ABSTRACT

Title of Dissertation: ANALYTIC $\mathrm{SO}^{\circ}(p,q)$ ACTIONS ON CLOSED, CONNECTED $(p+q-1)$-DIMENSIONAL MANIFOLDS

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This thesis provides a classification of analytic actions of the semiorthogonal group $\mathrm{SO}^{\circ}(p,q)$, where $p \geq 3$, on closed, connected $(p+q-1)$-dimensional manifolds. Adapting Uchida's construction of $\mathrm{SO}^{\circ}(p,q)$ actions on S^{p+q-1}, we explicitly construct analytic actions of $\mathrm{SO}^{\circ}(p,q)$ on $\mathrm{S}^{p} \times \mathrm{S}^{q-1}$ and $\mathrm{S}^{p-1} \times \mathrm{S}^{q}$, as well as actions on $\mathrm{SO}^{\circ}(p,q) \times_P \mathrm{S}^1$, where P is a maximal parabolic subgroup of $\mathrm{SO}^{\circ}(p,q)$. The central result of this thesis demonstrates that any analytic $\mathrm{SO}^{\circ}(p,q)$ action on a closed, connected $(p+q-1)$-dimensional manifold is covered by one of the constructed actions. For $q \neq 2$, the actions of $\mathrm{SO}^{\circ}(p,q)$ correspond to a particular class of vector fields on the circle, while for $q = 2$, they correspond to actions of $\mathrm{SO}^{\circ}(1,2)$ on either the sphere or the torus.

ANALYTIC $\mathrm{SO}^\circ(p, q)$ ACTIONS ON CLOSED, CONNECTED $(p+q-1)$-DIMENSIONAL MANIFOLDS

by

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Chapter 1: Introduction

Consider the Lie group $\mathrm{SO}^\circ(p,q)$, for $p \geq 3$. We are interested in analytic actions of $\mathrm{SO}^\circ(p,q)$ on closed, connected manifolds. When the dimension of the manifold is $< p+q-2$, there can be no nontrivial action of $\mathrm{SO}^\circ(p,q)$. When the dimension is $p+q-2$ there exists essentially only one action, namely the one on the homogeneous space $\mathrm{SO}^\circ(p,q)\big/P_{\mathrm{null}}$, where $P_{\mathrm{null}} \leq \mathrm{SO}^\circ(p,q)$ is a maximal parabolic subgroup, which is isomorphic to the stabiliser of an isotropic line in the standard representation of $\mathrm{SO}^\circ(p,q)$ on $\mathbb{R}^{p+q}$ with a scalar product of signature (p,q). We classify analytic $\mathrm{SO}^\circ(p,q)$ actions on closed, connected manifolds of dimension $p+q-1$. More specifically, we get a classification up to analytic isomorphism for the analytic $\mathrm{SO}^\circ(p,q)$ actions on S^{p+q-1}, $\mathrm{S}^p \times \mathrm{S}^{q-1}$, $\mathrm{S}^{p-1} \times \mathrm{S}^q$ and $\mathrm{SO}^\circ(p,q) \times_{P_{\mathrm{null}}} \mathrm{S}^1$, where a maximal compact subgroup acts in a standard way, and then we show that an arbitrary manifold of dimension $p+q-1$ with a nontrivial $\mathrm{SO}^\circ(p,q)$ action is $\mathrm{SO}^\circ(p,q)$-equivariantly covered by one of these spaces. Here, P_{null} acts on S^1 via an analytic flow, see Section 4.1 for the relevant definitions. Theorem 1.0.1 below gives the classification up to diffeomorphism of all manifolds admitting a nontrivial action. For the purposes of presentation, we assume here $p, q \geq 3$. For $q = 1$ or $q = 2$, see Chapter 5 and Chapter 6 respectively.

Theorem 1.0.1. *Suppose* $\mathrm{SO}^\circ(p,q)$*,* $p, q \geq 3$*, acts analytically on a closed, connected manifold* M *of dimension* $p+q-1$*. Consider* $\mathrm{SO}(p) \simeq \mathrm{SO}(p) \times \{1\} \leq \mathrm{SO}(p) \times \mathrm{SO}(q) \leq$

$\mathrm{SO}^\circ(p,q)$ *and* $\mathrm{SO}(q)$ *similarly.*

- *If both* $\mathrm{SO}(p)$ *and* $\mathrm{SO}(q)$ *have a fixed point, then* M *is covered by* S^{p+q-1}
- *If only* $\mathrm{SO}(p)$ *has a fixed point, then* M *is covered by* $\mathrm{S}^p \times \mathrm{S}^{q-1}$
- *If only* $\mathrm{SO}(q)$ *has a fixed point, then* M *is covered by* $\mathrm{S}^{p-1} \times \mathrm{S}^q$
- *If neither* $\mathrm{SO}(p)$ *nor* $\mathrm{SO}(q)$ *have a fixed point, then* M *is covered by* $\mathrm{SO}^\circ(p,q) \times_{P_{\mathrm{null}}} \mathrm{S}^1$*. If* $P_{\mathrm{null}} = M_P A_P N_P$ *is the Langlands decomposition of* P_{null}*, then* P_{null} *acts on* S^1 *by a flow via* A_P*.*

We are actually able to say more about the actions, see Theorem 4.0.1. There are four orbit types that appear in these actions. Specifically, the orbits are diffeomorphic to the homogeneous spaces $\mathrm{SO}^\circ(p,q)\big/\mathrm{SO}^\circ(p-1,q)$, $\mathrm{SO}^\circ(p,q)\big/\mathrm{SO}^\circ(p,q-1)$, $\mathrm{SO}^\circ(p,q)\big/G_{\mathrm{null}}$ or $\mathrm{SO}^\circ(p,q)\big/P_{\mathrm{null}}$. Here, $G_{\mathrm{null}} \leq \mathrm{SO}^\circ(p,q)$ is the stabiliser of a null vector in the standard representation of $\mathrm{SO}^\circ(p,q)$ on $\mathbb{R}^{p+q}$ and we call these orbits *nullcone orbits*. Topologically these are $\mathbb{R}^q \times \mathrm{S}^{p-1}$, $\mathbb{R}^p \times \mathrm{S}^{q-1}$, a component of the nullcone in $\mathbb{R}^{p+q}$ with respect to a scalar form of signature (p,q), and $\mathrm{S}^{p-1} \times \mathrm{S}^{q-1}$ respectively.

Firstly, in Chapter 2, we define analytic actions of $\mathrm{SO}^\circ(p,q)$, $p,q \geq 3$, on $\mathrm{S}^p \times \mathrm{S}^{q-1}$ where the maximal compact subgroup $\mathrm{SO}(p) \times \mathrm{SO}(q)$ acts in a standard way, see the beginning of Chapter 2. These actions correspond to analytic flows satisfying certain conditions, see Definition 2.0.1, on the fixed point set of $\mathrm{SO}(p-1) \times \mathrm{SO}(q-1)$, which is a principal isotropy group with codimension 1 orbits for the action of $\mathrm{SO}(p) \times \mathrm{SO}(q)$. We then get a classification of those $\mathrm{SO}^\circ(p,q)$ actions on $\mathrm{S}^p \times \mathrm{S}^{q-1}$, using a result of [7], see Appendix A. The $\mathrm{SO}^\circ(p,q)$ actions on $\mathrm{S}^{p-1} \times \mathrm{S}^q$ are defined similarly, while the actions on S^{p+q-1} are defined in [16]. The actions of $\mathrm{SO}^\circ(p,q)$ on $\mathrm{SO}^\circ(p,q) \times_{P_{\text{null}}} \mathrm{S}^1$ are defined in Section 4.1.

Subsequently, we consider an analytic action of $\mathrm{SO}^\circ(p,q)$ on a closed, connected manifold M of dimension $p+q-1$. An important step towards classifying these actions is analysing the action of a maximal compact subgroup of $\mathrm{SO}^\circ(p,q)$ and determining the orbit types that can appear, see Chapter 3 and in particular Proposition 3.3.1. Then, on the fixed point set of $\mathrm{SO}(p-1) \times \mathrm{SO}(q-1)$, which is a finite union of circles, we get an analytic flow, like above. If there are no nullcone orbits, then this flow helps us define an $\mathrm{SO}^\circ(p,q)$ action on S^{p+q-1}, $\mathrm{S}^p \times \mathrm{S}^{q-1}$ or $\mathrm{S}^{p-1} \times \mathrm{S}^q$ depending on the existence of fixed points of $\mathrm{SO}(p)$ and $\mathrm{SO}(q)$, and get an $\mathrm{SO}^\circ(p,q)$-equivariant covering map to M. In the presence of a nullcone orbit, which is equivalent to neither $\mathrm{SO}(p)$ nor $\mathrm{SO}(q)$ having fixed points, M is $\mathrm{SO}^\circ(p,q)$-equivariantly covered by $\mathrm{SO}^\circ(p,q) \times_{P_{\text{null}}} \mathrm{S}^1$.

Finally, we consider the analytic actions of $\mathrm{SO}^\circ(p,1)$ and $\mathrm{SO}^\circ(p,2)$, $p \geq 3$. While the results established are analogous to the $q \geq 3$ case, there are some important differences. In the case of $\mathrm{SO}^\circ(p,1)$ actions on a manifold M, there can be more than two disjoint orbits diffeomorphic to ${}^{\mathrm{SO}^\circ(p,1)}\!/\!{}_{\mathrm{SO}^\circ(p-1,1)}$ and the absence of $\mathrm{SO}(p)$ fixed points is no longer sufficient to determine the topology of M. In the case of $\mathrm{SO}^\circ(p,2)$ actions, the fixed point set of $\mathrm{SO}(p-1) \times \mathrm{SO}(1) \simeq \mathrm{SO}(p-1)$ is no longer 1-dimensional, but it is 2-dimensional. Therefore, for the classification of the $\mathrm{SO}^\circ(p,2)$ analytic actions, we employ Schneider's classification of the analytic $\mathrm{SL}_2(\mathbb{R})$ actions on 2-dimensional manifolds in [13]. For more details, see Chapter 5 and Chapter 6.

1.1 Background and Motivation

In [16], Uchida studied smooth actions of $\mathrm{SO}^\circ(p,q)$ on S^{p+q-1} with the extra assumption that the action of a maximal compact subgroup on S^{p+q-1} is the standard orthogonal one. He showed that such actions are in one to one correspondence with pairs (Φ, f) where Φ is a smooth flow on S^1 and $f : \mathrm{S}^1 \to \mathbb{R}\mathrm{P}^1$ is smooth, and Φ and f satisfy certain conditions. We will recall Uchida's construction in the next chapter. Here, S^1 is diffeomorphic to the fixed point set of H, which is a principal isotropy group with codimension 1 orbits for the action of the maximal compact subgroup. The function f encodes the isotropy algebra at a point in S^1 for the given action of $\mathrm{SO}^\circ(p,q)$, see Lemma 2.2.1 and Remark 4. Note that Uchida's result gives a correspondence between smooth $\mathrm{SO}^\circ(p,q)$ actions and pairs (Φ, f), but only classifies them up to homeomorphism.

Such a pair (Φ, f) was first introduced by Asoh in [2] in order to study actions of $\mathrm{SL}_2(\mathbb{C})$ on S^3. It was then modified by Uchida in [16] to study the actions mentioned above and modified even further by Mukoyama in [10] to study smooth $\mathrm{Sp}(2,\mathbb{R})$ actions on S^4. Uchida in [17] used Mukoyama's approach to study smooth $\mathrm{SO}^\circ(p,2)$ actions on S^{p+1}, as well as $\mathrm{Sp}(p,q)$ actions on $\mathrm{S}^{4p+4q-1}$ in [18]. Mukoyama also studied smooth $\mathrm{SU}(p,q)$ actions on $\mathrm{S}^{2p+2q-1}$ and $\mathbb{R}\mathrm{P}^{p+q-1}$ in [11]. On the other hand, assuming analyticity, Schneider classified $\mathrm{SL}_2(\mathbb{R})$ actions on 2-dimensional compact manifolds in [13]. We note that $\mathrm{SL}_2(\mathbb{R})$, $\mathrm{SL}_2(\mathbb{C})$ and $\mathrm{Sp}(2,\mathbb{R})$ are locally isomorphic to $\mathrm{SO}^\circ(1,2)$, $\mathrm{SO}^\circ(1,3)$ and $\mathrm{SO}^\circ(2,3)$ respectively.

In a different direction, Uchida classified analytic $\mathrm{SL}_n(\mathbb{R})$ actions on S^n in [14]. Later, both the analytic and the smooth $\mathrm{SL}_n(\mathbb{R})$ actions on n-dimensional closed manifolds were classified by Fisher and Melnick in [4].

Schneider's actions are of three types. The first type comprises actions with only closed orbits diffeomorphic to S^1. The second comprises effective $\mathrm{SL}_2(\mathbb{R})$ actions that are classified by a continuous invariant and the third type comprises non effective actions that are classified by a finite set of discrete invariants. For such a non effective action there can be multiple open orbits separated by closed S^1 orbits. This picture is in accordance with Asoh's analytic actions and also with the analytic $\mathrm{SO}(p,1)$ actions for $p \geq 3$, see Chapter 5. On the other hand, for analytic $\mathrm{SO}(p,q)$ actions with $p \geq 3$, $q \geq 2$ there can be at most two open orbits or an arbitrary number of open orbits which are all *nullcone orbits*, i.e. orbits isomorphic to a component of a nullcone in $\mathbb{R}^{p+q}$ equipped with a scalar form of signature (p, q). In Uchida's result on S^{p+q-1}, which concerns smooth actions, there can be more than two open orbits with all but two of those open orbits being nullcone orbits.

1.2 Notation

Throughout the text we use the following notation

- If A is a Lie group, A° denotes its identity component
- $G := \mathrm{SO}^\circ(p,q) = \left\{X \in \mathrm{SL}_{p+q}(\mathbb{R}) : X\, I_{p,q}\, X^T = I_{p,q}\right\}$, $p \geq 3$,

  where $I_{p,q} = \begin{bmatrix} -I_p & \\ & I_q \end{bmatrix}$ where I_p and I_q are the identity $p \times p$ and $q \times q$ matrices respectively.
- $K := \left\{ \begin{bmatrix} A & \\ & B \end{bmatrix} \in G : A \in \mathrm{SO}(p), B \in \mathrm{SO}(q) \right\} \leq G$, $K \simeq \mathrm{SO}(p) \times \mathrm{SO}(q)$.

- $H := \left\{ \begin{bmatrix} 1 & & & \\ & \tilde{A} & & \\ & & 1 & \\ & & & \tilde{B} \end{bmatrix} \in K : \tilde{A} \in \mathrm{SO}(p-1), \tilde{B} \in \mathrm{SO}(q-1) \right\} \leq K$,

  $H \simeq \mathrm{SO}(p-1) \times \mathrm{SO}(q-1)$

- $\mathscr{M}(p,q) = \left\{ \begin{bmatrix} \cosh(\theta) & & \sinh(\theta) & \\ & I_{p-1} & & \\ \sinh(\theta) & & \cosh(\theta) & \\ & & & I_{q-1} \end{bmatrix} \right\} \leq G$

- $j_1 = \begin{bmatrix} -1 & & \\ & -1 & \\ & & I_{p+q-2} \end{bmatrix} \in \mathrm{SO}(p) \times \{I_q\} \leq K$

- $j_2 = \begin{bmatrix} I_p & & & \\ & -1 & & \\ & & -1 & \\ & & & I_{q-1} \end{bmatrix} \in \{I_p\} \times \mathrm{SO}(q) \leq K$, whenever $q \geq 2$

- If G is a group and X is a space on which G acts, then for $g \in G$ and $x \in X$ we will denote the action of g on x by $g \star x$. An exception to this will be when a matrix A acts on a vector v, where we will just write Av.

- If A is a Lie group, $B \leq A$ a Lie subgroup and X is a space on which B acts, then $A \times_B X$ is the quotient space of $A \times X$ module the equivalence relation $(a,x) \sim (ab, b^{-1}x)$. A acts on $A \times_B X$ by multiplication on the left: $\tilde{a} \star (a,x) = (\tilde{a}a, x)$

Chapter 2: Construction of $\mathrm{SO}^\circ(p,q)$ actions on $\mathrm{S}^p \times \mathrm{S}^{q-1}$ and $\mathrm{S}^{p-1} \times \mathrm{S}^q$

Assume $p, q \geq 3$ and consider $G := \mathrm{SO}^\circ(p,q)$. We see S^p as a subset of $\mathbb{R}^{p+1}$ with basis $\{e_1, \cdots, e_{p+1}\}$, and S^{q-1} as a subset of $\mathbb{R}^q$ with basis $\{\epsilon_1, \cdots, \epsilon_q\}$. We consider the following action of $K := \mathrm{SO}(p) \times \mathrm{SO}(q)$, which we call the *standard* action of K on $\mathrm{S}^p \times \mathrm{S}^{q-1}$: First, we embed $\mathrm{SO}(p)$ in $\mathrm{SO}(p+1)$ by

$$\kappa \mapsto \tilde{\kappa} := \begin{bmatrix} \kappa & \\ & 1 \end{bmatrix} \tag{2.1}$$

Then, K acts on $\mathrm{S}^p \times \mathrm{S}^{q-1}$ by matrix-vector multiplication in the obvious way: $\mathrm{SO}(p)$ acts on the first factor via its embedding in $\mathrm{SO}(p+1)$ and $\mathrm{SO}(q)$ acts on the second factor. Evidently, this action is analytic. In this chapter we will construct G actions on $\mathrm{S}^p \times \mathrm{S}^{q-1}$ that extend the standard K action. Note that for these actions $\mathrm{SO}(p) \times \{I_q\} \leq K$ has fixed points, while $\{I_p\} \times \mathrm{SO}(q) \leq K$ does not.

Let $\mathscr{F}$ be the fixed point set of the subgroup $H := \mathrm{SO}(p-1) \times \mathrm{SO}(q-1)$. Then, $\mathscr{F}$ comprises the following two circles:

$$\Big\{(\alpha e_1 + \beta e_{p+1}, \epsilon_1) : \alpha^2 + \beta^2 = 1\Big\} \qquad \text{and} \qquad \Big\{(\alpha e_1 + \beta e_{p+1}, -\epsilon_1) : \alpha^2 + \beta^2 = 1\Big\}$$

Set

$$\mathscr{S} = \Big\{(\alpha e_1 + \beta e_{p+1}, \epsilon_1) : \alpha^2 + \beta^2 = 1\Big\}$$

Recall, $j_1 = \begin{bmatrix} -1 & & \\ & -1 & \\ & & I_{p+q-2} \end{bmatrix} \in K$, which acts on $\mathscr{S}$ by

$$j_1 \star (\alpha e_1 + \beta e_{p+1}, \epsilon_1) \mapsto (-\alpha e_1 + \beta e_{p+1}, \epsilon_1)$$

Note that, the other connected component of $\mathscr{F}$ is equal to $j_2 \star \mathscr{S}$. Recall that by "$\star$" we denote the action of a group element, see the Notation section in the Introduction.

The main ingredient for our construction will be a special type of flows on S_1, see the next definition. Here, we see S_1 as the unit circle in $\mathbb{R}^2$, namely $\mathrm{S}^1 = \{\alpha\, e_1 + \beta\, e_2 : \alpha^2 + \beta^2 = 1\}$.

Definition 2.0.1. Assume Φ_θ is a nontrivial analytic flow on S^1 and let J_1 be the reflection with respect to the y-axis, namely $\mathrm{J}_1(\alpha\, e_1 + \beta\, e_2) = -\alpha\, e_1 + \beta\, e_2$. We will say that Φ_θ is a basic J_1-flow if:

- It has exactly two fixed points on S^1, none of which are $\pm e_2$.
- $\mathrm{J}_1\Phi_\theta(z) = \Phi_{-\theta}\,(\mathrm{J}_1 z)$, for $\theta \in \mathbb{R}$ and $z \in \mathrm{S}^1$.
- The Jacobian of Φ_θ is $-\frac{2}{n}$, respectively $\frac{2}{n}$, at the attracting, respectively repelling, fixed point, where $n \in \mathbb{N}$.

Note that then, if z_1, z_2 are the fixed points of Φ_θ, we have $\mathrm{J}_1 \star z_1 = z_2$. The actions we will construct in this chapter will correspond to basic J_1-flows. Note that, one of the fixed points of Φ_θ must be attracting and the other repelling, say z_1 is the former and z_2 the latter.

Remark 1. Suppose Φ_θ is a basic J_1-flow on S^1 and let ∂_{S^1} be a basic vector field of S^1. Then, Φ_θ generates an analytic vector field $X = g \cdot \partial_{\mathrm{S}^1}$. In [7], Hitchin classified analytic vector fields based on a number of local and global invariants, see Appendix A. For a basic J_1-flow, since we have two fixed points one of which is necessarily attracting and the other necessarily repelling, the only invariants that matter are the Jacobian at the fixed points, which is $\pm 2/n$ and the *global invariant* μ which can be thought of as

$$\mu = \int_{\mathrm{S}^1} \frac{1}{g}$$

Of course, in our case the right hand side does not make sense, but it can be defined using complex integration methods around the zeros of g, see [7].

Now, let

$$\rho_0 : \mathrm{S}^1 \to \mathscr{S} \tag{2.2}$$

be the analytic isomorphism defined by

$$\alpha\, e_1 + \beta\, e_2 \mapsto (\alpha\, e_1 + \beta\, e_{p+1}, \epsilon_1)$$

Definition 2.0.2. Suppose Φ_θ is a basic J_1-flow on S^1. Via the isomorphism ρ_0, see (2.2), Φ_θ induces an analytic flow Φ'_θ on $\mathscr{S}$. We will call analytic flows on $\mathscr{S}$ arising this way, induced basic j_1-flows.

Lemma 2.0.1. *Suppose Φ_θ is a nontrivial analytic flow on $\mathscr{S}$. Then, Φ_θ is an induced basic j_1-flow on $\mathscr{S}$ if and only if it satisfies the following relations:*

- *It has exactly two fixed points on $\mathscr{S}$, none of which are (e_{p+1}, ϵ_1) or $(-e_{p+1}, \epsilon_1)$.*
- *$j_1\Phi_\theta(z) = \Phi_{-\theta}(j_1 z)$, for $\theta \in \mathbb{R}$ and $z \in \mathscr{S}$.*
- *The Jacobian of Φ_θ is $-\frac{2}{n}$, respectively $\frac{2}{n}$, at the attracting, respectively repelling, fixed point, where $n \in \mathbb{N}$.*

Proof. The only if part is immediate. Now, if Φ_θ is a flow on $\mathscr{S}$, via ρ_0 of (2.2) we can define a flow, Φ'_θ, on S^1. If Φ_θ satisfies the three relation in the lemma, then it is easy to see that Φ'_θ is a basic J_1-flow that induces Φ_θ. ☐

Furthermore, the following function, defined in terms of an induced basic j_1-flow, will be useful to us later:

Remark 2. Let Φ_θ be an induced basic j_1-flow on $\mathscr{S}$. Via Φ_θ we can define a function, f_Φ on $\mathscr{S}$ in the following way:

- $f_\Phi(\pm e_{p+1}, \epsilon_1) := 0$
- $f_\Phi(z) := \tanh(\theta)$, for a point $z \in \mathscr{S}$ of the form $z = \Phi_\theta\left((\pm e_{p+1}, \epsilon_1)\right)$
- $f_\Phi(z_1) := 1$ and $f_\Phi(z_2) := -1$

Analogously, we can define a standard action of K on $\mathrm{S}^{p-1} \times \mathrm{S}^q$, *basic* J_2*-flows* on S^1, and *induced basic j_2-flows* on the connected component of the fixed point set of H:

$$\mathscr{S}_2 = \left\{(e_1, \alpha\epsilon_1 + \beta\epsilon_{q+1}) : \alpha^2 + \beta^2 = 1\right\}$$

The analogous construction of the one we will describe below will give G actions on $\mathrm{S}^{p-1} \times \mathrm{S}^q$ which extend the standard action of K. Note that for these actions, $\{I_p\} \times \mathrm{SO}(q)$ has fixed points, but $\mathrm{SO}(p) \times \{I_q\}$ does not. In the rest of this chapter we will deal with the $\mathrm{S}^p \times \mathrm{S}^{q-1}$ case. The statements and proofs for the $\mathrm{S}^{p-1} \times \mathrm{S}^q$ are completely analogous.

2.1 The Basic Construction

In this section we construct a G action on $\mathrm{S}^p \times \mathrm{S}^{q-1}$ given a basic J_1-flow on S^1. This construction is adapted from the construction in [16], see Section 2.3. Let

$$m(\theta) := \begin{bmatrix} \cosh(\theta) & & \sinh(\theta) & \\ & I_{p-1} & & \\ \sinh(\theta) & & \cosh(\theta) & \\ & & & I_{q-1} \end{bmatrix}$$

and set

$$\mathscr{M}(p,q) = \{m(\theta) \,:\, \theta \in \mathbb{R}\}$$

which is a one-parameter subgroup of G. Moreover, consider the isotropy group of the point $a\,e_1 + b\,e_{p+1} \in \mathbb{R}^{p+q}$, for $[a:b] \in \mathbb{RP}^1$, in the standard representation of G on $\mathbb{R}^{p+q}$, which we denote by $H_{[a:b]} \leq G$. The following equality is established in [16]:

$$G = K\,\mathscr{M}(p,q)\,\left(H_{[a:b]}\right)^{\circ} \tag{2.3}$$

for any $[a:b] \in \mathbb{RP}^1$.

Suppose $\widetilde{\Phi}_\theta$ is a basic J_1-flow on S^1, see Definition 2.0.1. Let Φ_θ be the induced basic j_1-flow on $\mathscr{S}$, see Definition 2.0.2. Recall that

$$\mathscr{S} = \Big\{(\alpha e_1 + \beta e_{p+1}, \epsilon_1) : \alpha^2 + \beta^2 = 1\Big\}$$

is a connected component of the fixed point set, $\mathscr{F}$, of H in the standard action of K on $\mathrm{S}^p \times \mathrm{S}^{q-1}$. Recall also the function f_Φ, see Remark 2. For $z \in \mathscr{S}$, let

$$U(z) = (H_{[f_\Phi(z):1]})^\circ \tag{2.4}$$

Then, equation (2.3) can be written as

$$G = K\,\mathscr{M}(p,q)\,U(z) \tag{2.5}$$

for any $z \in \mathscr{S}$. Now, let $g \in G$ and $(v, w) \in \mathrm{S}^p \times \mathrm{S}^{q-1}$. There exists $z \in \mathscr{S}$ and $k_0 = (\kappa_1, \kappa_2) \in K$ such that

$$k_0 \star z = (v, w)$$

Write $gk_0 = k\,m(\theta)\,u_z$, according to (2.5). Then, we define

$$g \star (v, w) := k \star \Phi_\theta(z) \tag{2.6}$$

We prove that this action is well-defined in Section 2.5 and we show that it is analytic in Appendix B. We will refer to G actions defined this way as *actions from the basic construc-*

tion. Note that $\Phi_\theta(z)$ is of the form $\Big(\alpha_{\Phi_\theta(z)}e_1 + \beta_{\Phi_\theta(z)}e_{p+1}, \epsilon_1\Big)$, for some $\alpha_{\Phi_\theta(z)}, \beta_{\Phi_\theta(z)} \in \mathbb{R}$ such that $\alpha^2_{\Phi_\theta(z)} + \beta^2_{\Phi_\theta(z)} = 1$.

Remark 3. Although the basic construction is described on $\mathrm{S}^p \times \mathrm{S}^{q-1}$ in terms of basic J_1-flows, there is an obvious analogue on $\mathrm{S}^{p-1} \times \mathrm{S}^q$ in terms of basic J_2-flows. We will will also refer to those actions as actions from the basic construction when it is clear we are talking about actions of G on $\mathrm{S}^{p-1} \times \mathrm{S}^q$.

Theorem 2.1.1. *Two actions from the basic construction are analytically isomorphic if and only if the corresponding basic* J_1*-flows are analytically isomorphic. Moreover, the basic* J_1*-flows are classified up to analytic isomorphism by the Jacobian at the fixed points, which is* $\pm 2/n$*, for* $n \in \mathbb{N}$*, and the global invariant* μ*.*

The first part of Theorem 2.1.1 is proved in Section 2.6. For the second part see Remark 1. Of course, there is an analogous theorem for the analytic $\mathrm{SO}^\circ(p,q)$ actions on $\mathrm{S}^{p-1} \times \mathrm{S}^q$:

Theorem 2.1.2. *Two* $\mathrm{SO}^\circ(p,q)$ *actions on* $\mathrm{S}^{p-1} \times \mathrm{S}^q$ *from the basic construction are analytically isomorphic if and only if the corresponding basic* J_2*-flows are analytically isomorphic. The basic* J_2*-flows are classified up to analytic isomorphism by the Jacobian at the fixed points and the global invariant* μ*.*

2.2 Properties of $\mathrm{SO}^\circ(p,q)$ actions on $\mathrm{S}^p \times \mathrm{S}^{q-1}$

Suppose we have an analytic action of G on $\mathrm{S}^p \times \mathrm{S}^{q-1}$ that extends the standard K action. We are going to extract some important data for such actions, which will be useful later. The results in this section also show that these kind of actions are action from the basic construction, see in particular Lemma 2.2.2. Recall that $H_{[a:b]}$ denotes the isotropy subgroup of the point $a\,e_1 + b\,e_{p+1} \in \mathbb{R}^{p+q}$ in the standard representation of G on $\mathbb{R}^{p+q}$ and let $\mathfrak{h}_{[a:b]}$ denote its Lie algebra.

Lemma 2.2.1. *[16, Lemma 1.7] Suppose $p, q \geq 3$. Let $\mathfrak{a}$ be a proper subalgebra of $\mathfrak{so}(p,q)$ which contains $\mathfrak{h} \simeq \mathfrak{so}(p-1) \oplus \mathfrak{so}(q-1)$. If*

$$\dim \mathfrak{so}(p,q) - \ \dim \mathfrak{a} \leq p+q-1$$

then $\mathfrak{a} = \mathfrak{h}_{[a:b]}$ for some $(a,b) \neq (0,0)$ or $\mathfrak{a} = \mathfrak{h}_{[1:\epsilon]} \oplus \theta^1$ for $\epsilon = \pm 1$, where the one-dimensional space θ^1 is generated by the matrix $E_{1,p+1} + E_{p+1,1}$

Remark 4. Recall that $\mathscr{F}$ is the fixed point set of H and that $\mathscr{F} = \mathscr{S} \bigcup j_2 \star \mathscr{S}$. The above lemma allows us to define a function $\tilde{f} : \mathscr{F} \to \mathbb{R}\mathrm{P}^1$ in the following way: for $z \in \mathscr{F}$, set $\tilde{f}(z) := [a_z : b_z]$, where $[a_z : b_z] \in \mathbb{R}\mathrm{P}^1$ is the unique point such that $\mathfrak{h}_{[a_z:b_z]} \leq \mathfrak{g}_z$, where $\mathfrak{g}_z$ is the Lie isotropy algebra at z. The function $\tilde{f}$ is analytic, see [16, p. 778]. This way of defining a function $\tilde{f}$ on the fixed point set of H will be used numerous times.

For the actions that we consider, since there is no point on $\mathscr{F}$ fixed by $\mathrm{SO}(q)$, the second coordinate of $\tilde{f}$ would always be non-zero. So, we can define an analytic function f by

$$f(z) = \frac{a}{b} \text{ if } \tilde{f}(z) = [a : b]$$

Additionally, it is easy to see that $\mathscr{M}(p,q)$ normalises H and hence, it preserves $\mathscr{F}$. The action of $\mathscr{M}(p,q)$ gives an analytic flow Φ_θ on $\mathscr{F}$. Abusing the notation, we will write $\mathrm{SO}(p)$

for $\mathrm{SO}(p) \times \{I_q\} \leq K$ and $\mathrm{SO}(q)$ for $\{I_p\} \times \mathrm{SO}(q) \leq K$. Let $j_1 = \begin{bmatrix} -I_2 & \\ & I_{p+q-2} \end{bmatrix} \in \mathrm{SO}(p)$ and $j_2 = \begin{bmatrix} I_{p+q-2} & \\ & -I_2 \end{bmatrix} \in \mathrm{SO}(q)$.

Remark 5. By straightforward matrix multiplications, we see that Φ_θ and f satisfy the following properties, for $z \in \mathscr{S}$ and $\theta \in \mathbb{R}$:

(A1) $j_i \star \Phi_\theta(z) = \Phi_{-\theta}(j_i \star z) \qquad (i = 1, 2)$

(A2) $f(j_i \star z) = -f(z) \qquad (i = 1, 2)$

(A3) $f\Big(\Phi_\theta(z)\Big) = \dfrac{f(z) + \tanh(\theta)}{1 + f(z)\tanh(\theta)}$

(A4) $f(z) = 0 \Leftrightarrow z = (\pm e_{p+1}, \epsilon_1)$ or $(\pm e_{p+1}, -\epsilon_1)$

Note that (A4) can be written alternatively as

(A4)$'$ $f(z) = 0 \Leftrightarrow z$ is fixed by $\mathrm{SO}(p)$

Lemma 2.2.2. *(i) Suppose Φ_θ is an induced basic j_1-flow on $\mathscr{S}$, see Definition 2.0.2. Then, f_Φ, see Definition 2, is analytic, and Φ_θ and f_Φ can be extended to $\mathscr{F}$ so that they satisfy relations* (A1)-(A4)*.*

(ii) Suppose Φ_θ and f are an analytic flow and function respectively on $\mathscr{F}$ such that they satisfy relations (A1)-(A4)*. Then, $\Phi'_\theta = \Phi_\theta\Big|_{\mathscr{S}}$ is an induced basic j_1-flow on $\mathscr{S}$ and $f_{\Phi'_\theta} = f$.*

Proof. (i) Suppose Φ_θ is an induced basic j_1-flow on $\mathscr{S}$. Firstly, we prove that f_Φ is analytic on $\mathscr{S}$. Observe that, f_Φ is evidently analytic on the orbits of $(\pm e_{p+1}, \epsilon_1)$. To prove the analyticity at a fixed point of Φ_θ, say z_1, we can use Poincaré's theorem, see [1, Section 22], to linearise the flow around z_1. The theorem implies that there exists a change of coordinates around z_1 such that, after using a chart centered at z_1, the flow has the form

$$\Phi_\theta(x) = x\, e^{tJ_\Phi(z_1)}$$

where J_Φ is the Jacobian of Φ_θ at z_1, which is nonzero. Then, let y be a point in the domain in which the above form of Φ_θ is valid. Note that, by (A3), f_Φ takes the form

$$f_\Phi\left(\Phi_\theta(y)\right) = \frac{f(y) + \tanh(\theta)}{1 + f(y)\tanh(\theta)}$$

Let $x = \Phi_\theta(y)$. Since $\Phi_\theta(y) = y\, e^{tJ_\Phi(z_1)}$, we can solve for θ:

$$\theta = \frac{1}{J_\Phi(z_1)}\,\ln\left(\frac{x}{y}\right)$$

Then, a straightforward calculation shows that

$$\tanh(\theta) = \frac{1 - \left(\frac{x}{y}\right)^{2/J_\Phi(z_1)}}{1 + \left(\frac{x}{y}\right)^{2/J_\Phi(z_1)}}$$

But the right hand side of this equality is an analytic function in a neighbourhood of $x = 0$. Hence, f_Φ is analytic at z_1. Similarly, it is shown that f_Φ is analytic at the other fixed point of Φ_θ as well. Subsequently, we extend Φ_θ to all of $\mathscr{F} = \mathscr{S} \bigcup j_2 \star \mathscr{S}$, by defining for $z \in j_2 \star \mathscr{S}$ and $\theta \in \mathbb{R}$:

$$\Phi_\theta(z) := j_2 \star (\Phi_{-\theta}(j_2 \star z))$$

Similarly, we extend f_Φ on $j_2 \star \mathscr{S}$ by defining

$$f_\Phi(z) := -f_\Phi(j_2 \star z)$$

Clearly, Φ_θ and f_Φ satisfy the relations (A1)-(A4). Evidently, Φ_θ and f_Φ are analytic on $\mathscr{F}$.

(ii) Suppose Φ_θ and f are an analytic flow and function such that they satisfy relations (A1)-(A4). The fact that the fixed points of Φ'_θ are not $(\pm e_{p+1}, \epsilon_1)$ and the relation $j_1 \star \Phi'_\theta(z) = \Phi'_{-\theta}(j_1 \star z)$ are immediate. The flow Φ'_θ generates a vector field X on $\mathscr{S}$. If $\partial_{\mathscr{S}}$ is a basic vector field of $\mathscr{S}$, the we can write

$$X = g \cdot \partial_{\mathscr{S}}$$

where g is an analytic function on $\mathscr{S}$. Then, property (A3) for $z = (e_{p+1}, \epsilon_1)$ shows that f satisfies the following differential equation

$$g\,\partial_{\mathscr{S}}(f) = 1 - f^2 \tag{2.7}$$

Note that this equation is valid at the fixed points of Φ'_θ also, since at those points $g = 0$ and $f = \pm 1$. The last equality follows from (A3) by taking $\theta \to \pm\infty$. Let z_1

be one of the fixed points of Φ_θ for which $f(z_1) = 1$. We can pick a chart centered at z_1, where (2.7) takes the form

$$g \cdot f' = 1 - f^2 \tag{2.8}$$

The functions in (2.8) are to be understood as composed with the chart, but we still use the same letters. Since f is an analytic function on $\mathscr{S}$ and since it is not constantly equal to 1, it cannot be constantly equal to 1 around z_1. That means that the order of vanishing of $1 - f$ in (2.8) at 0 cannot be infinite. By order of vanishing we mean the smallest number $n \in \mathbb{N}$ such that $(1-f)^{(n)}(0) = f^{(n)}(0) \neq 0$, but $(1-f)^{(k)}(0) = 0$ for $0 \leq k < n$, where $(1-f)^{(0)} = 1 - f$. But then, differentiating equation (2.8) $(n+1)$-times and evaluating at 0, we get that

$$g'(0) = -\frac{2}{n}$$

Of course, $g'(0)$ is equal to the Jacobian of Φ'_θ at z_1. By (A1), we get that $g \circ j_1 = g$, hence we see that at the other fixed point of Φ'_θ, which is $j_1(z_1)$, we have that the Jacobian of Φ'_θ is $J_\Phi(j_1(z_1)) = \frac{2}{n}$. Then, by Lemma 2.0.1, Φ'_θ is an induced basic j_1-flow.

$\square$

Remark 6. In the following, we will drop the subscript "Φ" from f_Φ and we will simply write f.

2.3 The Uchida construction

As it was mentioned in the Introduction, in [16], Uchida studied smooth actions of G on S^{p+q-1} extending the standard orthogonal action of K.

Remark 7. Uchida showed that such actions are in one to one correspondence with pairs $(\Phi, \tilde{f})$ where Φ is a smooth flow on S^1 and $\tilde{f} : \mathrm{S}^1 \to \mathbb{RP}^1$ is smooth, and ϕ and $\tilde{f}$ satisfy:

(i) $j_i \star \Phi_\theta(z) = \Phi_{-\theta}(j_i \star z) \qquad (i = 1, 2)$

(ii) $\tilde{f}(z) = [a : b] \Rightarrow \tilde{f}(j_i \star z) = [a : -b] \qquad (i = 1, 2)$

(iii) $\tilde{f}(z) = [a : b] \Rightarrow \tilde{f}\left(\Phi_\theta(z)\right) = [a\cosh(\theta) + b\sinh(\theta) : a\sinh(\theta) + b\cosh(\theta)]$

(iv) $\tilde{f}(z) = [0 : 1] \Leftrightarrow z = e_{p+1} \quad \text{and} \quad \tilde{f}(z) = [1 : 0] \Leftrightarrow z = e_1$

where, j_1 and j_2 act on S^1 as the reflections with respect to the y-axis and x-axis respectively, see [16, Theorem]. We will refer to the above conditions as *Uchida conditions*.

The correspondence roughly goes as follows. Given a G action on S^{p+q-1}, let $\mathscr{F}$ be the fixed point set of H. Then, $\mathscr{F}$ is diffeomorphic to S^1. The pair $(\Phi, \tilde{f})$ is obtained via $\mathscr{M}(p, q)$ and Lemma 2.2.1 respectively. For the reverse direction, suppose a pair $(\Phi, \tilde{f})$ satisfying the Uchida conditions is given. We can assume that $(\Phi, \tilde{f})$ are a flow and a function on $\mathscr{F}$, since $\mathscr{F}$ is diffeomorphic to S^1. For $z \in \mathscr{F}$ with $\tilde{f}(z) = [a : b]$, let

$$U'(z) = (H_{[a:b]})^\circ$$

Then, by equation (2.3),

$$G = K\,\mathscr{M}(p, q)\,U'(z)$$

for any $z \in \mathscr{F}$. Therefore, an action of G on S^{p+q-1} such that K acts in the standard way is defined in the following way: Let $v \in \mathrm{S}^{p+q-1}$ and $g \in G$. There exist $z \in \mathrm{S}^1$ and $k \in K$ such that

$$k \star z = v$$

Using (2.5), write

$$gk = k_1\, m(\theta)\, u$$

Then, the action is defined as

$$g \star v := k_1 \star \Phi_\theta(z)$$

where on the right hand side the action of k_1 is the standard orthogonal one.

Let $\mathscr{S}_3 = \{\alpha\, e_1 + \beta\, e_{p+1} : \alpha^2 + \beta^2 = 1\} \subseteq \mathrm{S}^{p+q-1}$. Then $\mathscr{S}_3 = \mathscr{F}$ for G actions on S^{p+q-1} extending the orthogonal K action.

Definition 2.3.1. Assume Φ_θ is a nontrivial analytic flow on S^1. We will say that Φ_θ is a basic $(\mathrm{J}_1, \mathrm{J}_2)$-flow if:

- It has exactly 4 fixed points on S^1, none of which are the points $\pm e_1$ or $\pm e_2$.
- $\mathrm{J}_i \Phi_\theta(z) = \Phi_{-\theta}(\mathrm{J}_i z)$, $i = 1, 2$, where J_1 is the reflection with respect to the y-axis and J_2 the reflection with respect to the x-axis.
- The Jacobian of Φ_θ at the fixed points is $\pm 2/n$, where $n \in \mathbb{N}$.

Note that the second condition in the above definition implies that if z_1 is a fixed point of Φ_θ, then the rest of the fixed points are $\mathrm{J}_i * z_1$, for $i = 1, 2$, and $\mathrm{J}_1\mathrm{J}_2 \star z_1$. Additionally, the same condition implies that if the Jacobian of Φ_θ at z_1 is $2/n$, the $\mathrm{J}_1\mathrm{J}_2 \star z_1$ has the same Jacobian, while the Jacobian equals $-2/n$ at $\mathrm{J}_i \star z_1$, $i = 1, 2$. Similarly to Definition 2.0.2, we can define *induced basic* (j_1, j_2)*-flows* on $\mathscr{S}_3$. Furthermore, similarly to Lemma 2.2.2 it can be shown that starting from an induced basic (j_1, j_2)-flow on $\mathscr{S}_3$ we can get a pair $(\Phi_\theta, \tilde{f})$ satisfying the Uchida conditions, where $\tilde{f} : \mathscr{S}_3 \to \mathbb{R}\mathrm{P}^1$. Consequently, by the results in [16], we get an analytic G action on S^{p+q-1} extending the orthogonal K action. We will refer to an action of G on S^{p+q-1} arising this way as an *action from the Uchida construction corresponding to a basic* $(\mathrm{J}_1, \mathrm{J}_2)$*-flow*. Similar arguments to those we will use to prove Theorem 2.1.1 can be used to prove a similar result in this case, see also Remark 1:

Theorem 2.3.1. (see also [16, Theorem]) *Two actions from the Uchida construction corresponding to basic* $(\mathrm{J}_1, \mathrm{J}_2)$*-flows are analytically isomorphic if and only if the corresponding basic* $(\mathrm{J}_1, \mathrm{J}_2)$*-flows are analytically isomorphic. Moreover, the basic* $(\mathrm{J}_1, \mathrm{J}_2)$*-flows are classified up to analytic isomorphism by the Jacobian at the fixed points, which is* $\pm 2/n$*, for* $n \in \mathbb{N}$*, and the global invariant* μ*.*

Note that this theorem strengthens Uchida's result about G actions on S^{p+q-1} extending the orthogonal K action, in the analytic setting.

2.4 Adapted lemmas from Uchida

Suppose Φ_θ is an induced basic j_1-flow on $\mathscr{S}$ and $f = f_\Phi$, see Definition 2.0.2 and Remark 2. Recall that by Lemma 2.2.2, Φ and f satisfy relation (A1)-(A4) from Remark 5.

For $z \in \mathscr{S}$, let

$$P(z) = \frac{1}{f(z)^2 + 1} \left(f(z)\, e_1 + \, e_{p+1}\right)^T \cdot \left(f(z)\, e_1 + \, e_{p+1}\right) \in \mathbb{R}^{(p+q)\times(p+q)} \tag{2.9}$$

Then, define the following subgroup of G

$$U\left(P(z)\right) = \left\{g \in G : gP(z)g^T = P(z)\right\}$$

Recall $U(z)$ by (2.4) and note that

$$U\left(P(z)\right)^{\circ} = U(z) \tag{2.10}$$

see [16]. For $\theta \in \mathbb{R}$, let

$$\lambda(\theta, z) = \frac{1}{f^2(z) + 1} \left[\left(f(z)\cosh(\theta) + \sinh(\theta)\right)^2 + \left(f(z)\sinh(\theta) + \cosh(\theta)\right)^2\right] \tag{2.11}$$

Straightforward matrix multiplication and (A3) show that, for $z \in \mathscr{S}$,

$$m(\theta)P(z)m(\theta) = \lambda(\theta, z)P\left(\Phi_\theta(z)\right) \tag{2.12}$$

and hence

$$m(-\theta)U(z)m(\theta) = U\left(\Phi_\theta(z)\right) \tag{2.13}$$

See also [16]. We show that the following adaptation of lemmas from [16] hold in our case too.

Lemma 2.4.1. (see also [16, Lemma 3.5]) *Suppose* $kP(z)k^T = P(w)$ *for some* $k \in K$ *and* $z, w \in \mathscr{F}$*. Then*

(1) If $f(z) \neq 0$*, then* $f(z) = f(w)$ *and* $k \in H \bigcup j_1 j_2 H$ *or* $f(z) = f(j_i(w))$ *and* $k \in j_1 H \bigcup j_2 H$*.*

(2) If $f(z) = 0$*, then* $f(z) = f(w)$ *and* $k \in U(z) \bigcup j_1 j_2 U(z)$*.*

The proof is the same as that of [16, Lemma 3.5].

Lemma 2.4.2. (see also [16, Lemma 3.6]) *If* $z \in \mathscr{S}$ *and* $f(\Phi_\theta(z)) = f(j_i(z))$*, then* $|f(z)| \neq 1$ *and* $\Phi_\theta(z) = j_1(z)$*.*

Proof. $f\Big(\Phi_\theta(z)\Big) = f(j_i(z)) = -f(z)$ by (A2) and $f\Big(\Phi_\theta(z)\Big) = \frac{f(z)+\tanh(\theta)}{1+f(z)\tanh(\theta)}$ by (A3), see Remark 5. Therefore, $f(z) \neq \pm 1$. Hence, $|f(z)| < 1$ and so, there exists $\tau \in \mathbb{R}$ such that $f(z) = \tanh(\tau)$. But also, $\tanh(\tau) = f\Big(\Phi_\tau(\pm e_{p+1}, \epsilon_1)\Big)$. Since $f(z) \neq \pm 1$ it is in the Φ-orbit of either (e_{p+1},ϵ_1) or $(-e_{p+1},\epsilon_1)$. Say it is in the former's orbit. Then, we have

$$\begin{aligned}
&f\Big(\Phi_{-\tau}(z)\Big)=0\\
&\Rightarrow \Phi_{-\tau}(z)=(e_{p+1},\epsilon_1)\\
&\Rightarrow z=\Phi_{\tau}(e_{p+1},\epsilon_1)\\
&\Rightarrow j_1*z=\Phi_{-\tau}(j_1*(e_{p+1},\epsilon_1))=\Phi_{-\tau}(e_{p+1},\epsilon_1)\\
&\Rightarrow f(\Phi_{\theta}(z))=\Phi_{-\tau}(e_{p+1},\epsilon_1)\\
&\Rightarrow f(\Phi_{\theta+\tau}(z))=f(e_{p+1},\epsilon_1)=0\\
&\Rightarrow \Phi_{\theta+\tau}(z)=(e_{p+1},\epsilon_1)=\Phi_{-\tau}(z)\\
&\Rightarrow \Phi_{2\tau+\theta}(z)=z\\
&\Rightarrow 2\tau+\theta=0\\
&\Rightarrow \tau=-\frac{\theta}{2}
\end{aligned}$$

But then,

$$j_1*z=\Phi_{\theta/2}(e_{p+1},\epsilon_1)=\Phi_{\theta}(z)$$

The argument is the same if z is in the orbit of $(-e_{p+1},\epsilon)$. □

Lemma 2.4.3. ([16, Lemma 3.7]) *If $j_i m(\theta)\in U(z)$, then $|f(z)|\neq 1$ and $i=1$.*

The proof is the same as in [16]. Note that in our case $i=2$ cannot happen. Indeed, then we would get

$$(1+f^2(z))=(f^2(z)-1)\cosh(\theta)$$

which is impossible, since the left hand side is always positive, while the right hand side is always negative.

2.5 Well-definedness of the $\mathrm{SO}^\circ(p,q)$ actions

Firstly, we show that the definition of the action in (2.6) of an element g of G, assuming $z \in \mathscr{S}$ and $k_0 = (\kappa_1, \kappa_2) \in K$ are fixed, is independent of the expression of gk_0 using (2.5). To that end, suppose that

$$gk_0 = km(\theta)u = k'm(\theta')u'$$

for $k, k' \in K$, $\theta, \theta' \in \mathbb{R}$ and $u, u' \in U(z)$. We have

$$\begin{gathered} km(\theta)P(z)m(\theta)k^T = k'm(\theta')P(z)m(\theta')(k')^T \\ \Rightarrow \lambda(\theta, z)kP\Big(\Phi_\theta(z)\Big)k^T = \lambda(\theta', z)k'P\Big(\Phi_{\theta'}(z)\Big)(k')^T \end{gathered}$$

by (2.12). By taking the trace of both sides, we get

$$\lambda(\theta, z) = \lambda(\theta', z) \tag{2.14}$$

and

$$kP\Big(\Phi_\theta(z)\Big)k^T = k'P\Big(\Phi_{\theta'}(z)\Big)(k')^T \tag{2.15}$$

Now, (2.15) gives $(k')^T k P\Big(\Phi_\theta(z)\Big) k^T k' = P\Big(\Phi_{\theta'}(z)\Big)$, which by Lemma 2.4.1 and Lemma 2.4.2 implies

$$\begin{cases} f\Big(\Phi_\theta(z)\Big) = f\Big(\Phi_{\theta'}(z)\Big) & \text{or} \\ f\Big(\Phi_\theta(z)\Big) = -f\Big(\Phi_{\theta'}(z)\Big) \end{cases}$$

$$\Rightarrow \begin{cases} f\Big(\Phi_{\theta-\theta'}(z)\Big) = f\Big(z\Big) & \text{or} \\ f\Big(\Phi_{\theta+\theta'}(z)\Big) = -f\Big(z\Big) = f(j_1(z)) \end{cases} \tag{2.16}$$

We deal first with the first case of (2.16). Suppose $f\left(\Phi_{\theta-\theta'}(z)\right) = f\Big(z\Big)$. If $f(z) = 1$, then by (2.14) we get:

$$\Big(\cosh(\theta) + \sinh(\theta)\Big)^2 = \Big(\cosh(\theta') + \sinh(\theta')\Big)^2$$

$$\Rightarrow (e^\theta)^2 = (e^{\theta'})^2$$

$$\Rightarrow \theta = \theta'$$

Similarly, if $f(z) = -1$, we again get $\theta = \theta'$. On the other hand, if $|f(z)| \neq 1$, then we have

$$\begin{aligned}
f\Big(\Phi_{\theta-\theta'}(z)\Big) &= f\Big(z\Big) \\
\xrightarrow{\theta_0=\theta-\theta'} \frac{f(z)+\tanh(\theta_0)}{1+f(z)\tanh(\theta_0)} &= f(z) \\
\Rightarrow f(z)+\tanh(\theta_0) &= f(z)+f^2(z)\tanh(\theta_0) \\
\Rightarrow \tanh(\theta_0) &= f^2(z)\tanh(\theta_0) \\
\Rightarrow \Big(1-f^2(z)\Big)\tanh(\theta_0) &= 0 \\
\xrightarrow{|f(z)|\neq 1} \tanh(\theta_0) &= 0 \\
\Rightarrow \theta_0 &= 0 \\
\Rightarrow \theta &= \theta'
\end{aligned}$$

Having now that $\theta = \theta'$, we have that

$$k^{-1}k' = m(\theta)u(u')^{-1}m(\theta) \in m(\theta)U(z)m(\theta)^{-1} = U\Big(\Phi_\theta(z)\Big)$$

by (2.13). Now, suppose an element $\tilde{k} = (\kappa_1, \kappa_2) \in K$ is in $U(\zeta)$ for some $\zeta \in \mathrm{S}^1$. Then,

$$\kappa_1\left(f(\zeta)e_1\right) = f(\zeta)e_1 \text{ and } \kappa_2 e_{p+1} = e_{p+1}$$

Thus, $\kappa_2 \in \mathrm{SO}(q-1)$ and, if $f(\zeta) \neq 0$, then $\kappa_1 \in \mathrm{SO}(p-1)$, while if $f(\zeta) = 0$, then $\zeta = (\pm e_{p+1}, \epsilon_1)$ In any case we see that

$$K \cap U(\zeta) = \mathrm{Stab}_K(\zeta)$$

Therefore, $k^{-1}k' \in \mathrm{Stab}_K\left(\Phi_\theta(z)\right)$ and hence

$$k \star \Phi_\theta(z) = k' \star \Phi_\theta(z)$$

Now, we consider the second case of (2.16), namely the case $f\Big(\Phi_{\theta+\theta'}(z)\Big) = f(j_1(z))$. By equation (2.15) and Lemma 2.4.1, we get that

$$k^{-1}k' \in j_1H \cup j_2H$$

Hence, there exists $h \in H$ for which $k' = kj_ih$. Then

$$\begin{aligned} km(\theta)u &= k'm(\theta')u' \\ \Rightarrow m(\theta)u &= j_ihm(\theta')u' \\ \Rightarrow m(\theta)u &= j_im(\theta')hu' \\ \Rightarrow m(\theta)u &= m(-\theta')j_ihu' \\ \Rightarrow j_im(\theta+\theta') &= hu'u^{-1} \end{aligned}$$

Hence, $j_im(\theta+\theta') \in U(z)$ and by Lemma 2.4.3, $i = 1$ and $|f(z)| \neq 1$, which then implies

$\Phi_{\theta+\theta'}(z) = j_1(z)$. As a result, we have

$$\begin{aligned} k' \star \Phi_{\theta'}(z) &= kj_1 h \star \Phi_{\theta'}(z) \\ &= kj_1 \star \Phi_{\theta'}(z) \\ &= kj_1 \star \Phi_{-\theta+(\theta+\theta')}(z) \\ &= kj_1 m(-\theta) \star \Phi_{\theta+\theta'}(z) \\ &= km(\theta) j_1 m(\theta+\theta') \star z \\ &= km(\theta) \star z \\ &= k \star \Phi_\theta(z) \end{aligned}$$

Hence, the action in (2.6) is well defined when $k_0 = (\kappa_1, \kappa_2) \in K$ and $z \in \mathscr{S}$ are fixed.

Next, suppose $(v, w) \in \mathrm{S}^p \times \mathrm{S}^{q-1}$. We now show that the action in (2.6) is independent of the way (v, w) is written as $(v, w) = k_0 \star z$ for $k_0 \in K$ and $z \in \mathscr{S}$. To the end, assume $(v, w) = k_0 \star z = k_0' \star z'$, for $k_0 = (\kappa_1, \kappa_2)$, $k_0' = (\kappa_1', \kappa_2') \in K$ and $z = (\alpha_z e_1 + \beta_z e_{p+1}, \epsilon_1)$, $z' = (\alpha_{z'} e_1 + \beta_{z'} e_p + 1, \epsilon_1) \in \mathscr{S}$. Moreover, for a $g \in G$, let $gk_0 = km(\theta)u$ and $gk_0' = k'm(\theta')u'$. Recall that for $\kappa \in \mathrm{SO}(p)$, $\tilde{\kappa} = \begin{bmatrix} \kappa & \\ & 1 \end{bmatrix}$, see equation (2.1). Now,

$$\begin{aligned} & k_0 \star z = k_0' \star z' \\ \Rightarrow & \begin{cases} \tilde{\kappa_1}(\alpha_z e_1 + \beta_z e_{p+1}) = \tilde{\kappa_1'}(\alpha_{z'} e_1 + \beta_{z'} e_{p+1}) \\ \kappa_2 \epsilon_1 = \kappa_2' \epsilon_1 \end{cases} \end{aligned}$$

From that we deduce that $\beta_z = \beta_{z'}$ and that $\alpha_z = \pm\alpha_{z'}$, since $\alpha_z^2 + \beta_z^2 = \alpha_{z'}^2 + \beta_{z'}^2 = 1$. Hence, $z = j_1^\eta z'$, where $\eta = 0$ or 1. Hence, $(k_0')^{-1} k_0 j_1^\eta$ fixes z' and therefore, $(k_0')^{-1} k_0 j_1^\eta \in U(z')$, which implies $k_0' = k_0 j_1^\eta u''$, for some $u'' \in U(z')$. Then,

$$\begin{aligned} gk_0' &= gk_0 j_1^\eta u'' \\ &= km(\theta) u j_1^\eta u'' \\ &= km(\theta) j_1^\eta \Big(j_1^\eta u j_1^\eta u'' \Big) \\ &= kj_1^\eta m\Big((-1)^\eta \theta \Big) \Big(j_1^\eta u j_1^\eta u'' \Big) \end{aligned}$$

Now, $P(j_i(z)) = j_i P(z) j_i$ and therefore, $j_1^\eta u j_1^\eta \in U(z')$. But then, $kj_1^\eta m\Big((-1)^\eta\theta\Big)\Big(j_1^\eta u j_1^\eta u''\Big)$ and $k'm(\theta')u'$ are different expressions for gk_0' using (2.5) for the point $z' \in \mathscr{S}$ and as we've already seen the action defined in (2.6) agrees for the two expressions, namely:

$$kj_1^\eta \star \Phi_{(-1)^\eta\theta}(z') = k' \star \Phi_{\theta'}(z')$$

If $\eta = 0$, then $z = z'$ and $k' \star \Phi_{\theta'}(z') = k \star \Phi_\theta(z)$, while if $\eta = 1$, then $z = j_1 z'$ and $k' \star \Phi_{\theta'}(z') = kj_1 \star \Phi_{-\theta}(z') = k \star \Phi_\theta(j_1 z') = k \star \Phi_\theta(z)$.

Finally, we see that this definition does indeed give an action of G. Let $g, g' \in G$, $(v, w) \in \mathrm{S}^p \times \mathrm{S}^{q-1}$ and $(v, w) = (\kappa_1, \kappa_2) \star z$, and write $g(\kappa_1, \kappa_2) = km(\theta)u$ and $g'k =$

$k'm(\theta')u'$, where $u \in U(z)$ and $u' \in U(\Phi_\theta(z))$. Then,

$$\begin{aligned} g'g(\kappa_1, \kappa_2) &= g'km(\theta)u \\ &= k'm(\theta')u'm(\theta)u \\ &= k'm(\theta' + \theta)\Big(m(-\theta)u'm(\theta)\Big)u \end{aligned}$$

But, by (2.13), $m(-\theta)u'm(\theta) \in U(z)$, therefore

$$\begin{aligned} g' \star \Big(g \star z\Big) &= g' \star \Big(k \star \Phi_\theta(z)\Big) \\ &= k' \star \Phi_{\theta'}(\Phi_\theta(z)) \\ &= k' \star \Phi_{\theta'+\theta}(z) \\ &= g'g \star z \end{aligned}$$

It is also easy to see that this action extends the action of K on $\mathrm{S}^p \times \mathrm{S}^{q-1}$.

2.6 Isomorphic Actions

Here we prove Theorem 2.1.1

Proof. (of Theorem 2.1.1) Suppose that for two G-actions from the basic construction, the basic J_1-flows on S^1 that they correspond to, say $\widetilde{\Phi}^1_\theta$ and $\widetilde{\Phi}^2_\theta$, are analytically isomorphic. Let Φ^1_θ be the induced basic j_1-flow of $\widetilde{\Phi}^1_\theta$ on $\mathscr{S}$, see Definition 2.0.2. Similarly, consider the induced basic j_1-flow Φ^2_θ of $\widetilde{\Phi}^2_\theta$. Recall $\mathscr{S} = \{(\alpha e_1 + \beta e_{p+1}, \epsilon_1)\} \subseteq \mathrm{S}^p \times \mathrm{S}^{q-1}$. Now, since $\widetilde{\Phi}_\theta^1$ and $\widetilde{\Phi}_\theta^2$ are analytically isomorphic, it is immediate that Φ_θ^1 and Φ_θ^2 are also analytically isomorphic. Equivalently, the vector fields that Φ_θ^1 and Φ_θ^2 generate, say X and Y respectively, are isomorphic. We note that, by the result in [7], analytic vector fields on the circle are classified, see Appendix A for more details, see also Remark 1. Then, there exists an analytic isomorphism, Ψ, of $\mathscr{S}$ such that

$$\Psi \circ \Phi_\theta^1 = \Phi_\theta^2 \circ \Psi$$

Moreover, we can assume that Ψ satisfies $\Psi \circ j_1 = j_1 \circ \Psi$, see Appendix A. Let $f_1 = f_{\Phi^1}$ and $f_2 = f_{\Phi^2}$, see Remark 2. We claim that Ψ also satisfies $f_1 = f_2 \circ \Psi$. Indeed, if $X = g_X \cdot \partial_1$, where ∂_1 is a basic vector of $\mathscr{S}$, then by (A1) and (A3) of Remark 5, it is easy to see that

$$\begin{cases} g_X \circ j_1 = g_X \\ g_X \cdot (f_1)' = (f_1)^2 - 1 \end{cases}$$

and similarly for Y.

Let $z = (\pm e_{p+1}, \epsilon_1)$. Then, $f_1(z) = 0$ and $g_X(z) \neq 0$. By $\Psi \circ j_1 = j_1 \circ \Psi$, $z' = \Psi(z)$ is also fixed by j_1, i.e. it is either (e_{p+1}, ϵ_1) or $(-e_{p+1}, \epsilon_1)$ and hence, $f_2(z') = 0$ and $g_Y(z') \neq 0$. Let I be a neighbourhood of z and see I as an interval on the real line. Similarly for an I' around z', such that $\Psi : I \to I'$ analytic isomorphism.

In terms of g_X and g_Y, the relation $\Psi \circ \Phi^1 = \Phi^2 \circ \Psi$ becomes

$$g_X \cdot \Psi' = g_Y \circ \Psi$$

This is equivalent to

$$g_X \circ \Psi^{-1} = g_Y \cdot \left(\Psi^{-1}\right)'$$

Now,

$$\begin{aligned} & g_X \cdot f_1' = f_1^2 - 1 \\ \Rightarrow & g_X \circ \Psi^{-1} \cdot f_1' \circ \Psi^{-1} = f_1^2\left(\Psi^{-1}\right) - 1 \\ \Rightarrow & g_Y \cdot \left(\Psi^{-1}\right)' \cdot f_1' \circ \Psi^{-1} = f_1^2\left(\Psi^{-1}\right) - 1 \\ \Rightarrow & g_Y \cdot \left(f_1 \circ \Psi^{-1}\right)' = f_1^2\left(\Psi^{-1}\right) - 1 \end{aligned}$$

Therefore, both f_2 and $f_1 \circ \Psi^{-1}$ solve the equation

$$\begin{cases} u' = \frac{1}{g_Y}\left(u^2 - 1\right) \\ u(z') = 0 \end{cases}$$

where by u' we denote the derivative of u. Therefore they're equal around z', and being analytic, they're equal. Therefore, we have an analytic isomorphism

$$\Psi : \mathscr{S} \to \mathscr{S}$$

such that

$$\Psi \circ \Phi_\theta^1 = \Phi_\theta^2 \circ \Psi \quad \text{and}$$

$$f_1 = f_2 \circ \Psi$$

Now, we can define a map

$$\tilde{\Psi} : \mathrm{S}^p \times \mathrm{S}^{q-1} \to \mathrm{S}^p \times \mathrm{S}^{q-1}$$

by

$$\tilde{\Psi}\left(k \star \Phi^1_\theta(z)\right) \mapsto k \star \Phi^2_\theta\left(\Psi(z)\right)$$

- $\tilde{\Psi}$ is well defined:

  Suppose $k\star\Phi^1_\theta(z) = k'\star\Phi^1_{\theta'}(z')$. If $\Phi^1_\theta(z) = (\pm e_{p+1}, \epsilon_1)$, then by $\Phi^1_\theta(z) = k^{-1}k'\star\Phi^1_{\theta'}(z')$, we get that $k^{-1}k' \in \mathrm{SO}(p) \times \mathrm{SO}(q-1)$ and $\Phi^1_{\theta'}(z') = \Phi^1_\theta(z) = (\pm e_{p+1}, \epsilon_1)$. Then, by $0 = f_1(\pm e_{p+1}, \epsilon_1) = f_2\left(\Psi(\pm e_{p+1}, \epsilon_1)\right)$ we have $\Psi(\pm e_{p+1}, \epsilon_1) = (e_{p+1}, \epsilon_1)$ or $(-e_{p+1}, \epsilon_1)$, hence $k\star\Psi(\pm e_{p+1}, \epsilon_1) = k'\star\Psi(\pm e_{p+1}, \epsilon_1)$. Finally, $k\star\Psi(\pm e_{p+1}, \epsilon_1) = k'\star\Psi(\pm e_{p+1}, \epsilon_1)$ implies $k \star \Psi(\Phi^1_\theta(z)) = k' \star \Psi(\Phi^1_{\theta'}(z'))$. Therefore, $k \star \Phi^2_\theta\left(\Psi(z)\right) = k' \star \Phi^2_{\theta'}\left(\Psi(z')\right)$. If $\Phi^1_\theta(z) \neq (\pm e_{p+1}, \epsilon_1)$, then $\Phi^1_\theta(z') = \Phi^1_\theta(z)$ or $j_1 \star \Phi^1_\theta(z)$, and $k^{-1}k' \in H$ or $j_1 H$ respectively. In the former case the result is immediate. In the latter, it follows by the equations $\Phi^i_\theta \circ j_1 = j_1 \circ \Phi^i_\theta$ and $\Psi \circ j_1 = j_1 \circ \Psi$.

- $\tilde{\Psi}$ is onto:

  Let $(v, w) \in \mathrm{S}^p \times \mathrm{S}^{q-1}$. There exist $k \in K$, $z \in \mathscr{S}$ and $\theta \in \mathbb{R}$ such that $(v, w) = k \star \Phi^2_\theta(z)$. Then, $(v, w) = \tilde{\Psi}\left(k \star \Phi^1\left(\Psi^{-1}(z)\right)\right)$

- $\tilde{\Psi}$ is 1-1: The proof proceeds similarly to the well definedness of $\tilde{\Psi}$.

- $\tilde{\Psi}$ is G-equivariant: The proof is identical to that of Lemma 4.2.3 in Section 4.2.

- $\tilde{\Psi}$ is a local analytic isomorphism: The proof is identical to that of Lemma 4.2.4 in Section 4.2.

Therefore, the two G actions are analytically isomorphic, if the analytic isomorphism Ψ on $\mathscr{S}$ between Φ_θ^1 and Φ_θ^2 exists. On the other hand, it is immediate that an analytic, G-equivariant isomorphism between two such G-actions will result in isomorphic induced basic j_1-flows on $\mathscr{S}$ and hence, the corresponding basic J_1-flows on S^1 are also isomorphic. □

Chapter 3: Extendable $\mathrm{SO}(p) \times \mathrm{SO}(q)$ actions

We consider $G = \mathrm{SO}^{\circ}(p,q)$ and its maximal compact subgroup $K = \mathrm{SO}(p) \times \mathrm{SO}(q)$. We are interested in the analytic actions of K on a closed, connected manifold M of dimension $p+q-1$, that extend to an analytic action of G. To that end, we are going to look at the possible $\mathrm{SO}(p)$ and $\mathrm{SO}(q)$ orbits. Uchida has classified the subgroups of O_p of codimension at most $2p-2$, see [15] and [14]. We show that Uchida's result can be applied here. We assume $p \geq q$ and that K acts on a manifold M as above. Assume also that the action extends to G. Let $x \in M$.

Notation: We denote by $\mathscr{O}^p$ the $\mathrm{SO}(p)$-orbit of x and by $\mathscr{O}^q$ its $\mathrm{SO}(q)$-orbit.

Lemma 3.0.1. $\dim \mathscr{O}^p < 2p-1$

Proof. Since $p \geq q$, $\dim \mathscr{O}^p \leq p+q-1 \leq 2p-1$. If $\dim \mathscr{O}^p = 2p-1$, then it follows $q = p$. Consider $\mathrm{Stab}_{\mathrm{SO}(q)}(x)$, which acts trivially on $\mathrm{T}_x\mathscr{O}^p = \mathrm{T}_xM$. If $\mathrm{Stab}_{\mathrm{SO}(q)}(x)$ has dimension ≥ 1, then the action of K is not locally effective and so it cannot extend to an action of G. If $\mathrm{Stab}_{\mathrm{SO}(q)}(x)$ has dimension $= 0$, then $\mathscr{O}^q$ has dimension $q(q-1)/2 = p(p-1)/2$. For $p = q \geq 5$ this is bigger than the dimension of the manifold and for $p = q = 4$ or 3, it is impossible for $\mathscr{O}^p$ to be of dimension $2p-1$ since that is bigger than the dimension of $\mathrm{SO}(p)$. □

So, we suppose we have $\dim \mathscr{O}^p \leq 2p-2$ and therefore, we can apply Uchida's result.

Let $V := \mathrm{T}_x\mathscr{O}^p \bigcap \mathrm{T}_x\mathscr{O}^q$. In particular, $V \leq \mathrm{T}_x\mathscr{O}^p$ is a trivial subrepresentation of the isotropy representation of $H' = \mathrm{Stab}_{\mathrm{SO}(p)}(x)$. Therefore, the dimension of V is bounded by the dimension of a maximal subspace of $\mathrm{T}_x\mathscr{O}^p$ on which H' acts trivially via the isotropy representation. In turn, the dimension of $\mathrm{T}_x\mathscr{O}^q$ is bounded by

$$\dim \mathrm{T}_x\mathscr{O}^q \leq \dim M - \dim \mathrm{T}_x\mathscr{O}^p + \dim V \tag{3.1}$$

At the same time q must satisfy

$$\dim\mathscr{O}^p \leq p + q - 1 \quad \text{and} \quad q \leq p \tag{3.2}$$

We present Uchida's classification of the orbit types, $\mathrm{SO}(p)\Big/H'$, mentioned above, in Table 3.1 below. The last column comes from taking (3.2) into consideration.

For reference, we will call the last four rows of Table 3.1 the bottom part and the rest the top part. The embedding of H' in $\mathrm{SO}(p)$ in each case is the usual one, except for the penultimate row of the top part of Table 3.1 where it is the irreducible 5-dimensional representation of $\mathrm{SO}(3)$.

Table 3.1: Uchida Table

p	subgroup H'	dim$\mathscr{O}^p$	dim$M$	q
p	$\mathrm{SO}(p-2)$	$2p-3$	$\leq 2p-1$	$p-2 \leq q \leq p$
p	$\mathrm{SO}(p-2) \times \mathrm{SO}(2)$	$2p-4$	$\leq 2p-1$	$p-3 \leq q \leq p$
9	$\mathrm{Spin}(7)$	15	≤ 17	$7 \leq q \leq 9$
8	G_2	14	≤ 15	$7 \leq q \leq 8$
8	U_4	12	≤ 15	$5 \leq q \leq 8$
8	SU_4	13	≤ 15	$6 \leq q \leq 8$
7	G_2	7	≤ 13	$3 \leq q \leq 7$
7	U_3	12	≤ 13	$6 \leq q \leq 7$
7	$\mathrm{SO}(3) \times \mathrm{SO}(4)$	12	≤ 13	$6 \leq q \leq 7$
6	$\mathrm{SO}(3) \times \mathrm{SO}(3)$	9	≤ 11	$4 \leq q \leq 6$
6	U_3	6	≤ 11	$3 \leq q \leq 6$
6	SU_3	7	≤ 11	$3 \leq q \leq 6$
6	$\mathrm{U}_2 \times \mathrm{U}_1$	10	≤ 11	$5 \leq q \leq 6$
5	U_2	6	≤ 9	$3 \leq q \leq 5$
5	SU_2	7	≤ 9	$3 \leq q \leq 5$
5	$\mathrm{U}_1 \times \mathrm{U}_1$	8	≤ 9	$4 \leq q \leq 5$
5	$\mathrm{SO}(3)$	7	≤ 9	$3 \leq q \leq 5$
3	$\{1\}$	3	≤ 5	3
p	$\mathrm{SO}(p-1)$	$p-1$	$\leq 2p-1$	$3 \leq q \leq p$
8	$\mathrm{Spin}(7)$	7	≤ 15	$3 \leq q \leq 8$
4	SU_2	3	≤ 7	$3 \leq q \leq 4$
4	U_2	2	≤ 7	$3 \leq q \leq 4$

Lemma 3.0.2. *In all the cases for $\mathscr{O}^p$, the dimension of a maximal subspace of $T_x\mathscr{O}^p$ on which the respective subgroup H' acts trivially is at most 1. In particular, V is at most 1 dimensional.*

Proof. This is easy to check in all cases, except maybe for the penultimate row of the top part of Table 3.1. Let $v \in \mathrm{T}_x\mathscr{O}^p$, nonzero, such that H' acts trivially on v in the isotropy representation. Since we can identify $\mathrm{T}_x\mathscr{O}^p$ with $\mathfrak{so}_5 \big/ \mathfrak{h}'$, where $\mathfrak{h}'$ is the Lie algebra of H', we can choose an $X \in \mathfrak{so}_5$ such that $X \bmod \mathfrak{h}' \equiv v$ under this identification, and $e^{tX} \in N_{\mathrm{SO}(5)}(H')$, the normaliser of H' in SO(5). Now, if there exists a subspace $\tilde{V} \leq \mathrm{T}_x\mathscr{O}^p$ on which H' acts trivially and dim$\tilde{V} \geq 2$, then for $N_{\mathrm{SO}(5)}(H')$ we would have

$$\dim N_{\mathrm{SO}(5)}(H') \geq \dim H' + 2 \Rightarrow \dim N_{\mathrm{SO}(5)}(H') \geq 5$$

But then, $N_{\mathrm{SO}(5)}(H')$ is a subgroup of SO(5) of codimension at most 5. According to Table 3.1, the only possibility is $N_{\mathrm{SO}(5)}(H') \simeq \mathrm{SO}(4)$ and (by conjugating) we can assume $N_{\mathrm{SO}(5)}(H')$ is imbedded in SO(5) with the standard representation of SO(4). But then, since $\mathrm{SO}(3) \leq N_{\mathrm{SO}(5)}(H')$, the representation of SO(3) in SO(5) is not irreducible, which is a contradiction. So, the dimension of a subspace of $\mathrm{T}_x\mathscr{O}^p$ on which H' acts trivially is at most 1. $\square$

Immediate from Table 3.1 is the following:

Lemma 3.0.3. *$\mathscr{O}^q$ cannot be 1-dimensional.*

Moreover, we have

Lemma 3.0.4. *Suppose that* $\dim M$ - $\dim \mathscr{O}^p \leq q-1$. *Then, x cannot be fixed by* $\mathrm{SO}(q)$, *i.e. $\mathscr{O}^q$ cannot be 0-dimensional.*

Proof. Firstly, let $q \neq 4$. Assume SO_q stabilises x and consider its isotropy representation on $\mathrm{T}_x M$. Then, $\mathrm{SO}(q)$ acts trivially on $\mathrm{T}_x\mathscr{O}^p$ since $\mathrm{SO}(p)$ and $\mathrm{SO}(q)$ commute in G. Moreover, since it does not have a non-trivial representation of dimension $\leq q-1$, we conclude that its isotropy representation is trivial. But then, SO_q acts trivially on a neighbourhood of x in M, which contradicts our assumption that the action extends to $\mathrm{SO}^\circ(p,q)$ since $\mathrm{SO}^\circ(p,q)$ is simple and hence the action is locally effective.

For $q = 4$, there is the possibility that $\dim M$ - $\dim \mathscr{O}^p = 3$ and that the representation of SO(4) on a 3-dimensional complement of $\mathrm{T}_x\mathscr{O}^p$ is not trivial. In that case, pick an SO(4) invariant metric on M and since SO(4) acts trivially on $\mathrm{T}_x\mathscr{O}^p$. The isotropy representation gives a homomorphism

$$\mathrm{SO}(4) \to \mathrm{O}_3$$

Then, this homomorphism has kernel with nontrivial dimension, which contradicts the local effectiveness of the action again. ☐

Lemma 3.0.5. *Suppose that $\mathscr{O}^p \simeq \mathrm{SO}(p)/H'$ and that, in the isotropy representation of H' on $\mathrm{T}_x\mathscr{O}^p$, the maximum dimension of a subspace on which H' acts trivially is δ, with $\delta = 0$ or 1. Then, the following situation is impossible:*

- $(2p-1) - \dim\mathscr{O}^p \leq 3 - \delta$

  and

- $\dim\mathscr{O}^p + 1 - p \geq 5$

Proof. The first condition forces $\dim\mathscr{O}^q \leq 3$. However, the second condition forces $q \geq 5$ which contradicts $1 \leq \dim\mathscr{O}^q \leq 3$, and $\mathrm{SO}(q)$ cannot fix x by Lemma 3.0.4. ☐

Note that if $\dim\mathscr{O}^p + \dim\mathscr{O}^q \geq \dim M$, then the K action is transitive. Indeed, the K-orbit of x is then open, closed and connected, therefore it equals M. It turns out that transitive actions cannot occur:

Lemma 3.0.6. *Suppose that the action of K on M is transitive. Then, the action does not extend to G.*

The proof of Lemma 3.0.6 is the content of Section 3.1.

Using Lemma 3.0.3, Lemma 3.0.4, Lemma 3.0.5 and Lemma 3.0.6 it is easy to check for most cases that if $\mathscr{O}^p$ is any from the top part of Table 3.1, then there is no possible $\mathscr{O}^q$ for which such a K action would extend to G. The only pairs $(\mathscr{O}^p, \mathscr{O}^q)$ which the above lemmas fail to eliminate are:

$$\left(\mathrm{SO}(7)\Big/\mathrm{G}_2\,,\,\mathrm{SO}(4)\Big/\mathrm{U}_2\right)\,,\,\left(\mathrm{SO}(6)\Big/\mathrm{U}_3\,,\,\mathrm{SO}(4)\Big/\mathrm{U}_2\right)\text{ and}$$
$$\left(\mathrm{SO}(4)\Big/(\mathrm{SO}(2)\times\mathrm{SO}(2))\,,\,\mathrm{SO}(4)\Big/\mathrm{U}_2\right)$$

These also do not extend to a G action as it will be shown in Section 3.2. As for when $\mathscr{O}^p$ is one from the bottom part of Table 3.1, the lemmas can be used to exclude the cases when $\mathscr{O}^q$ is from the top part of Table 3.1, but the cases where both $\mathscr{O}^p$ and $\mathscr{O}^q$ are from the bottom part still remain. These cases are treated in Section 3.2 as well.

3.1 Transitive $\mathrm{SO}(p) \times \mathrm{SO}(q)$ actions

Here we see why a transitive $\mathrm{SO}(p) \times \mathrm{SO}(q)$ action on a manifold M of dimension $p+q-1$ does not extend to an action of $G = \mathrm{SO}^\circ(p,q)$. Assume it does. Then, we get a compact homogeneous space of G. Let H' be a subgroup of G such that $G\big/H' \cong M$. That means that H' is a cocompact subgroup of G. We will need the following version of a theorem of Witte:

Theorem. *([21, Main Theorem 1.2]) Let G be a connected, semisimple Lie group with finite center and $H' \leq G$ closed and cocompact. Then, there exists a parabolic subgroup P of G such that, if $P^\circ = LEAN$ is the refined Langlands decomposition (see below) of P°, then there exist closed, connected subgroup $Y \leq EA$ and a normal, closed, connected subgroup $X \leq L$ such that $(H')^\circ = XYN$*

The *refined Langlands decomposition* of P°, as defined in [21], is the following: Let $P^\circ = MAN$ be a Langlands decomposition of P°. Then, let L be the product of all the noncompact simple factors of P° and E be the maximal compact factor of P°.

We will follow [9] for the definitions and notation about the roots and the root spaces of G, and [20] about the definitions and notation about parabolic subgroups. We show below that, because of the dimension of the manifolds we consider, H' could only be in a maximal parabolic of G. There are two different kind of maximal parabolic subgroups of G. In the standard representation of G on $\mathbb{R}^{p+q}$ with a scalar product of type (p, q), the first kind preserves a null line and the second kind preserves a maximal isotropic subspace.

3.1.1 The first kind of maximal parabolics

If

$$\mathfrak{so}(p,q) = \left\{ \begin{bmatrix} X & A \\ A^T & Y \end{bmatrix} \right\}$$

with X, Y antisymmetric matrices, then a maximal abelian subspace, $\mathfrak{a}$, of the symmetric part is the collection of matrices with $X = 0, Y = 0$ and A being of the form

$$A = \begin{bmatrix} & \vdots & \\ & & a_q \\ & \iddots & \\ a_1 & & \end{bmatrix}$$

with everything except for the a_i's being 0. Let f_i denote the restricted root that picks out the a_i component of an element of $\mathfrak{a}$.

First the case $p = q$. A fundamental root system is

$$\mathscr{Y} = \{f_1 - f_2, \cdots, f_{p-1} - f_p, f_{p-1} + f_p\}$$

and the restricted root spaces $\mathfrak{g}^{\pm f_i \pm f_j}$, $i \neq j$, have dimension 1. Subsets of $\mathscr{Y}$ correspond to parabolic subgroups P_Θ of G and vice versa, see [20]. Firstly, we take

$$\Theta = \{f_2 - f_3, \cdots, f_{p-1} - f_p, f_{p-1} + f_p\}$$

Denote by Σ^+ the space of positive roots, where positivity is induced by $\mathscr{Y}$ in the standard way. Moreover, denote by $\langle\Theta\rangle^+$, respectively $\langle\Theta\rangle^-$, the space of positive, respectively negative, roots that also belong in the span of Θ. Next, define $\mathfrak{a}_\Theta$ to be the subspace of $\mathfrak{a}$ where all roots are 0, $\mathfrak{m}$ to be the centraliser of $\mathfrak{a}$ in $\mathfrak{k} = \mathfrak{so}(p) \oplus \mathfrak{so}(q)$, $\mathfrak{n}^\pm(\Theta) = \sum_\lambda \mathfrak{g}^\lambda$, for $\lambda \in \langle\Theta\rangle^\pm$, and $\mathfrak{n}_\Theta^+ = \sum_\lambda \mathfrak{g}^\lambda$, for $\lambda \in \Sigma^+ - \langle\Theta\rangle^+$. In our case, $\mathfrak{a}_\Theta$ is 1-dimensional since it is the span of the element with $a_2 = a_3 = \cdots = a_p = 0$, while $\mathfrak{m} = 0$ since $p = q$, see [9, Chapter VI, Section 4]. Moreover,

$$\mathfrak{n}^+(\Theta) = \sum_{2 \leq i < j \leq p} \mathfrak{g}^{f_i \pm f_j} \tag{3.3}$$

and

$$\mathfrak{n}_\Theta^+ = \sum_{2 \leq i \leq p} \mathfrak{g}^{f_1 \pm f_j} \tag{3.4}$$

Then, if $\mathfrak{p}_\Theta$ is the Lie algebra of the parabolic subgroup P_Θ, we have that

$$|\mathfrak{p}_\Theta| = |\mathfrak{m}| + |\mathfrak{a}| + |\mathfrak{n}_\Theta^+| + |\mathfrak{n}^+(\Theta)| + |\mathfrak{n}^-(\Theta)|$$

where, by $|\cdot|$ we mean the dimension of the respective vector space, see [20, Chapter 1, Section 1.2.4]. We note that,

$$|\mathfrak{n}^+(\Theta)| = |\mathfrak{n}^-(\Theta)|$$

Therefore, we have the following data

$\lvert\mathfrak{n}^+(\Theta)\rvert$	$= p^2 - 3p + 2$
$\lvert\mathfrak{n}_\Theta^+\rvert$	$= 2(p-1)$
$\lvert\mathfrak{p}_\Theta\rvert = \lvert\mathfrak{m}\rvert + \lvert\mathfrak{a}\rvert + \lvert\mathfrak{n}_\Theta^+\rvert + \lvert\mathfrak{n}^+(\Theta)\rvert + \lvert\mathfrak{n}^-(\Theta)\rvert$	$= 2p^2 - 3p + 2$

As for $p > q$, we have

$$\mathfrak{m} = \mathfrak{so}_{p-q}$$

As fundamental root system we take

$$\mathscr{Y} = \{f_1 - f_2, \cdots, f_{q-1} - f_q, f_q\}$$

The restricted root spaces $\mathfrak{g}^{\pm f_i \pm f_j}$ have dimension 1, while the $\mathfrak{g}^{\pm f_i}$'s have dimension $p-q$. As Θ we take

$$\Theta = \{f_2 - f_3, \cdots, f_{q-1} - f_q, f_q\}$$

Then, $\mathfrak{a}_\Theta$ is 1-dimensional, spanned by the vector with $a_1 = 1$. Moreover, we have

$$\mathfrak{n}^+(\Theta) = \sum_{2 \le i < j \le q} \mathfrak{g}^{f_i \pm f_j} \oplus \sum_{2 \le i \le q} \mathfrak{g}^{f_i}$$

and

$$\mathfrak{n}_\Theta^+ = \sum_{2 \le i \le q} \mathfrak{g}^{f_1 \pm f_i} \oplus \mathfrak{g}^{f_1}$$

Hence,

$\lvert\mathfrak{m}\rvert$	$= \frac{(p-q)(p-q-1)}{2}$
$\lvert\mathfrak{n}^+(\Theta)\rvert$	$= pq - 2q - p + 2$
$\lvert\mathfrak{n}^+_\Theta\rvert$	$= p + q - 2$
$\lvert\mathfrak{p}_\Theta\rvert = \lvert\mathfrak{m}\rvert + \lvert\mathfrak{a}\rvert + \lvert\mathfrak{n}^+_\Theta\rvert + \lvert\mathfrak{n}^+(\Theta)\rvert + \lvert\mathfrak{n}^-(\Theta)\rvert$	$= \frac{1}{2}(p^2+q^2) + pq - \frac{3}{2}p - \frac{3}{2}q + 2$

In both cases, the codimension then of P_Θ is $p+q-2$ and so, the codimension of H' in P_Θ is 1. P_Θ leaves invariant a null-line in the standard representation of G in $\mathbb{R}^{p+q}$ with a scalar product of signature (p,q) and hence, we know that $M^\circ_\Theta \simeq \mathrm{SO}^\circ(p-1,q-1)$, $A_\Theta \simeq \mathbb{R}$. Suppose $A_\Theta \leq (H')^\circ$. Then, by Witte's theorem, M°_Θ would have a codimension 1 normal subgroup. However, $\mathrm{SO}^\circ(p-1,q-1)$ is semisimple, so that is impossible. Therefore,

$$(H')^\circ \simeq M^\circ_\Theta N$$

Assume now that we have a transitive action of $\mathrm{SO}(p)\times\mathrm{SO}(q)$ such that $H' \leq P_\Theta$. If $\mathscr{O}^p \simeq \mathrm{SO}(p)\big/H_1$ and $\mathscr{O}^q \simeq \mathrm{SO}(q)\big/H_2$, since $Y := H_1 \times H_2 \leq H'$ is compact and connected, it necessarily lies in a maximal compact subgroup of $(H')^\circ$ which is isomorphic to $\mathrm{SO}(p-1)\times\mathrm{SO}(q-1)$. Therefore, Y can be conjugated into a subgroup of $\mathrm{SO}(p-1)\times\mathrm{SO}(q-1)$ and we can assume $Y \leq \mathrm{SO}(p-1)\times\mathrm{SO}(q-1)$. In order to get a contradiction we are going to need *Goursat's lemma*:

Lemma. *(Goursat's lemma) Let G_1, G_2 be groups and suppose $A \leq G_1 \times G_2$. Denote $p_1 : A \to G_1$ and $p_2 : A \to G_2$ the projections to first and second factor and let N_1 be the kernel of p_2 and N_2 the kernel of p_1. Then, N_1 and N_2 can be seen as normal subgroups of $p_1(A)$ and $p_2(A)$ respectively, and the image of A in $p_1(A)\big/N_1 \times p_2(A)\big/N_2$ is the graph of an isomorphism between $p_1(A)\big/N_1$ and $p_2(A)\big/N_2$*

Now, as far as the dimension of Y is concerned, on the one hand we have

$$\dim(Y) = \dim(\mathrm{SO}(p)) + \dim(\mathrm{SO}(q)) - (p + q - 1) \tag{3.5}$$

On the other hand, we also have

$$\dim(Y) = \dim(p_1(Y)) + \dim(N_2) = \dim(p_2(Y)) + \dim(N_1) \tag{3.6}$$

Since $p_1(Y) \leq \mathrm{SO}(p-1)$, by equality (3.5) above, we have

$$\dim(Y) \leq \dim(\mathrm{SO}(p-1)) + \dim(N_2) \tag{3.7}$$

By (3.5) and (3.7), we get that N_2 is a subgroup of $\mathrm{SO}(q)$ of codimension at most q that is also contained in $\mathrm{SO}(q-1)$. By Table 3.1, the only possiblility is $N_2 = \mathrm{SO}(q-1)$. Therefore,

$$N_2 = p_2(Y) = \mathrm{SO}(q-1)$$

Then,(3.5) and (3.6) imply that N_1 is a codimension p subgroup of $\mathrm{SO}(p)$ that is also contained in $\mathrm{SO}(p-1)$. By Table 3.1, this is impossible.

3.1.2 The second kind of maximal parabolics

If $p = q$, we take

$$\Theta = \{f_1 - f_2, \cdots, f_{p-1} - f_p\}$$

in which case $\mathfrak{a}_\Theta$ is 1-dimensional, spanned by the element $a_1 = a_2 = \cdots = a_p = 1$. We also have

$$\mathfrak{n}^+(\Theta) = \sum_{1 \le i < j \le p} \mathfrak{g}^{f_i - f_j}$$

and

$$\mathfrak{n}^+_\Theta = \sum_{1 \le i < j \le p} \mathfrak{g}^{f_i + f_j}$$

As a result, we get

$\lvert\mathfrak{n}^+(\Theta)\rvert$	$= \frac{1}{2}(p^2 - p)$
$\lvert\mathfrak{n}^+_\Theta\rvert$	$= \frac{1}{2}(p^2 - p)$
$\lvert\mathfrak{p}_\Theta\rvert = \lvert\mathfrak{m}\rvert + \lvert\mathfrak{a}\rvert + \lvert\mathfrak{n}^+_\Theta\rvert + \lvert\mathfrak{n}^+(\Theta)\rvert + \lvert\mathfrak{n}^-(\Theta)\rvert$	$= \frac{1}{2}(3p^2 - p)$

Then, the codimension of P_Θ in G is $\frac{1}{2}(p^2 - p)$. Therefore, if $H \le G$ is such that $\operatorname{codim}(H) = 2p - 1$, we would need

$$\begin{aligned} \operatorname{codim}(H) &\ge \operatorname{codim}(P_\Theta) \\ 2p - 1 &\ge \frac{1}{2}(p^2 - p) \\ 5p - 2 &\ge p^2 \end{aligned}$$

This can only happen for $p = 3$ or 4.

If $p > q$ we take as Θ

$$\Theta = \{f_1 - f_2, \cdots, f_{q-1} - f_q\}$$

Again $\mathfrak{a}$ is 1-dimensional, generated by the element with $a_1 = \cdots a_q = 1$. In addition, we have

$$\mathfrak{n}^+(\Theta) = \sum_{1\leq i<j\leq q} \mathfrak{g}^{f_i - f_j}$$

and

$$\mathfrak{n}_\Theta^+ = \sum_{1\leq i<j\leq q} \mathfrak{g}^{f_i + f_j} \oplus \sum_{1\leq i\leq q} \mathfrak{g}^{f_i}$$

Therefore, in this case we have

$\lvert\mathfrak{m}\rvert$	$= \frac{(p-q)(p-q-1)}{2}$
$\lvert\mathfrak{n}^+(\Theta)\rvert$	$= \frac{1}{2}(q^2 - q)$
$\lvert\mathfrak{n}_\Theta^+\rvert$	$= \frac{1}{2}(q^2 - q) + pq - q^2$
$\lvert\mathfrak{p}_\Theta\rvert = \lvert\mathfrak{m}\rvert + \lvert\mathfrak{a}\rvert + \lvert\mathfrak{n}_\Theta^+\rvert + \lvert\mathfrak{n}^+(\Theta)\rvert + \lvert\mathfrak{n}^-(\Theta)\rvert$	$= \frac{1}{2}\Big(p^2 + 2q^2 - p\Big)$

Hence, the codimencion of P_Θ is $\dfrac{1}{2}\Big(-q^2 + 2pq - q\Big)$. It is not difficult to see that for $p > q \geq 4$, this codimension is greater that $p + q - 1$ and so, an $H' \leq P_\Theta \leq G$ such that $\dim\Big(G/H'\Big) = p + q - 1$ cannot exist. For $(p, q) = (4, 3)$, we get that $\mathrm{codim}(P_\Theta) = 6$ $(= p + q - 1)$.

For $(p, q) = (4, 4)$, the transitive $\mathrm{SO}(4) \times \mathrm{SO}(4)$ cases are the ones with the following $\mathscr{O}^p$'s and $\mathscr{O}^q$'s:

$$\text{(i) } \mathscr{O}^p \simeq \mathrm{SO}(4)\Big/\mathrm{SU}_2 \quad \text{and} \quad \mathscr{O}^q \simeq \mathrm{SO}(4)\Big/\left(\mathrm{SO}(2) \times \mathrm{SO}(2)\right)$$

$$\text{(ii) } \mathscr{O}^p \simeq \mathrm{SO}(4)\Big/\mathrm{SO}(3) \quad \text{and} \quad \mathscr{O}^q \simeq \mathrm{SO}(4)\Big/\left(\mathrm{SO}(2) \times \mathrm{SO}(2)\right)$$

$$\text{(iii) } \mathscr{O}^p \simeq \mathrm{SO}(4)\Big/\mathrm{SO}(2) \quad \text{and} \quad \mathscr{O}^q \simeq \mathrm{SO}(4)\Big/\mathrm{U}_2$$

For $(p,q)=(4,3)$, the cases are:

$$\begin{array}{lll} \text{(iv) } \mathscr{O}^p \simeq \mathrm{SO}(4) \big/ (\mathrm{SO}(2)\times\mathrm{SO}(2)) & \text{and} & \mathscr{O}^q \simeq \mathrm{SO}(3)\big/\mathrm{SO}(2) \\ \text{(v) } \mathscr{O}^p \simeq \mathrm{SO}(4)\big/\mathrm{SU}_3 & \text{and} & \mathscr{O}^q \simeq \mathrm{SO}(4)\big/\mathrm{SO}(3) \end{array}$$

For $(p,q)=(3,3)$, the only case is:

$$\mathscr{O}^p \simeq \mathrm{SO}(3)\big/\mathrm{SO}(2) \quad \text{and} \quad \mathscr{O}^q \simeq \mathrm{SO}(3)$$

In the following, we assume that the action of $\mathrm{SO}(p)\times\mathrm{SO}(q)$ on M extends to an action of $G=\mathrm{SO}^\circ(p,q)$ and that $H' \leq P_\Theta$; recall $H'=\mathrm{Stab}_G(x)$.

For $(p,q)=(4,4)$, recall that P_Θ is the parabolic subgroup of G that corresponds to the subset $\Theta=\{f_1-f_2,\cdots,f_3-f_4\}$. Changing the quadratic form on $\mathbb{R}^8$ from

$$-x_1^2-x_2^2-x_3^2-x_4^2+x_5^2+x_6^2+-x_7^2+x_8^2$$

to

$$2x_1x_8+2x_2x_7+2x_3x_6+2x_4x_5$$

and computing the Lie algrebra of P_Θ in this new representation, we can see that P_Θ leaves invariant the subspace spanned by the first 4 basis vectors, which is a maximal isotropic 4-dimensional subspace. This computation is rather straightforward, but also prolix and so it is omitted. Consequently, if $P_\Theta=M_\Theta A_\Theta N_\Theta$ is the Langlands decomposition, we have that

$$M_\Theta \simeq \mathrm{SL}_4(\mathbb{R})$$

If the action of K extends to G and $H' \leq P_\Theta$, by Witte's theorem, $N_\Theta \leq H'$.

- If $A_\Theta \leq H'$, then again by Witte's theorem, there exists a closed, connected and *normal* subgroup Y of M_Θ, such that

$$(H')^\circ = Y A_\Theta N_\Theta$$

Counting dimensions, Y needs to be of codimension 1 in M_Θ. But, $M_\Theta \simeq \mathrm{SL}_4(\mathbb{R})$ is simple, which gives a contradiction.

- If $A_\Theta \nleq H'$, then by counting dimensions,

$$(H')^\circ = M_\Theta^\circ N_\Theta$$

Now, we have

$$\mathrm{SU}_2 \times \mathrm{SO}(2) \times \mathrm{SO}(2) \leq (H')^\circ$$

Since $\mathrm{SU}_2 \times \mathrm{SO}(2) \times \mathrm{SO}(2)$ is compact, it is necessarily contained in a maximal compact subgroup of M_Θ°; recall that N_Θ is unipotent. Hence, a conjugate of $\mathrm{SU}_2 \times \mathrm{SO}(2) \times \mathrm{SO}(2)$ is a subgroup of $\mathrm{SO}(4)$. But, $\mathrm{SU}_2 \times \mathrm{SO}(2) \times \mathrm{SO}(2)$ has rank 3, while $\mathrm{SO}(4)$ has rank 2, which gives us a contradiction for case (i). A similar argument gives a contradiction for case (ii).

For $(p, q) = (4, 3)$, in the case of (iv) : $\mathrm{SU}_2 \times \{1\} \leq H'$. By a similar analysis like above, we can see that

$$M_\Theta^\circ \simeq \mathrm{SL}_3(\mathbb{R})$$

- If $A_\Theta \leq H'$, like above we get a contradiction by the simplicity of M_Θ°.
- If $A_\Theta \nleq H'$, then $\mathrm{SU}_2 \leq (H')^\circ = M_\Theta^\circ N_\Theta$ is contained in a maximal compact of $M_\Theta^\circ \simeq \mathrm{SL}_3(\mathbb{R})$. Hence, a conjugate of SU_2 is a subgroup of $\mathrm{SO}(3)$. However, both SU_2 and $\mathrm{SO}(3)$ are 3-dimensional and connected, hence this conjugation gives an isomorphism between them, which is a contradiction. A similar argument gives a contradiction for case (v).

For $(p, q) = (3, 3)$, we have $\dim(G) = 15$ and $\dim(H') = 10$. Like before, we can see that

$$M_\Theta \simeq \mathrm{SL}_3(\mathbb{R})$$

We write again $P_\Theta = M_\Theta A_\Theta N_\Theta$ for the Langlands decomposition. Recall, $N_\Theta \leq H'$. Witte's theorem gives a closed, connected and *normal* $Y \trianglelefteq M_\Theta^\circ$ inside $(H')^\circ$. Since $M_\Theta \simeq \mathrm{SL}_3(\mathbb{R})$ is simple, Y is either all of M_Θ° or trivial. In the former case, the dimension of H' is at least 11, while in the latter it is at most 4. Both contradict the fact that $\dim(H') = 10$.

3.2 Non-extendable non-transitive cases

Here, we treat cases where the action of K does not extend to G, but we do not have an immediate contradiction by counting dimensions.

Lemma 3.2.1. *Suppose* $\mathrm{SO}(4) \times \mathrm{SO}(4)$ *acts on a manifold* M *of dimension 7. Let* $x \in M$ *and denote* $\mathscr{O}^p$ *the orbit of* x *with respect to* $\mathrm{SO}(4) \times \{1\}$ *and* $\mathscr{O}^q$ *the orbit of* x *with respect to* $\{1\} \times \mathrm{SO}(4)$. *Suppose*

$$\mathscr{O}^p \simeq \mathrm{SO}(4)\big/\mathrm{U}_2 \quad \text{and} \quad \mathscr{O}^q \simeq \mathrm{SO}(4)\big/\mathrm{U}_2$$

Then, the action is not locally effective. In particular, it cannot extend to an action of $\mathrm{SO}^\circ(4,4)$.

Proof. Assume it is locally effective. We will get a contradiction by looking at the isotropy representation of $\mathrm{U}_2 \times \mathrm{U}_2$ at x. The subspace $\mathrm{T}_x\mathscr{O}^p \oplus \mathrm{T}_x\mathscr{O}^q$ is an invariant subspace of T_xM with respect to the isotropy representation. By introducing a $\mathrm{U}_2 \times \mathrm{U}_2$ invariant metric on M, we can take the orthogonal complement of the aforementioned subspace and hence obtain a decomposition

$$\mathrm{T}_xM = \mathrm{T}_x\mathscr{O}^p \oplus \mathrm{T}_x\mathscr{O}^q \oplus W$$

where $W \simeq \mathbb{R}^3$ is also invariant under the action of $\mathrm{U}_2 \times \mathrm{U}_2$. Now, since $\mathrm{U}_2 \times \mathrm{U}_2$ acts by isometries, the isotropy representation gives us a map from $\mathrm{U}_2 \times \mathrm{U}_2$ into block matrices of the form

$$\begin{bmatrix} A & & \\ & B & \\ & & C \end{bmatrix}$$

where $A, B \in \mathrm{O}_2$ and $C \in \mathrm{O}_3$. However, $\dim(\mathrm{U}_2 \times \mathrm{U}_2) = 8$, while $\dim(\mathrm{O}_2 \times \mathrm{O}_2 \times \mathrm{O}_3) = 5$. This means that the map above must have nontrivial kernel, i.e. there exists a subgroup H of $\mathrm{U}_2 \times \mathrm{U}_2$ with dimension at least 1, that acts trivially on T_xM. This contradicts the assumption that $\mathrm{SO}(4) \times \mathrm{SO}(4)$ acts locally effectively. $\square$

Lemma 3.2.2. *Suppose* $\mathrm{SO}(4) \times \mathrm{SO}(q)$, $q \geq 3$, *acts on a manifold* M *of dimension* $q+3$. *Let* $x \in M$ *and denote* $\mathscr{O}^p$ *the orbit of* x *with respect to* $\mathrm{SO}(4) \times \{1\}$ *and* $\mathscr{O}^q$ *the orbit of* x

with respect to $\{1\} \times \mathrm{SO}(q)$. *Suppose*

$$\mathscr{O}^p \simeq \mathrm{SO}(4) \big/ \mathrm{U}_2$$

and that $\mathrm{SO}(q)$ *fixes* x. *Then, the action is not locally effective. In particular, it cannot extend to an action of* $\mathrm{SO}^\circ(4, q)$.

Proof. Consider a $\mathrm{U}_2 \times \mathrm{SO}(q)$ invariant metric and decompose $\mathrm{T}_x M$ as

$$\mathrm{T}_x M = \mathrm{T}_x \mathscr{O}^p \oplus W$$

where $W \simeq \mathbb{R}^{q+1}$ is invariant under the action of $\mathrm{U}_2 \times \mathrm{SO}(q)$; here, $\mathrm{T}_x \mathscr{O}^q$ is trivial. Now, $\mathrm{SO}(q)$ acts on W. If that action is trivial, then we get that $\mathrm{SO}(q)$ acts trivially on $\mathrm{T}_x M$ and this contradicts the local effectiveness of the action. Assume momentarily that $q \neq 4$. Then, the action of $\mathrm{SO}(q)$ on W decomposed as

$$W \simeq \mathbb{R}^q \oplus \mathbb{R}^1$$

with $\mathbb{R}^q$ being the standard representation of $\mathrm{SO}(q)$ and $\mathbb{R}^1$ being the trivial one. Since U_2 preserves W and sends $\mathrm{SO}(q)$-invariant subspaces to $\mathrm{SO}(q)$-invariant subspaces because U_2 and $\mathrm{SO}(q)$ commute, U_2 preserves $\mathbb{R}^q$ and $\mathbb{R}^1$. Therefore, we get a map from $\mathrm{U}_2 \times \mathrm{SO}(q)$ to block matrices of the form

$$\begin{bmatrix} A & & \\ & B & \\ & & 1 \end{bmatrix}$$

where $A \in \mathrm{O}_2$ and $B \in \mathrm{O}_q$. But $\dim(\mathrm{U}_2 \times \mathrm{SO}(q)) = 4 + \frac{1}{2}q(q-1)$, while $\dim(\mathrm{O}_2 \times \mathrm{O}_q) = 1 + \frac{1}{2}q(q-1)$. Therefore we get a subgroup of $\mathrm{U}_2 \times \mathrm{SO}(q)$ that acts trivially on $\mathrm{T}_x M$ and that contradicts the local effectiveness of our action. If $q \neq 4$, then the action of $\mathrm{SO}(4)$ on $\mathbb{R}^5$ could decompose differently, but that would only result in smaller blocks in the place of "B" above. Counting dimensions, we would arrive at a similar contradiction. $\square$

Lemma 3.2.3. *Suppose* $\mathrm{SO}(4) \times \mathrm{SO}(q)$, $q \geq 3$, *acts on a manifold* M *of dimension* $q+3$. *Let* $x \in M$ *and denote* $\mathscr{O}^p$ *the orbit of* x *with respect to* $\mathrm{SO}(4) \times \{I_q\}$ *and* $\mathscr{O}^q$ *the orbit of* x *with respect to* $\{I_4\} \times \mathrm{SO}(q)$. *Suppose*

$$\mathscr{O}^p \simeq \mathrm{SO}(4) \Big/ \mathrm{U}_2 \quad \text{and} \quad \dim(M) - \dim(\mathscr{O}^q) - \dim(\mathscr{O}^p) \leq 2$$

Then, the action is not locally effective. In particular, it cannot extend to an action of $\mathrm{SO}^\circ(4,q)$.

Proof. Indeed, consider $\mathrm{SU}_2 \leq \mathrm{U}_2$. Then, SU_2 fixes x. Looking at the isotropy representation, SU_2 acts trivially on $\mathrm{T}_x\mathscr{O}^p$ since it is 2-dimensional, see [8]. It also acts trivially on $\mathrm{T}_x\mathscr{O}^q$. Finally, introducing a U_2 invariant metric on M and taking the orthogonal complement of $\mathrm{T}_x\mathscr{O}^p \oplus \mathrm{T}_x\mathscr{O}^q$, which will also be SU_2-invariant, SU_2 acts trivially on it also by our assumption on the dimensions. Therefore, SU_2 acts trivially on $\mathrm{T}_x M$ and so the action is not locally effective. $\square$

Lemma 3.2.4. *Suppose* $\mathrm{SO}(9) \times \mathrm{SO}(7)$ *acts on a manifold* M *of dimension* 15. *Let* $x \in M$ *and denote* $\mathscr{O}^p$ *the orbit of* x *with respect to* $\mathrm{SO}(9) \times \{I_7\}$ *and* $\mathscr{O}^q$ *the orbit of* x *with respect* *to* $\{I_9\} \times \mathrm{SO}(7)$*. Suppose*

$$\mathscr{O}^p \simeq \mathrm{SO}(9)\Big/\mathrm{Spin}(7)$$

Then, the action is not locally effective. In particular, it does not extend to an action of $\mathrm{SO}^\circ(9,7)$*.*

Proof. Indeed, we have that $\dim\Big(\mathrm{SO}(9)\Big/\mathrm{Spin}(7)\Big) = 15$. Let $H'_7 \leq \mathrm{SO}(7)$ be the isotropy group of x, for the restricted action of $\{I_9\} \times \mathrm{SO}(7)$. Then, H'_7 acts trivially on $\mathrm{T}_x\mathscr{O}^p$ which is all of T_xM. If $\dim(H'_7) \geq 1$ then the action is not locally effective. If $\dim(H'_7) = 0$, then $\dim(\mathscr{O}^q) = \dim(\mathrm{SO}(7)) = 21 > 15 = \dim(M)$, a contradiction. $\square$

Lemma 3.2.5. *Suppose that* $\mathrm{SO}(p) \times \mathrm{SO}(q)$ *acts on* M*. Let* $x \in M$ *denote* $\mathscr{O}^p$ *the orbit of* x *with respect to* $\mathrm{SO}(p) \times \{I_q\}$ *and* $\mathscr{O}^q$ *the orbit of* x *with respect to* $\{I_p\} \times \mathrm{SO}(q)$*. If*

$$\dim\mathscr{O}^p = p - 1 \quad \textit{and} \quad \dim\mathscr{O}^q = q - 1$$

Let

$$\mathscr{O}^p \simeq \mathrm{SO}(p)\Big/H'_p \quad \textit{and} \quad \mathscr{O}^q \simeq \mathrm{SO}(q)\Big/H'_q$$

and suppose the isotropy representations of H'_p *and* H'_q *are irreducible, and* $-I_p \in H'_p$*, where* I_p *is the* $p \times p$ *identity matrix. Assume that the action extends to* $\mathrm{SO}^\circ(p,q)$*. Then, the* $\mathrm{SO}^\circ(p,q)$*-orbit of* x*, denoted* $\mathscr{O}$*, cannot be an open orbit.*

Proof. We can assume that M is endowed with an $\mathrm{SO}(p)\times\mathrm{SO}(q)$-invariant metric. Looking at the isotropy representation of $H'_p \times H'_q$, there exists an invariant line in T_xM because of the assumptions on the dimensions of the orbits, and $H'_p \times H'_q$ must act trivially on this line.

The Lie algebra $\mathfrak{so}(p,q)$ comprises matrices of the following form

$$\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}$$

where $X_1 \in \mathfrak{so}_p$, $X_4 \in \mathfrak{so}_q$, $X_3 = X_2^{\mathrm{T}}$ and X_2 is a $p \times q$ matrix. We identify $H'_p \times H'_q$ with $\left\{ \begin{bmatrix} h_1 & \\ & h_2 \end{bmatrix} : h_1 \in H'_p,\ h_2 \in H'_q \right\}$.

Then, if $h_0 = (h_1, h_2) \in H'_p \times H'_q$ and $X = \begin{bmatrix} X_1 & X_2 \\ X_2^{\mathrm{T}} & X_4 \end{bmatrix} \in \mathfrak{so}(p,q)$,

$$\mathrm{Ad}_{h_0}(X) = \begin{bmatrix} h_1 X_1 h_1^{\mathrm{T}} & h_1 X_2 h_2^{\mathrm{T}} \\ h_2 X_2^{\mathrm{T}} h_1^{\mathrm{T}} & h_2 X_4 h_2^{\mathrm{T}} \end{bmatrix}$$

Now, assume that $\mathscr{O}$ is an open orbit. Then, $\mathscr{O} \simeq \mathrm{SO}^\circ(p,q) \big/ G_x$, where G_x is the isotropy group of x with respect to the $\mathrm{SO}^\circ(p,q)$-action. If $\mathfrak{g}_x$ is the isotropy algebra at x, then $\mathrm{T}_x M$ can be identified with $\mathfrak{so}(p,q) \big/ \mathfrak{g}_x$ and the isotropy representation of G_x with the adjoint representation mod $\mathfrak{g}_x$ (see [6, Corollary 10.2.13]). Let $v \in \mathrm{T}_x M$ be a nonzero vector on which $H'_p \times H'_q$ acts trivially. There exists an $X^v \in \mathfrak{so}(p,q)$ such that $X^v \bmod \mathfrak{g}_x \equiv v$ under the identification mentioned above. Write

$$X^v = \begin{bmatrix} X_1^v & X_2^v \\ (X_2^v)^{\mathrm{T}} & X_4^v \end{bmatrix}$$

Since the isotropy representations of H'_p and H'_q are irreducible, we can assume that $X_1^v =$

$X_4^v = 0$. Since v is non zero, $X_2^v \neq 0$ and

$$\mathrm{Ad}_{h_0}(X^v) \equiv X^v \bmod \mathfrak{g}_x$$

for any $h_o \in H'_p \times H'_q$. But, $h_0 = (-I_p, I_q) \in H'_p \times H'_q$ and $\mathrm{Ad}_{h_0}(X^v) = -X^v$ which implies $-X^v \equiv X^v \bmod \mathfrak{g}_x$. But this is a contradiction, since $X^v \notin \mathfrak{g}_x$. $\square$

Now, suppose $K = \mathrm{SO}(p) \times \mathrm{SO}(q)$ acts on a manifold M, $x \in M$ and either $p = 8$ and $H'_p \simeq \mathrm{Spin}(7)$ or $p = 4$ and $H'_p \simeq \mathrm{SU}_2$, where H'_p is like in the lemma above. Furthermore, assume that the $\mathrm{SO}(q)$-orbit of x is any of the first three in the bottom part of Table 3.1 and that the K action extends to a G action. If $\mathrm{SO}(q)$ fixes x, note that the isotropy representation of $\mathrm{SO}(q)$ on $\mathrm{T}_x\mathscr{O}^q$ has to be the standard irreducible one; so, we can always find a point near x in the same G-orbit that is fixed by neither $\mathrm{SO}(p)$ nor $\mathrm{SO}(q)$. Then, $\mathscr{O}^p$ and $\mathscr{O}^q$ for this new point are some from Table 3.1. So we can assume that x is not fixed by $\mathrm{SO}(q)$. Since the respective $-I_p$ is in both SU_2 and $\mathrm{Spin}(7)$, see [19], by the lemma above the G-orbit of x cannot be open in M. Therefore, it is closed and hence compact. If G_x is the isotropy group of x with respect to the G action, G_x is cocompact and hence we can apply Witte's result and by similar reasoning as in Section 3.1.2 we can show that this is impossible. Therefore, such K actions do not extend to G.

3.3 Conclusion

The following proposition now follows from the results of this chapter:

Proposition 3.3.1. *Let $p, q \geq 3$ and suppose $\mathrm{SO}^\circ(p, q)$ acts analytically on a manifold*

M. Then, the restricted K action on M is not transitive and with respect to the restricted action of $\mathrm{SO}(p) \times \{I\} \leq K$*, respectively* $\{I\} \times \mathrm{SO}(q) \leq K$*, the orbit type of a point $x \in M$ is either* $\mathrm{SO}(p) \big/ \mathrm{SO}(p-1)$ *or x is fixed by* $\mathrm{SO}(p)$*, respectively* $\mathrm{SO}(q) \big/ \mathrm{SO}(q-1)$ *or fixed by* $\mathrm{SO}(q)$.

In particular, we have

Corollary 3.3.1. *Let $p, q \geq 3$ and suppose* $\mathrm{SO}^{\circ}(p,q)$ *acts analytically on M. The fixed point set of* $H = \mathrm{SO}(p-1) \times \mathrm{SO}(q-1)$ *is non-empty and 1-dimensional.*

This follows from the above Proposition and Lemma 2.2.1. Of course, all the components of the aforementioned fixed point set are isomorphic to S^1.

Chapter 4: Classification of analytic $\mathrm{SO}^\circ(p,q)$, $p,q \geq 3$ actions

In this chapter, we assume $G = \mathrm{SO}^\circ(p,q)$ acts analytically and not trivially on a manifold M of dimension $\dim(M) = p+q-1$. As we will see, the presence or absence of fixed points for the subgroups $\mathrm{SO}(p)$ and $\mathrm{SO}(q)$ plays an important role in studying these actions and has consequences on the topology and geometry of M. The following theorem is an improved version of Theorem 1.0.1 in the Introduction.

Theorem 4.0.1. *Suppose* $\mathrm{SO}^\circ(p,q)$, $p,q \geq 3$, *acts analytically on a closed, connected manifold* M *of dimension* $p+q-1$. *Consider* $\mathrm{SO}(p) \simeq \mathrm{SO}(p) \times \{1\} \leq \mathrm{SO}(p) \times \mathrm{SO}(q) \leq \mathrm{SO}^\circ(p,q)$ *and* $\mathrm{SO}(q)$ *similarly.*

- *Suppose that there exists a point in* M *that is fixed by* $\mathrm{SO}(p)$ *and that there exist no points fixed by* $\mathrm{SO}(q)$. *Then* M *is equivariantly covered by* $\mathrm{S}^p \times \mathrm{S}^{q-1}$, *where the action of* $\mathrm{SO}^\circ(p,q)$ *on* $\mathrm{S}^p \times \mathrm{S}^{q-1}$ *is one from the basic construction, see Section 2.1.*

- *Suppose that there exists a point in* M *that is fixed by* $\mathrm{SO}(q)$ *and that there exist no points fixed by* $\mathrm{SO}(p)$. *Then* M *is equivariantly covered by* $\mathrm{S}^{p-1} \times \mathrm{S}^q$, *where the action of* $\mathrm{SO}^\circ(p,q)$ *on* $\mathrm{S}^{p-1} \times \mathrm{S}^q$ *is one from the basic construction, see Section 2.1.*

- *Suppose both* $\mathrm{SO}(p)$ *and* $\mathrm{SO}(q)$ *have fixed points in* M. *Then,* M *is equivariantly covered by* S^{p+q-1}, *where the action of* $\mathrm{SO}^\circ(p,q)$ *on* S^{p+q-1} *is one from the Uchida* *construction corresponding to a basic* $(\mathrm{J}_1, \mathrm{J}_2)$*-flow, see Section 2.3.*

- *Suppose that neither* $\mathrm{SO}(p)$ *nor* $\mathrm{SO}(q)$ *have a fixed point. Then, the action is equivariantly covered by* $G \times_P S$ *with the standard left* G *action, where* $P \leq G$ *is a maximal parabolic subgroup isomorphic to the stabiliser of an isotropic line in the standard representation of* $\mathrm{SO}^\circ(p,q)$ *on* $\mathbb{R}^{p+q}$*. If* $P = M_P A_P N_P$ *is the Langlands decomposition of* P*,* P *acts on* S^1 *by a flow via* A_P*, see (4.1).*

The proof of the first two cases is in Section 4.2, the third case is shown in Section 4.3, while the last case is shown in Section 4.1.

Lemma 4.0.1. *G does not have a fixed point.*

Indeed, we actually show something stronger:

Lemma 4.0.2. *$K = SO(p) \times SO(q)$ does not have a fixed point.*

Proof. Assume it does, and let $x \in M$ be a fixed point of K. Note that, since $\mathrm{SO}^\circ(p,q)$ is simple, its action is locally effective, and so the K action is also locally effective. Suppose $p \geq q$ and that $p \neq 4$. Consider the isotropy representation of $\mathrm{SO}(p)$ at x. If $\mathrm{SO}(p)$ acted trivially on $\mathrm{T}_x M \simeq \mathbb{R}^{p+q-1}$, then it would act trivially on a neighbourhood of x in M and so the action would not be locally effective. Therefore, we have a non-trivial representation of $\mathrm{SO}(p)$ on $\mathbb{R}^{p+q-1}$. The only possible decomposition into irreducibles is:

$$\mathbb{R}^{p+q-1} = \mathbb{R}^p \oplus \mathbb{R}^{q-1}$$

where the representation on the first summand is the standard one and on the second it is trivial. Since $\mathrm{SO}(p)$ and $\mathrm{SO}(q)$ commute, the two summands are also invariant by

SO(q) and hence by K. Then, if we introduce a K-invariant metric on M, the isotropy representation of K at x gives a homomorphism

$$\rho : K \to \mathrm{SO}(p) \times \mathrm{SO}(q-1)$$

Counting dimensions, we see that the kernel of ρ is not 0-dimensional, which contradicts the local effectiveness of the action. If $p = 4$ and the decomposition of $\mathbb{R}^{4+q-1}$ into irreducibles is not like above, then it would have to be

$$\mathbb{R}^3 \oplus \mathbb{R}^3 \oplus \mathbb{R} \quad \text{or} \quad \mathbb{R}^3 \oplus \mathbb{R}^3$$

depending on whether $q = 4$ or 3 respectively. In these cases,

$$\rho : \mathrm{SO}(4) \times \mathrm{SO}(4) \to \mathrm{SO}(3) \times \mathrm{SO}(3)$$

or

$$\rho : \mathrm{SO}(4) \times \mathrm{SO}(3) \to \mathrm{SO}(3) \times \mathrm{SO}(3)$$

respectively. Counting dimensions and considering the kernel of ρ, we arrive at a contradiction like before. □

Recall $H = \mathrm{SO}(p-1) \times \mathrm{SO}(q-1)$ and

$$\mathscr{F} = \text{fixed point set of } H$$

Now, by Corollary 3.3.1, $\mathscr{F} \neq \emptyset$ and $\mathscr{F} = \bigcup_i S_i$, where the S_i's are the connected components of $\mathscr{F}$ and with each S_i diffeomorphic to S^1. Let Σ be one of these components. By Lemma 4.0.2 and Lemma 2.2.1, we can define an analytic function $f : \Sigma \to \mathbb{R}\mathrm{P}^1$, see Remark 4. We also get an analytic flow Φ_θ or equivalently, a vector field X^Φ on Σ by the action of $\mathscr{M}(p, q)$.

Lemma 4.0.3. *If there exists a point $\zeta \in \Sigma$ such that $f(\zeta) = [a : b] \neq [\pm 1, 1]$, then there exists a point $\zeta' \in \Sigma$ which is a fixed point for $SO(p)$ or $SO(q)$.*

Proof. Set $f(z) = [a : b] \neq [\pm 1 : 1]$. Now, Φ_θ and f satisfy property (iii) from the Uchida conditions in Remark 7, see [16]. Namely,

$$f\left(\Phi_\theta(\zeta)\right) = [a \cosh(\theta) + b \sinh(\theta) : a \sinh(\theta) + b \cosh(\theta)]$$

Letting $\theta \to +\infty$, it is immediate that there exists a point $\zeta' \in \Sigma$ such that $f(\zeta') = [0 : 1]$ or $[1 : 0]$. Then, ζ' is a fixed point for $\mathrm{SO}(p)$ or $\mathrm{SO}(q)$ respectively, by the definition of f. $\square$

Remark 8. If we assume that neither $\mathrm{SO}(p)$ nor $\mathrm{SO}(q)$ has a fixed point in M and in particular in Σ, then $f(\zeta) = [\pm 1 : 1]$ for all $\zeta \in \Sigma$. Then, since f is analytic, it is constantly equal to $[1 : 1]$ or $[-1 : 1]$ on all of Σ. We note that in that case, the orbit of a point in Σ at which X^Φ is not zero, is analytically isomorphic to a component of a nullcone in $\mathbb{R}^{p+q}$ equipped with a scalar form of signature (p, q). We call these, *nullcone orbits*. Therefore, the existence of a nullcone orbit forces all the orbits to either be nullcone orbits or diffeomorphic to $\mathrm{S}^{p-1} \times \mathrm{S}^{q-1}$ as the next section shows and implies that there are no fixed points for $\mathrm{SO}(p)$ or $\mathrm{SO}(q)$.

4.1 $\mathrm{SO}^\circ(p,q)$ actions with neither $\mathrm{SO}(p)$ nor $\mathrm{SO}(q)$ fixed points

Let $\mathscr{F} := \mathrm{Fix}(\mathrm{SO}(p-1)\times\mathrm{SO}(q-1))$. The connected components of $\mathscr{F}$ are diffeomorphic to circles. Let Σ be one component of $\mathscr{F}$. On Σ, by Lemma 2.2.1, we can define an analytic function $f : \Sigma \to \mathbb{R}\mathrm{P}^1$, see Remark 4. Suppose neither $\mathrm{SO}(p)$ nor $\mathrm{SO}(q)$ has a fixed point in M. Then, by Remark 8, $f \equiv [1:1]$ or $[-1:1]$. Without loss of generality, we assume

$$f \equiv [1:1]$$

Then, for every $\zeta \in \Sigma$, the isotropy group of ζ is contained in or equal to a maximal parabolic subgroup P of G of the first kind, see Section 3.1.1.

Let S^1 be the usual unit circle in $\mathbb{R}^2$. Identify Σ and S^1 by an analytic isomorphism

$$\psi : \mathrm{S}^1 \to \Sigma$$

Recall that $\mathscr{M}(p,q)$ is the subgroup of G consisting of the matrices

$$m(\theta) := \begin{bmatrix} \cosh(\theta) & & \sinh(\theta) & \\ & I_{p-1} & & \\ \sinh(\theta) & & \cosh(\theta) & \\ & & & I_{q-1} \end{bmatrix}$$

for $\theta \in \mathbb{R}$. Now, Σ is invariant under the action of $\mathscr{M}(p,q)$, and on Σ we have a flow

$\Phi^{\Sigma}_{\theta}(\zeta) = m(\theta) \star \zeta$, where $\star$ denotes the action of G. Define a flow on S^1 by

$$\Phi_\theta(z) := \psi^{-1}\Big(m(\theta) \star \psi(z)\Big)$$

Let $\pi : P \to (\mathbb{R}, +)$ be the following homomorphism: Let $P = M_P A_P N_P$ be the Langlands decomposition of P. Here, $A_P = \mathscr{M}(p, q)$ and $N_P = U(z)$, for any $z \in \Sigma$ see (2.5). Define π so that it picks out the "A_P-part". Namely, if $p \in P$, we write $p = p_{M_P} \cdot p_{A_P} \cdot p_{N_P}$ according to the Langlands decomposition above. Then, $p_{A_P} = m(\theta)$ for some $m(\theta) \in \mathscr{M}(p, q)$ and we define

$$\pi(p) := \theta$$

It is easy to see that π is an analytic homomorphism between P and $(\mathbb{R}, +)$; recall that for a parabolic subgroup, $N_P \trianglelefteq P$, and M_P and A_P commute.

Now, we define the following action of P on S^1:

$$\begin{aligned} P \times \mathrm{S}^1 &\to \mathrm{S}^1 \\ (p, z) &\mapsto \Phi_{\pi(p)}(z) \end{aligned} \tag{4.1}$$

and consequently we get an action of G on the associated bundle $G \times_P \mathrm{S}^1$ where G acts by left multiplication on the first factor. It is easy to see that this action is well defined. Note also, that for $p \in P$ and $\zeta \in \Sigma$,

$$p \star \zeta = m(\pi(p)) \star \zeta \tag{4.2}$$

Remark 9. Suppose we are given two different actions of P on S^1 via flows Φ^1 and Φ^2 like above. If the vector fields that these flows generate are isomorphic, then it is immediate that the two actions of G on $G \times_P \mathrm{S}^1$ are also isomorphic. Since the analytic vector fields on the circle are classified in [7], see Appendix A, such analytic G actions on $G \times_P \mathrm{S}^1$ are also classified via the corresponding flows Φ.

Now, let Ψ be the analytic map

$$\begin{aligned} \Psi : G \times_P \mathrm{S}^1 &\to M \\ [g, z] \quad &\mapsto g \star \psi(z) \end{aligned}$$

where $[g, z]$ denotes the equivalence class of (g, z) in $G \times_P \mathrm{S}^1$. In the following, let $\zeta = \psi(z)$ for $z \in \Sigma$.

- Ψ is well defined:

A different representative of the class $[g, z]$ is of the form $[pg^{-1}, p \star z]$. We have, $\Psi([gp^{-1}, p \star z]) = (gp^{-1}) \star \psi(p \star z)$. But, $p \cdot z = \Phi_{\pi(p)}(z) = \psi^{-1}(m(\pi(p)) \star \zeta) = \psi^{-1}(p \star \zeta)$, and so $\psi(p \star z) = p \star \zeta$

- Ψ is G-equivariant:

We have $G = KP$; this can be seen from (2.5). Let $[\tilde{g}, z] \in G \times_P \Sigma$ and $g \in G$, and write

$g\tilde{g} = kp$, according to the above decomposition. We have

$$\begin{aligned}\Psi(g[\tilde{g}, z]) &= \Psi([g\tilde{g}, z]) = \Psi([kp, z]) \\ &= \Psi([k, p \star z]) = \Psi([k, \Phi_{\pi(p)}(z)]) \\ &= k \star \psi(\Phi_{\pi(p)}(z)) = k \star (m(\pi(p)) \star \zeta) \quad \text{(by the definition of } \Phi \text{ and (4.2))} \\ &= k \star (p \star \zeta) = g\tilde{g} \star \zeta \\ &= g \star \Psi([\tilde{g}, z])\end{aligned}$$

- Ψ is a local analytic isomorphism:

By the equivariancy, it suffices to look at a point of the form $[e, z]$ for some $z \in \mathrm{S}^1$. By using $(e, z) \in G \times \mathrm{S}^1$ as a representative for $[e, z]$, we have an identification

$$\mathrm{T}_{[e,z]}(G \times_P \mathrm{S}^1) \simeq \mathfrak{g}/\mathfrak{p} \oplus \mathrm{T}_z\mathrm{S}^1$$

where $\mathfrak{g}$ and $\mathfrak{p}$ are the Lie algebras of G and P respectively. Now, for an element $0 + v$, for $v \in \mathrm{T}_z\mathrm{S}^1$, we can find a curve of the form

$$\gamma : t \mapsto [e, z_t]$$

such that $\gamma(0) = [e, z]$ and $\gamma'(0) = v$. Then, since $\Psi\Big|_{\{e\}\times_P \mathrm{S}^1} = \psi$, we have that $\mathrm{D}\Psi(v) = \mathrm{D}\psi(v) \in \mathrm{T}_{\psi(z)}\Sigma$ and $\mathrm{D}\psi(v) \neq 0$ since ψ is an isomorphism.

On the other hand, for an element $w + 0$, for $w \in \mathfrak{g}/\mathfrak{p}$, we can find a representative $\tilde{w} = \tilde{w}_1 + \tilde{w}_2 \in \mathfrak{g}$ of w such that $\tilde{w}_1 \in \mathfrak{so}_p$ and $\tilde{w}_2 \in \mathfrak{so}_q$. Consider the curves of $G \times_P \mathrm{S}^1$,

$c : t \mapsto [e^{t\tilde{w}_i}, z]$, for $i = 1, 2$. Then, $c(0) = [e, z]$ and $c'(0) = w_i$, where w_i is the class of $\tilde{w}_i$ in $\mathfrak{g}/\mathfrak{p}$. and consequently

$$\mathrm{D}\Psi(w_i) = \frac{d}{dt}\bigg|_0 (e^{t\tilde{w}_i} \star \zeta)$$

But, $e^{t\tilde{w}_i} \star \zeta \in \mathscr{O}_{\mathrm{SO}(p)}(\zeta)$ or $\mathscr{O}_{\mathrm{SO}(q)}(\zeta)$ where $\mathscr{O}_{\mathrm{SO}(p)}(\zeta)$ and $\mathscr{O}_{\mathrm{SO}(q)}(\zeta)$ are the $\mathrm{SO}(p)$ and $\mathrm{SO}(p)$ orbits of ζ respectively. Those orbits are $(p-1)$-dimensional and $(q-1)$-dimensional respectively by Proposition 3.3.1. Moreover, they intersect $\mathscr{F}$ transversely. By considering all possible $w + 0 \in \mathfrak{g}/\mathfrak{p} \oplus \mathrm{T}_z\mathrm{S}^1$, we see that $p + q - 1 = \dim\ (\ \mathrm{D}\Psi(\mathfrak{g}/\mathfrak{p})\ )$. As a result, because $\mathscr{O}_{\mathrm{SO}(p)}(\zeta)$, $\mathscr{O}_{\mathrm{SO}(p)}(\zeta)$ and $\mathscr{F}$ intersect transversely, $(\mathrm{D}\Psi)_{[e,z]}$ is an isomorphism, and hence Ψ is a local analytic isomorphism.

- Ψ is a covering map:

Since Ψ is a local analytic isomorphism, the image of Ψ is an open set. Additionally, Ψ's domain is compact, and so the image is also closed. As a result, since M is a closed manifold, Ψ is onto M. Being a local diffeomorphism between compact sets, Ψ is a covering map. Hence, the fourth case of Theorem 4.0.1 is proved.

4.2 $\mathrm{SO}^\circ(p, q)$ actions with only $\mathrm{SO}(p)$ fixed points or only $\mathrm{SO}(q)$ fixed points

Suppose G acts analytically on a manifold M of dimension $\dim(M) = p + q - 1$. In this section, we assume that $\mathrm{SO}(p)$ has a fixed point, but $\mathrm{SO}(q)$ does not. We are going to show that M is equivariantly, analytically covered by $\mathrm{S}^p \times \mathrm{S}^{q-1}$ with an action from the basic construction. Recall that $H = \mathrm{SO}(p-1) \times \mathrm{SO}(q-1)$ and

$$\mathscr{F} = \text{fixed point set of } H$$

Then, $\mathscr{F} \neq \emptyset$ and $\mathscr{F} = \bigcup_i S_i$ with each S_i one dimensional, see the discussion in the beginning of this chapter. Let $\Sigma := \mathrm{S}_{i_0}$ be a connected component of $\mathscr{F}$. On Σ we get an analytic flow, Φ^Σ, defined by the action of the one parameter subgroup $\mathscr{M}(p,q)$. Furthermore, by Lemma 2.2.1 and Remark 4, we get an analytic function $f'_\Sigma : \Sigma \to \mathbb{R}\mathrm{P}^1$. The existence of an $\mathrm{SO}(p)$ fixed point, implies that S_{i_0} contains a point fixed by $\mathrm{SO}(p)$. Indeed, otherwise $f'_\Sigma \equiv [1:1]$ or $[-1:1]$ and then, by Section 4.1, M is covered by $G \times_P \mathrm{S}^1$ and that contradicts the existence of an $\mathrm{SO}(p)$ fixed point. We note also that since $j_1 = \begin{bmatrix} -I_2 & \\ & \\ & I_{p+q-2} \end{bmatrix} \in \mathrm{SO}(p)$ is an involution from Σ to itself with at least one fixed point, it has exactly two fixed points. Moreover, since we also assume that $\mathrm{SO}(q)$ does not have a fixed point, therefore $f'_\Sigma \neq [1:0]$. Let $\mathscr{U}_1 = \{[a:b] \in \mathbb{R}\mathrm{P}^1 : b \neq 0\} \subseteq \mathbb{R}\mathrm{P}^1$ and let $\chi_1 : \mathscr{U}_1 \to \mathbb{R}$ be the function defined by

$$[a:b] \mapsto \frac{a}{b}$$

Then, $f_\Sigma = \chi_1 \circ f'_\Sigma$. Note that since f'_Σ and Φ^Σ satisfy property (iii) from the Uchida conditions in Remark 7, see [16], we have that if $f'(z) = [a:b]$, for $z \in \Sigma$, then $|a| < |b|$. Indeed, otherwise Uchida condition (iii) implies that, taking $\theta \to +\infty$, we would get point $\tilde{z} \in \Sigma$ with $f'_\Sigma(\tilde{z}) = [1:0]$, namely $\tilde{z}$ would be a fixed point of $\mathrm{SO}(q)$, which is impossible.

Hence,

$$|f_\Sigma| \leq 1$$

We note that f_Σ and Φ^Σ satisfy relation (A1)-(A3) and (A4)$'$ of Remark 5.

Now, consider the space $\mathrm{S}^p \times \mathrm{S}^{q-1}$, where $\mathrm{S}^p \subseteq \mathbb{R}^{p+1}$ and $\mathrm{S}^{q-1} \subseteq \mathbb{R}^q$ and we denote the standard bases of $\mathbb{R}^{p+1}$ and $\mathbb{R}^q$ by $\{e_1, \cdots, e_{p+1}\}$ and $\{\epsilon_1, \cdots, \epsilon_q\}$ respectively. Let

$$\mathscr{S} = \Big\{(\alpha e_1 + \beta e_{p+1}, \epsilon_1) : \alpha^2 + \beta^2 = 1\Big\}$$

Recall from the beginning of Chapter 2 that we have a standard action of K on $\mathrm{S}^p \times \mathrm{S}^{q-1}$. The action of j_1 on $\mathscr{S}$ is

$$j_1(\alpha e_1 + \beta e_{p+1}, \epsilon_1) = (-\alpha e_1 + \beta e_{p+1}, \epsilon_1)$$

Lemma 4.2.1. *There exists an analytic isomorphism*

$$\psi : \Sigma \to \mathscr{S}$$

such that

$$\psi \circ j_1 = j_1 \circ \psi$$

Proof. We consider a j_1-invariant metric on Σ; hence, j_1 acts as an isometry. Let $x \in \Sigma$ be a fixed point of j_1. Since j_1 is an involution and x is an isolated fixed point, we have

$$(Dj_1)_x = -Id$$

We identify $\mathrm{T}_x\Sigma$ with $\mathbb{R}$ and consider

$$\exp_x : \mathbb{R} \to \Sigma$$

We can assume that $\exp_x$ is 2π-periodic. Since j_1 is an isometry, we have

$$\exp_x \left(Dj_1(t)\right) = j_1 \circ \exp_x(t)$$

for any $t \in \mathbb{R}$. Now, the map

$$\tilde{\psi} : \mathbb{R}\big/2\pi\mathbb{Z} \to \Sigma$$

defined by

$$t + 2\pi\mathbb{Z} \mapsto \exp_x(t)$$

is analytic, and if we identify its domain with $[-\pi, \pi]$, we have

$$\tilde{\psi} : [-\pi, \pi] \to \Sigma \qquad \text{such that}$$

$$\tilde{\psi}(-t) = j_1 \circ \tilde{\psi}(t) \tag{4.3}$$

Then, we identify $[-\pi, \pi]$ with S^1 by $t \mapsto e^{it}$, which is also analytic. Let $\rho_0 : \mathrm{S}^1 \to \mathscr{S}$ be the analytic isomorphism from (2.2). Finally, set $\psi = \rho_0 \circ \tilde{\psi}^{-1}$. Equation (4.3) above implies

$$\psi \circ j_1 = j_1 \circ \psi$$

$\square$

Using ψ we can define a function and a flow on $\mathscr{S}$ in the following way:

$$f : \mathscr{S} \to \mathbb{R} \quad \text{by} \quad f = f_\Sigma \circ \psi^{-1}$$

and

$$\Phi : \mathbb{R} \times \mathscr{S} \to \mathscr{S} \quad \text{by} \quad \Phi_\theta(z) = \psi^{-1} \circ \Phi_\theta^\Sigma \left(\psi(z)\right)$$

Then, Φ_θ and f are analytic on $\mathscr{S}$ and it is easy to see that they satisfy the relations (A1)-(A4) from Remark 5. Therefore, $f = f_\Phi$, see Remark 2, and Φ_θ is an induced basic j_1-flow on $\mathscr{S}$, see Lemma 2.2.2. From Φ_θ we get an analytic action of G on $\mathrm{S}^p \times \mathrm{S}^{q-1}$ by the basic construction.

We now define the map

$$F : \mathrm{S}^p \times \mathrm{S}^{q-1} \to M$$

in the following way. Let $x \in \mathrm{S}^p \times \mathrm{S}^{q-1}$. There exist $(\kappa_1, \kappa_2) \in K$ and $z \in \mathscr{S}$ such that $x = (\kappa_1, \kappa_2) \star z$. Then, define

$$F\left((\kappa_1, \kappa_2) \star z\right) := (\kappa_1, \kappa_2) \star \psi(z)$$

F is well defined and locally an analytic isomorphism.

Lemma 4.2.2. *F is well defined*

Proof. Let $z = (\alpha e_1 + \beta e_{p+1}, \epsilon_1)$ and $z' = (\alpha' e_1 + \beta' e_{p+1}, \epsilon_1)$ be elements of $\mathscr{S}$ and $k =$

$(\kappa_1, \kappa_2), k' = (\kappa_1', \kappa_2') \in K$. Suppose

$$k \star z = k' \star z'$$

Then, we have $\kappa_2^{-1}\kappa_2' \in \mathrm{SO}(q-1)$, $\beta = \beta'$ and $\alpha = \pm\alpha'$.

- If $\alpha = \alpha' \neq 0$, then $z = z'$ and $\kappa_1^{-1}\kappa_1' \in \mathrm{SO}(p-1)$. Hence, $k = k'h$, for some $h \in H$ and $k \star \psi(z) = k'h \star \psi(z) = k' \star \psi(z)$
- If $\alpha = -\alpha' \neq 0$, then $z = j_1(z')$ and $\kappa_1^{-1}\kappa_1' \in j_1\mathrm{SO}(p-1)$. Hence, $k = k'j_1h$ for some $h \in H$ and $k \star \psi(z) = k'j_1h \star \psi\left(j_1(z')\right) = k'j_1j_1 \star \psi(z') = k' \star \psi(z')$
- If $\alpha = \alpha' = 0$, then $z = z' = (\pm e_{p+1}, \epsilon_1)$ and $\psi(z) = \zeta$, where $\zeta \in \Sigma$ is fixed by $\mathrm{SO}(p)$, since $f_\Sigma(\zeta) = f(z) = [0 : 1]$. Moreover, $k = k'(g_1, g_2)$, with $(g_1, g_2) \in \mathrm{SO}(p) \times \mathrm{SO}(q-1)$. Then, $k \star \psi(z) = k'(g_1, g_2) \star \psi(z) = k'\psi(z)$

□

Lemma 4.2.3. *F is G-equivariant.*

Proof. Let $g \in G$ and $x = k_0 \star z \in \mathrm{S}^p \times \mathrm{S}^{q-1}$, where $k_0 = (\kappa_1, \kappa_2) \in K$ and $z \in \mathscr{S}$. Write

$$gk_0 = km(\theta)u$$

for $k \in K$, $\theta \in \mathbb{R}$ and $u \in U(z)$, according to (2.5). Note that $U(z) = U(\psi(z))$. Then

$$\begin{aligned}
F(g \star x)) &= F(k \star \Phi_\theta(z)) \\
&= k \star \psi(\Phi_\theta(z)) \\
&= k \star \Phi^\Sigma_\theta(\psi(z)) \\
&= km(\theta)u \star \psi(z) \\
&= gk_0 \star \psi(z) \\
&= g \star (k_0 \star \psi(z)) \\
&= g \star F(z)
\end{aligned}$$

□

Lemma 4.2.4. *F is a local analytic isomorphism.*

Proof. By the G-equivariance and the fact that $G \star \mathscr{S} = \mathrm{S}^p \times \mathrm{S}^{q-1}$, it suffices to show that F is a local analytic isomorphism around a $z \in \mathscr{S}$.

- If $f(z) \neq \pm 1$, then $f_\Sigma(F(z)) \neq 1$ and so, the G-orbits of both z and $F(z)$ are open. Then, G-equivariance of F implies that F is analytic at z and $(\mathrm{D}F)_z$ is an isomorphism and so we have the result.

- If $f(z) = \pm 1$, then the map

$$K \times \mathscr{S} \to \mathrm{S}^p \times \mathrm{S}^{q-1}$$

$$(k, z) \mapsto k \star z$$

is a submersion at $((I_p, I_q), z)$, and so F is analytic at z. Moreover, the dimension of the orbits of z and $\psi(z)$ are $p+q-2$. Now, G-equivariance gives us that $(DF)_z$ is onto the tangent space of $\psi(z)$. On the other hand, the orbit of $\psi(z)$, $\mathscr{O}(\psi(z))$, is locally isomorphic to G/P where P is a maximal parabolic for which G/P is diffeomorphic to $\mathrm{S}^{p-1} \times \mathrm{S}^{q-1}$. In this model, the fixed point set of H is 0-dimensional. Therefore, in M, $\mathscr{F}$ and $\mathscr{O}(\psi(z))$ intersect transversely. Hence, if $\gamma(t)$ is a non constant curve in $\mathscr{S}$ through z, we have

$$(DF)_z(\gamma'(0)) = (D\psi)_z(\gamma'(0)) \neq 0$$

since ψ is an isomorphism, and therefore

$$(DF)_z(\gamma'(0)) \in \mathrm{T}_{\psi(z)}M \Big\backslash \mathrm{T}_{\psi(z)}\mathscr{O}(\psi(z))$$

Hence, $(DF)_z$ is onto all of $\mathrm{T}_{\psi(z)}M$, hence it is an isomorphism and we have the desired result. □

Lemma 4.2.5. $M = F\left(\mathrm{S}^p \times \mathrm{S}^{q-1}\right)$

Proof. Since F is a local isomorphism, its image is open in M. Since $\mathrm{S}^p \times \mathrm{S}^{q-1}$ is compact, the image of F is also closed. The result follows, since M is connected. □

Finally, since F is a local isomorphism that is onto M, and both M and $\mathrm{S}^p \times \mathrm{S}^{q-1}$ are compact, it is a covering map. Hence, we have proved the first case of Theorem 4.0.1. Of course, changing the roles of $\mathrm{SO}(p)$ and $\mathrm{SO}(q)$, in an analogous manner we get the second case as well.

4.3 $\mathrm{SO}^\circ(p,q)$ actions with both $\mathrm{SO}(p)$ and $\mathrm{SO}(q)$ fixed points

Recall $H = \mathrm{SO}(p-1) \times \mathrm{SO}(q-1)$ and

$$\mathscr{F} = \text{fixed point set of } H$$

Assume that both $\mathrm{SO}(p)$ and $\mathrm{SO}(q)$ have fixed points in M. As it was noted in the beginning of Chapter 4, by Corollary 3.3.1, $\mathscr{F}$ is non empty and all its connected components are 1-dimensional. Let Σ be one of $\mathscr{F}$'s connected components. On Σ, by Lemma 2.2.1, there exists an analytic function

$$f_\Sigma : \Sigma \to \mathbb{R}\mathrm{P}^1$$

See Remark 4. If $f \equiv [\pm 1 : 1]$, then it was shown in Section 4.1 that M is covered by $G \times_P \mathrm{S}^1$, which contradicts the existence of $\mathrm{SO}(p)$ and $\mathrm{SO}(q)$ fixed points. Then, by Lemma 4.0.3, we may assume that there exists an $\mathrm{SO}(p)$ fixed point on Σ. Let $x \in M$ be a fixed point of $\mathrm{SO}(q)$. First of all, we note that x must be on Σ. Indeed, if it is not, the proof of the first case of Theorem 4.0.1, see Section 4.2, shows that M is covered by $\mathrm{S}^p \times \mathrm{S}^{q-1}$ on which the action of K is the standard one. Since x is fixed by $\mathrm{SO}(q)$, the fiber of x would be a discrete, $\mathrm{SO}(q)$-invariant subset of $\mathrm{S}^p \times \mathrm{S}^{q-1}$. However, evidently there are not any such sets. Being involutions on a circle with at least one fixed point, j_1 and j_2 have exactly two fixed points. Since they also commute, they either have the same fixed points on Σ or their fixed point sets on Σ are disjoint.

Consider S^1 and let J_1 and J_2 be the reflections with respect to the y-axis and the x-axis respectively.

<u>**Case 1:**</u> j_1 and j_2 have disjoint fixed point sets on Σ.

Lemma 4.3.1. *There exists an analytic isomorphism*

$$\psi : \Sigma \to S^1$$

such that

$$\psi \circ J_i = j_i \circ \psi$$

for $i = 1, 2$.

Proof. Introduce a j_1, j_2 invariant metric on Σ. The proof of Lemma 4.2.1 shows that we can identify Σ with S^1 in such a way that j_1 is the reflection J_1. Namely, there exists an analytic isomorphism

$$\sigma_1 : \Sigma \to \mathrm{S}^1$$

such that

$$\sigma_1 \circ j_1 = \mathrm{J}_1 \circ \sigma_1$$

Via σ_1, we can assume that j_2 also acts on S^1. Now, as in Lemma 4.2.1, we will identify S^1 with a quotient of the tangent space of q, where $q, \tilde{q} \in \mathrm{S}^1$ are the two fixed points of j_2. The fixed points of j_1 and j_2 alternate, therefore

$$-e_1 = \exp_q(\phi_0)$$

for a $\phi_0 \in (0, \pi)$ or a $\phi_0 \in (-\pi, 0)$. Say, $\phi_0 \in (0, \pi)$. But, $\exp_q((0, \pi))$ is the one of the arcs between q and $\tilde{q}$ and these two points are symmetric with respect to the y-axis, because the relation $j_1 \star \sigma_1^{-1}(q) = \sigma_1^{-1}(\tilde{q})$ implies $\mathrm{J}_1(q) = \tilde{q}$. Therefore, the distance between q and $-e_1$ is the same as the distance between $\tilde{q}$ and $-e_1$, which shows that

$$\phi_0 = \frac{\pi}{2}$$

Therefore, by the proof of Lemma 4.2.1 again, we can find an analytic isomorphism

$$\sigma_2 : \mathrm{S}^1 \to \mathrm{S}^1$$

such that $\sigma \circ j_2 = \mathrm{J}_2 \circ \sigma$. Hence the fixed points of j_2 are $(0, \pm 1)$. We need to check that $\sigma_2 \circ \mathrm{J}_1 = \mathrm{J}_1$. But, let $z = \exp_q(\phi_z)$, and say $0 < \phi_z < \frac{\pi}{2}$. Write $\phi_z = \frac{\pi}{2} - \epsilon$ for $\epsilon > 0$. Then, $\operatorname{dist}(z, -e_1) = \operatorname{dist}(J_1(z), -e_1)$ and $J_1(z) \in \exp_q((0, \pi))$. Therefore, $\mathrm{J}_1(z) = \exp_q(\frac{\pi}{2} + \epsilon)$. Hence, in the tangent space of q, the induced action of J_1 via the exponential map is reflection with respect to $\frac{\pi}{2}$ on $(0, \pi)$ and similarly, reflection with respect to $-\frac{\pi}{2}$ on $(-\pi, 0)$. Hence, $\sigma_2 \circ \mathrm{J}_1 = \mathrm{J}_1$. Then, it is easy to see that

$$\psi = \sigma_2 \circ \sigma_1$$

is the required isomorphism. □

Now, consider the sphere $\mathrm{S}^{p+q-1} \subseteq \mathbb{R}^{p+q}$ and denote the standard basis of $\mathbb{R}^{p+q}$ by $\{e_1, \cdots e_{p+q}\}$. Let

$$\mathscr{S} = \{\alpha e_1 + \beta e_{p+1} : \alpha^2 + \beta^2 = 1\} \subseteq \mathrm{S}^{p+q-1}$$

In the standard orthogonal action of K on S^{p+q-1}, $j_1 = \begin{bmatrix} -I_2 & \\ & I_{p+q-2} \end{bmatrix} \in \mathrm{SO}(p)$ and

$j_2 = \begin{bmatrix} I_{p+q-2} & \\ & -I_2 \end{bmatrix} \in \mathrm{SO}(q)$ act by

$$j_1\left(\alpha e_1 + \beta e_{p+1}\right) = -\alpha e_1 + \beta e_{p+1}$$

$$j_2\left(\alpha e_1 + \beta e_{p+1}\right) = \alpha e_1 - \beta e_{p+1}$$

Using Lemma 4.3.1, we get an analytic isomorphism $\psi' : \Sigma \to \mathrm{S}^1$, such that

$$\psi' \circ j_i = \mathrm{J}_i \circ \psi'$$

for $i = 1, 2$. Let $\rho : \mathrm{S}^1 \to \mathscr{S}$ be the analytic isomorphism defined by

$$\rho(\alpha e_1 + \beta e_2) = \alpha e_1 + \beta e_{p+1}$$

and let

$$\psi = \rho \circ \psi'$$

Then, $\rho : \Sigma \to \mathscr{S}$ and

$$\psi \circ j_i = j_i \circ \psi$$

Now, similarly to the case of Section 4.2, we get a flow and a function on $\mathscr{S}$, which satisfy the Uchida conditions, see Remark 7. Hence the flow is an induced basic (j_1, j_2)-flow.

By [16], we get an action of $\mathrm{SO}^\circ(p,q)$ on S^{p+q-1} from the Uchida construction corresponding to a basic $(\mathrm{J}_1,\mathrm{J}_2)$-flow, see Section 2.3. Then, essentially the same steps as in Section 4.2 show that this action covers equivariantly the action of $\mathrm{SO}^\circ(p,q)$ on M.

<u>Case 2:</u> j_1 and j_2 have the same fixed points on Σ.

First we note that $j_1\Big|_\Sigma = j_2\Big|_\Sigma$. Indeed, by considering a j_1, j_2 invariant metric on Σ, j_1 and j_2 have the same fixed points, which are isolated, and if x is one of those fixed points, $(\mathrm{D}j_1)_x = (\mathrm{D}j_2)_x = -\mathrm{Id}$. Therefore, they are equal on Σ. On Σ we also get an analytic flow Φ_θ on Σ induced by the action of the subgroup $\mathscr{M}(p,q)$. Because there are no nullcone orbits and because of Uchida condition (iii), see Remark 7, between two points fixed by Φ_θ there must be a point fixed by $\mathrm{SO}(p)$ or $\mathrm{SO}(q)$ and vice versa. Therefore, on Σ there are two points fixed by Φ_θ. Note that $\mathrm{SO}(p)$ or $\mathrm{SO}(q)$ cannot have a common fixed point, by Lemma 4.0.2.

A slight modification of the proof of Lemma 4.2.1 shows that there exists an analytic isomorphism $\psi : \Sigma \to \mathbb{R}\mathrm{P}^1$ such that if $j_0 : \mathbb{R}\mathrm{P}^1 \to \mathbb{R}\mathrm{P}^1$ is the map

$$[a : b] \mapsto [-a : b]$$

then

$$\psi \circ j_i = j_0 \circ \psi$$

for $i = 1, 2$. The fixed points of j_0 are $[1 : 0]$ and $[0 : 1]$. Moreover, via ψ, Φ_θ induces a flow, Φ'_θ, on $\mathbb{R}\mathrm{P}^1$ and let ζ_1, ζ_2 be the points fixed by Φ'_θ. Let

$$\pi : \mathrm{S}^1 \to \mathbb{R}\mathrm{P}^1$$

be the standard covering map. Let $z_1, z_3 \in \mathrm{S}^1$ be the points that map to ζ_1 and $z_2, z_4 \in \mathrm{S}^1$ the points that map to ζ_2. Note that because of the property

$$\Phi_\theta(j_1 \star z) = j_1 \star \Phi_{-\theta}(z)$$

for $\theta \in \mathbb{R}$ and $z \in \Sigma$, we have

$$\Phi'_\theta(j_0 \star \zeta) = j_0 \star \Phi'_{-\theta}(\zeta)$$

for $\theta \in \mathbb{R}$ and $\zeta \in \mathbb{R}\mathrm{P}^1$. Hence, if $\zeta_1 = [a : b]$ then $\zeta_2 = [-a : b]$. Therefore, we can assume that $z_2 = \mathrm{J}_2(z_1)$, $z_3 = \mathrm{J}_1 \circ \mathrm{J}_2(z_1)$, and $z_4 = \mathrm{J}_1(z_1)$. Note that $\mathrm{J}_1 \circ \mathrm{J}_2$ is the antipodal map on S^1.

We will now define a flow on S^1 that will cover the flow Φ'_θ. Let $\mathscr{S}_1$ be the connected component of $\mathrm{S}^1 \backslash \{z_1, z_3\}$ that contains z_2. Then

$$\pi\Big|_{\mathscr{S}_1} : \mathscr{S}_1 \to \mathbb{R}\mathrm{P}^1 \backslash \{\zeta_1\}$$

is an analytic isomorphism. Define Φ''_θ on $\mathscr{S}_1$ by

$$\Phi''_\theta(z) = \left(\pi\Big|_{\mathscr{S}_1}\right)^{-1} \circ \Phi'_\theta\left(\pi(z)\right)$$

Then, on $\mathrm{J}_1 \circ \mathrm{J}_2\left(\mathscr{S}_1\right)$, which is the connected component of $\mathrm{S}^1 \backslash \{z_1, z_3\}$ that contains z_4, define Φ''_θ by

$$\Phi''_\theta(z) = \left(\mathrm{J}_1 \circ \mathrm{J}_2\right) \circ \Phi''_\theta\left(\mathrm{J}_1 \circ \mathrm{J}_2(z)\right)$$

Note that, on $\{z_1, z_2, z_3, z_4\}$ Φ''_θ is constant. It is immediate that Φ''_θ is an analytic flow on S^1 such that

$$\pi \circ \Phi''_\theta(z) = \Phi'_\theta \circ \pi(z)$$

for $\theta \in \mathbb{R}$ and $z \in \mathrm{S}^1$. Now, define a function

$$f_{\mathrm{S}^1} := f_\Sigma \circ \pi$$

It is a straightforward calculation that Φ''_θ and f_{S^1} satisfy the Uchida conditions, see Remark 7.

As in Case 1 above, there exists an analytic action of $\mathrm{SO}^\circ(p,q)$ on S^{p+q-1} from the Uchida construction corresponding to a basic $(\mathrm{J}_1, \mathrm{J}_2)$-flow, see Section 2.3, that covers the action of $\mathrm{SO}^\circ(p,q)$ on M. Hence, we have proved the third case of Theorem 4.0.1.

Chapter 5: Classification of analytic $\mathrm{SO}^\circ(p,1)$, $p \geq 3$, actions

Here we consider the case $q = 1$, $p \geq 3$, namely an analytic action of $G = \mathrm{SO}^\circ(p,1)$ on a closed, connected manifold M of dimension p. We see that these actions are covered by actions of G on either S^p or $\mathrm{S}^{p-1} \times \mathrm{S}^1$. While the topology of these spaces is analogous to the corresponding spaces of the $p, q \geq 3$ case, the geometry can differ. The fixed point set of $H = \mathrm{SO}(p-1)$ is again going to be a union of circles. In $\mathrm{SO}(p)$ there exists the involution j_1 which forces the fixed points of $\mathrm{SO}(p)$ on one of these circles to be at most 2. However, there is no longer a j_2. That means that there could be more than two points whose isotropy group is the standard copy of $\mathrm{SO}^\circ(p-1,1)$, and so more than two disjoint orbits diffeomorphic to

$$\mathrm{SO}^\circ(p,1)\Big/\mathrm{SO}^\circ(p-1,1)$$

Another notable difference is that absence of K fixed points does not imply the existence of nullcone orbits. The actions of $K = \mathrm{SO}(p)$ on S^p and $\mathrm{S}^{p-1} \times \mathrm{S}^1$ are via rotations with respect to an axis in the former case and the standard orthogonal action on the first factor in the latter.

Furthermore, a result analogous to Lemma 2.2.1 holds for the case $q = 1$ as well. Lemma 5.0.1 below is implied in Uchida's work, but a proof is not provided. We define the following subspaces of $\mathfrak{so}(p,1)$

$$E = \text{Span}\left\{\begin{bmatrix} & & 1 \\ & \mathbf{0}_{p-1} & \\ 1 & & \end{bmatrix}\right\}$$

where $\mathbf{0}_{p-1}$ is the $(p-1)\times(p-1)$ zero matrix, and

$$\mathfrak{t}[a:b] := \left\{\begin{bmatrix} 0 & -bu^T & 0 \\ bu & 0 & au \\ 0 & au^T & 0 \end{bmatrix} : u \in \mathbb{R}^{p-1}\right\}$$

where $[a:b] \in \mathbb{R}\mathrm{P}^1$. Then, we have:

Lemma 5.0.1. *Suppose $p \geq 3$. Let $\mathfrak{a}$ be a proper subalgebra of $\mathfrak{so}(p,1)$ which contains $\mathfrak{h} \simeq \mathfrak{so}(p-1)$ and suppose dim $\mathfrak{so}(p,1)-$ dim $\mathfrak{a} \leq p$. Then, $\mathfrak{a} = \mathfrak{t}[a:b]$ for some $[a:b] \in \mathbb{R}\mathrm{P}^1$ or $\mathfrak{a} = E \oplus \mathfrak{t}(1:\pm 1)$.*

Proof. Consider the following subspaces of $\mathfrak{so}(p,1)$:

$$T_1 = \text{Span}\left\{\begin{bmatrix} 0 & -v^T & 0 \\ v & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : v \in \mathbb{R}^2\right\}$$

and

$$T_2 = \operatorname{Span}\left\{ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & w \\ 0 & w^T & 0 \end{bmatrix} : w \in \mathbb{R}^2 \right\}$$

Then, $\mathfrak{so}(p,1) = \mathfrak{so}(p-1) \oplus E \oplus T_1 \oplus T_2$. Moreover, if we consider the adjoint action of $\mathrm{SO}(p-1)$ on $\mathfrak{so}(p,1)$, all the above summands are invariant and irreducible. In addition:

- $[T_2 \oplus E, T_2 \oplus E] = \mathfrak{so}(p)$
- $[T_1, E] = T_2$
- $[T_1, T_1] = 0$

Now, $\mathfrak{a}$ is also $\mathrm{SO}(p-1)$ invariant and $\mathfrak{a} \cap \Big(T_1 \oplus T_2\Big) \neq \{0\}$ because of $\dim \mathfrak{so}(p,1) - \dim \mathfrak{a} \leq p$. Therefore $\mathfrak{a} \cap \Big(T_1 \oplus T_2\Big) = \mathfrak{t}[a:b]$. Suppose $\mathfrak{a} \cap E \neq \{0\}$ which implies $\mathfrak{a} \cap E = E$. In that case, $\mathfrak{a} \cap \Big(T_1 \oplus T_2\Big) = \mathfrak{t}[a:b]$, with $a, b \neq 0$. Indeed, if $b = 0$, then $T_2 \leq \mathfrak{a}$ and hence $\mathfrak{a} = \mathfrak{so}(p,1)$, since $[T_2 \oplus E, T_2 \oplus E] = \mathfrak{so}(p)$. On the other hand, if $a = 0$, then $T_1 \leq \mathfrak{a}$ hence, $[T_1, E] = T_2 \leq \mathfrak{a}$ and as before we conclude $\mathfrak{a} = \mathfrak{so}(p,1)$. Therefore, if $\mathfrak{a} \cap E = E$, then $\mathfrak{a} = \mathfrak{so}(p-1) \oplus E \oplus \mathfrak{t}[a:b]$, with $[a:b] \in \mathbb{R}P^1$. Assume that $[a:b] \neq [1:\pm 1]$. Then

$$\begin{bmatrix} 0 & -bu^T & 0 \\ bu & 0 & au \\ 0 & au^T & 0 \end{bmatrix} \cdot \begin{bmatrix} & & 1 \\ & \mathbf{0}_{p-1} & \\ 1 & & \end{bmatrix} - \begin{bmatrix} & & 1 \\ & \mathbf{0}_{p-1} & \\ 1 & & \end{bmatrix} \cdot \begin{bmatrix} 0 & -bu^T & 0 \\ bu & 0 & au \\ 0 & au^T & 0 \end{bmatrix} = \begin{bmatrix} 0 & -au^T & 0 \\ au & 0 & bu \\ 0 & bu^T & 0 \end{bmatrix} \in \mathfrak{a}$$

$$\xRightarrow{\times \frac{b}{a}} \begin{bmatrix} 0 & -bu^T & 0 \\ bu & 0 & \frac{b^2}{a}u \\ 0 & \frac{b^2}{a}u^T & 0 \end{bmatrix} \in \mathfrak{a}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{a^2-b^2}{a}u \\ 0 & \frac{a^2-b^2}{a}u^T & 0 \end{bmatrix} \in \mathfrak{a}$$

$$\Rightarrow T_2 \leq \mathfrak{a} \qquad \Big(\text{since } [a:b] \neq [1:\pm 1]\Big)$$

As before, we arrive at the contradiction that $\mathfrak{a} = \mathfrak{so}(p,1)$. □

Given a G action on a manifold M, Lemma 5.0.1 above gives us an analytic function $f : \mathscr{F} \to \mathbb{R}\mathrm{P}^1$ as in Remark 4.

5.1 Construction of $\mathrm{SO}^\circ(p,1)$ actions

5.1.1 $(p,1)$-basic construction (I)

Suppose that we have a basic J_1-flow, see Definition 2.0.1, on S^1 and identify S^1 with

$$\mathscr{S} = \left\{ \alpha e_1 + \beta e_{p+1} : \alpha^2 + \beta^2 = 1 \right\} \subseteq \mathrm{S}^p$$

Then, the basic J_1-flow on S^1 induces an induced basic j_1-flow on $\mathscr{S}$, see Definition 2.0.2. Note that, $\mathscr{S}$ is the fixed point set of H in the standard K action on S^p. Then, analogously to the basic construction in Section 2.1, see also [17], we can construct an analytic action of G on S^p. In that action, the restricted action of $\mathscr{M}(p,1)$ on $\mathscr{S}$ is the induced basic j_1-flow we started with. We will refer to actions of G on S^p arising this way as *actions from the* $(p,1)$*-basic construction* (I). By similar arguments as in Chapter 2 we get:

Theorem 5.1.1. (see also [17, Theorem]) *The analytic actions of* $\mathrm{SO}^\circ(p,1)$ *on* S^p *from the* $(p,1)$*-basic construction* (I) *are analytically isomorphic if and only if the corresponding basic* J_1*-flows are analytically isomorphic.*

5.1.2 $(p,1)$-basic construction (II)

Here we construct G actions on $\mathrm{S}^{p-1} \times \mathrm{S}^1$. However, we need to modify the definition of the flows used as the main data. This is due to the fact mentioned above, that there can now be more than two orbits with isotropy group $\mathrm{SO}^\circ(p-1,1)$.

Definition 5.1.1. Assume Φ_θ is a nontrivial analytic flow on S^1. We will say that Φ_θ is a simple flow if:

- The fixed points of Φ_θ alternate between attracting and repelling.
- The Jacobian of Φ_θ at a fixed point z_i is $J_\Phi(z_i) = \frac{2}{n_i}$, respectively $J_\Phi(z_i) = -\frac{2}{n_i}$, if z_i is an attracting, respectively repelling, fixed point, where $n_i \in \mathbb{N}$.

Note that in the case of a simple flow, the number of fixed points must be even, see [7]. Moreover, we can define a function f_Φ as in Chapter 2. Let $p_i \in \mathscr{S}$ such that p_i is between z_i and z_{i+1} for $1 \leq i \leq n-1$, and p_n is between z_n and z_1. Define a function f_Φ in the following way:

- $f_\Phi(p_i) := 0$, $1 \leq i \leq n$
- $f_\Phi(z) := \tanh(\theta)$, for $z = \Phi_\theta\left(p_i\right)$, $\theta \in \mathbb{R}$, in the Φ_θ-orbit of a p_i
- $f_\Phi(z_i) := 1$, respectively $f_\Phi(z_i) := -1$, if z_i is attracting, respectively repelling

The function f_Φ is analytic, see the proof of Lemma 2.2.2. The fixed point set of $H = \mathrm{SO}(p-1)$ in the standard action of K on $\mathrm{S}^{p-1} \times \mathrm{S}^1$ is

$$\mathscr{F} = \{\pm e_1\} \times \mathrm{S}^1$$

Let

$$\mathscr{S} = \{e_1\} \times \mathrm{S}^1$$

Then, identifying S^1 with $\mathscr{S}$, we can construct G actions on $\mathrm{S}^{p-1} \times \mathrm{S}^1$ in a way analogous to the basic construction in Section 2.1 using simple flows. These actions extend the standard K action and the restricted $\mathscr{M}(p,1)$ action on $\mathscr{S}$ is the flow induced by the simple flow we started with, via the identification of S^1 with $\mathscr{S}_1$.

We will refer to the actions on $\mathrm{S}^{p-1} \times \mathrm{S}^1$ obtained this way as actions *from the* $(p,1)$*-basic construction* (II). These actions can be classified via the vector fields that the corresponding simple flows Φ_θ induce S^1, similarly to Theorem 2.1.1. In particular, they can be classified by the Jacobian of Φ_θ at the fixed points and the global invariant μ, see the Appendix A as well as [7].

Theorem 5.1.2. *Let $p \geq 3$. Two $\mathrm{SO}^{\circ}(p,1)$-actions on $\mathrm{S}^{p-1} \times \mathrm{S}^1$ from the $(p,1)$-basic construction* (II) *are analytically isomorphic if and only if the corresponding simple flows are analytically isomorphic. Moreover, the simple flows on S^1 are classified up to analytic isomorphism by the number of fixed points, the Jacobian at the fixed points, and the global invariant μ.*

Remark 10. When proving the analyticity of the actions from the $(p,1)$-basic construction (II), a small modification is required in the existence part of equation (B.12) in Appendix B. As we mentioned above, there can be more than two points where $f_{\Phi} = 0$. Hence, using the notation of Appendix B, there are not just two points that play the role of (e_{p+1}, ϵ_1) and $(-e_{p+1}, \epsilon_1)$. However, for a $z \in S_+$ with $0 < f(z) < 1$, there exists a $z_0 \in S_+$ such that $f_{\Phi}(z_0) = 0$ and z is in the orbit of z_0. Then, the proof proceeds the same.

5.2 Classification

Now, suppose we have a G action on a manifold M of dimension p, let $x \in M$ be a point that is not fixed by $\mathrm{SO}(p)$. Looking at the restricted $K \simeq \mathrm{SO}(p)$ action, let $\mathscr{O}^p$ denote the $\mathrm{SO}(p)$-orbit of x. Then, $\dim \mathscr{O}^p \leq p$. Using Uchida's classification of subgroups of $\mathrm{SO}(p)$ of codimension at most p ,see [14], and a similar analysis as in Chapter 3, we can see that necessarily $\mathscr{O}^p \simeq \mathrm{SO}(p) \big/ \mathrm{SO}(p-1)$. Therefore, we can conclude that the fixed point set of H, $\mathscr{F}$, is nonempty and that its connected components are 1-dimensional.

Lemma 5.2.1. *G does not have any fixed points.*

Proof. Suppose $x \in M$ is a fixed point of G. Firstly, we show that x is an isolated fixed point. Indeed, first of all, x is also a fixed point of $\mathrm{SO}(p)$. Note that, since $G = \mathrm{SO}^{\circ}(p,1)$

is simple, its action is locally faithful, and so the SO(p) action is also locally faithful. Since SO(p) is compact, its action is linearisable at x, and by choosing an SO(p)-invariant metric for M, we get a representation

$$\rho_1 : \mathrm{SO}(p) \to \mathrm{O}(p)$$

This representation must be either trivial or irreducible. Because if there was a k-dimensional, $1 \leq k < p$, ρ_1-invariant subspace of $\mathbb{R}^p$, $1 \leq k < p$, then (changing basis if necessary) we'd get a homomorphism

$$\mathrm{SO}(p) \hookrightarrow \mathrm{O}(k) \times \mathrm{O}(p-k)$$

Since the action is locally faithful, Kerρ_1 must be 0-dimensional. Comparing dimension we get $\dfrac{p(p-1)}{2} \leq \dfrac{k(k-1)}{2} + \dfrac{(p-k)(p-k-1)}{2}$, which yields a contradiction. Now, if ρ_1 is trivial, that means the action of SO(p) is trivial, and consequently the action of G is trivial. So, ρ_1 is irreducible and x is an isolated fixed point of SO(p) and hence of G.

Consider the isotropy representation

$$\rho : \mathfrak{so}(p,1) \to \mathfrak{gl}_p(\mathbb{R})$$

Since $\mathfrak{so}(p,1)$ is simple, and the representation is not trivial, it has to be irreducible. Complexifying and using that $\mathfrak{so}(p,1) \cong \mathfrak{so}(p+1)(\mathbb{C})$, we get an irreducible representation

$$\rho_1^{\mathbb{C}} : \mathfrak{so}(p+1)(\mathbb{C}) \to \mathfrak{gl}_p(\mathbb{C})$$

See [12]. Using the Weyl character formula, [5, p.408-410], we can see that the dimension of an irreducible representation is $\binom{p+1}{k}$ if $p+1$ is odd, and $\binom{p+1}{k}$ or $\frac{1}{2}\binom{p+1}{k}$ if $p+1$ is even. Comparing p to these binomial coefficients, we get a contradiction. $\square$

In particular, G does not have a fixed point on $\mathscr{F}$ and hence by Lemma 5.0.1 we can define an analytic function

$$f : \mathscr{F} \to \mathbb{R}\mathrm{P}^1$$

and by the action of $\mathscr{M}(p,1)$ on $\mathscr{F}$ we get a flow

$$\Phi : \mathbb{R} \times \mathscr{F} \to \mathscr{F}$$

Then, as in Chapter 4 we can show the following:

Theorem 5.2.1. *Suppose* $\mathrm{SO}^\circ(p,1)$ *acts analytically on a closed connected manifold* M *of dimension* p*. Then, if* $\mathrm{SO}(p)$ *has a fixed point,* M *is equivariantly covered by* S^p*, where the* $\mathrm{SO}^\circ(p,1)$ *action on* S^p *is one from the* $(p,1)$*-basic construction* (I)*. If* $\mathrm{SO}(p)$ *does not have a fixed point, then*

- *If there exists a nullcone orbit, then* M *is equivariantly covered by* $\mathrm{SO}^\circ(p,1) \times_P \mathrm{S}^1$ *with the standard left* $\mathrm{SO}^\circ(p,1)$ *action, where where* $P \leq \mathrm{SO}^\circ(p,1)$ *is a maximal parabolic subgroup isomorphic to the stabiliser of an isotropic line in the standard representation of* $\mathrm{SO}^\circ(p,1)$ *on* $\mathbb{R}^{p+1}$*. If* $P = M_P A_P N_P$ *is a Langlands decomposition of* P*,* P *acts on* S^1 *by a flow via* A_P*, see* (4.1)*.*
- *If there does not exist a nullcone orbit, then* M *is equivariantly covered by* $\mathrm{S}^{p-1} \times \mathrm{S}^1$*, where the* $\mathrm{SO}^\circ(p,1)$ *action on* $\mathrm{S}^{p-1} \times \mathrm{S}^1$ *is one from* $(p,1)$*-basic construction* (II)*.*

Chapter 6: Classification of analytic $\mathrm{SO}^o(p, 2)$, $p \geq 3$, actions

Here, we consider the case $q = 2$ and $p \geq 3$, namely the analytic actions of $G = \mathrm{SO}(p, 2)$ on manifolds of dimension $p + 1 = p + q - 1$. Uchida in [17] studied these kind of actions on the sphere S^{p+1}. Modifying Uchida's methods, we get actions of G on $\mathrm{S}^p \times \mathrm{S}^1$ and $\mathrm{S}^{p-1} \times \mathrm{S}^2$ where $K = \mathrm{SO}(p) \times \mathrm{SO}(2)$ acts in a standard way, see below. Then we see that an action of G on a closed manifold of dimension $p+1$ is equivariantly covered by one of those above.

Consider the following actions of K on $\mathrm{S}^p \times \mathrm{S}^1$ and $\mathrm{S}^{p-1} \times \mathrm{S}^2$. For $\mathrm{S}^p \times \mathrm{S}^1$, we embed $\mathrm{SO}(p)$ in $\mathrm{SO}(p + 1)$ as top left block matrices, similar to (2.1) in Chapter 2. Then, via $\mathrm{SO}(p+1)$, $\mathrm{SO}(p)$ acts on S^p. The action of $\mathrm{SO}(2)$ on S^1 is the standard one and that gives the action of K on $\mathrm{S}^p \times \mathrm{S}^1$. Similarly, we get the action of K on $\mathrm{S}^{p-1} \times \mathrm{S}^2$. We will refer to these actions as *standard* actions of K on these spaces.

The main difference with the $q \geq 3$ case is that the fixed point set of $H = \mathrm{SO}(p-1) \times \mathrm{SO}(1) \simeq \mathrm{SO}(p - 1)$ is no longer 1-dimensional, but 2-dimensional. Abusing the notation we will just write $H = \mathrm{SO}(p-1)$. We will need some more terminology. Following [17], let

N be the subgroup of G with Lie algebra

$$\mathfrak{n} = \left\{ \begin{bmatrix} 0 & & \alpha & \beta \\ & O_{p-1} & & \\ \alpha & & 0 & \gamma \\ \beta & & -\gamma & 0 \end{bmatrix} : \alpha, \beta, \gamma \in \mathbb{R} \right\} \leq \mathfrak{so}(p, 2)$$

where O_{p-1} is the zero square matrix of dim $p-1$ and any omitted value is assumed to be equal to 0. Note that, $\mathrm{SO}(2) \leq N$, where by $\mathrm{SO}(2)$ we mean the subgroup $\{I_p\} \times \mathrm{SO}(2)$. Moreover, let

$$N^+ = N \bigcup j_1 N$$

Note that $j_1 j_2$ commutes with N.

Let $\mathscr{F}$ be the fixed point set of H and $f : \mathscr{F} \to \mathbb{R}\mathrm{P}^2$. For $Y \in \mathscr{F}$ with $f(Y) = [a : b : c]$, let

$$U^\circ(Y) = H^\circ_{[a:b:c]} \tag{6.1}$$

where $H_{[a:b:c]}$ is the stabiliser of the point $ae_1 + be_{p+1} + ce_{p+2}$ in the standard representation of G on $\mathbb{R}^{p+2}$. The following equality also holds

$$G = KNU^\circ(Y)$$

See [17]. Moreover, the subgroup

$$\mathscr{M}(p,2) = \left\{ \begin{bmatrix} \cosh(\theta) & & \sinh(\theta) & \\ & I_{p-1} & & \\ \sinh(\theta) & & \cosh(\theta) & \\ & & & 1 \end{bmatrix} \right\}$$

is in N and we have the following equality

$$N = \mathrm{SO}(2)\,\mathscr{M}(p,2)\,(U^{\circ}(Y) \cap N) \tag{6.2}$$

for any $Y \in \mathscr{F}$, as is established in [17]. Note that (6.2) is equivalent to

$$N = \mathrm{SO}(2)\,\mathscr{M}(p,2)\,(H^{\circ}_{[a:b:c]} \cap N) \tag{6.3}$$

Note that

$$N \simeq \mathrm{SO}^{\circ}(1,2)$$

the obvious isomorphism $\varphi : N \to \mathrm{SO}^{\circ}(1,2)$ being

$$\begin{bmatrix} x_1 & & x_2 & x_3 \\ & O_{p-1} & & \\ x_4 & & x_5 & x_6 \\ x_7 & & x_8 & x_9 \end{bmatrix} \mapsto \begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{bmatrix}$$

It is well known that $\mathrm{SO}^{\circ}(1,2)$ is isomorphic to $\mathrm{PSL}_2(\mathbb{R})$ and hence, N is also isomorphic to $\mathrm{PSL}_2(\mathbb{R})$. Let $\mathscr{M}(1,2)$ be the subgroup of $\mathrm{SO}^\circ(1,2)$

$$\mathscr{M}(1,2) = \left\{ \begin{bmatrix} \cosh(\theta) & \sinh(\theta) & \\ \sinh(\theta) & \cosh(\theta) & \\ & & 1 \end{bmatrix} : \theta \in \mathbb{R} \right\}. \tag{6.4}$$

Then, φ gives an isomorphism between $\mathscr{M}(p,2)$ and $\mathscr{M}(1,2)$. We will denote an element of either group by $m(\theta)$. It will be clear from the context every time what group it belongs to.

Moreover, via φ, we get an equation analogous to (6.3) for $\mathrm{SO}^\circ(1,2)$. Let $\widetilde{H}_{[a:b:c]}$ be the stabiliser of the point $ae_1 + be_2 + ce_3$ in the standard representation of $\mathrm{SO}^\circ(1,2)$ on $\mathbb{R}^3$. Then, we have

$$\mathrm{SO}^\circ(1,2) = \mathrm{SO}(2)\,\mathscr{M}(1,2)\,\widetilde{H}^\circ_{[a:b:c]} \tag{6.5}$$

Now, let

$$j_1' = \begin{bmatrix} -1 & & \\ & 0 & \\ & & 0 \end{bmatrix}$$

and consider the group

$$\mathrm{SO}^\circ(1,2)^+ = \mathrm{SO}^\circ(1,2) \cup j_1'\,\mathrm{SO}^\circ(1,2) \tag{6.6}$$

Defining $\varphi(j_1) = j_1'$, it is immediate that φ extends to an isomorphism between N^+ and

$\mathrm{SO}^{\circ}(1,2)^{+}$

$$\varphi : N^{+} \xrightarrow{\simeq} \mathrm{SO}^{\circ}(1,2)^{+} \tag{6.7}$$

In the case of $\mathrm{SO}^{\circ}(p,2)$ actions there exists a lemma similar to Lemma 2.2.1. We denote by $\mathfrak{h}_{[a:b:c]}$ the Lie algebra of the group $H_{[a:b:c]}$ defined earlier.

Lemma 6.0.1. *[17, Lemma 1.2] Suppose $p \geq 3$ and let $\mathfrak{a}$ be a proper Lie subalgebra of $\mathfrak{so}(p,2)$ which contains $\mathfrak{h} \simeq \mathfrak{so}(p-1)$. If*

$$\dim \mathfrak{so}(p,2) - \dim \mathfrak{a} \leq p+1$$

then

$$\mathfrak{a} = \mathfrak{h}_{[a:b:c]}$$

for some $(a,b,c) \neq (0,0,0)$, or

$$\mathfrak{a} = \mathfrak{h}_{[a:b:c]} \oplus R^{1}$$

for some $(a,b,c) \neq (0,0,0)$ such that $a^2 = b^2 + c^2$, where the space R^1 is generated by the matrix $b\left(E_{1,p+1} + E_{p+1,1}\right) + c\left(E_{1,p+2} + E_{p+2,1}\right)$.

Remark 11. Similarly to Remark 4, Lemma 6.0.1 above allows us to define an analytic function $f : \mathscr{F} \to \mathbb{R}\mathrm{P}^2$, where $\mathscr{F}$ is the fixed point set of H, by the property that for $Y \in \mathscr{F}$, $f(Y) \in \mathbb{R}\mathrm{P}^2$ is the unique point such that $\mathfrak{h}_{f(Y)} \leq \mathfrak{g}_Y$, where $\mathfrak{g}_Y$ is the isotropy algebra of Y with respect to the G action.

6.1 Extracting the main data

In this section, we begin with a G action on $\mathrm{S}^p \times \mathrm{S}^1$, $\mathrm{S}^{p-1} \times \mathrm{S}^2$, or S^{p+1} extending the standard K action in each case. We will show that to such G actions correspond an $\mathrm{SO}^\circ(1,2)^+$ action on $\mathbb{T}^2$ in the first case, an $\mathrm{SO}^\circ(1,2)$ action on S^2 in the second case, and an $\mathrm{SO}^\circ(1,2)^+$ action on S^2 in the third case. In Section 6.2 we will show that these actions are all we need in order to construct and classify the G actions that extend the standard K action in each case, see in particular Theorems 6.2.1, 6.2.2, and 6.2.3.

Remark 12 (See [17])**.** Let M be S^{p+1}, $\mathrm{S}^p \times \mathrm{S}^1$, or $\mathrm{S}^{p-1} \times \mathrm{S}^2$. Suppose G acts analytically on M in a way that extends the standard K action in each case. Consider the fixed point set, $\mathscr{F}$, of H. Let ϕ be the restricted action of N^+ on $\mathscr{F}$ and $f : \mathscr{F} \to \mathbb{RP}^2$ as in Remark 11. Then, the pair (ϕ, f) satisfies:

(B1) ϕ is an analytic action of N^+ on $\mathscr{F}$, such that the restricted action of $N^+ \bigcap K$ is the restriction of the standard action of K.

(B2) f is an analytic function from $\mathscr{F} \to \mathbb{RP}^2$, which is N^+-equivariant and satisfies: $N^+ \bigcap U^\circ(Y) \subset N^+_Y$

Moreover, let $\mathscr{R} = \Big\{[a : b : c] \in \mathbb{RP}^2 : c = 0\Big\} \subseteq \mathbb{RP}^2$ and $\rho : \mathscr{R} \to \mathbb{RP}^1$ the obvious isomorphism. Consider $\mathscr{S} = f^{-1}(\mathscr{R})$. Then, $\mathscr{S}$ is a one dimensional submanifold of $\mathscr{F}$ which is j_1 and j_2 invariant and it is transverse to each SO(2)-orbit, see [17]. Let Φ_θ be the flow that $\mathscr{M}(p, 2)$ induces on $\mathscr{S}$ and $f_0 = \rho \circ f\Big|_{\mathscr{S}}$. Then, Φ_θ is an analytic flow on $\mathscr{S}$, $f_0 : \mathscr{S} \to \mathbb{RP}^1$ is analytic, and they satisfy

(i) $j_i \star \Phi_\theta(z) = \Phi_{-\theta}(j_i \star z) \qquad (i = 1, 2)$

(ii) $f_0(z) = [a : b : 0] \Rightarrow f_0(j_i \star z) = [a : -b : 0] \qquad (i = 1, 2)$

(iii) $f_0(z) = [a : b : 0] \Rightarrow f_0 \left(\Phi_\theta(z)\right) = [a\cosh(\theta) + b\sinh(\theta) : a\sinh(\theta) + b\cosh(\theta) : 0]$

(iv) $f_0(z) = [0 : 1 : 0] \Leftrightarrow j_1 \star z = z$ and $f_0(z) = [1 : 0 : 0] \Leftrightarrow j_2 \star z = z$

These are analogous to the Uchida conditions from Chapter 2, see Remark 7.

6.1.1 Case 1: G actions on $\mathrm{S}^p \times \mathrm{S}^1$

Here we will see that a G action on $\mathrm{S}^p \times \mathrm{S}^1$ extending the standard K action correspond to a special type of $\mathrm{SO}^\circ(1,2)^+$, see Definition 6.6, actions on $\mathbb{T}^2$ defined below.

Definition 6.1.1. Assume that ϕ is an $\mathrm{SO}^\circ(1,2)^+$ action on $\mathbb{T}^2 = \mathrm{S}^1 \times \mathrm{S}^1$. We will say that ϕ is a $\mathbb{T}^2$-basic action if

- j_1' acts as the reflection with respect to the y-axis on the first factor and trivially on the second factor of $\mathbb{T}^2$.
- $\mathrm{SO}(2)$ acts trivially on the first factor and in the usual way on the second factor.
- ϕ has two closed circle orbits none of which are the circles fixed by j_1'.

Suppose that G acts analytically on $\mathrm{S}^p \times \mathrm{S}^1$ and the restricted K action is the standard one. Here, the fixed point set, $\mathscr{F}$, of H is

$$\mathscr{F} = \left\{\alpha e_1 + \beta e_{p+1} : \alpha^2 + \beta^2 = 1\right\} \times \mathrm{S}^1 \simeq \mathbb{T}^2$$

Lemma 6.0.1 implies the existence of a function $f : \mathscr{F} \to \mathbb{R}\mathrm{P}^2$. Denote the restricted action of the subgroup N^+ on $\mathscr{F}$ by ϕ_1. Then, the pair (ϕ_1, f) satisfies properties (B1)-(B2).

Recall that N^+ is isomorphic to $\mathrm{SO}^\circ(1,2)^+$, see the beginning of this chapter. Therefore, identifying $\mathscr{F}$ with $\mathbb{T}^2$ in the obvious way and using the isomorphism between N^+ and $\mathrm{SO}^\circ(1,2)^+$, we get an action ϕ of $\mathrm{SO}^\circ(1,2)^+$ on $\mathbb{T}^2$. Note that $N^+ \cap K = \mathrm{SO}(2) \cup j_1\mathrm{SO}(2)$, hence ϕ is a $\mathbb{T}^2$-basic action. It is immediate that starting with two analytically isomorphic G actions will result in analytically isomorphic $\mathbb{T}^2$-basic action of $\mathrm{SO}^\circ(1,2)^+$ on $\mathbb{T}^2$.

Remark 13. Schneider, in [13], classified $\mathrm{SL}_2(\mathbb{R})$ actions on $\mathbb{T}^2$. Fix a Borel subgroup Σ of $\mathrm{SL}_2(\mathbb{R})$ and an infinitesimal generator, say α, of the diagonal subgroup. Then, the $\mathrm{SL}_2(\mathbb{R})$ actions come down to a 1-dimensional, closed submanifold W of $\mathbb{T}^2$ and an action of Σ on W. Furthermore,

$$\mathbb{T}^2 \simeq \mathrm{SL}_2(\mathbb{R}) \times_\Sigma W$$

and the action is the left $\mathrm{SL}_2(\mathbb{R})$ action on $\mathrm{SL}_2(\mathbb{R}) \times_\Sigma W$ (See [13, Theorem 1]). Two such actions of $\mathrm{SL}_2(\mathbb{R})$ are isomorphic, if the vector fields on W generated by $\alpha \in \mathfrak{sl}_2(\mathbb{R})$ are isomorphic.

Here, $\mathrm{SL}_2(\mathbb{R}) \times_\Sigma W$ is the quotient space of $\mathrm{SL}_2(\mathbb{R}) \times W$ modulo the equivalence relation

$$(\mu, w) \sim (\mu \cdot \sigma, \sigma^{-1} \star w)$$

where $\mu \in \mathrm{SL}_2(\mathbb{R}), \sigma \in \Sigma$ and $w \in W$.

Since $\mathrm{SO}^\circ(1,2)$ is isomorphic to $\mathrm{PSL}_2(\mathbb{R})$, by Schneider's result the analytic actions of $\mathrm{SO}^\circ(1,2)$ on $\mathbb{T}^2$ are also classified. It is also immediate that two analytic $\mathrm{SO}^\circ(1,2)^+$ actions on $\mathbb{T}^2$ are isomorphic if and only if the restricted $\mathrm{SO}^\circ(1,2)$ actions are isomorphic via an isomorphism that commutes with j_1'.

6.1.2 Case 2: G actions on $\mathrm{S}^{p-1} \times \mathrm{S}^2$

Suppose we have a G action on $\mathrm{S}^{p-1} \times \mathrm{S}^2$ extending the standard K action. We will see that such an action corresponds to a special type of $\mathrm{SO}^\circ(1,2)$ actions on S^2.

Definition 6.1.2. Assume ϕ is an analytic $\mathrm{SO}^\circ(1,2)$ action on S^2. We will say that ϕ is an S^2-basic action if:

- the action of $\mathrm{SO}(2)$ on S^2 is rotation with respect to the z-axis.
- ϕ has one closed circle orbit and no fixed points.

For a G action on $\mathrm{S}^{p-1} \times \mathrm{S}^2$ like above, the fixed point set, $\mathscr{F}$, of H is

$$\mathscr{F} = \{\pm e_1\} \times \mathrm{S}^2$$

Let

$$\mathscr{F}_1 = \{e_1\} \times \mathrm{S}^2$$

and $\mathscr{F}_2$ be the other connected component of $\mathscr{F}$. We note that the action of N on $\mathscr{F}_2$ is determined by the action of N on $\mathscr{F}_1$ by the equation

$$n \star (j_1 \star v) = j_1 \left(j_2 n j_2\right) \star v$$

for $n \in N$, $v \in \mathscr{F}_1$. By Lemma 6.0.1, there exists a function $f : \mathscr{F} \to \mathbb{R}\mathrm{P}^2$ and if we denote the restricted action of the subgroup N^+ on $\mathscr{F}$ by ϕ_1, then the pair (ϕ_1, f) satisfy properties (B1)-(B2). Let $\mathscr{S} = f^{-1}(\mathscr{R})$, where $\mathscr{R} = \Big\{[a : b : c] \in \mathbb{R}\mathrm{P}^2 : c = 0\Big\} \subseteq \mathbb{R}\mathrm{P}^2$. As in Remark 12, it can be shown that $\mathscr{S}$ is a closed, one dimensional submanifold of $\mathscr{F}$, which is transverse to the SO(2) orbits. By the last condition, the points $(e_1, \pm\epsilon_1)$ are in $\mathscr{S}$, since they are fixed by SO(2). Therefore there exists a connected component, $\mathscr{S}_1$, of $\mathscr{S}$ such that $\mathscr{S}_1 \subseteq \mathscr{F}_1$. Hence, $\mathscr{S}_1$ is a one dimensional closed submanifold of $\mathscr{F}_1$, transverse to the SO(2) orbits. Now, $f \neq [0 : 1 : 0]$ since there are no SO(p) fixed points in the kind of G actions considered in this section. Hence, the flow, Φ_θ that the action of $\mathscr{M}(p, 2)$ induces on $\mathscr{S}$ has two fixed points, otherwise j_2 would have more than two fixed points, which is impossible for an involution on a one dimensional closed submanifold. Since the points $(e_1, \pm\epsilon_1)$ are in $\mathscr{S}$, $\mathscr{S}$ is the union of the Φ_θ orbits of these points, and the fixed points. Note that j_2 must map the fixed points of Φ_θ to each other, since $j_2 \star \Phi_\theta(z) = \Phi_{-\theta}(j_2 \star z)$ for $z \in \mathscr{S}_1$. Hence, on $\mathscr{F}_1$, $\mathrm{SO}^\circ(1, 2)$ has one closed orbit, namely the orbit of the fixed points of Φ_θ on $\mathscr{S}_1$.

Now, let ϕ_2 be the restricted N action on $\mathscr{F}_1$ and identify $\mathscr{F}_1$ with S^2 in the obvious way. Recall that N is isomorphic to $\mathrm{SO}^\circ(1, 2)$ and that $N \cap K = \mathrm{SO}(2)$. Then, it is immediate that we get an action ϕ of $\mathrm{SO}^\circ(1, 2)$ on S^2 which is an S^2-basic action.

Remark 14. Recall that $\mathrm{SO}^\circ(1, 2)$ is isomorphic to $\mathrm{PSL}_2(\mathbb{R})$ and hence covered by $\mathrm{SL}_2(\mathbb{R})$. The analytic $\mathrm{SL}_2(\mathbb{R})$ actions on S^2 are classified in [13], by the number of closed orbits and the *normal invariant* at each closed orbit, see [13, Theorem 6]. Therefore, also the $\mathrm{SO}^\circ(1, 2)$ on S^2 are classified. The normal invariant would be equal to the Jacobian of Φ_θ at a fixed point on $\mathscr{S}_1$, in the setting above.

6.1.3 Case 3: G actions on S^{p+1}

Finally, we consider G actions on S^{p+1} extending the standard K action, namely the actions studied in [17]. Here, the fixed point set of H is

$$\mathscr{F} = \{\alpha\, e_1 + \beta\, e_{p+1} + \gamma\, e_{p+1} \,:\, \alpha^2 + \beta^2 + \gamma^2 = 1\} \simeq \mathrm{S}^2$$

Recall $\mathrm{SO}^{\circ}(1,2)^{+}$ from Definition 6.6.

Definition 6.1.3. Let ϕ be an analytic $\mathrm{SO}^{\circ}(1,2)^{+}$ action on S^2. We will say that ϕ is an $(\mathrm{S}^2)^{+}$-basic action if:

- j_1' acts on S^2 as the reflection with respect to the xy-plane.
- $\mathrm{SO}(2)$ acts on S^2 as rotations with respect to the z-axis.
- ϕ has two closed circle orbit and no fixed points.

Suppose we have a G action on S^{p+1} extending the standard K action. Similar to Case 2, via the isomorphism $N^{+} \simeq \mathrm{SO}^{\circ}(1,2)^{+}$ and the obvious identification of $\mathscr{F}$ with S^2, it can be shown that to the G action corresponds an $(\mathrm{S}^2)^{+}$-basic action ϕ of $\mathrm{SO}^{\circ}(1,2)^{+}$ on S^2.

Remark 15. Two $(\mathrm{S}^2)^{+}$-basic actions are analytically isomorphic if and only if the restricted $\mathrm{SO}^{\circ}(1,2)$ actions are analytically isomorphic via an isomorphism that commutes with j_1'. We also note again that $\mathrm{SO}^{\circ}(1,2)$ is isomorphic to $\mathrm{PSL}_2(\mathbb{R})$ and hence covered by $\mathrm{SL}_2(\mathbb{R})$ and the analytic $\mathrm{SL}_2(\mathbb{R})$ actions on S^2 are classified in [13].

6.2 Construction of $\mathrm{SO}^\circ(p,2)$ actions

In this section we construct $\mathrm{SO}^\circ(p,2)$ actions on S^{p+1}, $\mathrm{S}^p \times \mathrm{S}^1$, and $\mathrm{S}^{p-1} \times \mathrm{S}^2$ that extend the standard K action in each case. We are going to use the following two lemmas. Uchida shows the following results in the case of smooth actions of G on the sphere S^{p+1} whose restricted K action is the standard orthogonal one. However, the arguments and techniques of Chapter 2 can be applied to prove them in the analytic setting, see also Appendix D.

Lemma 6.2.1. *Let M be S^{p+1}, $\mathrm{S}^p \times \mathrm{S}^1$, or $\mathrm{S}^{p-1} \times \mathrm{S}^2$. Assume K acts on M in the standard way in each case and let $\mathscr{F}$ be the fixed point set of H. Suppose:*

(B1) *ϕ is an analytic action of N^+ on $\mathscr{F}$, such that the restricted action of $N^+ \bigcap K$ is the restriction of the standard action of K.*

(B2) *f is a function from $\mathscr{F} \to \mathbb{R}P^2$, which is N^+-equivariant and satisfies: $N^+ \bigcap U^\circ(Y) \subset N^+_Y$*

Here, $\mathbb{R}P^2$ is seen as a N^+-space, via the identification $N \simeq SO^\circ(1,2)$ and by defining the action of j_1 by $j_1[a:b:c] = [-a:b:c]$. For the definition of $U^\circ(Y)$ see (6.1). Then, there exists an analytic G action on M extending the standard K action such that the restricted N^+ action on $\mathscr{F}$ is ϕ and f is exactly the function from Remark 11 for this G action.

Lemma 6.2.2. *Let M be S^{p+1}, $\mathrm{S}^p \times \mathrm{S}^1$, or $\mathrm{S}^{p-1} \times \mathrm{S}^2$. Let $\mathscr{F}$ be the fixed point set of H and suppose $\mathscr{S}$ is a connected, one dimensional submanifold of $\mathscr{F}$ which is j_1 and j_2 invariant in the case $M = \mathrm{S}^{p+1}$, j_1 invariant in the case $M = \mathrm{S}^p \times \mathrm{S}^1$, and j_2 invariant in the case*

$M = \mathrm{S}^{p-1} \times \mathrm{S}^2$. Assume further that in any case $\mathscr{S}$ is transverse to each $SO(2)$-orbit. Consider $\mathscr{R} = \left\{[a : b : c] \in \mathbb{R}P^2 : c = 0\right\}$ and let $\rho : \mathscr{R} \to \mathbb{R}P^1$ be the obvious isomorphism. Suppose Φ_θ is an analytic flow on $\mathscr{S}$ and $f_0 : \mathscr{S} \to \mathbb{R}P^1$ an analytic function such that they satisfy

(i) $j_i \star \Phi_\theta(z) = \Phi_{-\theta}(j_i \star z) \qquad (i = 1, 2)$

(ii) $f_0(z) = [a : b] \Rightarrow f_0(j_i \star z) = [a : -b] \qquad (i = 1, 2)$

(iii) $f_0(z) = [a : b] \Rightarrow f_0\left(\Phi_\theta(z)\right) = [a\, cosh(\theta) + b\, sinh(\theta) : a\, sinh(\theta) + b\, cosh(\theta)]$

(iv) $f_0(z) = [0 : 1] \Leftrightarrow j_1 \star z = z \quad \text{and} \quad f_0(z) = [1 : 0] \Leftrightarrow j_2 \star z = z$

Define an action ϕ of N^+ on $\mathscr{F}$ in the following way: let $n \in N^+$ and $Z \in \mathscr{F}$. There exist $k_1 \in SO(2)$ and $Y \in \mathscr{S}$ such that $k_1 \star Y = Z$. We then have $nk_1 = j_1^\sigma n'$, where $\sigma = 0$ or 1 and $n' \in N$. Write

$$n = k\, m(\theta)\, u$$

with $k \in SO(2)$, $\theta \in \mathbb{R}$, and $u \in H^\circ_{[1:1:0]} \cap N$, according to (6.5). Then define

$$n \star Z := k \star \Phi_\theta(Y)$$

Finally, define a function $f : \mathscr{F} \to \mathbb{R}P^2$ by

$$f(k \star z) := k \star \left(\rho^{-1} \circ f_0(z)\right) \tag{6.8}$$

for $k \in SO(2)$ and $z \in \mathscr{S}$. Then, ϕ is an analytic action of N^+ on $\mathscr{F}$, f is analytic, and ϕ and f satisfy conditions (B1)-(B2) *in Remark 12.*

Remark 16. Let M be as in the lemmas above and suppose we have a pair (ϕ, f) as in Lemma 6.2.1. By that lemma we get a G action on M extending the standard K action. By Remark 12, we get a pair (ϕ', f'). Then, $\phi' = \phi$ and $f' = f$. And conversely, starting from a G action on M, we get a pair (ϕ, f) satisfying (B1)-(B2), and the G action coming from Lemma 6.2.1 is the one we started with. The same property holds for G actions on M extedning the standard K action and triplets $(\mathscr{S}, \Phi_\theta, f_0)$ as in Lemma 6.2.2.

In the constructions below we will have an action of N^+ on $\mathscr{F}$ and our goal will be to apply Lemma 6.2.1 in order to get an action of G. To use that lemma we will need to construct the function f. To that end, we will find in every case a submanifold $\mathscr{S}$ and construct a function f_0, as in Lemma 6.2.2 so that we can apply Lemma 6.2.2.

6.2.1 $(p, 2)$-basic construction (I)

Here, we show how from a $\mathbb{T}^2$-basic action of $\mathrm{SO}^\circ(1,2)^+$ on $\mathbb{T}^2 = \mathrm{S}^1 \times \mathrm{S}^1$ we can construct an action of G on $\mathrm{S}^p \times \mathrm{S}^1$ extending the standard K action. Note that, for the standard K action the fixed point set of H is

$$\mathscr{F} = \left\{ \alpha e_1 + \beta e_{p+1} : \alpha^2 + \beta^2 = 1 \right\} \times \mathrm{S}^1$$

Let ϕ_1 be a $\mathbb{T}^2$-basic action of $\mathrm{SO}^\circ(1,2)^+$ on $\mathbb{T}^2$, see Definition 6.1.1. Consider the function $\tau : \mathrm{S}^1 \to \mathrm{S}^p$ defined by

$$(x, y) \mapsto x\, e_1 + y\, e_{p+1}$$

Then,

$$\psi = \tau \times \mathrm{Id} : \mathbb{T}^2 \to \mathscr{F} \tag{6.9}$$

is an analytic isomorphism. Via the isomorphism ψ, we get an action ϕ_2 of $\mathrm{SO}^\circ(1,2)^+$ on $\mathscr{F}$ defined by the equation

$$\phi_2(n, \psi(p)) = \psi\left(\phi_1(n,p)\right)$$

for $p \in \mathbb{T}^2$ and $n \in \mathrm{SO}^\circ(1,2)^+$. Note that, since ϕ_1 is a $\mathbb{T}^2$-basic action, ϕ_2 has two closed orbits of the form $\{\zeta\} \times \mathrm{S}^1$ and $\{j_1' \star \zeta\} \times \mathrm{S}^1$, for some $\zeta = \alpha e_1 + \beta e_{p+1} \neq \pm e_{p+1}$, with $\alpha^2 + \beta^2 = 1$.

Claim 1: There exist $z_0 \in \{e_{p+1}\} \times \mathrm{S}^1$ and $z_0' \in \{-e_{p+1}\} \times \mathrm{S}^1$ such that their isotropy Lie algebras with respect to ϕ_2 are generated by the element $X_0 = \begin{bmatrix} & & 1 \\ & 0 & \\ 1 & & \end{bmatrix} \in \mathfrak{so}(1,2)$.

Claim 2: As $\theta \to \infty$, $m_\theta \star z_0$ approaches a fixed point of $\mathscr{M}(1,2)$ in a closed ϕ_2-orbit. As $\theta \to -\infty$, $m_\theta \star z_0$ approaches a fixed point of $\mathscr{M}(1,2)$ in the other closed ϕ_2-orbit. The same thing holds for $m_\theta \star z_0'$.

Claims 1 and 2 are proved in Appendix C. Let $\mathscr{S}_0$ be the $\mathscr{M}(1,2)$-orbit of z_0, namely

$$\mathscr{S}_0 = \{m(\theta) \star z_0\}$$

By Claim 2, $\bar{\mathscr{S}}_0$ comprises $\mathscr{S}_0$ and two fixed points of $\mathscr{M}(1,2)$, one in each closed ϕ_2-orbit. Moreover, by a slight variation of the proof of Lemma C.0.3 in Appendix C, it is easy to see that $\mathscr{S}_0$ is transverse to the SO(2)-orbits. Let

$$\mathscr{S}_0' = \{m(\theta) \star z_0'\}$$

By Claim 2, as $\theta \to +\infty$, $m(\theta) \star z_0'$ converges to a fixed point of $\mathscr{M}(1,2)$ in one of the closed orbits. We can assume that this point is in the closure of $\mathscr{S}_0$. If it is not, we just consider $j_2 \star z_0'$ instead of z_0', which is easily seen to also satisfy Claims 1 and 2. Note that in the closed orbits, j_2 interchanges the two fixed points of $\mathscr{M}(1,2)$. By the j_1' invariance of $\mathscr{S}_0'$, as $\theta \to -\infty$, $m(\theta) \star z_0'$ converges to the point in the intersection of the other closed ϕ_2-orbit and the closure of $\mathscr{S}_0$. Finally, let

$$\mathscr{S} = \overline{\mathscr{S}_0} \cup \overline{\mathscr{S}_0'} \tag{6.10}$$

It can be shown that $\mathscr{S}$ is an analytic submanifold by using the local form of the infinitesimal generators of the induced $\mathrm{SL}_2(\mathbb{R})$ action around the $\mathscr{M}(1,2)$ fixed points, see [13, Theorem 2]. The action of $\mathscr{M}(1,2)$ on $\mathscr{S}$ induces an analytic flow Φ_θ. The Jacobian of Φ_θ at the fixed points is non zero, see [13, Proposition 6.1]. Since $\mathscr{S}$ is a closed, one dimensional manifold, one of the fixed points of Φ_θ is attracting and the other is repelling. We also define a function $f_0 : \mathscr{S} \to \mathbb{R}\mathrm{P}^1$ by

- $f_0(z_0) := [0 : 1]$ and $f_0(z_0') := [0 : 1]$.
- $f_0\left(\Phi_\theta(z)\right) := [\tanh(\theta) : 1]$, for $\theta \in \mathbb{R}$ and $z = z_0$ or z_0'.
- $f_0(z_1) := [1 : 1]$ and $f_0(z_2) := [1 : -1]$, where z_1, respectively z_2, is the attracting, respectively repelling, fixed points of Φ_θ.

By a proof similar to that of Lemma 2.2.2, it can be shown that f_0 is analytic and that $\mathscr{S}$, Φ_θ, and f_0 satisfy the hypotheses of Lemma 6.2.2. Therefore, applying that lemma, we get a pair (ϕ, f) of an analytic N^+ action, ϕ, on $\mathscr{F}$ and an analytic function $f : \mathscr{F} \to \mathbb{R}\mathrm{P}^2$ satisfying relations (B1)-(B2). Then, by Lemma 6.2.1 we obtain an analytic G action on $\mathrm{S}^p \times \mathrm{S}^1$ extending the standard K action. We will refer to such actions as *actions from the* $(p, 2)$-*basic construction* (I). We note also, that since N^+ and $\mathrm{SO}^\circ(1, 2)^+$ are isomorphic, ϕ induces an action of $\mathrm{SO}^\circ(1, 2)^+$ on $\mathscr{F}$. That action, by Remark 16, is ϕ_2.

Theorem 6.2.1. *Let $p \geq 3$. Two analytic* $\mathrm{SO}^\circ(p, 2)$ *actions on* $\mathrm{S}^p \times \mathrm{S}^1$ *from the* $(p, 2)$-*basic construction* (I) *are analytically isomorphic if and only if the corresponding* $\mathbb{T}^2$-*basic actions of* $\mathrm{SO}^\circ(1, 2)^+$ *on* $\mathbb{T}^2$*, see Definitions 6.6 and 6.1.1, are analytically isomorphic.*

Proof. The only if part is easy. For the other implication, let

$$\mathscr{F} = \left\{ \alpha e_1 + \beta e_{p+1} : \alpha^2 + \beta^2 = 1 \right\} \times \mathrm{S}^1$$

be the fixed point set of H. For an analytic $\mathrm{SO}^\circ(p, 2)$ action on $\mathrm{S}^p \times \mathrm{S}^1$ from the $(p, 2)$-basic construction (I), by Remark 12, we get a pair (ϕ^1, f_{ϕ^1}) of an analytic N^+-action on $\mathscr{F}$ and an analytic function $f_{\phi^1} : \mathscr{F} \to \mathbb{R}\mathrm{P}^2$ satisfying (B1)-(B2), as well as a triplet $(\mathscr{S}^1, \Phi^1_\theta, f^1_0)$ of a closed one dimensional submanifold $\mathscr{S}^1 \subseteq \mathscr{F}$, a flow Φ^1_θ on $\mathscr{S}^1$, and a function $f^1_0 : \mathscr{S}^1 \to \mathbb{R}\mathrm{P}^1$. Suppose we have a second $\mathrm{SO}^\circ(p, 2)$ action on $\mathrm{S}^p \times \mathrm{S}^1$ from the $(p, 2)$-basic construction (I), to which similarly correspond a pair (ϕ^2, f_{ϕ^2}) and a triplet $(\mathscr{S}^2, \Phi^2_\theta, f^2_0)$. Assume that the corresponding $\mathbb{T}^2$-basic actions of $\mathrm{SO}^\circ(1, 2)^+$ on $\mathbb{T}^2$, say ϕ'_1 and ϕ'_2, are isomorphic. Identifying $\mathbb{T}^2$ with $\mathscr{F}$ via the function ψ of (6.9), we get two actions of $\mathrm{SO}^\circ(1, 2)^+$ on $\mathscr{F}$ which are isomorphic. Finally, since $\mathrm{SO}^\circ(1, 2)^+$ is isomorphic to N^+, we get two N^+-actions, ϕ_1'' and ϕ_2'', which are isomorphic. However, note that by Remark 16, $\phi_1'' = \phi_1$ and $\phi_2'' = \phi_2$.

Hence, ϕ_1 and ϕ_2 are isomorphic. Let ψ' be the isomorphism between them. Namely, $\psi' : \mathscr{F} \to \mathscr{F}$ is an analytic isomorphism such that, $\psi'\left(\phi_1''(n,p)\right) = \phi_2''\left(n, \psi'(p)\right)$, for $n \in N^+$ and $p \in \mathscr{F}$. Then $\psi'(\mathscr{S}^1) = \mathscr{S}^2$ and the flows Φ_θ^1 and Φ_θ^2 are isomorphic. Then, as in the proof of Theorem 2.1.1 in Section 2.6, it can be shown that

$$f_0^2 \circ \psi' = f_0^1$$

Since $\mathscr{S}^1$ and $\mathscr{S}^2$ are transverse to every SO(2) orbit, and ψ', f_{ϕ^1}, and f_{ϕ^2} are SO(2) equivariant, f_{ϕ^1}, respectively f_{ϕ^2}, is determined by f_0^1, respectively f_0^2. See also (6.8). Therefore,

$$f_{\phi^2} \circ \psi = f_{\phi^1}$$

The proof then concludes similarly to the proof of Theorem 2.1.1 □

We note again, see Remark 13, that analytic $\mathrm{SL}_2(\mathbb{R})$ action on $\mathbb{T}^2$ are classified in [13], therefore, since $\mathrm{SO}^\circ(1,2) \simeq \mathrm{PSL}_2(\mathbb{R})$, the $\mathrm{SO}^\circ(1,2)$ actions on $\mathbb{T}^2$ are classified. Two $\mathrm{SO}^\circ(1,2)^+$ are analytically isomorphic if and only if the restricted $\mathrm{SO}^\circ(1,2)$ are isomorphic via an isomorphism that commutes with the action of j_1.

6.2.2 $(p,2)$-basic construction (II)

Let ϕ_1 be an S^2-basic action of $\mathrm{SO}^\circ(1,2)$ on S^2, see Definition 6.1.2. We will show how to construct an $\mathrm{SO}^\circ(p,2)$ action on $\mathrm{S}^{p-1} \times \mathrm{S}^2$ extending the standard K action starting from ϕ_1. Let

$$\mathscr{F} = \{\pm e_1\} \times \mathrm{S}^2$$

be the fixed point set of H in the standard K action on $\mathrm{S}^{p-1} \times \mathrm{S}^2$ and $\mathscr{F}_1 = \{e_1\} \times \mathrm{S}^2$, $\mathscr{F}_2 = \{-e_1\} \times \mathrm{S}^2$. Let $\mathrm{pr}_2 : \mathrm{S}^{p-1} \times \mathrm{S}^2 \to \mathrm{S}^2$ be the projection to the second factor, and

$$\psi = \left(\mathrm{pr}_2 \Big|_{\mathscr{F}_1} \right)^{-1}$$

Then, $\psi : \mathrm{S}^2 \to \mathscr{F}_1$ is an analytic isomorphism and via ψ, we get an analytic action, ϕ_2, of $\mathrm{SO}^\circ(1,2)$ on $\mathscr{F}_1$. Consider the points $(e_1, \pm\epsilon_3) \in \mathscr{F}_1$ which are fixed by $\mathrm{SO}(2)$ and let $\mathscr{S}$ be the union of the closures of their orbits under the action of $\mathscr{M}(1,2)$. Recall that $\mathrm{SL}_2(\mathbb{R})$ covers $\mathrm{SO}^\circ(1,2) \simeq \mathrm{PSL}_2(\mathbb{R})$ and hence, ϕ_1 induces an $\mathrm{SL}_2(\mathbb{R})$ action on $\mathscr{F}_1 \simeq \mathrm{S}^2$. By Schneider's description of the $\mathrm{SL}_2(\mathbb{R})$ actions in [13], $\mathscr{S}$ is a one dimensional submanifold which is transverse to each $\mathrm{SO}(2)$-orbit. It is also easy to see that $\mathscr{S}$ is j_2-invariant. The action of $\mathscr{M}(1,2) \leq \mathrm{SO}^\circ(1,2)$ induces a flow Φ_θ on $\mathscr{S}$. Since ϕ_1 is an S^2-basic action, ϕ_2 has only one closed orbit on $\mathscr{F}_1$, and therefore Φ_θ has two fixed points, see [13, p. 521]. Define a function $f_0 : \mathscr{S} \to \mathbb{R}\mathrm{P}^1$ by

- $f_0(e_1, \pm\epsilon_3) := [1:0]$
- $f_0\left(\Phi_\theta(z)\right) := [\tanh(\theta) : 1]$, for $z_0 = (e_1, \epsilon_3)$ or $(e_1, -\epsilon_3)$
- $f_0(z_1) := [1:1]$ and $f_0(z_2) := [-1:1]$, where z_1, respectively z_2, is the attracting, respectively repelling, fixed point of Φ_θ.

As in Lemma 2.2.2, it can be shown the f_0 is analytic and similarly to the $(p,2)$-basic construction (I), $\mathscr{S}$, Φ_θ, and f_0 satisfy the hypotheses of Lemma 6.2.2. Hence, we get an analytic pair (ϕ, f) of an N^+ action on $\mathscr{F}$ and a function $f : \mathscr{F} \to \mathbb{R}\mathrm{P}^2$ that satisfy relations (B1)-(B2). Note that, since $\mathrm{SO}^\circ(1,2) \simeq N$, the restriction of ϕ to N induces an $\mathrm{SO}^\circ(1,2)$ action on $\mathscr{F}_1$. That action is exactly ϕ_2, see Remark 16. Using the pair (ϕ, f) and similar arguments as in Chapter 2, we get an analytic action of $\mathrm{SO}^\circ(p,2)$ on $\mathrm{S}^{p-1} \times \mathrm{S}^2$ extending the standard K action.

Theorem 6.2.2. *Let* $p \geq 3$. *Two analytic* $\mathrm{SO}^\circ(p,2)$ *actions on* $\mathrm{S}^{p-1} \times \mathrm{S}^2$ *from the* $(p,2)$-*basic construction* (II) *are analytically isomorphic if and only if the corresponding* S^2-*basic actions of* $\mathrm{SO}^\circ(1,2)$ *actions on* S^2, *see Definition 6.1.2, are isomorphic.*

The proof is similar to that of Theorem 6.2.1. We note again, that the $\mathrm{SO}^\circ(1,2)$ actions on S^2 are classified by the results in [13], see Remark 14.

6.2.3 $(p,2)$-basic construction (III)

Let ϕ_1 be an $(\mathrm{S}^2)^+$-basic action of $\mathrm{SO}^\circ(1,2)^+$ on S^2, see Definition 6.1.3. Starting from ϕ_1, we will construct an $\mathrm{SO}^\circ(p,2)$ action on S^{p+1} extending the standard K action. The are the kind of actions Uchida studied in [17] in the smooth setting. In the standard K action, the fixed point set of H is

$$\mathscr{F} = \{\alpha\, e_1 + \beta\, e_{p+1} + \gamma\, e_{p+1} \,:\, \alpha^2 + \beta^2 + \gamma^2 = 1\}$$

Let $\psi : \mathrm{S}^2 \to \mathscr{F}$ be defined by

$$(\alpha\, e_1 + \beta\, e_{p+1} + \gamma\, e_3) \mapsto \alpha\, e_1 + \beta\, e_{p+1} + \gamma\, e_{p+1}$$

Then, ψ is an analytic isomorphism. Via ψ, we get an action, ϕ_2, of $\mathrm{SO}^\circ(1,2)^+$ on $\mathscr{F}$. Let $\mathscr{S}_1$ be the $\mathscr{M}(1,2)$-orbit of the point $e_1 \in \mathscr{F}$. It is easy to see that $\mathscr{S}_1$ is j_2-invariant since e_1 is fixed by j_2. Now, $\mathrm{SL}_2(\mathbb{R})$ covers $\mathrm{SO}^\circ(1,2) \simeq \mathrm{PSL}_2(\mathbb{R})$ and hence, we get an $\mathrm{SL}_2(\mathbb{R})$ action on $\mathscr{F}_1 \simeq \mathrm{S}^2$ induced by ϕ_2. By Schneider's description of the $\mathrm{SL}_2(\mathbb{R})$ action on S^2, see [13], we can find a point $z = \alpha\, e_{p+1} + \beta\, e_{p+2} \in \mathrm{S}^2$ such that

$$\lim_{\theta \to +\infty} m(\theta) \star z = \lim_{\theta \to +\infty} m(\theta) \star e_1$$

for $m(\theta) \in \mathscr{M}(1,2)$. Let $\mathscr{S}_2$ be the $\mathscr{M}(1,2)$-orbit of z. Finally, consider

$$\mathscr{S} = \mathscr{S}_1 \cup j_1'(\mathscr{S}_1) \cup \mathscr{S}_2 \cup j_2\left(\mathscr{S}_2\right)$$

Then, $\mathscr{S}$ is a one dimensional submanifold of S^2, it is j_1' and j_2-invariant, and it is transverse to each $\mathrm{SO}(2)$-orbit, see [13]. The action of $\mathscr{M}(1,2)$ on $\mathscr{S}$ induces an analytic flow, Φ_θ, on $\mathscr{S}$. Since ϕ_1 is an $(\mathrm{S}^2)^+$-basic action, ϕ_2 has two closed orbits in $\mathscr{F}$ and hence, Φ_θ has four fixed points on $\mathscr{S}$, see also [13]. Then, we can define a function $f_0 : \mathscr{S} \to \mathbb{R}\mathrm{P}^1$ by

- $f_0(\pm e_1) := [1:0]$, $f_0(z) := [0:1]$, and $f(j_2 \star z) := [0:1]$
- $f_0\left(\Phi_\theta(e_1)\right) := [\cosh(\theta) : \sinh(\theta)]$, $f_0\left(\Phi_\theta(z)\right) := [\sinh(\theta) : \cosh(\theta)]$ and in the rest of $\mathscr{S}$ f_0 is defined so that it satisfies $j_1' \star \Phi_\theta(z) = \Phi_{-\theta}(j_1' \star z)$ and $j_2 \star \Phi_\theta(z) = \Phi_{-\theta}(j_2 \star z)$
- $f_0(\zeta) := [1:1]$, respectively $[1:-1]$, if $\zeta \in \mathscr{S}$ is an attracting, respectively repelling, fixed point of Φ_θ.

Similarly to Lemma 2.2.2, it can be shown the f_0 is analytic. Consequently, analogously to the $(p,2)$-basic construction (I), $\mathscr{S}$, Φ_θ, and f_0 satisfy the hypotheses of Lemma 6.2.2, which gives an analytic pair (ϕ, f) of an N^+ action on $\mathscr{F}$ and a function $f : \mathscr{F} \to \mathbb{R}\mathrm{P}^2$ that satisfy relations (B1)-(B2). Note that, since $\mathrm{SO}^\circ(1,2)$ and N are isomorphic, ϕ induces an $\mathrm{SO}^\circ(1,2)$ action on $\mathscr{F}$. That action is ϕ_2, see also Remark 16. Using the pair (ϕ, f) and analogous arguments as in Chapter 2, we get an analytic action of $\mathrm{SO}^\circ(p,2)$ on $\mathrm{S}^{p-1} \times \mathrm{S}^2$ extending the standard K action. The following theorem strengthens [17, Theorem 4.12] in the analytic setting.

Theorem 6.2.3. (See also [17, Theorem 4.12]) *Let $p \geq 3$. Analytic $\mathrm{SO}^\circ(p,2)$ actions on S^{p+1} from the $(p,2)$-basic construction* (III) *are analytically isomorphic if and only if the corresponding $\left(\mathrm{S}^2\right)^+$-basic actions of $\mathrm{SO}^\circ(1,2)^+$ actions on S^2, see* (6.6) *and Definition 6.1.3, are isomorphic.*

The proof is similar to that of Theorem 6.2.1. We note, that the $\mathrm{SO}^\circ(1,2)^+$ actions on S^2 are classified by the results in [13], see Remark 15.

6.3 Classification

Theorem 6.3.1. *Suppose $\mathrm{SO}^\circ(p,2)$, $p \geq 3$, acts analytically on a closed, connected manifold M of dimension $p+1$. Consider $\mathrm{SO}(p) \simeq \mathrm{SO}(p) \times \{1\} \leq \mathrm{SO}(p) \times \mathrm{SO}(2) \leq \mathrm{SO}^\circ(p,2)$ and $\mathrm{SO}(2)$ similarly. For $p > 3$, we have:*

- *If only $\mathrm{SO}(p)$ has a fixed point, then M is equivariantly covered by $\mathrm{S}^p \times \mathrm{S}^1$, where the action of $\mathrm{SO}^\circ(p,2)$ on $\mathrm{S}^p \times \mathrm{S}^1$ is one from the $(p,2)$-basic construction* (I)*, see Section 6.2.1.*

- *If only* $\mathrm{SO}(2)$ *has a fixed point, then* M *is equivariantly covered by* $\mathrm{S}^{p-1} \times \mathrm{S}^2$*, where the action of* $\mathrm{SO}^\circ(p,2)$ *on* $\mathrm{S}^{p-1} \times \mathrm{S}^2$ *is one from the* $(p,2)$*-basic construction* (II)*, see Section 6.2.2.*

- *If both* $\mathrm{SO}(p)$ *and* $\mathrm{SO}(2)$ *have a fixed point, then* M *is equivariantly covered by* S^{p+1}*, where the action of* $\mathrm{SO}^\circ(p,2)$ *on* S^{p+1} *is one from the* $(p,2)$*-basic construction* (III)*, see Section 6.2.3.*

- *If neither* $\mathrm{SO}(p)$ *nor* $\mathrm{SO}(2)$ *have a fixed point, then* M *is equivariantly covered by* $\mathrm{SO}^\circ(p,2) \times_P \mathrm{S}^1$ *with the left* $\mathrm{SO}^\circ(p,2)$ *action, where* $P \leq \mathrm{SO}^\circ(p,2)$ *is a maximal parabolic subgroup isomorphic to the stabiliser of an isotropic line in the standard representation of* $\mathrm{SO}^\circ(p,2)$ *on* $\mathbb{R}^{p+2}$*. If* $P = M_P A_P N_P$ *is a Langlands decomposition of* P*,* P *acts on* S^1 *by a flow via* A_P*, see* (4.1)*.*

For $\mathrm{SO}^\circ(3,2)$*, the above applies if the action is not transitive and covered by* $\mathrm{SO}^\circ(3,2) \big/ P_{\min}$*, where* $P_{\min} \leq \mathrm{SO}^\circ(3,2)$ *is a minimal parabolic subgroup.*

Proof. Suppose $G = \mathrm{SO}^\circ(p,2)$ acts analytically on a manifold M of dimension $p+1$. In the case of $\mathrm{SO}^\circ(3,2)$ the homogeneous space $\mathrm{SO}^\circ(3,2) \big/ P_{\min}$, where $P_{\min}$ is a minimal parabolic subgroup of $\mathrm{SO}^\circ(3,2)$ has dimension $4 = p+1$. So, suppose that either $p > 3$ or in the case of $p = 3$ that the action is not covered by $\mathrm{SO}^\circ(3,2) \big/ P_{\min}$. Then, the methods of Chapter 3 can be used to show that the fixed point set of H, $\mathscr{F}$, is non-empty and 2-dimensional. We will split the proof in three cases depending on the existence of fixed points of $\mathrm{SO}(p)$ or $\mathrm{SO}(2)$.

Similarly to Remark 12, from this action of G we get the restricted action, ϕ_1, of N^+ on $\mathscr{F}$ and a function $f_1 : \mathscr{F} \to \mathbb{R}\mathrm{P}^2$, by Lemma 6.0.1. Additionally, ϕ_1 and f_1 satisfy properties (B1)-(B2) from Remark 12, see [17]. Let $\mathscr{R} = \{[a:b:c] \in \mathbb{R}\mathrm{P}^2 : c = 0\} \subseteq \mathbb{R}\mathrm{P}^2$ and consider $S = f_1^{-1}(\mathscr{R}) \subset \mathscr{F}$.

Case I: Suppose that $\mathrm{SO}(p)$ has a fixed point and that $\mathrm{SO}(2)$ does not have any fixed points. Since $N \leq G$ acts on $\mathscr{F}$, and $N \simeq \mathrm{PSL}_2(\mathbb{R})$, see the beginning of this chapter, $\mathrm{SL}_2(\mathbb{R})$ also acts on $\mathscr{F}$. However, $\mathrm{SO}(2)$ does not have fixed points, hence by [13], $\mathscr{F}$ has to be diffeomorphic to a torus. As in Section 4.2, we can find an analytic isomorphism

$$\psi : S \to \mathrm{S}^1 \quad \text{s.t.} \quad \psi \circ j_1 = j_1 \circ \psi$$

where j_1 acts on S^1 by reflection with respect to the y-axis. Suppose the image of ψ is the first factor of $\mathbb{T}^2 = \mathrm{S}^1 \times \mathrm{S}^1$. Then, define:

$$\Psi : \mathscr{F} \to \mathbb{T}^2$$

by

$$R_\theta z \mapsto R_\theta \psi(z)$$

for $z \in S$ and $R_\theta \in \mathrm{SO}(2)$, where $\mathrm{SO}(2)$ acts on the second factor of $\mathbb{T}^2$ in the standard way. Recall that S is transverse to all the $\mathrm{SO}(2)$ orbits. Now, if we identify $\mathbb{T}^2$ with the set

$$\hat{\mathscr{F}} = \left\{ \alpha e_1 + \beta e_{p+1} : \alpha^2 + \beta^2 = 1 \right\} \times \mathrm{S}^1$$

in the obvious way, then, via Ψ and (ϕ_1, f_1), we get a pair (ϕ, f) of an analytic N^+ action on $\hat{\mathscr{F}}$ and an analytic function $f : \hat{\mathscr{F}} \to \mathbb{R}\mathrm{P}^2$ satisfying (B1)-(B2) of Lemma 6.2.1and hence an analytic action of G on $\mathrm{S}^p \times \mathrm{S}^1$ that extends the standard K action. Note that this is an action from the basic construction corresponding to ϕ', where ϕ' is $\mathbb{T}^2$-basic action of $\mathrm{SO}^\circ(1,2)^+$ on $\mathbb{T}^2$ induced by Ψ and the isomorphism between N^+ and $\mathrm{SO}^\circ(1,2)^+$, see (6.7). For $p \in \mathrm{S}^p \times \mathrm{S}^1$, there exist $k \in K$ and $Y \in \hat{\mathscr{F}}$ such that $k \star Y = p$. Then, we define

$$\begin{aligned} F :& \mathrm{S}^p \times \mathrm{S}^1 \to M \\ & k \star Y \mapsto k \star \Psi(Y) \end{aligned}$$

As in Section 4.2 it can be shown that all the resulting maps are analytic and F is a covering map.

Case II: In the case that neither $\mathrm{SO}(p)$ nor $\mathrm{SO}(q)$ has a fixed point, looking at $S \subset \mathscr{F}$ we get that

$$f_1\Big|_J = [1 : \pm 1 : 0]$$

where $J \subseteq S$ is a nontrivial open subset. Without loss of generality, we may assume that the value of f_1 on J is $(1:1:0)$. Since, f is analytic,

$$f_1\Big|_J \equiv [1:1:0]$$

That means that the isotropy group of every point on S is contained in or equal to P, where $P \leq G$ is a maximal parabolic. As in Section 4.1, it can be shown that such an action is covered by an associated bundle $G \times_P \mathrm{S}^1$. Note that the absence of $\mathrm{SO}(p)$ and $\mathrm{SO}(2)$ fixed points is equivalent to the existence of a nullcone orbit.

Case III: Now, suppose that $\mathrm{SO}(2)$ has a fixed point. Recall that via $N \simeq \mathrm{PSL}_2(\mathbb{R})$,

$\mathrm{SL}_2(\mathbb{R})$ also acts on $\mathscr{F}$. According to Schneider's results, the connected components of $\mathscr{F}$ are diffeomorphic to either $\mathbb{R}\mathrm{P}^2$ or S^2, see [13]. Suppose a component is diffeomorphic to $\mathbb{R}\mathrm{P}^2$, in particular [13, Section 4, Corollary], and denote this component by $\mathscr{F}_1$. Assume that j_1 does not preserve this component. By Schneider's description of the $\mathrm{SL}_2(\mathbb{R})$ actions on $\mathbb{R}\mathrm{P}^2$, there exists an action of $\mathrm{SL}_2(\mathbb{R})$ and hence an action, ϕ_2, of N on S^2 that covers the action ϕ_1 of N on $\mathscr{F}_1$ equivariantly. Let $\pi : \mathrm{S}^2 \to \mathscr{F}_1$ be that N-equivariant covering map. Let ρ_1 be the obvious inclusion

$$\rho_1 : \mathrm{S}^2 \to \{e_1\} \times \mathrm{S}^2$$

Via ρ_1, ϕ_2 induces an action ϕ' of N on $\{e_1\} \times \mathrm{S}^2 \subseteq \mathrm{S}^{p-1} \times \mathrm{S}^2$. Now, on $\{-e_1\} \times \mathrm{S}^2$ consider the action of N defined by

$$n \star (j_1 v) = j_1 \left(j_2 n j_2 \star v \right)$$

for $n \in N$ and $v \in \{e_1\} \times \mathrm{S}^2$. Then, $\{-e_1\} \times \mathrm{S}^2$ covers N-equivariantly the component $j_1(\mathscr{F}_1)$ and we get an action ϕ of N^+ on $\{\pm e_1\} \times \mathrm{S}^2$, extending ϕ'. Now, let

$$f_2 : \{\pm e_1\} \times \mathrm{S}^2 \to \mathbb{R}\mathrm{P}^2$$

be defined by $f_2 = f_1 \circ \pi \circ \rho_1^{-1}$ on $\{e_1\} \times \mathrm{S}^2$ and by $f_2 = f_1 \circ \pi \circ \rho_1^{-1} \circ j_1$ on $\{-e_1\} \times \mathrm{S}^2$. Then, ϕ and f_2 satisfy (B1)-(B2) of Lemma 6.2.1. As in Section 4.2 it can be shown that there exists a G action on $\mathrm{S}^{p-1} \times \mathrm{S}^2$ from the $(p, 2)$-basic construction (II), which covers the action on M, G-equivariantly. A similar argument applies when the component of $\mathscr{F}$ is diffeomorphic to S^2 and j_1 does not preserve this component.

Now, suppose again that a component of $\mathscr{F}$ is diffeomorphic to $\mathbb{R}\mathrm{P}^2$ and denote this component by $\mathscr{F}_1$, but now we assume that j_1 preserves $\mathscr{F}_1$. Now, S is a 1-dimensional closed submanifold of $\mathscr{F}_1$. Let S_0 be a connected component of S which contains a fixed point of SO(2), say Y. We claim that j_1 preserves S_0. Indeed, otherwise $j_1 Y$ would also be an SO(2) fixed point and there would be at least two points fixed by SO(2). But that is impossible for an $\mathrm{SL}_2(\mathbb{R})$ action on $\mathbb{R}\mathrm{P}^2$, see [13]. Therefore j_1 acts on S_0. Note that, being an involution on a manifold diffeomorphic to S^1, j_1 has either none or two fixed points. Let $Y \in \mathscr{F}_1$ be the fixed point of SO(2). Since j_1 commutes with SO(2) in G, $j_1 Y$ is also fixed by SO(2) and hence it must equal Y. So, j_1 has at least one fixed point on S_0 and therefore it has exactly two. Suppose j_1 acts on $\mathbb{R}\mathrm{P}^1$ by

$$j_1 \star [a : b] = [-a : b]$$

A slight variation in the proof of Lemma 4.2.1 shows that there exists an analytic isomorphism

$$\psi : \mathbb{R}\mathrm{P}^1 \to S_0$$

which is j_1 equivariant. Define an action of $\mathscr{M}(p, 2)$ on $\mathbb{R}\mathrm{P}^1$ by

$$m(\theta) \star \psi^{-1}(x) := \psi^{-1}(m(\theta) \star x)$$

and a function $f_2 : \mathbb{R}\mathrm{P}^1 \to \mathbb{R}\mathrm{P}^1$ by

$$f_2 := f_1 \circ \psi$$

Imposing SO(2)-equivariance, we can extend ψ to a map $\hat{\psi}$,

$$\hat{\psi} : \mathbb{R}\mathrm{P}^2 \to \mathscr{F}_1$$

and using f_2 and the action of $\mathscr{M}(p, 2)$ we can define an action $\hat{\phi}$ of N^+ on $\mathbb{R}\mathrm{P}^2$. The proofs are completely analogous to the arguments in Chapter 2, using (6.2). Then, $\hat{\psi}$ is N^+-equivariant.

Now, as before we can find an action, ϕ', of N on S^2 such that it covers the N action $\hat{\phi}$ on $\mathbb{R}\mathrm{P}^2$. On S^2 we consider the action of j_1 by

$$j_1 \star (a\, e_1 + b\, e_2 + c\, e_3) = -a\, e_1 + b\, e_2 + c\, e_3$$

Then the projection to $\mathbb{R}\mathrm{P}^2$ is also j_1 equivariant. Composing with $\hat{\psi}$, we get a covering map

$$\tilde{\psi} : \mathrm{S}^2 \to \mathscr{F}_1$$

which is N^+-equivariant. Composing this map with f_1, we get a map

$$f_2' = f_1 \circ \tilde{\psi} : \mathrm{S}^2 \to \mathbb{R}\mathrm{P}^2$$

Let

$$\hat{\mathscr{F}} = \{\alpha\, e_1 + \beta\, e_{p+1} + \gamma\, e_{p+1} \; : \; \alpha^2 + \beta^2 + \gamma^2 = 1\} \subseteq \mathrm{S}^{p+1}$$

and let $\rho_2 : \mathrm{S}^2 \to \hat{\mathscr{F}}$ be the isomorphism defined by

$$\rho_2\,(\alpha\, e_1 + \beta\, e_{p+1} + \gamma\, e_3) = \alpha\, e_1 + \beta\, e_{p+1} + \gamma\, e_{p+1}$$

Then, via ρ_2 we get an action, ϕ, of N^+ on $\hat{\mathscr{F}}$. Finally, consider the function

$$f = f_2' \circ \rho_2^{-1}$$

As in the first part of Case III, (ϕ, f) satisfy relations (B1)-(B2) from Lemma 6.2.1, and therefore, there exists a G action on S^{p+1} from the $(p, 2)$-basic construction (III), such that S^{p+1} equivariantly covers M. Similar arguments apply when the component of $\mathscr{F}$ is diffeomorphic to S^2 and j_1 preserves this component. $\square$

Appendix A: Hitchin's equivalence of vector fields on the circle

Suppose X, Y are two vector fields on the circle $\mathrm{S}^1 \subseteq \mathbb{R}^2$, such that

- X analytic
- X has a finite number of zeros, each of finite order
- If we assume $\mathrm{S}^1 = \{(\alpha, \beta) \in \mathbb{R}^2 : \alpha^2 + \beta^2 = 1\}$ and J_1 is the reflection $(\alpha, \beta) \mapsto (-\alpha, \beta)$, then $D\mathrm{J}_1(X_p) = -X_{\mathrm{J}_1(p)}$ for $p \in \mathrm{S}^1$. In particular, $X_p = 0$ implies $X_{\mathrm{J}_1(p)} = 0$
- $(0, \pm 1)$ are not zeros of X

and similarly for Y. Examples of such vectors fields are the ones induced by basic J_1-flows, see Definition 2.0.1.

In [7], Hitchin classified smooth vector fields on the circle that vanish only up to finite order. Since here we are interested in analytic vector fields, we will focus only on those. As it can be easily inferred from that paper, two analytic vector fields are *isomorphic*, i.e. there exists an analytic isomorphism, Ψ, of the circle such that X and Y are Ψ-related, if and only if they have the same set of invariants as defined in [7]. Let ∂_{S^1} be a basic vector field of S^1 and for an analytic vector field X on S^1, write $X = g \cdot \partial_{\mathrm{S}^1}$, where g is an analytic function on S^1. The invariant introduced in [7] for X are the following:

- The number of zeros of X, in some counter-clockwise order.

- The order of vanishing of g at each zero.

- A numerical invariant for each zero, called *residue*. In the case of a simple zero $z_0 \in \mathrm{S}^1$, the residue is just the action of the dual form or equivalently, the number $1/g'(z_0)$.

- A numerical *global invariant* μ. In the case that X is non vanishing, μ is just the integral $\int_{\mathrm{S}^1} 1/g$. If X has zeros in S^1, then μ is defined using complex integration methods around those zeros.

- An *orientation* $\sigma \in \{\pm 1\}$, depending on whether on the arc between the first and second zero the orientation that X induces is the same or the opposite of that of S^1.

Suppose X and Y have the same set of invariants. The construction of the isomorphism Ψ in [7] proceeds along the following lines: Based on the local invariants and the orientation, the zeros of X correspond to zeros of Y. Starting from the first zero of X after $(0,1)$ in counter-clockwise order, there is a diffeomorphism, in fact analytic isomorphism, between a neighbourhood of that zero and a neighbourhood of the corresponding zero of Y, giving an equivalence between X and Y. Subsequently, this diffeomorphism on the whole arc between the first and second zero and up to a neighbourhood of the second zero. Then, this process is repeated for the second and third zeros etc.

Assume that the diffeomorphism, Ψ, has been constructed from a neighbourhood of the first zero after $(0,1)$ up to a neighbourhood of the last zero before $(0,-1)$. We would like the diffeomorphism constructed eventually, call it again Ψ, to satisfy $\Psi \circ \mathrm{J}_1 = \mathrm{J}_1 \circ \Psi$. Call the last zero of X before $(0,-1)$, x_1 and the first one after $(0,-1)$, x_2. Then, $x_2 = \mathrm{J}_1(x_1)$. Assume ψ_1 is the local diffeomorphism at x_1 that was used in the construction of Ψ. Namely,

$\psi_1 : U \to V$, with U neighbourhood of x_1 and V a neighbourhood of the corresponding zero of Y, say y_1, and such that $(\psi_1)_*(X) = Y$ and also $\Psi\Big|_U = \psi_1$.

In Hitchin's construction of Ψ, the diffeomorphism is first extended to the arc between two zeros, and then the local diffeomorphism at the latter zero is *chosen* so that it is compatible with Ψ. Since we would like Ψ to satisfy $\Psi \circ \mathrm{J}_1 = \mathrm{J}_1 \circ \Psi$, we will first choose the local diffeomorphism at x_2 and then show that the extension of Ψ is compatible. As the local diffeomorphism at x_2, we choose

$$\psi_2 := \mathrm{J}_1 \circ \psi_1 \circ \mathrm{J}_1 : \mathrm{J}_1(U) \to \mathrm{J}_1(V)$$

Now, J_1 is a diffeomorphism around x_1, so $\mathrm{J}_1(U)$ is a neighbourhood of x_2 and similarly for $\mathrm{J}_1(V)$. We need to check that $(\psi_2)_*(X) = Y$. For a point $p \in \mathrm{J}_1(U)$, we have $p = \mathrm{J}_1(\tilde{p})$ for some $\tilde{p} \in U$. Then:

$$\begin{aligned}
(D\psi_2)_p(X_p) &= D\mathrm{J}_1 \circ D\psi_1 \circ D\mathrm{J}_1(X_{\mathrm{J}_1(\tilde{p})}) \\
&= D\mathrm{J}_1 \circ D\psi_1\Big(- X_{\tilde{p}}\Big) \quad \text{(by our assumptions on } X\text{)} \\
&= -D\mathrm{J}_1\Big(Y_{\psi_1(\tilde{p})}\Big) \\
&= Y_{\mathrm{J}_1 \circ \psi_1(\tilde{p})} \\
&= Y_{\psi_2(p)}
\end{aligned}$$

Now, following [7], if $X = g_X \cdot \partial_1$, where ∂_1 is the basic vector field of the circle, consider the metric $\dfrac{1}{g_X^2} dx^2$ on $(x_1 \overset{\frown}{:} x_2)$, where by $(x_1 \overset{\frown}{:} x_2)$ we mean the arc from x_1 to x_2 in counter-clockwise order. Similarly for Y and $(y_1 \overset{\frown}{:} y_2)$, where $(y_1 \overset{\frown}{:} y_2)$ is the arc that corresponds to $(x_1 \overset{\frown}{:} x_2)$. Let $p \in (x_1 \overset{\frown}{:} x_2)$ where ψ_1 is defined and extend Ψ on $(x_1 \overset{\frown}{:} x_2)$ by

$$\Psi = \exp_{\psi_1(p)} \circ D\psi_1 \circ \exp_p^{-1}$$

We need to show that this extension agrees with ψ_2 on their common domain. Note that, on their common domain, Ψ and ψ_2 are orientation preserving isometries, so they differ by a constant. Therefore, it suffices to show that they agree at a point. We will show they agree on $\mathrm{J}_1(p)$. Also note that since they are isometries, J_1 satisfies:

$$\exp_{\mathrm{J}_1(p)} \circ D\mathrm{J}_1 \circ \exp_p^{-1} \circ \mathrm{J}_1(p) = p$$

while ψ_1 and ψ_2 satisfy:

$$\exp_{\psi_2(\mathrm{J}_1(p))} \circ D\psi_2 = \psi_1 \circ \exp_{\mathrm{J}_1(p)}$$

So, we want:

$$\begin{aligned}
&\Psi\Big(\mathrm{J}_1(p)\Big) = \psi_2\Big(\mathrm{J}_1(p)\Big) \\
\Leftrightarrow\ &\exp_{\psi_1(p)} \circ D\psi_1 \circ \exp_p^{-1}(\mathrm{J}_1(p)) = \mathrm{J}_1\Big(\psi_1(p)\Big) \\
\Leftrightarrow\ &\mathrm{J}_1 \circ \exp_{\psi_1(p)} \circ D\psi_1 \circ \exp_p^{-1}(\mathrm{J}_1(p)) = \psi_1(p)
\end{aligned}$$

We have:

$$\begin{aligned}
&\exp_{\mathrm{J}_1(p)} \circ D\mathrm{J}_1 \circ \exp_p^{-1} \circ \mathrm{J}_1(p) = p \\
\Rightarrow &\psi_1 \circ \exp_{\mathrm{J}_1(p)} \circ D\mathrm{J}_1 \circ \exp_p^{-1} \circ \mathrm{J}_1(p) = \psi_1(p) \\
\Rightarrow &\exp_{\psi_2 \circ \mathrm{J}_1(p)} \circ D\psi_2 \circ D\mathrm{J}_1 \circ \exp_p^{-1}(\mathrm{J}_1(p)) = \psi_1(p) \\
\Rightarrow &\exp_{\mathrm{J}_1(\psi_1(p))} \circ D\mathrm{J}_1 \circ D\psi_1 \circ \exp_p^{-1}(\mathrm{J}_1(p)) = \psi_1(p) \\
\Rightarrow &\mathrm{J}_1 \circ \exp_{\psi_1(p)} \circ D\psi_1 \circ \exp_p^{-1}(\mathrm{J}_1(p)) = \psi_1(p)
\end{aligned}$$

Therefore, $\Psi\Big|_{\mathrm{J}_1(U)\cap(x_1 \overset{\frown}{:} x_2)} = \psi_2$. Moreover, by the same argument, since $\Psi\Big(\mathrm{J}_1(p)\Big) = \Psi\Big(\mathrm{J}_1(p)\Big)$, we get that

$$\Psi \circ \mathrm{J}_1 = \mathrm{J}_1 \circ \Psi$$

on $(x_1 \overset{\frown}{:} x_2)$.

Then, for the next zeros of X after x_2, Ψ can be extended by defining it to be equal to $\mathrm{J}_1 \circ \Psi \circ \mathrm{J}_1$. A similar calculation will hold when Ψ is defined between the two zeros before and after $(1, 0)$. As a result, we get a diffeomorphism $\Psi : S^1 \to S^1$ such that $(\Psi)_*(X) = Y$ and $\mathrm{J}_1 \circ \Psi = \Psi \circ \mathrm{J}_1$.

In the presence of two reflections

Now, let $\mathrm{J}_2 : \mathrm{S}^1 \to \mathrm{S}^1$ be the map $(\alpha, \beta) \mapsto (\alpha, -\beta)$. For a vector field X, we assume it satisfies the same conditions as before, in addition to

- $D\mathrm{J}_i(X_p) = -X_{\mathrm{J}_i(p)}$, for $i = 1, 2$ and $p \in \mathrm{S}^1$

- $(\pm 1, 0)$ are not zeros of X

We assume the same things about Y. Suppose an isomorphism between X and Y has been constructed from a neighbourhood of the first zero of X after $(0, 1)$, always in counter-clockwise order, up to a neighbourhood of the last zero before $(0, 1)$. By a similar calculation like above, we can extend the isomorphism, say Ψ, to the second quadrant in a way that it commutes with J_2. Namely, it satisfies

$$\Psi \circ \mathrm{J}_2 = \mathrm{J}_2 \circ \Psi$$

on a subset of the upper semicircle. Now, using what was shown above, we get an isomorphism $\widetilde{\Psi} : \mathrm{S}^1 \to \mathrm{S}^1$ such that

$$\widetilde{\Psi} \circ \mathrm{J}_1 = \mathrm{J}_1 \circ \widetilde{\Psi}$$

and $\widetilde{\Psi}$ extends Ψ. We do not know yet whether the same is true for J_2. However, both $\Psi \circ \mathrm{J}_2$ and $\mathrm{J}_2 \circ \Psi$ are analytic isomorphisms and there is a neighbourhood around the first zero of X after $(1, 0)$ in counter-clockwise direction where these two function are equal. By their analyticity, we get that:

$$\Psi \circ \mathrm{J}_2 = \mathrm{J}_2 \circ \Psi$$

Appendix B: Analyticity of the actions in the basic construction

Following a similar approach as in [16], we will show analyticity of the action defined in (2.6) by writing $G \times \left(\mathrm{S}^p \times \mathrm{S}^{q-1}\right)$ as a union of three open sets and showing that the action map is analytic when restricted to any of them. Two of these open sets are going to be G times the G-orbit of the two fixed points of $\mathrm{SO}(p)$, namely $(\pm e_{p+1}, \epsilon_1)$, while the third one will be an open set that accounts for the closed G-orbits in $\mathrm{S}^p \times \mathrm{S}^{q-1}$.

Let $\mathscr{O}$ be the orbit of (e_{p+1}, ϵ_1) under the action defined in (2.6).

Proposition B.0.1. *The G action defined in (2.6) is analytic on $\mathscr{O}$.*

In order to prove the proposition, we will define a G-equivariant, analytic isomorphism from $\mathscr{O}$ to $\mathscr{O}_1$, where $\mathscr{O}_1$ is the orbit of e_{p+1} under the standard projective G action on S^{p+q-1}, which we see as a subset of $\mathbb{R}^{p+q}$ with its standard basis; we will however rename the standard basis as $\{e_1, \cdots e_p, \epsilon_1, \cdots, \epsilon_q\}$. We will break the proof of the proposition in a series of lemmas.

For $x \in \mathbb{R}^p$ and $y \in \mathbb{R}^q$, we write $x \oplus y$ for the vector $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^{p+q}$. Note that

$$\mathscr{O}^1 = \{x \oplus y \in \mathrm{S}(\mathbb{R}^p \oplus \mathbb{R}^q) : \|x\| < \|y\|\}$$

Recall $\mathscr{S} = \left\{(\alpha e_1 + \beta e_{p+1}, \epsilon_1) : \alpha^2 + \beta^2 = 1\right\} \subseteq \mathrm{S}^p \times \mathrm{S}^{q-1}$. Let $\mathscr{I}$ be the intersection of $\mathscr{S}$

and $\mathscr{O}$, namely

$$\mathscr{I} = \Big\{\Phi_\theta(e_{p+1}, \epsilon_1) : \theta \in \mathbb{R}\Big\}$$

By property (A3) of Remark 5, f is an analytic isomorphism between $\mathscr{I}$ and $(-1, 1)$. Recall that $f(e_{p+1}, \epsilon_1) = 0$. We can assume that

$$f > 0 \text{ on } \{z \in \mathscr{I} : z = (\alpha_z e_1 + \beta_z e_{p+1}, \epsilon_1) \text{ with } \alpha_z > 0\} \tag{B.1}$$

For $(v, w) \in \mathscr{O}$, we write v as $v = \begin{bmatrix} v_0 \\ \\ v_{p+1} \end{bmatrix}$, with $v_0 \in \mathbb{R}^p$ and $v_{p+1} \in \mathbb{R}$. Define a function $F : \mathscr{O} \to B_1^p(0) \times \mathrm{S}^{q-1}$ by

$$F(v, w) = \begin{cases} \frac{f\Big(\|v_0\|e_1 + v_{p+1}e_{p+1}, \epsilon_1\Big)}{\|v_0\|} v_0 \oplus w & \text{if } v \neq e_{p+1} \\ \\ 0 \oplus w & \text{if } v = e_{p+1} \end{cases}$$

where $B_1^p(0)$ is the unit open ball in $\mathbb{R}^p$ centered at 0. Define $\phi : \mathscr{O} \to \mathscr{I}$ by

$$(v, w) \mapsto (\|v_0\|e_1 + v_{p+1}e_{p+1}, \epsilon_1)$$

so that $F(v, w) = \Big(\dfrac{f\Big(\phi(v, w)\Big)}{\|v_0\|} v_0 \oplus w\Big)$.

Lemma B.0.1. *The function $F : \mathscr{O} \to B_1^p(0) \times S^{q-1}$ is analytic.*

Proof. It is easy to see that F analytic at any point with $v \neq e_{p+1}$, since f is analytic.

Around (e_{p+1}, ϵ_1), we parametrise $\mathscr{S}$ by

$$\begin{aligned} \psi :&(-\delta, \delta) \to \mathscr{V} && \text{(B.2)} \\ s &\mapsto (se_1 + \sqrt{1-s^2}e_{p+1}, \epsilon_1) && \text{(B.3)} \end{aligned}$$

for some $0 < \delta < 1$ and $\mathscr{V} \subseteq \mathscr{S}$ a neighbourhood of (e_{p+1}, ϵ_1). Set

$$\tilde{f}(s) := f \circ \psi = f\Big(se_1 + \sqrt{1-s^2}e_{p+1}, \epsilon_1\Big) \tag{B.4}$$

Then, $\tilde{f}$ is an analytic function on $(-\delta, \delta)$. Moreover, $\tilde{f}(0) = 0$ and

$$\begin{aligned} f\Big(-se_1 + \sqrt{1-s^2}e_{p+1}, \epsilon_1\Big) &= f\Big(j_1(se_1 + \sqrt{1-s^2}e_{p+1}, \epsilon_1)\Big) \\ &= -f\Big(se_1 + \sqrt{1-s^2}e_{p+1}, \epsilon_1\Big) \end{aligned}$$

by (A2). Hence, $\tilde{f}(-s) = -\tilde{f}(s)$. Consider the function on $(-\delta, \delta)$ defined by

$$\mathscr{H}(s) = \begin{cases} \frac{\tilde{f}(s)}{s}\,, s \neq 0 \\ \\ \tilde{f}'(0)\,, s = 0 \end{cases}$$

Since $\tilde{f}$ is analytic and $\tilde{f}(0) = 0$, the function $\mathscr{H}$ is analytic, and it is an even function. Smoothness of $\mathscr{H}$ suffices to give us the existence of a smooth function $\tilde{\mathscr{H}}$, defined around 0 such that

$$\tilde{\mathscr{H}}(s^2) = \mathscr{H}(s)$$

See [3, Ch VIII, §14 , Problem 6]. In fact, $\tilde{\mathscr{H}}$ can actually be taken analytic. Indeed, since $\mathscr{H}$ is analytic, there exists a power series centered at 0 that converges to $\mathscr{H}(s)$ for s close to 0. However, $\mathscr{H}$ is an even function, which implies that all its derivatives of the form $\mathscr{H}^{(2n+1)}$ for $n \in \mathbb{N}$, are odd functions and so $\mathscr{H}^{(2n+1)}(0) = 0$. Therefore, all the odd powers in the Taylor series of $\mathscr{H}$ around 0 vanish. Hence, there are real numbers $\{c_2, c_4, \cdots, c_{2n}, \cdots\}$ and $0 < \delta' < \delta$ such that for any $|s| < \delta'$, the series $\sum\limits_{n\geq 1} c_{2n}s^{2n}$ converges to $\mathscr{H}(s)$. Now, consider the power series $\sum_{n\geq 1} c_{2n}s^n$. Then,

$$\begin{aligned}\limsup_{n\to\infty}(\sqrt[n]{|c_{2n}|}) &= \limsup_{n\to\infty}\left([\sqrt[2n]{|c_{2n}|}]^2\right)\\ &= \left(\limsup_{n\to\infty}(\sqrt[2n]{|c_{2n}|})\right)^2\end{aligned}$$

which converges, since $\dfrac{1}{\limsup_{n\to\infty}(\sqrt[2n]{|c_{2n}|})}$ is the radius of convergence of $\sum_{n\geq 1} c_{2n}s^{2n}$. Hence, the series $\sum_{n\geq 1} c_{2n}s^n$ converges and defines an analytic function around 0. Then, we can take $\tilde{\mathscr{H}}$ to be defined by this series and by the uniqueness of Taylor series, we see that $\tilde{\mathscr{H}}(s^2) = \mathscr{H}(s)$.

Now, the function $v \mapsto \tilde{\mathscr{H}}(\|v_0\|^2)$ is an analytic function around 0. Hence, the function $v \mapsto \mathscr{H}(\|v_0\|)$ is an analytic function around 0. But

$$\mathscr{H}(\|v_0\|) = \frac{f\Big(\phi(v,w)\Big)}{\|v_0\|}$$

Consequently, the function

$$v \mapsto \frac{f\Big(\phi(v,w)\Big)}{\|v_0\|}v_0$$

is analytic around $v = e_{p+1}$, which in turn implies that the function F is analytic around

(e_{p+1}, ϵ_1). ☐

Subsequently, we define $F_0 : \mathscr{O} \to \mathscr{O}^1$ by

$$F_0(v,w) = \frac{1}{\sqrt{1 + f^2\Big(\phi(v,w)\Big)}} F(v,w)$$

Lemma B.0.2. *The function $F_0 : \mathscr{O} \to \mathscr{O}^1$ is a G-equivariant, analytic isomorphism.*

Proof. By Lemma B.0.1, F_0 is analytic. Before we calculate its inverse, we show that F_0 is G-equivariant. Firstly, we show equivarience with respect to K. Let $k = (\kappa_1, \kappa_2) \in K$ and $(v,w) \in \mathscr{O}$. Recall $\tilde{\kappa} = \begin{bmatrix} \kappa & \\ & 1 \end{bmatrix}$ for $\kappa \in \mathrm{SO}(p)$; see (2.1). Then,

$$k \star (v,w) = (\kappa_1, \kappa_2) \star (v,w) = (\tilde{\kappa}_1 v, \kappa_2 w) = \Big(\begin{bmatrix} \kappa_1 v_0 \\ \\ v_{p+1} \end{bmatrix}, \kappa_2 w \Big)$$

Now, $\phi\Big(\begin{bmatrix} \kappa_1 v_0 \\ \\ v_{p+1} \end{bmatrix}, \kappa_2 w \Big) = (\|\kappa_1 v_0\| e_1 + v_{p+1} e_{p+1}, \epsilon_1) = (\|v_0\| e_1 + v_{p+1} e_{p+1}, \epsilon_1) = \phi(v,w)$ since $\kappa_1 \in \mathrm{SO}(p)$. Therefore,

$$\begin{aligned} F_0\Big((\kappa_1, \kappa_2) \star (v,w)\Big) &= \frac{1}{\sqrt{1 + f^2\Big(\phi\Big((\kappa_1,\kappa_2) \star (v,w)\Big)\Big)}} (\kappa_1 v_0 \oplus \kappa_2 w) \\ &= \frac{1}{\sqrt{1 + f^2\Big(\phi(v,w)\Big)}} (\kappa_1 v_0 \oplus \kappa_2 w) \\ &= (\kappa_1, \kappa_2) \star F_0(v,w) \end{aligned}$$

where the action in the right hand side of the last equality is the orthogonal action of K on S^{p+q-1}. Therefore, we have K-equivarience. Now, we let $\theta \in \mathbb{R}$ and write $m(\theta) \star (e_{p+1}, \epsilon_1) = \Phi_\theta(e_{p+1}, \epsilon_1) = (\alpha_\theta e_1 + \beta_\theta e_{p+1}, \epsilon_1)$. Then,

$$F_0\Big(\Phi_\theta(e_{p+1}, \epsilon_1)\Big) = \frac{1}{\sqrt{1+\tanh^2(|\theta|)}}\Big(\frac{\tanh(|\theta|)}{|\alpha_\theta|}\alpha_\theta e_1 \oplus \epsilon_1\Big)$$

Recall that $f\left(\Phi_\theta(e_{p+1}, \epsilon_1)\right) = \tanh\left(\theta\right)$ by relation (A3) in Remark 5 in Chapter 2. If $\theta > 0$, we have that $f\left(\Phi_\theta(e_{p+1}, \epsilon_1)\right) = \tanh\left(\theta\right) > 0$. Then $\alpha_\theta > 0$ by assumption (B.1) and so

$$\frac{\tanh(|\theta|)}{|\alpha_\theta|}\alpha_\theta e_1 = \frac{\tanh(\theta)}{\alpha_\theta}\alpha_\theta e_1 = \tanh(\theta) e_1$$

On the other hand, if $\theta < 0$, then $\alpha_\theta < 0$ and then also

$$\frac{\tanh(|\theta|)}{|\alpha_\theta|}\alpha_\theta e_1 = \frac{\tanh(-\theta)}{-\alpha_\theta}\alpha_\theta e_1 = \tanh(\theta) e_1$$

Hence

$$F_0\Big(\Phi_\theta(e_{p+1}, \epsilon_1)\Big) = \frac{1}{\sqrt{1+\tanh^2(|\theta|)}}\Big(\tanh\left(\theta\right) e_1 \oplus \epsilon_1\Big) \tag{B.5}$$

Computing now the action of $m(\theta)$ on $F_0(e_{p+1}, \epsilon_1) = \epsilon_1$, we get

$$\begin{aligned}
\begin{bmatrix} \cosh(\theta) & & \sinh(\theta) & \\ & I_{p-1} & & \\ \sinh(\theta) & & \cosh n(\theta) & \\ & & & I_{q-1} \end{bmatrix} \epsilon_1 &= \begin{bmatrix} \cosh(\theta) & & \sinh(\theta) & \\ & I_{p-1} & & \\ \sinh(\theta) & & \cosh(\theta) & \\ & & & I_{q-1} \end{bmatrix} \begin{bmatrix} \\ \\ 1 \\ \\ \end{bmatrix} \\
&= \begin{bmatrix} \sinh(\theta) \\ \\ \cosh(\theta) \\ \\ \end{bmatrix} \\
&= \sinh(\theta) e_1 \oplus \cosh(\theta) \epsilon_1
\end{aligned}$$

The Euclidean norm of $\sinh(\theta) e_1 \oplus \cosh(\theta) \epsilon_1$ is $\cosh(\theta)\sqrt{1 + \tanh^2(\theta)}$. Hence,

$$\begin{aligned}
m(\theta) \star F_0(e_{p+1}, \epsilon_1) &= m(\theta) \star \epsilon_1 \\
&= \frac{1}{\cosh(\theta)\sqrt{1 + \tanh^2(\theta)}} \Big(\sinh(\theta) e_1 \oplus \cosh(\theta) \epsilon_1 \Big) \\
&= \frac{1}{\sqrt{1 + \tanh^2(\theta)}} \Big(\tanh(\theta) e_1 \oplus \epsilon_1 \Big) \\
&= F_0\left(\Phi_\theta(e_{p+1} \epsilon_1) \right) \;\; \text{(by (B.5))} \\
&= F_0\Big(m(\theta) \star (e_{p+1}, \epsilon_1) \Big)
\end{aligned}$$

where the action in either side of the first equality is the projective action of G on S^{p+q-1}.

Now, we can show G-equivarience. Let $g \in G$ and, according to (2.5), write $g = km(\theta)u$, with $u \in U(e_{p+1}, \epsilon_1) = H^{\circ}_{[0:1]}$, see (2.4). Note that then, u fixes $F_0(e_{p+1}, \epsilon_1)$. Then

$$\begin{aligned} F_0(g \star (e_{p+1}, \epsilon_1)) &= F_0(km(\theta) \star (e_{p+1}, \epsilon_1)) \\ &= (km(\theta)) \star F_0(e_{p+1}, \epsilon_1) \\ &= (km(\theta)) \star \epsilon_1 \\ &= (km(\theta)u) \star \epsilon_1 \\ &= g \star F_0(e_{p+1}, \epsilon_1) \end{aligned}$$

Finally, we define the inverse of F_0. Let $x \oplus y \in \mathscr{O}^1 = \left\{ x \oplus y \in \mathrm{S}^{p+q-1} : \|x\| < \|y\| \right\}$. Recall that f restricted on $\mathscr{I} = \left\{ \Phi_\theta(e_{p+1}, \epsilon_1) : \theta \in \mathbb{R} \right\}$ is invertible. Let $\mathrm{pr}_{\mathrm{S}^p} : \mathrm{S}^p \times \mathrm{S}^{q-1} \to \mathrm{S}^p$ be the projection to the first factor. Let $\hat{f}$ be the function

$$\hat{f} = \mathrm{pr}_{\mathrm{S}^p} \circ \left(f\Big|_{\mathscr{I}} \right)^{-1} \tag{B.6}$$

Namely, if

$$f\left(\alpha\, e_1 + \beta\, e_{p+1}, \epsilon_1\right) = s$$

then

$$\hat{f}(s) = \alpha\, e_1 + \beta\, e_{p+1}$$

Define

$$v(x \oplus y) = \begin{bmatrix} \frac{\langle \hat{f}\left(\frac{\|x\|}{\|y\|}\right), e_1 \rangle}{\|x\|} x \\ \\ \langle \hat{f}\left(\frac{\|x\|}{\|y\|}\right), e_{p+1} \rangle \end{bmatrix} \text{ and } w(x \oplus y) = \frac{1}{\|y\|} y \tag{B.7}$$

where $\langle \cdot, \cdot \rangle$ is the euclidean inner product in $\mathbb{R}^{p+1}$. Note that, for $x \oplus y \in \mathscr{O}^1$ we have $\|x\| < \|y\|$ and hence, $y \neq 0$. Therefore, w is well defined and analytic on $\mathscr{O}^1$. Define

$$\begin{aligned} F_1 :& \mathscr{O}^1 \to \mathscr{O} \\ & x \oplus y \mapsto \Big(v(x \oplus y), w(x \oplus y)\Big) \end{aligned}$$

Suppose $x \neq 0$. Then, $\Big(f\Big|_{\mathscr{I}}\Big)^{-1}\Big(\frac{\|x\|}{\|y\|}\Big) = \Big(\alpha e_1 + \beta e_{p+1}, \epsilon_1\Big)$ with $\alpha > 0$. Let $F_1(x \oplus y) = (v, w)$. We have $v_0 = \frac{\alpha}{\|x\|}x$, $v_{p+1} = \beta$ and $w = \frac{1}{\|y\|}y$. Then,

$$f\Big(\phi(v, w)\Big) = \frac{\|x\|}{\|y\|}$$

As a result,

$$\begin{aligned} F_0(v, w) &= \frac{1}{\sqrt{1 + \frac{\|x\|^2}{\|y\|^2}}}\left(\frac{\frac{\|x\|}{\|y\|}}{\alpha}\frac{\alpha}{\|x\|}x \oplus \frac{1}{\|y\|}y\right) \\ &= \frac{\|y\|}{\sqrt{\|y\|^2 + \|x\|^2}}\Big(\frac{1}{\|y\|}x \oplus \frac{1}{\|y\|}y\Big) \\ &= \frac{1}{\sqrt{\|y\|^2 + \|x\|^2}}\Big(x \oplus y\Big) \end{aligned}$$

But, $x \oplus y \in \mathscr{O}^1 \subset \mathrm{S}^{p+q-1}$ and so $\|y\|^2 + \|x\|^2 = 1$. Hence,

$$F_0(v, w) = F_0\Big(F_1(x \oplus y)\Big) = x \oplus y \tag{B.8}$$

If $x = 0$, then $y \in \mathrm{S}^{q-1}$. Hence, $v(0 \oplus y) = e_{p+1}$ and $w(0 \oplus y) = y$, by (B.7). Then,

$F_1(0 \oplus y) = (e_{p+1}, y)$. Since $f(e_{p+1}, \epsilon_1) = 0$, by the definition of F and F_0, we have

$$F_0(e_{p+1}, y) = 0 \oplus y$$

Therefore,

$$F_0\left(F_1(0 \oplus y)\right) = 0 \oplus y \tag{B.9}$$

As a result, by equations (B.8) and (B.9) we have that

$$F_0\Big(F_1(x \oplus y)\Big) = x \oplus y$$

for any $x \oplus y \in \mathscr{O}^1$. Similarly, we can see that $F_1\Big(F_0(v, w)\Big) = (v, w)$ and so $F_1 = F_0^{-1}$.

As for the analyticity of F_0^{-1}, it follows easily at points with $x \neq 0$. In order to deal with the case $x = 0$, set $v_0 = \dfrac{\langle \hat{f}\left(\frac{\|x\|}{\|y\|}\right), e_1\rangle}{\|x\|}x$ and $v_{p+1} = \langle \hat{f}\left(\frac{\|x\|}{\|y\|}\right), e_{p+1}\rangle$. Now, around (e_{p+1}, ϵ_1) in $\mathscr{S}$, we use the chart given by the inverse of ψ from (B.2), namely

$$\begin{aligned} \psi^{-1} :& \mathscr{V} \to (-\delta, \delta) \\ & \Big(se_1 + \sqrt{1-s^2}e_{p+1}, \epsilon_1\Big) \mapsto s \end{aligned}$$

Consider the function

$$h' = \psi^{-1} \circ \Big(f\Big|_{\mathscr{I}}\Big)^{-1}$$

We note that $h' = \tilde{f}^{-1}$, see (B.4). Since $\tilde{f}$ is an odd function, h' is also an odd analytic function. In addition, $h'(0) = 0$. Then, the function

$$s \mapsto \frac{h'(s)}{s}$$

is an even, analytic function on $(-\delta, \delta)$. Using the same argument as in the proof of Lemma B.0.1, we conclude that the function

$$x \oplus y \mapsto \frac{\left(f\Big|_{\mathscr{I}}\right)^{-1}\left(\frac{\|x\|}{\|y\|}\right)}{\frac{\|x\|}{\|y\|}}$$

is an analytic function around $0 \oplus y \in \mathscr{O}^1$. Therefore,

$$x \oplus y \mapsto \frac{\left(f\Big|_{\mathscr{I}}\right)^{-1}\left(\frac{\|x\|}{\|y\|}\right)}{\|x\|}$$

is an analytic function around $0 \oplus y \in \mathscr{O}^1$. Hence v_0 is an analytic function. As for v_{p+1}, around $x = 0$, it is equal to the function $v_0 \mapsto \sqrt{1 - v_0^2}$. However, $v_0 \neq 1$ around $x = 0$ and we just saw that it is analytic. Hence, v_{p+1} is also analytic. $\square$

Proposition B.0.1 now follows from Lemma B.0.2. Similarly, it can be shown that the G action restricted to the orbit of $(-e_{p+1}, \epsilon_1)$ is analytic.

Now, let

$$\mathscr{S}_+ = \{z \in \mathscr{S} : f(z) > 0\} = \{z = (\alpha_z e_1 + \beta_z e_{p+1}, \epsilon_1) : \alpha_z > 0\}$$

Also, define

$$D_+ = \Big\{(\theta, z) \in \mathbb{R} \times \mathrm{S}^1 : \Phi_\theta(z) \in \mathscr{S}_+\Big\}$$

and

$$W_+ = \left\{(g, z) \in G \times \mathscr{S}_+ : \pm Tr\Big(gP(z)g^T\Big) \neq \frac{1 - f^2(z)}{1 + f^2(z)}\right\}$$

as in [16]. See equation (2.9) for the definition of the matrices $P(z)$. Let $g \in G$ and $z \in \mathscr{S}$. Then we can write $g = k\, m(\theta)\, u$ according to (2.6). By definition of $P(z)$ in (2.9) and equations (2.10), (2.11), and (2.12) it follows easily that

$$\begin{aligned} Tr\Big(gP(z)g^T\Big) &= \lambda(\theta, z) \\ &= \frac{1}{f^2(z)+1}\Big[(f(z)\cosh(\theta) + \sinh(\theta))^2 + (f(z)\sinh(\theta) + \cosh(\theta))^2\Big] \\ &= \frac{1}{f^2(z)+1}\Big[\Big(f^2(z)+1\Big)\left(\cosh^2(\theta)\sinh^2(\theta)\right) + 4f(z)\sinh(\theta)\cosh(\theta)\Big] \\ &= \cosh^2(\theta) + \sinh^2(\theta) + \frac{4f(z)\sinh(\theta)\cosh(\theta)}{f^2(z)+1} \end{aligned} \tag{B.10}$$

Using the identities $\cosh^2(\theta) + \sinh^2(\theta) = \cosh(2\theta)$ and $2\cosh(\theta)\sinh(\theta) = \sinh(2\theta)$, for $\theta \in \mathbb{R}$, the above trace is also equal to

$$Tr\Big(gP(z)g^T\Big) = \cosh(\theta) + \frac{2f(z)\sinh(2\theta)}{f^2(z)+1} \tag{B.11}$$

Lemma B.0.3. *[16, Lemma 4.7] For any $(g, z) \in W_+$, there exist unique $kH \in K/H$ and $\theta \in \mathbb{R}$ such that*

$$(\theta, z) \in D_+ \text{ and } g = km(\theta)u \tag{B.12}$$

for some $u \in U(z)$. Moreover, the function

$$\Delta : W_+ \to K/H \times D_+$$

$$(g, z) \mapsto (kH, \theta, z)$$

is analytic.

Proof. • **Existence and uniqueness in** (B.12):

For the existence, write $g = km(\theta')u'$ for some $\theta' \in \mathbb{R}$ and $u' \in U(z)$. If $(\theta', z) \in D_+$ we are done, so suppose this is not the case. If $f(z) = 1$ then $\Phi_\theta(z) = 1$ for all θ. In particular for θ' we have that $f(\Phi_{\theta'}(z)) = f(z) = 1 > 0$ and hence, $(z, \theta') \in D_+$. Since we assumed this is not the case, we have that $f(z) \neq 1$ and therefore, $0 < f(z) < 1$. Recall that f cannot be greater than 1 since $f = f_\Phi$ for a basic j_1-flow, see Lemma 2.2.2. Now, z belongs to either the orbit of (e_{p+1}, ϵ_1) or $(-e_{p+1}, \epsilon_1)$. Assume we have the former case; the other one proceeds analogously. There exists $\tau \in \mathbb{R}$ such that

$$z = \Phi_\tau(e_{p+1}, \epsilon_1) \quad \text{and} \quad f(z) = \tanh(\tau)$$

Then, for $\theta' \in \mathbb{R}$ we have:

$$\begin{aligned} \Phi_{\theta'}(z) &= \Phi_{\theta'+\tau}(e_{p+1}, \epsilon_1) \\ \Rightarrow f(\Phi_{\theta'}(z)) &= f(\Phi_{\theta'+\tau}(e_{p+1}, \epsilon_1)) \\ &= \tanh(\theta' + \tau) \end{aligned}$$

Since we have assumed that $(\theta', z) \notin D_+$, it follows that $\tanh(\theta'+\tau) \leq 0 \Rightarrow \theta'+\tau \leq 0$. We claim that $\theta'+\tau < 0$. Indeed, assume $\theta'+\tau = 0$. Then $f(z) = -\tanh(\theta')$. Therefore, by (B.10) we have

$$\begin{aligned}
Tr\left(gP(z)g^T\right) &= \cosh^2(\theta') + \sinh^2(\theta') + \frac{-4\tanh(\theta')}{1+\tanh^2(\theta')}\sinh(\theta')\cosh(\theta') \\
&= \frac{\left(\cosh^2(\theta') + \sinh^2(\theta')\right)\left(1+\tanh^2(\theta')\right) - 4\sinh^2(\theta')}{1+\tanh^2(\theta')} \\
&= \frac{\left(\cosh^2(\theta') - \sinh^2(\theta')\right) - \sinh^2(\theta')\left(\tanh^2(\theta') - 1\right)}{1+\tanh^2(\theta')} \\
&= \frac{1-\tanh^2(\theta')}{1+\tanh^2(\theta')} \\
&= \frac{1-f^2(z)}{1+f^2(z)}
\end{aligned}$$

In the above calculation we also used the identities $\cosh^2(\theta') - \sinh^2(\theta') = 1$ and $\tanh^2(\theta') - 1 = \dfrac{1}{\cosh^2(\theta')}$. However, we started with the assumption that $(g, z) \in W_+$ and so the above equality leads to a contradiction. Therefore, $\theta'+\tau < 0$. This implies that

$$\begin{aligned}
& f\left(\Phi_{-\theta'-2\tau}(z)\right) = f\left(\Phi_{-\theta'-\tau}(e_{p+1}, \epsilon_1)\right) = \tanh(-\theta'-\tau) = -\tanh(\theta'+\tau) > 0 \\
& \Rightarrow \Phi_{-\theta'-2\tau}(z) \in \mathscr{S}_+
\end{aligned}$$

Set $\theta = -\theta' - 2\tau$. Then, $(\theta, z) \in D_+$. In addition, since $j_1 \in U(e_{p+1}, \epsilon_1)$, we have

$$j_1 m(-2\tau) = m(\tau) j_1 m(-\tau) \in U(z)$$

Set $u = j_1 m(-2\tau)u' \in U(z)$. Then, we can write

$$g = (kj_1)\, m(\theta)u$$

Note that $kj_1 \in K$. As for uniqueness, suppose

$$g = km(\theta)u = k'm(\theta')u'$$

with $(\theta, z), (\theta', z) \in D_+$, $u, u' \in U(z)$, and $k, k' \in K$. That means,

$$k \star \Phi_\theta(z) = k' \star \Phi_{\theta'}(z)$$

Write $\Phi_\theta(z) = (\alpha_z e_1 + \beta_z e_{p+1}, \epsilon_1)$, $\Phi_{\theta'}(z) = (\alpha'_z e_1 + \beta'_z e_{p+1}, \epsilon_1)$, $k = (\kappa_1, \kappa_2)$ and $k' = (\kappa'_1, \kappa'_2)$. Then,

$$k \star \Phi_\theta(z) = \left(\begin{bmatrix} \kappa_1 & \\ & 1 \end{bmatrix} \begin{bmatrix} a_z \\ 0 \\ \vdots \\ \beta_z \end{bmatrix}, \kappa_2 \epsilon_1 \right)$$

and similarly for $k' \star \Phi_{\theta'}(z)$. Since $k \star \Phi_\theta(z) = k' \star \Phi_{\theta'}(z)$, we see that $\beta_z = \beta'_z$ and since $\Phi_\theta(z), \Phi_{\theta'}(z) \in \mathscr{S}_+$, we have $\alpha_z, \alpha'_z > 0$. Since $\alpha_z e_1 + \beta_z e_{p+1} \in \mathrm{S}^p$ and $\alpha'_z e_1 + \beta'_z e_{p+1} \in \mathrm{S}^p$, we conclude that $\Phi_\theta(z) = \Phi_{\theta'}(z)$. Consequently, $k^{-1}k' \in \mathrm{Stab}_K(\Phi_\theta(z))$. But, $\mathrm{Stab}_K(\Phi_\theta(z)) = H$ and so $k^{-1}k' \in H$. Finally, if $f(z) \neq 1$, $\Phi_\theta(z) = \Phi_{\theta'}(z)$ implies that $\theta = \theta'$. On the other hand, if $f(z) = 1$, observe that

$$\begin{aligned}
&\mathrm{Tr}\left(gP(z)g^T\right) = \mathrm{Tr}\left(m(\theta)P(z)m(\theta)\right) = \mathrm{Tr}\left(m(\theta')P(z)m(\theta')\right) \\
\Rightarrow&\lambda(\theta, z) = \lambda(\theta', z) \\
\Rightarrow&e^{2\theta} = e^{2\theta'} \\
\Rightarrow&\theta = \theta'
\end{aligned}$$

See equation (2.12).

- **Analyticity of Δ:**

Recall

$$\Delta : W_+ \to K/H \times D_+$$
$$(g, z) \mapsto (kH, \theta, z)$$

Set $\Delta_1(g, z) := \mathrm{pr}_1\left(\Delta(g, z)\right)$ and $\Delta_2(g, z) := \mathrm{pr}_2\left(\Delta(g, z)\right)$, where pr_1 and pr_2 are the projections to the first and second factor respectively. Namely, if $\Delta(g, z) = (kH, \theta, z)$, then $\Delta_1(g, z) = kH$ and $\Delta_2(g, z) = \theta$. Consider the function

$$\gamma : W_+ \times \mathbb{R} \to \mathbb{R}$$
$$(g, z, \theta) \mapsto \cosh(2\theta) + \frac{2f(z)}{1 + f^2(z)}\sinh(2\theta) - \mathrm{Tr}\left(gP(z)g^T\right)$$

Then, by (B.11) we have

$$\gamma\left(g, z, \Delta_2(g, z)\right) = 0$$

and

$$\frac{\partial \gamma}{\partial \theta}(g,z,\theta) = 2\text{sinh}(2\theta) + \frac{4f(z)}{1+f^2(z)}\text{cosh}(2\theta)$$

Suppose for (g,z,θ) we have $\frac{\partial \gamma}{\partial \theta}(g,z,\theta) = 0$. Then, $\frac{2f(z)}{1+f^2(z)} = -\tanh(2\theta)$. Then, by (B.11)

$$\begin{aligned}
\operatorname{Tr}\left(gP(z)g^T\right) &= \cosh(2\theta) + \frac{2f(z)}{1+f^2(z)}\sinh(2\theta) \\
&= \cosh(2\theta) + (-\tanh(2\theta))\sinh(2\theta) \\
&= \cosh(2\theta) - \frac{\sinh^2(2\theta)}{\cosh(2\theta)} \\
&= \frac{1}{\cosh(2\theta)}
\end{aligned}$$

On the other hand

$$\begin{aligned}
\left(\frac{1-f^2(z)}{1+f^2(z)}\right)^2 &= \frac{1-2f^2(z)+f^4(z)}{\left(1+f^2(z)\right)^2} \\
&= 1 - \frac{4f^2(z)}{\left(1+f^2(z)\right)^2} \\
&= 1 - \tanh^2(2\theta) \\
&= \frac{1}{\cosh^2(2\theta)}
\end{aligned}$$

Hence, we get $\operatorname{Tr}\Big(gP(z)g^T\Big) = \pm\dfrac{1-f^2(z)}{1+f^2(z)}$, but that is impossible since $(g,z) \in W_+$. So,

$$\frac{\partial \gamma}{\partial \theta}\left(g,z,\Delta_2(g,z)\right) \neq 0$$

and we can apply the analytic Implicit Function Theorem to get analyticity of Δ_2.

As for Δ_1, firstly we define

$$\begin{aligned}
\delta_1 &: W_+ \to \mathbb{R}^{p+q} \\
(g,z) &\mapsto \frac{1}{\sqrt{1+f^2(z)}} g\left(e_1 + f(z)\epsilon_1\right)
\end{aligned}$$

where, $g\left(e_1 + f(z)\epsilon_1\right)$ is just matrix-vector multiplication. Recall that for $\mathbb{R}^{p+q}$ we denote the standard basis by $\{e_1, \cdots, e_p, \epsilon_1, \cdots, \epsilon_q\}$. Obviously, δ_1 is an analytic function. Then, we set $\Delta_1(g,z) = kH$ and $\Delta_2(g,z) = \theta$ and define

$$\begin{aligned}
\alpha &= \frac{1}{\sqrt{1+f^2(z)}} \left(\cosh(\theta) + f(z)\sinh(\theta)\right) \\
\beta &= \frac{1}{\sqrt{1+f^2(z)}} \left(\sinh(\theta) + f(z)\cosh(\theta)\right)
\end{aligned}$$

Note, that $\alpha, \beta > 0$. Indeed, since $(g,z) \in W_+$ (in particular $z \in \mathscr{S}_+$) and $(\theta, z) \in D_+$ we have

$$f\left(\Phi_\theta(z)\right) > 0$$

But $f\left(\Phi_\theta(z)\right) = \frac{f(z)+\tanh(\theta)}{1+f(z)\tanh(\theta)}$. Hence

$$\frac{f(z)+\tanh(\theta)}{1+f(z)\tanh(\theta)} > 0$$

which is equivalent to

$$\left(\left(\sinh(\theta) + f(z)\cosh(\theta)\right)\left(\cosh(\theta) + f(z)\sinh(\theta)\right) > 0\right. \tag{B.13}$$

It is not difficult to see that the function

$$\theta \mapsto \frac{f(z)+\tanh(\theta)}{1+f(z)\tanh(\theta)}$$

is an increasing function. Therefore, for a fixed $z \in \mathscr{S}_+$, the $\theta \in \mathbb{R}$ for which $(\theta, z) \in D_+$ form an interval, say I_z, which always contains 0. The function $\theta \mapsto ((\sinh(\theta) + f(z)\cosh(\theta))$ is continuous as a function of θ on I_z and never zero, because of equation (B.13). Since at $\theta = 0$ the function is positive, it is positive in all of I_z. Hence, $\beta > 0$. Then, also $\alpha > 0$ by (B.13).

Since $g = km(\theta)u$, we also note that

$$\delta_1(g, z) = k\left(\alpha e_1 + \beta \epsilon_1\right)$$

Now, we consider the maps

$$\begin{aligned} \sigma_1 : K/H &\to \mathbb{R}^{p+q} \\ kH &\mapsto k\left(e_1 + \epsilon_1\right) \end{aligned}$$

and

$$\begin{aligned} \sigma_2 : \mathbb{R}^{p+q} \backslash \Big((\{0\} \times \mathbb{R}^q) \cup (\mathbb{R}^p \times \{0\}) \Big) &\to \mathbb{R}^{p+q} \\ x \oplus y &\mapsto \frac{x}{\|x\|} \oplus \frac{y}{\|y\|} \end{aligned}$$

σ_1 is an embedding of K/H into $\mathbb{R}^{p+q}$, and both σ_1 and σ_2 are analytic. Denote

$$\mathbb{R}^{p+q}_{\neq 0} = \mathbb{R}^{p+q} \backslash \Big((\{0\} \times \mathbb{R}^q) \cup (\mathbb{R}^p \times \{0\}) \Big)$$

Then, we have the commutative diagramme

$$\begin{tikzcd}
\mathbb{R}^{p+q}_{\neq 0} \arrow[r, "\sigma_2"] & \mathbb{R}^{p+q} \\
W_+ \arrow[u, "\delta_1"] \arrow[r, "\Delta_1"] & K/H \arrow[u, "\sigma_1"']
\end{tikzcd}$$

Since δ_1, σ_1 and σ_2 are analytic, and σ_1 is an embedding, Δ_1 is analytic. $\square$

Define the set

$$W_0 := \{(g, k \star z) : (gk, z) \in W_+\}$$

Lemma B.0.4. *W_0 is an open subset of $G \times \Big(S^p \times S^{q-1}\Big)$*

Proof. Evidently, W_+ is open in $G \times \mathscr{S}_+$. Let $(g, k \star z) \in W_0$, so $(gk, z) \in W_+$. Then, there exist $U \subseteq G$, $V \subseteq \mathscr{S}_+$, both open such that

$$(gk, z) \in U \times V \subseteq W_+$$

Let

$$\tilde{U} = Uk^{-1} \text{ and } \tilde{V} = k \star V$$

Then, $(g, k \star z) \in \tilde{U} \times \tilde{V}$ and $\tilde{U} \times \tilde{V}$ is open. If $(\tilde{g}, \tilde{z}) \in \tilde{U} \times \tilde{V}$, then there exist $g_U \in U$

and $z_V \in V$ such that

$$\begin{aligned} \tilde{g} &= g_U k^{-1} \text{ and } \tilde{z} = k \star z_V \\ \Rightarrow \tilde{g}k &= g_U \in U \text{ and } z_V \in V \\ &\Rightarrow (\tilde{g}k, z_V) \in U \times V \subseteq W_+ \\ &\Rightarrow (\tilde{g}, \tilde{z}) \in W_0 \end{aligned}$$

Therefore, $\tilde{U} \times \tilde{V} \subseteq W_0$ and hence, W_0 is open. □

Proposition B.0.2. *The action map of the G action defined in (2.6) is analytic on W_0.*

Proof. The action map restricted to W_0 is

$$g \star (k \star z) = \Delta_1(gk, z) \star \Phi_{\Delta_2(gk,z)}(z)$$

and hence the action is analytic, when restricted to W_0, by Lemma B.0.3. □

Finally, we only have to observe that that

$$G \times \left(\mathrm{S}^p \times \mathrm{S}^{q-1}\right) = (G \times \mathscr{O}) \cup (G \times \tilde{\mathscr{O}}) \cup W_0$$

where $\mathscr{O}$ and $\tilde{\mathscr{O}}$ are the G-orbits of (e_{p+1}, ϵ_1) and $(-e_{p+1}, \epsilon_1)$ respectively. Each of the sets on the right hand side is open, and the action map is analytic when restricted to either of them. Hence, the action of G on $\mathrm{S}^p \times \mathrm{S}^{q-1}$ defined by (2.6) is analytic.

Appendix C: Proof of Claims 1 and 2 of 6.2.1

Here we prove Claims 1 and 2 from 6.2.1. We briefly recall the setting. We have a $\mathbb{T}^2$-basic action, ϕ_1, of $\mathrm{SO}^\circ(1,2)^+$ on $\mathbb{T}^2$, see Definitions 6.6 and 6.1.1. We identify $\mathbb{T}^2$ with

$$\mathscr{F} = \Big\{\alpha e_1 + \beta e_{p+1} : \alpha^2 + \beta^2 = 1\Big\} \times \mathrm{S}^1$$

via ψ, see (6.9), and hence, we get an action ϕ_2 of $\mathrm{SO}^\circ(1,2)^+$ on $\mathscr{F}$. In this action, $\mathrm{SO}(2)$ acts by rotations on the second factor and trivially on the first, j_1' acts by

$$j_1' \star (\alpha e_1 + \beta e_{p+1}, \zeta) = (-\alpha e_1 + \beta e_{p+1}, \zeta)$$

and ϕ_2 has two closed orbits none of which are the j_1' fixed circles $\{\pm e_{p+1}\} \times \mathrm{S}^1$. The proof of Claim 1 is an immediate consequence of the following lemma:

Lemma C.0.1. *The isotropy algebras with respect to ϕ_2 of the points in the j_1'-fixed circles, namely $\{\pm e_{p+1}\} \times \mathrm{S}^1$, are generated by elements of the form*

$$\begin{bmatrix} 0 & \alpha & \beta \\ \alpha & 0 & 0 \\ \beta & 0 & 0 \end{bmatrix}$$

Proof. First of all, we note that since ϕ_1 is a $\mathbb{T}^2$-basic action, the Lie algebras of the points in the statement of the Lemma are 1-dimensional. Let $z \in \{e_{p+1}\} \times \mathrm{S}^1$. The proof for $\{-e_{p+1}\} \times \mathrm{S}^1$ proceeds similarly. Suppose that the Lie isotropy algebra of z is generated by the element of $\mathfrak{so}(1,2)$,

$$X = \begin{bmatrix} 0 & \alpha & \beta \\ \alpha & 0 & \gamma \\ \beta & -\gamma & 0 \end{bmatrix}$$

We can assume that $\alpha^2 + \beta^2 = 1$. Observe that $(\alpha, \beta) \neq (0,0)$, since otherwise z would be fixed by SO(2), which is impossible. For $t \in \mathbb{R}$, let

$$R_t = \begin{bmatrix} 1 & & \\ & \cos(t) & -\sin(t) \\ & \sin(t) & \cos(t) \end{bmatrix} \tag{C.1}$$

where omitted entries are equal to 0. Acting on z with R_t we get a point $z_t = R_t \star z \in \{e_{p+1}\} \times \mathrm{S}^1$ whose isotropy algebra is generated by

$$R_t\, X\, R_t^{-1} = \begin{bmatrix} 0 & \alpha\cos(t) - \beta\sin(t) & \beta\cos(t) + \alpha\sin(t) \\ \alpha\cos(t) - \beta\sin(t) & 0 & \gamma \\ \beta\cos(t) + \alpha\sin(t) & -\gamma & 0 \end{bmatrix}$$

Therefore, there exists a $t_0 \in \mathbb{R}$ such that the Lie isotropy algebra of the point z_{t_0} is generated by

$$X_{t_0} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & \gamma \\ 1 & -\gamma & 0 \end{bmatrix}$$

Now, acting on z_{t_0} with an $m(\theta) = \begin{bmatrix} \cosh(\theta) & \sinh(\theta) & \\ \sinh(\theta) & \cosh(\theta) & \\ & & 1 \end{bmatrix} \in \mathscr{M}(1,2)$, we get a point whose Lie isotropy algebra is generated by

$$m(\theta)X_{t_0}m(\theta)^{-1} = \begin{bmatrix} 0 & 0 & 1+\gamma\tanh(\theta) \\ 0 & 0 & \gamma+\tanh(\theta) \\ 1+\gamma\tanh(\theta) & -(\gamma+\tanh(\theta)) & 0 \end{bmatrix}$$

Then, necessarily $|\gamma| < 1$. Indeed, firstly if $|\gamma| > 1$, then on the one hand $\gamma + \tanh(\theta) \neq 0$ for all $\theta \in \mathbb{R}$, while on the other hand there exists $\theta_1 \in \mathbb{R}$ such that $1 + \gamma\tanh(\theta_1) = 0$. Hence, the point $m(\theta) \star z_{t_0}$ is fixed by SO(2) which gives a contradiction. Suppose now that $\gamma = 1$, namely

$$X_{t_0} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

In that case

$$m(\theta)X_{t_0}m(\theta)^{-1} = \begin{bmatrix} 0 & 0 & 1+\tanh(\theta) \\ 0 & 0 & 1+\tanh(\theta) \\ 1+\tanh(\theta) & -(1+\tanh(\theta)) & 0 \end{bmatrix}$$

which belongs in $\langle X_{t_0}\rangle$. Therefore, either $\mathscr{M}(1,2)$ fixes z_{t_0} or every point in the $\mathscr{M}(1,2)$-orbit of z_{t_0} has isotropy algebra $\langle X_{t_0}\rangle$. The former case cannot occur by our assumption ϕ_2, namely that the closed orbits do not coincide with $\{\pm e_{p+1}\} \times \mathrm{S}^1$. Consequently, we assume the latter case. Take $\theta > 0$ and consider the points $\zeta_1 = m(\theta) \star z_{t_0}$ and $\zeta_2 = j_1' \star \zeta_1$. By the relation, in $\mathrm{SO}^\circ(1,2)^+$,

$$j_1'\, m(\theta) = m(-\theta)\, j_1'$$

and the fact that $j_1' \star z_{t_0} = z_{t_0}$, we see that ζ_2 is also in the $\mathscr{M}(1,2)$-orbit of z_{t_0} and hence, its isotropy algebra is $\langle X_{t_0}\rangle$. Now, for an element $n \in N$, by the relations $j_1'j_2 = j_2j_1'$, $(j_1')^2 = j_2^2 = Id$, and the fact that $j_1'j_2$ commutes with $\mathrm{SO}(1,2)$, it is immediate that

$$j_1'nj_1' = j_2nj_2 \tag{C.2}$$

Hence, since

$$\zeta_2 = j_1' \star \zeta_1$$

using (C.2), it is easy to see that

$$j_2 X_{t_0} j_2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

must be in the Lie isotropy algebra of ζ_2. However, that isotropy algebra must be $\langle X_{t_0} \rangle$, hence we get a contradiction. The case $\gamma = -1$ proceeds similarly.

Subsequently, since $|\gamma| < 1$, there exists $\theta_0 \in \mathbb{R}$ such that

$$\tanh(\theta_0) = -\gamma$$

Let $\zeta_0 = m(\theta_0) \star z_{t_0}$. Then, the isotropy algebra of ζ_0 contains the element

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Now, if $\{e^{sA} : s \in \mathbb{R}\} \leq \mathrm{SO}^{\circ}(1,2)$ is the 1-parameter subgroup of $\mathrm{SO}^{\circ}(1,2)$ corresponding to A, we claim that e^{sA} fixes $j_1' \star \zeta_0$. Indeed, by (C.2), $j_1' e^{sA} j_1' = j_2 e^{sA} j_2$ and

$$\begin{aligned}\left.\frac{d}{ds}\right|_{s=0} \left(j_2 e^{sA} j_2\right) &= j_2 A j_2 \\ &= \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \\ &= -A\end{aligned}$$

which belongs in the Lie isotropy algebra of ζ_0. Hence, $j_1' e^{sA} j_1' \star \zeta_0 = \zeta_0$, which implies that $e^{sA} \star (j_1' \star \zeta_0) = j_1' \star \zeta_0$. Therefore, A belongs in the isotropy algebra of $j_1' \star \zeta_0$. Using the relation $j_1' m(\theta) = m(-\theta) j_1'$, we have

$$\begin{aligned} j_1' \star \zeta_0 &= j_1' m(\theta_0) \star z_{t_0} \\ &= m(-\theta_0) j_1' \star z_{t_0} \\ &= m(-\theta) \star z_{t_0} \end{aligned}$$

But then, the isotropy algebra of $j_1' \star \zeta_0$ must also contain the element

$$m(-\theta_0) X_{t_0} m(-\theta)^{-1} = \begin{bmatrix} 0 & 0 & 1+\gamma^2 \\ 0 & 0 & 2\gamma \\ 1+\gamma^2 & -2\gamma & 0 \end{bmatrix}$$

As a result, if $\gamma \neq 0$, by considering the element $A - \dfrac{1}{1+\gamma^2}\left(m(-\theta_0) X_{t_0} m(-\theta)^{-1}\right)$ we have that $j_1' \star \zeta_0$ is fixed by SO(2) which is a contradiction. Therefore, $\gamma = 0$ and the lemma is proved. $\square$

In order to prove Claim 2, we are going to need the following two lemmas.

Lemma C.0.2. *There exists no point in the ϕ_2-orbit of z_0 whose Lie isotropy algebra is generated by the element*

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

Proof. Assume z_1 is such an element. Since z_1 is in the orbit of z_0, there exists an element $g \in N^+$ such that

$$z_0 = g \star z_1$$

By equation (6.5) we can write

$$g = (j_1')^\sigma \, R_t \, m(\theta) \, u$$

with $\sigma = 0$ or 1, $R_t \in \mathrm{SO}(2)$ for some $t \in \mathbb{R}$, see (C.1), $\theta \in \mathbb{R}$, and $u \in \widetilde{H}^\circ_{[1:1:0]}$. Recall that $\widetilde{H}_{[a:b:c]}$ is the isotropy group of the point $ae_1 + be_{p+1} + ce_3$ in the standard representation of $\mathrm{SO}^\circ(1,2)$ on $\mathbb{R}^3$. Let

$$X_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

Note that $uX_1u^{-1} = \lambda X_1$ for some $\lambda \in \mathbb{R}$, because $u \in \widetilde{H}^\circ_{[1:1:0]}$ and X_1 generates the Lie algebra of $\widetilde{H}^{\circ}_{[1:1:0]}$. Subsequently, since the isotropy algebra of z_1 is generated by X_1, the isotropy algebra of z_0 is generated by gX_1g^{-1}. However, the isotropy algebra of z_0 contains

$$X_0 = \begin{bmatrix} & & 1 \\ & 0 & \\ 1 & & \end{bmatrix}$$

A straightforward matrix multiplication shows that we have a contradiction. □

Lemma C.0.3. *Let z be a point in the closed orbits ϕ_2. If the Lie isotropy algebra of z contains the element* $X_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$*, then z is fixed by $\mathscr{M}(1,2)$.*

Proof. Assume that z is not fixed by $\mathscr{M}(1,2)$. For an element $X \in \mathfrak{so}(1,2)$, consider the *fundamental vector field* $\mathscr{V}^X$ on $\mathscr{F}$ which is defined at a point $p \in \mathscr{F}$ by

$$\mathscr{V}^X_p := \left.\frac{d}{ds}\right|_{s=0} e^{sX} \star p$$

where $\{e^{sX} : s \in \mathbb{R}\} \leq \mathrm{SO}^{\circ}(1,2)$ is the 1-parameter subgroup of $\mathrm{SO}^{\circ}(1,2)$ corresponding to X. Let

$$H = \begin{bmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 0 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 0 & & \\ & 0 & 1 \\ & -1 & 0 \end{bmatrix}$$

Note that H and K are elements of $\mathfrak{so}(1,2)$ and that H, respectively K, generates the Lie algebra of $\mathscr{M}(1,2)$, respectively $\mathrm{SO}(2)$. By our assumptions on z, its ϕ_2-orbit is one dimensional and the fundamental vector field $\mathscr{V}^H$ is not 0. Therefore,

$$\mathscr{V}_z^H = \lambda \mathscr{V}_z^K$$

for some $\lambda \neq 0$. Consequently,

$$\left[\mathscr{V}^H, \mathscr{V}^K\right]_z = 0 \tag{C.3}$$

But $\left[\mathscr{V}^H, \mathscr{V}^K\right]$ is also a fundamental vector field that corresponds to the element $[H, K] \in \mathfrak{so}(1, 2)$. Then, equation (C.3) implies that $[H, K]$ is in the Lie isotropy algebra of z. By the assumption of the Lemma, X_1 is also in the same isotropy algebra, hence $[[H, K], X_1]$ is in the same algebra. A simple matrix computation shows that

$$[X_1, [H, K]] = H$$

However, this contradicts the assumption that $\mathscr{M}(1, 2)$ does not fix z. Hence, the Lemma is proved. □

Now we can prove Claim 2. Recall that z_0 is a point in $\{e_{p+1}\} \times \mathrm{S}^1$, whose isotropy algebra is generated by the element

$$X_0 = \begin{bmatrix} & & 1 \\ & 0 & \\ 1 & & \end{bmatrix}$$

and $\mathscr{S}_0$ is the $\mathscr{M}(1,2)$-orbit of z_0, namely

$$\mathscr{S}_0 = \{m(\theta) \star z_0\}$$

Consider the ϕ_2-orbit of z_0, $\mathscr{O}$, and its closure, $\bar{\mathscr{O}}$. Since ϕ_1 is a $\mathbb{T}^2$-basic action, $\bar{\mathscr{O}}$ comprises $\mathscr{O}$ and the two closed orbits of ϕ_2. Suppose that $m(\theta) \star z_0$ does not converge to an $\mathscr{M}(1,2)$ fixed point in one of the two closed orbits. Then, since $\bar{\mathscr{O}}$ is compact, there exists a sequence θ_n, $n \in \mathbb{N}$, such that $\theta_n \to +\infty$ and $z_n = m(\theta_n) \star z_0$ converges to a point ζ in $\bar{\mathscr{O}}$, different from the fixed points of $\mathscr{M}(1,2)$. Note that the element

$$m(\theta_n) X_0 m(\theta_n)^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & \tanh(\theta) \\ 1 & -\tanh(\theta) & 0 \end{bmatrix}$$

is in the Lie isotropy algebra of z_n for the action ϕ_2. Therefore, the isotropy algebra of ζ contains the element

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

By Lemma C.0.2, ζ is not in $\mathscr{O}$ and hence, by Lemma C.0.3 ζ must be fixed by $\mathscr{M}(1,2)$, which is a contradiction. As a result, as $\theta \to +\infty$, $m(\theta) \star z_0$ converges to a fixed point of $\mathscr{M}(1,2)$ in one of the closed orbits. Since $\mathscr{S}_0$ is j_1' invariant, as $\theta \to -\infty$, $m(\theta) \star z_0$ converges to a fixed point of $\mathscr{M}(1,2)$ in the other closed circle orbit, since j_1' maps one to the other.

Appendix D: Proof of Lemma 6.2.1

Lemma 6.2.1 follows essentially from the arguments and techniques of Chapter 2 and Appendix B. A small modification is required when showing the analyticity of the G actions. We will show how the arguments need to change for the case $M = \mathrm{S}^p \times \mathrm{S}^1$ in Lemma 6.2.1. The other cases are shown similarly.

In particular, a modification is needed in the part where, if $Y = (e_{p+1}, z) \in \mathscr{F}$ is a point fixed by $\mathrm{SO}(p)$, we identify the orbit of Y with the orbit of $e_{p+1} \in \mathbb{R}^{p+2}$ under the standard G action on $\mathbb{R}^{p+2}$, by a G-equivariant analytic isomorphism, see Lemma B.0.2. Let

$$S = f^{-1}(\mathscr{R}) \tag{D.1}$$

where $\mathscr{R} = \left\{[a : b : c] \in \mathbb{RP}^2 : c = 0\right\} \subset \mathbb{RP}^2$. Then, S is a 1-dimensional submanifold of $\mathscr{F}$ which is transverse to every $\mathrm{SO}(2)$ orbit, see [17]. For any $Y = (v, w) \in \mathscr{F}$, because f is N^+-equivariant, there exists $\theta_Y \in \mathbb{R}$ such that $(v, R_{\theta_Y} w) \in S$, where we see $R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$.

Finally, for $Y \in S$, let $\tilde{f}(Y) = \frac{a_Y}{b_Y}$. In order to apply the same argument as in

Appendix B, we need to show that the function

$$\mathscr{H}(v,w) = \tilde{f}\left(\|v_0\|e_1 + \sqrt{1-\|v_0\|}e_{p+1}, R_{\theta_{(\tilde{v},w)}}w\right)$$

is an analytic function, where $(v,w) \in \mathrm{S}^p \times \mathrm{S}^1$ and $v = \begin{bmatrix} v_0 \\ \\ v_{p+1} \end{bmatrix}$, $v_0 \in \mathbb{R}^p$ and $\tilde{v} = \|v_0\|\, e_1 + \sqrt{1-\|v_0\|}\, e_{p+1}$, hence $(\tilde{v},w) \in \mathscr{F}$.

Lemma D.0.1. *The above $\mathscr{H}(v,w)$ is analytic.*

Proof. Let $Y \in \mathscr{F}$ and note that, since $\mathrm{SO}(2)$ does not have any fixed points in this case,

$$f(Y) \neq [1:0:0]$$

Therefore, if we write $f(Y) = (a_Y, b_Y, c_Y)$, we have that $(b_Y, c_Y) \neq (0,0)$. Assume that $b_Y \neq 0$. Then, it is easy to see that

$$\theta_Y = -\arctan\left(\frac{c_Y}{b_Y}\right)$$

and so it is an analytic function of Y. Therefore, $\mathscr{H}$ is evidently analytic at point with $v_0 \neq 0$.

If $v_0 = 0$, then $(v,w) = (\pm e_{p+1}, z)$. We can assume that $v = e_{p+1}$, that $(e_{p+1}, w) \in S$ and that $f(e_{p+1}, w) = [0:1:0]$. We can parametrise a neighbourhood around (e_{p+1}, w) in

$\mathscr{F}$ by

$$(s,t) \mapsto \left(s\,e_1 + \sqrt{1-s}\,e_{p+1}, w_t\right)$$

Let

$$f\left(s\,e_1 + \sqrt{1-s}\,e_{p+1}, w_t\right) = (a(s,t) : b(s,t) : c(s,t))$$

and set

$$\theta(s,t) = -\arctan\left(\frac{c(s,t)}{b(s,t)}\right)$$

Then, $f\left(s\,e_1 + \sqrt{1-s}\,e_{p+1}, R_{\theta(s,t)}w_t\right) = \left(a(s,t) : \tilde{b}(s,t) : 0\right)$. Let

$$f_2(s,t) := \frac{a(s,t)}{\tilde{b}(s,t)}$$

Since f is N^+-equivariant and in particular j_1-equivariant,

$$\theta(-s,t) = \theta(s,t)$$

and hence

$$f_2(-s,t) = -f_2(s,t)$$

f_2 is an analytic function around (e_{p+1}, w). Using our parametrisation, if we write

$$f_2(s,t) = \sum_{i,j} \alpha_{i,j} s^i t^j$$

by the equation above we get that $\alpha_{i,j} = 0$ for i even. Therefore,

$$\frac{1}{s} f_2(s,t) = \sum_{i \, \text{odd}, \, j} \alpha_{i,j} s^{i-1} t^j$$

Finally, we consider the function

$$\mathscr{H}(s,t) = \sum_{i,j} \alpha_{i,j} s^{\frac{i-1}{2}} t^j$$

Then, $\mathscr{H}$ is analytic at $(0,0)$ and

$$\mathscr{H}(s^2,t) = \frac{1}{s} f_2(s,t)$$

Hence, the function $(v, w_t) \mapsto \mathscr{H}(\|v_0\|, t) = \frac{1}{\|v_0\|} f_2(\|v_0\|, t)$ is analytic.

$\square$

Then, we can proceed analogously to Appendix B to get analyticity of the action of G.

Bibliography

[1] V. I. Arnold. *Geometrical Methods in the Theory of Ordinary Differential Equations.* Springer New York, NY, New York, 1988.

[2] T. Asoh. On smooth SL$(2, \mathbf{C})$ actions on 3-manifolds. *Osaka Journal of Mathematics*, 24(2):271 – 298, 1987.

[3] J. Dieudonné. *Foundations of Modern Analysis*, volume 10 of *Pure and Applied Mathematics.* Academic Press, 1960.

[4] D. Fisher and K. Melnick. Smooth and analytic actions of $SL(n, \mathbf{R})$ and $SL(n, \mathbf{Z})$ on closed n-dimensional manifolds. *Kyoto Journal of Mathematics*, 64(4):873 – 904, 2024.

[5] W. Fulton and J. Harris. *Representation Theory.* Springer New York, NY, 2004.

[6] J. Hilgert and K.-H. Neeb. *Structure and Geometry of Lie Groups.* Springer New York, NY, 2012.

[7] N. Hitchin. Vector fields on the circle. In M. Francaviglia, editor, *Mechanics, Analysis and Geometry: 200 Years After Lagrange*, North-Holland Delta Series, pages 359–378. Elsevier, Amsterdam, 1991.

[8] G. Itzkowitz, S. Rothman, and H. Strassberg. A note on the real representations of SU(2,C). *Journal of Pure and Applied Algebra*, 69(3):285–294, 1991.

[9] A. W. Knapp. *Lie Groups: Beyond an Introduction.* Birkhäuser, Boston, 2002.

[10] K. Mukoyama. Smooth Sp(2,R)-actions on the 4-sphere. *Tohoku Mathematical Journal, Second Series*, 48(4):543–560, 1996.

[11] K. Mukoyama. Smooth *SU(p,q)*-actions on the $(2p{+}2q\text{-}1)$-sphere and on the complex projective $(p{+}q\text{-}1)$-space. *Kyushu Journal of Mathematics*, 55(2):213–236, 2001.

[12] A. Onishchik and E. S. I. I. for Mathematical Physics. *Lectures on Real Semisimple Lie Algebras and Their Representations.* ESI lectures in mathematics and physics. European Mathematical Society, 2004.

[13] C. R. Schneider. SL(2,R) Actions on Surfaces. *American Journal of Mathematics*, 96(4):511–528, 1974.

[14] F. Uchida. Classification of real analytic SL(n, $\mathbf{R}$) actions on n-sphere. *Osaka Journal of Mathematics*, 16(3):561 – 579, 1979.

[15] F. Uchida. Real Analytic SL(n, R) Actions on Spheres. *Tohoku Mathematical Journal*, 33:145–175, 1981.

[16] F. Uchida. On smooth $\mathrm{SO}_0(p, q)$-actions on S^{p+q-1}. *Osaka Journal of Mathematics*, 26(4):775 – 787, 1989.

[17] F. Uchida. On smooth $\mathrm{SO}_0(p, q)$-actions on S^{p+q-1}, II. *Tohoku Mathematical Journal*, 49:185–202, 1997.

[18] F. Uchida. On smooth $\mathbf{Sp}(p, q)$-actions on $S^{4p+4q-1}$. *Osaka Journal of Mathematics*, 39(2):293 – 314, 2002.

[19] V. Varadarajan. Spin(7)-subgroups of SO(8) and Spin(8). *Expositiones Mathematicae - EXPO MATH*, 19:163–177, 12 2001.

[20] G. Warner. *Harmonic Analysis on Semi-Simple Lie Groups I.* Springer Berlin, Heidelberg, 1972.

[21] D. Witte. Cocompact subgroups of semisimple lie groups. In G. Benkart and J. M. Osborn, editors, *Lie algebras and related topics*, number 110 in Contemporary Mathematics, pages 309–313, Providence RI, 1990. American Mathematical Society.