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Linear Feedback Stabilization of Nonlinear Systems with an Uncontrollable Critical Mode

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Linear Feedback Stabilization of Nonlinear Systems with an Uncontrollable Critical Mode

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Subtitle

Stabilization of an equilibrium point of a nonlinear system by linear feedback is studied in two critical cases, corresponding to the presence of either a zero eigenvalue or a pair of pure imaginary eigenvalues for the linearized system. In each case, the critical eigenvalues are assumed uncontrollable, and stabilization is studied using bifurcation-theoretic calculations.

ABSTRACT

Linear feedback stabilization of nonlinear systems is studied for systems whose linearization at an equilibrium point possesses a simple critical mode that is uncontrollable. The results complement previous work on the synthesis of nonlinear stabilizing control laws. The present work addresses continuous-time systems for which the linearization has either a simple zero eigenvalue or a pair of simple pure imaginary eigenvalues. Both the stability analysis and stabilizing control design employ results on stability of bifurcations of parametrized systems.

Keywords: Stabilizers, nonlinear systems, controllability, feedback control.

1 Introduction

Feedback stabilization of nonlinear control systems is a subject which has received significant attention in the control literature in recent years. Local, semiglobal, and global stabilization problems have been considered. In this paper we address problems of local stabilization, using linear feedback, of an equilibrium point of a nonlinear system

$$\dot{x} = f(x, u). \tag{1}$$

Here, $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the control, and f is sufficiently smooth in x and u with $f(0,0) = 0$.

Thus, we consider the question of existence of a linear feedback

$$u = u(x) = kx \tag{2}$$

($k \in \mathbb{R}^{1\times n}$) rendering the origin of Eq. (1) locally asymptotically stable. In addressing this question, we take an approach which is by nature constructive. By standard theory, stabilizability of the linearization of (1) is sufficient for local stabilizability of the nonlinear system (1). Thus, in this paper, as in many previous studies, we assume that the linearization of (1) at the origin has uncontrollable critical eigenvalues. An eigenvalue is critical if it lies on the imaginary axis. More specifically, we consider two critical cases. In the first, the linearization has an uncontrollable zero eigenvalue. In the second, an uncontrollable complex conjugate pair of purely imaginary eigenvalues occurs. The recent book (Bacciotti, 1992) and the review paper (Sontag, 1990) provide overviews of many results on nonlinear stabilization. The papers (Andreini *et al.*, 1989), (Bacciotti and Boieri, 1990) address linear feedback stabilization of nonlinear systems with uncontrollable critical modes.

Since the right side of (1) is smooth, we may write

$$\dot{x} = f(x,0) + ug_1(x) + u^2g_2(x) + u^3g_3(x) + \cdots. \tag{3}$$

We can also expand the zero-input system corresponding to (1):

$$\begin{aligned} \dot{x} &= f(x,0) \\ &= Ax + B(x,x) + C(x,x,x) + \cdots. \end{aligned} \tag{4}$$

Denote the Taylor expansions of the input kernel functions $g_i(x),\ i = 1,2,\ldots$, appearing in Eq. (3) as

$$g_i(x) = b_i + A_ix + B_i(x,x) + C_i(x,x,x) + \cdots. \tag{5}$$

In (4) and (5), $B(x,x)$, $C(x,x,x)$, and $B_i(x,x)$, $C_i(x,x,x)$, $i = 1,2,\ldots$, are vector quadratic and cubic forms induced by symmetric bilinear and trilinear forms, and the dots denote higher order terms (Abed and Fu, 1986), (Abed and Fu, 1987), (Iooss and Joseph, 1990). To conform with standard notation in linear control theory, we shall henceforth denote b_1 by b, viz.,

$$b := b_1.$$

As noted by Brockett (1983), the only situations in which the smooth local stabilizability question for (1) cannot be addressed based only on considerations for the linearized system

$$\dot{x} = Ax + bu \tag{6}$$

are those in which (6) possesses at least one *uncontrollable critical mode*. That is, the cases in which the determination of existence of a stabilizing smooth feedback requires one to consider the influence of the nonlinear terms in (1) are those in which the matrix A has at least one uncontrollable eigenvalue with zero real part. These are referred to as the *critical cases* in nonlinear stabilization. This terminology derives from the classical nonlinear stability literature, in which a critical case corresponds to the vanishing of the real part of at least one system eigenvalue (Zubov, 1964).

This having been said, we wish to note that there are cases in which a system is stabilizable by linear methods, but a stabilizing controller designed using nonlinear considerations is preferable to a traditionally designed linear controller. For example, a system may possess eigenvalues on the imaginary axis which are *nearly uncontrollable*. In such a case, a linear feedback giving an adequate margin of stability may have an unacceptably large gain, whereas there might exist a nonlinear controller of low gain which also stabilizes the system adequately. There is evidence (Marino and Kokotovic, 1986) that high gain linear feedback which achieves a large margin of stability for the linearized system may compromise stability of the nonlinear system. A careful investigation of these issues would require a separate effort.

Several approaches have been used in addressing stabilization problems for (1) in critical cases. Following the introductory work of Aeyels (1985), center manifold reduction has been frequently employed in the literature on feedback stabilization; see, for instance, (Behtash and Sastry, 1988), (Andreini *et al.*, 1989), (Bacciotti and Boieri, 1990), and (Boothby and Marino, 1989). An alternative but mathematically equivalent approach was introduced by the authors in (Abed and Fu, 1986), (Abed and Fu, 1987). Here, a bifurcation-theoretic framework is employed, and stabilization of an equilibrium point in a critical case is tied to the stabilization of bifurcated solution branches. This approach is also employed in the present paper. This approach affords the advantage of calculations performed directly on the original system dynamic equations.

From a control-theoretic viewpoint, it is interesting to consider the feasibility of employing the common linear feedback laws in the local feedback stabilization of nonlinear systems. This work is also motivated by Theorem 3 of Abed and Fu (1987), which asserts the following: Under certain generic conditions (see (Abed and Fu, 1987)), the local feedback stabilization problem for (1) whose Jacobian A has an uncontrollable zero eigenvalue is not solvable by a smooth feedback control with vanishing linear part.

In essence, the purely nonlinear stabilizing feedback laws in (Abed and Fu, 1986), (Abed and Fu, 1987) are constructed by ensuring negativity of certain so-called bifurcation stability coefficients, the values of which can be determined using formulae given directly in terms of the system dynamic equations. It appears that these bifurcation stability coefficients play a role similar to that of dominant open-loop pole(s) in linear control theory. Thus the problem investigated here may be viewed as a corresponding "nonlinear pole" assignability problem.

The strategy for studying stabilizability employed in previous papers of the authors (e.g., (Abed and Fu, 1986), (Abed and Fu, 1987), (Fu, 1990), (Fu and Abed, 1993)) will also be used here to study existence and synthesis of linear stabilizing feedback controllers. The analysis is more involved than that in (Abed and Fu, 1986), (Abed and Fu, 1987), however. This is because linear feedback modifies the Jacobian matrix A, unlike the case with purely nonlinear feedback. This results in two important differences between the work in (Abed and Fu, 1986), (Abed and Fu, 1987) and the present endeavor: First, the linear feedback gain vector k must now be restricted to ensure that the nominally stable eigenvalues remain stable for the closed-loop system. Second, in (Abed and Fu, 1986), (Abed and Fu, 1987) the pertinent closed-loop bifurcation stability coefficients are *polynomial* functions of the control gain. In contrast, we shall be faced with *rational* functions of the gain k in the present paper.

If it happens that $b = 0$, which occurs in particular if (1) is affine in the control, then A is invariant under any linear feedback. Under this condition, the complications just noted do not occur. The simplified results for this situation follow easily from the more general results presented here.

The remainder of this paper proceeds as follows. In Section 2 the basic problem set-up is given, including expressions for the stability coefficients of the open- and closed-loop systems. In Section 3 the stability of the linearized closed-loop system is considered. Section 3 also addresses the computation of some relevant matrix inverses as well as of the closed-loop eigenvectors associated with the critical eigenvalues. The main results of the paper are presented in Sections 4 and 5: Section 4 contains the results in the case of an uncontrollable zero eigenvalue, while Section 5 addresses critical systems with an uncontrollable pair of pure imaginary eigenvalues. Concluding remarks are collected in Section 6.

Notation. In what follows, $\|\cdot\|$ stands for the Euclidean norm: $\|v\| = (v^T v)^{\frac{1}{2}}$ for $v \in \mathbb{R}^n$ and $\|v\| = (v^H v)^{\frac{1}{2}}$ for $v \in \mathbb{C}^n$. The real and imaginary parts of a scalar expression are denoted by $\mathrm{Re}\{\cdot\}$ and $\mathrm{Im}\{\cdot\}$, respectively. A superscript T (resp., H) indicates the transpose (resp., Hermitian transpose) of a matrix or vector. With $k \in \mathbb{R}^{1\times n}$, denote by $\mathcal{R}_{\hat{k}}(F(\cdot))$ the range of a function of k over a domain $\|k\| \leq \hat{k}$:

$$\mathcal{R}_{\hat{k}}(F(\cdot)) := \{F(k) : \ \|k\| \leq \hat{k}\}. \tag{7}$$

2 Stability Criteria and Stabilization Strategy

As mentioned above, this paper addresses the stabilization by linear feedback of the origin of Eq. (1) in two critical cases. In this preliminary section, we give the basic problem set-up and recall formulae for quantities known as *stability coefficients*. The use of the stability coefficients in stability assessment for the origin in the critical cases studied here is summarized. The effect of linear feedback on the values of the stability coefficients is determined. Finally, we outline an overall strategy for the investigation of stabilizability by linear feedback.

Consider the zero-input system (4). In this paper, the Jacobian A of this system is assumed to satisfy either of the following two hypotheses:

(S) A has an eigenvalue $\lambda_1 = 0$, with the remaining eigenvalues $\lambda_2, \ldots, \lambda_n$ in the open left half complex plane;

(H) A has a pair of simple, pure imaginary eigenvalues $\lambda_1 = i\omega_c$ and $\lambda_2 = -i\omega_c$, with the remaining eigenvalues $\lambda_3, \ldots, \lambda_n$ in the open left half complex plane.

For ease of reference, say that Case (S) (resp., Case (H)) prevails for Eq. (1) if A satisfies hypothesis (S) (resp., (H)).

It is convenient to denote by λ_c the critical eigenvalue in either of the critical cases, (S) or (H). That is, $\lambda_c := 0$ in Case (S) and $\lambda_c := i\omega_c$ in Case (H). The row and column vectors l and r are the left and right eigenvectors, respectively, of the Jacobian matrix A corresponding to the critical eigenvalue λ_c. For definiteness, set the first component of l to 1 in Case (S) and to $\frac{1}{2}$ in Case (H), and further normalize r to satisfy [1]

$$lr = 1. \tag{8}$$

[1] It may be necessary to first interchange the first component of the state vector with another component, to ensure the nonvanishing of the first component of l. This normalization turns out to be more convenient here than that used in (Abed and Fu, 1986), (Abed and Fu, 1987), as will be seen in Section 3.2. In the cited references, the first component of r was set to unity and then l was chosen to satisfy $lr = 1$ both in Case (S) and Case (H). This modification does not affect the form of the stability criteria.

In Case (S), two stability coefficients β_{1_S}, β_{2_S} will be employed in the analysis. These coefficients and their connection to the stability of the origin of (4) are identified next.

Fact 1. ((Abed and Fu, 1987), (Iooss and Joseph, 1990)) In Case (S), denote

$$\beta_{1_S} := lB(r,r), \tag{9}$$

$$\beta_{2_S} := l[-2B(r,(A^TA+l^Tl)^{-1}A^TB(r,r))+C(r,r,r)]. \tag{10}$$

Then the origin of (4) is locally asymptotically stable (resp., unstable) if $\beta_{1_S}=0$ and $\beta_{2_S}<0$ (resp., $\beta_{1_S}\neq 0$, or $\beta_{1_S}=0$ and $\beta_{2_S}>0$). ∎

Two stability coefficients were introduced above for the assessment of stability in Case (S). In the stability analysis of the origin of (4) in Case (H), a single stability coefficient, denoted β_{2_H}, occurs. A formula for this coefficient, and its connection to the stability of the origin of (4) are recalled next.

Fact 2. ((Abed and Fu, 1986), (Howard, 1979)) In Case (H), denote

$$\beta_{2_H} := 2\mathrm{Re}\{l[-B(r,A^{-1}B(r,\bar{r}))-\frac{1}{2}B(\bar{r},(A-2i\omega_cI)^{-1}B(r,r))+\frac{3}{4}C(r,r,\bar{r})]\}. \tag{11}$$

Then the origin of (4) is locally asymptotically stable (resp., unstable) if $\beta_{2_H}<0$ (resp., $\beta_{2_H}>0$). ∎

We note in passing that the quantities β_{1_S}, β_{2_S} and β_{2_H} can be interpreted as coefficients in asymptotic expansions of dominant Liapunov exponents for bifurcated solutions of parametrized embeddings of Eq. (4). Various derivations of these coefficients have been given in the literature. These employ, for instance, the projection method (Abed and Fu, 1987), (Iooss and Joseph, 1990), harmonic balance (Howard, 1979), and Liapunov functions (Fu and Abed, 1993). Note also that in the nongeneric situations in which the coefficients of Eqs. (9)-(11) vanish, typically stability can still be assessed by examination of further (higher order) coefficients in the appropriate asymptotic expansions.

Due to the foregoing considerations, assignability of bifurcation stability coefficients by feedback plays a role in critical nonlinear systems (1) analogous to that played by pole assignability in linear time-invariant systems.

Under either Case (S) or Case (H), the basic assumptions we make on (1) are that: (i) the origin is unstable for the zero-input system (4); and (ii) the critical mode is uncontrollable for the linearized system (6). Indeed, the results of this paper can easily be re-construed to apply to the case in which the origin is locally asymptotically stable for (4) or that in which the stability of the origin is unknown. In these situations, assignability of the pertinent stability coefficients takes the place of the pure stabilizability question. Achieving a certain degree of stability or simply guaranteeing stability despite inability to determine the stability properties of the open-loop system become the issues of concern.

Assumption (ii) above is equivalent, by the well known Popov-Belevitch-Hautus eigenvector test for controllability (Kailath, 1980), to

$$lb=0. \tag{12}$$

Generically, the local stability or instability of the origin of Eq. (4) in Cases (S) and (H) is determined by terms in the vector field up to cubic order. This follows from Facts 1 and 2 above. Thus, next we record how these terms in the vector field are affected by linear state feedback $u = kx$.

Substituting the linear feedback (2) into (1), and using an asterisk to indicate quantities after feedback, we obtain the closed-loop system

$$\dot{x} = A^*x + B^*(x,x) + C^*(x,x,x) + \cdots \tag{13}$$

where the linear, quadratic and cubic terms in the closed-loop vector field are given by

$$A^* = A + bk, \tag{14}$$

$$B^*(x,x) = B(x,x) + (kx)A_1x + b_2(kx)^2, \tag{15}$$

$$C^*(x,x,x) = C(x,x,x) + (kx)B_1(x,x) + (kx)^2A_2x + (kx)^3b_3, \tag{16}$$

respectively.

Note from Eqs. (14)-(16) that linear state feedback affects terms of every order in (1). In particular, with $b \neq 0$ such a feedback in general affects the system linearization: $A^* \neq A$. This implies that a large gain linear feedback may *destabilize* modes that are stable for the open-loop system. Hence, in the stabilization of a critical system using linear feedback, one must ensure that all eigenvalues of A^*, besides the immobile critical mode, are not moved out of the open left half complex plane.

Thus, to stabilize (1) using linear feedback, it suffices to find a vector k that, firstly, does not destabilize the nominally stable system eigenvalues, and secondly, ensures satisfaction of the stability criteria of Facts 1 and 2 above for the closed-loop system. The first of these requirements is equivalent to asking that k preserves the original criticality hypothesis (S) or (H). The second requirement we impose is that, for the closed-loop system, $\beta^*_{1_S} = 0$ and $\beta^*_{2_S} < 0$ in Case (S); and $\beta^*_{2_H} < 0$ in Case (H). Here, by the formulae in Facts 1 and 2, these stability coefficients for the closed-loop system are given in Case (S) by

$$\beta^*_{1_S} = l^*B^*(r^*,r^*), \tag{17a}$$

$$\beta^*_{2_S} = l^*[-2B^*(r^*,(A^{*T}A^* + l^{*T}l^*)^{-1}A^{*T}B^*(r^*,r^*)) + C^*(r^*,r^*,r^*)], \tag{17b}$$

and in Case (H) by

$$\begin{aligned}\beta^*_{2_H} = \; & 2\mathrm{Re}\{l^*[-B^*(r^*,A^{*-1}B^*(r^*,\bar{r}^*)) - \frac{1}{2}B^*(\bar{r}^*,(A^* - 2i\omega_cI)^{-1}B^*(r^*,r^*)) \\ & + \frac{3}{4}C^*(r^*,r^*,\bar{r}^*)]\}.\end{aligned} \tag{18}$$

Here, as in Eqs. (13)-(16), an asterisk indicates a closed-loop quantity.

3 The Closed-Loop Linearized System

In this section we present calculations relevant to the closed-loop linearized system. The concept of restricted matrix inverse is recalled and used in the calculation of the closed-loop critical eigenvectors l^* and r^*, in addition to other useful closed-loop matrix quantities.

3.1 Restricted Matrix Inverses

In Case (S), we deal with a singular Jacobian matrix A. This matrix has a (geometrically and algebraically) simple zero eigenvalue. In Case (H), the Jacobian matrix is not singular, but we will none the less find it convenient to solve certain associated linear algebraic equations with coefficient matrix $A - i\omega_c I$, which is singular with a simple zero eigenvalue. In this subsection, therefore, we consider linear algebraic equations

$$Mx = y \tag{19}$$

where $M \in \mathbb{C}^{n\times n}$ possesses a simple zero eigenvalue and $y \in \mathbb{C}^n$. The conclusions for M and y real in Eq. (19) follow as a special case. The notation l, r for the left and right eigenvectors of M, respectively, is as employed in the foregoing section. The particular normalization used in defining these eigenvectors does not affect the restricted inverses obtained.

Denote by $\mathcal{N}(M)$ the null space of M, a subspace of $\mathbb{C}^n$. Denote by $\mathcal{N}(M)^\perp$ the orthogonal complement of $\mathcal{N}(M)$ (a subspace of $\mathbb{C}^n$). Using the Fredholm Alternative, we find that, if both the domain and range of M are restricted to $\mathcal{N}(M)^\perp$, there results a unique solution to the equation $Mx = y$. Restriction to $\mathcal{N}(M)^\perp$ is tantamount to imposing the conditions $lx = ly = 0$. The corresponding solution operator is found to be given by the following *restricted inverse* on $\mathcal{N}(M)^\perp$:

$$\begin{aligned} M^- &:= (M|_{\mathcal{N}(M)^\perp})^{-1} : \mathcal{N}(M)^\perp \to \mathcal{N}(M)^\perp \\ &= (M^H M + l^H l)^{-1} M^H \end{aligned} \tag{20}$$

To illustrate the relevance of (20) in this work, note that formula (9), which gives β_{2_S} in Case (S), is rewritten using (20) in the following compact form:

$$\beta_{2_S} = l[-2B(r, A^- B(r,r)) + C(r,r,r)]. \tag{21}$$

3.2 Closed-Loop Inverses and Critical Eigenvectors

Observe that, since $lb = 0$ (the uncontrollability assumption), it follows from (14) that $lA^* = lA$. Taken along with the normalization of Section 2 for the eigenvectors, we thus have that $l^* = l$ in both Cases (S) and (H). To compute r^*, write $r^* = r + \tilde{r}$ and solve for $\tilde{r}$ subject to the normalization $l^* r^* = 1$ and to the associated eigenvector equation $A^* r^* = \lambda_c r^*$. (Recall that λ_c is either 0 or $i\omega_c$.) Since $l^* = l$, $l^* r^* = 1$ implies that $\tilde{r}$ satisfies $l\tilde{r} = 0$. It follows that the corresponding eigenvalue/eigenvector equations in Case (S) and in Case (H) lead to $A^*\tilde{r} = -(kr)b$ and $(A^* - i\omega_c I)\tilde{r} = -(kr)b$, respectively.

Next we consider solving the equations at the end of the last paragraph for $\tilde{r}$. Since $lb = 0$ and $l\tilde{r} = 0$, the Fredholm Alternative applies. Using also the fact that $l^* = l$, the closed-loop system restricted matrix inverses in Case (S) and Case (H) are found to be given by

$$A^{*-} = A^- - \frac{1}{1 + kA^- b} A^- bkA^-, \tag{22}$$

$$(A^* - i\omega_c I)^- = (A - i\omega_c I)^- - \frac{1}{1 + k(A - i\omega_c I)^- b}(A - i\omega_c I)^- bk(A - i\omega_c I)^- \tag{23}$$

respectively, provided that $1 + kA^{-}b \neq 0$ and $1 + k(A - i\omega_c I)^{-}b \neq 0$. These formulae are easily verified. Their statement was initially motivated by the Sherman-Morrison Formula ((Golub and Van Loan, 1983), Eq. (2.1.4)).

Motivated by the Eqs. (22) and (23), introduce the following class of rational functions:

$$\mathcal{F}(k;p,q) := \frac{kp}{1+kq}, \quad k \in \mathbb{R}^{1\times n}, \quad p,q \in \mathbb{C}^{n\times 1}. \tag{24}$$

It follows that the closed-loop critical eigenvector r^* of A^* is given by[2]

$$r^* = \begin{cases} r - \mathcal{S}(k)A^{-}b & \text{in Case (S)} \\ r - \mathcal{H}(k)(A - i\omega_c I)^{-}b & \text{in Case (H)} \end{cases} \tag{25}$$

where

$$\mathcal{S}(k) := \mathcal{F}(k;r,A^{-}b) = \frac{kr}{1+kA^{-}b} \in \mathbb{R} \tag{26}$$

and

$$\mathcal{H}(k) := \mathcal{F}(k;r,(A - i\omega_c I)^{-}b) = \frac{kr}{1+k(A - i\omega_c I)^{-}b} \in \mathbb{C}. \tag{27}$$

It is easily verified that $kr^* = \mathcal{S}(k)$ in Case (S) and $kr^* = \mathcal{H}(k)$ in Case (H). These observations will be useful in Appendix A in the computation of $\beta^*_{2_S}$ and $\beta^*_{2_H}$, respectively.

In Case (S), by (25) we have that r^* is an affine vector function of the real fraction $\mathcal{S}(k)$. However, in the next section the necessary condition for stabilization $\beta^*_{1_S} = 0$ (cf. Fact 1) will be imposed, resulting in $\mathcal{S}(k)$ being constrained to take one of only three possible values. Hence, r^* is similarly constrained by the condition $\beta^*_{1_S} = 0$. In contrast, in Case (H), no such equality constraint is imposed. Hence, from Eq. (25), in Case (H), r^* is an affine vector function in the complex fraction $\mathcal{H}(k)$, a free variable.

Several matrix inverses associated with the closed-loop linearized system will be used in the calculations in Case (H). One is the *restricted* inverse $(A^* - i\omega_c I)^{-}$ of Eq. (23). However, in applying Eq. (18) expressions for the actual inverses A^{*-1} and $(A^* - 2i\omega_c I)^{-1}$ will be needed. Since both A and $A - 2i\omega_c I$ are invertible in Case (H), direct application of the Sherman-Morrison formula ((Golub and Van Loan, 1983), Eq. (2.1.4)) yields

$$A^{*-1} = A^{-1} - \frac{A^{-1}bkA^{-1}}{1+kA^{-1}b}, \tag{28}$$

$$(A^* - 2i\omega_c I)^{-1} = (A - 2i\omega_c I)^{-1} - \frac{(A - 2i\omega_c I)^{-1}bk(A - 2i\omega_c I)^{-1}}{1+k(A - 2i\omega_c I)^{-1}b}, \tag{29}$$

respectively, provided that $1 + kA^{-1}b \neq 0$ and $1 + k(A - 2i\omega_c I)^{-1}b \neq 0$.

Finally, we introduce the following notation, which will be used in the statement of results on stabilization by linear feedback in Case (S) and Case (H). Denote by $\hat{k}$ the largest scalar for which the matrix $A + bk$ retains the same number of stable eigenvalues as the matrix A, for all k with $\|k\| \leq \hat{k}$. Thus, for all k with norm less than $\hat{k}$, the linear state feedback $u = kx$ does not destabilize any of the open-loop stable eigenvalues.

[2]Note that if l and r were specified in the way that one first normalizes $[r]_1 = 1$ and then normalizes l to $lr = 1$, then since (25) would not guarantee that $[r^*]_1 = 1$, it would have been necessary to take the new normalizing factor into account for applying (17) and (18). The present way of specifying l and r makes use of the result $l^* = l$ which guarantees $[l^*]_1 = 1$ in Case (S) and $[l^*]_1 = \frac{1}{2}$ in Case (H), and thus does away with the need for renormalization.

4 Linear Feedback Stabilization in the Case of One Zero Eigenvalue

This section contains the main results of this paper for Case (S); hypothesis (S) is in force throughout the section. The linear stabilizability of the origin of (1) is addressed first, followed by a procedure by which a stabilizing linear feedback can be constructed explicitly. In studying the linear stabilizability problem, it is natural to assume that the origin of the open-loop system does not satisfy the conditions for asymptotic stability ${\beta_1}_S = 0, {\beta_2}_S < 0$ of Fact 1.

In the remainder of this section, the objective is to find a linear feedback $u = kx$ ensuring that ${\beta_1^*}_S = 0$ and ${\beta_2^*}_S < 0$, without moving any of the open-loop stable eigenvalues of the linearization out of the open left half complex plane. The last requirement can also be expressed by asking that matrix A^* is itself consistent with hypothesis (S). Derivations of the expressions for the closed-loop bifurcation stability coefficients ${\beta_1^*}_S$ and ${\beta_2^*}_S$ in terms of the gain vector k are detailed in Appendix A.

4.1 Necessary Conditions for Stabilizability

From Fact 1, any smooth feedback which stabilizes the origin of (1) must result in a vanishing closed-loop stability coefficient ${\beta_1^*}_S$. Such a feedback must satisfy further conditions to be guaranteed locally stabilizing. Here, we shall ask that it also achieve ${\beta_2^*}_S < 0$, and that it not result in destabilization of the open-loop stable eigenvalues.

In this subsection, we determine necessary conditions for linear stabilizability by considering the possibility of achieving ${\beta_1^*}_S = 0$ through a linear feedback $u = kx$, and then proceed to consider the possibility of achieving, in addition, the condition ${\beta_2^*}_S < 0$.

Substituting $x := r^*$ from (25) (for Case (S)) in the formula (15) for $B^*(x, x)$, and grouping terms of like order in $\mathcal{S}(k)$ (introduced in (26)), we obtain

$$B^*(r^*, r^*) = B(r, r) + \mathcal{S}(k)v_1 + \mathcal{S}^2(k)v_2, \tag{30}$$

where the v_1 and v_2 are real coefficient vectors determined in Appendix A (see Eqs. (A.3) and (A.4)). Hence, using (17a), the necessary condition for stabilization ${\beta_1^*}_S = 0$ now takes the form

$$\begin{aligned} 0 &= {\beta_1^*}_S(\mathcal{S}) \\ &= {\beta_1}_S + \mathcal{S}(k)(lv_1) + \mathcal{S}^2(k)(lv_2). \end{aligned} \tag{31}$$

From Fact 1 and Eq. (17a), we see that (1) is not linearly stabilizable if there is no real solution $\mathcal{S}(k)$ to Eq. (31). Conversely, for any gain vector k which linearly stabilizes (1), the resulting $\mathcal{S} = \mathcal{S}(k)$ given by (26) must solve (31). When (31) is truly a quadratic equation, i.e., when $lv_2 \neq 0$, we denote by Δ the discriminant of (31):

$$\Delta := (lv_1)^2 - 4{\beta_1}_S(lv_2). \tag{32}$$

Clearly, if $\Delta < 0$, then (31) has no real solutions, and Eq. (1) is not linearly stabilizable. If Δ is defined (i.e., $lv_2 \neq 0$) and $\Delta > 0$, then (31) has precisely two real solutions. If

Δ is not defined (i.e., $lv_2 = 0$) and if $lv_1 \neq 0$, then there is only one solution, namely $\mathcal{S}(k) = -\beta_{1_S}/(lv_1)$. If both lv_2 and lv_1 vanish, then (31) has either no solution or else any value for $\mathcal{S}(k)$ is a solution, depending on whether or not $\beta_{1_S} \neq 0$. Disregarding the degenerate case in which all of lv_1, lv_2 and β_{1_S} vanish, we have that the possible real solutions of (31) are any of $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$ defined as follows:

$$\mathcal{S}_1, \mathcal{S}_2 = \frac{-(lv_1) \pm \Delta^{1/2}}{2(lv_2)}, \tag{33}$$

$$\mathcal{S}_3 = -\frac{\beta_{1S}}{(lv_1)}. \tag{34}$$

Below, we will need to consider the special cases in which $\mathcal{S}_1, \mathcal{S}_2$ apply and either $\Delta = 0$ or $\beta_{1_S} = 0$, and the case in which $\mathcal{S}_3$ applies and $\beta_{1_S} = 0$.

The preceding remarks clearly delineate certain necessary conditions for linear stabilizability of (1) under hypothesis (S). The next theorem collects several of these. Note that these conditions are also sufficient conditions for existence of a linear feedback rendering $\beta^*_{1_S} = 0$.

Theorem 1. (Necessary Condition for Linear Stabilizability) Let (1) satisfy hypothesis (S). Suppose the origin of (1) is stabilizable by a linear feedback $u = kx$. Then either $\beta_{1_S} = 0$ or there is a linear feedback $u = kx$ rendering $\beta^*_{1_S} = 0$. More precisely, either $\beta_{1_S} = 0$, or one of the following cases holds:

(a) $lv_2 \neq 0$ and $\Delta \geq 0$, or

(b) $lv_2 = 0$ and $lv_1 \neq 0$. ■

Theorem 1 is a constructive result, when taken along with the derivation preceding it. For example, we see that in case $\beta_{1_S} = 0$, any stabilizing linear feedback gain vector k must either satisfy $\mathcal{S}(k) = 0$ or $\mathcal{S}(k) = -(lv_1)/(lv_2)$ (the latter being a possibility only if $lv_2 \neq 0$).

4.2 Sufficient Conditions for Stabilizability

Suppose that (1) satisfies the necessary conditions given in Theorem 1, and let the feedback gain vector k be such that $\mathcal{S} = \mathcal{S}(k)$ is one of the $\mathcal{S}_i$, $i = 1, 2, 3$ of Eqs. (33), (34). Thus, having ensured $\beta^*_{1_S} = 0$, the formula (17b) for $\beta^*_{2_S}$ applies. The detailed calculations, given in Appendix A, result in the following expression:

$$\begin{aligned} \beta^*_{2_S} &= \beta^*_{2_S}(k; \mathcal{S}) \\ &= P_S(\mathcal{S}) + d(\mathcal{S})\phi(k; \mathcal{S}). \end{aligned} \tag{35}$$

In this expression,

$$P_S(\mathcal{S}) := (lw_0) + (lw_1)\mathcal{S} + (lw_2)\mathcal{S}^2 + (lw_3)\mathcal{S}^3, \tag{36}$$

$$d(\mathcal{S}) := (lv_1) + 2(lv_2)\mathcal{S}, \tag{37}$$

$$\begin{aligned} \phi(k; \mathcal{S}) &:= \mathcal{F}(k; A^- B^*(r^*, r^*), A^- b), \\ &= \mathcal{F}(k; A^-\{B(r, r) + \mathcal{S}v_1 + \mathcal{S}^2 v_2\}, A^- b) \end{aligned} \tag{38}$$

where the coefficient vectors w_0, w_1, w_2, w_3 appearing in (36) are specified in Eqs. (A.5)-(A.8) of Appendix A. (Recall that v_1 and v_2 are defined in Eqs. (A.3) and (A.4) of Appendix A.) Thus $P_S(\mathcal{S})$ is a cubic polynomial in $\mathcal{S}$, $d(\mathcal{S})$ is affine in $\mathcal{S}$, and $\phi(k;\mathcal{S})$ is rational in the elements of k. The real fractional map $\mathcal{F}$ was defined in (24).

Note that we have used the notation $\beta^*_{2_S}(k;\mathcal{S})$ and $\phi(k;\mathcal{S})$, even though $\mathcal{S}$ depends on k (see Eq. (26)). This notation is useful, however, since it is a reminder of the two-stage nature of the control design procedure of this paper. First, one determines, if possible, the values of $\mathcal{S}$ which would ensure $\beta^*_{1_S} = 0$. Since any particular value of $\mathcal{S}$ is to be achieved by appropriate choice of k, this places a restriction on the allowable feedback gain vectors k. The next step is to find, if possible, a gain vector k from among those achieving $\mathcal{S}(k) = \mathcal{S}$, which also results in a negative value for $\beta^*_{2_S}$, and which does not at the same time destabilize any of the open-loop stable eigenvalues.

In what follows, we focus on the achievable values of $\beta^*_{2_S}(k;\mathcal{S})$ where k is restricted such that $\mathcal{S}(k)$ is one of the $\mathcal{S}_i$, $i = 1,2,3$, given above. That is, we study the assignability of $\beta^*_{2_S}$ by linear feedback laws achieving $\beta^*_{1_S} = 0$.

Consider the goal of finding a gain vector k for which $\beta^*_{1_S} = 0$ and $\beta^*_{2_S} < 0$. In the foregoing, we have determined that this goal is closely related to that of finding a gain vector k which results in certain desired values for $\mathcal{S}(k)$ and $\phi(k;\mathcal{S})$. The latter are fractional maps of the form (24). We are thus led to consider the simultaneous solvability of equations involving such fractional maps. This is addressed in detail in Appendix B, and a specific result is given in Lemma B.1. From Lemma B.1 and the expressions above for $\beta^*_{1_S}$, $\beta^*_{2_S}$, the next result follows.

Theorem 2. (Stability Coefficient Assignability) Let (1) satisfy hypothesis (S) and suppose $lb = 0$. Fix a $\mathcal{S} \in \mathbb{R}$ for which $\beta^*_{1_S}(\mathcal{S}) = 0$ (see Eq. (31)). The function $\beta^*_{2_S}(k;\mathcal{S})$ defined on the domain of those vectors $k \in \mathbb{R}^{1\times n}$ for which $\mathcal{S}(k) = \mathcal{S}$ is onto if either of the following holds:

(a) $lv_2 \neq 0$ and $\Delta > 0$,

(b) $lv_2 = 0$ and $lv_1 \neq 0$, or

(c) $\beta_{1_S} = lv_1 = lv_2 = 0$ and either $lw_3 \neq 0$, or $lw_3 = lw_2 = 0$ and $lw_1 \neq 0$. ■

Note that it is also straightforward to address in the same fashion other special cases. However, to limit the complexity of the presentation, we prefer not to discuss these explicitly.

As observed previously, sufficient conditions such as those given in Theorem 2 do not alone imply local stabilizability of the origin of Eq. (1) by linear state feedback. Besides stability coefficient assignability, a linear feedback also must not destabilize open-loop stable eigenvalues. The next result addresses linear stabilizability, and involves consideration of the issue of nondestabilization of the open-loop stable modes. To characterize the linear stabilizability in a more explicit fashion, we first note Eq. (35) and the notation (38). Theorem 3 below results immediately. The theorem statement uses the notation $\hat{k}$ introduced at the end of Section 3, and the notation $\mathcal{R}_{\hat{k}}(F(\cdot))$ introduced at the end of Section 1. This is in order to take into account the stable mode nondestabilization requirement.

Theorem 3. (Sufficient Condition for Linear Stabilizability) Let (1) satisfy hypothesis (S) and suppose $lb = 0$. Then the origin of Eq. (1) is stabilizable by linear state feedback if there is a k with $\|k\| \leq \hat{k}$ that satisfies $kA^{-}b \neq -1$ and $\mathcal{S}(k) = \mathcal{S}_i \in \mathcal{R}_{\hat{k}}(\mathcal{S}(\cdot))$ (defined in Eqs. (33), (34)) for some $i \in \{1,2,3\}$; and for this k and this $\mathcal{S}_i$, $\phi(k;\mathcal{S}_i) \in \mathcal{R}_{\hat{k}}(\phi(\cdot,\mathcal{S}_i))$ satisfies the inequality

$$P_S(\mathcal{S}_i) + d(\mathcal{S}_i)\phi(k;\mathcal{S}_i) < 0. \tag{39}$$

■

It is possible, though tedious and not very informative, to translate the conditions of Theorem 3 into more explicit conditions for the linear stabilizability of (1). This would involve explicit limits on the allowed norm of the control gain vector under which the nominally stable modes would be guaranteed to remain stable for the closed-loop system. Obtaining such upper bounds from, say, Liapunov matrix equations would be feasible but would lead to conservative results. Given such bounds, one would simply check if any of the gain vectors of Theorem 3 which result in appropriate values of the stability coefficients satisfy the bounds. We do not present calculations on this issue here, since they are intricate and are not of primary importance in this work. We note, however, that one approach to obtaining the bounds is to change coordinates of the linearized system so as to separate the uncontrollable critical mode from the remaining modes. Also, note that it may be best to obtain such bounds using extensive numerical search.

4.3 Construction of Stabilizing Linear Feedbacks

Next, we combine the results of the previous two subsections to result in a procedure for the construction of a stabilizing linear feedback $u = kx$ which achieves a *prescribed value* $\beta^*_{2_S}(k) = \beta$. For brevity, only case (a) of the statement of Theorem 2 is addressed. Construction procedures for other cases follow similarly.

Our strategy for constructing stabilizing linear feedback is as follows. First, the family of linear feedback gain vectors k achieving a prescribed value assignment $\beta^*_{2_S} = \beta < 0$ (with $\beta^*_{1_S} = 0$ fulfilled) is generated via a set of linear algebraic equations. Then, those k that preserve the stability of the open-loop stable modes can be selected from the obtained family. If for the current choice of $\beta < 0$ there exists no such solution k, decrease the modulus $|\beta|$ and restart the procedure.

Given a desired value $\beta < 0$ for $\beta^*_{2_S}$, the problem is equivalent to finding those k which solve two equations: $\mathcal{S}(k) = \mathcal{S}_i$ and

$$\phi(k;\mathcal{S}_i) = \frac{\beta - P_S(\mathcal{S}_i)}{d(\mathcal{S}_i)} =: \phi_{\mathcal{S}_i,\beta}. \tag{40}$$

(Note that $d(\mathcal{S}_i) \neq 0$, do to the fact that $\Delta > 0$, which holds because case (a) of Theorem 2 is in force.)

Applying Lemma B.1, we obtain the linear system

$$k\Psi_S = [\mathcal{S}_i, \phi_{\mathcal{S}_i,\beta}] \tag{41}$$

where the coefficient matrix Ψ_S is given by

$$\Psi_S = \Psi_S(\mathcal{S}_i, \beta) := [r - \mathcal{S}_i A^- b, A^-(B^*(r^*, r^*) - \phi_{\mathcal{S}_i,\beta} b)]. \tag{42}$$

Note that rank $[\Psi_S(\mathcal{S}_i, \beta)] = 2$ for any $\mathcal{S}_i$ and β such that $B^*(r^*, r^*) \neq \phi_{\mathcal{S}_i,\beta} b$, since $lr = 1$ whereas $lA^- x = 0$ for any $x \in E^s$. Also, note that if it happens that $P_S(\mathcal{S}_i) < 0$, then (39) holds with $\phi(k; \mathcal{S}_i) = 0$, giving a linear equation to be solved along with (31).

Procedure S below summarizes the main steps in constructing a stabilizing linear feedback under hypothesis (S) assuming the situation specified in case (a) of the statement of Theorem 2.

Procedure S (Construction of stabilizing linear feedbacks in Case (S))

Step 1. Solve Eq. (31) for $\mathcal{S}_1$ and $\mathcal{S}_2$, and compute v_1, v_2, w_1, w_2, w_3, $B^*(r^*, r^*)$, $P_S(\mathcal{S}_i)$ and $d(\mathcal{S}_i)$ for $i = 1$ and 2. Select any $\beta < 0$.

Step 2. Compute $\phi_{\mathcal{S}_i,\beta}$ for $i = 1$ and 2.

Step 3. Solve Eq. (41) for k, first for $i = 1$ then for $i = 2$. If one of these solutions k satisfies $\|k\| \leq \hat{k}$, then this k gives a stabilizing linear feedback $u = kx$. Otherwise, select another $\beta < 0$ of smaller magnitude, and go to Step 2. ■

5 Linear Feedback Stabilization in the Case of One Pair of Imaginary Eigenvalues

In this section, results on linear feedback stabilizability of the origin of (1) in Case (H) are given. The development entails an analysis of the assignability of $\beta_{2_H}^*$ and implications for linear stabilizability. A procedure for the synthesis of stabilizing linear feedback laws which achieve a prescribed value $\beta < 0$ of $\beta_{2_H}^*$ is also given. The analysis proceeds in a fashion similar to that for Case (S) in Section 4. However, in Case (H) there is no equality condition analogous to the condition $\beta_{1_S}^* = 0$ in Case (S).

5.1 Stability Coefficient Assignability

A formula giving the closed-loop stability coefficient $\beta_{2_H}^*$ in terms of the linear state feedback vector k is obtained in Appendix A. This formula is:

$$\beta_{2_H}^* = P_H(\mathcal{H}) + d_1(\mathcal{H})\phi_1(k; \mathcal{H}) + \mathrm{Re}\{d_2(\mathcal{H})\phi_2(k; \mathcal{H})\} \tag{43}$$

with $\mathcal{H} = \mathcal{H}(k)$ the complex fraction given by Eq. (27), $P_H(\mathcal{H})$ a real-valued cubic polynomial in $\mathcal{H}$, $d_1(\mathcal{H})$ a real coefficient, $d_2(\mathcal{H})$ a complex coefficient, $\phi_1(k; \mathcal{H})$ a real fraction, and

$\phi_2(k;\mathcal{H})$ a complex fraction. These are given by

$$\begin{aligned}
P_H(\mathcal{H}) &:= \beta_{2_H} + 2\mathrm{Re}\{\mathcal{H}(lw_1) + \bar{\mathcal{H}}(lw_2) + \mathcal{H}^2(lw_3) + \mathcal{H}\bar{\mathcal{H}}(lw_4) + \mathcal{H}^2\bar{\mathcal{H}}(lw_5)\}, && (44)\\
d_1(\mathcal{H}) &:= \mathrm{Re}\{(lv_{01}) + \mathcal{H}(lv_{11})\}, && (45)\\
d_2(\mathcal{H}) &:= (lv_{02}) + \bar{\mathcal{H}}(lv_{12}), && (46)\\
\phi_1(k;\mathcal{H}) &:= \mathcal{F}(k; A^{-1}B^*(r^*,\bar{r}^*), A^{-1}b), \quad \text{and} && (47)\\
\phi_2(k;\mathcal{H}) &:= \mathcal{F}(k; (A-2i\omega_c I)^{-1}B^*(r^*,r^*), (A-2i\omega_c I)^{-1}b), && (48)
\end{aligned}$$

respectively; See Appendix A for the values of coefficient vectors w_i, $i = 1,\ldots,5$ and v_{01}, v_{02}, v_{02}, v_{12} as well as the vectors $B^*(r^*,\bar{r}^*)$ and $B^*(r^*,r^*)$. Note that although $\mathcal{H}$ does not appear explicitly in the formulae (47) and (48) for $\phi_1(k;\mathcal{H})$ and $\phi_2(k;\mathcal{H})$, it affects these quantities through the expressions for $B^*(r^*,\bar{r}^*)$ and $B^*(r^*,r^*)$, respectively. Compare with Eq. (38) for Case (S).

Apparently, the fraction $\mathcal{H}$ plays a role similar to that of $\mathcal{S}$ in Case (S), and analogies occur between P_H and P_S; between $(d_1(\mathcal{H}), d_2(\mathcal{H}))$ and $d(\mathcal{S})$; and between $(\phi_1(k;\mathcal{H}), \phi_2(k;\mathcal{H}))$ and $\phi(k;\mathcal{S})$. However, because of the absence of a necessary stabilizability condition analogous to $\beta^*_{1_S} = 0$, $\mathcal{H}$ is a free variable. In Case (S), the variable $\mathcal{S}$ was subject to the requirement $\mathcal{S} = \mathcal{S}_i$. The difference between Eq. (35) for $\beta^*_{2_S}(k)$ and Eq. (43) for $\beta^*_{2_H}(k)$ results in added freedom in achieving value assignments $\beta^*_{2_H} = \beta$. Indeed, since $\mathcal{H}$ need not be fixed, $\beta^*_{2_H}$ can be modified via three 'channels,' namely $\mathcal{H}$, $\phi_1(k;\mathcal{H})$ and $\phi_2(k;\mathcal{H})$. In parallel to Theorems 3 and 4, we have

Theorem 4. ($\beta^*_{2_H}$ Assignability) Let (1) satisfy hypothesis (H) and suppose $lb = 0$. The function $\beta^*_{2_H}(\cdot)$ is onto if lw_i, $i = 1,2,3,5$, as well as lv_{01}, lv_{02}, lv_{11} and lv_{12} do not vanish simultaneously. If $lw_i = 0$ for $i = 1,2,3,5$ and $lv_{01} = lv_{11} = lv_{02} = lv_{12} = 0$, then

$$\beta^*_{2_H}(k) = \beta_{2_H} + |\mathcal{H}(k)|^2\mathrm{Re}\{lw_4\} \tag{49}$$

which has a global maximum (resp., global minimum) β_{2_H} if $\mathrm{Re}\{lw_4\} \leq 0$ (resp., if $\mathrm{Re}\{lw_4\} \geq 0$). ■

Theorem 5 below is analogous to Theorem 3 of Section 4. The theorem statement uses the notation $\hat{k}$ introduced at the end of Section 3, the notation $\mathcal{R}_{\hat{k}}(F(\cdot))$ introduced at the end of Section 1, the notation and the quantities $\mathcal{H}(k)$, $\phi_1(k;\mathcal{H})$ and $\phi_2(k;\mathcal{H})$ given by (27), (47) and (48), respectively.

Theorem 5. (Sufficient Condition for Linear Stabilizability in Case (H)) Let (1) satisfy hypothesis (H) and suppose $lb = 0$. Then the origin of Eq. (1) is stabilizable by linear state feedback if there is a k with $\|k\| \leq \hat{k}$ that satisfies $kA^{-1}b \neq -1$ and $k(A - 2i\omega_c I)^{-1}b \neq -1$; and with $\mathcal{H} := \mathcal{H}(k)$, we have $\mathcal{H} \in \mathcal{R}_{\hat{k}}(\mathcal{H}(\cdot))$ and that $\phi_1(k;\mathcal{H}) \in \mathcal{R}_{\hat{k}}(\phi_1(\cdot))$ and $\phi_2(k;\mathcal{H}) \in \mathcal{R}_{\hat{k}}(\phi_2(\cdot))$ satisfy the inequality

$$P_H(\mathcal{H}) + d_1(\mathcal{H})\phi_1(k;\mathcal{H}) + \mathrm{Re}\{d_2(\mathcal{H})\phi_2(k;\mathcal{H})\} < 0. \tag{50}$$

■

5.2 Construction of Stabilizing Linear Feedbacks

In this subsection, which parallels Section 4.3, a procedure is given for generating stabilizing linear feedback laws $u(x) = kx$ for (1).

An equation similar to (41) in Case (S) is sought to characterize the set of vectors k which achieve the value assignment $\beta^*_{2_H} = \beta$. For brevity, as in Case (S), only the simplest (nondegenerate) case is considered, namely that in which $d_1(\mathcal{H})$ and $d_2(\mathcal{H})$ are nonvanishing functions of $\mathcal{H}$. That is, we assume that at least one of the coefficients appearing in each of Eqs. (45) and (46) does not vanish. Consider the assignment equation

$$P_H(\mathcal{H}(k)) + d_1(\mathcal{H}(k))\phi_1(k;\mathcal{H}(k)) + \mathrm{Re}\{d_2(\mathcal{H}(k))\phi_2(k;\mathcal{H}(k))\} = \beta. \tag{51}$$

By the reasoning preceding Theorem 4, first we fix an $\mathcal{H} \in \mathbb{C}$ such that $|d_1(\mathcal{H})| + |d_2(\mathcal{H})| \neq 0$ so that $\beta^*_{2_H}$ can be modified also by either $\phi_1(k;\mathcal{H}(k))$ or $\phi_2(k;\mathcal{H}(k))$. Thus, $P_H(\mathcal{H})$ becomes a fixed constant, and $\phi_1(k;\mathcal{H}(k))$ and $\phi_2(k,\mathcal{H}(k))$ reduce to $\phi_1(k;\mathcal{H})$ and $\phi_2(k;\mathcal{H})$, respectively. Now, Eq. (51) can be treated as a linear equation in the two unknowns $\phi_1 := \phi_1(k;\mathcal{H}) \in \mathbb{R}$ and $\phi_2 := \phi_2(k;\mathcal{H}) \in \mathbb{C}$:

$$d_1(\mathcal{H})\phi_1 + \mathrm{Re}\{d_2(\mathcal{H})\phi_2\} = \beta - P_H(\mathcal{H}). \tag{52}$$

It follows that (51) now consists of three equations in k, namely $\mathcal{H}(k) = \mathcal{H}$, $\phi_1(k;\mathcal{H}) = \phi_1$ and $\phi_2(k;\mathcal{H}) = \phi_2$ (with (ϕ_1,ϕ_2) any solution pair of Eq. (52)), each involving expressions of the form of the fraction introduced in (24). Finally, we use Lemma B.2 (including the $(\cdot)'$-notation defined in Eq. (B.4) for complex fractions $\mathcal{H}(k) = \mathcal{H}$ and $\phi_2(k;\mathcal{H}) = \phi_2$) to obtain the system of *real* linear equations (see Eq. (B.5)):

$$k\Psi_H = [\mathrm{Re}\{\mathcal{H}\}, \mathrm{Im}\{\mathcal{H}\}; \phi_1; \mathrm{Re}\{\phi_2\}, \mathrm{Im}\{\phi_2\}] \tag{53}$$

where the coefficient matrix Ψ_H is given by

$$\begin{aligned}\Psi_H &= \Psi_H(\mathcal{H};\phi_1;\phi_2)\\ &:= [(r)', ((A - i\omega_c I)^{-}b)'; A^{-1}(B^*(r^*,\bar{r}^*) - \phi_1 b);\\ &\qquad ((A - 2i\omega_c I)^{-1}B^*(r^*,r^*))', ((A - 2i\omega_c I)^{-1}b)'].\end{aligned} \tag{54}$$

Procedure H below is analogous to Procedure S given at the end of Section 4. It summarizes the main steps in constructing a stabilizing linear feedback under hypothesis (H). The simplest situation addressed in the statement of Theorem 4 is assumed, namely that in which not all of the quantities lw_i, $i = 1,2,3,5$ and lv_{01}, lv_{02}, lv_{11}, lv_{12} vanish simultaneously.

Procedure H (Construction of stabilizing linear feedbacks in Case (H))

Step 1. Compute v_{01}, v_{02}, v_{11}, v_{12} and lw_i for $i = 1,\ldots,5$. Select any $\beta < 0$.

Step 2. Select $\mathcal{H} \in \mathbb{C}$ such that $d_1(\mathcal{H})$ and$d_2(\mathcal{H})$ do not both vanish. Compute $P_H(\mathcal{H})$ using Eq. (44). Select a solution pair $\phi_1 \in \mathbb{R}$, $\phi_2 \in \mathbb{C}$ to Eq. (52).

Step 3. Solve Eq. (53) for k. If a solution k satisfies $\|k\| \leq \hat{k}$, then this k gives a stabilizing linear feedback $u = kx$. Otherwise, select another $\beta < 0$ of smaller magnitude, and go to Step 2. ■

6 Conclusions

Linear feedback stabilizability and stabilization of nonlinear systems with uncontrollable critical modes have been studied. The paper focused on the two basic critical cases, those in which the system linearization at the origin possesses either a simple zero eigenvalue or a pair of simple pure imaginary eigenvalues. Construction procedures facilitating synthesis of stabilizing linear feedback laws were also given. In general, the conditions obtained combine sufficient conditions for assignability of certain associated stability coefficients with a requirement that open-loop stable eigenvalues not be destabilized. The critical system is generically stabilizable by a linear static state feedback $u = kx$, under the condition that the open-loop stable eigenvalues have a sufficiently large margin of stability. In the case of systems (1) for which the control u enters the vector field $f(x,u)$ linearly, the results presented here readily simplify, since destabilization of stable modes is no longer a possibility. The specialized results for these so-called *linear-analytic* systems were not given in this paper, in the interest of brevity. An interesting topic for further research concerns the trade-off between between linear and nonlinear feedback in the local stabilization of systems for which the critical eigenvalues are nearly uncontrollable.

Appendix A. Closed-Loop System Stability Coefficients

Here, we compute the stability coefficients for the closed-loop system (13) under hypotheses (S) and (H). These are $\beta^*_{1_S}$ and $\beta^*_{2_S}$ in Case (S), and β^*_{2H} in Case (H). To employ the general expressions for stability coefficients given in Facts 1 and 2, it is necessary that the bilinear and trilinear forms in the closed-loop dynamics (13) retain the symmetry properties of the analogous terms in the open-loop dynamics (4). To achieve this, we employ the symmetrization operation, as in (Abed and Fu, 1986), (Abed and Fu, 1987).

For the quadratic and cubic forms $B^*(x,x)$ and $C^*(x,x,x)$, respectively, of Eqs. (15) and (16), the associated bilinear and trilinear forms are defined as follows:

$$B^*(x,y) := B(x,y) + \tfrac{1}{2}[(kx)A_1y + (ky)A_1x] + b_2(kx)(ky), \tag{A.1}$$

$$\begin{aligned} C^*(x,y,z) := {} & C(x,y,z) + \tfrac{1}{3}[(kx)B_1(y,z) + (ky)B_1(z,x) + (kz)B_1(x,y)] \\ & + \tfrac{1}{3}[(kx)(ky)A_2z + (ky)(kz)A_2x + (kz)(kx)A_2y] + b_3(kx)(ky)(kz). \end{aligned} \tag{A.2}$$

We can now proceed to the calculations associated with obtaining the expressions for the stability coefficients $\beta^*_{1_S}$, $\beta^*_{2_S}$ occurring in the discussion of Case (S). To begin, we compute the coefficient vectors v_1 and v_2 first appearing in the text in Eq. (30). Letting $x := r^*$, with r^* as given in (25) (Case (S)), in Eq. (15) yields Eq. (30), with the coefficient vectors v_1 and v_2 given by

$$v_1 := -2B(r, A^-b) + A_1r, \tag{A.3}$$

$$v_2 := B(A^-b, A^-b) - A_1A^-b + b_2, \tag{A.4}$$

respectively. Now using this same substitution $x := r^*$ in Eq. (16) yields $C^*(r^*,r^*,r^*)$, a quantity required in applying Eq. (17b). The vector $A^{*-}B^*(r^*,r^*)$ which occurs in Eq. (17b) is easily obtained using Eqs. (22) and (30). Substituting this value and that of r^* into (A.1) gives the first term in (17b). Thus, $\beta^*_{2_S}$ is given by (35), along with the notation (36)-(38), where

$$w_0 := -2B(r, A^-B(r,r)) + C(r,r,r), \tag{A.5}$$

$$w_1 := B_1(r,r) - A_1A^-B(r,r) - 2B(r,A^-b) - 3C(r,r,A^-b), \tag{A.6}$$

$$w_2 := A_2 r - 2B_1(r, A^- B) + 3C(r, r, A^- b), \quad \text{and} \tag{A.7}$$
$$w_3 := b_3 - C(A^- b, A^- b, A^- b). \tag{A.8}$$

Next we obtain the closed-loop stability coefficient $\beta^*_{2_H}$ which is defined in Case (H). First, using r^* from Eq. (25) (Case (H)), substitute $x := r^*$ and $y := \bar{r}^*$ into Eq. (A.1), and substitute $x := r^*$ into (15), to obtain

$$B^*(r^*, \bar{r}^*) = B(r, \bar{r}) + \mathcal{H} v_1 + \bar{\mathcal{H}} \bar{v}_1 + \mathcal{H}\bar{\mathcal{H}} v_2, \tag{A.9}$$
$$B^*(r^*, r^*) = B(r, r) + \mathcal{H} v_3 + \mathcal{H}^2 v_4. \tag{A.10}$$

To state the values of the vectors v_1, v_2, v_3, v_4 appearing in (A.9) and (A.10), denote

$$\xi := (A - i\omega_c I)^- b. \tag{A.11}$$

Then the v_i, $i = 1, \ldots, 4$ are given by

$$v_1 := -B(\bar{r}, \xi) + \tfrac{1}{2} A_1 \bar{r}, \tag{A.12}$$
$$v_2 := B(\xi, \bar{\xi}) - \tfrac{1}{2} A_1(\xi + \bar{\xi}) + b_2, \tag{A.13}$$
$$v_3 := -2B(r, \xi) + A_1 r, \quad \text{and} \tag{A.14}$$
$$v_4 := B(\xi, \xi) - A_1 \xi + b_2. \tag{A.15}$$

Next, substitute $x := r^*$, $y := r^*$ and $z := \bar{r}^*$ into Eq. (A.2) to obtain $C^*(r^*, r^*, \bar{r}^*)$. Then, premultiply the quantities $B^*(r^*, \bar{r}^*)$ and $B^*(r^*, r^*)$ by A^{*-1} and $(A^* - 2i\omega_c)^{-1}$, respectively. (These matrix inverses are given in Eqs. (28) and (29), respectively.) Substitute the obtained products along with r^* and $\bar{r}^*$ in place of the proper vector arguments in (A.1) to obtain the first two terms in the expression (18) for $\beta^*_{2_H}$. The final expression (43) follows by adding these two vectors to $C^*(r^*, r^*, \bar{r}^*)$, obtained earlier. The notation used in Eqs. (44)-(48) (needed in the expression (43)) is as follows:

$$v_{01} := -2B(r, A^{-1} b) + A_1 r, \tag{A.16}$$
$$v_{11} := 2B(z, A^{-1} b) - A_1(z + A^{-1} b) + 2b_2, \tag{A.17}$$
$$v_{02} := -B(\bar{r}, A^{-1} b) + \tfrac{1}{2} A_1 \bar{r}, \tag{A.18}$$
$$v_{12} := B(\bar{z}, A^{-1} b) - \tfrac{1}{2} A_1(z + A^{-1} b) + b_2, \tag{A.19}$$
$$\begin{aligned} w_1 := {} & -B(r, A^{-1} v_1) - 2B(z, B(r, \bar{r})) - \tfrac{1}{2} B(\bar{r}, (A - 2i\omega_c I)^{-1} v_2) \\ & -\tfrac{3}{2} C(r, \bar{r}, z) + A_1 a + \tfrac{1}{2} B_1(r, \bar{r}), \end{aligned} \tag{A.20}$$
$$\begin{aligned} w_2 := {} & -B(r, A^{-1} \bar{v}_1) - B(\bar{z}, b) \\ & -\tfrac{3}{4} C(r, r, \bar{z}) + \tfrac{1}{2} A_1 b + \tfrac{1}{4} B_1(r, r), \end{aligned} \tag{A.21}$$
$$\begin{aligned} w_3 := {} & B(z, A^{-1} v_1) - \tfrac{1}{2} B(\bar{r}, (A - 2i\omega_c I)^{-1} v_4) \\ & +\tfrac{3}{4} C(\bar{r}, z, z) - \tfrac{1}{2} A_1 A^{-1} v_1 - \tfrac{1}{2} B_1(\bar{r}, z) + \tfrac{1}{4} A_2 \bar{r}, \end{aligned} \tag{A.22}$$
$$\begin{aligned} w_4 := {} & -B(r, A^{-1} v_2) + \tfrac{1}{2} B(z, A^{-1} \bar{v}_1) + \tfrac{3}{2} C(r, z, \bar{z}) \\ & -\tfrac{1}{2} A_1 A^{-1} \bar{v}_1 - \tfrac{1}{4} A_1 (A - 2i\omega_c I)^{-1} v_2 - \tfrac{1}{2} B_1(r, \bar{z}) + \tfrac{1}{2} A_2 r, \quad \text{and} \end{aligned} \tag{A.23}$$
$$\begin{aligned} w_5 := {} & B(z, A^{-1} v_2) + \tfrac{1}{2} B(\bar{z}, (A - 2i\omega_c)^{-1} v_4) \\ & +\tfrac{3}{4} C(z, z, \bar{z}) - \tfrac{1}{2} A_1 A^{-1} v_2 - \tfrac{1}{4} A_1 (A - 2i\omega_c I)^{-1} v_4 \\ & +\tfrac{1}{4} B_1(z, z + 2\bar{z}) - \tfrac{1}{4} A_2(2z + \bar{z}) + \tfrac{3}{4} b_3. \end{aligned} \tag{A.24}$$

Appendix B. Solution Sets of Certain Rational Equations

In this Appendix, we consider the solution of systems of equations in which certain rational functions of the unknown vectors appear. Each rational function is a ratio of linear functions of the unknown vectors. The systems of equations are related to systems of linear algebraic equations, and the structure of the solution set is studied.

Consider the following special class of functions F given by the rational functions of a real vector k, which are parametrized by real vectors u and v:

$$\mathcal{F}(k;u,v) := \frac{ku}{1+kv}. \tag{B.1}$$

The range of $\mathcal{F}(k;u,v)$, and the set of solutions k to a system $\mathcal{F}(k;u,v) = \theta$ ($\theta \in \mathbb{R}$), are important in the stabilizability analysis and stabilization procedures in the paper.

First, we note that for any θ the equation $\mathcal{F}(k;u,v) = \theta$ is equivalent to the linear equation $k(u-\theta v) = \theta$. Hence, for any θ such that $u-\theta v \neq 0$, the solution set of $\mathcal{F}(k;u,v) = \theta$ coincides with that of the linear equation $k(u-\theta v) = \theta$. (Note: thus, if u, v are linearly independent, the equation $\mathcal{F}(k;u,v) = \theta$ is solvable for any θ; if $u = \alpha v$ for some $\alpha \neq \theta$, then $u - \theta v = (\alpha - \theta)v \neq 0$ and the rational equation is solvable for any $\theta \neq \alpha$.) Applying this idea to a system of equations each in the form of (B.1): $\mathcal{F}_i(k;u,v) = \theta_i$, $i = 1, \dots, n$, we have

Lemma B.1. The solution set of the simultaneous equations $\mathcal{F}_i(k;u_i,v_i) = \theta_i$, $i = 1, \dots, n$ defined in the form of (B.1) via a set of vectors $\{u_1, v_1, \dots, u_m, v_m\}$, coincides with the solution set of the linear system

$$k[u_1 - \theta_1 v_1, \dots, u_m - \theta_m v_m] = [\theta_1, \dots, \theta_m]. \tag{B.2}$$

■

Note that if $kv_i = -1$ for some solution k of (B.2) for some i, then $ku_i = 0$ necessarily. Also note that (B.2) is guaranteed solvable for any $\theta_1, \dots, \theta_m$ if the set $\{u_1 - \theta_1 v_1, \dots, u_m - \theta_m v_m\}$ is linearly independent.

The conclusion above, in particular whether or not a single rational function of type $\mathcal{F}(\cdot;u,v)$ is onto, is in doubt if an additional constraint is imposed on k, such as $|k| \leq \hat{k}$.

Direct application of Lemma B.1 in Case (H) is not possible, due to the presence of complex quantities. Therefore some modification in the statement of Lemma B.1 is in order. Consider Eq. (B.1), with the function $\mathcal{F}(k;u,v)$ now defined for vectors $u, v \in \mathbb{C}^n$. For brevity, denote by $[\cdot]^{\mathrm{r}}$ and $[\cdot]^{\mathrm{i}}$ the real and imaginary parts of a vector $[\cdot]$. Rewrite $\mathcal{F}(k;u,v)$ as:

$$\mathcal{F}(k;u,v) = \frac{(ku^{\mathrm{r}}) + i(ku^{\mathrm{i}})}{(1 + kv^{\mathrm{r}}) + i(kv^{\mathrm{i}})}. \tag{B.3}$$

Let

$$\begin{cases} (u)' := u^{\mathrm{r}} - \mathrm{Re}\{\theta\}v^{\mathrm{r}} + \mathrm{Im}\{\theta\}v^{\mathrm{i}}, \\ (v)' := u^{\mathrm{i}} - \mathrm{Re}\{\theta\}v^{\mathrm{i}} - \mathrm{Im}\{\theta\}v^{\mathrm{r}}. \end{cases} \tag{B.4}$$

It can be shown that for any $\theta \in \mathbb{C}$ for which the set $\{(u)', (v)'\}$ is linearly independent, or $\mathrm{Im}\{\theta\}(u)' = \mathrm{Re}\{\theta\}(v)'$, the equation $\mathcal{F}(k;u,v) = \theta$ has solutions coinciding with the solutions of the system of linear equations $k[(u)', (v)'] = [\mathrm{Re}\{\theta\}, \mathrm{Im}\{\theta\}]$.

The same idea applies to a set of rational equations $\mathcal{F}_i(k; u_i, v_i) = \theta_i$, $i = 1, \ldots, m \leq n/2$, all in the form of (B.1) defined via vectors u_i, v_i. That is, to consider each equation $\mathcal{F}_i(k; u_i, v_i) = \theta_i$ as in (B.3) and to define $(u_i)'$ and $(v_i)'$ as in (B.4). We thus have

Lemma B.2. The solution set of the rational system $\mathcal{F}_i(k; u_i, v_i) = \theta_i \in \mathbb{C}$, $i = 1, \ldots, m \leq n/2$ coincides with the solution set of the linear system

$$k[(u_1)', (v_1)'; \ldots; (u_m)', (v_m)'] = [\mathrm{Re}\{\theta_1\}, \mathrm{Im}\{\theta_1\}; \ldots; \mathrm{Re}\{\theta_m\}, \mathrm{Im}\{\theta_m\}]. \tag{B.5}$$

■

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