

NONLINEAR FILTERING WITH HOMOGENIZATION

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Abstract: The problem of nonlinear filtering is studied for class of diffusions whose statistics depend periodically on the process state and a small parameter ϵ . The problem exhibits a homogenization property as $\epsilon \downarrow 0$.

Key Words: Nonlinear filtering, homogenization, asymptotic analysis.

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1. Introduction and Summary

Large scale dynamical systems encountered in engineering practice often have subsystems which are substantially homogeneous. For example, electric power systems are generally composed of several operating areas. Within each area there may be many generating units of essentially the same type and size. It is natural to base control or state estimation algorithms for the subsystems on aggregate variables, e.g., the “system frequency” in power systems [1][2]. Similarly, in econometric systems, especially in large cartels containing many identical firms, it is natural to try to estimate or control the performance of a “typical” firm based on measurements of the aggregate dynamics of the cartel [3]. The dynamics of interconnected networks of neural-like elements have also been described in this way [4]-[6]. In most cases individual elements in the larger network may be subject to local, random fluctuations and the network interconnections may be random (e.g., the models in [4]-[6]). The thermal and mechanical properties of large lattice structures may also be usefully treated from this perspective [7][8].

In this paper we shall give a detailed treatment of the filtering problem for the system

$$\begin{aligned} dx^\epsilon(t) &= g\left[\frac{x^\epsilon(t)}{\epsilon}\right]x^\epsilon(t)dt + \sigma\left[\frac{x^\epsilon(t)}{\epsilon}\right]dw(t) \\ dz^\epsilon(t) &= h\left[\frac{x^\epsilon(t)}{\epsilon}\right]x^\epsilon(t)dt + dv(t) \\ x^\epsilon(0) &= \xi, \quad z^\epsilon(0) = 0, \quad 0 \leq t \leq T, \quad \epsilon > 0 \end{aligned} \tag{1.1}$$

which may be regarded as a prototype of stochastic dynamical systems with many components. Here ξ is an \mathbb{R}^n - valued (Gaussian) random variable, g , σ , and h are periodic on the (unit) torus in \mathbb{R}^n , and $w(t)$ and $v(t)$ are independent, standard vector-valued Wiener processes which are independent of ξ . The filtering problem for (1.1) is to estimate $x^\epsilon(t)$, i.e., compute its conditional density, given $Z_t^\epsilon = \sigma\{z^\epsilon(s), 0 \leq s \leq t\}$, the σ -algebra of observations. We are interested in the behavior of this filtering problem in

the limit as $\epsilon \downarrow 0$.¹

The vector $x^\epsilon(t)$ may be regarded as the composite state of the overall system formed from the lexicographical listing of the states of each of the components of the system. The periodicity of g , σ , and h is a regularity property of the array; and the small parameter ϵ represents a natural, non-dimensional “distance” or “coupling” variable characterizing component interactions. In a subsequent paragraph we shall describe a formal analysis of a prototype system of this type.

One would expect the system (1.1) to be well approximated as $\epsilon \downarrow 0$ by a similar system with $g(x/\epsilon)$, $\sigma(x/\epsilon)$, and $h(x/\epsilon)$ replaced by their averages \bar{g} , $\bar{\sigma}$, and \bar{h} over the torus. This is the case, although the precise nature of the average is difficult to guess from a cursory inspection of (1.1). The filtering problem for the limiting system is just the Kalman-Bucy filtering problem which has a simple, closed form solution. By constructing an asymptotic expansion for the conditional density of $x^\epsilon(t)$ given Z_t^ϵ , we can obtain a family of finite dimensional linear filters which provide increasingly accurate, e.g., $O(\epsilon)$, $O(\epsilon^2)$, ..., etc., approximations of the conditional density of $x^\epsilon(t)$ based on the Kalman estimator. The technique used to derive the result is “homogenization” of a linear stochastic partial differential equation for the (un-normalized) conditional density of $x^\epsilon(t)$ given Z_t^ϵ . The theory of homogenization, which has been widely used in physics and applied mathematics, is described in [10][11].

While the system (1.1) is obviously only an example of a larger class of problems, we shall see that its analysis has all the essential difficulties of more general problems. Before starting the analysis it is useful to describe how a problem like (1.1) might arise in “practice”.

Consider the prototype system:

$$dx_i(t) = a[x_i(t), u(t)]dt + \frac{1}{N} \sum_{j=1}^N b[x_j(t)]dw_{ij}(t) \quad (1.2)$$

$$i = 1, 2, \dots, N, \quad t \geq 0$$

¹A version of this problem was first discussed in [9].

Here a and b are smooth functions of their (vector-valued) arguments, w_{ij} and w_{kl} are standard (vector) Wiener processes which are independent for $(i, j) \neq (k, l)$ and $u(t)$ is a vector of control variables. The functions a and b are the same for all the subsystems - so the overall system with state $x(t) = [x_1(t), \dots, x_N(t)]^T$ has a homogeneous structure. The coupling is random and normalized by $1/N$ to reflect the assumption that each subsystem has $O(1)$ coupling to the remainder of the system (as opposed to $O(N)$, $O(1/N)$, etc.), no matter how large the latter is.

Associated with (1.2), we define

$$\begin{aligned} S(t) &= \sum_{i=1}^N x_i(t) = \text{“the aggregate output”} \\ \sigma(t) &= \frac{1}{N} \sum_{i=1}^N x_i(t) = \text{“the average output”} \end{aligned} \quad (1.3)$$

Suppose that in the process of controlling the system, we observe not $x(t)$, but the aggregate $S(t)$ through the measurement

$$dz(t) = h[S(t)]dt + dv(t) \quad (1.4)$$

with h smooth and $v(t)$ a standard Wiener process. Suppose further that the control $u(t)$ is defined by $u(t) = f[\hat{S}(t)]$ with $\hat{S}(t)$ an estimate of $S(t)$ derived from $z(s)$, $s \leq t$. We would like to analyze (1.2) - (1.4) in the limit as $N \rightarrow \infty$; and, more precisely, show that this analysis involves the asymptotic analysis of systems scaled like (1.1).

Defining $\delta x_i(t) = x_i(t) - \sigma(t)$, we have $\sum_{i=1}^N \delta x_i(t) = 0$. The aggregate output

$S(t)$ satisfies

$$\begin{aligned} dS(t) &= \sum_{i=1}^N a(x_i(t), u(t))dt + \frac{1}{N} \sum_{j=1}^N b(x_j(t)) \sum_{i=1}^N dw_{ij}(t) \\ &= Na(\sigma(t), u(t))dt + O(\delta |x_i(t)|^2)dt \\ &\quad + b(\sigma(t)) \frac{1}{N} \sum_{i,j=1}^N dw_{ij}(t) + b_x(\sigma(t)) \frac{1}{N} \sum_{i=1}^N \delta x_i(t) \sum_{j=1}^N dw_{ij}(t) \end{aligned} \quad (1.5)$$

$$+ O(|\delta x_i(t)|^2) d\tilde{w}(t)$$

where $\tilde{w}(t)$ is a vector Wiener process defined from the components of $w_{ij}(t)$. Neglecting $O(|\delta x_i(t)|^2)$ terms, we have

$$\begin{aligned} d\sigma(t) = & a(\sigma(t), u(t))dt + b(\sigma(t)) \frac{1}{N^2} \sum_{i,j=1}^N dw_{ij}(t) \\ & + b_x(\sigma(t)) \frac{1}{N} \sum_{i=1}^N \delta x_i(t) \frac{1}{N} \sum_{j=1}^N dw_{ij}(t) \end{aligned} \quad (1.6)$$

To treat the last term, we use the formal argument in [6] which goes as follows: As $N \rightarrow \infty$ a “local chaos” condition prevails in which each subsystem with state $\delta x_i(t)$ behaves “independently” of every other subsystem, and, in effect, of the noises $(\sum_{j=1}^N dw_{ij}(t)/N)$, $i=1, \dots, N$. That is, a law of large numbers applies to the last term as $N \rightarrow \infty$. Since

$$\frac{1}{N} \sum_{i=1}^N \delta x_i(t) = 0$$

by the definition of $\sigma(t)$, the last term in (1.6) is zero. (In a more general situation, this term would approach zero as $N \rightarrow \infty$.) Notice

$$\tilde{w}(t) = \sum_{i,j=1}^N w_{ij}(t)/N^2$$

in the second term behaves like a standard Wiener process for each N . Thus, for large N we obtain the approximate model

$$d\sigma(t) = a(\sigma(t), u(t))dt + b(\sigma(t))d\tilde{w}(t) \quad (1.7)$$

Now let $\hat{a}(\sigma, S) \triangleq a(\sigma, \tilde{S})$ and assume that S and \tilde{S} have the same order behavior in N for N large. Defining $\epsilon = 1/N$, we have two descriptions of the aggregate behavior of (1.2) for N large:

$$d\sigma(t) = \hat{a}(\sigma(t), \frac{1}{\epsilon}\sigma(t))dt + b(\sigma(t))d\tilde{w}(t)$$

$$dz(t) = h\left(\frac{1}{\epsilon}\sigma(t)\right)dt + dv(t) \quad (1.8a)$$

$$dS(t) = \frac{1}{\epsilon} \hat{a}(\epsilon S(t), S(t))dt + \frac{1}{\epsilon} b(\epsilon S(t))d\tilde{w}(t)$$

$$dz(t) = h(S(t))dt + dv(t) \quad (1.8b)$$

So to analyze the aggregate behavior of the original system (1.2) as $N \rightarrow \infty$, we can study (1.8a) or (1.8b) as $\epsilon \downarrow 0$. If \hat{a} , b , and h have a periodic or randomly recurrent dependence on their arguments, then the analysis of (1.8a,b) involves a homogenization problem.

The literature in mathematical physics and engineering contains many examples of systems scaled like (1.8) which can be effectively treated using homogenization theory. The structural mechanical and thermal properties of lattice structures may be treated in this way by deriving continuum models for the macroscopic behavior of lattices with a regular infrastructure [7][8]. The transport of liquid through a porous medium may frequently be analyzed by replacing the description of the porous, heterogeneous medium in the analysis with a model of a homogeneous medium whose transport parameters are systematic averages of the material properties of the original medium [13]. Similarly, in nuclear reactor designs it is important to be able to estimate the neutron production in a reactor core composed of a periodic array of fuel rods. Homogenization methods have been used to estimate the neutron transport properties of the medium by approximating periodic core structure with a homogeneous core from which the neutron population can be more easily computed [14]. It is possible that the filtering methods developed here could be adapted to play a role in verifying these models from measurements. Homogenization methods have not been developed in control theory, other than the limited results in [8][15] - [17].

Summary. In the next section we precisely state the filtering problem to be treated. In section 3 we introduce and analyze a duality form which is useful in representing the conditional density that is the essential element of the filtering problem. In section 4 we carry out the asymptotic analysis of the filtering problem by first homo-

genizing the deterministic duality expression (Proposition 4.1) using standard techniques, and then applying this to prove a limit theorem for the conditional density (Theorem 4.1). This theorem is our main result which constitutes homogenization of the filtering problem. In section 5 we prove three estimates which are required in the probabilistic and asymptotic analysis.

2. The Filtering Problem

Let (Ω, F, P) be a probability space on which are defined two independent Wiener processes $\tilde{w}(t)$ and $z(t)$ with values in \mathbb{R}^n and \mathbb{R}^d , respectively. Let ξ be a Gaussian random variable with values in \mathbb{R}^n which has mean x_0 and covariance P_0 . Suppose ξ is independent of $\tilde{w}(t)$ and $z(t)$. Let F^t , $t \geq 0$, be a family of σ -algebras with $F^\infty = F$, such that $\tilde{w}(t)$ and $z(t)$ are adapted to F^t and ξ is F^0 - measurable. Let $Z^t = \sigma\{z(s), s \leq t\}$. Let Y be the unit torus in \mathbb{R}^n and

$$\begin{aligned} g(y) &\in L(\mathbb{R}^n; \mathbb{R}^n) \\ \sigma(y) &\in L(\mathbb{R}^n; \mathbb{R}^n); \quad \text{invertible} \\ h(y) &\in L(\mathbb{R}^n; \mathbb{R}^d) \end{aligned} \tag{2.1}$$

which are defined on the torus Y , and which are sufficiently smooth there.

Let $x^\epsilon(t)$ be the solution of the Ito equation

$$\begin{aligned} dx^\epsilon(t) &= \sigma\left[\frac{x^\epsilon(t)}{\epsilon}\right] d\tilde{w}(t) \\ x^\epsilon(0) &= \xi, \quad 0 \leq t \leq T \end{aligned} \tag{2.2}$$

and note that $x^\epsilon(t)$ is independent of $z(t)$. Consider the processes

$$\begin{aligned} w^\epsilon(t) &= - \int_0^t (\sigma^{-1}g)\left(\frac{x^\epsilon}{\epsilon}\right) x^\epsilon ds + z(t) \\ v^\epsilon(t) &= - \int_0^t h\left(\frac{x^\epsilon}{\epsilon}\right) x^\epsilon ds + z(t) \end{aligned} \tag{2.3}$$

and

$$\mu^\epsilon(t) = \exp\left\{\int_0^t h\left(\frac{x^\epsilon}{\epsilon}\right) x^\epsilon dz + \int_0^t (\sigma^{-1}g)\left(\frac{x^\epsilon}{\epsilon}\right) x^\epsilon d\tilde{w}\right\}$$

$$- \frac{1}{2} \int_0^t \left| h\left(\frac{x^\epsilon}{\epsilon}\right) x^\epsilon \right|^2 ds - \frac{1}{2} \int_0^t \left| \sigma^{-1} g\left(\frac{x^\epsilon}{\epsilon}\right) x^\epsilon \right|^2 ds \} \quad (2.4)$$

For any finite T we have

$$E \mu^\epsilon(T) < \infty \quad E [\mu^\epsilon(T)]^2 < C \quad (2.5)$$

(See section 5.)

Because of (2.5) we can use the change of probability given by the Girsanov transformation

$$\frac{dP^\epsilon}{dP} \Big|_{F^T} = \mu^\epsilon(T) \quad (2.6)$$

Under the probability P^ϵ the processes $w^\epsilon(t)$ and $v^\epsilon(t)$ are independent standard Wiener processes. Since $w^\epsilon(t)$ and $v^\epsilon(t)$ are independent of F^0 under P^ϵ , ξ is independent of $w^\epsilon(t)$ and $v^\epsilon(t)$. Further, since $\{\mu^\epsilon(t), F^t\}$ is a martingale, ξ has the same distribution under P^ϵ as under P .

In the space $(\Omega, F, P^\epsilon, F^t)$ we can write

$$\begin{aligned} dx^\epsilon &= g\left(\frac{x^\epsilon}{\epsilon}\right) x^\epsilon dt + \sigma\left(\frac{x^\epsilon}{\epsilon}\right) dw^\epsilon \\ x^\epsilon(0) &= \xi \end{aligned} \quad (2.7)$$

$$dz^\epsilon = h\left(\frac{x^\epsilon}{\epsilon}\right) x^\epsilon dt + dv^\epsilon(t)$$

where w^ϵ and v^ϵ are standard F^t - Wiener processes which are mutually independent. Moreover, ξ is a F^0 - Gaussian random variable with mean x_0 and covariance matrix P_0 . The *filtering problem* associated with (2.7) consists in computing

$$\pi^\epsilon(t)(\psi) = E^\epsilon[\psi(x^\epsilon(t)) \mid Z^t] \quad (2.8)$$

for ψ any Borel bounded test function on \mathbb{R}^n . It is easy to check that

$$\begin{aligned} \pi^\epsilon(t)(\psi) &= \frac{E[\psi(x^\epsilon(t)) \mu^\epsilon(t) \mid Z^t]}{E[\mu^\epsilon(t) \mid Z^t]} \\ &\triangleq \frac{p^\epsilon(t)(\psi)}{p^\epsilon(t)(1)} \end{aligned} \quad (2.9)$$

where

$$p^\epsilon(t)(\psi) \triangleq E[\psi(x^\epsilon(t))\mu^\epsilon(t) | Z^t] \quad (2.10)$$

Our purpose here is to study the behavior of the conditional density $p^\epsilon(t)(\psi)$ as $\epsilon \rightarrow 0$.

3. A Duality Form and an Expression for the Conditional Density

By introducing a certain duality formula it is possible to obtain an expression for the conditional density which is convenient for the homogenization and convergence analysis.

Let β be a deterministic function in $L^\infty(0, T; \mathbb{R}^d)$ and

$$\rho(t) = e^{\{\int_0^t \beta \cdot dz - \frac{1}{2} \int_0^t |\beta|^2 ds\}} \quad (3.1)$$

It is known that $\forall T$, the set of random variables, $\{\rho(T)\}$, obtained by varying β in $L^\infty(0, T; \mathbb{R}^d)$ is dense in $L^2(\Omega, Z^T, P; \mathbb{R}^d)$.

Let ψ be a smooth, bounded function on \mathbb{R}^n and let $\beta(t)$ be a smooth, bounded deterministic function on $[0, T]$ with values in \mathbb{R}^d . We introduce the deterministic function $V^\epsilon(x, t)$ which is the solution of²

$$\begin{aligned} \frac{\partial V^\epsilon}{\partial t} + a_{ij}(\frac{x}{\epsilon}) \frac{\partial^2 V^\epsilon}{\partial x_i \partial x_j} + g_{ij}(\frac{x}{\epsilon}) x_j \frac{\partial V^\epsilon}{\partial x_i} \\ + V^\epsilon h_{ij}(\frac{x}{\epsilon}) x_j \beta_i(t) = 0 \end{aligned} \quad (3.2)$$

$$V^\epsilon(x, T) = \psi(x), \quad T \geq t \geq 0$$

Because the coefficients are smooth, (3.2) has a solution in $C^{2,1}(\mathbb{R}^n \times [0, T])$. Moreover, it satisfies the growth conditions (see section 5)

$$|V^\epsilon(x, t)| \leq C_\delta e^{\delta |x|^2}$$

²Here and in the following we use the convention that repeated indices are summed over their range.

$$|DV^\epsilon(x, t)| \leq C_{\delta, \epsilon} e^{2\delta|x|^2} \quad (3.3)$$

where $\delta > 0$ can be chosen arbitrarily small. Note that the first constant C_δ in (3.3) can be chosen independent of ϵ , but not δ .

Using the function $V^\epsilon(x, t)$, it is possible to obtain a convenient expression for $p^\epsilon(t)(\psi)$.

Proposition 3.1 *Under the assumptions (2.1) we have, for any β , the equality*

$$E[p^\epsilon(T)(\psi)\rho(T)] = E[V^\epsilon(\xi, 0)] = \int_{\mathbb{R}^n} V^\epsilon(x, 0)\pi_0(x)dx \quad (3.4)$$

where

$$\pi_0(x) \triangleq \frac{1}{[(2\pi)^n \det P_0]^{1/2}} e^{-\frac{1}{2}(x-x_0)^T P_0^{-1}(x-x_0)} \quad (3.5)$$

Proof. From (2.10) we have

$$\begin{aligned} E[p^\epsilon(T)(\psi)\beta(T)] &= E[\psi(x^\epsilon(T))\mu^\epsilon(T)\beta(T)] \\ &= E[V^\epsilon(x^\epsilon(T), T)\mu^\epsilon(T)\beta(T)]. \end{aligned} \quad (3.6)$$

But

$$dV^\epsilon(x^\epsilon(t), t) = \left(\frac{\partial V^\epsilon}{\partial t} + a_{ij} \frac{\partial^2 V^\epsilon}{\partial x_i \partial x_j} \right) dt + \frac{\partial V^\epsilon}{\partial x_i} \sigma_{ij} \tilde{dw}_j \quad (3.7)$$

and

$$\begin{aligned} d(\mu^\epsilon \rho) &= \rho \mu \left[h\left(\frac{x^\epsilon}{\epsilon}\right) x^\epsilon \bullet dz + (\sigma^{-1} g)\left(\frac{x^\epsilon}{\epsilon}\right) x^\epsilon \bullet \tilde{dw} \right] \\ &\quad + \rho \mu \beta \bullet dz + \rho \mu \beta \bullet h\left(\frac{x^\epsilon}{\epsilon}\right) x^\epsilon dt \end{aligned} \quad (3.8)$$

Using this and (3.2), we have

$$d[V^\epsilon(x^\epsilon(t), t)\mu^\epsilon(t)\rho(t)] = \mu^\epsilon(t)\rho(t)[\sigma^*\left(\frac{x^\epsilon}{\epsilon}\right)DV^\epsilon(x^\epsilon(t), t)]$$

$$\begin{aligned}
& + V^\epsilon(x^\epsilon(t), t)(\sigma^{-1}g)(\frac{x^\epsilon(t)}{\epsilon})x^\epsilon(t)) \bullet d\tilde{w} \\
& + V^\epsilon(x^\epsilon(t), t)(h(\frac{x^\epsilon(t)}{\epsilon})x^\epsilon(t) + \beta(t)) \bullet dz].
\end{aligned} \tag{3.9}$$

Because of the estimates (3.3) one can take the expectation of the stochastic integrals obtained by integrating (3.9). Integrating and taking the expectation gives

$$EV^\epsilon(\xi, 0) = E[V^\epsilon(x^\epsilon(T), T)\mu^\epsilon(T)\rho(T)] \tag{3.10}$$

which is the desired result.

QED

Remark. Note that (3.4) is well defined if ψ is Borel bounded and $\beta \in L^\infty(0, T; \mathbb{R}^d)$. In this case the function V^ϵ is not $C^{2,1}(\mathbb{R}^n \times [0, T])$; but this is not essential for the right hand side of (3.4) to be well defined. Thus, by regularization, it follows that (3.4) also holds when ψ is Borel bounded and $\beta \in L^\infty(0, T; \mathbb{R}^d)$.

4. Homogenization

Our objective is to derive a homogenization representation of the conditional distribution $p^\epsilon(t)(\psi)$ as $\epsilon \rightarrow 0$. The representation is based on the homogenization of (3.2) which is a relatively classical problem [10].

Proposition 4.1. *Under the assumption (1.1) we have the estimate*

$$|V^\epsilon(x, t) - V_0(x, t)| \leq \epsilon K_\delta e^{(\delta|x|^2)} \tag{4.1}$$

where $\delta > 0$ can be chosen arbitrarily small, and

$$\frac{\partial V_0}{\partial t} + \bar{a}_{ij} \frac{\partial^2 V_0}{\partial x_i \partial x_j} + \bar{g}_{ij} x_j \frac{\partial V_0}{\partial x_i} + V_0 \bar{h}_{ij} x_j \beta_i = 0$$

$$V_0(x, T) = \psi(x) \quad T \geq t \geq 0 \quad (4.2)$$

with

$$\bar{a}_{ij} = \int_Y a_{ij}(y) m(y) dy \quad (4.3)$$

and \bar{g}_{ij} and \bar{h}_{ij} are similarly defined in terms of the unique density $m(y)$ satisfying

$$\frac{\partial^2}{\partial y_i \partial y_j} [a_{ij}(y) m(y)] = 0 \quad (4.4)$$

$$m \text{ periodic on } Y, m > 0, m \in C^2, \int_Y m(y) dy = 1$$

(c.f. [10], p. 530).

By adapting the procedure used to derive this result (see the proof below), we can construct the homogenization properties of the conditional distribution (2.10) in the non-linear filtering problem. This is the main result of the paper.

First, consider the “limiting filtering problem” defined as follows: Let

$$\begin{aligned} dx &= \bar{g} \, x dt + \bar{\sigma} dw \\ dz &= \bar{h} \, x dt + dv \\ x(0) &= \xi, \quad z(0) = 0, \quad 0 \leq t \leq T \end{aligned} \quad (4.5)$$

where $\bar{\sigma} \triangleq (2\bar{a})^{1/2}$ and let

$$p^0(T)(\psi) = E[\psi(x(T)) \nu^0(T) \mid Z^T] \quad (4.6)$$

where

$$\nu^0(t) = e^{\{\int_0^t \bar{h} x \cdot dz - \frac{1}{2} \int_0^t |\bar{h} x|^2 ds\}} \quad (4.7)$$

in which z is a standard Wiener process. ((4.5) follows from a Girsanov transformation as used in (2.7).) In fact, we have the well-known formula

$$p^0(T)(\psi) = e^{(-\rho(T))} \cdot \int_{\mathbb{R}^n} \frac{\psi(y) e^{[-\frac{1}{2}(y - \hat{x}(T))^T P^{-1}(T)(y - \hat{x}(T))]} (2\pi)^{n/2} (\det P(T))^{1/2}} dy \quad (4.8)$$

in which

$$\rho(t) = \frac{1}{2} \int_0^t |\bar{h}x|^2 ds - \frac{1}{2} \int_0^t \bar{h}x \bullet dz \quad (4.9)$$

and $\hat{x}(t)$ is the state of the Kalman filter

$$\begin{aligned} d\hat{x} &= \bar{g}\hat{x} dt + P\bar{h}^T(dz - \bar{h}\hat{x} dt) \quad \hat{x}(0) = x_0 \\ \frac{d}{dt} P + P\bar{h}^T \bar{h}P - \bar{\sigma}\bar{\sigma}^T - (\bar{g}P + P\bar{g}^T) &= 0 \quad P(0) = P_0 \end{aligned} \quad (4.10)$$

As in Proposition 3.1, we can show that

$$E[p^0(T)(\psi)\rho(T)] = E[V_0(\xi, 0)] = \int_{\mathbb{R}^n} V_0(x, 0)\pi_0(x)dx. \quad (4.11)$$

Using this, we can state the following:

Theorem 4.1. *Under the assumptions (2.1) and (3.10) we have*

$$p^\epsilon(T)(\psi) \xrightarrow{\epsilon \rightarrow 0} p^0(T)(\psi) \quad (4.12)$$

weakly in $L^2(\Omega, Z^T, P)$ for every bounded, uniformly continuous ψ .

Proof. First note that we can assume, without loss of generality, that ψ is smooth and bounded. Indeed, let $\xi_T \in L^2(\Omega, Z^T, P)$, then using (3.5)

$$\begin{aligned} |E[p^\epsilon(T)(\psi)\xi_T]| &= |E[\psi(x^\epsilon(T))\mu^\epsilon(T)\xi_T]| \\ &\leq \|\psi\|_{L^\infty} \|\xi_T\|_{L^2} \sqrt{E\mu^\epsilon(T)^2} \\ &\leq C \|\psi\|_{L^\infty} \|\xi_T\|_{L^2}. \end{aligned} \quad (4.13)$$

Since ψ is uniformly continuous and bounded, it can be approximated in the *sup* norm by a sequence of smooth, bounded functions. This and the uniform estimate (4.13)

means that it suffices to establish (4.12) for smooth ψ 's.

Note also that the estimate (4.13) proves that $p^\epsilon(T)(\psi)$ is bounded in $L^2(\Omega, Z^T, P)$. Therefore, it is sufficient to prove that

$$E[p^\epsilon(T)(\psi)\rho(T)] \xrightarrow{\epsilon \rightarrow 0} E[p^0(T)(\psi)\rho(T)] \quad (4.14)$$

for any β , since the corresponding set of $\rho(T)$'s is dense in $L^2(\Omega, Z^T, P)$, as we have already noted.

But from formulas (3.4) and (4.11), the assertion (4.14) is equivalent to

$$\int_{\mathbb{R}^n} V^\epsilon(x, 0) \pi_0(x) dx \xrightarrow{\epsilon \rightarrow 0} \int_{\mathbb{R}^n} V_0(x, 0) \pi_0(x) dx \quad (4.15)$$

Since this is immediate from Proposition 4.1, the Theorem is proved.

QED

Remark. Theorem 4.1 implies the convergence of the conditional probability itself in a very weak sense. Indeed we have for any $\xi_T \in L^2(\Omega, Z^T, P)$

$$\begin{aligned} E^\epsilon \pi^\epsilon(T)(\psi) \xi_T &= E \psi(x^\epsilon(T)) \mu^\epsilon(T) \xi_T = E p^\epsilon(T)(\psi) \xi_T \\ E^0 \pi^0(T)(\psi) \xi_T &= E p^0(T)(\psi) \xi_T \end{aligned} \quad (4.16)$$

where

$$\pi^0(T)(\psi) = \frac{p^0(T)(\psi)}{p^0(T)(1)} \quad (4.17)$$

denotes the limit conditional probability, and E^0 refers to the probability on Ω for which z satisfies (4.5). Therefore we can assert that

$$E^\epsilon \pi^\epsilon(T)(\psi) \xi_T \rightarrow E^0 \pi^0(T)(\psi) \xi_T \quad \forall \psi, \forall \xi_T. \quad (4.18)$$

It would be nice to prove stronger convergence results, but it must be kept in mind that

the processes (1.1) themselves converge just in law and not in a stronger sense (c.f. [11] p. 405).

Proof of Proposition 4.1. Formally, the method is as follows: We consider an expansion of the form

$$V^\epsilon(x, t) = V_0(x, t) + \epsilon V_1(x, \frac{x}{\epsilon}, t) + \epsilon^2 V_2(x, \frac{x}{\epsilon}, t) + \tilde{V}^\epsilon(x, t) \quad (4.19)$$

Introducing $y = x/\epsilon$ and using the change of coordinates

$$\frac{\partial}{\partial x_i} \rightarrow \frac{\partial}{\partial x_i} + \frac{1}{\epsilon} \frac{\partial}{\partial y_i} \quad (4.20)$$

we obtain

$$\begin{aligned} & \frac{\partial V_0}{\partial t} + \epsilon \frac{\partial V_1}{\partial t} + \epsilon^2 \frac{\partial V_2}{\partial t} + \frac{\partial \tilde{V}^\epsilon}{\partial t} \\ & + a_{ij}(y) \frac{\partial^2 V_0}{\partial x_i \partial x_j} + g_{ij}(y) x_j \frac{\partial V_0}{\partial x_i} + V_0 h_{ij}(y) x_j \beta_i \\ & + \frac{1}{\epsilon} a_{ij}(y) \frac{\partial^2 V_1}{\partial y_i \partial y_j} + 2 a_{ij}(y) \frac{\partial^2 V_1}{\partial y_i \partial x_j} + \epsilon a_{ij}(y) \frac{\partial^2 V_1}{\partial x_i \partial x_j} \\ & + g_{ij}(y) x_j (\epsilon \frac{\partial V_1}{\partial x_i} + \frac{\partial V_1}{\partial y_i}) + \epsilon V_1 h_{ij}(y) x_j \beta_i \\ & + a_{ij}(y) \frac{\partial^2 V_2}{\partial y_i \partial y_j} + 2\epsilon a_{ij}(y) \frac{\partial^2 V_2}{\partial y_i \partial x_j} + \epsilon^2 a_{ij}(y) \frac{\partial^2 V_2}{\partial x_i \partial x_j} \\ & + g_{ij}(y) x_i (\epsilon^2 \frac{\partial V_2}{\partial x_i} + \frac{\partial V_2}{\partial y_i}) + \epsilon^2 V_2 h_{ij}(y) x_j \beta_i - A^\epsilon \tilde{V}^\epsilon = 0 \end{aligned} \quad (4.21)$$

where we have set

$$A^\epsilon V = -a_{ij}(y) \frac{\partial^2 V}{\partial x_i \partial x_j} - g_{ij}(y) x_j \frac{\partial V}{\partial x_i} - h_{ij}(y) x_j \beta_i V \quad (4.22)$$

with $y = x/\epsilon$. We choose

$$V_1(x, y, t) = V_1(x, t) \quad (4.23)$$

and

$$\begin{aligned} \frac{\partial V_0}{\partial t} + a_{ij}(y) \frac{\partial^2 V_0}{\partial x_i \partial x_j} + g_{ij}(y) x_j \frac{\partial V_0}{\partial x_i} \\ + V_0 h_{ij}(y) x_j \beta_i + a_{ij}(y) \frac{\partial^2 V_2}{\partial y_i \partial y_j} = 0 \end{aligned} \quad (4.24)$$

To deal with the latter, we introduce $m(y)$ the unique solution of

$$\frac{\partial^2}{\partial y_i \partial y_j} (a_{ij}(y) m(y)) = 0 \quad (4.25)$$

$$m \text{ periodic on } Y, \quad m > 0, \quad m \in C^2, \quad \int_Y m(y) dx = 1$$

(c.f. [10], p. 530). Then the solvability condition (Fredholm Alternative) for (4.24) is

$$\begin{aligned} \frac{\partial V_0}{\partial t} + \bar{a}_{ij} \frac{\partial^2 V_0}{\partial x_i \partial x_j} + \bar{g}_{ij} x_j \frac{\partial V_0}{\partial x_i} + V_0 \bar{h}_{ij} x_j \beta_i = 0 \\ V_0(x, T) = \psi(x), \quad T \geq t \geq 0 \end{aligned} \quad (4.26)$$

where we have set

$$\bar{a}_{ij} = \int_Y a_{ij}(y) m(y) dy \quad (4.27)$$

and similarly defined \bar{g}_{ij} and \bar{h}_{ij} .

If we choose

$$V_1(x, t) \equiv 0 \quad (4.28)$$

then $\tilde{V}^\epsilon(x, t)$ is the solution of

$$-\frac{\partial \tilde{V}^\epsilon}{\partial t} + A^\epsilon \tilde{V}^\epsilon = \epsilon (2a_{ij} \frac{\partial^2 V_2}{\partial y_i \partial x_j} + g_{ij} x_j \frac{\partial V_2}{\partial y_i})$$

$$+ \epsilon^2 \left(\frac{\partial V_2}{\partial t} + a_{ij} \frac{\partial^2 V_2}{\partial x_i \partial x_j} + g_{ij} x_j \frac{\partial V_2}{\partial x_i} + V_2 h_{ij} x_j \beta_i \right) \quad (4.29)$$

$$\tilde{V}^\epsilon(x, T) = 0, \quad T \geq t \geq 0$$

To estimate \tilde{V}^ϵ , we proceed as follows: First, we derive an explicit formula for $V_0(x, t)$. Consider the Gaussian process

$$d\xi = \bar{g}\xi dt + \bar{\sigma}db, \quad \xi(t) = x \quad (4.30)$$

where $\bar{\sigma} \triangleq (2\bar{a})^{1/2}$. Using this

$$V_0(x, t) = E \left\{ \psi(\xi_{x,t}(T)) e^{|\int_t^T \bar{h} \xi_{x,t}(s) \cdot \theta(s)|} \right\} \quad (4.31)$$

and we can easily check that

$$\begin{aligned} |V_0(x, t)| &\leq K_\delta e^{(\delta|x|^2)} \\ |DV_0(x, t)| &\leq K_\delta e^{(\delta|x|^2)} \end{aligned} \quad (4.32)$$

for some K_δ and any $\delta > 0$. An additional calculation shows that

$$\left| \frac{\partial^2 V_0}{\partial x_i \partial x_j} \right| \leq K_\delta e^{(\delta|x|^2)} \quad (4.33)$$

These estimates mean that

$$\left| \frac{\partial V_0}{\partial t} \right| \leq K_\delta e^{(\delta|x|^2)} \quad (4.34)$$

From (4.24) - (4.26) we can assert that

$$\begin{aligned} V_2(x, y, t) &= \chi_{ij}(y) \frac{\partial^2 V_0}{\partial x_i \partial x_j} + \eta_{ij}(y) x_j \frac{\partial V_0}{\partial x_i} \\ &\quad + \varsigma_{ij}(y) x_j \beta_i V_0 \end{aligned} \quad (4.35)$$

for some smooth, bounded functions χ_{ij} , η_{ij} , and ς_{ij} on Y . Since the higher order derivatives of V_0 also satisfy the bounds (4.32) - (4.34), we can deduce from (4.29) and

(4.35) that

$$-\frac{\partial \tilde{V}^\epsilon}{\partial t} + A^\epsilon \tilde{V}^\epsilon = \epsilon f^\epsilon \quad \tilde{V}^\epsilon(x, T) = 0 \quad (4.36)$$

where

$$|f^\epsilon(x, t)| \leq K_\delta e^{(\delta|x|^2)} \quad (4.37)$$

Using standard results, we can write

$$\tilde{V}^\epsilon(x, t) = \epsilon E \left\{ \int_t^T f^\epsilon(x^\epsilon(s), s) \left(e^{\left| \int_t^s h \left(\frac{x^\epsilon}{\epsilon} \right) x^\epsilon \cdot \beta dr \right|} \right) ds \right\} \quad (4.38)$$

And, by using arguments similar to those which led to the first estimate in (3.3), we obtain

$$|\tilde{V}^\epsilon(x, t)| \leq \epsilon K_\delta e^{(\delta|x|^2)} \quad (4.39)$$

where $\delta > 0$ can be chosen arbitrarily small. Combining this estimate with the expression (4.35) for V_2 completes the proof.

QED

5. Some Necessary Estimates

It remains to verify two key estimates used in the probabilistic and asymptotic analyses, i.e., inequalities (2.5) and (3.3), respectively.

Bounds on the expectation of $\mu^\epsilon(t)$.

Recall the definition of $\mu^\epsilon(t)$ in (2.4) based on (2.1) - (2.3).

Lemma 5.1. *For any finite T one has*

$$(a) \quad E \mu^\epsilon(T) < \infty \quad (5.1)$$

$$(b) \quad E[\mu^\epsilon(T)]^2 < C \quad (5.2)$$

Proof. (a) This is a consequence of the following condition (see [18])

$$Ee^{\delta |x^\epsilon(t)|^2} \leq C, \quad \forall t \in [0, T] \quad (5.3)$$

To check (5.3), consider the backward Cauchy problem ($a \triangleq \frac{1}{2}\sigma\sigma^*$)

$$\begin{aligned} \frac{\partial u}{\partial s} + a_{ij}(\frac{x}{\epsilon}) \frac{\partial^2 u}{\partial x_i \partial x_j} &= 0, \quad s \leq t \\ u(x, t) &= e^{\delta |x|^2} \end{aligned} \quad (5.4)$$

Then

$$Ee^{\delta |x^\epsilon(t)|^2} = Eu(\xi, 0). \quad (5.5)$$

Consider the function

$$\begin{aligned} \zeta(x, s) &= e^{[P(s)|x|^2 + \rho(s)]}, \quad P(s) \geq 0 \\ P(t) &= \delta, \quad \rho(t) = 0 \end{aligned} \quad (5.6)$$

We have

$$\begin{aligned} \frac{\partial \zeta}{\partial s} + a_{ij}(\frac{x}{\epsilon}) \frac{\partial^2 \zeta}{\partial x_i \partial x_j} &= \zeta \left[\frac{d}{dt} \rho + \frac{d}{dt} P |x|^2 \right. \\ &\quad \left. + 2 \operatorname{tr} a P + 4 |ax|^2 P^2 \right] \\ &\leq \zeta \left[\frac{d}{dt} \rho + \left(\frac{d}{dt} P + 4 ||a|| P^2 \right) + 2 |\operatorname{tr} a| P \right] \\ &\leq \zeta \left[\frac{d}{dt} \rho + \left(\frac{d}{dt} P + 4 ||a|| P^2 \right) + 2n ||a|| P \right] \end{aligned} \quad (5.7)$$

Choosing P and ρ so that

$$\frac{d}{dt} P + 4 ||a|| P^2 = 0, \quad \frac{d}{dt} \rho + 2n ||a|| P = 0 \quad (5.8)$$

we have

$$P(s) = \frac{\delta}{[1-4||a||\delta(t-s)]}$$

$$e^{\rho(s)} = \frac{1}{[1-4||a||\delta(t-s)]^{n/2}} \quad (5.9)$$

By the maximum principle, $\zeta(x, s) \geq u(x, s)$. Hence,

$$Ee^{\delta|x^\epsilon(t)|} \leq E\left[\frac{e^{(\delta|s|^2/[1-4||a||\delta t])}}{(1-4||a||\delta t)^{n/2}}\right]$$

$$= \frac{e^{\delta[(1-4||a||\delta t)I - 2P_0]^{-1}|x_0|^2}}{\sqrt{|(1-4||a||\delta t)I - 2\delta P_0|}} \quad (5.10)$$

Therefore, sufficient conditions for (5.3) to hold are

$$1 - 4||a||\delta T > 0$$

$$(1 - 4||a||\delta T)I > 2\delta P_0. \quad (5.11)$$

which hold if δ is sufficiently small. These conditions are independent of ϵ .

(b) to ensure (5.2), we proceed as follows: For $s > 1$ we write

$$\mu^\epsilon(T)^2 = e^{\{2 \int_0^T (hx^\epsilon \bullet dz + (\sigma^{-1}g)x^\epsilon \bullet d\bar{w} - 2s \int_0^T |(h + \sigma^{-1}g)x^\epsilon|^2 dt\}}$$

$$\bullet e^{\{(2s-1) \int_0^T |(h + \sigma^{-1}g)x^\epsilon|^2 dt\}}$$

From this we have

$$E\mu^\epsilon(T)^2 \leq (Ee^{\{\frac{s(2s-1)}{s-1} \int_0^T |(h + \sigma^{-1}g)x^\epsilon|^2 dt\}})_{(s-1)/s} \quad (5.13)$$

Note that $s(2s-1)/(s-1)$ has a minimum on $[1, \infty)$ at some $s_0 > 1$. Thus, it suffices to check that

$$Ee^{\{\frac{s_0(2s_0-1)}{s_0-1} \int_0^T |(h + \sigma^{-1}g)x^\epsilon(t)|^2 dt\}} < \infty, \quad t \in [0, T]. \quad (5.14)$$

This is similar to (5.3) except that the parameter δ is fixed. Taking

$$\delta \triangleq \frac{s_0(2s_0-1)}{s_0-1} T \|h + \sigma^{-1}g\|^2 \quad (5.15)$$

we require (5.11) which reads

$$1 > 4 \|a\| T^2 \frac{s_0(2s_0-1)}{s_0-1} \|h + \sigma^{-1}g\|^2 \triangleq \delta_0 \quad (5.16)$$

$$(1 - \delta_0)I > 2\delta P_0.$$

These conditions restrict the size of T , and the extent to which they are necessary is not clear.

QED

Growth conditions on the dual function.

It remains to verify the estimates (3.3). One way to do this is to use a probabilistic formula for $V^\epsilon(x, t)$. Consider the equation

$$dx^\epsilon = g\left(\frac{x^\epsilon}{\epsilon}\right)x^\epsilon dt + \sigma\left(\frac{x^\epsilon}{\epsilon}\right)db \quad x^\epsilon(t) = x \quad (5.17)$$

on a probability space (not necessarily the original one) where $b(s)$ is a standard Wiener process. Then

$$V^\epsilon(x, t) = E \left\{ \psi(x^\epsilon(T)) e^{\int_t^T h\left(\frac{x^\epsilon}{\epsilon}\right)x^\epsilon \beta ds} \right\}. \quad (5.18)$$

Therefore,

$$\begin{aligned} |V^\epsilon(x, t)| &\leq K \int_t^T E e^{C|x^\epsilon(s)|} ds \\ &\leq K_\delta \int_t^T E e^{\delta|x^\epsilon(s)|^2} ds \end{aligned} \quad (5.19)$$

where $\delta > 0$ may be chosen arbitrarily small. A calculation similar to (5.7) shows that

$$E e^{\delta |x^\epsilon(t)|^2} \leq k_\delta^t(0) e^{P_\delta^t(0) |x|^2} \quad (5.20)$$

where

$$\begin{aligned} \frac{d}{dt} P_\delta^t + 4(P_\delta^t)^2 ||a|| + 2||g|| P_\delta^t &= 0 \\ P_\delta^t(t) &= \delta, \quad t \geq s \end{aligned} \quad (5.21)$$

$$\frac{\frac{d}{dt} k_\delta^t}{k_\delta^t} + 2P_\delta^t n ||a|| = 0, \quad k_\delta^t(t) = 1$$

Now

$$\begin{aligned} P_\delta^t(s) &= \frac{\delta}{\exp[-2||g||(t-s) - 4||a||\delta(t-s)] \frac{(1-\exp[-2||g||(t-s)])}{2||g||(t-s)}} \\ &= \frac{2\delta||g||(t-s)}{(1 - \exp[-2||g||(t-s)]) \left(\frac{2||g||(t-s)\exp[-2||g||(t-s)]}{1-\exp[1-2||g||(t-s)]} - 4||a||\delta(t-s) \right)} \end{aligned} \quad (5.22)$$

Since the function $x \exp(-x)/(1-\exp(-x))$ is decreasing on $[0, \infty]$, one has

$$\frac{2||g||(t-s)\exp[-2||g||(t-s)]}{1-\exp[-2||g||(t-s)]} > \frac{2||g||T\exp[-2||g||T]}{1-\exp[2||g||T]} \quad (5.23)$$

If we choose $\delta > 0$ so that

$$4||a||\delta T < \frac{2||g||\exp[-2||g||T]}{1-\exp[-2||g||T]} \quad (5.24)$$

then

$$|P_\delta^t(s)| \leq \frac{2\delta||g||T}{2||g||T\exp[-2||g||T] - 4||a||\delta T(1-\exp[-2||g||T])} \quad (5.25)$$

And from this the first estimate in (3.3) follows.

To prove the second estimate in (3.3), one may proceed by differentiating the expression (5.18). Namely,

$$\begin{aligned}
\frac{\partial V^\epsilon}{\partial x_i} &= E \left\{ \frac{\partial \psi}{\partial x_k} \frac{\partial x_k^\epsilon(T)}{dx_i} e^{|\int_t^T h(\frac{x^\epsilon}{\epsilon}) x^\epsilon \beta ds|} \right. \\
&\quad \left. + \psi(x^\epsilon(T)) (e^{|\int_t^T h(\frac{x^\epsilon}{\epsilon}) x^\epsilon \beta ds|} \right) \\
&\quad \cdot \int_t^T \left(\frac{1}{\epsilon} \frac{\partial h_{ik}}{\partial x_l} x_k^\epsilon + h_{jl} \right) \frac{\partial x_l^\epsilon}{\partial x_i}(s) ds \Big\}
\end{aligned} \tag{5.26}$$

and from (5.17)

$$\begin{aligned}
d \left(\frac{\partial x_k^\epsilon}{\partial x_i} \right) &= \left(\frac{1}{\epsilon} \frac{\partial g_{kj}}{\partial x_l} \frac{\partial x_l^\epsilon}{\partial x_i} x_j^\epsilon + g_{kj} \frac{\partial x_j^\epsilon}{\partial x_i} \right) ds \\
&\quad + \frac{1}{\epsilon} \frac{\partial \sigma_{kl}}{\partial x_j} \frac{\partial x_j^\epsilon}{\partial x_i} db_l \\
\frac{\partial x_k^\epsilon}{\partial x_i}(t) &= \delta_{ki}, \quad s \leq t \leq 0
\end{aligned} \tag{5.27}$$

It follows from (5.27) that

$$\begin{aligned}
E \left(\left| \frac{\partial x_k}{\partial x_i}(s) \right|^4 \right) &\leq C \left[1 + E \int_t^s \left| x^\epsilon(r) \right|^2 dr \right] \\
&\leq C(1 + |x|^2).
\end{aligned} \tag{5.28}$$

Hence,

$$E \left(\left| \frac{\partial x_k^\epsilon(s)}{\partial x_i} \right|^2 \right) \leq C(1 + |x|) \tag{5.29}$$

and from this one can readily deduce the second estimate in (3.3).

QED

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