)
c     and xk=x(k*deltat) from the model :
c     dx(t)=f(x(t)).dt + sig.dw(t) , x(0)=x0 N(m0,sig0^2)
c     dy(k)=g(x(t)).dt + rho.dv(t) , y(0)=0
c     w(t),v(t) standard N(0,t) , deltat=iT/N , sqd its sqrt
c     ggnqf(dseed) generates a N(0,1)-variate Zk(dseed) .
c     the value of dseed is internally changed by ggnqf for
c     a future call.
c     ****************************************************
c
      real xk,xfk,yk,yyk
      real ggnqf,Zk,Qk
      double precision dseed0,dseed1,dseed2
      common /const/deltat,sqd,xm0,sig0,sig,c,rho,N,iT
      common /param/a,xmuu,xmub,dzetau,dzetab
      if(km1.gt.0) go to 98
      xk=sig0*ggnqf(dseed0)+xm0
      yk=0.
      go to 99
   98 Zk=ggnqf(dseed1)
      Qk=ggnqf(dseed2)
      tk=(km1-1)*deltat
      fk=a*xk+exp(-tk)*sin(50*tk)*tanh(xk)
      gk=c*xk+xk*exp(-xk*xk)/(tk*tk+1)
      xkp1=xk+fk*deltat+sig*sqd*Zk
      ykp1=yk+gk*deltat+rho*sqd*Qk
      xk=xkp1
      yk=ykp1
   99 gfg=0.
      return
      end

c     *******************
c     SOUBROUTINE KALFILT :
c     *******************

      subroutine kalfilt(km1,dseed0,dseed1,dseed2,xk,xfk,yyk,u)
c
c     ***************************************************
c     Using observations from from the model in subroutine
c     observy this subroutine generates xfk=xf(k*deltat)
```

```
c     where xf(t) is the kalfilt (constant gain filter) :
c     dxf(t)=f(xf(t)).dt + sig/rho[dy(t) - c.xf(t).dt]
c     xf(0)=E(x0)=m0
c     kalfilt is asymptotically optimal as rho--->0, f cone
c     bounded and observations linear.
c      (kalfilt also returns the true state xk)
c     *****************************************************
c
      real xk,xfk,yk,yyk
      real ggnqf,Zk,Qk
      dimension u(5000)
      double precision dseed0,dseed1,dseed2
      common /const/deltat,sqd,xm0,sig0,sig,c,rho,N,iT
      common /param/a,xmuu,xmub,dzetau,dzetab
      if(km1.gt.0) go to 78
      xfk=xm0
      yyk=0.
      call observy(km1,dseed0,dseed1,dseed2,xk,yk)
      go to 79
   78 fk=a*xfk
      call observy(km1,dseed0,dseed1,dseed2,xk,yk)
      yykp1=yk
      dyyk=yykp1-yyk
      gain=c*u(km1)/(rho**2)
      xfkp1=xfk+fk*deltat+gain*(dyyk-c*xfk*deltat)
      xfk=xfkp1
      yyk=yykp1
   79 return
      end
c
c
c     ******************
c     SUBROUTINE BOFILT:
c     ******************
c
      subroutine bofilt(km1,dseed0,dseed1,dseed2,xk,xfk,yyk,u)
c
c     ****************************************************
c     Using observations from from the model in subroutine
c     observy this subroutine generates xfk=xf(k*deltat)
c     where xf(t) is the kalfilt (constant gain filter) :
c     dxf(t)=f(xf(t)).dt + sig/rho[dy(t) - c.xf(t).dt]
c     xf(0)=E(x0)=m0
c     kalfilt is asymptotically optimal as rho--->0, f cone
c     bounded and observations linear.
c      (kalfilt also returns the true state xk)
c     *****************************************************
c
      dimension u(5000)
      real xk,xfk,yk,yyk
      real ggnqf,Zk,Qk
      double precision dseed0,dseed1,dseed2
      common /const/deltat,sqd,xm0,sig0,sig,c,rho,N,iT
      common /param/a,xmuu,xmub,dzetau,dzetab
      if(km1.gt.0) go to 78
      xfk=xm0
      yyk=0.
      call observy(km1,dseed0,dseed1,dseed2,xk,yk)
      go to 79
   78 tk=(km1-1)*deltat
```

```
      fk=a*xfk+exp(-tk)*sin(50*tk)*tanh(xfk)
      gk=c*xfk+xfk*exp(-xfk*xfk)/(tk*tk+1)
      call observy(km1,dseed0,dseed1,dseed2,xk,yk)
      yykp1=yk
      dyyk=yykp1-yyk
c
      bofgain=c*u(km1)/(rho**2)
      xfkp1=xfk+fk*deltat+bofgain*(dyyk-gk*deltat)
c
      xfk=xfkp1
      yyk=yykp1
 79   return
      end
c
c     ***************
c     SUBROUTINE RIC:
c     ***************
c
      subroutine ric(kswitch,kr,u)
      dimension u(5000),x(4),dx(4)
      common /const/deltat,sqd,xm0,sig0,sig,c,rho,N,iT
      common /param/a,xmuu,xmub,dzetau,dzetab
      h=deltat
      delmu=(xmub-xmuu)/2
      if(kswitch.eq.0) goto 17
      p1=xmuu
      p2=dzetab
      goto 18
 17   p1=xmub
      p2=dzetau
 18   nn=1
      x(1)=sig0**2
      u(1)=x(1)
      t=0.0
      k=0
      m=0
c     write the ode
  1   q1tt=4*(rho*exp(-t)*sin(50*t)*delmu/sig)**2
      alp=a+kr*exp(-t)*sin(50*t)*p1
      bet=c+kr*p2/(t*t+1)
      r2=rho**2
      dx(1)=sig**2+2.0*alp*x(1)-(bet**2+kswitch*q1tt)*(x(1)**2)/r2
      call runta(nn,k,ii,x,dx,t,h)
      go to (1,2),ii
  2   m=m+1
      u(m+1)=x(1)
      if (t.le.iT) go to 1
      return
      end

       subroutine runta(nn,k,ii,x,dx,t,h)
       dimension y(4),z(4),x(4),dx(4)
       k=k+1
       go to (1,2,3,4,5),k
  2    do 10 j=1,nn
       z(j)=dx(j)
       y(j)=x(j)
 10    x(j)=y(j)+0.5*h*dx(j)
 25    t=t+0.5*h
  1    ii=1
```

```
      return
 3    do 15 j=1,nn
      z(j)=z(j)+2.0*dx(j)
15    x(j)=y(j)+0.5*h*dx(j)
      ii=1
      return
 4    do 20 j=1,nn
      z(j)=z(j)+2.0*dx(j)
20    x(j)=y(j)+h*dx(j)
      go to 25
 5    do 30 j=1,nn
30    x(j)=y(j)+(z(j)+dx(j))*h/6.0
      ii=2
      k=0
      return
      end
```