ABSTRACT

Title of dissertation: UNIVERSAL BOUNDS ON COARSENING RATES FOR SOME MODELS OF PHASE TRANSITIONS

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In this thesis, we prove one-sided, universal bounds on coarsening rates for three models of phase transitions by following a strategy developed by Kohn and Otto (Comm. Math. Phys. 229(2002),375-395).

Our analysis for the phase-field model is performed in a regime in which the ratio between the transition layer thickness and the length scale of the pattern is small, and is also small compared to the square of the ratio between the pattern scale and the system size. The analysis extends the Kohn-Otto method to deal with both temperature and phase fields.

For the mean-field models, we consider two kinds of them: one with a coarsening rate $l \sim t^{1/3}$ and the other with $l \sim t^{1/2}$. The $l \sim t^{1/2}$ rate is proved using a new dissipation relation which extends the Kohn-Otto method. In both cases, the dissipation relations are subtle and their proofs are based on a residual lemma (Lagrange identity) for the Cauchy-Schwarz inequality.

The monopole approximation is a simplification of the Mullins-Sekerka model in the case when all particles are non-overlapping spheres and the centers of the particles do not move. We derive the monopole approximation and prove its well-posedness by considering a gradient flow restricted on collections of finitely many non-overlapping spheres. After that, we prove one-sided universal bounds on the coarsening rate for the monopole approximation.

UNIVERSAL BOUNDS ON COARSENING RATES FOR SOME MODELS OF PHASE TRANSITIONS

by

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This dissertation is dedicated to my parents.

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Chapter 1

Introduction

1.1 Models of phase transitions

In the late stages of heterogeneously nucleated phase transitions, a two-phase mixture is created, composed of particles of one phase dispersed in a matrix of the other. Initially the pattern of the phases is very complicated, the particles are small and their total surface area is large. According to thermodynamics, the system evolves in order to decrease the total surface area and conserve the total mass or volume of the particles. Smaller particles shrink and disappear and larger ones grow. It is widely observed that some typical length scale that characterizes the particle size increases and the length scale behaves as a temporal power law (see, e.g. [1]).

The dynamics that determine this power law behavior is not very well understood [3]. The first heuristic explanation goes back to Lifshitz, Slyozov [16] and Wagner [28] in terms of mean-field models. We will give detailed discussion about mean-field models later. Their classical theory suggests that the distribution function of particle sizes approaches a universal self-similar solution where the critical radius R_c, which is the averaged radius in $\mathbb{R}^3$, follows the temporal power law $R_c \sim t^{1/3}$. However, Niethammer and Pego [20] proved that mathematically, the size distribution function does not necessarily converge to the predicted universal similarity solution, and the long-time behavior need not be self-similar.

Thus, the question is whether anything can be said universally about the coarsening rates. We cannot expect all solutions to coarsen at the same rate, due to the likely presence of finescale unstable equilibria for example, and anyway, in the infinite-time limit the system should typically approach a stable equilibrium and stop coarsening. One would like to be able to show that the expected power-law behavior is typical in some sense. What is "typical" is not clear, but a related question is whether it is possible that some solutions coarsen faster than expected. We know of no heuristic reason that would prevent such behavior.

Before we go into any detailed discussion about the universality of these coarsening rates, let's first review the models for phase transitions.

1.1.1 Sharp-interface models

The first kind of models for phase transitions is sharp-interface models. In these models, phases are considered sharply separated by an interface, and we are interested in the evolution of the interface. Such models include the Mullins-Sekerka type models, surface diffusion models and the mean curvature flow type models. The Mullins-Sekerka model consists of three equations:

$$-\Delta u = 0 \quad \text{outside } \Gamma(t) \tag{1.1}$$

$$[n \cdot \nabla u]_-^+ = V \quad \text{on } \Gamma(t), \tag{1.2}$$

$$u = \kappa + \beta V \quad \text{on } \Gamma(t). \tag{1.3}$$

Here $\Gamma(t)$ is the boundary of the particles, u is a chemical potential, n is the outer normal to Γ, $[n \cdot \nabla u]_-^+$ is the jump of the normal derivative of u across Γ, κ is the mean curvature and V is the normal velocity of Γ. Note that (1.3) is the Gibbs-Thomson law modified by a kinetic drag term βV.

1.1.2 Diffuse-interface models

The second kind of models for phase transitions is diffuse-interface models. In these models, an order parameter is used to indicate the local microscopic order of the material and varies continuously from -1(one phase, such as solid) to 1(the other phase, such as liquid). Such models include Cahn-Hilliard type models, phase field models, and Allen-Cahn type models. The phase-field model consists of two equations for two continuous field variables: the temperature u and the order parameter ϕ. We will consider one specific phase-field model:

$$\varepsilon u_t + \frac{l}{2}\phi_t = K\Delta u, \tag{1.4}$$

$$\alpha\varepsilon\phi_t = \varepsilon\Delta\phi - \frac{1}{\varepsilon}g(\phi) + 2u. \tag{1.5}$$

Here l, K and α are non-dimensional parameters that respectively represent latent heat, thermal diffusivity, and a relaxation time. The function $g(\phi)$ is the derivative of the double well potential $G(\phi) = \frac{1}{4}(\phi^2 - 1)^2$ which is minimized at $\phi = \pm 1$. The small parameter ε measures the thickness of the transition layers between the two phases $\{\phi \approx +1\}$ and $\{\phi \approx -1\}$ and is also related to relaxation and diffusion times and the energetic contributions of temperature fluctuations compared to phase changes. We supply more details concerning the non-dimensionalization procedure and the interpretation of parameters in Appendix A.

1.1.3 Mean-field models

Mean-field models give another approach. In mean-field models, particles of one phase exchange mass by some interaction through a mean field $\theta(t)$ which is determined as a function of time t by the conservation of mass. There are many mechanisms that can dominate the mass transfer process [25]. We will consider two of them in this thesis that correspond to two kinds of mean-field models.

In the first model, particle growth is controlled by bulk or volume diffusion, with or without kinetic drag at the interface. Each particle radius R obeys the growth law

$$\dot{R} = \frac{1}{R+\beta}\big(\theta(t) - \frac{1}{R}\big), \tag{1.6}$$

where $\beta \geqslant 0$ is a constant. The particle size distribution $f(t,R)$ satisfies the transport equation

$$\partial_t f + \partial_R \left(\frac{1}{R+\beta}(\theta - \frac{1}{R}) f \right) = 0. \tag{1.7}$$

To conserve the total mass, the mean field θ satisfies

$$\theta(t) = \frac{\int_0^\infty (R+\beta)^{-1} R^{n-2} f(t,R)\, dR}{\int_0^\infty (R+\beta)^{-1} R^{n-1} f(t,R)\, dR}, \tag{1.8}$$

where n is the dimension of space. When $\beta = 0$, (1.6)-(1.8) is the classical model by Lifshitz, Slyozov [16] and Wagner [28]. Equation (1.6) is an approximation to the Mullins-Sekerka sharp interface model (1.1)-(1.3) in the situation that the particles are sparsely located in a domain Ω. In [17] and [18], Niethammer rigorously derived the model (1.6)-(1.8) in $\mathbb{R}^3$ for $\beta = 0$ and $\beta > 0$, respectively, from a model similar to (1.1)-(1.3) under the condition that the total capacity of the particles was small.

The second mean-field model arises formally from (1.1)-(1.3) by taking $\beta \to \infty$ after rescaling time by β. In this model, particle growth is controlled by the attachment reaction at the interface [1]. Now each particle radius R obeys the law

$$\dot{R} = \theta(t) - \frac{1}{R}. \tag{1.9}$$

The corresponding transport equation of the particle size distribution becomes

$$\partial_t f + \partial_R\big((\theta - \frac{1}{R})f\big) = 0. \tag{1.10}$$

In this case, the mean field θ satisfies

$$\theta(t) = \frac{\int_0^\infty R^{n-2} f(t,R)\, dR}{\int_0^\infty R^{n-1} f(t,R)\, dR}. \tag{1.11}$$

Equation (1.9) is the normalized mean curvature flow for a collection of spheres; i.e., it is a special case of the following sharp interface model:

$$V = -\kappa + \frac{1}{|\Gamma|}\int_\Gamma \kappa\, dS. \tag{1.12}$$

1.1.4 The monopole approximation of the Mullins-Sekerka model

In this subsection, let's consider an approximation of the Mullins-Sekerka model (1.1)-(1.3) with $\beta = 0$ in a simplified situation. When one phase consists of only a small fraction of the total volume, that phase breaks into finitely many small particles which are almost spherical. We regard these particles as spheres. It is observed that the centers of these particles are almost spatially fixed. To simplify the model, we consider the centers to be spatially fixed. In this case, there is a simple heuristic argument about the Mullins-Sekerka model and it results in the so-called monopole approximation (see, e.g. [27]).

Imagine that at each particle center x_i, there is a source or sink of magnitude A_i which needs to be determined later. Considering equation (1.1) of the Mullins-Sekerka model, we can require u to be constant inside each particle. Outside the particles, we can write down a harmonic function

$$u = \theta + \sum \frac{A_i}{|x - x_i|}. \tag{1.13}$$

Here θ is a mean field which is a spatial constant and will be determined.

On the boundary of the i^{th} particle, rather than apply pointwise Gibbs-Thomson condition (1.3), it is preferable to consider (1.3) as an average over each particle surface. By the mean value property of harmonic functions, the averaged version of (1.3) is

$$\frac{1}{R_i} = \theta + \frac{A_i}{R_i} + \sum_{j \neq i} \frac{A_j}{|x_j - x_i|}. \tag{1.14}$$

The normal velocity of each particle surface is determined by the time derivative of its radius, i.e., $v = \dot{R}_i$ on the surface of the i^{th} particle. Equation (1.2) becomes

$$-\frac{A_i}{R_i^2} = \dot{R}_i. \tag{1.15}$$

The conservation of the total volume then gives us

$$\sum A_i = 0. \tag{1.16}$$

The system (1.14) + (1.16) should determine the values of A_i and θ since the number of equations equals the number of unknowns. Hence the normal velocity $\dot{R}_i$ is determined through (1.15) and the evolution of the whole system is determined.

This is the so-called monopole approximation of the Mullins-Sekerka model.

1.2 Heuristic arguments about coarsening rates

The coarsening rates for the models of phase transitions can be predicted by heuristic reasoning based on scaling invariance. For example, when $\beta = 0$, the Mullins-Sekerka model (1.1)-(1.3) is invariant under the scaling

$$x = \lambda \hat{x}, \quad t = \lambda^3 \hat{t}, \quad \hat{u}(\hat{x}) = \lambda u(x), \quad \hat{V}(\hat{x}) = \lambda^2 V(x). \tag{1.17}$$

If one expects that over long times the behavior of the coarsening system will appear scale invariant in some statistical sense, then this kind of scaling invariance suggests that a characteristic length scale $l(t)$ ought to satisfy $l(t) = \lambda l(t/\lambda^3)$, so that $l(t)$ will be given by a temporal power law

$$l(t) \sim t^{1/3}. \tag{1.18}$$

When $\beta \neq 0$, under the scaling (1.17), equations (1.1)-(1.3) keeps its form if β is replaced by $\hat{\beta} = \beta/\lambda$. Then the system (1.1)-(1.3) is not invariant since we assume β to be a constant. However, this suggests that as the length scale becomes large, the influence of kinetic drag can be neglected and should not influence the ultimate coarsening rate for the Mullins-Sekerka model.

For the phase-field model (1.4) and (1.5), the sharp-interface limit as $\varepsilon \to 0$ is the Mullins-Sekerka model of the following form [5]:

$$\Delta u = 0 \qquad \text{outside } \Gamma(t) \tag{1.19}$$

$$\left[n \cdot \nabla u \right]_{-}^{+} = -\frac{l}{K} V \qquad \text{on } \Gamma(t) \tag{1.20}$$

$$\delta s \; u = -\sigma\kappa - \alpha\sigma V \qquad \text{on } \Gamma(t) \tag{1.21}$$

where $\Gamma(t) \approx \{x|\phi(x,t) = 0\}$ is the interface between the two phases, δs is the difference of the entropy between the two phases, and σ is the surface tension. The same scaling argument (1.17) suggests a $t^{1/3}$ coarsening rate for solutions of the phase-field model, at least when ε is small.

The mean-field models we are considering are invariant under the scalings

$$R = \eta\hat{R}, \quad t = \eta^3\hat{t}, \quad \theta = \eta^{-1}\hat{\theta} \qquad \text{for (1.6) if } \beta = 0; \tag{1.22}$$

$$R = \eta\hat{R}, \quad t = \eta^2\hat{t}, \quad \theta = \eta^{-1}\hat{\theta} \qquad \text{for (1.9).} \tag{1.23}$$

This kind of scaling invariance suggests that a characteristic length scale $l(t)$ will be given by a temporal power law

$$l(t) \sim t^{1/3} \qquad \text{for (1.6),} \tag{1.24}$$

$$l(t) \sim t^{1/2} \qquad \text{for (1.9).} \tag{1.25}$$

As discussed at the beginning of this chapter, our question is the universality of the coarsening rates.

Recently, Kohn and Otto [12] introduced a powerful method to answer this question. They obtain rigorous, universally valid time-averaged upper bounds on coarsening rates, in the setting of Cahn-Hilliard equations, which are diffuse-interface models for phase transitions (see also [13, 14, 15] for subsequently related results). Kohn and Otto consider the standard Cahn-Hilliard equation, whose sharp-interface limit is the Mullins-Sekerka model (1.1)-(1.3) with $\beta = 0$, and the Cahn-Hilliard equation with degenerate mobility, whose sharp interface limit is the surface diffusion model. Scaling invariance suggests that these two models have coarsening rates

$l \sim t^{1/3}$ and $l \sim t^{1/4}$ respectively. Define $⨍_0^T := \frac{1}{T}\int_0^T$ to indicate the time-averaged integral. The results of Kohn and Otto, in their simplest form, are estimates of the following forms:

(i) $⨍_0^T E^2(t)\, dt \geqslant C_2 ⨍_0^T (t^{-1/3})^2\, dt \quad$ for $T \geqslant C_3 L(0)^3$ (standard Cahn-Hilliard);

(ii) $⨍_0^T E^3(t)\, dt \geqslant C_2 ⨍_0^T (t^{-1/4})^3\, dt \quad$ for $T \geqslant C_3 L(0)^4$ (Cahn-Hilliard with degenerate mobility).

Here E is the volume-averaged free energy, which is a decreasing function of time and scales as inverse to length, L is a 'length scale' that is dual to E, C_2 and C_3 are universal positive constants that depend only on the dimension of space n. So, these estimates are time-averaged version of $E \geqslant C_2 t^{-1/3}$ and $E \geqslant C_2 t^{-1/4}$ respectively, which correspond to upper bounds on the length scale E^{-1}. These results show that, in a time-averaged sense, it is impossible for solutions to coarsen at a rate faster than the expected power law.

1.3 Outline of the thesis

Our goal in this paper is to prove universal time-averaged upper bounds on corresponding coarsening rates for

(i) the phase-field model (1.4) and (1.5);

(ii) the mean-field models; and

(iii) the monopole approximation of the Mullins-Sekerka model.

Again, we find that no solution can coarsen at a rate faster than that expected from scaling.

In Chapter 2, we will work on the phase-field model. The analysis is performed in a regime corresponding to the late stages of phase separation, in which the ratio between the transition layer thickness and the length scale of the pattern is small, and is also small compared to the square of the ratio between the pattern scale and system size. The analysis extends the method of Kohn and Otto to deal with both temperature and phase field. The results in this chapter have been accepted for publication in *Interfaces and Free Boundaries* [7].

In Chapter 3, we will work on the mean-field models. The mean-field models that we study have three aspects that distinguish them from the phase-field model in Chapter 2 and those models considered in [12, 14, 13]:

(i) Mean-field models concern the evolution of dilute systems; i.e., the second phase consists of only a small fraction of the whole mixture. Kohn and Otto's analysis for the Cahn-Hilliard equations breaks down in this extreme case.

(ii) There is no spatial information and hence no pattern scale in mean-field models. This requires a different definition and interpretation of the dual length scale L.

(iii) For the normalized mean curvature flow (1.12), there is no result available for the corresponding diffuse-interface model — the conserved Allen-Cahn equation (see [24] for an asymptotic analysis of this correspondence).

To handle these differences, we will need to define all relevant quantities in terms of the distribution of particle radii. For the interface-reaction–controlled model, we will establish a new dissipation relation that extends the Kohn-Otto method and enables us to prove bounds that correspond to a coarsening rate of the form $l \sim t^{1/2}$. The proof of the dissipation relations in both mean-field models requires a different technique from previous works. A key ingredient in our proofs is the use of residual lemma (Lagrange identity) for the Cauchy-Schwarz inequality to compare the dissipation rates of E and L. These results have been accepted for publication in *SIAM Journal on Mathematical Analysis* [6].

In Chapter 4, we will derive the monopole approximation of the Mullins-Sekerka model in 3D by considering the restriction of a gradient flow structure. Using this structure, we show that the monopole approximation has a unique solution when the spherical particles are non-overlapping. We also give a one-sided estimate for the $l \sim t^{1/3}$ coarsening rate. In this case, The interpolation inequality is easy to prove due to the fact that all particles are spherical. The dissipation relation is also easy to prove because of the restricted gradient flow structure. The monopole approximation simplifies the Mullins-Sekerka model in the sense that every particle is considered spherical. On the other hand, it includes some spatial information, which is different from the mean-field model, and hence may be helpful to understand what spatial correlations are.

Chapter 2

Coarsening rate for a phase-field model

2.1 Introduction of the main result

In this chapter, we will consider the coarsening rate of the solutions for the phase-field model (2.1)-(2.2):

$$\varepsilon u_t + \frac{l}{2}\phi_t = K\Delta u, \tag{2.1}$$

$$\alpha\varepsilon\phi_t = \varepsilon\Delta\phi - \frac{1}{\varepsilon}g(\phi) + 2u. \tag{2.2}$$

We will consider the coarsening dynamics in a large cubic cell $Q := [0,a]^n \subset \mathbf{R}^n$ and with periodic boundary conditions to avoid boundary effects. As in [12], we will always consider volume-averaged integrals denoted by

$$⨍ f := \frac{1}{\mathrm{vol}(Q)}\int_Q f,$$

as our goal is to obtain universal bounds independent of the size of Q. Our bounds will be valid when the transition layer thickness ε is small compared to a characteristic length scale $\hat{L}$ and the ratio $\varepsilon/\hat{L} \ll (\hat{L}/a)^2$, and therefore we are able to consider very complicated patterns of phases when $\hat{L}(t) \ll a$.

As long as the initial values are continuous and $\varepsilon < \alpha K$, the initial-value problem for the phase field system (2.1)-(2.2) is globally well posed and the solution is classical, see [4]. By (2.1) and the periodic boundary condition,

$$\frac{d}{dt}⨍(\varepsilon u + \frac{l}{2}\phi) = ⨍(\varepsilon u_t + \frac{l}{2}\phi_t) = ⨍ K\Delta u = 0.$$

So $\fint(\varepsilon u + \frac{l}{2}\phi)$ is conserved, and we will focus on the case $\fint(\varepsilon u + \frac{l}{2}\phi) = 0$, i.e.,

$$\varepsilon\bar{u} + \frac{l}{2}\bar{\phi} = 0, \tag{2.3}$$

where $\bar{u} = \fint u$ and $\bar{\phi} = \fint \phi$. Hence we only consider those initial data that satisfy (2.3). The phase field system (2.1)–(2.2) dissipates a volume-averaged negative entropy $S(t)$ (cf. [22]), which is defined by

$$S(t) := \fint \frac{\varepsilon}{2}|\nabla\phi|^2 + \frac{1}{\varepsilon}G(\phi) + \frac{2\varepsilon}{l}u^2. \tag{2.4}$$

The time derivative of S is

$$\begin{aligned}
\dot{S} &= \fint \big(-\varepsilon\Delta\phi + \frac{1}{\varepsilon}g(\phi)\big)\phi_t + \frac{4\varepsilon}{l}uu_t \\
&= \fint \big(2u - \alpha\varepsilon\phi_t\big)\phi_t + \frac{4}{l}\big(K\Delta u - \frac{l}{2}\phi_t\big)u \\
&= \fint -\frac{4K}{l}|\nabla u|^2 - \alpha\varepsilon\phi_t^2.
\end{aligned}$$

So $\dot{S} \leqslant 0$ and $S(t)$ is a decreasing function of t. Note that in the sharp-interface limit, $S(t)$ corresponds to the volume-averaged *area* of the interface between the phases, and so scales as inverse to length, cf. [5, 11].

The method of Kohn and Otto involves three key steps. The first is to find a *dissipation relation* that bounds the growth rate of a suitable measure of length scale in terms of the dissipation of a dual quantity, which is negative entropy in this case. Here, as a measure of length scale we will employ the H^{-1} Sobolev norm of the scaled energy density $\varepsilon u + \frac{l}{2}\phi$. We define

$$L(t) := \left(\fint |\nabla v|^2\right)^{1/2}, \tag{2.5}$$

where v is a periodic function that satisfies

$$\Delta v = \varepsilon u + \frac{l}{2}\phi. \tag{2.6}$$

By (2.3), v exists and is uniquely determined up to a spatial constant, so L is well defined. Taking the time derivative of $L^2(t) = \fint |\nabla v|^2$, we get

$$\begin{aligned} L\dot{L} &= \fint \nabla v \nabla v_t = \fint (-\Delta v_t) v = \fint \left(-\varepsilon u_t - \frac{l}{2}\phi_t\right) v \\ &= \fint -K\Delta u \, v = \fint K \nabla u \nabla v \leqslant K\left(\fint |\nabla v|^2\right)^{1/2} \left(\fint |\nabla u|^2\right)^{1/2}. \end{aligned}$$

So

$$|\dot{L}| \leqslant K\left(\fint |\nabla u|^2\right)^{1/2} \leqslant K\left(\frac{l}{4K}(-\dot{S})\right)^{1/2},$$

that is,

$$|\dot{L}|^2 \leqslant \frac{Kl}{4}(-\dot{S}) \tag{2.7}$$

This will prove to be the required dissipation relation.

The second key step involves proving an *interpolation inequality*, of the form

$$L(t)S(t) \geqslant C_1, \tag{2.8}$$

valid under certain conditions for all $t \geq 0$. The constant $C_1 > 0$ depends only on K, l, the dimension of space n, and the form of the double-well potential, and doesn't depend on the domain Q, the parameter ε or the size of S and L. We shall find that (2.8) is valid under the conditions

$$\frac{\varepsilon}{\hat{L}} \ll 1, \qquad \frac{\varepsilon}{\hat{L}} \ll \left(\frac{\hat{L}}{a}\right)^2, \tag{2.9}$$

where $\hat{L}^{-1}$ is an upper bound for $S(0)$ and may be regarded as a length scale.

The third step in the Kohn-Otto method is an elementary ODE argument (Lemma 3 in [12]). The dissipation relation (2.7) and the interpolation inequality (2.8) together with the ODE lemma in [12] lead directly to our main result.

Theorem 2.1. *Provided that the conditions (2.9) hold, there exist positive constants C_2 and C_3 such that for any solutions $u(t,x)$ and $\phi(t,x)$ of the equations (2.1) and (2.2), if the initial data satisfy (2.3) and $\hat{L}S(0) \leqslant 1$, then*

$$\fint_0^T S(t)^2 dt \geqslant C_2 \fint_0^T (t^{-1/3})^2 dt \quad \text{for } T \geqslant C_3 L(0)^3. \tag{2.10}$$

The constants C_2 and C_3 depend only on K, l, n and the form of the double-well potential G, and not on ε, α, $L(0)$, or $S(0)$.

The estimate (2.10) is a time-averaged version of the (unproven) pointwise estimate $S(t) \geqslant Ct^{-1/3}$, which corresponds to an upper bound on the length scale $1/S(t)$ with the expected power-law behavior. Theorem 3.1, adapted from [12], provides time-averaged estimates on some other integral combinations of $S(t)$ and $L(t)$. By tracking the constants in the arguments of [12], we find $C_2 = \frac{1}{6}(3m)^{1/3}$ and $C_3 = 8/(3m)$ where $m = \min\{\frac{1}{4}C_1^2, C_1^4/(Kl)^2\}$.

At this point, it only remains to prove the interpolation inequality (2.8).

2.2 The interpolation inequality

In this section, we will prove the interpolation inequality (2.8) under the assumptions indicated above. Define periodic functions w and ψ such that

$$\Delta w = u - \bar{u} \qquad \text{and} \qquad \Delta\psi = \phi - \bar{\phi}. \tag{2.11}$$

w and ψ are determined up to a spatial constant, which we fix by requiring $\bar{w} = 0$, $\bar{\psi} = 0$. By (2.4) we have

$$\fint \frac{2\varepsilon}{l} u^2 \leqslant S, \tag{2.12}$$

so we get

$$\left(\fint |u-\bar{u}|^2\right)^{1/2} \leqslant \left(\fint u^2\right)^{1/2} \leqslant \sqrt{\frac{l}{2\varepsilon}}\sqrt{S}, \tag{2.13}$$

The periodicity of w guarantees that $\int \nabla w = 0$. By Poincaré's inequality, together with an integration by parts justified by the periodicity of w,

$$\begin{aligned}\left(\fint |\nabla w|^2\right)^{1/2} &\leqslant Ca\left(\fint \sum_{i,j}\left|\frac{\partial^2 w}{\partial x_i \partial x_j}\right|^2\right)^{1/2} = Ca\left(\fint |\Delta w|^2\right)^{1/2} \\ &= Ca\left(\fint |u-\bar{u}|^2\right)^{1/2} \leqslant Ca\sqrt{\frac{l}{2\varepsilon}}\sqrt{S}, \end{aligned} \tag{2.14}$$

where C is a positive constant which depends only on the dimension of space.

By (2.11) and (2.3),

$$\Delta(\varepsilon w + \frac{l}{2}\psi) = \varepsilon(u-\bar{u}) + \frac{l}{2}(\phi - \bar{\phi}) = \varepsilon u + \frac{l}{2}\phi. \tag{2.15}$$

Comparing (2.15) with (2.6), we get

$$\begin{aligned} L(t) &= \left(\fint |\varepsilon\nabla w + \frac{l}{2}\nabla\psi|^2\right)^{1/2} \\ &\geqslant \frac{l}{2}\left(\fint |\nabla\psi|^2\right)^{1/2} - \varepsilon\left(\fint |\nabla w|^2\right)^{1/2} \\ &\geqslant \frac{l}{2}\left(\fint |\nabla\psi|^2\right)^{1/2} - Ca\sqrt{\frac{l\varepsilon}{2}}S^{1/2}, \end{aligned}$$

so

$$L(t)S(t) \geqslant \frac{l}{2}\left(\fint |\nabla\psi|^2\right)^{1/2}\left(\fint \frac{\varepsilon}{2}|\nabla\phi|^2 + \frac{1}{\varepsilon}G(\phi)\right) - Ca\sqrt{\frac{l\varepsilon}{2}}S^{3/2}.$$

Let us now define

$$L_1(t) = \left(\fint |\nabla\psi|^2\right)^{1/2}, \tag{2.16}$$

$$S_1(t) = \fint \frac{\varepsilon}{2}|\nabla\phi|^2 + \frac{1}{\varepsilon}G(\phi). \tag{2.17}$$

Then

$$L(t)S(t) \geqslant \frac{l}{2}L_1S_1 - Ca\sqrt{\frac{l\varepsilon}{2}}\,S^{3/2}. \tag{2.18}$$

Now it is time to prove the interpolation inequality relating $L(t)$ and $S(t)$.

Lemma 2.2. *Given any constant $M > 0$, provided $\varepsilon_0 M$ and $\varepsilon_0 a^2 M^3$ are sufficiently small, there exists a positive constant C_1 such that whenever $0 < \varepsilon < \varepsilon_0$ and $S(0) < M$, we have*

$$L(t)S(t) \geqslant C_1 \quad \textit{for all } t \geqslant 0. \tag{2.19}$$

Proof. The proof is similar to that of Lemma 1 in [12]. But our length scales L_1 and L are different from that in [12] and need a somewhat different treatment. For the sake of completeness and since we want to track every constant, especially the parameter ε, we reproduce every detail here.

Since $1 = (1-\phi^2) + \phi^2$, and

$$⨍(1-\phi^2) \leqslant \left(⨍(1-\phi^2)^2\right)^{1/2} \leqslant (4\varepsilon S_1)^{1/2}, \tag{2.20}$$

the remaining work is to estimate $⨍\phi^2$ in terms of L_1, S_1 and S.

Next, we will use the Modica-Mortola inequality. Define

$$W(\phi) = \int_0^\phi |1-t^2|\,dt. \tag{2.21}$$

We have

$$\frac{\partial W}{\partial \phi} = |1-\phi^2| = 2\sqrt{G(\phi)},$$

so

$$⨍|\nabla(W(\phi))| = ⨍|\nabla\phi|\frac{\partial W}{\partial\phi} \leqslant ⨍\frac{\varepsilon}{2}|\nabla\phi|^2 + \frac{1}{2\varepsilon}|\frac{\partial W}{\partial\phi}|^2 \leqslant 2S_1. \tag{2.22}$$

We will use a smooth mollifier ρ which is radially symmetric, non-negative and supported in the unit ball with $\int_{\mathbf{R}^n} \rho = 1$. Let the subscript δ denote the convolution with the kernel

$$\frac{1}{\delta^n}\rho\left(\frac{\cdot}{\delta}\right).$$

The parameter δ will be optimized later. We split $⨍ \phi^2$ into two parts:

$$⨍ \phi^2 \leqslant 2⨍ (\phi - \phi_\delta)^2 + 2⨍ \phi_\delta^2. \tag{2.23}$$

Noting that

$$|\phi_1 - \phi_2|^2 \leqslant 8|W(\phi_1) - W(\phi_2)|$$

for all ϕ_1 and ϕ_2, we get the following estimate for the first term of (2.23),

$$\begin{aligned} 2⨍ (\phi - \phi_\delta)^2 &\leqslant 2 \sup_{|h|\leqslant\delta} ⨍ (\phi(x) - \phi(x+h))^2 dx \\ &\leqslant 16 \sup_{|h|\leqslant\delta} ⨍ |W(\phi(x)) - W(\phi(x+h))|\, dx \\ &\leqslant 16\delta ⨍ |\nabla(W(\phi))| \leqslant 32\delta S_1. \end{aligned} \tag{2.24}$$

For the second term of (2.23), we need to deal with large and small values of $|\phi_\delta|$:

$$⨍ \phi_\delta^2 = ⨍ (\phi_\delta^2 - \min\{\phi_\delta^2, 4\}) + ⨍ \min\{\phi_\delta^2, 4\}. \tag{2.25}$$

Since $F(\phi) := \phi^2 - \min\{\phi^2, 4\}$ is convex in ϕ, by Jensen's inequality and the fact that $\int \rho(y)dy = 1$,

$$F(\phi_\delta(x)) = F\left(\int \rho(y)\phi(x - \delta y)dy\right) \leqslant \int \rho(y)F(\phi(x - \delta y))dy.$$

So the first term of (2.25) is

$$\begin{aligned}
\fint (\phi_\delta^2 - \min\{\phi_\delta^2, 4\}) &\leqslant \fint \int \rho(y) F(\phi(x-\delta y))\,dy\,dx \\
&= \int \rho(y) \fint \Big[\phi^2(x-\delta y) - \min\{\phi^2(x-\delta y), 4\}\Big]\,dx\,dy \\
&= \fint (\phi^2(x) - \min\{\phi^2(x), 4\})\,dx \\
&\leqslant \fint \frac{1}{2}(1-\phi^2)^2 \leqslant 2\varepsilon S_1. \qquad (2.26)
\end{aligned}$$

For the second term of (2.25), we have

$$\fint \min\{\phi_\delta^2, 4\} \leqslant 2 \fint |\phi_\delta|. \qquad (2.27)$$

We know that

$$\fint |\phi_\delta| = \sup\bigg\{\fint \phi_\delta(x)\zeta(x)\,dx : \zeta \text{ is } Q\text{-periodic and } |\zeta(x)| \leqslant 1 \text{ a.e.}\bigg\}.$$

For any ζ that is Q-periodic and $|\zeta(x)| \leqslant 1$ a.e.,

$$\zeta_\delta(x) = \int \frac{1}{\delta^n}\rho\left(\frac{x-y}{\delta}\right)\zeta(y)dy.$$

So

$$\nabla\zeta_\delta(x) = \frac{1}{\delta}\int \frac{1}{\delta^n}\nabla\rho\left(\frac{x-y}{\delta}\right)\zeta(y)dy = \frac{1}{\delta}\int \nabla\rho(y)\zeta(x-\delta y)dy,$$

and hence

$$\sup|\nabla\zeta_\delta| \leqslant \beta\frac{1}{\delta}\sup|\zeta| \leqslant \beta\frac{1}{\delta},$$

where $\beta = \int |\nabla \rho|$.

$$\begin{aligned}
\fint \phi_\delta(x)\zeta(x)\,dx &= \fint \phi(x)\zeta_\delta(x) \\
&= \fint \big(\Delta\psi - \frac{2\varepsilon}{l}\bar{u}\big)\zeta_\delta(x) \quad \text{(by (2.11) and (2.3))} \\
&= -\fint \nabla\psi\nabla\zeta_\delta(x)\,dx - \frac{2\varepsilon}{l}\bar{u}\fint \zeta_\delta \\
&\leqslant \Big(\fint |\nabla\psi|^2\Big)^{1/2}\Big(\fint |\nabla\zeta_\delta|^2\Big)^{1/2} + \frac{2\varepsilon}{l}|\bar{u}|\fint |\zeta_\delta| \\
&\leqslant \frac{\beta}{\delta}L_1 + \sqrt{\frac{2\varepsilon S}{l}}. \qquad (2.28)
\end{aligned}$$

Taking supremum over all such ζ, we get

$$\fint |\phi_\delta| \leqslant \frac{\beta}{\delta}L_1 + \sqrt{\frac{2\varepsilon S}{l}}. \qquad (2.29)$$

Combining these estimates, we get

$$\fint \phi^2 \leqslant 32\delta S_1 + 4\varepsilon S_1 + 4\frac{\beta}{\delta}L_1 + 4\sqrt{\frac{2\varepsilon S}{l}}. \qquad (2.30)$$

Since δ is arbitrary, we minimize the right hand side over all $\delta > 0$ and get

$$\fint \phi^2 \leqslant 16\sqrt{2\beta}\sqrt{L_1 S_1} + 4\varepsilon S_1 + 4\sqrt{\frac{2\varepsilon S}{l}}. \qquad (2.31)$$

Combining this estimate with (2.20), we obtain

$$1 \leqslant 16\sqrt{2\beta}\sqrt{L_1 S_1} + 4\varepsilon S_1 + 4\sqrt{\frac{2\varepsilon S}{l}} + \sqrt{4\varepsilon S_1}. \qquad (2.32)$$

Now, since S is a decreasing function of t and $S_1(t) \leqslant S(t)$ for all $t > 0$, we have

$$S_1(t) \leqslant S(t) \leqslant M \quad (t > 0).$$

Provided $\varepsilon_1 M$ is sufficiently small (depending only on l), we have

$$4\varepsilon S_1 + 4\sqrt{\frac{2\varepsilon S}{l}} + \sqrt{4\varepsilon S_1} < \frac{1}{2} \qquad (0 < \varepsilon \leqslant \varepsilon_1),$$

so

$$16\sqrt{2\beta}\sqrt{L_1S_1} \geqslant \frac{1}{2},$$

and hence

$$L_1S_1 \geqslant \hat{C}_1, \tag{2.33}$$

where $\hat{C}_1 = 1/(2048\beta)$. On the other hand, provided $\varepsilon_2 M \cdot (aM)^2$ is sufficiently small, (depending only on l and n), we have

$$Ca\sqrt{\frac{l\varepsilon}{2}}S^{3/2} \leqslant \frac{l}{4}\hat{C}_1 \quad (0 < \varepsilon \leqslant \varepsilon_2,\ t > 0). \tag{2.34}$$

Let $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\}$ and $C_1 = \frac{l}{4}\hat{C}_1$. By (2.18),

$$L(t)S(t) \geqslant \frac{l}{2}L_1(t)S_1(t) - Ca\sqrt{\frac{l\varepsilon}{2}}S^{3/2} \geqslant C_1 \quad (0 < \varepsilon \leqslant \varepsilon_0,\ t > 0). \tag{2.35}$$

2.3 Upper bounds

Applying the ODE argument of [12] (see Lemma 3.7 in chapter 3 for a similar argument) without change, we get the main result.

Theorem 2.3. *Under the assumptions of Lemma 2.2, for any $0 \leqslant \theta \leqslant 1$ and $0 < r < 3$ satisfying $\theta r > 1$ and $(1-\theta)r < 2$, there exists positive constants C_2 and C_3, depending only on K, l, θ, r and the dimension of space, such that for all $0 < \varepsilon \leqslant \varepsilon_0$*

$$\fint_0^T S^{\theta r}L^{-(1-\theta)r}dt \geqslant C_2\fint_0^T (t^{-1/3})^r dt,\ \textit{if}\ T \geqslant C_3 L(0)^3. \tag{2.36}$$

Proof. The inequalities (2.7) and (2.19) give us

$$(\dot{L})^2 \leqslant \frac{Kl}{4}(-\dot{S}) \quad \text{and} \quad LS \geqslant C_1, \quad (0 < \varepsilon \leqslant \varepsilon_0,\ t > 0).$$

The theorem is then an immediate consequence of Lemma 3 in [12]. In particular, we obtain (2.10) by choosing $\theta = 1, r = 2$.

Chapter 3

Coarsening rate for mean-field models

3.1 Strategy and main results

Our goal in this chapter is to prove universal time-averaged upper bounds on corresponding coarsening rates for the mean-field models — see (3.4) and (3.5) below. Again, we find that no solution can coarsen at a rate faster than that expected from scaling.

Let us describe our strategy for obtaining bounds on coarsening rates for the mean-field models (1.7) and (1.10) and state our main results. We work at first with a collection of finitely many particles undergoing coarsening with growth laws (1.6) and (1.9), respectively, for each particle. Such a system of particles has a discrete size distribution. We will apply a strategy similar to that of Kohn and Otto [12] to get time-averaged bounds for such discrete systems, and then pass to limits in section 3.4 to establish the bounds for arbitrary size distributions that have finite $(n+1)^{st}$ moment.

As discussed in chapter 2, Kohn and Otto's strategy involves two quantities that measure length scales, and three key steps. The first quantity is a volume-averaged free energy or negative entropy, that decreases with time and scales as inverse to length. The second quantity scales like length, but its physical interpretation is not as clear. What is important is that, in a sense to be made precise, the second quantity is dual to the first one while at the same time being controlled by it.

In our situation, thermodynamics suggests that a natural quantity that is decreasing is the surface energy, which is proportional to the total surface area S of all the particles. Analogous to the cases considered in [12, 14, 13] and in chapter 2, we will consider a kind of volume average of the surface area, which gives a quantity scaling as inverse to length. Because the total volume V of the particles is conserved, it is reasonable to consider the ratio S/V. For a finite particle system, we therefore define

$$E := \frac{\sum R_i^{n-1}}{\sum R_i^n}, \tag{3.1}$$

where n is the dimension of space and the sum goes over all surviving particles. E can also be considered as the volume-weighted average of curvatures $\{1/R_i\}$. In sections 3.2 and 3.3, we will prove that E is indeed decreasing in both models considered.

We need a length scale L that is dual to E. Since radius is dual to curvature, we define L to be the volume-weighted average of the radii $\{R_i\}$, i.e.,

$$L := \frac{\sum R_i^{n+1}}{\sum R_i^n}. \tag{3.2}$$

The first step of the Kohn-Otto method is to establish an *interpolation inequality* that expresses the duality of E and L. With the definitions (3.1) and (3.2) this is easy. By the Cauchy-Schwarz inequality,

$$\sum R_i^n = \sum R_i^{(n-1)/2} R_i^{(n+1)/2} \leqslant \left(\sum R_i^{n-1} \sum R_i^{n+1}\right)^{1/2},$$

and this immediately yields the required interpolation inequality:

$$EL \geqslant 1. \tag{3.3}$$

The second step is to obtain a *dissipation inequality* that controls $\dot{L}$ in terms of $\dot{E}$. In sections 3.2 and 3.3 below, we will prove that

$$\begin{aligned}
&|\dot{L}|^2 \leqslant C_1\,(-\dot{E}) \quad \text{for volume-diffusion–controlled growth (1.6),} \\
&|\dot{L}|^2 \leqslant D_1\,(-\dot{E})\,L \quad \text{for interface-reaction–controlled growth (1.9),}
\end{aligned}$$

where C_1 and D_1 are positive constants depending only on the dimension of space n. We remark that in the cases considered in chapter 2 and in [12, 14, 13], the difficult part is proving the interpolation inequalities; the dissipation relations are rather easy to prove. By contrast, in the situation of the mean-field models considered here, under definitions (3.1) and (3.2) the interpolation inequality is a simple consequence of the Cauchy-Schwarz inequality and it is the dissipation relations that need careful treatment.

The third step is an ODE argument. For the case of volume-diffusion–controlled growth, Lemma 3 in [12] and the two inequalities $EL \geqslant 1$ and $|\dot{L}|^2 \leqslant C_1(-\dot{E})$ directly give us appropriate time-averaged bounds on coarsening rates. Those that involve only E, the volume-averaged surface area, take a simple form, saying that for any $1 < p < 3$, there exist positive constants C_2 and C_3, depending only on n, p and nothing else, such that

$$\fint_0^T E(t)^p \, dt \geqslant C_2 \fint_0^T \left(t^{-1/3}\right)^p \, dt \quad \text{for} \quad T \geqslant C_3 L(0)^3. \tag{3.4}$$

This is exactly a time-averaged version of $E \geqslant t^{-1/3}$, which corresponds to an upper bound on the "length scale" E^{-1}.

For the case of interface-reaction-controlled growth, we will establish an ODE lemma in section 3.3 to show that the inequalities $EL \geqslant 1$ and $|\dot{L}|^2 \leqslant D_1 L(-\dot{E})$ give us appropriate time-averaged estimates. In particular, for any $1 < p < 2$, there exist positive constants D_2 and D_3, depending only on n, p and nothing else, such that

$$\fint_0^T E(t)^p \, dt \geqslant D_2 \fint_0^T \left(t^{-1/2}\right)^p \, dt \quad \text{for} \quad T \geqslant D_3 L(0)^2. \tag{3.5}$$

This is a time-averaged version of $E \geqslant t^{-1/2}$.

Once these results for discrete systems are established, we will pass to the case of general size distributions in section 3.4 by applying the well-posedness and compactness results for a family of mean-field models established by Niethammer and Pego in [21]. All of our models under consideration are included in that work except for the 2D volume-diffusion–controlled growth model with $\beta = 0$. So this case is not included in our main theorems on coarsening rates for general size distributions.

The results in [21] enable us to approximate a general distribution by a sequence of discrete ones. These results, together with an extended moment compactness result proved here in an appendix, enable us to take limits in the estimates for the discrete sequence. This leads to our main results on coarsening rates for general size distributions.

We consider such size distributions to belong to $\mathcal{P}_n$, the set of Borel probability measures on $[0, \infty)$ with finite n^{th} moment. Topologically we regard $\mathcal{P}_n$ as a subset of the Banach space of finite Radon measures on $[0, \infty)$, which is dual to $C_0([0, \infty))$, the space of continuous functions on $[0, \infty)$ that vanish at infinity. A *measure-valued solution* of the transport equation (1.7) or (1.10) is a weak-star continuous map $t \mapsto \nu_t$ taking $[0, \infty) \to \mathcal{P}_n$ that is a solution in the sense of distributions on $(0, \infty) \times (0, \infty)$. Based on the results in [21], we will see that for each initial size distribution $\mu \in \mathcal{P}_n$, there is a unique measure-valued solution with initial value $\nu_0 = \mu$ that preserves the n^{th} moment (total volume). The corresponding mean field is given for a.e. $t > 0$ by

$$\theta(t) = \int_0^\infty \frac{R^{n-2}}{R+\beta} d\nu_t(R) \Big/ \int_0^\infty \frac{R^{n-1}}{R+\beta} d\nu_t(R) \tag{3.6}$$

in the case of volume-diffusion–controlled growth and

$$\theta(t) = \int_0^\infty R^{n-2} d\nu_t(R) \Big/ \int_0^\infty R^{n-1} d\nu_t(R) \; . \tag{3.7}$$

in the case of interface-reaction–controlled growth. The quantities corresponding to (3.1) and (3.2) are defined by

$$E(t) := \int_0^\infty R^{n-1}\, d\nu_t(R) \Big/ \int_0^\infty R^n\, d\nu_t(R) \; , \tag{3.8}$$

$$L(t) := \int_0^\infty R^{n+1}\, d\nu_t(R) \Big/ \int_0^\infty R^n\, d\nu_t(R) \; . \tag{3.9}$$

Our main results take the following form.

Theorem 3.1. (**Volume-diffusion–controlled growth**) *Let $n \geqslant 2$ be an integer and $\beta \geqslant 0$, with $\beta > 0$ if $n = 2$, and let p be real with $1 < p < 3$. Then there exist positive constants C_2 and C_3, depending on p, n and nothing else, such that whenever ν is a measure-valued solution of the transport equation (1.7) and ν_0 has* *finite* n^{th} *and* $(n+1)^{st}$ *moments, we have*

$$\fint_0^T E(t)^p\,dt \geqslant C_2 \fint_0^T \left(t^{-1/3}\right)^p dt \quad \text{for } T \geqslant C_3\, L(0)^3. \tag{3.10}$$

Theorem 3.2. (**Interface-reaction–controlled growth**) *Let $n \geqslant 2$ be an integer and let p be real with $1 < p < 2$. Then there exist positive constants D_2 and D_3, depending on p, n and nothing else, such that whenever ν is a measure-valued solution of the transport equation (1.10) and ν_0 has finite n^{th} and $(n+1)^{st}$ moments, we have*

$$\fint_0^T E(t)^p\,dt \geqslant D_2 \fint_0^T \left(t^{-1/2}\right)^p dt \quad \text{for } T \geqslant D_3\, L(0)^2. \tag{3.11}$$

3.2 Discrete systems I: volume-diffusion–controlled growth

In this section, our aim is to prove the coarsening estimate (3.4) for any collection of finitely many spherical particles in $\mathbb{R}^n$ that undergoes coarsening controlled by volume diffusion with or without kinetic drag. The following growth law holds for each particle:

$$\dot{R}_i = \frac{1}{R_i + \beta}\big(\theta - \frac{1}{R_i}\big), \quad (1 \leqslant i \leqslant N(t)), \tag{3.12}$$

where R_i is the radius of the i^{th} particle, $N(t)$ is the number of surviving particles at time t, θ is the mean field, and the dot denotes the time derivative.

By the conservation of total mass,

$$0 = \frac{d}{dt}\sum R_i^n = n \sum R_i^{n-1}\dot{R}_i = n \sum \frac{R_i^{n-1}}{R_i+\beta}\big(\theta - \frac{1}{R_i}\big). \tag{3.13}$$

Here the sum goes over all surviving particles. So

$$\theta = \frac{\sum (R_i+\beta)^{-1} R_i^{n-2}}{\sum (R_i+\beta)^{-1} R_i^{n-1}}. \tag{3.14}$$

The right hand side of equation (3.12) is smooth as long as there is no particle disappearing. The conservation of total mass guarantees that the solution for (3.12) and (3.14) cannot blow up in finite time. So the solution is smooth and unique from time $t_0 = 0$ up to t_1 when some particles disappear. Restarting from t_1 with the remaining particles, we again get a smooth solution until a next time t_2 when some other particles disappear. In this way, we can find finitely many times $\{t_i\}$ such that the solution for (3.12) and (3.14) globally exists, is unique, and is smooth in each time interval $(t_i, t_{i+1}), i = 0, 1, \cdots$.

By definition (3.1),

$$E = \frac{\sum R_i^{n-1}}{\sum R_i^n}. \tag{3.15}$$

Notice that E is non-increasing in time — we have

$$\begin{aligned}\dot{E} &= \frac{n-1}{\sum R_i^n} \sum R_i^{n-2} \dot{R}_i = \frac{n-1}{\sum R_i^n} \sum \frac{R_i^{n-2}}{R_i + \beta} \big(\theta - \frac{1}{R_i}\big) \\ &= \frac{n-1}{\sum R_i^n} \Big[\frac{\big(\sum (R_i+\beta)^{-1} R_i^{n-2}\big)^2}{\sum (R_i+\beta)^{-1} R_i^{n-1}} - \sum \frac{R_i^{n-3}}{R_i+\beta} \Big] \leqslant 0,\end{aligned} \tag{3.16}$$

since, by the Cauchy-Schwarz inequality,

$$\sum \frac{R_i^{n-2}}{R_i+\beta} = \sum \Big[\frac{R_i^{(n-1)/2}}{(R_i+\beta)^{1/2}} \frac{R_i^{(n-3)/2}}{(R_i+\beta)^{1/2}} \Big] \leqslant \Big(\sum \frac{R_i^{n-1}}{R_i+\beta} \sum \frac{R_i^{n-3}}{R_i+\beta} \Big)^{1/2}.$$

By definition (3.2),

$$L = \frac{\sum R_i^{n+1}}{\sum R_i^n}. \tag{3.17}$$

Inequality (3.3) gives us the required interpolation inequality $EL \geqslant 1$. Next we establish a dissipation relation that controls $\dot{L}$ in terms of $\dot{E}$. Taking the time derivative of L, we get

$$\begin{aligned}\dot{L} = \frac{n+1}{\sum R_i^n}\sum R_i^n \dot{R}_i \ &= \ \frac{n+1}{\sum R_i^n}\sum \frac{R_i^n}{R_i+\beta}\Big(\theta - \frac{1}{R_i}\Big)\\ &= \frac{n+1}{\sum R_i^n}\Big[\frac{\sum(R_i+\beta)^{-1}R_i^n\cdot\sum(R_i+\beta)^{-1}R_i^{n-2}}{\sum(R_i+\beta)^{-1}R_i^{n-1}} - \sum\frac{R_i^{n-1}}{R_i+\beta}\Big].\end{aligned} \tag{3.18}$$

We can infer $\dot{L} \geqslant 0$ using again the Cauchy-Schwarz inequality, but we will not need this fact. We want to prove a dissipation inequality

$$|\dot{L}|^2 \leqslant C_1\big(-\dot{E}\big) \tag{3.19}$$

for some constant C_1 depending only on n. Choosing $C_1(n) = (n+1)^2/(n-1)$, and plugging in the expressions (3.16) and (3.18), (3.19) becomes

$$\begin{aligned}&\Big[\sum\frac{R_i^n}{R_i+\beta}\ \sum\frac{R_i^{n-2}}{R_i+\beta} - \Big(\sum\frac{R_i^{n-1}}{R_i+\beta}\Big)^2\Big]^2\\ &\quad\leqslant \sum R_i^n\sum\frac{R_i^{n-1}}{R_i+\beta}\cdot\Big[\sum\frac{R_i^{n-1}}{R_i+\beta}\sum\frac{R_i^{n-3}}{R_i+\beta} - \Big(\sum\frac{R_i^{n-2}}{R_i+\beta}\Big)^2\Big].\end{aligned} \tag{3.20}$$

Lemma 3.3. *Inequality (3.20) holds for any sequence of positive numbers* $\{R_i\}_{i=1}^N$.

To prove lemma 3.3, we need the following lemma from [2].

Lemma 3.4. (Lagrange identity)

$$\Big(\sum_{i=1}^N x_i^2\Big)\Big(\sum_{i=1}^N y_i^2\Big) - \Big(\sum_{i=1}^N x_i y_i\Big)^2 = \sum_{\substack{i,j=1\\ i<j}}^N (x_i y_j - x_j y_i)^2 \tag{3.21}$$

for any sequences of real numbers $\{x_i\}_{i=1}^N$ *and* $\{y_i\}_{i=1}^N$.

Proof of Lemma 3.3: The proof consists of several careful applications of the Lagrange identity and the Cauchy-Schwarz inequality. Taking $x_i = (R_i^n/(R_i+\beta))^{1/2}$

and $y_i = (R_i^{n-2}/(R_i+\beta))^{1/2}$ in (3.21), we get

$$
\begin{aligned}
I &:= \sum \frac{R_i^n}{R_i+\beta} \sum \frac{R_i^{n-2}}{R_i+\beta} - \Big(\sum \frac{R_i^{n-1}}{R_i+\beta}\Big)^2 \\
&= \sum_{\substack{i,j=1\\ i<j}}^{N} \Big[\Big(\frac{R_i^n}{R_i+\beta}\Big)^{1/2}\Big(\frac{R_j^{n-2}}{R_j+\beta}\Big)^{1/2} - \Big(\frac{R_j^n}{R_j+\beta}\Big)^{1/2}\Big(\frac{R_i^{n-2}}{R_i+\beta}\Big)^{1/2}\Big]^2 \\
&= \sum_{\substack{i,j=1\\ i<j}}^{N} \frac{R_i^{n-2}R_j^{n-2}}{(R_i+\beta)(R_j+\beta)}(R_i-R_j)^2 \\
&\leqslant \Bigg\{\sum_{\substack{i,j=1\\ i<j}}^{N} \frac{R_i^{n-1}R_j^{n-1}(R_i-R_j)^2}{(R_i+\beta)(R_j+\beta)}\Bigg\}^{1/2} \cdot \Bigg\{\sum_{\substack{i,j=1\\ i<j}}^{N} \frac{R_i^{n-3}R_j^{n-3}(R_i-R_j)^2}{(R_i+\beta)(R_j+\beta)}\Bigg\}^{1/2}. \qquad (3.22)
\end{aligned}
$$

Taking $x_i = R_i^{n/2}$ and $y_i = (R_i^{n-1}/(R_i+\beta))^{1/2}$ in (3.21), we get

$$
\begin{aligned}
&\sum R_i^n \sum \frac{R_i^{n-1}}{R_i+\beta} \\
&\quad = \Big[\sum\Big(\frac{R_i^{2n-1}}{R_i+\beta}\Big)^{1/2}\Big]^2 + \sum_{\substack{i,j=1\\ i<j}}^{N} \Big[R_i^{n/2}\Big(\frac{R_j^{n-1}}{R_j+\beta}\Big)^{1/2} - R_j^{n/2}\Big(\frac{R_i^{n-1}}{R_i+\beta}\Big)^{1/2}\Big]^2 \\
&\quad \geqslant \sum_{\substack{i,j=1\\ i<j}}^{N} \frac{R_i^{n-1}R_j^{n-1}}{(R_i+\beta)(R_j+\beta)}\Big[R_i^{1/2}(R_i+\beta)^{1/2} - R_j^{1/2}(R_j+\beta)^{1/2}\Big]^2. \qquad (3.23)
\end{aligned}
$$

Taking $x_i = (R_i^{n-1}/(R_i+\beta))^{1/2}$ and $y_i = (R_i^{n-3}/(R_i+\beta))^{1/2}$ in (3.21), we get

$$
\begin{aligned}
&\sum \frac{R_i^{n-1}}{R_i+\beta} \sum \frac{R_i^{n-3}}{R_i+\beta} - \Big(\sum \frac{R_i^{n-2}}{R_i+\beta}\Big)^2 \\
&\quad = \sum_{\substack{i,j=1\\ i<j}}^{N} \Big[\Big(\frac{R_i^{n-1}}{R_i+\beta}\Big)^{1/2}\Big(\frac{R_j^{n-3}}{R_j+\beta}\Big)^{1/2} - \Big(\frac{R_j^{n-1}}{R_j+\beta}\Big)^{1/2}\Big(\frac{R_i^{n-3}}{R_i+\beta}\Big)^{1/2}\Big]^2 \\
&\quad = \sum_{\substack{i,j=1\\ i<j}}^{N} \frac{R_i^{n-3}R_j^{n-3}}{(R_i+\beta)(R_j+\beta)}(R_i-R_j)^2. \qquad (3.24)
\end{aligned}
$$

So

$$
\begin{aligned}
II :=& \sum R_i^n \sum \frac{R_i^{n-1}}{R_i+\beta}\Big[\sum \frac{R_i^{n-1}}{R_i+\beta}\sum \frac{R_i^{n-3}}{R_i+\beta} - \Big(\sum \frac{R_i^{n-2}}{R_i+\beta}\Big)^2\Big] \\
\geqslant& \sum_{\substack{i,j=1\\ i<j}}^{N} \frac{R_i^{n-1}R_j^{n-1}}{(R_i+\beta)(R_j+\beta)}\Big[R_i^{1/2}(R_i+\beta)^{1/2} - R_j^{1/2}(R_j+\beta)^{1/2}\Big]^2 \\
&\cdot \sum_{\substack{i,j=1\\ i<j}}^{N} \frac{R_i^{n-3}R_j^{n-3}}{(R_i+\beta)(R_j+\beta)}(R_i-R_j)^2. \qquad (3.25)
\end{aligned}
$$

Comparing (3.22) and (3.25), $I^2 \leqslant II$ is an immediate consequence of the inequality

$$
(R_i-R_j)^2 \leqslant \big[R_i^{1/2}(R_i+\beta)^{1/2} - R_j^{1/2}(R_j+\beta)^{1/2}\big]^2 \quad \text{for all } \ i,j. \qquad (3.26)
$$

Inequality (3.26) holds since

$$
\begin{aligned}
&\big[R_i^{1/2}(R_i+\beta)^{1/2} - R_j^{1/2}(R_j+\beta)^{1/2}\big]^2 - (R_i-R_j)^2 \\
&\quad = \beta(R_i+R_j) + 2R_iR_j - 2R_i^{1/2}R_j^{1/2}(R_i+\beta)^{1/2}(R_j+\beta)^{1/2},
\end{aligned}
$$

and

$$
\big[\beta(R_i+R_j)+2R_iR_j\big]^2 - \big[2R_i^{1/2}R_j^{1/2}(R_i+\beta)^{1/2}(R_j+\beta)^{1/2}\big]^2 = \beta^2(R_i-R_j)^2 \geqslant 0.
$$

□

The dissipation inequality (3.19) follows from Lemma 3.3. Applying Lemma 3 in [12], we directly get the following estimates.

Theorem 3.5. *For any $0 \leqslant \lambda \leqslant 1$ and $0 < r < 3$ satisfying $\lambda r > 1$ and $(1-\lambda)r < 2$, there exist positive constants C_2 and C_3, depending only on λ, r and the dimension of space n, such that for any solution $\{R_i\}$ of equations (3.12) and (3.14), we have*

$$
⨍_0^T E^{\lambda r}L^{-(1-\lambda)r}\,dt \geqslant C_2 ⨍_0^T \big(t^{-1/3}\big)^r\,dt, \quad for \quad T \geqslant C_3\,L(0)^3, \qquad (3.27)
$$

where E and L are defined in terms of (3.1) and (3.2), respectively.

Proof: Lemma 3.3 guarantees that the dissipation relation (3.19) holds. Together with the interpolation inequality (3.3), we have

$$EL \geqslant 1 \quad \text{and} \quad |\dot{L}|^2 \leqslant C_1(-\dot{E}).$$

Lemma 3 in [12] leads directly to (3.5). $\square$

Taking $\lambda = 1$ and $r = p$ for $1 < p < 3$ in Theorem 3.5 yields (3.4).

3.3 Discrete systems II: interface-reaction–controlled growth

Our aim in this section is to prove the coarsening estimate (3.5) for any collection of finitely many spherical particles in $\mathbb{R}^n$ that undergoes coarsening controlled by interface reactions. Each particle obeys the growth law

$$\dot{R}_i = \theta - \frac{1}{R_i}, \quad (1 \leqslant i \leqslant N(t)), \tag{3.28}$$

where R_i is the radius of the i^{th} particle and θ is the mean field.

By the conservation of total mass,

$$0 = \frac{d}{dt}\sum R_i^n = n\sum R_i^{n-1}\dot{R}_i = n\sum R_i^{n-1}(\theta - \frac{1}{R_i}), \tag{3.29}$$

so

$$\theta = \frac{\sum R_i^{n-2}}{\sum R_i^{n-1}}. \tag{3.30}$$

Solutions of the system (3.28) and (3.30) have the same global existence and piecewise smooth properties as that of (3.12) and (3.14). Taking the time derivative of

$E = \sum R_i^{n-1} / \sum R_i^n$, we have

$$\begin{aligned}\dot{E} &= \frac{n-1}{\sum R_i^n} \sum R_i^{n-2} \dot{R}_i = \frac{n-1}{\sum R_i^n} \sum R_i^{n-2} (\theta - \frac{1}{R_i}) \\ &= \frac{n-1}{\sum R_i^n} \Big[\frac{\big(\sum R_i^{n-2}\big)^2}{\sum R_i^{n-1}} - \sum R_i^{n-3} \Big] \leqslant 0, \end{aligned} \tag{3.31}$$

since

$$\sum R_i^{n-2} = \sum \Big[R_i^{(n-1)/2} R_i^{(n-3)/2} \Big] \leqslant \Big(\sum R_i^{n-1} \Big)^{1/2} \Big(\sum R_i^{n-3} \Big)^{1/2}. \tag{3.32}$$

Taking the time derivative of $L = \sum R_i^{n+1} / \sum R_i^n$, we have

$$\dot{L} = \frac{n+1}{\sum R_i^n} \sum R_i^n \dot{R}_i = \frac{n+1}{\sum R_i^n} \Big[\frac{\sum R_i^n \sum R_i^{n-2}}{\sum R_i^{n-1}} - \sum R_i^{n-1} \Big]. \tag{3.33}$$

Again, by the Cauchy-Schwarz inequality, we can infer $\dot{L} \geqslant 0$.

As described in section 3.1, we will need a dissipation inequality that relates $\dot{L}$ and $\dot{E}$. We claim that

$$|\dot{L}|^2 \leqslant D_1 L(-\dot{E}) \tag{3.34}$$

for some positive constant D_1 depending only on n. Choosing $D_1(n) = (n+1)^2/(n-1)$, and plugging in the expression (3.31) and (3.33), inequality (3.34) becomes

$$\begin{aligned} &\Big[\sum R_i^n \sum R_i^{n-2} - \Big(\sum R_i^{n-1} \Big)^2 \Big]^2 \\ &\quad \leqslant \sum R_i^{n-1} \sum R_i^{n+1} \Big[\sum R_i^{n-1} \sum R_i^{n-3} - \Big(\sum R_i^{n-2} \Big)^2 \Big]. \end{aligned} \tag{3.35}$$

Lemma 3.6. *Inequality (3.35) holds for any sequence of positive numbers* $\{R_i\}_{i=1}^N$.

Proof. Similar to the proof of lemma 3.3, we will apply the Lagrange identity (3.21) and the Cauchy-Schwarz inequality. Taking $x_i = R_i^{n/2}$ and $y_i = R_i^{(n-2)/2}$ in (3.21), we have

$$
\begin{aligned}
I &:= \sum R_i^n \sum R_i^{n-2} - \Big(\sum R_i^{n-1}\Big)^2 \\
&= \sum_{\substack{i,j=1\\ i<j}}^N \Big[R_i^{n/2} R_j^{(n-2)/2} - R_j^{n/2} R_i^{(n-2)/2}\Big]^2 \\
&= \sum_{\substack{i,j=1\\ i<j}}^N R_i^{n-2} R_j^{n-2} (R_i - R_j)^2 \\
&\leqslant \Big[\sum_{\substack{i,j=1\\ i<j}}^N R_i^{n-1} R_j^{n-1} (R_i - R_j)^2\Big]^{1/2} \Big[\sum_{\substack{i,j=1\\ i<j}}^N R_i^{n-3} R_j^{n-3} (R_i - R_j)^2\Big]^{1/2}. \qquad (3.36)
\end{aligned}
$$

Taking $x_i = R_i^{(n-1)/2}$ and $y_i = R_i^{(n+1)/2}$ in (3.21), we have

$$
\begin{aligned}
\sum R_i^{n-1} \sum R_i^{n+1} &= \Big(\sum R_i^n\Big)^2 + \sum_{\substack{i,j=1\\ i<j}}^N \Big(R_i^{(n-1)/2} R_j^{(n+1)/2} - R_j^{(n-1)/2} R_i^{(n+1)/2}\Big)^2 \\
&= \Big(\sum R_i^n\Big)^2 + \sum_{\substack{i,j=1\\ i<j}}^N R_i^{n-1} R_j^{n-1} (R_j - R_i)^2. \qquad (3.37)
\end{aligned}
$$

Taking $x_i = R_i^{(n-1)/2}$ and $y_i = R_i^{(n-3)/2}$ in (3.21), we have

$$
\begin{aligned}
\sum R_i^{n-1} \sum R_i^{n-3} - \Big(\sum R_i^{n-2}\Big)^2 &= \sum_{\substack{i,j=1\\ i<j}}^N \Big[R_i^{(n-1)/2} R_j^{(n-3)/2} - R_j^{(n-1)/2} R_i^{(n-3)/2}\Big]^2 \\
&= \sum_{\substack{i,j=1\\ i<j}}^N R_i^{n-3} R_j^{n-3} (R_i - R_j)^2. \qquad (3.38)
\end{aligned}
$$

So

$$
\begin{aligned}
II &:= \sum R_i^{n-1} \sum R_i^{n+1} \Big[\sum R_i^{n-1} \sum R_i^{n-3} - \Big(\sum R_i^{n-2}\Big)^2\Big] \\
&\geqslant \sum_{\substack{i,j=1\\ i<j}}^N R_i^{n-1} R_j^{n-1} (R_j - R_i)^2 \sum_{\substack{i,j=1\\ i<j}}^N R_i^{n-3} R_j^{n-3} (R_j - R_i)^2 \\
&\geqslant I^2. \qquad (3.39)
\end{aligned}
$$

$\square$

At this point we have established the desired interpolation and dissipation inequalities. The third step toward our coarsening estimates is an ODE lemma.

Lemma 3.7. (**ODE lemma**) *Let $E(t)$ and $L(t)$ be two continuous and piecewise smooth positive functions. Assume that for some T_1, $0 \leqslant T_1 \leqslant \infty$, $E(t)$ satisfies*

$$\dot{E} < 0 \text{ a.e. on } (0, T_1), \quad \dot{E} = 0 \text{ on } (T_1, \infty). \tag{3.40}$$

If $E(t)$ and $L(t)$ satisfy

$$EL \geqslant 1 \quad \text{and} \quad |\dot{L}|^2 \leqslant D_1 L(-\dot{E}), \tag{3.41}$$

then for any $0 \leqslant \lambda \leqslant 1$ and $r > 0$ satisfying

$$r < 3, \quad \lambda r > 1 \quad \text{and} \quad (1-\lambda) r < 2, \tag{3.42}$$

we have

$$⨍_0^T E(t)^{\lambda r} L(t)^{1-(1-\lambda)r}\, dt \geqslant D_2 ⨍_0^T (t^{-1/2})^{r-1}\, dt \ \text{ for } T \geqslant D_3 L(0)^2, \tag{3.43}$$

where D_2 and D_3 are positive constants depending only on λ, r and D_1.

We remark that this lemma is key for obtaining bounds on coarsening rates for the $t^{1/2}$ growth law. We will extend the ideas in the proof of Lemma 3 in [12] to establish this result. A special case of Lemma 3.7 is to take $r = p+1$ and $\lambda = \frac{p}{p+1}$ for $1 < p < 2$. In this case, we obtain (3.5):

$$⨍_0^T E(t)^p\, dt \geqslant D_2 ⨍_0^T (t^{-1/2})^p\, dt \ \text{ for } T \geqslant D_3 L(0)^2, \tag{3.44}$$

where D_2 and D_3 are positive constants depending only on p and D_1.

Proof of Lemma 3.7: (1). If $T_1 = 0$, then $\dot{E} = 0$ on $(0, \infty)$. By assumption (3.41), we get $\dot{L} = 0$ on $(0, \infty)$. Hence $E(t) = E(0)$ and $L(t) = L(0)$ for all $t \in (0, \infty)$. By (3.42), $\lambda r > 1$ and $0 \leqslant \lambda \leqslant 1$ imply that $r > 1/\lambda \geqslant 1$. Hence we have $1 < r < 3$. Then

$$\begin{aligned}
\fint_0^T E(t)^{\lambda r} L(t)^{1-(1-\lambda)r}\, dt &= E(0)^{\lambda r} L(0)^{1-(1-\lambda)r} \\
&\geqslant L(0)^{1-r} \\
&\geqslant \frac{2}{3-r} T^{(1-r)/2} \quad \text{ if } T \geqslant \left(\frac{2}{3-r}\right)^{2/(r-1)} L(0)^2 \\
&= D_2' \fint_0^T (t^{-1/2})^{r-1}\, dt \quad \text{ if } T \geqslant D_3' L(0)^2 \qquad (3.45)
\end{aligned}$$

where

$$D_2' = 1 \quad \text{ and } \quad D_3' = \left(\frac{2}{3-r}\right)^{2/(r-1)}.$$

(2). Now we consider the case when $T_1 > 0$. $\dot{E}(t) < 0$ on $(0, T_1)$ implies that E is a strictly decreasing function of t on $(0, T_1)$. Hence $E(t)$ is invertible on $(0, T_1)$ and we regard $t \in (0, T_1)$ as a function of ε, with ε denoting the independent variable to distinguish it from $E = E(t)$ and avoid confusion. Note that ε ranges from $E(0)$ to $E(\infty) := \lim_{t\to\infty} E(t)$, since $\dot{E}(t) = 0$ for $t \in (T_1, \infty)$ implies that $E(t) = E(T_1)$ for any $t > T_1$. Consequently for $t \in (0, T_1)$, $L(t)$ can be viewed as a function of ε. So

$$\frac{dL}{dt} = \frac{dL}{d\varepsilon}\frac{dE}{dt} \quad \text{ for } t \in (0, T_1) \qquad (3.46)$$

and $|\dot{L}|^2 \leqslant D_1 L(-\dot{E})$ implies that

$$\left|\frac{dL}{d\varepsilon}\right|^2 (-\dot{E}) \leqslant D_1 L(\varepsilon). \qquad (3.47)$$

Multiplying both sides by a positive function $f(\varepsilon)$ and integrating from 0 to T, we have if $T < T_1$,

$$\int_0^T f(E(t))L(t)\,dt \geqslant \frac{1}{D_1}\int_{E_T}^{E_0} f(\varepsilon)\Big(\frac{dL}{d\varepsilon}\Big)^2\,d\varepsilon; \tag{3.48}$$

and if $T \geqslant T_1$,

$$\int_0^T f(E(t))L(t)\,dt \geqslant \int_0^{T_1} f(E(t))L(t)\,dt \geqslant \frac{1}{D_1}\int_{E_T}^{E_0} f(\varepsilon)\Big(\frac{dL}{d\varepsilon}\Big)^2\,d\varepsilon, \tag{3.49}$$

where $E_0 = E(0)$ and $E_T = E(T)$. Taking $f(\varepsilon) = \varepsilon^{\lambda r}L(\varepsilon)^{-(1-\lambda)r}$, we get

$$\int_0^T E(t)^{\lambda r}L(t)^{1-(1-\lambda)r}\,dt \geqslant \frac{1}{D_1}\int_{E_T}^{E_0} \varepsilon^{\lambda r}L(\varepsilon)^{-(1-\lambda)r}\Big(\frac{dL}{d\varepsilon}\Big)^2\,d\varepsilon. \tag{3.50}$$

We will change variables so that the right hand side becomes an integral of a square of some gradient. Consider

$$\hat{\varepsilon} = \frac{1}{\lambda r - 1}\varepsilon^{-(\lambda r-1)}, \quad \hat{L} = \frac{1}{1-r(1-\lambda)/2}L^{1-r(1-\lambda)/2}. \tag{3.51}$$

Our requirements $\lambda r > 1$ and $(1-\lambda)r < 2$ guarantee that $\hat{\varepsilon} > 0, \hat{L} > 0$ and $\hat{\varepsilon} \to \infty, \hat{L} \to \infty$ as $\varepsilon \to 0$ and $L \to \infty$, respectively. Also, we have

$$\frac{d\hat{\varepsilon}}{d\varepsilon} = -\varepsilon^{-\lambda r} \tag{3.52}$$

and

$$\Big(\frac{d\hat{L}}{d\hat{\varepsilon}}\Big)^2\,d\hat{\varepsilon} = \Big(\frac{d\hat{L}}{dL}\Big)^2\Big(\frac{dL}{d\varepsilon}\Big)^2\Big(\frac{d\varepsilon}{d\hat{\varepsilon}}\Big)^2\Big(\frac{d\hat{\varepsilon}}{d\varepsilon}\Big)d\varepsilon = -\varepsilon^{\lambda r}L^{-(1-\lambda)r}\Big(\frac{dL}{d\varepsilon}\Big)^2\,d\varepsilon \tag{3.53}$$

Hence

$$\int_{E_T}^{E_0} \varepsilon^{\lambda r}L^{-(1-\lambda)r}\Big(\frac{dL}{d\varepsilon}\Big)^2\,d\varepsilon = \int_{\hat{E}_0}^{\hat{E}_T}\Big(\frac{d\hat{L}}{d\hat{\varepsilon}}\Big)^2\,d\hat{\varepsilon}. \tag{3.54}$$

The right hand side is bounded below by its minimum over all functions $\hat{L}(\hat{\varepsilon})$ with the same endpoint values

$$\hat{L}_0 := \hat{L}(\hat{E}_0) = \frac{1}{1 - r(1-\lambda)/2} L(0)^{1-r(1-\lambda)/2}$$

and

$$\hat{L}_T := \hat{L}(\hat{E}_T) = \frac{1}{1 - r(1-\lambda)/2} L(t)^{1-r(1-\lambda)/2}$$

and the minimum is achieved when $\hat{L}$ is a linear function of $\hat{\varepsilon}$. So

$$\int_0^T E^{\lambda r}(t) L^{1-(1-\lambda)r}(t)\, dt \geqslant \frac{1}{D_1} \frac{(\hat{L}_T - \hat{L}_0)^2}{\hat{E}_T - \hat{E}_0}. \tag{3.55}$$

(2a) If $L(T) \geqslant 2L(0)$, then

$$\hat{L}_0 \leqslant 2^{r(1-\lambda)/2-1} \hat{L}_T < \hat{L}_T.$$

Hence

$$\hat{L}_T - \hat{L}_0 \geqslant \big(1 - 2^{r(1-\lambda)/2-1}\big) \hat{L}_T,$$

and consequently

$$\begin{aligned}
&\int_0^T E^{\lambda r}(t) L^{1-(1-\lambda)r}(t)\, dt \\
&\geqslant \frac{1}{D_1} \frac{(\hat{L}_T - \hat{L}_0)^2}{\hat{E}_T - \hat{E}_0} \geqslant \frac{\big(1 - 2^{r(1-\lambda)/2-1}\big)^2}{D_1} \frac{\hat{L}_T^2}{\hat{E}_T} \\
&\geqslant \frac{\big(1 - 2^{r(1-\lambda)/2-1}\big)^2}{D_1} \frac{(\lambda r - 1)}{(1 - r(1-\lambda)/2)^2}\ E^{\lambda r - 1} L^{2-(1-\lambda)r} \\
&= \hat{D}_2\ (EL)^{((2\lambda - 1)r + 1)/(r-1)} \cdot \big(E^{\lambda r} L^{1-(1-\lambda)r}\big)^{-(r-3)/(1-r)},
\end{aligned} \tag{3.56}$$

where

$$\hat{D}_2 := \frac{(\lambda r - 1)}{D_1} \Big(\frac{1 - 2^{r(1-\lambda)/2-1}}{1 - r(1-\lambda)/2}\Big)^2.$$

Since $1 < r < 3$ and $\lambda r > 1$,

$$(2\lambda - 1)r + 1 = 2\lambda r + 1 - r > 3 - r > 0.$$

Thus

$$\frac{(2\lambda - 1)r + 1}{r - 1} > 0 \quad \text{and} \quad \frac{r - 3}{1 - r} > 0.$$

So $EL \geqslant 1$ implies $(EL)^{((2\lambda-1)r+1)/(r-1)} \geqslant 1$ and hence

$$\int_0^T E^{\lambda r}(t) L^{1-(1-\lambda)r}(t)\, dt \geqslant \hat{D}_2 \big(E^{\lambda r} L^{1-(1-\lambda)r}\big)^{-(r-3)/(1-r)} \quad \text{if } \ L_T \geqslant 2L_0. \tag{3.57}$$

Define

$$h(T) := \int_0^T E^{\lambda r}(t) L^{1-(1-\lambda)r}(t)\, dt. \tag{3.58}$$

Then $h'(T) = E^{\lambda r}(T) L^{1-(1-\lambda)r}(T)$ and (3.57) can be rewritten as

$$h(T) \geqslant \hat{D}_2 (h'(T))^{-(r-3)/(1-r)} \quad \text{if } L_T \geqslant 2L_0, \tag{3.59}$$

or equivalently

$$h'(T)\big(h(T)\big)^{(r-1)/(3-r)} \geqslant \hat{D}_2^{(r-1)/(3-r)} \quad \text{if } L_T \geqslant 2L_0. \tag{3.60}$$

(2b) If $L(T) < 2L(0)$, then by $EL \geqslant 1$,

$$E_T \geqslant L_T^{-1} \geqslant \frac{1}{2} L_0^{-1},$$

$$E_T^{\lambda r} L_T^{1-(1-\lambda)r} = \Big(E_T L_T\Big)^{\lambda r} L_T^{1-r} \geqslant L_T^{1-r} \geqslant L_0^{1-r}\, 2^{1-r}.$$

So

$$h'(T) \geqslant L_0^{1-r}\, 2^{1-r} \quad \text{if } \ L(T) \leqslant 2L(0). \tag{3.61}$$

Combining (3.60) and (3.61), we have

$$h'(T) \Big[h(T) + L_0^{3-r}\Big]^{(r-1)/(3-r)} \geqslant \min\{2^{1-r}, \hat{D}_2^{(r-1)/(3-r)}\} =: m \quad \text{for all } T.$$

Thus

$$\frac{d}{dt}\Big[h(t)+L_0^{3-r}\Big]^{2/(3-r)} = \frac{2}{3-r}h'(t)\Big[h(t)+L_0^{3-r}\Big]^{(r-1)/(3-r)} \geqslant \frac{2m}{3-r}. \tag{3.62}$$

Integration over time from 0 to T, we get

$$\begin{aligned}
h(T) &\geqslant \Big[\frac{2m}{3-r}T + L_0^2\Big]^{(3-r)/2} - L_0^{3-r} \\
&\geqslant \Big(\frac{2m}{3-r}\Big)^{(3-r)/2} T^{(3-r)/2} - L_0^{3-r} \\
&\geqslant \frac{1}{2}\Big(\frac{2m}{3-r}\Big)^{(3-r)/2} T^{(3-r)/2} \qquad \text{if } T > 2^{2/(3-r)}\frac{(3-r)}{2m}L_0^2.
\end{aligned} \tag{3.63}$$

Equivalently,

$$⨍_0^T E(t)^{\lambda r} L(t)^{1-(1-\lambda)r}\, dt \geqslant D_2'' ⨍_0^T (t^{-1/2})^{r-1}\, dt \quad \text{for } T > D_3'' L_0^2 \tag{3.64}$$

where

$$D_2'' = \frac{3-r}{4}\Big(\frac{2m}{3-r}\Big)^{(3-r)/2} \quad \text{and} \quad D_3'' = 2^{2/(3-r)}\frac{(3-r)}{2m}.$$

(3). Combining (1) and (2), we conclude that

$$⨍_0^T E(t)^{\lambda r} L(t)^{1-(1-\lambda)r}\, dt \geqslant D_2 ⨍_0^T (t^{-1/2})^{r-1}\, dt \quad \text{for } T > D_3 L_0^2 \tag{3.65}$$

where

$$D_2 = \min\{D_2', D_2''\} \quad \text{and} \quad D_3 = \max\{D_3', D_3''\}. \qquad \square$$

We claim the following estimate for the collection of particles that undergoes coarsening determined by equation (3.3).

Theorem 3.8. *For any $0 \leqslant \lambda \leqslant 1$ and $0 < r < 3$ satisfying $\lambda r > 1$ and $(1-\lambda)r < 2$, there exist positive constants D_2 and D_3, depending only on λ, r and the dimension of space n, such that for any solution $\{R_i\}$ of equations (3.28) and (3.30), we have*

$$⨍_0^T E(t)^{\lambda r} L(t)^{1-(1-\lambda)r}\, dt \geqslant D_2 ⨍_0^T (t^{-1/2})^{r-1}\, dt \quad \textit{for } T \geqslant D_3 L(0)^2, \tag{3.66}$$

where E and L are defined in terms of (3.1) and (3.2), respectively.

Proof. As we discussed at the beginning of this section, solutions $\{R_i\}$ of equations (3.28) and (3.30) are continuous and piecewise smooth. Hence E and L defined by (3.1) and (3.2) are continuous and piecewise smooth. Furthermore, by (3.31), $\dot{E} \leqslant 0$ and $\dot{E} = 0$ if and only if all R_i are equal. Notice that if all R_i are equal, then the system (3.28) and (3.30) reaches an equilibrium point and the solution stops coarsening. Consequently, if $\dot{E} = 0$ at some time t_1, then $\dot{E}(t) = 0$ for all $t \geqslant t_1$. Hence, $\dot{E}$ satisfies the condition (3.40) of lemma 3.7.

On the other hand, the interpolation inequality (3.3) and the dissipation relation (3.34) says

$$EL \geqslant 1 \quad \text{and} \quad |\dot{L}|^2 \leqslant D_1 L(-\dot{E}).$$

The theorem is then an immediate consequence of Lemma 3.7. $\square$

3.4 Coarsening rates for particle systems with general size distributions

Now it is time to consider our mean-field models with more general size distributions. Definitions (3.1) and (3.2) imply that, in the more general case, E and L should be defined in terms of the $(n-1)^{st}$, n^{th} and $(n+1)^{st}$ moments of the size distributions. So it is necessary to require the initial size distributions to be in $\mathcal{P}_{n+1}$, the set of Borel probability measures on $[0, \infty)$ with finite $(n+1)^{st}$ moments. By Hölder's inequality, it is immediate to see that $\mathcal{P}_{n+1}$ is a subset of $\mathcal{P}_n$.

In [21], Niethammer and Pego proved well-posedness and compactness results for a family of mean-field models. All of our models under consideration are included in that work except for the 2D volume-diffusion–controlled growth model with $\beta = 0$. Their results guarantee the existence and uniqueness of measure-valued solutions of equation (1.7) or (1.10). A measure-valued solution is a weak-star continuous map $t \mapsto \nu_t$ taking $[0, \infty) \to \mathcal{P}_n$ that is a solution in the sense of distributions on $(0, \infty) \times (0, \infty)$, i.e., for all $\phi \in C_c^\infty([0, \infty) \times (0, \infty))$ (smooth functions with compact support),

$$\int_0^\infty \int_0^\infty \Big(\partial_t \phi + \frac{1}{R + \beta} (\theta(t) - \frac{1}{R}) \partial_R \phi \Big) \, d\nu_t \, dt + \int_0^\infty \phi(0, \cdot) \, d\nu_0 = 0 \tag{3.67}$$

in the case of volume-diffusion–controlled growth (equation (1.7)), or

$$\int_0^\infty \int_0^\infty \Big(\partial_t \phi + (\theta(t) - \frac{1}{R}) \partial_R \phi \Big) \, d\nu_t \, dt + \int_0^\infty \phi(0, \cdot) \, d\nu_0 = 0 \tag{3.68}$$

in the case of interface-reaction-controlled growth (equation (1.10)).

Our main results are estimates in terms of these measure-valued solutions.

Theorem 3.9. (**Volume-diffusion–controlled growth**) *Let $n \geqslant 2$ be an integer and $\beta \geqslant 0$, with $\beta > 0$ if $n = 2$. For any $0 \leqslant \lambda \leqslant 1$ and $0 < r < 3$ satisfying $\lambda r > 1$ and $(1 - \lambda) r < 2$, there exist positive constants C_2 and C_3 depending only on λ, r and the dimension of space n such that whenever ν is a measure-valued solution of the transport equation (1.7) and the initial value ν_0 has finite n^{th} and $(n+1)^{st}$ moments, we have*

$$⨍_0^T E(t)^{\lambda r} L(t)^{-(1-\lambda) r} \, dt \geqslant C_2 ⨍_0^T (t^{-1/3})^r \, dt \quad \textit{for } T \geqslant C_3 L(0)^3, \tag{3.69}$$

where $E(t)$ and $L(t)$ are defined by (3.8) and (3.9), respectively, and the mean field $\theta(t)$ is defined by (3.6).

Taking $r = p$ and $\lambda = 1$ for $1 < p < 3$ in Theorem 3.9 gives Theorem 3.1.

Theorem 3.10. (**Interface-reaction-controlled growth**) *Let $n \geqslant 2$ be an integer. For any $0 \leqslant \lambda \leqslant 1$ and $0 < r < 3$ satisfying $\lambda r > 1$ and $(1-\lambda)r < 2$, there exist positive constants D_2 and D_3 depending only on λ, r and the dimension of space n such that whenever ν is a measure-valued solution of the transport equation (1.10) and the initial value ν_0 has finite n^{th} and $(n+1)^{st}$ moments, we have*

$$\fint_0^T E(t)^{\lambda r} L(t)^{1-(1-\lambda)r}\, dt \geqslant D_2 \fint_0^T (t^{-1/2})^{r-1}\, dt \quad \text{for } T \geqslant D_3 L(0)^2, \tag{3.70}$$

where $E(t)$ and $L(t)$ are defined by (3.8) and (3.9), respectively, and the mean field $\theta(t)$ is defined by (3.7).

Taking $r = p+1$ and $\lambda = p/(p+1)$ for $1 < p < 2$ in Theorem 3.10 gives Theorem 3.2.

The remaining part of this section is devoted to proving the theorems above. To do this, we will need a change of variables as is done in [21]. In that paper, rather than directly working on distributions of particle radii R, Niethammer and Pego change the problems into equivalent ones expressed in terms of rescaled particle volumes $x(:= R^n)$ and work with a size-ranking function for particle volumes.

According to equations (1.6) and (1.9), the particle volume x satisfies the following growth law:

$$\dot{x} = a(x)\theta - b(x), \tag{3.71}$$

where

$$a(x) = \tfrac{nx^{1-1/n}}{x^{1/n}+\beta}, \quad b(x) = \tfrac{nx^{1-2/n}}{x^{1/n}+\beta} \quad \text{for volume-diffusion–controlled case,} \tag{3.72}$$

$$a(x) = nx^{1-1/n}, \quad b(x) = nx^{1-2/n} \quad \text{for interface-reaction–controlled case,} \tag{3.73}$$

and $\theta(t) = \int b(x)\, d\nu_t(x) / \int a(x)\, d\nu_t(x)$. Here ν is the measure-valued solution in the sense of distributions for the transport equation

$$\partial_t u + \partial_x \big((a(x)\theta - b(x))u \big) = 0. \tag{3.74}$$

The results of Niethammer and Pego are established by a further change of variables [21] (see also [20]). For any size distribution of particles which is a probability measure μ on $[0, \infty)$, they define a *size-ranking function* $x = \hat{x}(\mu) : (0, 1] \to [0, \infty)$ by

$$x(\varphi) = \begin{cases} \sup\{y \mid \mu([y, \infty)) > \varphi\} & \text{for} \quad 0 < \varphi < 1, \\ 0 & \text{for} \quad \varphi = 1. \end{cases} \tag{3.75}$$

This is the right-continuous inverse of the tail distribution function $\varphi(x) = \mu([x, \infty))$. The map $\hat{x}$ gives a 1-1 correspondence between the set of Borel probability measures on $[0, \infty)$ and the set of right-continuous decreasing functions x on $(0, 1]$ with $x(1) = 0$.

The following space for size ranking is introduced in [21]:

$$L^1_d = \{x : (0, 1] \to \mathbb{R} \mid x \in L^1((0, 1)), x(1) = 0, \text{ and } x \text{ is decreasing}$$
$$\text{and right continuous on}(0, 1]\}.$$

It is a closed subspace of $L^1((0, 1))$. We will also perform our estimates in this space.

By [10], 2.5.18(3), for any continuous function $f : (0, \infty) \to \mathbf{R}$ with compact support,

$$\int_0^1 f(x(\varphi))\, d\varphi = \int_0^\infty f(y)\, d\mu(y). \tag{3.76}$$

For any positive number $\alpha > 0$, $y \mapsto y^\alpha$ can be approximated by a monotonically increasing sequence of such functions, so by the monotone convergence theorem,

$$\int_0^1 x(\varphi)^\alpha\, d\varphi = \int_0^\infty y^\alpha\, d\mu(y), \tag{3.77}$$

where both sides may be infinite. Hence $\mu \in \mathcal{P}_\alpha$ (Borel probability measures with finite α^{th} moment) if and only if x is right-continuous decreasing on $(0, 1]$ with $x(1) = 0$ and $\int_0^1 x(\varphi)^\alpha\, d\varphi < \infty$.

The growth law (3.71) can be rewritten as an integral equation:

$$x(t, \varphi) = x(0, \varphi) + \int_0^t (a(x(s, \varphi))\theta(s) - b(x(s, \varphi)))\, ds \tag{3.78}$$

with

$$\theta(t) = \int_0^{\bar{\varphi}(t)} b(x(t, \varphi))\, d\varphi \Big/ \int_0^1 a(x(t, \varphi))\, d\varphi \quad \text{for a.e. } t > 0, \tag{3.79}$$

where $\bar{\varphi}(t) := \sup\{\varphi | x(t, \varphi) > 0\}$.

Theorem 2.3 of [21] established the existence and uniqueness of the initial value problem for (3.78) and (3.79) under some assumptions ((H1)-(H5) in [21]) which our problems satisfy except for the 2D volume-diffusion–controlled growth model with $\beta = 0$. This theorem claims that for any $x_0 \in L^1_d$, there exists a unique function $x \in C([0, \infty), L^1_d)$ such that (3.78) and (3.79) hold with $x(0, \varphi) = x_0(\varphi)$. This is equivalent to the existence and uniqueness (Theorem 2.1 of [21]) of a weak-star continuous solution $\nu : [0,\infty) \to \mathcal{P}_1$ for the transport equation (3.74) in the sense of distributions on $(0,\infty)\times(0,\infty)$ with initial value $\nu_0 = \hat{x}^{-1}(x_0)$.

Proposition 6.1 of [21] established an L^1 compactness result for (3.78) and (3.79), namely, given $T \in (0,\infty)$, for a compact sequence of initial values $\{x_{0k}\} \subset L^1_d$, the corresponding sequence of solutions x_k is compact in $C([0,T], L^1_d)$ and any limit x is again a solution of (3.78) and (3.79).

Based on this result, in the appendix, we prove an L^p compactness result for (3.78) and (3.79) for any $1 < p < \infty$, namely, given $T \in (0,\infty)$, for a sequence of initial values $\{x_{0k}\} \subset L^1_d \cap L^p((0,1))$ which is compact in $L^p((0,1))$, the corresponding sequence of solutions x_k is compact in $C([0,T], L^p((0,1)))$ and any limit x is again a solution of (3.78) and (3.79).

Given $x_0 \in L^1_d \cap L^{(n+1)/n}((0,1))$, for any positive integer N, we divide the interval $(0,1)$ uniformly into N subintervals and define a function $x_{0N}(\varphi)$ by

$$x_{0N}(\varphi) = N \int_{(i-1)/N}^{i/N} x_0(\psi)\, d\psi \ (=: x^i_{0N}) \quad \text{for } \frac{i-1}{N} \leqslant \varphi < \frac{i}{N}, \quad (i = 1, \cdots, N). \tag{3.80}$$

Then $x_{0N} \in L^1_d \cap L^{(n+1)/n}((0,1))$ is piecewise constant, and $x_{0N} \to x_0$ in $L^{(n+1)/n}((0,1))$ as $N \to \infty$.

By the above compactness and uniqueness results, the solutions $\{x_N\}$ for (3.78) and (3.79) with initial values $\{x_{0N}\}$ converge in the space $C([0,T], L^{(n+1)/n}((0,1)))$ to the solution x for (3.78) and (3.79) with initial value x_0.

For any N, $\{x^i_{0N}\}_{i=1}^N$ gives a discrete collection of particles and the corresponding collection of radii $\{R_i := (x^i_{0N})^{1/n}\}$ undergoes coarsening determined by (1.6) or

(1.9). Hence the estimates (3.27) and (3.66) claimed in Theorems 3.5 and 3.8 hold for

$$E_N(t) = \frac{\sum R_i(t)^{n-1}}{\sum R_i(t)^n} = \int_0^1 x_N(t,\varphi)^{(n-1)/n}\, d\varphi \Big/ \int_0^1 x_N(t,\varphi)\, d\varphi$$

and

$$L_N(t) = \frac{\sum R_i(t)^{n+1}}{\sum R_i(t)^n} = \int_0^1 x_N(t,\varphi)^{(n+1)/n}\, d\varphi \Big/ \int_0^1 x_N(t,\varphi)\, d\varphi.$$

We will establish the convergence results for $E_N(t)$ and $L_N(t)$ in Lemma 3.12. To do this, let's first prove a general convergence result for L^p functions.

Lemma 3.11. *For nonnegative functions $f_k, f \in L^p(\Omega)$ $(k = 1, 2, \cdots)$ with $1 < p < \infty$ and Ω a bounded open subset of $\mathbb{R}^n$, if*

$$\int_\Omega |f_k(y)^p - f(y)^p|\, dy \to 0 \quad \text{as } k \to \infty, \tag{3.81}$$

then

$$\int_\Omega |f_k(y) - f(y)|^p\, dy \to 0 \quad \text{as } k \to \infty. \tag{3.82}$$

Proof. The convergence (3.81) implies that $\{f_k\}$ is bounded in $L^p(\Omega)$. Notice that $L^p(\Omega)$ is a reflexive Banach space since $1 < p < \infty$. So there exist a subsequence $\{f_{k_j}\}$ and $w \in L^p(\Omega)$ such that f_{k_j} converges weakly to w: $f_{k_j} \rightharpoonup w$ as $j \to \infty$. Hence

$$||w||_{L^p(\Omega)} \leqslant \liminf_{j\to\infty} ||f_{k_j}||_{L^p(\Omega)} = ||f||_{L^p(\Omega)}. \tag{3.83}$$

By $f_{k_j}^p \to f^p$ in $L^1(\Omega)$, there exists a further subsequence, denoted the same, such that $f_{k_j}(y) \to f(y)$ for a.e. $y \in \Omega$. Hence by Fatou's lemma and Hölder's inequality,

$$\int_\Omega f^p = \int_\Omega \liminf_{j\to\infty} f^{p-1} f_{k_j} \leqslant \liminf_{j\to\infty} \int_\Omega f^{p-1} f_{k_j} = \int_\Omega f^{p-1} w \leqslant \Big(\int_\Omega f^p\Big)^{1-\frac{1}{p}} \Big(\int_\Omega w^p\Big)^{1/p}.$$

So

$$||f||_{L^p(\Omega)} \leqslant ||w||_{L^p(\Omega)}. \tag{3.84}$$

Comparing inequalities (3.83) and (3.84), we get $||f||_{L^p(\Omega)} = ||w||_{L^p(\Omega)}$. So

$$f_{k_j} \rightharpoonup w \quad \text{in} \;\; L^p(\Omega), \tag{3.85}$$

$$||f_{k_j}||_{L^p(\Omega)} \to ||w||_{L^p(\Omega)}. \tag{3.86}$$

Thus (see, e.g. [8])

$$f_{k_j} \to w \quad \text{in} \;\; L^p(\Omega), \tag{3.87}$$

and there exists a further subsequence of f_{k_j} that converges a.e. to w. Since $f_{k_j} \to f$ a.e. in Ω, we have $w = f$ and hence

$$f_{k_j} \to f \quad \text{in} \;\; L^p(\Omega). \tag{3.88}$$

The above argument works for every weakly convergent subsequence and hence the whole sequence f_k converges strongly to f in $L^p(\Omega)$. $\square$

Lemma 3.12. *For any $t > 0$, we have*

$$E_N(t) \to E(t) := \int_0^1 x(t,\varphi)^{(n-1)/n}\, d\varphi \Big/ \int_0^1 x(t,\varphi)\, d\varphi \quad \textit{as} \;\; N \to \infty, \tag{3.89}$$

$$L_N(t) \to L(t) := \int_0^1 x(t,\varphi)^{(n+1)/n}\, d\varphi \Big/ \int_0^1 x(t,\varphi)\, d\varphi \quad \textit{as} \;\; N \to \infty. \tag{3.90}$$

Proof. Fix $t > 0$. By the conservation of total mass and the convergence of initial value $x_{0N} \to x_0$ in $L^1((0,1))$,

$$\int_0^1 x_N(t,\varphi)\, d\varphi = \int_0^1 x_{0N}(\varphi)\, d\varphi \to \int_0^1 x_0(\varphi)\, d\varphi = \int_0^1 x(t,\varphi)\, d\varphi \quad \text{as } N \to \infty. \tag{3.91}$$

By the compactness of $\{x_N\}$ in $C\big([0,T], L^p((0,1))\big)$ for all $T > 0$ and all $p > 1$,

$$\int_0^1 x_N(t,\varphi)^{(n+1)/n}\, d\varphi \to \int_0^1 x(t,\varphi)^{(n+1)/n}\, d\varphi \quad \text{as } N \to \infty. \tag{3.92}$$

The convergence of $L_N(t)$ to $L(t)$ is an immediate consequence of (3.91) and (3.92).

Define $f_N = x_N(t,\varphi)^{(n-1)/n}$, $f = x(t,\varphi)^{(n-1)/n}$ and $p = n/(n-1)$, equation (3.91) implies that $f_N^p \to f^p$ as $N \to \infty$. So Lemma 3.11 implies $f_N \to f$ in $L^p((0,1))$ and consequently $f_N \to f$ in $L^1((0,1))$. Hence

$$\int_0^1 x_N(t,\varphi)^{(n-1)/n}\, d\varphi \to \int_0^1 x(t,\varphi)^{(n-1)/n}\, d\varphi. \tag{3.93}$$

The convergence of $E_N(t)$ to $E(t)$ is an immediate consequence of (3.91) and (3.93).

$\square$

To enable us to take limit in the estimates (3.27) and (3.66) claimed in Theorems 3.5 and 3.8, we will prove the following boundedness lemma for $E_N(t)$ and $L_N(t)$ and then apply Lebesgue's dominated convergence theorem.

Lemma 3.13. *Given $T > 0$, there exist positive constants M_1, m_2, M_2 depending only on n and T such that*

$$0 < E_N(t) \leqslant M_1, \quad m_2 < L_N(t) < M_2$$

uniformly in N and $0 \leqslant t \leqslant T$, with M_1, m_2, M_2 positive constants depending only on n and T.

Proof. By equation (3.91), there exist positive constants $\hat{m}_1$ and $\hat{M}_1$ such that for all N and all $t \geqslant 0$,

$$\hat{m}_1 \leqslant \int_0^1 x_N(t,\varphi)\, d\varphi \leqslant \hat{M}_1, \quad \hat{m}_1 \leqslant \int_0^1 x(t,\varphi)\, d\varphi \leqslant \hat{M}_1. \tag{3.94}$$

Then by Hölder's inequality,

$$\int_0^1 x_N(t,\varphi)^{(n-1)/n}\,d\varphi \leqslant \Big(\int_0^1 x_N(t,\varphi)\,d\varphi\Big)^{(n-1)/n} \leqslant \hat{M}_1^{(n-1)/n}. \tag{3.95}$$

Hence

$$E_N(t) = \int_0^1 x_N(t,\varphi)^{(n-1)/n}\,d\varphi \Big/ \int_0^1 x_N(t,\varphi)\,d\varphi \leqslant \hat{M}_1^{(n-1)/n}/\hat{m}_1 =: M_1. \tag{3.96}$$

By Hölder's inequality,

$$\hat{m}_1 \leqslant \int_0^1 x_N(t,\varphi)\,d\varphi \leqslant \Big\{\int_0^1 x_N(t,\varphi)^{(n+1)/n}\,d\varphi\Big\}^{n/(n+1)}. \tag{3.97}$$

So

$$L_N(t) = \int_0^1 x_N(t,\varphi)^{(n+1)/n}\,d\varphi \Big/ \int_0^1 x_N(t,\varphi)\,d\varphi \geqslant \hat{m}_1^{(n+1)/n}/\hat{M}_1 =: m_2. \tag{3.98}$$

In the appendix, we will prove that there exists a positive increasing function $G(t)$ such that $\int_0^1 x_N(t,\varphi)^{(n+1)/n}\,d\varphi \leqslant G(t) \leqslant G(T)$. So for all $0 \leqslant t \leqslant T$,

$$L_N(t) = \int_0^1 x_N(t,\varphi)^{(n+1)/n}\,d\varphi \Big/ \int_0^1 x_N(t,\varphi)\,d\varphi \leqslant G(T)/\hat{m}_1 =: M_2. \tag{3.99}$$

□

The above boundedness results and Lebesgue's dominated convergence theorem guarantee that we can take limit as $N \to \infty$ in the estimates for coarsening rates for discrete systems (Theorems 3.5 and 3.8). This procedure gives us the estimates in Theorems 3.9 and 3.10, with E and L defined as in Lemma 3.12, for the coarsening rates for solutions of (3.78)+(3.79) with initial value $x_0 \in L^1_d \cap L^{(n+1)/n}((0,1))$.

Our ultimate goal is to get estimates for coarsening rates for measure-valued solutions of the transport equations (1.7) and (1.10), respectively. To do this, we will establish the one-one correspondence between these measure-valued solutions, which are distributions of particle radii, and volume size-ranking solutions for (3.78)+(3.79). The estimates for coarsening rates for these measure-valued solutions are immediate consequences of this one-one correspondence and the estimates for these size-ranking solutions.

For any initial particle radius distribution $\mu(R) \in \mathcal{P}_{n+1}$, we define a particle volume distribution $\hat{\mu}(x) = (T\mu)(x)$ by requiring

$$\int_0^\infty f(x)\, d\hat{\mu}(x) = \int_0^\infty f(R^n)\, d\mu(R) \tag{3.100}$$

for all continuous function f with compact support. Then $\hat{\mu} \in \mathcal{P}_{(n+1)/n}$.

The size-ranking function $x_0(\varphi) = \hat{x}(\hat{\mu})$ defined as in (3.75) belongs to $L^1_d \cap L^{(n+1)/n}((0,1))$. Hence the solution $x(t,\varphi)$ of problem (3.78)+(3.79) with $x(0,\cdot) = x_0(\cdot)$ belongs to L^1_d and we can get the estimates as in Theorems 3.9 and 3.10 by the procedure described above.

It is proved in [21] that the mapping (3.75) is invertible and under the assumptions (H1)-(H5), the weak-star continuous mapping $\hat{\nu} : [0,\infty) \to \mathcal{P}_1$ related with $x(t,\varphi)$ through (3.75) is the unique measure-valued solution of the transport equation (3.74) in the sense of distributions with initial value $\hat{\mu}$. For any $t \in [0,\infty)$, we define a Borel measure ν_t by requiring

$$\int_0^\infty f(R)\, d\nu_t(R) = \int_0^\infty f(x^{1/n})\, d\hat{\nu}_t(x) \tag{3.101}$$

for all continuous function f with compact support. Then $\nu : [0,\infty) \to \mathcal{P}_n$ is weak-star continuous and is a measure-valued solution of the transport equation (1.7) or (1.10), with initial value $\nu_0 = \mu$. Again, since we can approximate a power function

$y \mapsto y^\alpha (\alpha > 0)$ by a monotonically increasing sequence of continuous functions with compact support, we get the following moment equivalence identity for ν and $\hat{\nu}$:

$$\int_0^\infty R^\alpha \, d\nu_t(R) = \int_0^\infty x^{\alpha/n} \, d\hat{\nu}_t(x) \quad \text{for any } \alpha > 0. \tag{3.102}$$

On the other hand, if we have a measure-valued solution $\nu : [0, \infty) \to \mathcal{P}_n$ for (1.7) or (1.10) with a given initial value μ, we can define $\hat{\nu} : [0, \infty) \to \mathcal{P}_1$ by (3.101) and $\hat{\nu}$ will be a measure-valued solution for (3.74) with initial value $\hat{\mu}$ defined by (3.100). The uniqueness of $\hat{\nu}$ then implies the uniqueness of ν.

The above analysis, together with the moment equivalence statements (3.77) and (3.102), gives us theorems 3.9 and 3.10 on coarsening rates for mean-field models with general initial distributions of particle radii.

Chapter 4

Monopole approximation of the Mullins-Sekerka model

4.1 Introduction

The Mullins-Sekerka model is a standard sharp-interface model that describes the evolution of a two phase mixture. Denoting G as the region consisting of one of the phases, we have the following system

$$\Delta u = 0 \quad \text{in } \mathbb{R}^3 \setminus \partial G, \tag{4.1}$$

$$[\nabla u \cdot n] = v \quad \text{on } \partial G, \tag{4.2}$$

$$u = \kappa \quad \text{on } \partial G. \tag{4.3}$$

Here u is a chemical potential, n is the normal to the interface ∂G, $[\nabla u \cdot n]$ is the jump of the normal derivative of u across ∂G, v is the normal velocity of the interface ∂G, and κ is the mean curvature of ∂G.

In the case when the volume fraction of the phase G is small, G breaks into a collection of small particles which are approximately spheres. Also, the centers of these spheres are approximately fixed in space. So it is reasonable to assume all the particles are spheres with centers not moving. We will derive the monopole approximation of the Mullins-Sekerka model in this case, and give an estimate for the upper bound on the coarsening rate of this approximation.

To achieve this goal, we first formally establish the gradient flow structure of a general Mullins-Sekerka model defined in the whole space. Then we will rigorously establish the gradient flow structure of the Mullins-Sekerka model for a collection of finitely many spherical particles with fixed centers. After that, we will construct an exact solution for the Mullins-Sekerka model for the latter case and will show that this solution is exactly the monopole approximation. Finally, we will apply the strategy established by Kohn and Otto [12] to get an upper bound on the coarsening rate for the monopole approximation.

4.2 Gradient flow structure: general situation

It is shown in [19] that the Mullins-Sekerka model has a gradient flow structure when it is considered in a finite box with periodic boundary conditions. We will extend their argument to show that the model also has a gradient flow structure when it is considered in the whole space $\mathbb{R}^3$. This argument is formal since we do not consider the regularity of solutions of the Mullins-Sekerka model. Another way to do this argument is to assume that the solutions are smooth enough. We need the following vector space

$$\mathcal{H} := \left\{ u \in L^6(\mathbb{R}^3) : \int_{\mathbb{R}^3} |\nabla u|^2 < \infty \right\}. \tag{4.4}$$

Define an inner porduct

$$(u, v)_{\mathcal{H}} := \int_{\mathbb{R}^3} \nabla u \cdot \nabla v \; dx \tag{4.5}$$

and a corresponding norm

$$||u||_{\mathcal{H}} := ||\nabla u||_{L^2(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3} |\nabla u|^2 \right)^{1/2}. \tag{4.6}$$

By the Gagliardo-Nirenberg-Sobolev inequality (see, e.g. [9]), there exists a constant C such that

$$||u||_{L^6(\mathbb{R}^3)} \leqslant C\, ||\nabla u||_{L^2(\mathbb{R}^3)} \qquad \text{for all } u \in \mathcal{H}. \tag{4.7}$$

Lemma 4.1. *$\mathcal{H}$ is a Hilbert space under inner product (4.5) and $C_c^\infty(\mathbb{R}^3)$(the set of infinitely differentiable functions with compact support) is dense in $\mathcal{H}$.*

Proof. (1). To show that $\mathcal{H}$ is a Hilbert space, we need only show that $\mathcal{H}$ is complete under norm (4.6). Suppose $\{u_n : n = 1, 2, \cdots\}$ is a Cauchy sequence under norm (4.6), i.e., $\{\nabla u_n\}$ is Cauchy in $L^2(\mathbb{R}^3, \mathbb{R}^3)$. Then there exists $\mathbf{w} \in L^2(\mathbb{R}^3, \mathbb{R}^3)$ such that

$$||\nabla u_n - \mathbf{w}||_{L^2(\mathbb{R}^3)} \to 0 \qquad \text{as } n \to \infty. \tag{4.8}$$

By (4.7),

$$||u_n - u_m||_{L^6(\mathbb{R}^3)} \leqslant C\, ||\nabla u_n - \nabla u_m||_{L^2(\mathbb{R}^3)} \to 0 \qquad \text{as } n, m \to \infty. \tag{4.9}$$

So $\{u_n\}$ is a Cauchy sequence in $L^6(\mathbb{R}^3)$ and there is $u \in L^6(\mathbb{R}^3)$ such that

$$||u_n - u||_{L^6(\mathbb{R}^3)} \to 0 \qquad \text{as } n \to \infty. \tag{4.10}$$

For any $\varphi \in C_c^\infty(\mathbb{R}^3)$, we have

$$\int_{\mathbb{R}^3} \mathbf{w}\varphi = \lim_{n\to\infty} \int_{\mathbb{R}^3} \nabla u_n \varphi = -\lim_{n\to\infty} \int_{\mathbb{R}^3} u_n \nabla\varphi = -\int_{\mathbb{R}^3} u \nabla\varphi. \tag{4.11}$$

So $\mathbf{w} = \nabla u$ in the sense of distributions. Hence $u \in \mathcal{H}$ and $||u_n - u||_{\mathcal{H}} \to 0$ as $n \to \infty$.

(2). To show that $C_c^\infty(\mathbb{R}^3)$ is dense in $\mathcal{H}$, we define a smooth cutoff function χ that satisfies $0 \leqslant \chi \leqslant 1$ and

$$\chi(x) = \begin{cases} 1, & \text{if } |x| < 1, \\ \\ 0, & \text{if } |x| > 2. \end{cases} \tag{4.12}$$

Given $u \in \mathcal{H}$, we define

$$u_k(x) = u(x)\chi(x/k). \tag{4.13}$$

Then

$$u_k(x) = u(x) \ \text{ for } |x| < k \quad \text{ and } \quad u_k(x) = 0 \ \text{ for } |x| > 2k. \tag{4.14}$$

Hence $u_k \in H_0^1(B(0,2k))$ and it can be approximated by functions $\varphi_k^i \in C_c^\infty(B(0,2k))$ under the norm of $H_0^1(B(0,2k))$. Consequently

$$||\varphi_k^i - u_k||_{\mathcal{H}} \to 0 \text{ as } i \to \infty. \tag{4.15}$$

So we need only prove that

$$||u_k - u||_{\mathcal{H}} \to 0 \text{ as } k \to \infty. \tag{4.16}$$

Note that

$$\nabla u_k(x) = \chi(x/k)\nabla u(x) + \frac{1}{k}u(x)\nabla\chi(x/k) = \nabla u(x) \quad \text{ if } |x| < k. \tag{4.17}$$

So

$$
\begin{aligned}
||u_k - u||_{\mathcal{H}}^2 &= \int_{\mathbb{R}^3} |\nabla u_k(x) - \nabla u(x)|^2 \, dx \\
&= \int_{|x| \geqslant k} \left| \nabla u(x)(\chi(x/k) - 1) + \frac{1}{k} u(x) \nabla \chi(x/k) \right|^2 \, dx \\
&\leqslant 2 \int_{|x| \geqslant k} \Big((\chi(x/k) - 1)^2 |\nabla u(x)|^2 + \frac{1}{k^2} u(x)^2 |\nabla \chi(x/k)|^2 \Big) \, dx \\
&\leqslant 2 \int_{|x| \geqslant k} |\nabla u(x)|^2 \, dx + \frac{2}{k^2} \int_{|x| \geqslant k} |u(x)|^2 \, |\nabla \chi(x/k)|^2 \, dx \\
&=: I + II
\end{aligned}
\tag{4.18}
$$

The first term I approaches 0 as $k \to \infty$ since $\int_{\mathbb{R}^3} |\nabla u|^2 < \infty$. For the second term, by Hölder's inequality,

$$
\begin{aligned}
II &\leqslant \frac{2}{k^2} \Big(\int_{|x| \geqslant k} |u(x)|^6 \, dx \Big)^{1/3} \Big(\int_{|x| \geqslant k} |\nabla \chi(x/k)|^3 \, dx \Big)^{2/3} \\
&= 2 \Big(\int_{|x| \geqslant k} |u(x)|^6 \, dx \Big)^{1/3} \Big(\int_{1 \leqslant |x| \leqslant 2} |\nabla \chi(x)|^3 \, dx \Big)^{2/3} \\
&\leqslant C_1 \Big(\int_{|x| \geqslant k} |u(x)|^6 \, dx \Big)^{1/3} \\
&\to 0 \quad \text{ as } k \to \infty
\end{aligned}
\tag{4.19}
$$

since $u \in L^6(\mathbb{R}^3)$. Here C_1 is a constant depending on nothing but the choice of χ. So equation (4.16) holds. Together with the smooth approximation (4.15), we can find smooth functions $\psi_k \in C_c^\infty(\mathbb{R}^3)$ such that

$$
||\psi_k - u||_{\mathcal{H}} \to 0 \quad \text{ as } k \to \infty. \tag{4.20}
$$

□

Define a manifold $\mathcal{M}$ to be all bounded sets $G \in \mathbb{R}^3$ with smooth boundary, and a submanifold $\mathcal{M}_0$ to be all bounded sets $G \in \mathbb{R}^3$ with smooth boundary that have a fixed volume, i.e., $\mathcal{L}^3(G) = const$. The tangent space $T_G\mathcal{M}$ consists of all possible normal velocities of ∂G for $G \in \mathcal{M}$. And the tangent space $T_G\mathcal{M}_0$ is described by the admissible normal velocities of ∂G for $G \in \mathcal{M}_0$, i.e.,

$$T_G\mathcal{M}_0 = \big\{v : \partial G \to \mathbb{R} : \int_{\partial G} v \, dS = 0\big\}. \tag{4.21}$$

We want to define a metric on the tangent space $T_G\mathcal{M}$. To do this, let's first define a Neumann problem solver

$$J_G : T_G\mathcal{M} \to \mathcal{H}. \tag{4.22}$$

For any $v \in T_G\mathcal{M}$, define $J_G v := u \in \mathcal{H}$ to be the unique solution for the Neumann boundary problem

$$-\Delta u = 0 \quad \text{in } \mathbb{R}^3 \setminus \partial G, \tag{4.23}$$

$$[\nabla u \cdot n] = v \quad \text{on } \partial G. \tag{4.24}$$

Equivalently, $u \in \mathcal{H}$ is the unique solution for the variational problem

$$\int_{\mathbb{R}^3} \nabla u \cdot \nabla w = -\int_{\partial G} vw \quad \text{for all } w \in \mathcal{H}. \tag{4.25}$$

Lemma 4.2. *Problem (4.25) has a unique solution $u \in \mathcal{H}$.*

Proof. The left hand side of (4.25) is a coercive bilinear form on $\mathcal{H}$. For any given $v \in T_G\mathcal{M}$, we have

$$\begin{aligned}
\left|-\int_{\partial G} vw\right| &\leqslant ||v||_{H^{-1/2}(\partial G)} \, ||w||_{H^{1/2}(\partial G)} \\
&\leqslant C||v||_{H^{-1/2}(\partial G)}\Big(||w||_{L^2(G)} + ||\nabla w||_{L^2(G)}\Big) \quad \text{(by trace theorem)} \\
&\leqslant C||v||_{H^{-1/2}(\partial G)}\Big(||w||_{L^6(G)} + ||\nabla w||_{L^2(G)}\Big) \quad \text{(by Hölder's inequality)} \\
&\leqslant C||v||_{H^{-1/2}(\partial G)}||w||_{\mathcal{H}} \quad \text{by (4.7).}
\end{aligned} \tag{4.26}$$

So for any given $v \in T_G\mathcal{M}$, the right hand side of (4.25) is a bounded linear functional on $\mathcal{H}$. By the Lax-Milgram theorem, there exists a unique $u \in \mathcal{H}$ such that (4.25) holds. $\square$

The Neumann problem solver J_G gives a 1–1 correspondence between the tangent space $T_G\mathcal{M}$ and a subspace $\mathcal{U}_G$ of $\mathcal{H}$:

$$\mathcal{U}_G := \{u \in \mathcal{H} : \Delta u = 0 \quad \text{in } \mathbb{R}^3 \setminus \partial G\}. \tag{4.27}$$

Now we can define a metric on $T_G\mathcal{M}$ by

$$g(v^1, v^2) = \int_{\mathbb{R}^3} \nabla u^1(x) \cdot \nabla u^2(x)\, dx, \tag{4.28}$$

where $u^i = J_G v^i$ for $i = 1, 2$.

Define a functional A to be the total interfacial area

$$A(G) := |\partial G| = \int_{\partial G} 1\, dS. \tag{4.29}$$

Denote dA_G the cotangent vector at point $G \in \mathcal{M}$ generated by the differential of A. It is converted into a tangent vector $\text{grad}A_G = \hat{v} \in T_G\mathcal{M}$ by the metric g:

$$< dA_G, \tilde{v} >= g(\hat{v}, \tilde{v}) \quad \text{for all } \tilde{v} \in T_G\mathcal{M}. \tag{4.30}$$

We want to find this $\hat{v} \in T_G\mathcal{M}$. It is well known that the differential of A is given by its curvature

$$< dA_G, \tilde{v} >= \int_{\partial G} \kappa \tilde{v}\, dS. \tag{4.31}$$

Comparing (4.28) and (4.31), we need to find a function $\hat{u} \in \mathcal{U}_G$ such that

$$\int_{\partial G} \kappa \tilde{v}\, dS = \int_{\mathbb{R}^3} \nabla \hat{u} \cdot \nabla \tilde{u} \quad \text{for all } \tilde{v} \in T_G\mathcal{M} \tag{4.32}$$

where $\tilde{u} = J_G \tilde{v}$. Equivalently, we want to find $\hat{u} \in \mathcal{U}_G$ such that

$$\int_{\partial G} \kappa \tilde{v} \, dS = \int_{\mathbb{R}^3} \nabla \hat{u} \cdot \nabla \tilde{u} \quad \text{for all } \tilde{u} \in \mathcal{U}_G \tag{4.33}$$

where $\tilde{v} = [\nabla \tilde{u} \cdot n]$ on ∂G.

Lemma 4.3. *Problem (4.33) has a unique solution in $\mathcal{U}_G$.*

Proof. The right hand side is a coercive bilinear form on $\mathcal{U}_G$. We need only show that the left hand side is a bounded linear functional on $\mathcal{U}_G$. For any $\tilde{u} \in \mathcal{U}_G$,

$$\left| \int_{\partial G} \kappa \tilde{v} \, dS \right| = \left| \int_{\partial G} \kappa \left[\nabla \tilde{u} \cdot n \right] \, dS \right| \leqslant ||\kappa||_{H^{1/2}(\partial G)} \, || \left[\nabla \tilde{u} \cdot n \right] ||_{H^{-1/2}(\partial G)}. \tag{4.34}$$

Note that

$$[\nabla \tilde{u} \cdot n] = (\nabla \tilde{u} \cdot n)_{\partial G^+} - (\nabla \tilde{u} \cdot n)_{\partial G^-}, \tag{4.35}$$

where ∂G^+ and ∂G^- indicate that $\nabla \tilde{u} \cdot n$ is defined by values of $\tilde{u}$ in $\mathbb{R}^3 \setminus G$ and G, respectively. It is known that for any bounded open set $\Omega \subset \mathbb{R}^3$ with smooth boundary and any $\mathbf{w} \in L^2(\Omega; \mathbb{R}^3)$ with $\mathrm{div}\mathbf{w} \in L^2(\Omega)$, $\mathbf{w} \cdot n|_{\partial\Omega}$ is well defined and

$$||\mathbf{w} \cdot n||_{H^{-1/2}(\partial\Omega)} \leqslant C\big(||\mathbf{w}||_{L^2(\Omega)} + ||\mathrm{div}\mathbf{w}||_{L^2(\Omega)}\big), \tag{4.36}$$

where C is a generic constant depending only on Ω (see, e.g. [26]). Then

$$\begin{aligned} ||(\nabla \tilde{u} \cdot n)_{\partial G^-}||_{H^{-1/2}(\partial G)} &\leqslant C\big(||\nabla \tilde{u}||_{L^2(G)} + ||\mathrm{div} \nabla \tilde{u}||_{L^2(G)}\big) \\ &= C||\nabla \tilde{u}||_{L^2(G)} \leqslant C||\tilde{u}||_{\mathcal{H}}. \end{aligned} \tag{4.37}$$

Here C is a generic constant depending only on G. It remains to get an estimate for $||(\nabla \tilde{u} \cdot n)_{\partial G^+}||_{H^{-1/2}(\partial G)}$. We can not directly apply estimates as above because now we have an unbounded domain $\mathbb{R}^3 \setminus G$. Choose a ball $B_R = B(0,R)$ big enough such that $G \subset \bar{G} \subset B_R$. Define a smooth cutoff function φ such that φ is compactly supported in $B_{2R} = B(0,2R)$ and $\varphi = 1$ in B_R. Then

$$\nabla(\tilde{u}\varphi) = (\nabla\tilde{u})\varphi + \tilde{u}\nabla\varphi \in L^2(B_{2R} \setminus G) \tag{4.38}$$

and

$$\mathrm{div}\nabla(\tilde{u}\varphi) = (\Delta\tilde{u})\varphi + 2\nabla\tilde{u}\cdot\nabla\varphi + \tilde{u}\Delta\varphi = 2\nabla\tilde{u}\cdot\nabla\varphi + \tilde{u}\Delta\varphi \in L^2(B_{2R} \setminus G). \tag{4.39}$$

It can be seen that

$$\nabla(\tilde{u}\varphi)\cdot n = 0 \quad \text{ on } \partial B_{2R} \tag{4.40}$$

and

$$\nabla(\tilde{u}\varphi)\cdot n = (\nabla\tilde{u}\cdot n)_{\partial G^+} \quad \text{ on } \partial G. \tag{4.41}$$

So

$$\begin{aligned}
||(\nabla\tilde{u}\cdot n)_{\partial G^+}||_{H^{-1/2}(\partial G)} &= ||\nabla(\tilde{u}\varphi)\cdot n||_{H^{-1/2}(\partial(B_{2R}\setminus G))} \\
&\leqslant C\big(||\nabla(\tilde{u}\varphi)||_{L^2(B_{2R}\setminus G)} + ||\mathrm{div}\nabla(\tilde{u}\varphi)||_{L^2(B_{2R}\setminus G)}\big) \\
&\leqslant C\big(||\tilde{u}||_{L^2(B_{2R})} + ||\nabla\tilde{u}||_{L^2(B_{2R})}\big) \\
&\leqslant C\big(||\tilde{u}||_{L^6(B_{2R})} + ||\nabla\tilde{u}||_{L^2(B_{2R})}\big) \\
&\leqslant C||\tilde{u}||_{\mathcal{H}},
\end{aligned} \tag{4.42}$$

where C depends only on G, R and the choice of φ.

Combining (4.37) and (4.42), we conclude that

$$||\,[\nabla\tilde{u}\cdot n]\,||_{H^{-1/2}(\partial G)} \leqslant C||\tilde{u}||_{\mathcal{H}} \quad \text{ for all } \tilde{u} \in \mathcal{U}_G. \tag{4.43}$$

Inequalities (4.34) and (4.43) guarantee that the left hand side of (4.33) is a bounded linear functional on $\mathcal{U}_G$. This completes our proof. $\square$

Note. Problem (4.33) is equivalent to

$$
\begin{aligned}
-\Delta\hat{u} &= 0 \quad \text{in } \mathbb{R}^3 \setminus \partial G, && (4.44)\\
\hat{u} &= -\kappa \quad \text{on } \partial G. && (4.45)
\end{aligned}
$$

Define

$$\text{grad}A_G = [\nabla\hat{u} \cdot n] \in T_G\mathcal{M}. \tag{4.46}$$

Then by (4.28), (4.31) and (4.33), we have

$$g(\text{grad}A_G, \tilde{v}) = \int_{\mathbb{R}^3} \nabla\hat{u} \cdot \nabla\tilde{u} =< dA_G, \tilde{v} > \qquad \text{for all } \tilde{v} \in T_G\mathcal{M}, \tag{4.47}$$

where $\tilde{u} = J_G\tilde{v}$. The above equation shows that $\text{grad}A_G$ is really the gradient of A_G.

So far we have found a gradient $\text{grad}A_G \in T_G\mathcal{M}$ for any $G \in \mathcal{M}$. Our goal is to restrict the functional A onto the submanifold $\mathcal{M}_0$ and define a corresponding gradient $\text{grad}_0 A_G$ on $T_G\mathcal{M}_0$ for any $G \in \mathcal{M}_0$. We claim that $\text{grad}_0 A_G$ is the projection of $\text{grad}A_G$ onto $T_G\mathcal{M}_0$ for every $G \in \mathcal{M}_0$.

Let's first consider the orthogonal complement of $T_G\mathcal{M}_0$ in $T_G\mathcal{M}$. Any $v \in T_G\mathcal{M}$ that is orthogonal to $T_G\mathcal{M}_0$ should satisfy

$$0 = g(v, \tilde{v}) \quad \text{for all } \tilde{v} \in T_G\mathcal{M}_0. \tag{4.48}$$

That is, for ψ satisfying

$$
\begin{aligned}
-\Delta\psi &= 0 \quad \text{in} \mathbb{R}^3 \setminus \partial G, && (4.49)\\
[\nabla\psi \cdot n] &= v \quad \text{on} \partial G, && (4.50)
\end{aligned}
$$

we need

$$0=\int_{\partial G}\psi\tilde{v}\,dS \quad \text{for all } \tilde{v}\in T_G\mathcal{M}_0. \tag{4.51}$$

By the definition of $T_G\mathcal{M}_0$, we immediately conclude that the above equation holds if and only if ψ is a constant on ∂G. We choose $\psi_0\in\mathcal{H}$ to be the one such that

$$\begin{aligned} -\Delta\psi_0 &= 0 \quad \text{in } \mathbb{R}^3\setminus\partial G, & (4.52)\\ \psi_0 &= 1 \quad \text{on } \partial G. & (4.53)\end{aligned}$$

The orthogonal complement of $T_G\mathcal{M}_0$ is then the span of $[\nabla\psi_0\cdot n]$.

Define θ to be

$$\theta=-\frac{\int_{\partial G}[\nabla\hat{u}\cdot n]}{\int_{\partial G}[\nabla\psi_0\cdot n]}. \tag{4.54}$$

Then the total flux of $\hat{u}+\theta\psi_0$ on ∂G is 0. Define

$$\mathrm{grad}_0 A_G=[\nabla(\hat{u}+\theta\psi_0)\cdot n]\in T_G\mathcal{M}_0. \tag{4.55}$$

Then

$$g(\mathrm{grad}_0 A_G,\tilde{v})=g(\mathrm{grad}A_G,\tilde{v})=\int_{\mathbb{R}^3}\nabla\hat{u}\cdot\nabla\tilde{u}=\int_{\partial G}\kappa\tilde{v}=<dA_G,\tilde{v}> \tag{4.56}$$

for all $\tilde{v}\in T_G\mathcal{M}_0$. This means $\mathrm{grad}_0 A_G$ is really the projection of $\mathrm{grad}A_G$ onto $T_G\mathcal{M}_0$ for all $G\in\mathcal{M}_0$ and it is the gradient of A_G on $T_G\mathcal{M}_0$.

Now let's consider the gradient flow

$$v=-\mathrm{grad}_0 A_G \quad \text{on } \partial G. \tag{4.57}$$

Define

$$u=-\hat{u}-\theta\psi+\theta. \tag{4.58}$$

Then u satisfies

$$-\Delta u = 0 \quad \text{in } \mathbb{R}^3 \setminus G, \tag{4.59}$$

$$u = \kappa \quad \text{on } \partial G, \tag{4.60}$$

$$[\nabla u \cdot n] = v \quad \text{on } \partial G. \tag{4.61}$$

Equations (4.59)-(4.61) imply that u solves the Mullins-Sekerka model (4.1)-(4.3) with the normal velocity of ∂G being $v = -\text{grad}_0 A_G$. In this sense, we say that the Mullins-Sekerka model has a gradient flow structure.

Special attention should be paid to the spatial constant θ. Since both $\hat{u}$ and ψ_0 are functions in $\mathcal{H}$, they are L^6 integrable over the whole space and hence have limit zero at infinity. So by the definition of u (equation (4.58)), θ is the limit of u at infinity. In this sense, we can say that θ is a mean field.

4.3 Gradient flow restricted to spherical particles

Now we consider the restriction of the above gradient flow on a submanifold $\mathcal{N}$ of $\mathcal{M}$ consisting of collections of finitely many non-overlapping spherical particles with centers spatially fixed. In this section all arguments are rigorous since we do not need to worry about the regularity of solutions any more. Let N be the number of particles. Suppose the spheres are labeled as P_i with centers x_i and radii R_i. Now $G = \cup_i P_i$. The tangent space $T_G\mathcal{N}$ consists of all N-component vectors $\mathbf{v} = (v_i)$ with $v_i \in \mathbb{R}$ being the normal velocity of ∂P_i. Define a submanifold $\mathcal{N}_0$ of $\mathcal{N}$ to be collections of N non-overlapping spheres with spatially fixed centers and fixed total volume. Then the tangent space of $\mathcal{N}_0$ is

$$T_G\mathcal{N}_0 := \Big\{\mathbf{v} = (v_i) : \sum_i R_i^2 v_i = 0\Big\}. \tag{4.62}$$

We want to apply a metric similar to g defined by (4.28) in section 4.2 onto our tangent space $T_G\mathcal{N}$. To do this, we need consider some Neumann and Dirichlet boundary problems and let us first define some Hilbert space in which we will consider the solvability of those problems. For any given $G = \cup P_i$, define

$$\mathcal{W}_G := \big\{u \in \mathcal{H} : \Delta u = 0 \text{ in } \mathbb{R}^3 \setminus \cup P_i, \quad [\nabla u \cdot n] = c_i \text{ on } \partial P_i, \quad c_i \in \mathbb{R}\big\} \tag{4.63}$$

It is easy to see that $\mathcal{W}_G$ is a N dimensional Hilbert space, where N is the number of spheres in G.

For any $\mathbf{v} \in T_G\mathcal{N}$, define a Neumann problem solver

$$J_G^{\mathcal{N}} : \mathbf{v} \mapsto u \tag{4.64}$$

where $u \in \mathcal{W}_G$ solves the Neumann problem

$$-\Delta u \;=\; 0 \quad \text{in } \mathbb{R}^3 \setminus \cup P_i, \tag{4.65}$$

$$[\nabla u \cdot n] \;=\; v_i \quad \text{on } \partial P_i. \tag{4.66}$$

Equivalently, the variational form of the above system is

$$\int_{\mathbb{R}^3} \nabla u \cdot \nabla w = -\sum \int_{\partial P_i} v_i w \quad \text{for all } w \in \mathcal{W}_G. \tag{4.67}$$

Lemma 4.4. *Problem (4.67) has a unique solution in $\mathcal{W}_G$.*

Proof. The left hand side is a coercive bilinear form on the Hilbert space $\mathcal{H}$ and consequently it is coercive on the subspace $\mathcal{W}_G$. The right hand side is a bounded linear functional on $\mathcal{H}$ and hence it is a bounded linear functional on $\mathcal{W}_G$. The Lax-Milgram theorem then gives us the existence and uniqueness of the solution. $\square$

The metric can now be defined through this Nuemann solver $J_G^{\mathcal{N}}$. Using the same notation as in 4.2, we define

$$g(\mathbf{v}^1, \mathbf{v}^2) = \int_{\mathbb{R}^3} \nabla u^1 \cdot \nabla u^2, \tag{4.68}$$

where $u^\alpha = J_G^{\mathcal{N}} \mathbf{v}^\alpha$ for $\alpha = 1, 2$.

For any $G = \cup P_i$, define a rescaled interfacial area functional

$$A_G^* = \frac{1}{2} \left\{ 4\pi \sum R_i^2 \right\}, \tag{4.69}$$

where the factor 1/2 is needed for later convenience. The differential of A_G^* with respect to a virtual velocity $\tilde{\mathbf{v}} = (\tilde{v}_i) \in T_G\mathcal{N}$ is

$$< dA_G^*, \tilde{\mathbf{v}} >= 4\pi \sum R_i \tilde{v}_i = \sum \int_{\partial P_i} \frac{1}{R_i} \tilde{v}_i \; dS. \tag{4.70}$$

We want to find the gradient of A_G^* deduced by its differential through the metric g. That is, we want to find $\text{grad}A_G^* = \hat{\mathbf{v}} \in T_G\mathcal{N}$ such that $< dA_G^*, \tilde{\mathbf{v}} >= g(\hat{\mathbf{v}}, \tilde{\mathbf{v}})$ for all $\tilde{\mathbf{v}} \in T_G\mathcal{N}$. Comparing the right hand sides of (4.68) and (4.70), we need only find $\hat{u} \in \mathcal{W}_G$ such that

$$\sum \int_{\partial P_i} \frac{1}{R_i} \tilde{v}_i = \int_{\mathbb{R}^3} \nabla \hat{u} \cdot \nabla \tilde{u} \quad \text{for all } \tilde{u} \in \mathcal{W}_G, \tag{4.71}$$

where $\tilde{v}_i = [\nabla \tilde{u} \cdot n]$ on ∂P_i and $\hat{\mathbf{v}}$ will be defined componently as $[\nabla \hat{u} \cdot n]$ on ∂P_i for each i.

Lemma 4.5. *Problem (4.71) has a unique solution in $\mathcal{W}_G$.*

Proof. Since $\mathcal{W}_G$ is a subspace of $\mathcal{U}_G$, the proof follows the same as that of lemma 4.3 without change. $\square$

Remark. $\hat{u}$ is in fact the unique solution in $\mathcal{W}_G$ for the following elliptic problem with an averaged Dirichlet boundary condition:

$$\begin{aligned} -\Delta \hat{u} &= 0, & (4.72) \\ \fint_{\partial P_i} \hat{u} &= -\frac{1}{R_i} \quad \text{for any } i. & (4.73) \end{aligned}$$

This can be seen through an integration by parts:

$$\int_{\mathbb{R}^3} \nabla \hat{u} \cdot \nabla \tilde{u} = -\sum \int_{\partial P_i} \hat{u} \, [\nabla \tilde{u} \cdot n] = -\sum \int_{\partial P_i} \hat{u} \tilde{v}_i. \tag{4.74}$$

Comparing (4.71) and (4.74), we have

$$\sum \int_{\partial P_i} (\frac{1}{R_i} + \hat{u}) \tilde{v}_i = 0 \quad \text{for all } (\tilde{v}_i) \in T_G \mathcal{N}, \tag{4.75}$$

which is exactly (4.73) since $\tilde{v}_i$ is arbitrary in $\mathbb{R}$.

Now $\text{grad} A^*_G \in T_G \mathcal{N}$ is well defined for each $G \in \mathcal{N}$. We want to restrict the functional A^* onto the submanifold $\mathcal{N}_0$ and define a corresponding gradient $\text{grad}_0 A^*_G \in T_G \mathcal{N}_0$ for each $G \in \mathcal{N}_0$. Similar to section 4.2, we claim that $\text{grad}_0 A^*_G$ is the projection of $\text{grad} A^*_G$ onto $T_G \mathcal{N}_0$ for every $G \in \mathcal{N}_0$.

Any $(v_i) \in T_G \mathcal{N}$ that is orthogonal to $T_G \mathcal{N}_0$ should satisfy

$$0 = g((v_i), (\tilde{v}_i)) \quad \text{for all } (\tilde{v}_i) \in T_G \mathcal{N}_0. \tag{4.76}$$

That is, for ψ satisfying

$$\begin{aligned} -\Delta \psi &= 0 \quad \text{in } \mathbb{R}^3 \setminus \cup P_i, & (4.77) \\ [\nabla \psi \cdot n] &= v_i \quad \text{on } \partial P_i \text{ for each } i, & (4.78) \end{aligned}$$

we need

$$0 = \sum \int_{\partial P_i} \psi \tilde{v}_i \, dS \quad \text{for all } \tilde{v} \in T_G\mathcal{N}_0. \tag{4.79}$$

By the definition of $T_G\mathcal{N}_0$, we conclude that the above equation is equivalent to requiring that there exists a constant c such that

$$\int_{\partial P_i} \psi \, dS = cR_i^2 \quad \text{for all } i. \tag{4.80}$$

In other words, we need $⨍_{\partial P_i} \psi \, dS = c$ for every i. We choose $\psi_0 \in \mathcal{W}$ to be the one such that

$$-\Delta\psi_0 = 0 \quad \text{in } \mathbb{R}^3 \setminus \partial G, \tag{4.81}$$

$$⨍_{\partial P_i} \psi_0 \, dS = 1 \quad \text{for any } i. \tag{4.82}$$

The orthogonal complement of $T_G\mathcal{N}_0$ in $T_G\mathcal{N}$ is then the span of ψ_0.

Define θ to be

$$\theta = -\frac{\sum \int_{\partial P_i} [\nabla \hat{u} \cdot n]}{\sum \int_{\partial P_i} [\nabla \psi_0 \cdot n]}. \tag{4.83}$$

The total flux of $\hat{u} + \theta\psi_0$ on $\cup P_i$ is 0. By the definition of $\mathcal{W}$, we have that $[\nabla \hat{u} \cdot n]$ and $[\nabla \psi_0 \cdot n]$ are both constants on each ∂P_i. Define

$$\text{grad}_0 A_G^* = [\nabla(\hat{u} + \theta\psi_0) \cdot n] \in T_G\mathcal{N}_0. \tag{4.84}$$

Then

$$\begin{aligned} g(\text{grad}_0 A_G^*, (\tilde{v}_i)) &= g(\text{grad} A_G^*, (\tilde{v}_i)) = \int_{\mathbb{R}^3} \nabla \hat{u} \cdot \nabla \tilde{u} = \sum \int_{\partial P_i} \frac{1}{R_i} \tilde{v}_i \\ &= < dA_G^*, (\tilde{v}_i) > \end{aligned} \tag{4.85}$$

for all $(\tilde{v}_i) \in T_G\mathcal{N}_0$. This means that $\text{grad}_0 A_G^*$ is really the projection of $\text{grad} A_G^*$ onto $T_G\mathcal{N}_0$ and it is the gradient of A_G^* on $T_G\mathcal{N}_0$.

Now let's consider the gradient flow

$$\dot{R}_i = -\mathrm{grad}_0 A_G^* \quad \text{on each } \partial P_i. \tag{4.86}$$

Define

$$u = -\hat{u} - \theta\psi_0 + \theta. \tag{4.87}$$

Then u satisfies

$$-\Delta u = 0 \quad \text{in } \mathbb{R}^3 \setminus \cup P_i, \tag{4.88}$$

$$⨍_{\partial P_i} u \, dS = \frac{1}{R_i} \quad \text{for any } i, \tag{4.89}$$

$$[\nabla u \cdot n] = \dot{R}_i \quad \text{on each } \partial P_i. \tag{4.90}$$

Equations (4.88)-(4.90) imply that u solves a Mullins-Sekerka type problem with a modified Gibbs-Thomson condition (4.89).

Now let us try to write down an expression of u in terms of $R_i, \dot{R}_i$ and the mean field θ. This expression will lead to the monopole approximation of the Mullins-Sekerka model.

For each index i, we want to find $u_i \in \mathcal{W}_G$ such that

$$\Delta u_i = 0 \quad \text{in } \mathbb{R}^3 \setminus \partial P_i, \tag{4.91}$$

$$[\nabla u_i \cdot n] = \dot{R}_i \quad \text{on } \partial P_i. \tag{4.92}$$

Recall that x_i is the center of P_i. This u_i can be written as a monopole with an undetermined magnitude A_i:

$$u_i = \begin{cases} \frac{A_i}{|x-x_i|} & \text{in } \mathbb{R}^3 \setminus P_i, \\ \\ \frac{A_i}{R_i} & \text{in } P_i. \end{cases} \tag{4.93}$$

To make $[\nabla u_i \cdot n] = \dot{R}_i$ on ∂P_i, we need

$$-\frac{A_i}{R_i^2} = \dot{R}_i, \tag{4.94}$$

i.e., $A_i = -R_i^2 \dot{R}_i$. The superposition of these u_i gives the unique solution $u^* = \sum_i u_i \in \mathcal{W}_G$ that satisfies

$$\Delta u^* = 0 \quad \text{in } \mathbb{R}^3 \setminus \cup_i \partial P_i, \tag{4.95}$$

$$[\nabla u^* \cdot n] = \dot{R}_i \quad \text{on } \partial P_i \text{ for each } i. \tag{4.96}$$

Hence

$$u^* = \begin{cases} \sum_j \frac{A_j}{|x-x_j|} & \text{in } \mathbb{R}^3 \setminus \cup_j P_j, \\ \\ \sum_{j \neq i} \frac{A_j}{|x-x_j|} + \frac{A_i}{R_i} & \text{in } P_i \text{ for each } i, \end{cases} \tag{4.97}$$

and

$$u^* = -\hat{u} - \theta\psi. \tag{4.98}$$

Consequently,

$$u = \theta + u^*. \tag{4.99}$$

Since $A_j/|x-x_j|$ is harmonic away from the point x_j, the mean value property of harmonic functions and equation (4.89) give us

$$\frac{1}{R_i} = \theta + \frac{A_i}{R_i} + \sum_{j \neq i} \frac{A_j}{|x_i - x_j|} \quad \text{for all } i. \tag{4.100}$$

The above analysis in this section guarantee that for any given configuration $\{B(x_i, R_i), i = 1, \cdots, N\}$ which are N non-overlapping spherical particles, there exist a unique mean field θ, unique normal velocities $\dot{R}_i$ and hence unique corresponding monopole magnitudes A_i such that function u is defined as (4.99), equations

(4.100) hold for $i = 1, \cdots, N$ together with the conservation of total volume

$$\sum A_i = -\sum R_i^2 \dot{R}_i = 0. \tag{4.101}$$

The above analysis exactly gives us the monopole approximation of the Mullins-Sekerka model, see subsection 1.1.4 in chapter one.

4.4 The coarsening rate for the monopole approximation

In this sections, we consider the coarsening rate for the monopole approximation of the Mullins-Sekerka model in the setting of finitely many disjoint spherical particles. We will apply the strategy of Kohn and Otto [12]. This strategy involves two quantities of length scaling and three key steps.

The first quantity is a volume-averaged interfacial area which scales as inverse to length and is defined as

$$E = \frac{\sum R_i^2}{\sum R_i^3}. \tag{4.102}$$

The second quantity scales as length and describes the pattern of the system. Define ϕ to be the unique solution in $\mathcal{H}$ that satisfies

$$-\Delta\phi = \chi_{\cup B(x_i, R_i)}, \tag{4.103}$$

$$[\nabla\phi \cdot n]|_{\cup\partial B(x_i, R_i)} = 0, \tag{4.104}$$

where $\chi_{\cup B(x_i,R_i)}$ is the characteristic function of $\cup B(x_i, R_i)$, i.e.,

$$\chi_{\cup B(x_i,R_i)} = \begin{cases} 1 & \text{in } \cup B(x_i, R_i), \\ 0 & \text{in } \mathbb{R}^3 \setminus \cup B(x_i, R_i). \end{cases} \tag{4.105}$$

The time derivative of ϕ satisfies

$$-\Delta\phi_t = 0 \quad \text{in } \mathbb{R}^3 \setminus \cup\partial B(x_i, R_i), \tag{4.106}$$

$$[\nabla\phi_t \cdot n]|_{\partial B(x_i,R_i)} = \dot{R}_i \quad \text{for each } i. \tag{4.107}$$

ϕ is a superposition of $\{\phi_i\}$, with each $\phi_i \in \mathcal{H}$ satisfying

$$-\Delta\phi_i = \chi_{B(x_i,R_i)}, \tag{4.108}$$

$$[\nabla\phi_i \cdot n]|_{\partial B(x_i,R_i)} = 0. \tag{4.109}$$

Then

$$\phi_i = \begin{cases} -\frac{|x-x_i|^2}{6} + \frac{R_i^2}{2} & \text{in } B(x_i, R_i), \\ \\ \frac{R_i^3}{3|x-x_i|} & \text{in } \mathbb{R}^3 \setminus B(x_i, R_i). \end{cases} \tag{4.110}$$

$$\begin{aligned} \int_{\mathbb{R}^3} |\nabla\phi|^2 &= -\int_{\mathbb{R}^3} \phi\Delta\phi \\ &= -\int_{\mathbb{R}^3} \sum_i \phi_i \sum_j \Delta\phi_j \\ &= \sum_j \int_{B(x_j,R_j)} \sum_i \phi_i \\ &= \sum_j \Big\{ \int_{B(x_j,R_j)} \phi_j + \sum_{i\neq j} \int_{B(x_j,R_j)} \phi_i \Big\} \\ &= \sum_j \Big\{ \int_{B(x_j,R_j)} \Big(\frac{R_j^2}{2} - \frac{|x-x_i|^2}{6} \Big) + \sum_{i\neq j} \phi_i(x_j)|B(x_j,R_j)| \Big\} \\ &= \sum_j \Big\{ \frac{8\pi}{15} R_j^5 + \frac{4\pi}{9} \sum_{i\neq j} \frac{R_i^3 R_j^3}{|x_i - x_j|} \Big\} \end{aligned} \tag{4.111}$$

Define L by the following

$$L^2 = \int_{\mathbb{R}^3} |\nabla\phi|^2 \Big/ \sum_j R_j^3 = \sum_j \Big\{ \frac{8\pi}{15} R_j^5 + \frac{4\pi}{9} \sum_{i\neq j} \frac{R_i^3 R_j^3}{|x_i - x_j|} \Big\} \Big/ \sum_j R_j^3. \tag{4.112}$$

Then

$$E^2L^2 \geqslant \frac{8\pi\left(\sum_j R_j^5\right)\left(\sum_j R_j^2\right)^2}{15\left(\sum_j R_j^3\right)^3} \geqslant \frac{8\pi}{15}. \tag{4.113}$$

Hence

$$EL \geqslant C_1 \tag{4.114}$$

with $C_1 = \left(8\pi/15\right)^{1/2}$.

Taking the time derivative of L^2, we get

$$\begin{aligned}
L\dot{L} &= \int_{\mathbb{R}^3\setminus\cup\partial P_i} \nabla\phi\nabla\phi_t \Big/ \sum_j R_j^3 \\
&= \left\{\int_{\mathbb{R}^3\setminus\cup\partial P_i} \phi\left(-\Delta\phi_t\right) - \sum_j \int_{\partial B_j} [\nabla\phi_t \cdot n]\phi\right\} \Big/ \sum_j R_j^3 \\
&= -\sum_j \int_{\partial B_j} \dot{R}_j\phi \Big/ \sum_j R_j^3 \\
&= \int_{\mathbb{R}^3} \nabla u\nabla\phi \Big/ \sum_j R_j^3 \\
&\leqslant \left\{\int_{\mathbb{R}^3} |\nabla u|^2 \Big/ \sum_j R_j^3\right\}^{1/2} \left\{\int_{\mathbb{R}^3} |\nabla\phi|^2 \Big/ \sum_j R_j^3\right\}^{1/2} \\
&= (-2\pi\dot{E})^{1/2}L.
\end{aligned} \tag{4.115}$$

Hence

$$|\dot{L}|^2 \leqslant 2\pi(-\dot{E}). \tag{4.116}$$

By applying lemma 3 of [12], we get the following theorem on the coarsening rate of the 3D monopole approximation.

Theorem 4.6. *For any $1 < p < 3$, there exist positive constants C_2 and C_3 depending only on p such that for any solution $\{R_i\}$ of the monopole approximation of the*

3D Mullins-Sekerka model, we have

$$\fint_0^T E(t)^p \, dt \geqslant C_2 \fint_0^T \left(t^{-1/3}\right)^p \, dt, \quad for \quad T \geqslant C_3 \, L(0)^3, \tag{4.117}$$

where E and L are defined in terms of (4.102) and (4.112), respectively.

Remark. Although (4.114) is always true, it is meaningful only when the ratio between

$$I := \sum_j \sum_{i \neq j} \frac{R_i^3 R_j^3}{|x_i - x_j|} \quad \text{and} \quad II := \sum_j R_j^5$$

is not big. Let's compare the two terms in the simplest situation when all particles occupy a square region Ω with side length l, Ω being divided into small squares of side length d and the particles being located on vertices of small squares. d is the distance between two nearest particles. Suppose furthermore the radii of particles are all equal to R. In this situation, the total particle number $N = (l/d)^3$ and

$$I \sim N^{5/3} \frac{R^6}{d}$$

and

$$II \sim N R^5.$$

Then

$$\frac{I}{II} \sim \frac{R}{d} N^{2/3} \sim \frac{l^2 R}{d^3}. \tag{4.118}$$

It is shown in [19] that the screening length ξ_{scr} is approximately

$$\xi_{scr} \approx \left(\frac{d^3}{R}\right)^{1/2}. \tag{4.119}$$

So

$$\frac{I}{II} \sim \left(\frac{l}{\xi_{scr}}\right)^2. \tag{4.120}$$

To make I/II small, we need ξ_{scr} to be bigger than or at least comparable to the system size l. This is the physical regime in which our estimate in theorem 4.6 is meaningful.

Appendix A

Non-dimensionalization for the phase-field model

To help clarify what physical conditions yield the system (1.4)–(1.5), we briefly discuss the non-dimensionalization procedure here. We begin from a dimensional version of the standard phase field system, derived following [23]. We start with a bulk free energy density f at the phase transition temperature T_0 given by

$$f(T_0, \phi) = \beta_0 G(\phi) = \frac{\beta_0}{4}(\phi^2 - 1)^2, \tag{A.1}$$

and bulk energy density given in terms of temperature T and order parameter ϕ by

$$e(T, \phi) = c_0 T + b_0 \phi. \tag{A.2}$$

Here c_0 is heat capacity and $2b_0$ is latent heat. The thermodynamic relation $\partial(f/T)/\partial(1/T) = e$ yields

$$f(T, \phi) = c_0 T \log \frac{T_0}{T} + b_0 \phi \left(1 - \frac{T}{T_0}\right) + \beta_0 \frac{T}{T_0} G(\phi). \tag{A.3}$$

The phase field system obtained from the kinetic derivation of [22], after linearizing the contribution of temperature to the phase-field evolution equation, is

$$c_0 T_t + b_0 \phi_t = K_0 \Delta T, \tag{A.4}$$

$$\alpha_0 \phi_t = \kappa_0 \Delta \phi - \frac{\beta_0}{T_0} G'(\phi) + \frac{b_0}{T_0^2}(T - T_0). \tag{A.5}$$

Here K_0 is the heat conductivity, and α_0 and κ_0 can be determined from the quantities

$$x_1 = \left(\frac{\kappa_0 T_0}{\beta_0}\right)^{1/2}, \qquad t_r = \frac{\alpha_0 T_0}{\beta_0}, \tag{A.6}$$

which respectively represent a domain wall thickness and a relaxation time for the phase field. For the system in this form, a Lyapunov function is the quantity

$$S_0 = \int_Q \frac{1}{2}\kappa_0|\nabla\phi|^2 + \frac{\beta_0}{T_0}G(\phi) + \frac{c_0}{2T_0^2}(T-T_0)^2, \tag{A.7}$$

which has dimensions of entropy, but is not identical to the (negative) entropy involved in the kinetic derivation of [22] due to the linearization step mentioned.

We non-dimensionalize according to

$$T - T_0 = u_0\hat{u}, \qquad x = x_0\hat{x}, \qquad t = t_0\hat{t}, \tag{A.8}$$

where u_0, x_0, t_0 represent typical temperature fluctuation, length and time scales, respectively. One then obtains the system (1.4)–(1.5) under the conditions that

$$\varepsilon = \frac{x_1}{x_0} = \sqrt{\frac{t_r}{\alpha t_0}} = \frac{b_0 u_0}{2\beta_0 T_0} = \frac{l}{2}\frac{c_0 u_0}{b_0} = K\frac{c_0 x_0^2}{K_0 t_0}. \tag{A.9}$$

These relations make clear the conditions under which the parameter ε is small while l, K, and α remain order one quantities: the domain wall thickness and phase relaxation time should be small compared to typical length and time scales; energetic contributions of temperature fluctuations should be small compared to those of phase changes; and the time scale t_0 should be long compared to the heat diffusion time $t_D = x_0^2 c_0/K_0$.

Appendix B

Compactness results for the L^p moments

In this appendix, we will establish a compactness result for solutions $x(t,\varphi)$ of problem (3.78) + (3.79) with $x(0,\cdot) = x_0(\cdot)$ for any $x_0 \in L^1_d \cap L^p((0,1))$ with

$1 < p < \infty$ under the same assumptions (H1)-(H4) as in [21]. Note that the two models we considered fall into this category except for the case $n = 2, \beta = 0$ of the volume-diffusion–controlled growth model (and this is the reason why we don't include this case in our estimates for coarsening rates with general size distribution).

Proposition B.1. *Fix $T \in (0, \infty)$ and consider a sequence $\{x_k\}_{k=1}^{\infty}$ of solutions to (3.78) + (3.79) for $0 \leqslant t \leqslant T$ with initial values $x_k(0, \varphi) = x_{0k}(\varphi)$ $(\varphi \in (0,1))$. Assume that the sequence of initial data $\{x_{0k}\} \subset L^1_d \cap L^p((0,1))$ is compact in $L^p((0,1))$ for some $1 < p < \infty$ with $c_1 := \inf_k \int_0^1 x_{0k} > 0$. Then $\{x_k\}$ is compact in $C([0,T], L^p((0,1)))$ and any limit x is again a solution of (3.78) + (3.79).*

Proof. By Hölder's inequality, the assumption that $\{x_{0k}\} \subset L^1_d \cap L^p((0,1))$ is compact in $L^p((0,1))$ implies that $\{x_{0k}\}$ is compact in $L^1((0,1))$. Hence by Proposition 6.1 in [21], $\{x_k\}$ is compact in $C([0,T], L^1_d)$ and any limit x is again a solution of (3.78) + (3.79). We will follow the strategy of the proof of Lemma 6.2 in [21] to prove that x_k is compact in $L^p((0,1))$.

It has been shown in [21] that $\theta(t)$ is uniformly bounded on [0,T]. The assumptions $(H1) - (H4)$ together with the boundedness of θ imply that there exists a positive constant C, depending only on T, such that

$$|a(x)\theta(t) - b(x)| \leqslant C(1 + x)$$

for all $x \geqslant 0$. By the generalized Arzela-Ascoli theorem, to show that $\{x_k\}$ is compact in $C([0,T], L^p((0,1)))$, we need prove the following three steps:

(1) uniform boundedness of $\{\int_0^1 x_k^p(t,\varphi)\, d\varphi\}$ for all $t \in [0,T]$ and all k,

(2) for fixed $t \in (0,T)$, $\{x_k(t,\cdot)\}$ is compact in $L^p((0,1))$,

(3) $\sup_k ||x_k(t_1,\cdot) - x_k(t_2,\cdot)||_{L^p((0,1))} \to 0 \quad \text{as} \quad |t_1 - t_2| \to 0.$

To show (1), define $F_\delta(t) = \int_\delta^1 x_k^p(t,\varphi)\, d\varphi$ for $\delta > 0$. Then $F_\delta < \infty$ since $x_k(t,\cdot)$ is decreasing and

$$\begin{aligned} F_\delta(t) &= \int_\delta^1 x_{0k}^p(\varphi)\, d\varphi + \int_\delta^1 \int_0^t p x_k^{p-1} \partial_s x_k(s,\varphi)\, ds\, d\varphi \\ &= \int_\delta^1 x_{0k}^p(\varphi)\, d\varphi + \int_\delta^1 \int_0^t p x_k^{p-1} (a(x_k)\theta - b(x_k))\, ds\, d\varphi \\ &\leqslant \int_\delta^1 x_{0k}^p(\varphi)\, d\varphi + Cp \int_\delta^1 \int_0^t x_k^{p-1}(1 + x_k)\, dt\, d\varphi. \end{aligned}$$

By Young's inequality,

$$x_k^{p-1} \leqslant \frac{p-1}{p} x_k^p + \frac{1}{p}. \tag{B.1}$$

So

$$\begin{aligned} F_\delta(t) &\leqslant \int_\delta^1 x_{0k}^p(\varphi)\, d\varphi + C \int_\delta^1 \int_0^t ((2p-1)x_k^p + 1)\, dt\, d\varphi \\ &\leqslant \int_\delta^1 x_{0k}^p(\varphi)\, d\varphi + CT + C(2p-1) \int_0^t \int_\delta^1 x_k^p\, d\varphi\, dt. \end{aligned}$$

By Gronwall's inequality,

$$F_\delta(t) \leqslant \int_\delta^1 x_{0k}^p(\varphi)\, d\varphi + CT + C(2p-1)e^{C(2p-1)t} \Big(\int_\delta^1 x_{0k}^p(\varphi)\, d\varphi + CT \Big). \tag{B.2}$$

The compactness of x_{0k} in $L^p((0,1))$ implies that there exists positive constant C_1 such that $\int_0^1 x_{0k}^p(\varphi)\, d\varphi \leqslant C_1$ for all k. So by taking $\delta \to 0$ in (B.2), we get

$$\int_0^1 x_k^p(t,\varphi)\, d\varphi \leqslant C_1 + CT + C(2p-1)e^{C(2p-1)t}(C_1 + CT) =: G(t) \leqslant G(T). \tag{B.3}$$

Here G is an increasing function of t and does not depend on k. Hence (1) is proved.

It is shown in the proof of Lemma 6.2 in [21] that, for fixed t, there exists a pointwise convergent subsequence, still denoted as x_k for simplicity. So to prove (2), we need only show that $\{x_k\}$ is equi-integrable. Since $x_k(t,\cdot)$ are decreasing, it is enough to show

$$\sup_k \int_0^\varepsilon x_k(t,\varphi)\,d\varphi \to 0 \quad \text{ as } \varepsilon \to 0. \tag{B.4}$$

$$\begin{aligned}
\int_0^\varepsilon x_k^p(t,\varphi)\,d\varphi &= \int_0^\varepsilon x_{0k}^p(\varphi)\,d\varphi + \int_0^\varepsilon \int_0^t p x_k^{p-1}(s,\varphi)\partial_s x_k(s,\varphi)\,ds\,d\varphi \\
&= \int_0^\varepsilon x_{0k}^p(\varphi)\,d\varphi + \int_0^\varepsilon \int_0^t p x_k^{p-1}(s,\varphi)(a(x_k)\theta - b(x_k))\,ds\,d\varphi \\
&\leqslant \int_0^\varepsilon x_{0k}^p(\varphi)\,d\varphi + Cp\int_0^\varepsilon \int_0^t x_k^{p-1}(1+x_k)\,ds\,d\varphi \\
&\leqslant \int_0^\varepsilon x_{0k}^p(\varphi)\,d\varphi + C\int_0^\varepsilon \int_0^t ((2p-1)x_k^p + 1)\,ds\,d\varphi \quad \text{by } (B.1) \\
&\leqslant \int_0^\varepsilon x_{0k}^p(\varphi)\,d\varphi + CT\varepsilon + C(2p-1)\int_0^t \int_0^\varepsilon x_k^p\,d\varphi\,ds.
\end{aligned} \tag{B.5}$$

By Gronwall's inequality,

$$\int_0^\varepsilon x_k^p(t,\varphi)\,d\varphi \leqslant \int_0^\varepsilon x_{0k}^p(\varphi)\,d\varphi + CT\varepsilon + (2p-1)Ce^{(2p-1)Ct}\Big(\int_0^\varepsilon x_{0k}^p(\varphi)\,d\varphi + CT\varepsilon\Big). \tag{B.6}$$

We can assume without loss of generality that $x_{0k} \to x_0$ in $L^p((0,1))$. Hence $\sup_k \int_0^\varepsilon x_{0k}^p(\varphi)\,d\varphi \to 0$ as $\varepsilon \to 0$. By (B.6), $\sup_k \int_0^\varepsilon x_k^p(t,\varphi)\,d\varphi \to 0$ as $\varepsilon \to 0$ and (2) is proved.

Now let's prove (3). Assume $t_1 < t_2$.

$$
\begin{aligned}
\int_0^1 |x_k^p(t_1,\varphi) - x_k^p(t_2,\varphi)|\, d\varphi &= p\int_0^1 \Big|\int_{t_1}^{t_2} x_k^{p-1}\partial_t x_k(t,\varphi)\, dt\Big|\, d\varphi \\
&= p\int_0^1 \Big|\int_{t_1}^{t_2} x_k^{p-1}(a(x_k)\theta(t) - b(x_k)); dt\Big|\, d\varphi \\
&\leqslant Cp\int_0^1\int_{t_1}^{t_2} x_k^{p-1}(1+x_k)\, dt\, d\varphi \\
&\leqslant C\int_0^1\int_{t_1}^{t_2} ((2p-1)x_k^p + 1)\, dt\, d\varphi \quad \text{by } (B.1) \\
&\leqslant C(2p-1)(G(T)+1)|t_2 - t_1| \quad \text{by } (B.3). \qquad \text{(B.7)}
\end{aligned}
$$

Thus (3) is true and the proposition is proved.

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