ABSTRACT

Title of Dissertation: USING THE Q-WEIBULL DISTRIBUTION FOR RELIABILITY ENGINEERING MODELING AND APPLICATIONS

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Modeling and improving system reliability require selecting appropriate probability distributions for describing the uncertainty in failure times. The q-Weibull distribution, which is based on the Tsallis non-extensive entropy, is a generalization of the Weibull distribution in the context of non-extensive statistical mechanics. The q-Weibull distribution can be used to describe complex systems with long-range interactions and long-term memory, can model various behaviors of the hazard rate, including unimodal, bathtub-shaped, monotonic, and constant, and can reproduce both short and long-tailed distributions. Despite its flexibility, the q-Weibull has not been widely used in reliability applications partly because parameter estimation is challenging. This research develops and tests an adaptive hybrid artificial bee colony approach for estimating the parameters of a q-Weibull distribution. This research demonstrates that the q-Weibull distribution has a superior performance over Weibull distribution in the characterization of lifetime data with

a non-monotonic hazard rate. Moreover, in terms of system reliability, the q-Weibull distribution can model dependent series systems and can be modified to model dependent parallel systems. This research proposes using the q-Weibull distribution to directly model failure time of a series system composed of dependent components that are described by Clayton copula and discusses the connection between the q-Weibull distribution and the Clayton copula and shows the equivalence in their parameters. This dissertation proposes a Nonhomogeneous Poisson Process (NHPP) with a q-Weibull as underlying time to first failure (TTFF) distribution to model the minimal repair process of a series system composed of multiple dependent components. The proposed NHPP q-Weibull model has the advantage of fewer parameters with smaller uncertainty when used as an approximation to the Clayton copula approach, which in turn needs more information on the assumption for the underlying distributions of components and the exact component cause of system failure. This dissertation also proposes a q-Fréchet distribution, dual distribution to q-Weibull distribution, to model a parallel system with dependent component failure times that are modeled as a Clayton copula. The q-Weibull and q-Fréchet distributions are successfully applied to predict series and parallel system failures, respectively, using data that is characterized by non-monotonic hazard rates.

USING THE Q-WEIBULL DISTRIBUTION FOR RELIABILITY ENGINEERING MODELING AND APPLICATIONS

by

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Dedication

To my parents,

my husband, my son

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List of Abbreviations and Nomenclature

ABC	Artificial bee colony
AHABC	Adaptive hybrid artificial bee colony
ML	Maximum likelihood
PDF	Probability density function
CDF	Cumulative distribution function
AIC	Akaike information criterion
BIC	Bayesian information criterion
KS	Kolmogorov-Smirnov
ES	Experimental settings
ACI	Asymptotic confidence intervals
BCI-P	Parametric bootstrap confidence intervals
BCI-NP	Non-parametric bootstrap confidence intervals
NHPP	Nonhomogeneous Poisson process
TTFF	Time to first failure
LHD	Load-haul-dump
$f_q(t)$	Probability density function of the q-Weibull distribution
$f_{qf}(t)$	Probability density function of the q-Fréchet distribution
$\exp_q(x)$	q-Exponential function
$F_q(x)$	Cumulative distribution function of the q-Weibull distribution

$F_f(t)$	Cumulative distribution function of the Fréchet distribution
$F_{qf}(t)$	Cumulative distribution function of the q-Fréchet distribution
$R_q(t)$	Reliability function of the q-Weibull distribution
$R_{qf}(t)$	Reliability function of the q-Fréchet distribution
$R_q(t\|t_0)$	Conditional reliability function of the q-Weibull distribution
$h_q(t)$	Hazard rate function of the q-Weibull distribution
$h_{qf}(t)$	Hazard rate function of the q-Fréchet distribution
$L(\cdot)$	Likelihood function
$\mathcal{L}(\cdot)$	Log-likelihood function
NS	Number of simplex searches
limit	Parameter of artificial bee colony algorithm
SN	Number of employed or onlooker bees
D	Dimension of the optimization problem
MCN	Maximum cycle number
MFE	Maximum number of function evaluations
$bias(\hat{\theta}, \theta)$	Bias for parameter θ
$MSE(\hat{\theta})$	Mean squared error for parameter θ
$I(\cdot)$	Information matrix
D^0	Kolmogorov-Smirnov test statistic
$o(x)$	Little o

X_i	Lifetime of component i
d	Number of components
C	Clayton copula
$\hat{C}$	Clayton survival copula
$R_i(t)$	Marginal reliability function of component i at time t
$F_i(t)$	Marginal distribution function of component i at time t
$R_s(t)$	Reliability function of a system at time t
$F_s(t)$	Cumulative distribution function of a system at time t
$f(t_i \mid t_{i-1})$	Conditional probability density function at the i_{th} failure time t_i, given the previous failure occurred at time t_{i-1}
$\lambda(t)$	Intensity function at time t
$H(t)$	Cumulative intensity function at time t
$H_q(t)$	Cumulative intensity function of the q-Weibull distribution at time t
$P(\cdot)$	Probability function
$h_{j_i}(t)$	Hazard rate function of component j_i at time t
$R_{j_i}(t)$	Reliability function of component j_i at time t
D_0	Modified Kolmogorov-Smirnov statistic
θ	Parameter of Clayton copula
q	Shape parameter of q-Weibull distribution

Chapter 1: Introduction

1.1 Background and Motivation

Modeling and improving system reliability require selecting appropriate probability distributions for describing the uncertainty in failure times. Development, choice, and application of a probability distribution to accurately describe failure times is not a trivial task, and reliability analysis depends crucially on it. Many probability distributions can be used to model failure times. However, for some systems and components, the classical distributions are not satisfactory. It's important to remark that deviation from a given distribution is not merely quantitative, but also qualitative, once the bathtub-shaped failure rate behavior cannot be described by the classic Weibull distribution, many reliability inferences (e.g., maintenance policies, risk and cost analyses) may be inaccurate if the reliability model cannot recognize non-monotonic failure rate behavior. Besides, some characteristics of complex systems including long-range correlations are not well described by the classical distributions. For simple systems, by assuming independence, there is a well-established theoretical framework with approaches based on classical distributions. In reality, this assumption usually cannot be satisfied, and there is a dependency relationship among components of the system. In this scenario, a generalized distribution capable of providing a better description of complex systems is welcome. This is the case of the family of q-distributions which emerge from the non-extensive statistical mechanics. Concepts related to non-extensive statistical mechanics have been applied to a variety of problems in diverse research areas of complex systems, including physics, chemistry, biology, mathematics, economics, among others. In the context of non-extensive statistical mechanics, the q-distribution is a generalization of the classic distribution in the same way that the non-extensive entropy [1] is a generalization of Boltzmann-Gibbs-Shannon (BGS) entropy (using a parameter q, known as entropic index), extending statistical mechanics to complex systems. For complex systems with dependence, the q-distributions may be used to improve the description of reliability engineering problems.

The Weibull distribution is one of the most frequently used distributions in reliability engineering. In this research, we focus on its generalization known as q-Weibull distribution in the context of non-extensive statistical mechanics, and this was done by Picoli et al. [2]. The q-Weibull distribution can be used to describe complex systems with long-range interactions and long-term memory [3]. Compared to the Weibull distribution, which can only describe monotonic hazard rate functions, the q-Weibull has its advantage of containing only three parameters with flexibility to model various behaviors of the hazard rate, including the unimodal, bathtub-shaped, monotonic (monotonically decreasing, monotonically increasing) and constant. The q-Weibull probabilistic model unifies monotonic and non-monotonic hazard rate functions by using one general formula, which is flexible and elegant for failure data fitting. Such flexibility is important to accurately perform reliability analyses when failure data are characterized by non-monotonic hazard rates. For example, the well-known bathtub curve, which is widely used in reliability engineering and that can be reproduced directly by q-Weibull model using a single set of three parameters instead of three Weibull models. Additionally, the q-Weibull model can reproduce both short and long-tailed distributions [2]. The performance of the q-Weibull distribution is expected to be superior over that of the classic Weibull distribution due to its flexibility to fit failure times data and the ability to describe complex systems.

This research seeks to contribute to the insertion of q-Weibull distribution to model reliability engineering problems. The q-Weibull distribution has already been introduced in the literature, but its application in reliability engineering is limited, and its benefits have not been recognized. It is partly because parameter estimation is challenging. In this work, the parameters of the q-Weibull are estimated by the maximum likelihood (ML) method. Due to the intricate system of nonlinear derivative equations related to the log-likelihood function, analytical solution is very difficult to be obtained. Given that parameter estimation and data fitting are crucial steps for reliability analyses, a numerical approach may be employed. This work employs an artificial bee colony (ABC) algorithm [4], which is a nature-based heuristic method that does not require derivative information to solve the q-Weibull distribution ML problem.

1.2 Research Objectives

Within the scope of this research, we seek answers to the following questions:

- How well the q-Weibull distribution fits failure times data compared with classic Weibull and other Weibull generalizations?

- How can the q-Weibull distribution be used to model dependent series systems?
- How can the q-Weibull distribution be modified to model dependent parallel systems?

Therefore, the main objectives of this research are as follows:

- Demonstrate that the q-Weibull distribution is a flexible and useful distribution to describe failure time data with a variety of hazard rate behaviors, in particular data with a non-monotonic hazard rate. That allows us to propose the q-Weibull distribution is a good candidate for the existing life distributions in modeling reliability data.
- Explore the ability of q-Weibull distribution to model a series system with dependent component failure times and estimating the model's parameters from failure time data. Specifically, explore the connection between q-Weibull distribution and Clayton copula and investigate the effect of parameter q on the systems' dependency.
- Propose a q-Fréchet distribution, dual distribution to q-Weibull distribution, to model a parallel system with dependent component failure times that are modeled as a Clayton copula.
- Develop an efficient approach based on an artificial bee colony algorithm to solve all the maximum likelihood problems.

1.3 Research Approach

To achieve the first objective, we adopt the ML method to estimate the q-Weibull distribution parameters. Numerical experiments are conducted to evaluate the ability of the q-Weibull distribution to model various behaviors of the hazard rate function. The accuracy and precision of the ML estimates of the q-Weibull parameters are evaluated by bias and MSE. Intervals estimates for the q-Weibull parameters are provided, including asymptotic intervals based on the ML theory, parametric and non-parametric bootstrapped confidence intervals. The proposed parameter estimation method is also applied to an example involving failure data characterized by a bathtub-shaped hazard rate function. For comparison purposes, we consider the standard Weibull and some alternative bathtub-shaped hazard rate models: the modified Weibull extension [5] and the ENH [6] models. A modified Kolmogorov-Smirnov (KS) goodness-of-fit test statistic and p-value are used to determine the goodness-of-fit of these models.

To achieve the second objective, we analytically derive that a q-Weibull distribution can approximate the distribution of the failure time of a series system with dependent component failure times that are modeled as a Clayton survival copula. Also, we derive the relationship between the parameter q in q-Weibull distribution and the parameter θ in Clayton copula, which measures the degree of dependence among the components of a system. For a series system with minimal repair, we develop a method for estimating the parameters of the Clayton copula given data about component failures, and we show that this process can be modeled as a nonhomogeneous Poisson process (NHPP). Thus, we propose the NHPP with q-Weibull as the underlying time to first failure (TTFF) distribution model to approximate the minimal repair process of a repairable series system composed of multiple dependent components characterized by Clayton copula. The maximum likelihood (ML) method is developed to estimate the model parameters, and asymptotic confidence intervals based on ML asymptotic theory are also developed.

A simulation study is conducted to validate the proposed NHPP q-Weibull model. In the simulation, a sampling method for conditional failure times of dependent subsystems modeled by Clayton copula is developed. A modified Kolmogorov-Smirnov (KS) goodness-of-fit test statistic and p-value are used to determine the goodness-of-fit of the proposed NHPP q-Weibull model. The proposed NHPP q-Weibull model and parameter estimation procedure are applied to a real failure times data set of a load-haul-dump (LHD) machine given by Kumar et al. [7]. The proposed model is compared with other commonly used minimal repair process models, including NHPP Weibull and NHPP S-PLP [8], and the independent models.

To achieve the third objective, we propose a q-Fréchet distribution to model a dependent parallel system with dependent component failure times that are modeled as a Clayton copula. We derive that the parameter q in q-Fréchet distribution approximates the parameter θ in Clayton copula, which measures the degree of dependence among the components. One example of dependence is illustrated as common cause failures when all components' hazard rates are affected by a common randomized environmental effect. We perform a simulation study to evaluate the q-Fréchet approximation. We also apply the proposed q-Fréchet model to a data set of 18 two-motor parallel systems' failure times.

Due to the intricate likelihood function, it is impractical to analytically obtain the ML estimates for the q-Weibull parameters, and the classic numerical optimization approach fails to efficiently find the global solution for the associated ML problem. To achieve the fourth objective, we use the heuristic optimization method of artificial bee colony (ABC) algorithm. To deal with the slow convergence of ABC, we propose an adaptive hybrid ABC (AHABC) algorithm that dynamically combines a local Nelder-Mead simplex search method with ABC to efficiently solve the q-Weibull distribution ML problem. Numerical experiments are conducted to evaluate the performance of the proposed AHABC algorithm to solve the q-Weibull ML problem, comprising different behaviors of the hazard rate and sample sizes. The proposed AHABC is compared with ABC and a similar algorithm in terms of accuracy and convergence speed in the context of the maximum likelihood problem for the q-Weibull distribution.

1.4 Dissertation Outline

This dissertation is arranged into the following chapters.

Chapter 2 presents a literature review of the background and related studies in q-Weibull distribution.

Chapter 3 demonstrates that the q-Weibull distribution is a flexible and useful distribution to describe failure time data with both monotonic and non-monotonic hazard rate behaviors. This chapter develops an adaptive hybrid ABC (AHABC) algorithm to obtain the ML estimates for the q-Weibull distribution parameters.

Chapter 4 presents analytical derivation and simulation validation that a q-Weibull distribution can approximate the distribution of the failure time of a series system with dependent component failure times that are modeled as a Clayton survival copula.

Chapter 5 proposes a q-Fréchet distribution, which can be used to approximate the distribution of the failure time of a parallel system with dependent component failure times that are modeled as a Clayton copula.

Chapter 6 presents a summary of conclusions, contributions, and recommendations for future research.

The source code for this dissertation can be found https://github.com/Mengumd/q-Weibull-distribution-in-reliability

Chapter 2: Literature Review

This chapter presents related work about the main topics of the dissertation: q-Weibull distribution, other Weibull generalizations, lifetime data fitting by q-Weibull distribution, dependent systems modeling, and the numerical solution of MLE using artificial bee colony algorithm.

2.1 Characterization of q-Weibull Distribution

The probability density function (PDF) of the q-Weibull distribution is as follows:

$$f_q(t) = (2-q)\frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1} \exp_q\left[-\left(\frac{t}{\eta}\right)^{\beta}\right], t \geq 0, \qquad (2\text{-}1)$$

where $\beta > 0$ and $q < 2$ are shape parameters, and $\eta > 0$ is a scale parameter. The q-Exponential function $\exp_q(x)$ is defined as:

$$exp_q(x) = \begin{cases} (1+(1-q)x)^{\frac{1}{1-q}}, & if\ 1+(1-q)x > 0, \\ 0, & otherwise. \end{cases} \qquad (2\text{-}2)$$

Therefore, the q-Weibull PDF can be rewritten as:

$$f_q(t) = (2-q)\frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1} \left[1-(1-q)\left(\frac{t}{\eta}\right)^{\beta}\right]^{\frac{1}{1-q}}, t \geq 0, \qquad (2\text{-}3)$$

where

$$t \in \begin{cases} [0, \infty), & \text{for } 1 < q < 2, \\ [0, t_{max}], & \text{for } q < 1, \end{cases} \tag{2-4}$$

with $t_{max} = \frac{\eta}{(1-q)^{1/\beta}}$.

In the limit $q \to 1$, $f_q(t)$ reduces to the Weibull PDF, for $\beta = 1$ it corresponds to the q-Exponential PDF, and when $q \to 1$ and $\beta = 1$ it becomes the Exponential distribution [9]. The q-Weibull cumulative distribution function (CDF) and reliability function are as follows:

$$F_q(t) = 1 - \left[1 - (1-q)\left(\frac{t}{\eta}\right)^{\beta}\right]^{\frac{2-q}{1-q}}, \tag{2-5}$$

$$R_q(t) = \left[1 - (1-q)\left(\frac{t}{\eta}\right)^{\beta}\right]^{\frac{2-q}{1-q}}. \tag{2-6}$$

Then, the hazard rate function is defined as:

$$h_q(t) = \frac{f_q(t)}{R_q(t)} = \frac{(2-q)\frac{\beta}{\eta^{\beta}} t^{\beta-1}}{1 - (1-q)\left(\frac{t}{\eta}\right)^{\beta}}. \tag{2-7}$$

Equation (2-7) can represent different types of hazard rate functions according to the values of the shape parameters [3]. Indeed, Assis et al. [3] provided the ranges of the shape parameters q and β related to each type of curve. $h_q(t)$ is monotonically decreasing for $1 < q < 2$ and $0 < \beta < 1$, monotonically increasing for $q < 1$ and $\beta > 1$, unimodal for $1 < q < 2$ and $\beta > 1$, and bathtub-shaped for $q < 1$ and $0 < \beta < 1$.

Figure 2-1 illustrates the different behaviors of $h_q(t)$ for $\eta = 5$ and specific values of the shape parameters q and β. Note that for $q = 0.5$ $(q < 1)$, $h_q(t)$ – as well as $f_q(t)$, $F_q(t)$ and $R_q(t)$ – has a limited support. For the cases $\beta = 0.5$ with $q = 0.5$ and $\beta = 1.5$ with $q = 0.5$ depicted in Figure 2-1, t_{max} is 20 and 7.937, respectively.

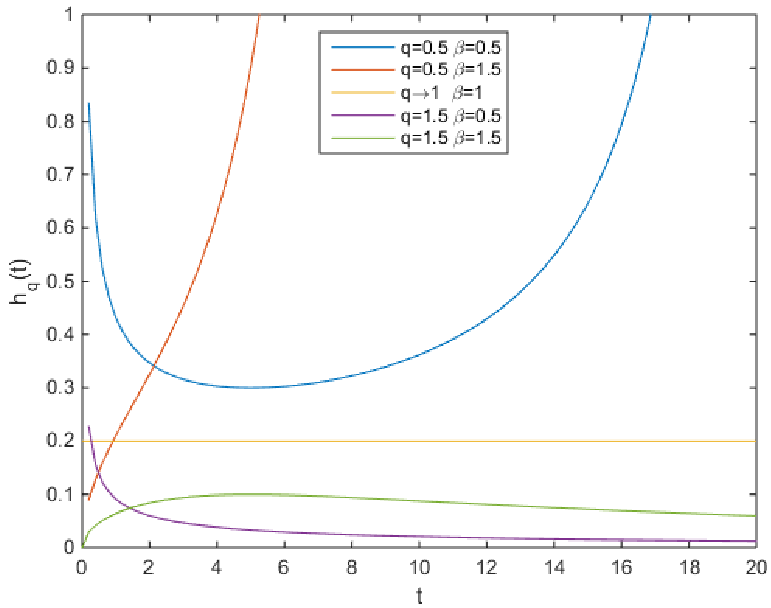

Figure 2-1: Behaviors of the q-Weibull hazard rate function for $\eta = 5$ and different values of the shape parameters q and β

Moreover, random samples may be generated according to the q-Weibull distribution by inverting $F_q(t)$. Indeed, the q-Weibull random number generator is obtained as:

$$t = \eta \left\{ \frac{\left[1 - U^{\frac{1-q}{2-q}}\right]}{1-q} \right\}^{\frac{1}{\beta}}, \tag{2-8}$$

where U is a uniform random number in [0, 1].

Suppose that an item has survived to the time t_0, then the q-Weibull conditional reliability function is given by:

$$R_q(t|t_0) = \frac{R_q(t_0 + t)}{R_q(t_0)} = \left[\frac{1 - (1-q)\left(\frac{t_0 + t}{\eta}\right)^{\beta}}{1 - (1-q)\left(\frac{t_0}{\eta}\right)^{\beta}}\right]^{\frac{2-q}{1-q}}. \tag{2-9}$$

Sometimes, it is convenient to rewrite the reliability function of q-Weibull distribution in Equation (2-6) as:

$$R_q(t) = \exp_{q'}\left[-\left(\frac{t}{\eta'}\right)^{\beta}\right] = \left[1 - (1-q')\left(\frac{t}{\eta'}\right)^{\beta}\right]^{\frac{1}{1-q'}}. \tag{2-10}$$

Where $q' = \frac{1}{2-q}$, $\eta' = \frac{\eta}{(2-q)^{\frac{1}{\beta}}}$. The parameter q' comes from the cumulative distribution function (CDF) form of q-Weibull distribution; the corresponding parameter from the probability density function (PDF) is q.

2.2 Generalizations of Weibull Distribution

The Weibull distribution has been modified or generalized in different ways to allow for non-monotonic hazard rate functions. For instance, Murthy et al. [10] provide a taxonomy to integrate the different Weibull models. There are some recent Weibull distribution extensions in the reliability engineering literature. Pham and Lai [11], and Almalki and Nadarajah [12] reviewed the generalizations and modifications of the Weibull distribution.

These models are capable of modeling a bathtub-shaped hazard rate functions and can be classified into two categories: i) methods that add parameters to an existing distribution to obtain classes of more flexible distributions as introduced by Olkin [13], and ii) methods that combine two or more distributions with one or more being Weibull. Examples include the IDB model (Hjorth [14]), the exponentiated Weibull (EW) distribution (Mudholkar and Srivastava [15]), the generalized Weibull (GW) (Mudholkar and Kollia [16]), the additive Weibull (AW) distribution (Xie and Lai [17]), the extended Weibull distribution (Marshall and Olkin [18]), the modified Weibull (MW) distribution (Lai et al. [19]), the modified Weibull extension (MWE) (Xie et al. [5]), the beta Weibull (BW) distribution (Lee et al. [20]), the flexible Weibull extension (FWE) (Bebbington et al. [21]), the generalized modified Weibull (GMW) distribution (Carrasco et al. [22]), the ENH distribution (Lemonte [6]), the additive modified Weibull (AMW) distribution (He et al. [23]), and the generalized modified Weibull power series (GMWPS) distribution (Bagheri et al. [24]). There are also models involving two or more Weibull distributions, for example, sectional method, competing risk approach, and multiplicative model introduced by Jiang and Murthy [25].

2.3 Lifetime Data Fitting by q-Weibull Distribution

q-Weibull distribution is a generalization of Weibull distribution in the context of non-extensive statistical mechanics, and it has been successfully applied to model lifetime data in the context of reliability engineering. For example, Costa et al. [26] used q-Weibull distribution to properly describe time-to-breakdown data of electronic devices; Sartori et al. [27] considered a q-Weibull distribution to describe the failure rate of a compression unit in a typical natural gas recovery plant based on time-to-failure data.

The q-Weibull distribution parameters have been estimated via the least squares estimation (LSE) procedure (see Picoli et al. [2]) or through square correlation coefficient R^2 maximization (in Sartori et al. [27] and Assis et al. [3]). Jose and Naik [28] provided the likelihood function, but claimed that it is very difficult to obtain the maximum likelihood (ML) estimates of the parameters due to the nonlinear set of equations. Alternatively, Jose and Naik [28] employed the method of moments stating, however, that the moment estimates are not easy to evaluate when all the parameters are unknown. Extensive simulation studies have shown the ML method is better than the LSE in reliability applications when data sets are typically small or moderate in size [29]. Since the distribution of ML parameter estimates are more accurate with smaller variance, we here adopt the ML method.

2.4 Dependent Systems Modeling by q-Distribution and Clayton Copula

This q-Weibull flexibility is related to the parameter q, which controls the shape of the distribution along with the parameter β, while the Weibull distribution has just one parameter β affecting its shape. Besides interpreting parameter q to be a shape parameter, this research goes further to explore the meaning of parameter q. The shape parameter q is related to the entropic index in the context of Tsallis statistics [1]. The q-entropy proposed by Tsallis [1] $S_q = \frac{1-\sum_{i=1}^{W} p_i^q}{1-q}$. Here, p_i is the probability of the ith state, W is the number of accessible states of the system, and q is a real parameter that rules the degree of generalization of the theory (when $q \to 1$, the standard Boltzmann-Gibbs entropy is recovered). The maximization of S_q subject to specific constraints generates q-distributions such as q-exponential, q-Gaussian and q-Weibull. Costa et al. [30] interpreted parameter q occurring in the Tsallis statistics to be entirely induced by the fluctuations of the parameter characterized by Gamma function. To the best of our knowledge, it has not been realized that parameter q is connected with copula, thus q-Weibull distribution is able to model complex system with dependence. Specifically, this research explores the connection of q-Weibull distribution with Clayton copula, which is one of the most important Archimedean copulas for the dependence structure of random vectors.

A copula is a useful tool for handling multivariate distributions with given univariate marginals. A copula is a distribution function, defined on the unit cube$[0, 1]^n$, with uniform one-dimensional marginals. For continuous multivariate distribution functions, the univariate marginals and multivariate dependence structure can be separated, and the dependence structure can be represented by a copula. The copula was first developed by Sklar [32], according to the Sklar's theorem [32], every multivariate distribution admits a representation in terms of a copula and a set of marginal distributions. The copula theory and its applications can be found, for example, in Nelsen [33]. In the context of reliability, the survival copula denoted by $\hat{C}$ is more effective. Clayton [34] is one of the first to propose a bivariate association model for survival analysis. Without knowing the concept of copulas, the implicit survival copula associated with the Clayton model is $\hat{C}(u_1, u_2)) =$

$\left(u_1^{1-\vartheta} + u_2^{1-\vartheta} - 1\right)^{-\frac{1}{\vartheta-1}}$, where $\vartheta > 0$, Oakes [35] explicitly showed this bivariate survival copula by reparameterization of Clayton model. This copula function is a special case of the multivariate Cook-Johnson [36] copula. For this reason, many people referred to this copula as Clayton or Cook-Johnson copula, in this research, we prefer to name it Clayton copula. In this research, we find that q-Weibull distribution can model a series system with dependence characterized by the Clayton copula, which is the multivariate survival copula $\hat{C}(u_1, u_2, \dots, u_d) = \left(\sum_{i=1}^{d} u_i^{-\theta} - d + 1\right)^{-1/\theta}$, where the case $\theta > 0$ can be used to construct a copula in any dimension; the case $\theta = 0$ constructs the independence copula in any dimension; in the case $\theta < 0$, for dimension $d \geq 2$, $\theta \geq -1/(d-1)$ [37].

2.5 *Numerical Solution of MLE: Artificial Bee Colony Algorithm*

In this research, we employ an artificial bee colony (ABC) algorithm, which is a nature-based heuristic method that does not require derivative information to solve the maximum likelihood problems. ABC was introduced by Karaboga [4] and is an optimization algorithm based on the intelligent foraging behavior of honey bee swarm for optimizing multidimensional and multimodal numerical functions. In ABC, a swarm of employed bees, onlooker bees, and scouts are generated, and the swarm moves in a search space of possible solutions for an optimization problem. The global minimum of the objective function can be obtained from the bee interactions. The performance of ABC has been compared to other well-known modern heuristic algorithms such as Genetic Algorithm (GA), Differential Evolution (DE), Particle Swarm Optimization (PSO), and Evolution

Strategies (ES) [38] [39] [40]. Results show that ABC is better than or at least comparable to these population-based algorithms with the advantage of employing fewer control parameters. Due to its simple structure, easy implementation and outstanding performance, ABC has received significant interest from researchers of different areas and has been successfully applied in many optimization problems [41].

However, the convergence performance of ABC for local search is slow due to its solution search method, which is good at exploration but poor at exploitation [42]. In order to improve its performance, some modified versions of ABC have been proposed in the literature. For instance, inspired by Particle Swarm Optimization (PSO), Zhu and Kwong [42] developed an improved ABC algorithm named gbest-guided ABC (GABC) by incorporating the information of global best solution into the solution search equation to improve exploitation. Kang et al. [43] proposed a Hooke-Jeeves ABC (HABC) algorithm that combines Hooke-Jeeves pattern search with ABC algorithm. In the HABC, the exploration phase is performed by ABC, and the exploitation stage is completed by pattern search. Karaboga and Gorkemli [44] adopted the Quick ABC (qABC), which models the behavior of onlooker bees more accurately and improves the performance of standard ABC in terms of local search ability. In order to achieve an optimization performance with higher convergence speed and an improved exploitation capacity, Shan et al. [45] used a self-adaptive hybrid artificial bee colony (SAHABC) algorithm inspired by self-adaptive mechanism, DE, and PSO algorithm. In the SAHABC, the search equation for employed bees is modified based on the self-adaptive mechanism, which is used to balance the exploration ability and the convergence speed of ABC, and DE mutation strategy, which uses the best solution to improve convergence performance. The search equation for onlooker bees is modified based on PSO to improve the exploitation ability. Kang et al. [46] proposed a hybrid simplex ABC algorithm (HSABCA) that combines Nelder-Mead simplex method with artificial bee colony algorithm for inverse analysis problems. The HSABCA was applied to parameter identification of concrete dam-foundation systems. The Nelder-Mead simplex algorithm proposed by Nelder and Mead [47] is an efficient local search method. It was also combined with other heuristic to improve the convergence accuracy and speed. For example, Fan and Zahara [48] proposed the hybrid NM-PSO algorithm based on the Nelder-Mead simplex search method and PSO for unconstrained optimization.

2.6 Summary

Using the q-Weibull distribution for reliability analysis is a step towards an efficient approach to handle equipment failure time data dismissing previous limitations in terms of modeling the whole failure rate behavior, specifically when unimodal or bathtub-shaped ones are presented. More than an alternative to the existing life distributions in modeling reliability data, the q-Weibull has the advantage of being originated from a theoretical background rooted in non-extensive statistical mechanics. The flexibility of q-Weibull distribution allows decisions about reliability, maintenance planning, and evaluation to be performed more accurately. It is proposed in this research that the q-Weibull distribution can be considered the main distribution in some situations for complex systems.

In this research, we adopt the ML method to estimate the model parameters due to the good statistical properties of the resulting estimators. The obtained estimators through ML are approximately unbiased, and its variance is nearly as small as the variance resulting from other estimators. However, the application of ML on q-Weibull distribution presents some challenges: the first derivative equations of the related log-likelihood function are highly nonlinear, and the equations do not have analytical solutions for the parameters' estimators. Such a difficulty can explain the limited number of applications based on the q-Weibull model given that parameter estimation, and data fitting are crucial steps for reliability analyses.

A method that does not depend on derivative, but also presents fast convergence is necessary for the q-Weibull distribution ML optimization problem. This research employs an artificial bee colony (ABC) algorithm, which is a nature-based heuristic method that does not require derivative information to solve the q-Weibull distribution ML problem. To deal with the slow convergence of ABC, this research proposes to develop an adaptive hybrid ABC (AHABC) algorithm that dynamically combines a local Nelder-Mead simplex search method with ABC for the ML estimation of the q-Weibull parameters. Differently from HSABCA proposed by Kang et al. [46], AHABC dynamically controls the exploration and exploitation, given that the parameter for Nelder-Mead local search is adaptively tuned according to the search status. AHABC is also different from SAHABC in terms of the hybrid strategy and adaptive mechanism. The proposed AHABC is an efficient manner to tackle the difficult ML problem related to the q-Weibull distribution comprising different behaviors of the hazard rate function.

With the new algorithm in hand, we can efficiently obtain the parameter estimation for q-Weibull distribution and compare the q-Weibull distribution with classic Weibull distribution and other Weibull generalizations in fitting failure times data.

In the literature, the reliability applications of q-Weibull distribution are limited. To the best of our knowledge, no previous work has shown that the shape parameter q in a q-Weibull distribution is equivalent to a parameter of the Clayton copula and that the q-Weibull distribution is able to model a system with dependent component failure times. Specifically, this research explores the connection between the q-Weibull distribution and the Clayton copula, which is one of the most important Archimedean copulas for the dependence structure of random vectors. This research investigates the effect of the parameter q on the dependence of the system. We expect that the performance of the q-Weibull distribution is superior over that of the classic Weibull distribution due to its flexibility to fit failure times data and the ability to describe complex systems.

Chapter 3: Lifetime Data Fitting

3.1 Overview

This chapter[1] proposes to demonstrate that the q-Weibull distribution is a flexible and useful distribution to describe failure time data with a variety of hazard rate behaviors, in particular data with non-monotonic hazard rate. We adopt the ML method to estimate the life distribution parameters. We propose to develop an adaptive hybrid ABC (AHABC) algorithm that dynamically combines a local Nelder-Mead simplex search method with ABC to efficiently solve the q-Weibull distribution ML problem. Numerical experiments are conducted to evaluate the performance of the proposed AHABC algorithm to solve the q-Weibull ML problem, comprising different behaviors of the hazard rate and sample sizes. The proposed method is also applied to an example involving failure data characterized by a bathtub-shaped hazard rate function.

[1] The full-text of this chapter entitled "On the q-Weibull distribution for reliability applications: An adaptive hybrid artificial bee colony algorithm for parameter estimation" has been published in the Journal of Reliability Engineering & System Safety. Volume 158, February 2017, Pages 93-105. https://doi.org/10.1016/j.ress.2016.10.012

3.2 *Maximum Likelihood Constrained Problem*

In this section, the parameters of the q-Weibull distribution are estimated via the ML method. Let $\underline{t} = (t_1, t_2, \dots, t_n)$ be an n-dimensional vector of observed failure times $t_i, i = 1, \dots, n$, independently drawn from a q-Weibull distribution. The likelihood function is given by:

$$L(\underline{t}|\eta,\beta,q) = \prod_{i=1}^{n} f_q(t_i)$$
$$= \prod_{i=1}^{n} (2-q)\frac{\beta}{\eta}\left(\frac{t_i}{\eta}\right)^{\beta-1}\left[1-(1-q)\left(\frac{t_i}{\eta}\right)^{\beta}\right]^{\frac{1}{1-q}}. \tag{3-1}$$

The log-likelihood function is as follows:

$$\mathcal{L}(\underline{t}|\eta,\beta,q) = n ln(2-q) + n ln(\beta) - n\beta\ln(\eta) + (\beta - 1)\sum_{i=1}^{n}\ln(t_i) + \frac{1}{1-q}\sum_{i=1}^{n}\ln\left[1-(1-q)\left(\frac{t_i}{\eta}\right)^{\beta}\right]. \tag{3-2}$$

Considering the constraints of parameters and the support, the constrained optimization problem is:

$$\max \quad n\ln(2-q) + n\ln(\beta) - n\beta\ln(\eta) + (\beta-1)\sum_{i=1}^{n}\ln(t_i) + \frac{1}{1-q}\sum_{i=1}^{n}\ln\left[1-(1-q)\left(\frac{t_i}{\eta}\right)^{\beta}\right], \tag{3-3}$$

$$\text{s.t.} \quad 2-q > 0, \tag{3-4}$$

$$1-(1-q)\left(\frac{t_i}{\eta}\right)^{\beta}>0\,, i=1,\dots,n, \tag{3-5}$$

$$\eta>0, \tag{3-6}$$

$$\beta>0. \tag{3-7}$$

The first derivatives of the log-likelihood function w.r.t. parameters are nonlinear, and analytical solutions are very difficult to be obtained. A heuristic based constrained optimization method can be applied to tackle this problem. In this research, the ML estimates $\hat{\eta}, \hat{\beta}$ and $\hat{q}$ are obtained using an adaptive hybrid artificial bee colony (AHABC) algorithm developed in the next section.

3.3 Proposed Adaptive Hybrid Artificial Bee Colony Algorithm

In the ABC algorithm, while onlookers and employed bees carry out the exploitation process in the search space, the scouts control the exploration process [4]. However, the original ABC is good at exploration but bad at exploitation for numerical benchmark functions optimization [42]. From our simulation experiments for ML estimation of the q-Weibull parameters by ABC (see Section 3.4.2), we also observe similar results: although ABC could find the global optimum, the estimates' variability is large due to the slow convergence speed of ABC for local search.

Thus, in order to make full use of ABC's exploration, and avoid its drawbacks, an adaptive hybrid ABC is proposed that incorporates a local search stage. The main idea of AHABC is that through adaptively tuning the parameters of hybrid ABC according to the search process, the hybrid ABC will gradually change from the global ABC search pattern to the local search pattern. The general AHABC framework is shown in Figure 3-1. The details of the proposed AHABC algorithm are presented in the following subsections.

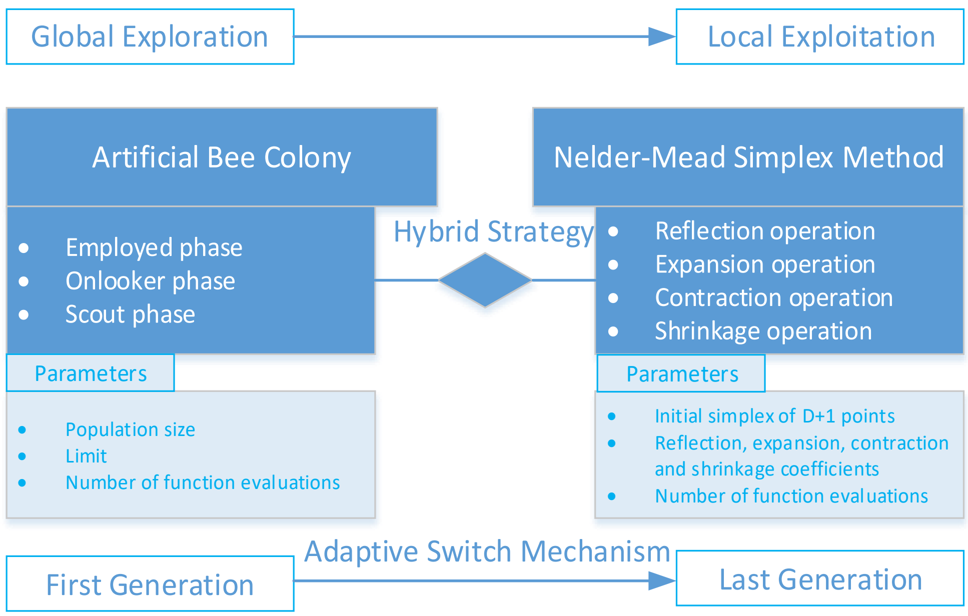

Figure 3-1: Framework of adaptive hybrid ABC

3.3.1 Hybrid Strategy

"Hybrid Strategy" is the method to combine ABC with a local search algorithm. There are two common types of hybrid strategies: i) selectively applying either ABC or local search, which means that for a certain population, the next generation is given by ABC or local search method; ii) merging the local search into ABC, which means that the local search is incorporated into ABC as an operation or a phase.

In the proposed algorithm, we adopt the second hybrid strategy. Nelder-Mead simplex search is chosen as the local search method and is added to ABC as an additional step after the original three phases and within every iteration. This method rescales the simplex by four procedures: reflection, expansion, contraction and shrinkage. The input of local search phase is the best $D + 1$ solutions in the population, where D is the dimension of the optimization problem, as shown in Figure 3-2. Then, three candidate solutions are generated and evaluated. If the best of these new solutions can outperform the worst solution in the current simplex, this new solution replaces the worst one (see Figure 3-2). Otherwise, the current simplex shrinks towards the best solution in the current simplex (see Figure 3-2). These solutions will be exploited by the Nelder-Mead simplex local search for a number of function evaluations NS.

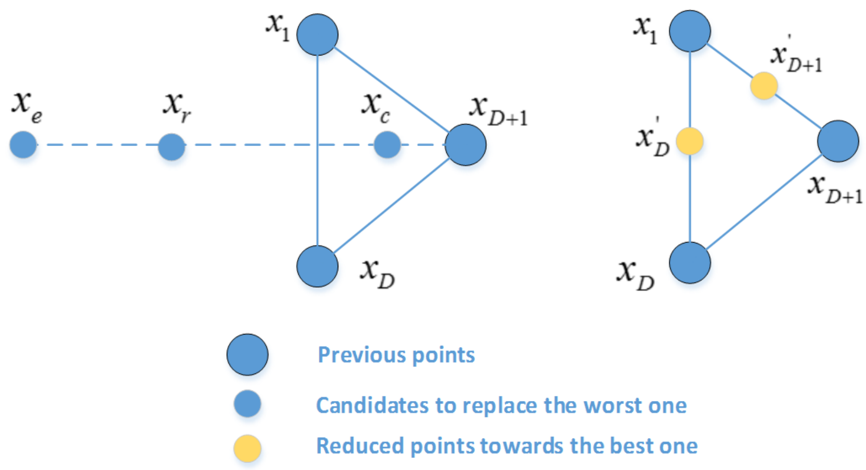

Figure 3-2: Scheme of simplex search

3.3.2 Adaptive Switch Mechanism

"Adaptive switch mechanism" determines how the hybrid algorithm changes from global exploration to local exploitation. The principle of "adaptive switch mechanism" is to gradually increase the use of local search by tuning algorithm parameters according to the search process. These tunable parameters are search space-related, i.e., changing their values will modify the search property (more global or more local).

In this paper, we adaptively increase the number of simplex searches, and the searching process becomes more local. The remaining challenge is how to determine the number of simplex searches NS. We propose the following formula:

$$NS = C * \text{limit} * \text{total number of scout bees}. \tag{3-8}$$

Firstly, since the total number of scout bees increases over ABC iterations, this definition of NS will guarantee that NS is non-decreasing, which means the search process will become more and more local. Secondly, the total number of scout bees is a symbol of search status. A large number of scout bees indicates that a significant portion of the solution space has been explored, that the exploration is becoming inefficient and local exploitation is becoming urgent. Also, the "limit" is an important ABC parameter, which controls the scout bee generation frequency. C is a coefficient that controls the amount of local search. For the q-Weibull distribution ML optimization problem, $C = 1$ provided an acceptable convergence speed (shown in Section 3.4.1). Thus, we use the product of limit and the total number of scout bees as the number of function evaluations within the local search phase of the AHABC. In summary, NS dynamically increases along the search process and it gradually changes from global to local.

3.3.3 Constraints

For the constraints (3-4) to (3-7) related to the q-Weibull ML problem, we adopt the "throw away" approach, which means that if the generated solution is not feasible, we throw it away and keep the current solution. This is a simplified Deb's rule [49] that involves domination rules between solutions. In our proposed algorithm, we do not allow infeasible solutions in the population, and once an infeasible one is generated, we consider it as inferior to its previous solution and throw it away.

3.3.4 Proposed Algorithm

The pseudo-code of the proposed AHABC algorithm is given in Figure 3-3.

01: Initialize a randomly distributed population of SN candidate solutions $x_{i,j}, i = 1 \ldots SN, j = 1 \ldots D$
02: Set number of function evaluations $k = 0$, number of scout bees $s = 0$
03: Evaluate the fitness of the initial population, $k = k + 1$ for each evaluation
04: **Repeat**
(The employed bees phase)
05: Each employed bee produces a candidate solution v_i
06: Evaluate the fitness of v_i, and apply the greedy selection to compare x_i and v_i $k = k + 1$ for each evaluation
07: Calculate the probability values p_i for solutions x_i
(The onlooker bees phase)
08: Each onlooker produces a candidate solutions v_i for x_i depending on p_i
09: Evaluate the fitness of v_i, and apply the greedy selection to compare x_i and v_i $k = k + 1$ for each evaluation
(The scout bees phase)
10: When a solution cannot be improved further through a predetermined number of cycles, called "limit", then that solution is abandoned and replaced with a new solution v_i randomly generated by a scout bee. Evaluate the fitness of v_i, $s = s + 1, k = k + 1$.
(Adaptively tune parameters)
11: Calculate the number of function evaluations for Nelder-Mead local search according to the formula $NS = C * limit * s$
12: Rank the solutions in the population according to fitness values
(Nelder-Mead local search phase)
13: Select the best $D + 1$ solutions to generate the initial simplex
14: Set number of function evaluations for local search $k' = 0$
15: **Repeat**
16: apply Nelder-Mead operators: reflection, expansion, contraction and shrinkage, $k' = k' + 1, k = k + 1$ for each function evaluation.
17: **Until** $k' = NS$ or termination criteria
18: Memorize the best solution achieved so far
19: **Until** $k = MFE$ or termination criteria

Figure 3-3: Pseudo-code of AHABC

There are three commonly used control parameters in the standard ABC: the number of food sources, which is equal to the number of employed or onlooker bees (SN); the value of $limit$, which can be obtained from the formula $limit = SN * D$ [4], where D is dimension of the optimization problem; and the maximum cycle number (MCN).

In the AHABC algorithm, one iteration cycle incorporates iterations of the Nelder-Mead local search. Instead of separately setting the iteration numbers for ABC and Nelder-Mead local search, we use only one parameter of maximum number of function evaluations (MFE), totaling all the ABC and Nelder-Mead local search function evaluations. The number of function evaluations for Nelder-Mead local search is set by Equation (3-8), which is adaptively tuned according to the search process.

There are three stop criteria employed in the AHABC algorithm:

Maximum number of function evaluations (MFE).

1) The global best solution is the same for $maxBestTrial$ times. In this case, the iteration number in which the best solution has been found is used.
2) The global best objective function value in two consecutive iterations are different, but such a difference is less than a predefined tolerance ε.

3.4 *Validation of AHABC by Numerical Experiments*

The proposed AHABC was coded in MATLAB environment, and simulation experiments were conducted to evaluate its performance. The experimental settings (ES) cover different behaviors of the q-Weibull hazard rate for reliability applications, as they involve different value combinations of the shape parameters q and β. Note that for all ES, $\eta = 5$. Table 3-1 shows the ES, the q and β values as well as the corresponding hazard rate function behavior.

Table 3-1: Experimental settings

ES	q	β	Behavior of hazard rate function
A	0.5	0.5	Bathtub-shaped
B	1.5	0.5	Decreasing
C	1	1	Constant
D	0.5	1.5	Increasing
E	1.5	1.5	Unimodal

Sample sizes of 100, 500 and 1000 are taken into consideration. Samples for ES-A, B, D, and E were generated by Equation (2-8), whereas ES-C samples were directly drawn from the inverse transform of the Exponential cumulative distribution [50]. The parameters' values used in the AHABC simulation experiments are shown in Table 3-2.

Table 3-2: AHABC parameters

Part of AHABC approach	Parameter	Value
ABC	SN	50
	$limit$	150
	MFE	200,000
	$maxBestTrial$	1000
	ε	1e-16
Nelder-Mead simplex method	α	1
	γ	2
	ρ	-0.5
	δ	0.5
Adaptive hybrid coefficient	C	1

The initial intervals for q, β and η are set to [-10, 1.9], [0.1, 10], [0.1,t_{mean}], respectively, where t_{mean} is the mean of the sample. The initial population of SN solutions is randomly generated between these intervals. In the initialization, we also adopt the "throw away" method to ensure that all the initial solutions are feasible.

3.4.1 Effect of Parameter C on AHABC

The effect of parameter C on AHABC is tested on ES-A with sample size $n = 100$. Parameter C is set to 0, 0.5, 1, 1.5, 2, 25, and 125. To assess the convergence performance of AHABC, we take the difference between the objective function value $\mathcal{L}\left(\underline{t}\middle|\hat{\eta},\hat{\beta},\hat{q}\right)$ and the true optimum value as the convergence performance. Since the true parameters of the sample are unknown, we take the best objective function value max $\{\mathcal{L}\left(\underline{t}\middle|\hat{\eta},\hat{\beta},\hat{q}\right)\}$ found among 30 replication runs as the true optimum value. The mean and the standard deviations of this difference for 30 replication runs are shown in Figure 3-4 and Figure 3-5, respectively.

The results reveal that a proper value of C can improve the performance of AHABC by providing faster convergence and more accurate solutions. It is observed that both for $C = 1$ and $C = 125$, satisfactory convergence can be obtained. For the sake of simplicity, $C = 1$ is adopted in the subsequent experiments. Thus, Equation (3-8) for the number of function evaluations for local search can be simplified to $NS = \text{limit} * \text{total number of scout bees}$.

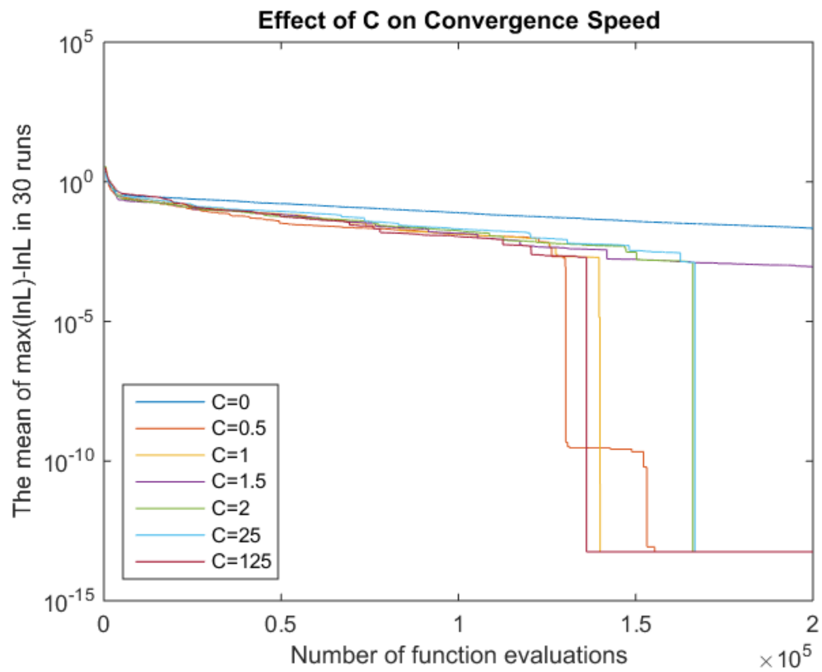

Figure 3-4: Effect of C on convergence speed

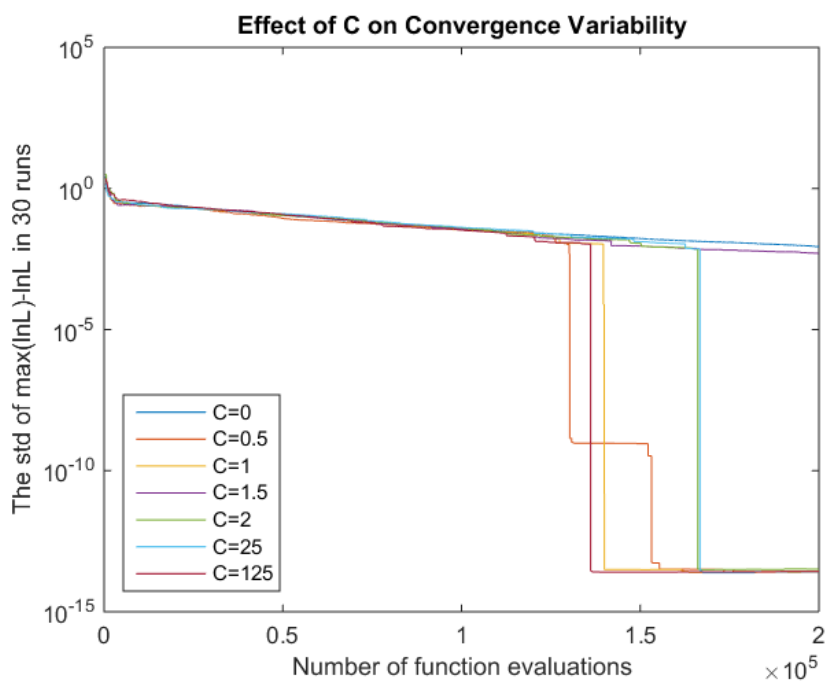

Figure 3-5: Effect of C on convergence variability

3.4.2 Comparison with ABC and SAHABC

The proposed AHABC algorithm is compared with the standard ABC and with SAHABC for the q-Weibull ML problem in terms of variability and convergence speed. AHABC uses the same parameters given in Table 3-2, ABC is fed with the parameters' values of the ABC part, also shown in Table 3-2, and SAHABC uses the parameters provided in Table 3-3. The algorithms are replicated 30 times for each sample (with $n = 100, 500, 1000$) and ES (A, B, C, D, E), which yields 15 different scenarios. The mean and standard deviations of ML estimates for parameters q, β, η, as well as log-likelihood function $\mathcal{L}$ over 30 runs are shown in Table 3-4.

For a given sample size and an ES, AHABC can provide accurate estimates for the parameters and the log-likelihood. Indeed, as we can see in Table 3-4, all the standard deviations for parameters estimates are in the order of 10^{-6} or less, and for the log-likelihood in the order of 10^{-12} or less. The mean values of the parameter estimates are close to the true values of the q-Weibull distribution shown in Table 3-1.

By comparing the results from AHABC, ABC, and SAHABC in Table 3-4, the best result for each scenario is highlighted in grey, and it is clear that most of the standard deviations for parameters and the log-likelihood by AHABC are smaller than those provided by ABC and SAHABC algorithms. These results indicate that AHABC can give more accurate estimates than both ABC and SAHABC. We also compare the convergence speed for ES-

A and E (see Figure 3-6 and Figure 3-7). AHABC converges faster than ABC and SAHABC in both cases. Therefore, one can expect the proposed AHABC to be more efficient and to provide better solutions than ABC and SAHABC for the q-Weibull ML optimization problem.

Table 3-3: SAHABC parameters

Parameters	Values
SN	50
limit	150
MCN	2,000
maxBestTrial	1,000
ε	1e-16

Table 3-4: ML estimates for 30 replications of AHABC, ABC and SAHABC

			AHABC		ABC		SAHABC	
Sample size	ES	Statistic	Mean	Std.	Mean	Std.	Mean	Std.
n=100	A	$\hat{q}$	0.4700	4.65E-08	0.5616	0.0471	0.4681	0.0112
		$\hat{\beta}$	0.5497	7.96E-09	0.5642	0.0069	0.5494	0.0017
		$\hat{\eta}$	5.4926	5.16E-07	4.5407	0.5390	5.5153	0.1304
		$\mathcal{L}$	-158.2313	2.42E-14	-158.2537	0.0066	-158.2317	0.0004
	B	$\hat{q}$	1.4236	1.36E-08	1.4236	4.58E-08	1.4236	2.8624e-08
		$\hat{\beta}$	0.4556	6.83E-09	0.4556	2.39E-08	0.4556	1.8592e-08
		$\hat{\eta}$	5.9228	5.95E-07	5.9229	1.62E-06	5.9228	1.2351e-06
		$\mathcal{L}$	-531.0407	3.00E-13	-531.0407	3.17E-13	-531.0407	1.3352e-13
	C	$\hat{q}$	0.9926	2.27E-08	0.9928	4.47E-05	0.9942	0.0041
		$\hat{\beta}$	0.9735	1.46E-08	0.9736	2.86E-05	0.9743	0.0026
		$\hat{\eta}$	4.9522	2.01E-07	4.9508	0.0004	4.9376	0.0393

		$\mathcal{L}$	-259.5912	4.95E-14	-259.5912	1.98E-07	-259.5916	0.0005
	D	$\hat{q}$	0.5583	2.65E-08	0.5711	0.0060	0.5549	0.0186
		$\hat{\beta}$	1.4949	1.96E-08	1.5013	0.0035	1.4932	0.0101
		$\hat{\eta}$	4.8668	1.03E-07	4.8141	0.0253	4.8817	0.0782
		$\mathcal{L}$	-186.6367	1.11E-13	-186.6379	0.0004	-186.6394	0.0050
	E	$\hat{q}$	1.5853	9.78E-09	1.5853	4.38E-08	1.5853	2.4355e-06
		$\hat{\beta}$	1.6157	3.21E-08	1.6157	1.31E-07	1.6157	6.4441e-06
		$\hat{\eta}$	4.4819	1.16E-07	4.4819	5.02E-07	4.4819	2.6417e-05
		$\mathcal{L}$	-387.1655	6.06E-14	-387.1655	2.05E-13	-387.1655	1.9181e-09
n=500	A	$\hat{q}$	0.5338	3.35E-08	0.5474	0.0021	0.5328	0.0148
		$\hat{\beta}$	0.5115	7.43E-09	0.5139	0.0004	0.5114	0.0028
		$\hat{\eta}$	4.2319	3.58E-07	4.0921	0.0193	4.2467	0.1491
		$\mathcal{L}$	-677.2932	1.48E-13	-677.2990	0.0014	-677.3011	0.0083
	B	$\hat{q}$	1.4665	2.08E-08	1.4665	4.90E-08	1.4665	4.0271e-08

		$\hat{\beta}$	0.4790	1.79E-08	0.4790	3.71E-08	0.4790	3.1588e-08
		$\hat{\eta}$	5.9459	8.30E-07	5.9459	2.11E-06	5.9459	1.2849e-06
		$\mathcal{L}$	-2807.6055	6.03E-13	-2807.6055	2.57E-12	-2807.6055	1.3695e-12
	C	$\hat{q}$	1.0429	1.06E-07	1.0429	5.65E-07	1.0427	0.0011
		$\hat{\beta}$	1.0509	6.27E-08	1.0509	4.61E-07	1.0508	0.0009
		$\hat{\eta}$	4.6741	9.26E-07	4.6741	4.93E-06	4.6767	0.0089
		$\mathcal{L}$	-1302.9980	7.24E-12	-1302.9980	2.40E-10	-1302.9982	0.0004
	D	$\hat{q}$	0.4999	3.44E-08	0.5072	0.0009	0.4972	0.0222
		$\hat{\beta}$	1.5091	2.66E-08	1.5131	0.0005	1.5081	0.0120
		$\hat{\eta}$	5.0255	1.34E-07	4.9949	0.0038	5.0391	0.0929
		$\mathcal{L}$	-924.4560	5.39E-13	-924.4581	0.0005	-924.4776	0.0236
	E	$\hat{q}$	1.5044	1.38E-08	1.5044	1.34E-07	1.5043	9.2122e-05
		$\hat{\beta}$	1.5687	3.98E-08	1.5687	3.01E-07	1.5687	0.0002
		$\hat{\eta}$	5.2296	1.80E-07	5.2296	1.51E-06	5.2298	0.0008

		$\mathcal{L}$	-1831.8860	9.76E-13	-1831.8860	3.19E-12	-1831.8860	1.5732e-05
n=1000	A	$\hat{q}$	0.5519	3.31E-08	0.5582	0.0020	0.5509	0.0109
		$\hat{\beta}$	0.5048	7.87E-09	0.5059	0.0004	0.5045	0.0024
		$\hat{\eta}$	4.5260	4.09E-07	4.4546	0.0224	4.5408	0.1242
		$\mathcal{L}$	-1438.8339	7.67E-13	-1438.8371	0.0009	-1438.8471	0.0208
	B	$\hat{q}$	1.5040	2.31E-08	1.5040	4.07E-08	1.5040	4.7864e-08
		$\hat{\beta}$	0.5035	2.28E-08	0.5035	3.28E-08	0.5035	4.7886e-08
		$\hat{\eta}$	4.5311	7.36E-07	4.5311	1.22E-06	4.5311	1.4466e-06
		$\mathcal{L}$	-5616.3389	1.51E-12	-5616.3389	6.27E-12	-5616.3389	5.8845e-12
	C	$\hat{q}$	0.9919	8.42E-07	0.9919	2.85E-06	0.9915	0.0027
		$\hat{\beta}$	0.9442	5.25E-07	0.9442	1.72E-06	0.9441	0.0014
		$\hat{\eta}$	4.5989	7.78E-06	4.5989	2.52E-05	4.6028	0.0234
		$\mathcal{L}$	-2532.4055	8.50E-10	-2532.4055	8.82E-09	-2532.4074	0.0032
	D	$\hat{q}$	0.5062	3.18E-08	0.5132	0.0008	0.5100	0.0193

		$\hat{\beta}$	1.5061	2.57E-08	1.5099	0.0004	1.5083	0.0099
		$\hat{\eta}$	4.9962	1.39E-07	4.9669	0.0031	4.9793	0.0814
		$\mathcal{L}$	-1848.9886	1.17E-12	-1848.9924	0.0008	-1849.0244	0.0532
	E	$\hat{q}$	1.5134	1.64E-08	1.5134	1.09E-07	1.5134	3.0101e-05
		$\hat{\beta}$	1.5076	4.18E-08	1.5076	2.66E-07	1.5077	9.6409e-05
		$\hat{\eta}$	4.7017	2.32E-07	4.7017	1.08E-06	4.7016	0.0002
		$\mathcal{L}$	-3657.4987	9.67E-13	-3657.4987	3.71E-12	-3657.4987	4.5104e-06

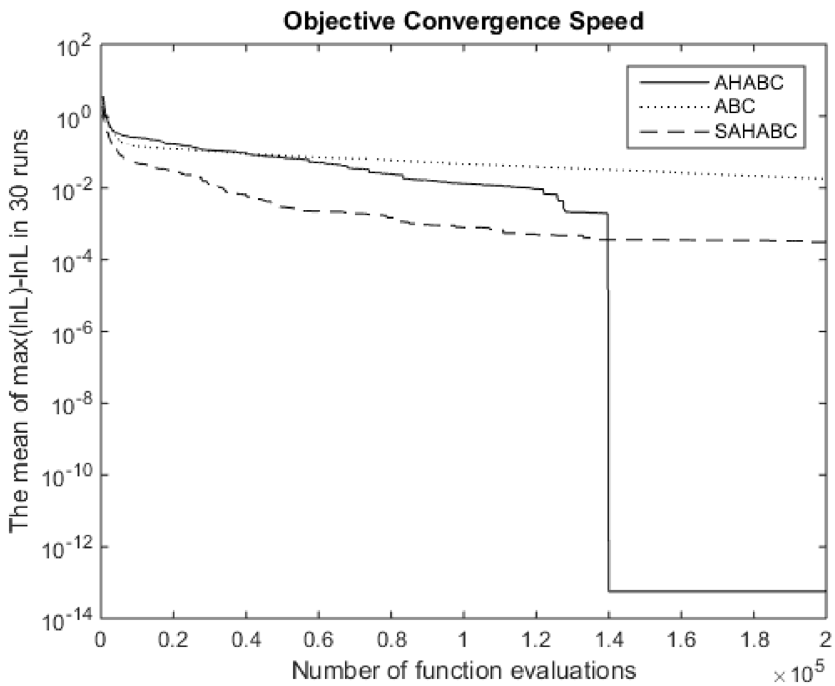

Figure 3-6: Convergence comparison of AHABC, ABC and SAHABC for ES-A, $n = 100$

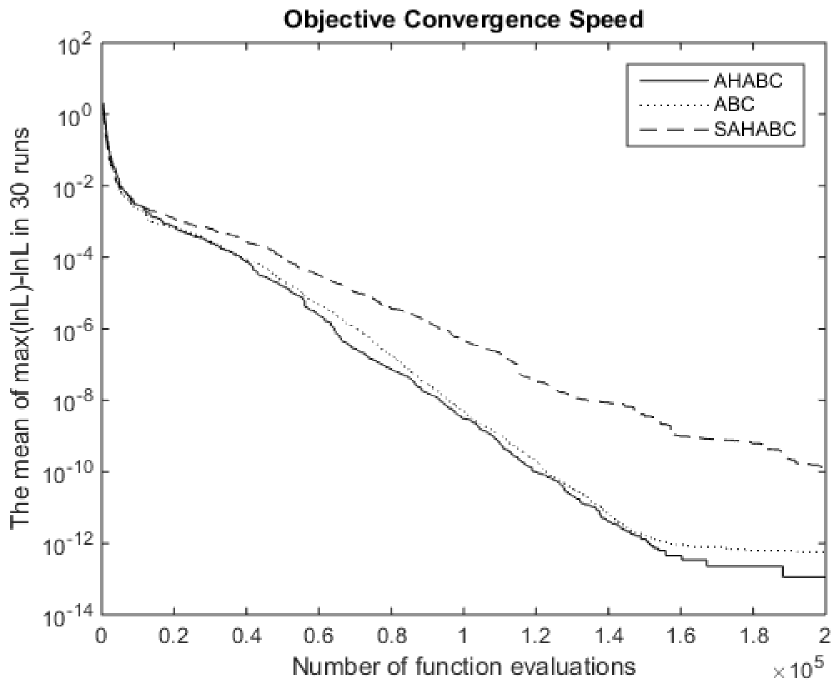

Figure 3-7: Convergence comparison of AHABC, ABC and SAHABC for ES-E, $n = 100$

3.4.3 Bias and Mean Squared Error

We also used the bias and MSE as additional criteria to evaluate the quality of the ML estimators via AHABC. For this purpose, we generate 1000 samples for each ES-A, B, C, D, and E for each sample size $n = 100, 500, 1000$. Then, AHABC algorithm was executed once for each sample. For each scenario, we have 1000 estimates for parameters q, β, η. Bias and MSE are computed according to the following equations:

$$bias(\hat{\theta}, \theta) = \frac{1}{m}(\sum_{i=1}^{m} \widehat{\theta_i}) - \theta \tag{3-9}$$

$$MSE(\hat{\theta}) = var(\hat{\theta}) + [bias(\hat{\theta}, \theta)]^2 \tag{3-10}$$

with $m = 1000$, $\hat{\theta} = \hat{q}, \hat{\beta}$ or $\hat{\eta}$, and $var(\hat{\theta})$ as the variance of the 1000 estimates.

Results of bias and MSE are shown in Table 3-5 and Table 3-6 respectively. From these results, for larger sample sizes n=500, and 1000, bias and MSE are very small for the q-Weibull parameters' estimates. Thus, AHABC is able to provide accurate and precise estimates for the q-Weibull parameters.

Table 3-5: Bias of ML estimates for q-Weibull parameters

ES	Statistic	n=100	n=500	n=1000
A	$\hat{q}$	-0.2802	-0.0431	-0.0227
	$\hat{\beta}$	-0.0059	-0.0018	-0.0010
	$\hat{\eta}$	11.6693	0.7646	0.3583
B	$\hat{q}$	-0.0087	-0.0035	-0.0036
	$\hat{\beta}$	0.0173	0.0021	0.0002
	$\hat{\eta}$	1.9045	0.3734	0.2226
C	$\hat{q}$	-0.0812	-0.0157	-0.0086
	$\hat{\beta}$	0.0044	-0.0018	-0.0023
	$\hat{\eta}$	1.0719	0.1756	0.0909
D	$\hat{q}$	-0.2689	-0.0496	-0.0278
	$\hat{\beta}$	-0.0171	-0.0080	-0.0046
	$\hat{\eta}$	1.1526	0.2032	0.1161
E	$\hat{q}$	-0.0078	0.0003	-0.0026
	$\hat{\beta}$	0.0518	0.0159	0.0016
	$\hat{\eta}$	0.2154	0.0420	0.0452

Table 3-6: MSE of ML estimates for q-Weibull parameters

ES	Statistic	n=100	n=500	n=1000
A	$\hat{q}$	0.4205	0.0214	0.0086
	$\hat{\beta}$	0.0046	0.0008	0.0004
	$\hat{\eta}$	1771.1883	5.1727	1.6128
B	$\hat{q}$	0.0099	0.0017	0.0010
	$\hat{\beta}$	0.0079	0.0012	0.0007
	$\hat{\eta}$	42.7344	2.6959	1.4148
C	$\hat{q}$	0.0689	0.0064	0.0029
	$\hat{\beta}$	0.0208	0.0032	0.0016
	$\hat{\eta}$	10.4633	0.6140	0.2621
D	$\hat{q}$	0.4851	0.0226	0.0085
	$\hat{\beta}$	0.0383	0.0071	0.0031
	$\hat{\eta}$	9.8914	0.3953	0.1515
E	$\hat{q}$	0.0105	0.0018	0.0009
	$\hat{\beta}$	0.0770	0.0119	0.0058
	$\hat{\eta}$	1.7203	0.2723	0.1409

3.5 Confidence Intervals

In order to construct confidence intervals for the parameters of the q-Weibull distribution, asymptotic confidence intervals (ACI), parametric bootstrap confidence intervals (BCI-P) and non-parametric bootstrap confidence intervals (BCI-NP) are developed. The related covariance matrix associated with the ML estimators for the q-Weibull distribution parameters can be estimated by the inverse of the observed information matrix $I\left(\underline{t}\middle|\hat{\eta},\hat{\beta},\hat{q}\right)$.

$$\widehat{var}\left(\underline{t}\middle|\hat{\eta},\hat{\beta},\hat{q}\right) = I^{-1}\left(\underline{t}\middle|\hat{\eta},\hat{\beta},\hat{q}\right) = -\frac{1}{\nabla^2\mathcal{L}\left(\underline{t}\middle|\hat{\eta},\hat{\beta},\hat{q}\right)}$$

$$= -\begin{bmatrix} \frac{\partial^2\mathcal{L}\left(\underline{t}\middle|\hat{\eta},\hat{\beta},\hat{q}\right)}{\partial\eta^2} & \frac{\partial^2\mathcal{L}\left(\underline{t}\middle|\hat{\eta},\hat{\beta},\hat{q}\right)}{\partial\eta\partial\beta} & \frac{\partial^2\mathcal{L}\left(\underline{t}\middle|\hat{\eta},\hat{\beta},\hat{q}\right)}{\partial\eta\partial q} \\ \frac{\partial^2\mathcal{L}\left(\underline{t}\middle|\hat{\eta},\hat{\beta},\hat{q}\right)}{\partial\beta\partial\eta} & \frac{\partial^2\mathcal{L}\left(\underline{t}\middle|\hat{\eta},\hat{\beta},\hat{q}\right)}{\partial\beta^2} & \frac{\partial^2\mathcal{L}\left(\underline{t}\middle|\hat{\eta},\hat{\beta},\hat{q}\right)}{\partial\beta\partial q} \\ \frac{\partial^2\mathcal{L}\left(\underline{t}\middle|\hat{\eta},\hat{\beta},\hat{q}\right)}{\partial q\partial\eta} & \frac{\partial^2\mathcal{L}\left(\underline{t}\middle|\hat{\eta},\hat{\beta},\hat{q}\right)}{\partial q\partial\beta} & \frac{\partial^2\mathcal{L}\left(\underline{t}\middle|\hat{\eta},\hat{\beta},\hat{q}\right)}{\partial q^2} \end{bmatrix}^{-1} \tag{3-11}$$

whose entries are

$$\frac{\partial^2\mathcal{L}\left(\underline{t}\middle|\hat{\eta},\hat{\beta},\hat{q}\right)}{\partial\eta^2} = \frac{\hat{\beta}}{\hat{\eta}^2}\left\{n - \sum_{i=1}^{n}\frac{1}{\left(\frac{\hat{\eta}}{t_i}\right)^{\hat{\beta}} - 1 + \hat{q}}\right\} - \hat{\beta}^2\hat{\eta}^{\hat{\beta}-2}\sum_{i=1}^{n}\frac{1}{t_i^{\hat{\beta}}\left[\left(\frac{\hat{\eta}}{t_i}\right)^{\hat{\beta}} - 1 + \hat{q}\right]^2} \tag{3-12}$$

$$\frac{\partial^2\mathcal{L}\left(\underline{t}\middle|\hat{\eta},\hat{\beta},\hat{q}\right)}{\partial\eta\partial\beta} = \frac{\partial^2\mathcal{L}\left(\underline{t}\middle|\hat{\eta},\hat{\beta},\hat{q}\right)}{\partial\beta\partial\eta}$$

$$= -\frac{1}{\hat{\eta}}\left\{n - \sum_{i=1}^{n}\frac{1}{\left(\frac{\hat{\eta}}{t_i}\right)^{\hat{\beta}} - 1 + \hat{q}}\right\} - \hat{\beta}\hat{\eta}^{\hat{\beta}-1}\sum_{i=1}^{n}\frac{\ln\left(\frac{\hat{\eta}}{t_i}\right)}{t_i^{\hat{\beta}}\left[\left(\frac{\hat{\eta}}{t_i}\right)^{\hat{\beta}} - 1 + \hat{q}\right]^2} \tag{3-13}$$

$$\frac{\partial^2\mathcal{L}\left(\underline{t}\middle|\hat{\eta},\hat{\beta},\hat{q}\right)}{\partial\eta\partial q} = \frac{\partial^2\mathcal{L}\left(\underline{t}\middle|\hat{\eta},\hat{\beta},\hat{q}\right)}{\partial q\partial\eta} = -\frac{\hat{\beta}}{\hat{\eta}}\sum_{i=1}^{n}\frac{1}{\left[\left(\frac{\hat{\eta}}{t_i}\right)^{\hat{\beta}} - 1 + \hat{q}\right]^2} \tag{3-14}$$

$$\frac{\partial^2 \mathcal{L}(\underline{t}|\hat{\eta},\hat{\beta},\hat{q})}{\partial \beta^2} = -\frac{n}{\hat{\beta}^2} + \hat{\eta}^{\hat{\beta}} \sum_{i=1}^{n} \frac{\ln\left(\frac{\hat{\eta}}{t_i}\right)\ln\left(\frac{t_i}{\hat{\eta}}\right)}{t_i^{\hat{\beta}}\left[\left(\frac{\hat{\eta}}{t_i}\right)^{\hat{\beta}} - 1 + \hat{q}\right]^2} \tag{3-15}$$

$$\frac{\partial^2 \mathcal{L}(\underline{t}|\hat{\eta},\hat{\beta},\hat{q})}{\partial \beta \partial q} = \frac{\partial^2 \mathcal{L}(\underline{t}|\hat{\eta},\hat{\beta},\hat{q})}{\partial q \partial \beta} = \sum_{i=1}^{n} \frac{\ln\left(\frac{t_i}{\hat{\eta}}\right)}{\left[\left(\frac{\hat{\eta}}{t_i}\right)^{\hat{\beta}} - 1 + \hat{q}\right]^2} \tag{3-16}$$

$$\begin{aligned}\frac{\partial^2 \mathcal{L}(\underline{t}|\hat{\eta},\hat{\beta},\hat{q})}{\partial q^2} = &-\frac{\mathrm{n}}{(2-\hat{q})^2} \\ &+ \frac{2}{(1-\hat{q})^3}\sum_{i=1}^{n} ln\left[1-(1-\hat{q})\left(\frac{t_i}{\hat{\eta}}\right)^{\hat{\beta}}\right] \\ &+ \frac{2}{(1-\hat{q})^2}\sum_{i=1}^{n} \frac{1}{\left(\frac{\hat{\eta}}{t_i}\right)^{\hat{\beta}} - 1 + \hat{q}} - \frac{1}{1-\hat{q}}\sum_{i=1}^{n} \frac{1}{\left[\left(\frac{\hat{\eta}}{t_i}\right)^{\hat{\beta}} - 1 + \hat{q}\right]^2}\end{aligned} \tag{3-17}$$

Once we have this covariance matrix, the asymptotic confidence intervals could be constructed for the q-Weibull distribution parameters. The asymptotic confidence intervals with $(1-\alpha)100\%$ confidence for η, β and q are given below:

$$\mathrm{CI}[\eta: (1-\alpha)100\%] = \left[\hat{\eta} + z_{\frac{\alpha}{2}}\sqrt{\widehat{var}_{11}}, \hat{\eta} + z_{1-\frac{\alpha}{2}}\sqrt{\widehat{var}_{11}}\right] \tag{3-18}$$

$$\mathrm{CI}[\beta: (1-\alpha)100\%] = \left[\hat{\beta} + z_{\frac{\alpha}{2}}\sqrt{\widehat{var}_{22}}, \hat{\beta} + z_{1-\frac{\alpha}{2}}\sqrt{\widehat{var}_{22}}\right] \tag{3-19}$$

$$\mathrm{CI}[q: (1-\alpha)100\%] = \left[\hat{q} + z_{\frac{\alpha}{2}}\sqrt{\widehat{var}_{33}}, \hat{q} + z_{1-\frac{\alpha}{2}}\sqrt{\widehat{var}_{33}}\right] \tag{3-20}$$

in which $z_{\frac{\alpha}{2}}$ and $z_{1-\frac{\alpha}{2}}$ are the $\frac{\alpha}{2}$ and $1-\frac{\alpha}{2}$ quantiles of the standard normal distribution, and $\widehat{var}_{11}, \widehat{var}_{22}$ and $\widehat{var}_{33}$ are the diagonal elements of the covariance matrix.

The bootstrap is a computer-based method for assigning measures of accuracy to sample estimates [51]. This technique allows us to generate confidence intervals for the parameters of the q-Weibull distribution by using simple sampling methods to infer the precision of the ML estimators.

The bootstrap approaches are classified as parametric and non-parametric, depending on how the samples are generated [52]. Given the original data set and the estimates of the parameters obtained from it, parametric and non-parametric bootstrap samples can be generated. For parametric bootstrap, the q-Weibull distribution uses the estimates to generate other B new samples by Equation (2-8). For the non-parametric bootstrap, B samples are generated by resampling with replacement from the original data set. Along with the original sample, a total of $B+1$ samples are obtained, and we apply the ML method via AHABC to these samples. By sorting the $B+1$ resulting estimates, the $\frac{\alpha}{2}100\%$ and $\left(1-\frac{\alpha}{2}\right)100\%$ percentiles are set as the lower and upper bounds to construct the confidence intervals with α level of significance.

Once again, $n = 100, 500, 1000$ and ES-A, B, C, D, and E. For all bootstrap experiments, $B = 999$. For BCI-P sampling, the estimates obtained from the first sample are used as q-Weibull parameters to generate B bootstrap samples by Equation (2-8). For BCI-NP sampling, in turn, the first sample is used to generate B bootstrap samples by resampling with replacement. Then, AHABC is applied to each sample to compute ML estimates. The

5th and 95th percentiles are obtained to construct the corresponding 90% confidence interval. The resulting confidence intervals for parameters η, β, q are presented in Table 3-7 to

Table 3-9 for sample sizes $n = 100, 500, 1000,$ respectively. The values in parentheses are the corresponding interval lengths.

From the results, it can be observed that all intervals contain the true values of parameters η, β, q. For larger sample sizes, asymptotic and bootstrap approaches tend to provide similar and more accurate interval estimates for the q-Weibull parameters. Note also that for the experimental settings A and B with n = 100 (Table 3-7), ACI provided negative lower bounds related to parameter eta. Despite being infeasible values for this parameter, the asymptotic approach does not guarantee valid bounds, and their results become more accurate and precise as the sample size increases.

Table 3-7: Interval estimates for the parameters, *n*=100

ES	True values of parameters	η			β			q		
		ACI	BCI-P	BCI-NP	ACI	BCI-P	BCI-NP	ACI	BCI-P	BCI-NP
A	$\eta = 5$ $\beta = 0.5$ $q = 0.5$	-4.240 15.226 (19.466)	2.413 55.818 (53.405)	1.200 837.669 (836.469)	0.406 0.693 (0.287)	0.431 0.676 (0.245)	0.396 0.734 (0.338)	-0.378 1.318 (1.696)	-1.028 0.819 (1.846)	-4.782 1.066 (5.849)
B	$\eta = 5$ $\beta = 0.5$ $q = 1.5$	-0.607 12.453 (13.060)	2.148 23.876 (21.728)	2.207 19.314 (17.107)	0.353 0.558 (0.205)	0.358 0.607 (0.249)	0.377 0.565 (0.188)	1.267 1.580 (0.313)	1.189 1.575 (0.386)	1.241 1.550 (0.309)
C	$\eta = 5$ $\beta = 1$ $q = 1$	1.967 7.938 (5.971)	3.092 11.048 (7.956)	2.863 10.877 (8.014)	0.749 1.198 (0.450)	0.777 1.222 (0.445)	0.741 1.342 (0.601)	0.669 1.316 (0.646)	0.461 1.218 (0.757)	0.525 1.278 (0.752)
D	$\eta = 5$ $\beta = 1.5$ $q = 0.5$	2.871 6.863 (3.992)	3.555 9.793 (6.238)	3.392 11.176 (7.784)	1.184 1.805 (0.621)	1.202 1.827 (0.625)	1.209 1.813 (0.604)	0.077 1.040 (0.962)	-0.523 0.893 (1.415)	-0.871 0.907 (1.777)
E	$\eta = 5$ $\beta = 1.5$ $q = 1.5$	2.910 6.054 (3.144)	3.158 6.665 (3.507)	3.194 6.811 (3.617)	1.191 2.041 (0.850)	1.268 2.284 (1.017)	1.277 2.170 (0.893)	1.457 1.714 (0.257)	1.428 1.718 (0.290)	1.417 1.699 (0.283)

Table 3-8: Interval estimates for the parameters, *n*=500

ES	True values of parameters	η			β			q		
		ACI	BCI-P	BCI-NP	ACI	BCI-P	BCI-NP	ACI	BCI-P	BCI-NP
A	$\eta = 5$ $\beta = 0.5$ $q = 0.5$	2.002 6.462 (4.460)	2.853 7.701 (4.848	2.376 8.090 (5.715)	0.464 0.559 (0.095)	0.467 0.556 (0.090)	0.462 0.570 (0.108)	0.320 0.748 (0.429)	0.269 0.683 (0.414)	0.250 0.746 (0.496)

B	$\eta = 5$ $\beta = 0.5$ $q = 1.5$	3.074 8.817 (5.744)	3.712 10.403 (6.691	3.642 9.736 (6.094)	0.428 0.530 (0.102)	0.429 0.535 (0.107)	0.437 0.535 (0.098)	1.401 1.534 (0.135)	1.381 1.529 (0.148)	1.393 1.529 (0.136)
C	$\eta = 5$ $\beta = 1$ $q = 1$	3.727 5.622 (1.895)	3.889 6.003 (2.115)	3.862 6.075 (2.213)	0.955 1.147 (0.192)	0.953 1.152 (0.199)	0.952 1.155 (0.203)	0.933 1.152 (0.219)	0.906 1.136 (0.230)	0.892 1.143 (0.252)
D	$\eta = 5$ $\beta = 1.5$ $q = 0.5$	4.243 5.809 (1.566)	4.347 6.293 (1.946)	4.371 6.084 (1.712)	1.377 1.641 (0.264)	1.365 1.655 (0.289)	1.373 1.642 (0.269)	0.314 0.686 (0.373)	0.196 0.664 (0.467)	0.249 0.656 (0.406)
E	$\eta = 5$ $\beta = 1.5$ $q = 1.5$	4.385 6.074 (1.689)	4.515 6.186 (1.671)	4.450 6.255 (1.805)	1.387 1.751 (0.364)	1.409 1.777 (0.367)	1.404 1.785 (0.381)	1.435 1.574 (0.139)	1.433 1.569 (0.136)	1.430 1.573 (0.143)

Table 3-9: Interval estimates for the parameters, *n*=1000

ES	True values of parameters	η			β			q		
		ACI	BCI-P	BCI-NP	ACI	BCI-P	BCI-NP	ACI	BCI-P	BCI-NP
A	$\eta = 5$ $\beta = 0.5$ $q = 0.5$	2.951 6.101 (3.150)	3.456 6.750 (3.294)	3.167 6.869 (3.702)	0.473 0.537 (0.064)	0.472 0.537 (0.065)	0.467 0.546 (0.079)	0.416 0.688 (0.272)	0.390 0.654 (0.265)	0.389 0.682 (0.293)
B	$\eta = 5$ $\beta = 0.5$ $q = 1.5$	2.957 6.105 (3.148)	3.215 6.650 (3.436)	3.341 6.423 (3.082)	0.463 0.544 (0.082)	0.466 0.547 (0.081)	0.464 0.549 (0.085)	1.456 1.552 (0.096)	1.452 1.550 (0.098)	1.455 1.549 (0.093)
C	$\eta = 5$ $\beta = 1$ $q = 1$	3.855 5.343 (1.488)	3.932 5.549 (1.617)	3.933 5.707 (1.774)	0.884 1.005 (0.121)	0.883 1.008 (0.126)	0.882 1.010 (0.129)	0.909 1.074 (0.165)	0.891 1.070 (0.179)	0.877 1.070 (0.193)
D	$\eta = 5$ $\beta = 1.5$	4.435 5.557	4.530 5.729	4.556 5.536	1.412 1.600	1.406 1.604	1.413 1.595	0.372 0.640	0.328 0.619	0.379 0.614

	$q = 0.5$	(1.121)	(1.199)	(0.981)	(0.188)	(0.199)	(0.182)	(0.268)	(0.292)	(0.235)
E	$\eta = 5$ $\beta = 1.5$ $q = 1.5$	4.154 5.250 (1.096)	4.229 5.235 (1.005)	4.159 5.349 (1.190)	1.386 1.629 (0.243)	1.397 1.637 (0.240)	1.391 1.639 (0.248)	1.466 1.560 (0.094)	1.467 1.555 (0.088)	1.462 1.562 (0.0998)

Based on the validation results presented in this section, the AHABC can provide accurate estimates for the q-Weibull parameters for all the ES-A, B, C, D, and E covering different behaviors of the q-Weibull hazard rate. Therefore, with the proposed AHABC, the q-Weibull distribution is used to tackle a real reliability problem in the next section.

3.6 *Application Example*

In this section, the proposed procedure to obtain the ML estimates of the q-Weibull parameters is illustrated through one application example involving lifetime data of engineering devices in reliability studies. The example deals with a data set of the time to the first failure of 500MW generators [53] that results in a bathtub-shaped hazard rate. For the data with a non-monotonic hazard rate, commonly used distributions like Weibull are barely suitable to fit the failure data. Thus, the use of the q-Weibull illustrates the ability of this distribution in dealing with non-monotonic hazard rate function, which encompasses a set of problems with relevant applications in the reliability context [54][55].

Table 3-10 shows the time to the first failure for a group of 36 generators of 500MW [53]. The AHABC is replicated 30 times and the estimated ML parameters, and the associated standard deviations are shown in Table 3-11.

Table 3-10: Time to first failure (1000's of hours) of 500 MW generators

0.058	0.070	0.090	0.105	0.113	0.121	0.153	0.159
0.224	0.421	0.570	0.596	0.618	0.834	1.019	1.104
1.497	2.027	2.234	2.372	2.433	2.505	2.690	2.877
2.879	3.166	3.455	3.551	4.378	4.872	5.085	5.272
5.341	8.952	9.188	11.399				

Table 3-11: ML estimates for 30 replications of AHABC

	Mean	Std.
$\hat{q}$	0.4318	2.5555e-08
$\hat{\beta}$	0.6697	4.8570e-09
$\hat{\eta}$	6.6087	2.9609e-07
$\mathcal{L}$	-68.0595	1.4211e-14

To check the goodness-of-fit, we use the Kolmogorov-Smirnov (KS) test, which compares the empirical and the cumulative distribution function (CDF). However, the traditional KS test is not applicable to our situation, where the parameters of the theoretical distribution have been estimated from the same data used to apply this goodness-of-fit test [56]. Therefore, a bootstrapped version of the KS test [57] has been developed and applied in this paper. The KS test statistic is computed as follows:

$$D^0 = max\left|\left|F_n(t_i) - F(t_i|\hat{q},\hat{\beta},\hat{\eta}))\right|,\left|F_n(t_{i-1}) - F(t_i|\hat{q},\hat{\beta},\hat{\eta})\right|\right|, \qquad (3\text{-}21)$$

where $F_n(t_i) = i/n$ is the empirical CDF and $F(t_0) = 0$, $F(t_i|\hat{q},\hat{\beta},\hat{\eta})$ is the theoretical CDF with estimated parameters. B bootstrap samples $\underline{t}^j = \{t_1^j, t_2^j, \dots, t_n^j\}, j = 1,2, \dots, B$ are generated using Equation (2-8) with $\hat{q},\hat{\beta},\hat{\eta}$. The ML estimates $\hat{q}^j,\hat{\beta}^j,\hat{\eta}^j$ for the j^{th} sample are obtained by the proposed AHABC. The test statistic D^j is computed with $F(t_i^j|\hat{q}^j,\hat{\beta}^j,\hat{\eta}^j)$ in place of $F(t_i|\hat{q},\hat{\beta},\hat{\eta})$. Then, we get $B+1$ observations of the KS test statistic D. The p-value is computed as the number of observations where D^j exceeds D^0 divided by $B+1$.

For comparison purposes, we consider the standard Weibull and some alternative bathtub-shaped hazard rate models: the modified Weibull extension [5] and the ENH [6] models, as shown in Table 3-12. We then apply the proposed AHABC procedure to obtain the ML estimates of the parameters not only for the q-Weibull but also for the modified Weibull extension and the ENH models. The fitted parameters and log-likelihoods are given in Table 3-13, which also gives the KS test statistic D^0 and p-value. Figure 3-8 presents the empirical and fitted CDFs for the example, and Figure 3-9 shows the hazard rate functions. Note that except for the standard Weibull that models the data as decreasing hazard rate, all the other models result in a bathtub-shaped hazard rate, which has also been observed by Bebbington et al. [54].

Compared to the standard Weibull, q-Weibull is more flexible to perform reliability analyses when failure data are characterized by non-monotonic hazard rates. Moreover, with the low KS test statistic and high p-value (see Table 3-13), the q-Weibull distribution is a good alternative to the other bathtub-shaped hazard rate models, namely the modified Weibull extension and the ENH.

Table 3-12: Some bathtub-shaped hazard rate models

Model	$h(t)$	Parameters
Modified Weibull Extension	$h(t) = \lambda\beta(t/\alpha)^{(\beta-1)}\exp\,[(t/\alpha)^{\beta}]$	$\alpha, \beta, \lambda > 0$
ENH	$\alpha\beta\lambda\dfrac{(1+\lambda t)^{\alpha-1}\exp[1-(1+\lambda t)^{\alpha}]\{1-\exp[1-(1+\lambda t)^{\alpha}]\}^{\beta-1}}{1-\{1-\exp\,[1-(1+\lambda t)^{\alpha}]\}^{\beta}}$	$\alpha, \beta, \lambda > 0$

Table 3-13: Results for the example

Model	ML estimates	log$L$	D^0	p
q-Weibull	$\hat{q} = 0.4318, \hat{\beta} = 0.6697, \hat{\eta} = 6.6087$	-68.0595	0.0983	0.5080
Weibull	$\hat{\beta} = 0.8156, \hat{\eta} = 2.3118$	-68.6906	0.1219	0.1880
Modified Weibull Extension	$\hat{\alpha} = 10.0923, \hat{\beta} = 0.6920, \hat{\lambda} = 0.2130$	-68.2628	0.1046	0.2900
ENH	$\hat{\alpha} = 1.6347, \hat{\beta} = 0.6415, \hat{\lambda} = 0.1430$	-68.3560	0.1021	0.3330

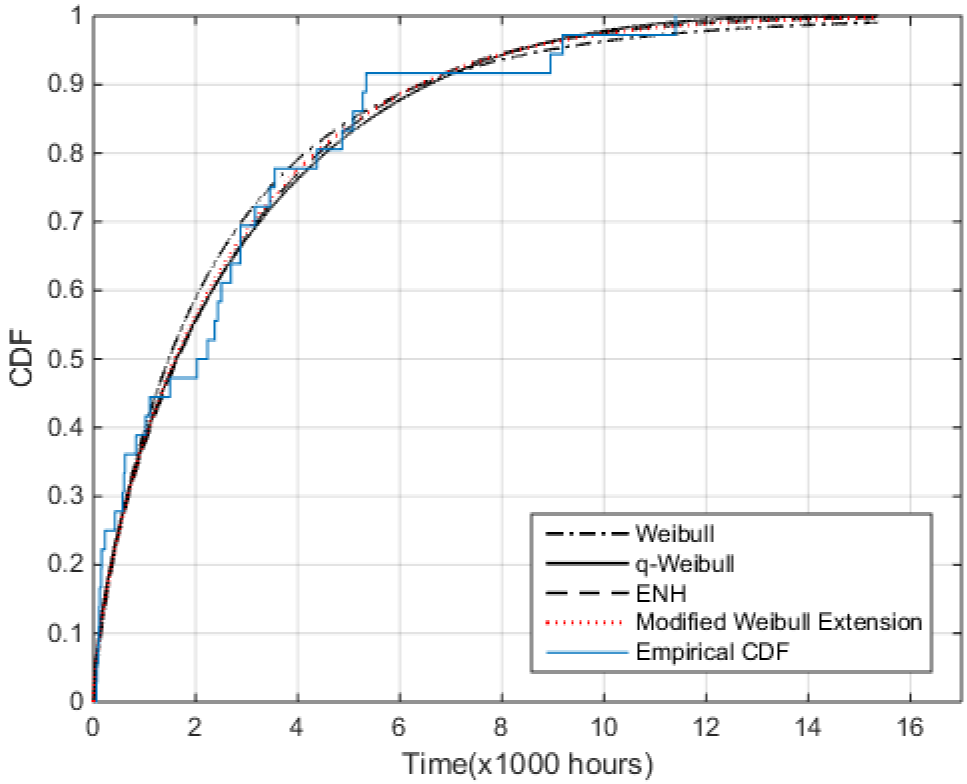

Figure 3-8: Empirical and fitted CDFs

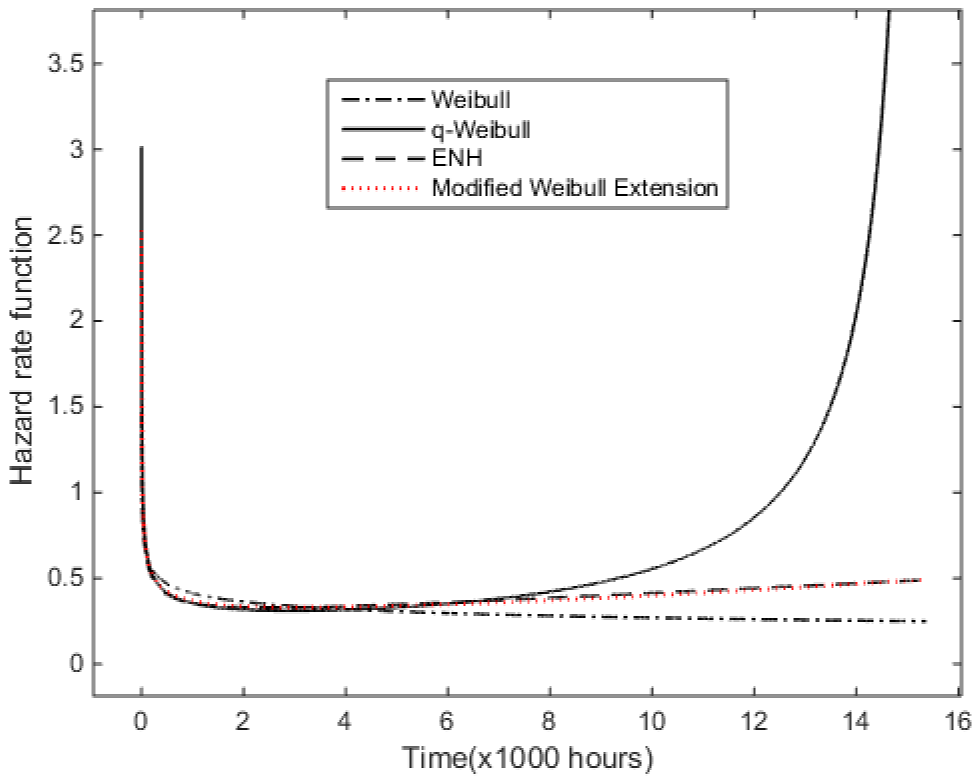

Figure 3-9: Hazard rate functions

For the sake of comparison, the estimates of q-Weibull and Weibull parameters shown in Table 3-13 are used for obtaining the conditional reliability (Equation (2-9)) as shown in Figure 3-10. Note that as t_0 increases, the Weibull provides higher conditional reliability, which is in accordance with the decreasing behavior of the hazard rate resulting from the application of the Weibull to this data set. On the other hand, the q-Weibull conditional reliability decreases more rapidly as t_0 increases. Given the nature of the reliability data set in Table 3-10, one can argue that the Weibull model results in an optimistic performance of the generators when compared to the q-Weibull distribution.

Note that these results are representative of the failure data set in Table 3-10, and different outcomes might be obtained for different sets of reliability data. However, based on the experimental results discussed in the previous section and the ones from this application example, the q-Weibull distribution is a flexible and capable model that might be considered as one more alternative distribution when engineers are faced with modeling of reliability data sets.

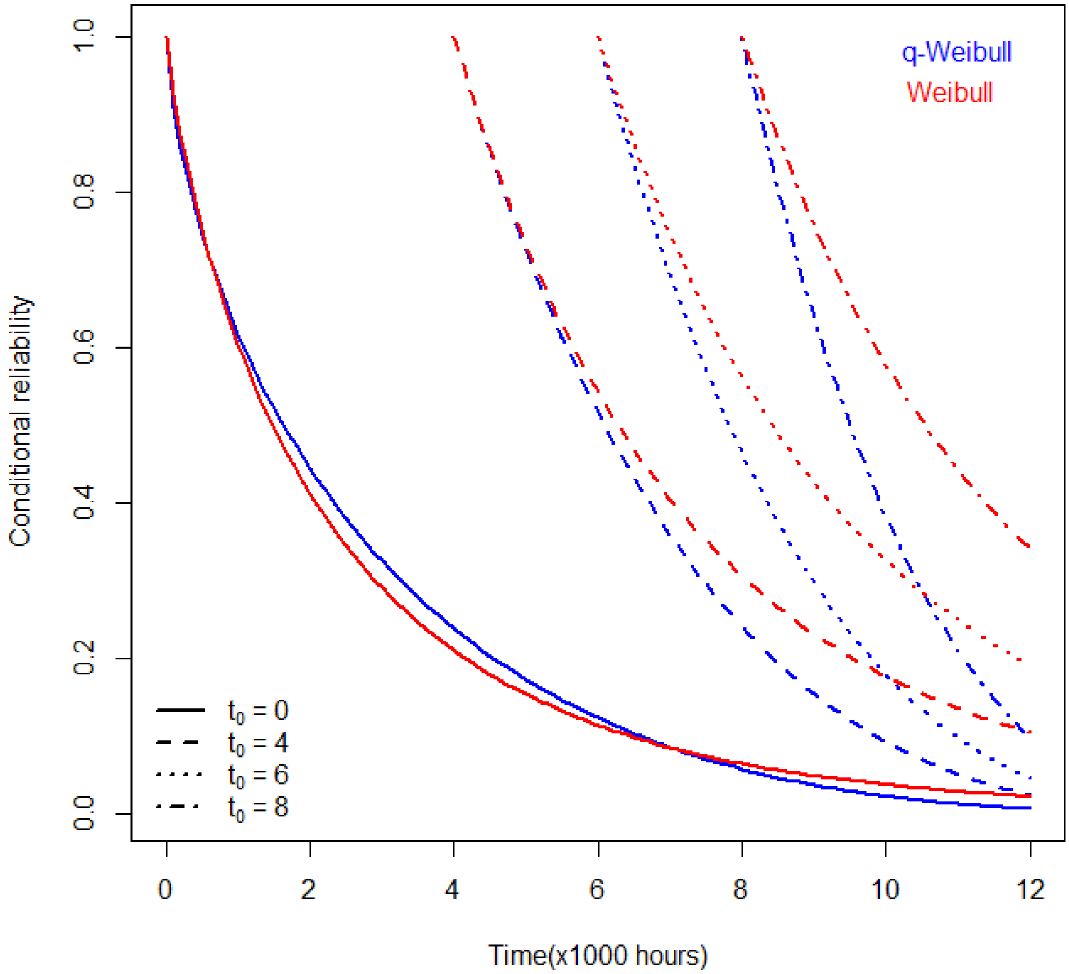

Figure 3-10: Conditional reliability of q-Weibull and Weibull

3.7 <u>Summary</u>

The q-Weibull distribution can describe various behaviors of the hazard rate - monotonically decreasing, monotonically increasing, constant, unimodal and bathtub-shaped - with a single set of parameters. This flexibility provided by the q-Weibull probabilistic model is important to describe accurately failure data characterized by both monotonic and non-monotonic hazard rate functions. Although there are other 3-parameter distributions with that flexibility (e.g., modified Weibull extension [5], ENH [6]), the q-Weibull distribution constitutes another alternative to the arsenal of options available for the reliability analyst.

However, it is impractical to analytically obtain the ML estimates for the q-Weibull parameters, and the classic numerical optimization approach fails to efficiently find the global solution for the associated ML problem. Thus, the q-Weibull distribution is flexible and useful in the context of reliability engineering as it allows for the modeling and analysis of a variety of failure data behaviors, in particular data with non-monotonic hazard rate functions. However, its intricate likelihood function imposes significant numerical difficulties in estimating its parameters, which has limited the number of applications of this distribution so far.

In this research, an adaptive hybrid artificial bee colony (AHABC) algorithm has been proposed to tackle this problem, which combines the global exploration of ABC and the local exploitation of Nelder-Mead simplex search. The exploitation ability of Nelder-Mead improves the local search performance of ABC.

Numerical results show that the proposed AHABC algorithm efficiently finds the optimal solution for the q-Weibull ML problem, comprising different behaviors of the hazard rate and sample sizes. The ML estimates of the q-Weibull parameters obtained via AHABC are accurate and precise with small bias and MSE. Using the proposed AHABC algorithm, intervals estimates for the q-Weibull parameters are provided, including asymptotic intervals based on the ML theory, parametric and non-parametric bootstrapped confidence intervals. Based on the results presented in Section 3.4.2, the proposed AHABC outperformed both ABC and similar algorithms in terms of accuracy and convergence speed in the context of the maximum likelihood problem for the q-Weibull distribution. The proposed method for the ML constrained q-Weibull problem was also applied to an example involving failure data characterized by a bathtub-shaped hazard rate function.

To conclude, the proposed AHABC for parameter estimation showed that the q-Weibull is a promising alternative distribution for reliability modeling and constitutes in another distribution model in the reliability engineers' toolbox.

Chapter 4: Modeling Dependent Series Systems

4.1 Overview

This chapter considers the problems of modeling a series system with dependent component failure times and estimating the model's parameters from failure time data. This chapter shows that a q-Weibull distribution can approximate the distribution of the failure time of a series system with dependent component failure times that are modeled as a Clayton survival copula. Moreover, the parameter q in q-Weibull distribution approximates the parameter θ in Clayton copula, which measures the degree of dependence among the components. For a series system with minimal repair, we develop a method for estimating the parameters of the Clayton copula given data about component failures, and we show that this process can be modeled as a nonhomogeneous Poisson process (NHPP).

Thus, we propose the NHPP with q-Weibull as the underlying time to first failure (TTFF) distribution model to approximate the minimal repair process of a system composed of multiple components with dependence characterized by Clayton copula. Furthermore, the proposed model is flexible and elegant to analyze the failure pattern of a complex repairable system showing monotonic and non-monotonic behaviors of the intensity function.

For a life test of a series system, one ideally would be able to observe and record the system failure times and the specific component that failed. In this case, we describe a procedure for estimating the parameters of the components' failure time distributions and the copula from the failure data. In some cases, however, information about the failed components is not available due to the reason that component sometimes cannot be identified when resources are restricted. Instead, a set of components for the failure may only be known. So accurately estimating the parameters (of the components' failure time distributions and the copula) from the failure data is unlikely. To address this scenario, we will use a simpler q-Weibull model to approximate the Clayton copula model.

Compared with the copula model, which needs more information on the assumption for the underlying distributions of components and the exact component cause of system failure, the simpler q-Weibull model can approximate the Clayton copula model without knowing this information.

We will perform a simulation study to evaluate the q-Weibull approximation. We also apply the proposed NHPP q-Weibull model to a data set of 44 LHD machine failure times given by Kumar et al. [7]. The data appear to have a bathtub-shaped failure intensity. Besides, we compare our model with Superposed-PLP (S-PLP) model by Pulcini [8] that superpositions two independent power law processes to fit this data set and also confirm the bathtub shape of the failure intensity.

4.2 *Modeling System Failure Time*

This section considers the first failure time of a series system with dependent components and shows that a q-Weibull distribution can approximate the time to first failure distribution. Herein, we use the notation $o(x)$ [58] to denote a function of x that satisfies the following property: $\lim_{x\to 0}\frac{o(x)}{x} = 0$.

4.2.1 Clayton Copula Model

Consider a system with d components connected in series. The dependence of these d failure times can be described by a Clayton copula.

Let the random vector $(X_1, X_2, \dots, X_d)$ represents the lifetimes of the d components. Let $R_i(x_i) = \Pr(X_i > x_i)\,, i = 1, \dots, d$ be the marginal reliability function. Assume the joint survival distribution function of the vector $(X_1, X_2, \dots, X_d)$ can be modeled as the Clayton survival copula:

$$\begin{aligned} \mathrm{P}\{X_1 > x_1, X_2 > x_2, \dots, X_d > x_d\} &= \hat{C}\big(R_1(x_1), R_2(x_2), \dots, R_d(x_d)\big) \\ &= \big[R_1^{-\theta}(x_1) + R_2^{-\theta}(x_2) + \cdots \\ &\quad + R_d^{-\theta}(x_d) - d + 1\big]^{-\frac{1}{\theta}}. \end{aligned} \tag{4-1}$$

where $\hat{C}$ is Clayton survival copula and $\theta \in [-1, \infty)\backslash\{0\}$.

Because the components are in series, the system's failure time is the minimum of all the components' failure times: $t = \min\{X_1, X_2, \dots, X_d\}$. That is, the system is operating at time t if and only if every component is operating at time t. Thus, from Equation (4-1), the reliability function of the series system at time t is given as follows:

$$\begin{aligned} R_s(t) &= P(\min\{X_1, X_2, \dots, X_d\} > t) \\ &= P\{X_1 > t, X_2 > t, \dots, X_d > t\} \\ &= \hat{C}\big(R_1(t), R_2(t), \dots, R_d(t)\big) \\ &= \left[R_1^{-\theta}(t) + R_2^{-\theta}(t) + \cdots + R_d^{-\theta}(t) - d + 1\right]^{-\frac{1}{\theta}}. \end{aligned} \tag{4-2}$$

4.2.2 q-Weibull Approximation

Consider a series system with identical components. Now, suppose that the reliability function for a component can be expressed as follows:

$$R_i(t) = 1 - \left(\frac{t}{\lambda}\right)^{\alpha} + o\left(\left(\frac{t}{\lambda}\right)^{\alpha}\right), \quad \text{as } \frac{t}{\lambda} \to 0. \tag{4-3}$$

This leads to the following expression:

$$R_i^{-\theta}(t) = 1 + \theta\left(\frac{t}{\lambda}\right)^{\alpha} + o\left(\left(\frac{t}{\lambda}\right)^{\alpha}\right), \quad \text{as } \frac{t}{\lambda} \to 0. \tag{4-4}$$

After substituting Equation (4-4) into Equation (4-2), the system reliability function can be expressed as follows:

$$R_s(t) = \left[d \cdot \left(1 + \theta\left(\frac{t}{\lambda}\right)^\alpha + o\left(\left(\frac{t}{\lambda}\right)^\alpha\right)\right) - d + 1\right]^{-\frac{1}{\theta}}$$

$$= \left[d \cdot \left(\theta\left(\frac{t}{\lambda}\right)^\alpha + o\left(\left(\frac{t}{\lambda}\right)^\alpha\right)\right) + 1\right]^{-\frac{1}{\theta}} \tag{4-5}$$

$$\approx \left[1 + \theta d\left(\frac{t}{\lambda}\right)^\alpha\right]^{-\frac{1}{\theta}}.$$

Now, set $q' = 1 + \theta$, $\beta = \alpha$, and $\eta' = \lambda d^{-\frac{1}{\alpha}}$. Substituting these into Equation (4-5) yields the following:

$$R_s(t) \approx \left[1 - (1 - q')\left(\frac{t}{\eta'}\right)^\beta\right]^{\frac{1}{1-q'}}.$$

This is the reliability function for a q-Weibull distribution, as shown in Equation (2-10). Thus, the system time-to-failure is approximately distributed as q-Weibull distribution.

The quality of this approximation depends upon the magnitude of t. For $\theta \in [-1, 0)$, to ensure that $d \cdot R_i^{-\theta}(t) - d + 1 > 0$ in Equation (4-2) that is $R_i^{-\theta}(t) \in (1 - \frac{1}{d},\ 1]$, the support ensures t is small. The approximate is more accurate when t is small (see Figure 4-1 to Figure 4-4). For $\theta \in (0, \infty)$, $R_i^{-\theta}(t) \in [0, 1]$, there is no restriction on t, the approximation is not accurate (see Figure 4-5 and Figure 4-6).

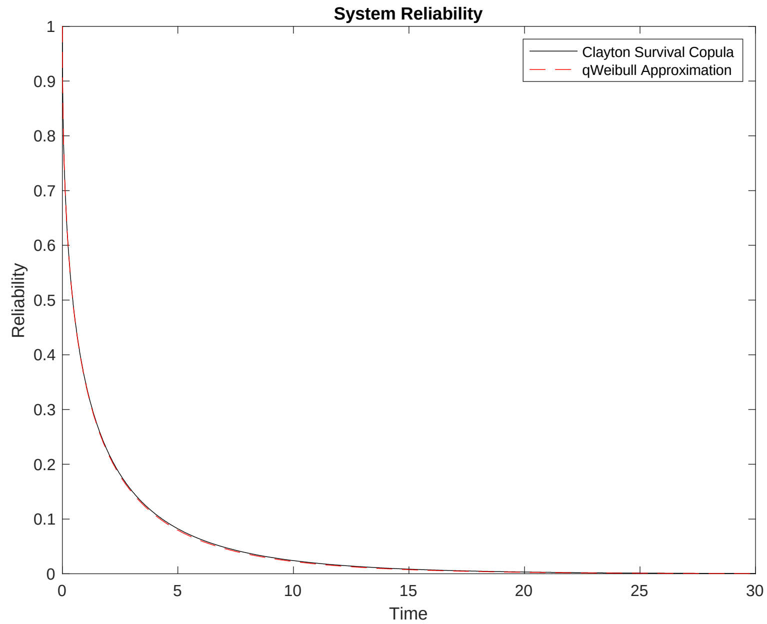

Figure 4-1: (d = 10, $\theta = -0.1$, $\lambda = 100$, $\alpha = 0.5$)

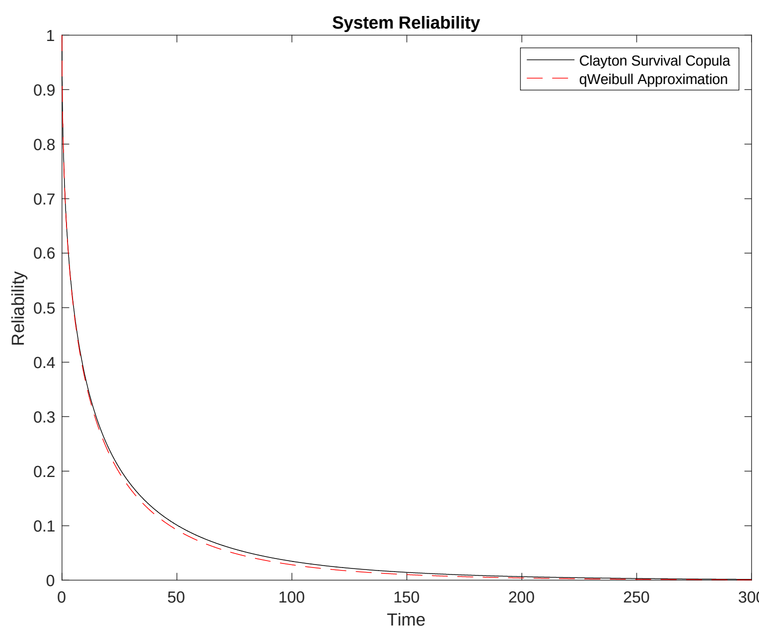

Figure 4-2: (d = 3, $\theta = -0.1$, $\lambda = 100$, $\alpha = 0.5$)

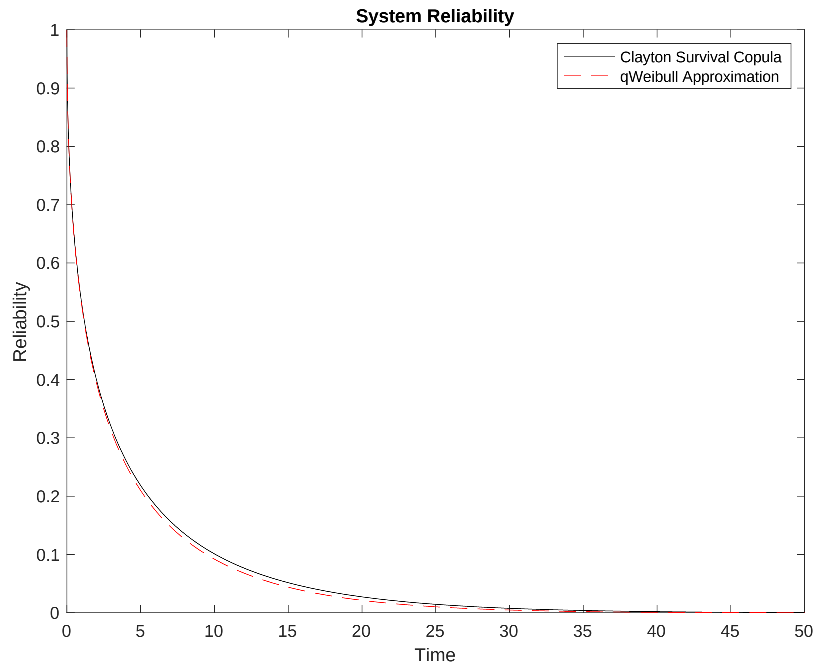

Figure 4-3: (d = 6, $\theta = -0.2$, $\lambda = 100$, $\alpha = 0.5$)

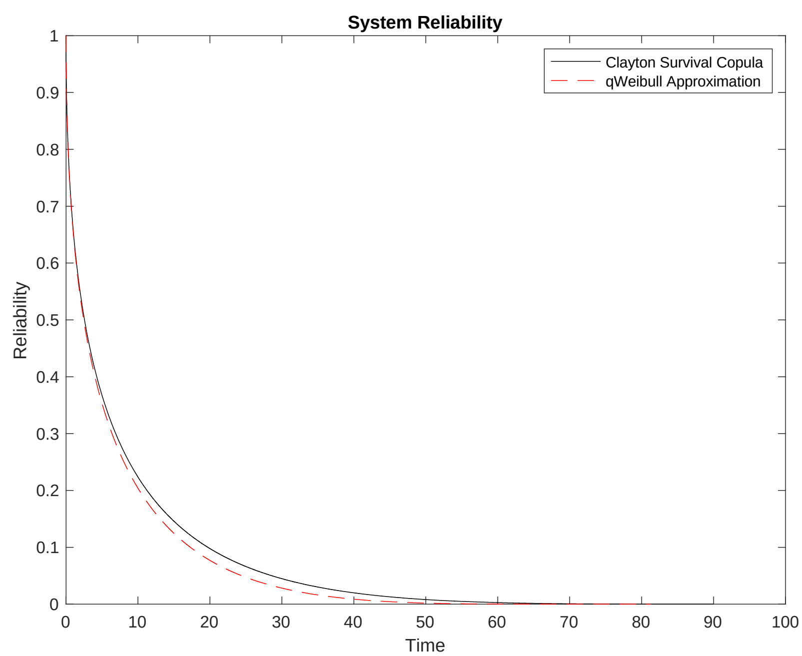

Figure 4-4: (d = 4, $\theta = -0.3$, $\lambda = 100$, $\alpha = 0.5$)

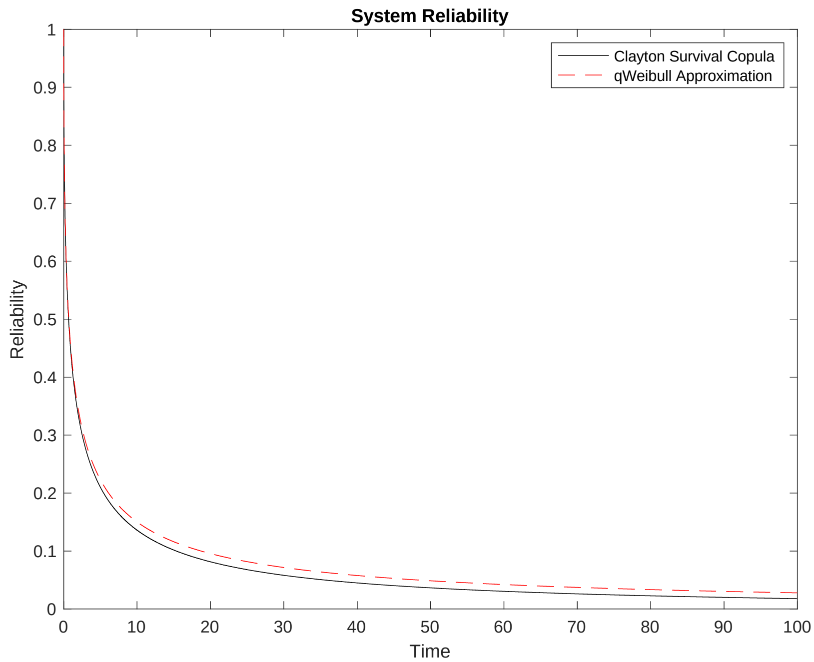

Figure 4-5: (d = 10, $\theta = 0.5$, $\lambda = 100$, $\alpha = 0.5$)

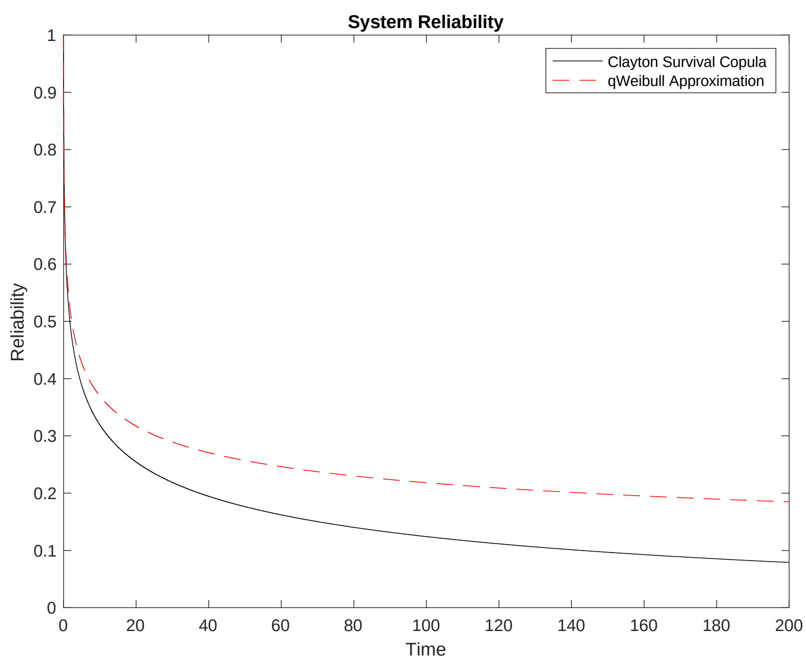

Figure 4-6: (d = 10, $\theta = 2$, $\lambda = 100$, $\alpha = 0.5$)

4.2.3 Heterogeneous Components

Consider a series system with non-identical components. Suppose that the reliability function for component i can be expressed as follows:

$$R_i(t) = 1 - \left(\frac{t}{\lambda_i}\right)^{\alpha_i} + o\left(\left(\frac{t}{\lambda_i}\right)^{\alpha_i}\right), \text{ as } \frac{t}{\lambda_i} \to 0 \tag{4-6}$$

Let m be the component with the smallest exponent such that $\alpha_m = \min\{\alpha_1, \alpha_2, \dots, \alpha_d\}$. The system reliability function is given by:

$$\begin{aligned}
R_s(t) &= \left[\sum_{i=1}^{d}\left(1 + \theta\left(\frac{t}{\lambda_i}\right)^{\alpha_i} + o\left(\left(\frac{t}{\lambda_i}\right)^{\alpha_i}\right)\right) - d + 1\right]^{-\frac{1}{\theta}} \\
&= \left[\sum_{i=1}^{d}\left(\theta\left(\frac{t}{\lambda_i}\right)^{\alpha_i} + o\left(\left(\frac{t}{\lambda_i}\right)^{\alpha_i}\right)\right) + 1\right]^{-\frac{1}{\theta}} \\
&\approx \left[1 + \theta \cdot \sum_{i=1}^{d}\left[\left(\frac{t}{\lambda_i}\right)^{\alpha_i}\right]\right]^{-\frac{1}{\theta}} \\
&= \left[1 + \theta \cdot \left(\frac{t}{\lambda_m}\right)^{\alpha_m}\left({\lambda_m}^{\alpha_m}\sum_{i=1}^{d}[{\lambda_i}^{-\alpha_i}(t)^{\alpha_i - \alpha_m}]\right)\right]^{-\frac{1}{\theta}} \\
&= \Bigg[1 + \theta \\
&\quad \cdot \left(\frac{t}{\lambda_m}\right)^{\alpha_m}\Bigg({\lambda_m}^{\alpha_m}\sum_{\alpha_i = \alpha_m}^{d}{\lambda_i}^{-\alpha_i} \\
&\quad + {\lambda_m}^{\alpha_m}\sum_{\alpha_i \neq \alpha_m}^{d}[{\lambda_i}^{-\alpha_i}(t)^{\alpha_i - \alpha_m}]\Bigg)\Bigg]^{-\frac{1}{\theta}} \\
&\approx \left[1 + \theta t^{\alpha_m}\sum_{\alpha_i = \alpha_m}^{d}{\lambda_i}^{-\alpha_m}\right]^{-\frac{1}{\theta}}
\end{aligned} \tag{4-7}$$

Now, set $q' = 1 + \theta$, $\beta = \alpha_m$, and $\eta' = \left(\sum_{\alpha_i=\alpha_m}^{d} \lambda_i^{-\alpha_m}\right)^{-\frac{1}{\alpha_m}}$. Substituting these into Equation (4-7) yields the following:

$$R_s(t) \approx \left[1 - (1 - q')\left(\frac{t}{\eta'}\right)^{\beta}\right]^{\frac{1}{1-q'}}.$$

This is the reliability function for a q-Weibull distribution, as shown in Equation (2-10). Thus, the system time-to-failure is approximately distributed as a q-Weibull distribution. Again, the approximation is more accurate when t is small (see Figure 4-7 to Figure 4-8).

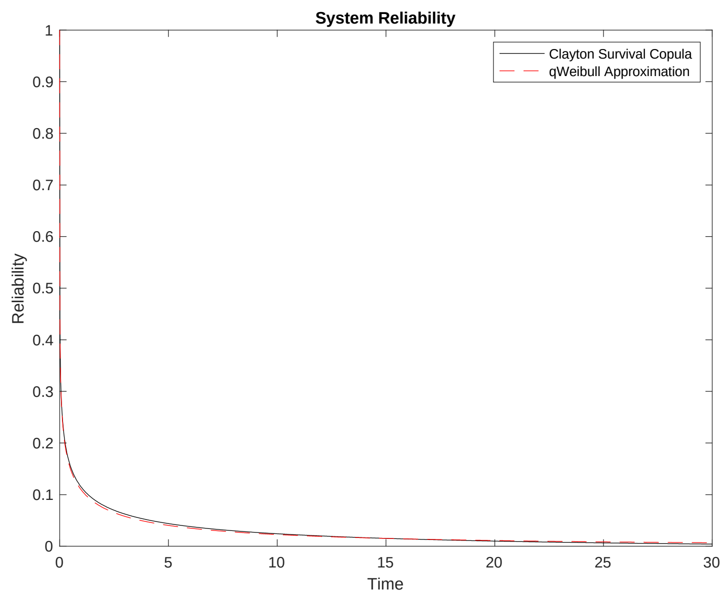

Figure 4-7: (d = 10, $\theta = -0.1$, $\lambda = 100$

$\alpha_i = [2, 2, 2, 2, 2, 0.2, 0.2, 0.2, 0.2, 0.2]$)

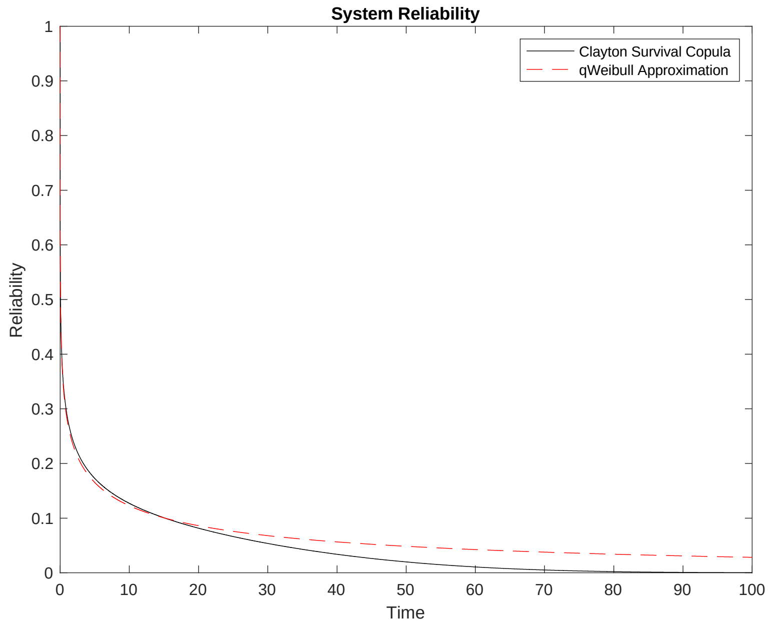

Figure 4-8: (d = 6, $\theta = -0.1$, $\lambda = 100$

$\alpha_i = [2, 2, 2, 0.2, 0.2, 0.2]$)

Therefore, for a series system with dependence characterized by Clayton copula, the failure time of the system follows q-Weibull distribution under the assumption that the component's failure time distribution satisfies the Equation (4-3). For example, Weibull distribution: $R_i(t) = exp\left(-\left(\frac{t}{\lambda}\right)^{\alpha}\right) = 1 - \left(\frac{t}{\lambda}\right)^{\alpha} + o\left(-\left(\frac{t}{\lambda}\right)^{\alpha}\right)$, uniform distribution on $[0, t_{max}]$: $R_i(t) = 1 - \frac{t}{t_{max}}$.

Thus, the system time-to-failure is approximately distributed as with a q-Weibull distribution.

4.3 Modeling the Minimal Repair Process of the Dependent Series System

This section discusses approaches for modeling the failure times of a series system with dependent components that undergo minimal repairs when a component fails. The data analysis problem is to estimate the parameters of the model from system failures data.

We model the dependent series system by NHPP q-Weibull model. For comparison purposes, we provide two other dependence modeling methods using Clayton copula models. One model considers information regarding the exact cause component for the system failure. The other model does not have this information.

4.3.1 Nonhomogeneous Poisson Process with Underlying q-Weibull Distribution

In this section, we propose the model of NHPP with q-Weibull as an underlying distribution for a minimal repair process of a series system. The NHPP can be used to model the failure process of repairable systems. The NHPP model presumes that, whenever a failure occurs, the system is repaired to the condition as it was right before the failure, which is the minimal repair or same-as-old repair assumption. For a system composed of many components having close reliability functions, this assumption is appropriate because only a few of the system's many components are repaired at a time, which yields only a small change of the system hazard rate [59]. We can consider the NHPP is a process in which each failed system is instantaneously replaced by an identical one having the same failure rate as the failed one [59]. The intensity of the NHPP coincides with the hazard rate function of the underlying time to first failure (TTFF) distribution. It also means that just after any repair action carried out at time *t*, the intensity is equal to the hazard rate of the TTFF distribution [60].

The NHPP has time-dependent intensity function $\lambda(t) > 0$. Let $N(t)$ be the number of events (failures) that occur in the interval $[0, t]$. Let $R(t)$ be the probability that no events (failures) occur in the interval $[0, t]$.

$$
\begin{aligned}
R(t) &= P\{N(t) = 0\} \\
&= \frac{\left(\int_0^t \lambda(\tau) d\tau\right)^0}{0!} e^{-\int_0^t \lambda(\tau) d\tau} \\
&= e^{-\int_0^t \lambda(\tau) d\tau}.
\end{aligned}
\tag{4-8}
$$

Consider a series of failures occurring at the time $t_1, t_2, \dots, t_n$ according to the NHPP with intensity $\lambda(t)$. Let t_k be the time to the k_{th} failures. Let $R(t_k, t)$ be the probability that no failure occurs in the interval (t_k, t):

$$
\begin{aligned}
R(t_k, t) &= P\{N(t) = 0\} \\
&= \frac{\left(\int_{t_k}^t \lambda(\tau) d\tau\right)^0}{0!} e^{-\int_{t_k}^t \lambda(\tau) d\tau} \\
&= e^{-\int_{t_k}^t \lambda(\tau) d\tau} \\
&= \frac{e^{-\int_0^t \lambda(\tau) d\tau}}{e^{-\int_0^{t_k} \lambda(\tau) d\tau}} \\
&= \frac{R(t)}{R(t_k)},
\end{aligned}
\tag{4-9}
$$

which is the conditional reliability function of a system having age at the time t_k. The NHPP is a process in which each failed component/system is instantaneously replaced by an identical, working one having the same age as the failed one. This model is a minimal repair condition. If t_k is equal to zero,

$$R(t) = e^{-\int_0^t \lambda(\tau) d\tau}, \tag{4-10}$$

which means that the intensity of the NHPP $\lambda(t)$ coincides with the failure rate function $h(t)$ of the underlying TTFF distribution. All future behavior of a repairable system is completely defined by this distribution.

Under the NHPP model, the probability that, the failure after the one at t_{i-1} will occur at $(t_i, t_i + \Delta t)$ can be approximated by [61]:

$$\lambda(t_i)\Delta t \cdot e^{-\int_{t_{i-1}}^{t_i} \lambda(\tau) d\tau}, \tag{4-11}$$

where the first multiplier is the probability of failure in $(t_i, t_i + \Delta t)$ and the second is the probability of a failure-free operation in the interval(t_{i-1}, t_i).

Let $f(t_i|t_{i-1})$ be the conditional probability density function of the i-th failure time t_i, given that the previous failure occurred at the time t_{i-1}:

$$f(t_i|t_{i-1}) = \lambda(t_i) e^{-\int_{t_{i-1}}^{t_i} \lambda(\tau) d\tau}, \quad t_i \geq t_{i-1}. \tag{4-12}$$

If the data are the observed failure times $t_1, t_2, \dots, t_n$, the likelihood function is the product of all the conditional probability density functions, which can be written as follows:

$$
\begin{aligned}
L &= f(t_1)\prod_{i=2}^{n} f(t_i|t_{i-1}) \\
&= \prod_{i=1}^{n} \lambda(t_i)\cdot e^{-\int_0^{t_1}\lambda(\tau)d\tau} e^{-\int_{t_1}^{t_2}\lambda(\tau)d\tau} \cdots e^{-\int_{t_{n-1}}^{t_n}\lambda(\tau)d\tau} \\
&= \prod_{i=1}^{n} \lambda(t_i)\cdot e^{-\int_0^{t_n}\lambda(\tau)d\tau} \\
&= \prod_{i=1}^{n} \lambda(t_i)\cdot e^{-H(t_n)}.
\end{aligned}
\tag{4-13}
$$

The corresponding log-likelihood function is given by:

$$lnL = \sum_{i=1}^{n} ln[\lambda(t_i)] - H(t_n), \tag{4-14}$$

Let $H(t)$ be the cumulative intensity function, which equals the cumulative hazard rate function:

$$H(t) = \int_0^t \lambda(\tau)d\tau = \int_0^t h(\tau)d\tau. \tag{4-15}$$

We propose to use the q-Weibull distribution as the underlying distribution. In this case, the intensity function $\lambda(t)$ equals the hazard rate function $h_q(t)$ of the q-Weibull distribution:

$$\lambda(t) = h_q(t) = \frac{\beta t^{\beta-1}\eta^{-\beta}}{1-(1-q)\left(\frac{t}{\eta}\right)^{\beta}} \tag{4-16}$$

The cumulative intensity function (cumulative hazard rate function) of the q-Weibull distribution is $H_q(t)$:

$$H_q(t) = \int_0^t h_q(\tau) d\tau = -\frac{1}{1-q} ln\left[1-(1-q)\left(\frac{t}{\eta}\right)^{\beta}\right]. \tag{4-17}$$

Substituting Equation (4-16) and (4-17) into Equation (4-13) yields the likelihood function for the NHPP with q-Weibull as underlying distribution as follows:

$$\begin{aligned} L &= \prod_{i=1}^{n} h_q(t_i) \cdot e^{-H_q(t_n)} \\ &= \prod_{i=1}^{n} \frac{\beta t_i^{\beta-1} \eta^{-\beta}}{1-(1-q)\left(\frac{t_i}{\eta}\right)^{\beta}} \cdot \left(1-(1-q)\left(\frac{t_n}{\eta}\right)^{\beta}\right)^{\frac{1}{1-q}}. \end{aligned} \tag{4-18}$$

Thus, the log-likelihood for NHPP with q-Weibull as underlying distribution is given by:

$$\begin{aligned} lnL &= \sum_{i=1}^{n} ln\left[h_q(t_i)\right] - H_q(t_n) \\ &= \sum_{i=1}^{n} ln\left[\frac{\beta t_i^{\beta-1} \eta^{-\beta}}{1-(1-q)\left(\frac{t_i}{\eta}\right)^{\beta}}\right] + \frac{1}{1-q} ln\left[1-(1-q)\left(\frac{t_n}{\eta}\right)^{\beta}\right]. \end{aligned} \tag{4-19}$$

For comparison, we will use copula to model the minimum repair process of the dependent series system. Depending on whether we know which component fails or not, we have two copula models in sections 4.3.2 and 4.3.3.

4.3.2 Clayton Copula with Unknown Components

We consider the case in which the component that caused the system failure is unknown. This model uses the same information as the q-Weibull model: the system's failure times

$t_1, t_2, \dots, t_n$ $(t_1 < t_2 < \cdots t_n)$. According to the reliability function for the series system in Equation (4-2), the hazard rate function for the series system can be expressed as follows:

$$\begin{aligned} h(t) &= -\frac{R_s'(t)}{R_s(t)} \\ &= \frac{R_1^{-\theta}(t)h_1(t) + R_2^{-\theta}(t)h_2(t) +, \dots, + R_d^{-\theta}(t)h_d(t)}{\hat{C}^{-\theta}\big(R_1(t), R_2(t), \dots, R_d(t)\big)}. \end{aligned} \tag{4-20}$$

The cumulative hazard function for the series system is $H(t)$:

$$\begin{aligned} H(t) &= -lnR_s(t) \\ &= -ln\hat{C}\big(R_1(t), R_2(t), \dots, R_d(t)\big). \end{aligned} \tag{4-21}$$

Here we substitute Equations (4-20) and (4-21) into the Equation (4-14) to determine the log-likelihood:

$$\begin{aligned} lnL &= \sum_{i=1}^{n} ln[h(t_i)] - H(t_n) \\ &= \sum_{i=1}^{n} \Big[ln\left(R_1^{-\theta}(t_i)h_1(t_i) + R_2^{-\theta}(t_i)h_2(t_i) +, \dots, + R_d^{-\theta}(t_i)h_d(t_i)\right) \\ &\qquad + \theta ln\left(\hat{C}\big(R_1(t_i), R_2(t_i), \dots, R_d(t_i)\big)\right)\Big] \\ &\qquad + ln\left(\hat{C}\big(R_1(t_n), R_2(t_n), \dots, R_d(t_n)\big)\right). \end{aligned} \tag{4-22}$$

4.3.3 Clayton Copula with Known Components

This section considers the case where the component that caused the system failure is known. The system's failure times are $t_1, t_2, \dots, t_n$ ($t_1 < t_2 < \cdots < t_n$). Let j_i be the component that fails at the time t_i. Each failure time is determined by the minimum of all the components' failure times.

Given the first $i-1$ failure times $t_1, t_2, \dots, t_{i-1}$, the probability that component j_i fails in the interval $(t_i, t_i + \Delta t)$ while no other component fails between t_{i-1} and t_i can be expressed as follows:

$$P\left(t_i < X_{j_\mathrm{i}} < t_\mathrm{i} + \Delta t, \cup_{l \neq j_\mathrm{i}} \{X_l > t_\mathrm{i}\} \middle| X_1 > t_{\mathrm{i}-1}, X_2 > t_{\mathrm{i}-1}, \dots, X_d > t_{\mathrm{i}-1}\right). \quad (4\text{-}23)$$

The probability density function for the failure time of component j_i, given that the $(i-1)_{th}$ failure occurred at t_{i-1}, is given by:

$$
\begin{aligned}
&f(t_i|t_{i-1}) \\
&= \lim_{\Delta t \to 0_+} \frac{P\left(t_i < X_{j_i} < t_i + \Delta t, \bigcup_{l \neq j_i} \{X_l > t_i\} \,\middle|\, X_1 > t_{i-1}, X_2 > t_{i-1}, \dots, X_d > t_{i-1}\right)}{\Delta t} \\
&= \lim_{\Delta t \to 0_+} \frac{P\left(t_i < X_{j_i} < t_i + \Delta t, \bigcup_{l \neq j_i} \{X_l > t_i\}\right)}{\Delta t \cdot P(X_1 > t_{i-1}, X_2 > t_{i-1}, \dots, X_d > t_{i-1})} \\
&= \lim_{\Delta t \to 0_+} \frac{P\left(X_{j_i} > t_i, \bigcup_{l \neq j_i} \{X_l > t_i\}\right) - P\left(X_{j_i} > t_i + \Delta t, \bigcup_{l \neq j_i} \{X_l > t_i\}\right)}{\Delta t \cdot P(X_1 > t_{i-1}, X_2 > t_{i-1}, \dots, X_d > t_{i-1})} \\
&= \lim_{\Delta t \to 0_+} \frac{\hat{C}\left(R_1(t_i), R_2(t_i), \dots, R_d(t_i)\right) - \hat{C}\left(R_{j_i}(t_i + \Delta t), \bigcup_{l \neq j_i} R_l(t_i)\right)}{\Delta t \cdot \hat{C}\left(R_1(t_{i-1}), R_2(t_{i-1}), \dots, R_d(t_{i-1})\right)} \\
&= -\frac{\partial \hat{C}\left(R_1(t_i), R_2(t_i), \dots, R_d(t_i)\right)}{\partial R_{j_i}(t_i)} \cdot \frac{\partial R_{j_i}(t_i)}{\partial t} \cdot \frac{1}{\hat{C}\left(R_1(t_{i-1}), R_2(t_{i-1}), \dots, R_d(t_{i-1})\right)} \\
&= \frac{\frac{\partial \hat{C}\left(R_1(t_i), R_2(t_i), \dots, R_d(t_i)\right)}{\partial R_{j_i}(t_i)} \cdot f_{j_i}(t_i)}{\hat{C}\left(R_1(t_{i-1}), R_2(t_{i-1}), \dots, R_d(t_{i-1})\right)},
\end{aligned}
\tag{4-24}
$$

where $i = 2,3, \dots, n$

and

$$
f(t_1) = \frac{\partial \hat{C}\left(R_1(t_1), R_2(t_1), \dots, R_d(t_1)\right)}{\partial R_{j_1}(t_1)} \cdot f_{j_1}(t_1). \tag{4-25}
$$

Plugging Clayton survival copula Equation (4-1) into Equations (4-24) and (4-25), we get:

$$
\begin{aligned}
&f(t_i|t_{i-1}) \\
&= \frac{-\frac{1}{\theta}\cdot\left[R_1^{-\theta}(t_i)+R_2^{-\theta}(t_i)+\cdots+R_d^{-\theta}(t_i)-d+1\right]_+^{-\frac{1}{\theta}-1}\cdot(-\theta)\cdot R_{j_i}^{-\theta-1}(t_i)\cdot f_{j_i}(t_i)}{\left[R_1^{-\theta}(t_{i-1})+R_2^{-\theta}(t_{i-1})+\cdots+R_d^{-\theta}(t_{i-1})-d+1\right]_+^{-\frac{1}{\theta}}} \\
&= \frac{\hat{C}\big(R_1(t_i),R_2(t_i),\dots,R_d(t_i)\big)\cdot R_{j_i}^{-\theta}(t_i)\cdot f_{j_i}(t_i)}{\hat{C}\big(R_1(t_{i-1}),R_2(t_{i-1}),\dots,R_d(t_{i-1})\big)\cdot\hat{C}^{-\theta}\big(R_1(t_i),R_2(t_i),\dots,R_d(t_i)\big)\cdot R_{j_i}(t_i)} \\
&= \frac{R_{j_i}^{-\theta}(t_i)}{\hat{C}^{-\theta}\big(R_1(t_i),R_2(t_i),\dots,R_d(t_i)\big)}\cdot h_{j_i}(t_i)\cdot\frac{\hat{C}\big(R_1(t_i),R_2(t_i),\dots,R_d(t_i)\big)}{\hat{C}\big(R_1(t_{i-1}),R_2(t_{i-1}),\dots,R_d(t_{i-1})\big)},
\end{aligned}
\tag{4-26}
$$

where $i = 2,3,\dots,n$

and

$$
\begin{aligned}
f(t_1) = \frac{R_{j_1}^{-\theta}(t_1)}{\hat{C}^{-\theta}\big(R_1(t_1),R_2(t_1),\dots,R_d(t_1)\big)}\cdot h_{j_1}(t_1) \\
\cdot\hat{C}\big(R_1(t_1),R_2(t_1),\dots,R_d(t_1)\big)\,.
\end{aligned}
\tag{4-27}
$$

The likelihood function for the system can be written as

$$
\begin{aligned}
L &= f(t_1)\prod_{i=2}^{n} f(t_i|t_{i-1}) \\
&= \prod_{i=1}^{n}\frac{h_{j_i}(t_i)\cdot R_{j_i}^{-\theta}(t_i)}{\hat{C}^{-\theta}\big(R_1(t_i),R_2(t_i),\dots,R_d(t_i)\big)} \\
&\qquad\cdot\hat{C}\big(R_1(t_n),R_2(t_n),\dots,R_d(t_n)\big),
\end{aligned}
\tag{4-28}
$$

where $h_{j_i}(t)$ and $R_{j_i}(t)$ are the hazard rate and reliability functions for the component j_i which caused the system to fail at time t_i .

The log-likelihood is given by:

$$lnL = \sum_{i=1}^{n} \Big[ln\left(h_{j_i}(t_i)\right) - \theta\, ln\left(R_{j_i}(t_i)\right) + \theta ln\left(\hat{C}\left(R_1(t_i), R_2(t_i), \dots, R_d(t_i)\right)\right)\Big] + ln\left(\hat{C}\left(R_1(t_n), R_2(t_n), \dots, R_d(t_n)\right)\right). \tag{4-29}$$

4.4 <u>*Simulation Experiments*</u>

To evaluate the accuracy of the models presented in Sections 4.2 and 4.3. We conducted simulation experiments of multiple series systems with dependent component failure times. In particular, the experiments were designed to show how well the q-Weibull model could estimate the system reliability function even when there was no information about the components that failed. The simulated systems included those with increasing hazard rates and those with bathtub-shaped hazard rates. Section 4.4.1 describes the process for sampling failure times. Section 4.4.2 presents the simulated systems that were considered. Section 4.4.3 presents the results.

4.4.1 Data Generating

Let $t_1, t_2, \dots, t_n$ ($t_1 < t_2 < \cdots < t_n$) represent the successive system's failure times, which refer to the corresponding components $j_1, j_2, \dots, j_n$ of the system. The system failure time t_i is determined by $t_i = min(t_1^i, t_2^i, \dots, t_d^i)$, where $t_1^i, t_2^i, \dots, t_d^i$ represent the components' failure times given that all the components survived at t_{i-1}.

As the commonly used sampling method for Clayton copula [62] is restricted to unconditional data sampling, here we develop a conditional data sampling method to generate components' failure times $(t_1^i, t_2^i, \dots, t_d^i)$ given the condition that all the components survived at t_{i-1}. Let $A = \{X_1 > t_{i-1}, X_2 > t_{i-1}, \dots, X_d > t_{i-1}\}$.

The components' failure times $(t_1^i, t_2^i, \dots, t_d^i)$ can be generated sequentially as follows:

$$
\begin{aligned}
&P\left(X_1 = t_1^i, X_2 = t_2^i, \dots, X_d = t_d^i \middle| A\right) \\
&= P\left(X_d = t_d^i \middle| X_{d-1} = t_{d-1}^i, \dots, X_2 = t_2^i, X_1 = t_1^i, A\right) \cdots P\left(X_2 = t_2^i \middle| X_1 = t_1^i, A\right) \\
&\cdot P\left(X_1 = t_1^i \middle| A\right)
\end{aligned}
\tag{4-30}
$$

To generate the failure time for the first component, the conditional probability density function that the first component fails in $(t_1^i, t_1^i + \Delta t)$ given that the previous system failure occurred at the time t_{i-1} can be derived as follows:

$$
\begin{aligned}
&f\left(X_1 = t_1^i \middle| A\right) = \lim_{\Delta t \to 0_+} \frac{P(X_1 = t_1^i, X_2 > t_{i-1}, \dots, X_d > t_{i-1})}{\Delta t \cdot P(X_1 > t_{i-1}, X_2 > t_{i-1}, \dots, X_d > t_{i-1})} \\
&= \lim_{\Delta t \to 0_+} \frac{P\left(X_1 > t_1^i, X_2 > t_{i-1}, \dots, X_d > t_{i-1}\right) - P(X_1 > t_1^i + \Delta t, X_2 > t_{i-1}, \dots, X_d > t_{i-1})}{\Delta t \cdot \hat{C}\left(R_1(t_{i-1}), R_2(t_{i-1}), \dots, R_d(t_{i-1})\right)} \\
&= \lim_{\Delta t \to 0_+} \frac{\hat{C}\left(R_1\left(t_1^i\right), R_2(t_{i-1}), \dots, R_d(t_{i-1})\right) - \hat{C}\left(R_1\left(t_1^i + \Delta t\right), R_2(t_{i-1}), \dots, R_d(t_{i-1})\right)}{\Delta t \cdot \hat{C}\left(R_1(t_{i-1}), R_2(t_{i-1}), \dots, R_d(t_{i-1})\right)} \\
&= -\frac{\hat{C}^{(1)}\left(R_1\left(t_1^i\right), R_2(t_{i-1}), \dots, R_d(t_{i-1})\right)}{\hat{C}\left(R_1(t_{i-1}), R_2(t_{i-1}), \dots, R_d(t_{i-1})\right)}.
\end{aligned}
\tag{4-31}
$$

Let $\hat{C}^{(m)}\big(R_1(x_1), R_2(x_2), \dots, R_d(x_d)\big)$ be the derivative of the copula $\hat{C}\big(R_1(x_1), R_2(x_2), \dots, R_d(x_d)\big)$ with respect to $x_1, x_2, \dots, x_m$. Take $t_0 = 0$. For $m = 2, 3, \dots, d$, and $i = 1, 2, \dots, n$, the conditional probability density function that the m_{th} component fails in $(t_m^i, t_m^i + \Delta t)$ is equal to

$$f\big(X_m = t_m^i \big| \cup_{l=1}^{m-1}\{X_l = t_l^i\}, A\big) = -\frac{\hat{C}^{(m)}\big(\cup_{l=1}^{m}\{R_l(t_l^i)\}, \cup_{l=m+1}^{d}\{R_l(t_{i-1})\}\big)}{\hat{C}^{(m-1)}\big(\cup_{l=1}^{m-1}\{R_l(t_l^i)\}, \cup_{l=m}^{d}\{R_l(t_{i-1})\}\big)}, \tag{4-32}$$

Then, the conditional reliability function based on the Clayton copula is given by:

$$\begin{aligned} R\big(X_1 = t_1^i \big| A\big) &= \frac{\hat{C}\left(R_1(t_1^i), R_2(t_{i-1}), \dots, R_d(t_{i-1})\right)}{\hat{C}\big(R_1(t_{i-1}), R_2(t_{i-1}), \dots, R_d(t_{i-1})\big)} \\ &= \frac{\left[R_1^{-\theta}(\mathrm{t}_1^i) + R_2^{-\theta}(\mathrm{t}_{i-1}) + \dots + R_d^{-\theta}(\mathrm{t}_{i-1}) - d + 1\right]^{-\frac{1}{\theta}}}{\left[R_1^{-\theta}(\mathrm{t}_{i-1}) + R_2^{-\theta}(\mathrm{t}_{i-1}) + \dots + R_d^{-\theta}(\mathrm{t}_{i-1}) - d + 1\right]^{-\frac{1}{\theta}}} \end{aligned} \tag{4-33}$$

For $m = 2, 3, \dots, d$, and $i = 1, 2, \dots, n$,

$$\begin{aligned} R\big(X_m = t_m^i \big| \cup_{l=1}^{m-1}\{X_l = t_l^i\}, A\big) &= \frac{\hat{C}^{(m-1)}\big(\cup_{l=1}^{m}\{R_l(t_l^i)\}, \cup_{l=m+1}^{d}\{R_l(t_{i-1})\}\big)}{\hat{C}^{(m-1)}\big(\cup_{l=1}^{m-1}\{R_l(t_l^i)\}, \cup_{l=m}^{d}\{R_l(t_{i-1})\}\big)} \\ &= \frac{\left[\sum_{l=1}^{m} R_l^{-\theta}(\mathrm{t}_l) + \sum_{l=m+1}^{d} R_l^{-\theta}(\mathrm{t}_{i-1}) - d + 1\right]^{-\frac{1}{\theta}-(m-1)}}{\left[\sum_{l=1}^{m-1} R_l^{-\theta}(\mathrm{t}_l) + \sum_{l=m}^{d} R_l^{-\theta}(\mathrm{t}_{i-1}) - d + 1\right]^{-\frac{1}{\theta}-(m-1)}} \end{aligned} \tag{4-34}$$

To generate the system failure times, firstly generate random values on [0, 1] for the conditional reliability, then use the inverse of reliability function in Equation (4-33) and (4-34) to sequentially generate $t_1^i, t_2^i, \dots, t_d^i$, the failure times for the components. Then, the i-th failure time for the system is determined by the minimum of the components' failure times $t_i = \min\{t_1^i, t_2^i, \dots, t_d^i\}$ for $i = 1, 2, \dots, n$. The pseudo-code of the proposed algorithm for generating the series system's failure times is given in Figure 4-9.

Initialize $t_0 = 0$ and $R_l(t_0) = 1$ for $l = 1, 2, \dots, d$

For $i = 1, 2, \dots, n$

    Draw a random value u on $[0, 1]$

    Generate $R_1^{-\theta}(t_1^i) = u^{-\theta} \cdot \left[\sum_{l=1}^{d} R_l^{-\theta}(\mathrm{t}_{i-1}) - d + 1\right] - \sum_{l=2}^{d} R_l^{-\theta}(\mathrm{t}_{i-1}) + d - 1$

    Generate t_1^i by taking the inverse of $R_1(x) = \exp[-(x/\lambda_1)^{\alpha_1}]$

    For $m = 2, \dots, d$

        Draw a random value u on $[0, 1]$

        Generate $R_m^{-\theta}(t_m^i) = u^{-\frac{\theta}{1+(m-1)\theta}} \cdot \left[\sum_{l=1}^{m-1} R_l^{-\theta}(\mathrm{t}_l^i) + \sum_{l=m}^{d} R_l^{-\theta}(\mathrm{t}_{i-1}) - d + 1\right] - \sum_{l=1}^{m-1} R_l^{-\theta}(\mathrm{t}_l^i) - \sum_{l=m+1}^{d} R_l^{-\theta}(\mathrm{t}_{i-1}) + d - 1$

        Generate t_m^i by taking the inverse of $R_m(x) = \exp[-(x/\lambda_m)^{\alpha_m}]$

    End

    $t_i = \min(t_1^i, t_2^i, \dots, t_d^i)$

End

Return $(t_1, t_2, \dots, t_n)$

Figure 4-9: Pseudo-code of the series system's failure times generating algorithm

4.4.2 Simulated Systems

Simulation experiments were conducted to evaluate the proposed NHPP with underlying q-Weibull and Clayton copula models. The algorithms were implemented in MATLAB. In these experiments, the Clayton copula parameter θ satisfies $\theta \in [-1, 0)$, and the equivalent parameter q in the PDF of the q-Weibull distribution is $q = 2 - \frac{1}{1+\theta}$. Thus, $q \in (-\infty, 1)$, which includes increasing and bathtub-shaped hazard rates $h_q(t)$. Indeed, $h_q(t)$

is monotonically increasing for $q < 1$ and $\beta > 1$, and bathtub-shaped for $q < 1$ and $0 < \beta < 1$ [63]. The components' failure times follow Weibull distributions with the reliability function $R_i(x) = exp[-(x/\lambda_i)^{\alpha_i}]$, which satisfies Equation (4-6). In these experiments, we set the components' scale parameter $\lambda_i = 5$, for all the components $i = 1, 2, \dots, d$. The maximum likelihood estimates for parameters of the underlying q-Weibull distribution were obtained by maximizing the log-likelihood function in Equation (4-19) via an adaptive hybrid artificial bee colony algorithm [64]. Table 4-1 and Table 4-2 show the parameters of the simulated systems from which we generated samples. The systems marked with a "*" in Table 4-1 and Table 4-2 were also used to compare the three models in sections 4.3.1, 4.3.2, and 4.3.3. For each simulated case, we ran 20 replications, and each replication had $n = 30$ failures. For experimental setting parameter θ, the correspongding dimension d satisfies $2 \le d \le 1 - \frac{1}{\theta}$ [37].

Table 4-1: Simulation settings for systems with increasing hazard rate

Copula parameter θ (equivalent q)	Number of components d	Components' shape parameters $\alpha_i, i = 1, 2, \dots, d$
$\theta = -0.3333$ ($q = 0.5$)	2	[2, 2]
		[2, 3]
	3	[2, 2, 2] *
		[2, 3, 5]
	4	[2, 2, 2, 2]
		[2, 2, 3, 5]

$\theta = -0.2857$ ($q = 0.6$)	2	[2, 2]
		[2, 3]
	3	[2, 2, 2] *
		[2, 3, 5]
	4	[2, 2, 2, 2]
		[2, 2, 3, 5]
$\theta = -0.1667$ ($q = 0.8$)	2	[2, 2]
		[2, 3]
	3	[2, 2, 2] *
		[2, 3, 5]
	5	[2, 2, 2, 2, 2]
		[2, 2, 3, 5, 5]
	6	[2, 2, 2, 2, 2, 2]
		[2, 2, 3, 3, 5, 5]

Table 4-2: Simulation settings for systems with a bathtub-shaped hazard rate.

Copula parameter θ (equivalent q)	Number of components d	Components' shape parameters α_i, $i = 1, 2, \dots, d$
$\theta = -0.3333$ ($q = 0.5$)	3	[2, 0.2, 0.2]
	4	[2, 2, 0.2, 0.2]

$\theta = -0.2857$ ($q = 0.6$)	3	[2, 0.2, 0.2]
	4	[2, 2, 0.2, 0.2] *
$\theta = -0.1667$ ($q = 0.8$)	3	[2, 0.2, 0.2]
	4	[2, 2, 0.2, 0.2] *
	5	[2, 2, 0.2, 0.2, 0.2]
	6	[2, 2, 2, 0.2, 0.2, 0.2]
$\theta = -0.0909$ ($q = 0.9$)	2	[1.2, 0.2]
	4	[2, 2, 0.2, 0.2]
	6	[2, 2, 2, 0.2, 0.2, 0.2]
	8	[2, 2, 2, 2, 0.2, 0.2, 0.2, 0.2]
	10	[2, 2, 2, 2, 2, 0.2, 0.2, 0.2, 0.2, 0.2]
	12	[2, 2, 2, 2, 2, 2, 0.2, 0.2, 0.2, 0.2, 0.2, 0.2]

4.4.3 Simulation Results

By simulation, we show two results, and one is that parameter q in q-Weibull distribution can approximate parameter θ in Clayton copula; the other one is the comparison of the q-Weibull model and the Clayton copula models.

- **q in the q-Weibull distribution can approximate θ in Clayton copula**

The ML estimated parameters for NHPP with the underlying q-Weibull distribution model (averages over 20 replications) are presented in Table 4-3 and Table 4-4.

Table 4-3: MLE parameters for systems with increasing hazard rate

Parameters for simulated systems			MLE parameters		
Copula parameter θ (equivalent q)	Number of components d	Components' shape parameters α_i, $i = 1, 2, \dots, d$	$\hat{q}$	$\hat{\beta}$	$\hat{\eta}$
$\theta = -0.3333$ ($q = 0.5$)	2	[2, 2]	0.4967	2.2878	4.9986
		[2, 3]	0.4952	2.6250	4.9107
	3	[2, 2, 2]	0.5139	3.3159	3.9025
		[2, 3, 5]	0.4767	3.7754	4.2483
	4	[2, 2, 2, 2]	0.4770	2.5762	3.2984
		[2, 2, 3, 5]	0.4733	2.5254	3.4405
$\theta = -0.2857$ ($q = 0.6$)	2	[2, 2]	0.5644	2.3926	4.8720
		[2, 3]	0.5663	2.1529	4.5009
	3	[2, 2, 2]	0.6219	2.9412	3.8951
		[2, 3, 5]	0.5279	2.7890	3.9746
	4	[2, 2, 2, 2]	0.5311	2.9951	3.6179
		[2, 2, 3, 5]	0.6081	4.2521	3.7457
$\theta = -0.1667$	2	[2, 2]	0.8202	2.4774	4.5530

$(q = 0.8)$		[2, 3]	0.8216	2.3907	3.9816
	3	[2, 2, 2]	0.7965	1.9926	3.2856
		[2, 3, 5]	0.7986	3.5096	3.8870
	5	[2, 2, 2, 2, 2]	0.8010	2.5192	2.7139
		[2, 2, 3, 5, 5]	0.8152	4.2286	3.2218
	6	[2, 2, 2, 2, 2, 2]	0.8298	2.4949	2.3182
		[2, 2, 3, 3, 5, 5]	0.8110	3.3100	2.8980

Table 4-4: MLE parameters for systems with a bathtub-shaped hazard rate

Parameters for simulated systems			MLE parameters		
Copula parameter θ (equivalent q)	Number of components d	Components' shape parameters α_i, $i = 1, 2, \ldots, d$	$\hat{q}$	$\hat{\beta}$	$\hat{\eta}$
$\theta = -0.3333$ $(q = 0.5)$	3	[2, 0.2, 0.2]	0.3626	0.5581	1.9567
	4	[2, 2, 0.2, 0.2]	0.3981	0.5951	1.6336
$\theta = -0.2857$ $(q = 0.6)$	3	[2, 0.2, 0.2]	0.4417	0.7399	2.9293
	4	[2, 2, 0.2, 0.2]	0.5160	0.7740	1.5874
$\theta = -0.1667$ $(q = 0.8)$	3	[2, 0.2, 0.2]	0.7913	0.8503	1.6090
	4	[2, 2, 0.2, 0.2]	0.7302	0.8534	1.2507
	5	[2, 2, 0.2, 0.2, 0.2]	0.7414	0.7342	0.8594

	6	[2, 2, 2, 0.2, 0.2, 0.2]	0.7047	0.5317	0.5003
$\theta = -0.0909$ ($q = 0.9$)	2	[1.2, 0.2]	0.9024	0.7373	3.2666
	4	[2, 2, 0.2, 0.2]	0.8841	0.9467	1.1295
	6	[2, 2, 2, 0.2, 0.2, 0.2]	0.8615	0.7669	0.6892
	8	[2, 2, 2, 2, 0.2, 0.2, 0.2, 0.2]	0.8663	0.5854	0.2955
	10	[2, 2, 2, 2, 2, 0.2, 0.2, 0.2, 0.2, 0.2]	0.8550	0.4048	0.0928
	12	[2, 2, 2, 2, 2, 2, 0.2, 0.2, 0.2, 0.2, 0.2, 0.2]	0.8400	0.3613	0.0828

The results in Table 4-3 and Table 4-4 suggest that the parameter q in the q-Weibull distribution can approximate the Clayton copula parameter θ in an equivalent form.

- **q-Weibull and Clayton copula models comparison**

We also compared the q-Weibull model, the Clayton copula models with and without information regarding the component cause for the system failure. The systems marked with a "*" in Table 4-1 and Table 4-2 were used for experimental settings. In each experiment, one sample of $n = 30$ failure times and the corresponding failed components were generated using the algorithm in Figure *4-9*, then the parameters were estimated through maximizing the log-likelihood functions in Equations (4-19), (4-22) and (4-29), respectively, for the three models. For the sake of clarity, *Clayton copula model 1* denotes the Clayton copula model that is estimated using information about which components failed, and *Clayton copula model 2* denotes the Clayton copula model that is estimated using no information about which components failed. In the data analysis, all the ML estimated parameters were obtained by an adaptive hybrid artificial bee colony algorithm [64]. We should mention that Clayton copula model 1 utilizes the information of both failure times and the corresponding failed components, while the q-Weibull model and Clayton copula model 2 only use the failure times and omit the information of corresponding failed components. Table 4-5 and Table 4-6 show the estimated parameters and log-likelihoods for the three models of the simulated systems with increasing hazard rate and bathtub-shaped hazard rate, respectively.

Table 4-5: Comparison between q-Weibull and copula models for systems with increasing hazard rate

Parameters for simulated systems	Estimated parameters	Log-likelihood
$\boldsymbol{\theta = -0.3333}$ ($q = 0.5$) (3 components) $[\alpha_1, \alpha_2, \alpha_3] = [2, 2, 2]$ $[\lambda_1, \lambda_2, \lambda_3] = [5, 5, 5]$	q-Weibull model: $\hat{q} = 0.4979$ ($\boldsymbol{\hat{\theta} = -0.3343}$), $\hat{\beta} = 2.0699$, $\hat{\eta} = 3.9533$	23.8943
	Clayton copula model 1: $\boldsymbol{\hat{\theta} = -0.2926}$ $[\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3] = [8.8988, 2.2566, 8.6938]$ $[\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3] = [5.7724, 2.8896, 6.1682]$	10.1252
	Clayton copula model 2: $\hat{\theta} = 42.1013$ $[\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3] = [1.2092, 1.0098\text{e+}08, 28.0569]$	41.8032

	$[\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3]$ = [4.081, 5.4970, 4.9764]	
$\boldsymbol{\theta = -0.2857}$ ($q = 0.6$) (3 components) $[\alpha_1, \alpha_2, \alpha_3] = [2, 2, 2]$ $[\lambda_1, \lambda_2, \lambda_3] = [5, 5, 5]$	q-Weibull model: $\hat{q} = 0.6814$ ($\boldsymbol{\hat{\theta} = -0.2416}$), $\hat{\beta} = 1.2065$, $\hat{\eta} = 2.3145$	21.5858
	Clayton copula model 1: $\boldsymbol{\hat{\theta} = -0.2406}$ $[\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3]$ = [0.9293, 2.4875, 1.4309] $[\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3]$ = [4.6923, 5.6968, 2.8343]	1.3052
	Clayton copula model 2: $\hat{\theta} = 36.3432$ $[\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3]$ = [2.8310e+15, 0.1868, 2.4731] $[\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3]$ = [5.9328, 0.2905, 3.1281]	48.7056
$\boldsymbol{\theta = -0.1667}$ ($q = 0.8$) (3 components) $[\alpha_1, \alpha_2, \alpha_3] = [2, 2, 2]$ $[\lambda_1, \lambda_2, \lambda_3] = [5, 5, 5]$	q-Weibull model: $\hat{q} = 0.8294$ ($\boldsymbol{\hat{\theta} = -0.1457}$), $\hat{\beta} = 1.5233$, $\hat{\eta} = 2.4425$	47.5582
	Clayton copula model 1: $\boldsymbol{\hat{\theta} = -0.1530}$ $[\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3]$ = [1.8296, 1.6244, 1.4890] $[\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3]$ = [5.0506, 3.9624, 3.9404]	12.4829
	Clayton copula model 2: $\hat{\theta} = 22.0311$ $[\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3]$ = [2.1215, 9.7726, 958.2601]	53.6636

	$[\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3]$ = [2.2702, 5.4771, 7.7224]	

Table 4-6: Comparison between q-Weibull and copula models for systems with bathtub-shaped hazard rate

Parameters for simulated systems	Estimated parameters	Log-likelihood
$\boldsymbol{\theta = -0.2857}$ ($q = 0.6$) (4 components) $[\alpha_1, \alpha_2, \alpha_3, \alpha_4]$ = [2, 2, 0.2, 0.2] $[\lambda_1, \lambda_2, \lambda_3, \lambda_4]$ $= [5, 5, 5, 5]$	q-Weibull model: $\hat{q} = 0.5938$ ($\boldsymbol{\hat{\theta} = -0.2889}$), $\hat{\beta} = 0.6944$, $\hat{\eta} = 1.3656$	32.7652
	Clayton copula model 1: $\boldsymbol{\hat{\theta} = -0.2264}$ $[\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\alpha}_4]$ = [1.6432, 2.0870, 67.0353, 0.1912] $[\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3, \hat{\lambda}_4]$ = [2.6412, 3.5363, 5.2235, 45.8162]	13.4870
	Clayton copula model 2: $\hat{\theta} = 31.8880$ $[\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\alpha}_4]$ = [3.8188, 35.9773, 0.7425, 9.5270e+05] $[\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3, \hat{\lambda}_4]$ = [2.4752, 4.5776, 0.4954, 4.9862]	47.7635
$\boldsymbol{\theta = -0.1667}$	q-Weibull model:	54.3851

($q = 0.8$)	$\hat{q} = 0.8359$ ($\boldsymbol{\hat{\theta} = -0.1410}$), $\hat{\beta} = 0.5507$, $\hat{\eta} = 0.2928$	
(4 components) $[\alpha_1, \alpha_2, \alpha_3, \alpha_4] = [2, 2, 0.2, 0.2]$ $[\lambda_1, \lambda_2, \lambda_3, \lambda_4] = [5, 5, 5, 5]$	Clayton copula model 1: $\boldsymbol{\hat{\theta} = -0.0926}$ $[\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\alpha}_4]$ = [2.1284, 2.5218, 0.2685, 0.0038] $[\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3, \hat{\lambda}_4]$ = [3.7955, 4.1438, 0.3812, 2.5892]	23.2597
	Clayton copula model 2: $\hat{\theta} = 14.7487$ $[\hat{\alpha}_1, \hat{\alpha}_2, \hat{\alpha}_3, \hat{\alpha}_4]$ = [787.0128, 16.6465, 0.3833, 2.2429] $[\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3, \hat{\lambda}_4]$ = [7662.5168, 6.2063, 0.0862, 1.8009]	60.0837

It can be observed that both the q-Weibull model and Clayton copula model 1 can approximate the parameter θ, while Clayton copula model 2 cannot. Specifically, our proposed q-Weibull model can successfully approximate the parameter θ using only the failure times data; it does not require knowing which components failed. In comparison, the Clayton copula model 2 cannot recover the dependence parameter θ with only the failure times data. Although, as shown in Table 4-5 and Table 4-6, the Clayton copula model 2 has a higher Log-likelihood than the proposed q-Weibull model does, it contains too many parameters given the limited data, which can easily cause overfitting and yield inaccurate parameter estimates. To recover the parameter θ, the Clayton copula model requires information about which components failed, which the Clayton copula model 1 does.

Based on the experimental results presented in this section, the q-Weibull distribution model is a good approximation to the series system with dependence describe by Clayton copula, covering increasing and bathtub-shaped behaviors of the q-Weibull intensity function. Therefore, the proposed NHPP with q-Weibull as an underlying distribution model could be used to tackle a real reliability problem.

4.5 *Modified Kolmogorov-Smirnov (KS) Goodness-of-Fit Test*

We used a modified Kolmogorov-Smirnov (KS) goodness-of-fit test to check the hypothesis that failure time data of a repairable system can be fitted by a NHPP q-Weibull model. Modified means that the parameters for the NHPP q-Weibull intensity function are replaced by their maximum likelihood estimates. The critical values of the modified KS statistics under the null hypothesis were generated via a Monte Carlo simulation following the approach proposed by Park and Kim[65]. In this approach, as shown in Figure 4-10, D_0 is the modified Kolmogorov-Smirnov statistic:

$$D_0 = \max\{D^+, D^-\}, \text{ where}$$

$$D^+ = \max_{1 \le i \le n} \left\{ \frac{i}{n} - \frac{\hat{H}(t_i)}{\hat{H}(t_n)} \right\},$$

$$D^- = \max_{1 \le i \le n} \left\{ \frac{\hat{H}(t_i)}{\hat{H}(t_n)} - \frac{i-1}{n} \right\}, \text{ and}$$

$\widehat{H}(t_i)$ is the cumulative intensity function with estimated parameters $\hat{q}, \hat{\beta}, \hat{\eta}$.

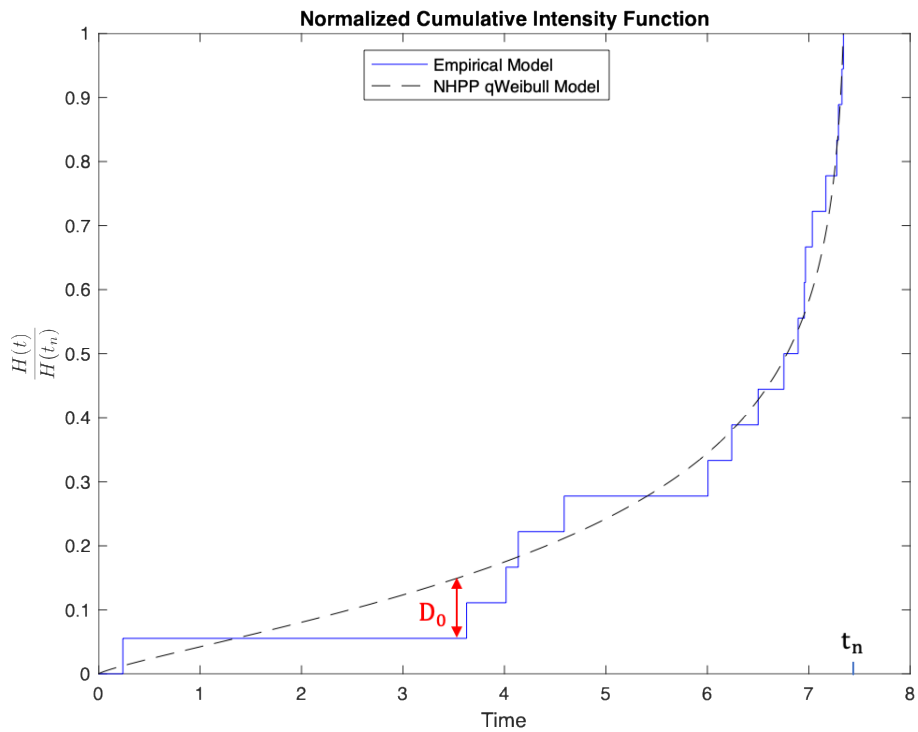

Figure 4-10: The normalized empirical and fitted cumulative intensity function and the illustration of D_0

Table 4-7: The goodness-of-fit test simulation settings

Case	Copula parameter θ (equivalent q)	Number of components d	Components' shape parameters α_i, $i = 1, 2, \ldots, d$
1	$\theta = -0.2857$ ($q = 0.6$)	3	[2, 0.2, 0.2]
2	$\theta = -0.3333$ ($q = 0.5$)	3	[2, 0.2, 0.2]

3	$\theta = -0.1667$ ($q = 0.8$)	3	[2, 0.2, 0.2]
4	$\theta = -0.1667$ ($q = 0.8$)	4	[2, 2, 0.2, 0.2]
5	$\theta = -0.2857$ ($q = 0.6$)	4	[2, 2, 0.2, 0.2]

We considered five cases in Table 4-7. For each case, we first generated a set of n = 30 system's failure times and used these to estimate the parameters $\hat{q}, \hat{\beta}, \hat{\eta}$ for the NHPP with underlying q-Weibull distribution model, results are shown in Table 4-8. Notice that the parameters were estimated from one sample, while those parameters in Table 4-3 and Table 4-4 were averages of 20 replications. We then generated 999 sets of samples $t^j = \{t_1^j, t_2^j, \dots, t_n^j\}, (t_1^j < t_2^j < \dots < t_n^j), j = 1, 2, \dots, 999$; each set had n = 30 samples from the q-Weibull distribution with the estimated parameters $\hat{q}, \hat{\beta}, \hat{\eta}$. For each set, we determined $\hat{q}^j, \hat{\beta}^j, \hat{\eta}^j$, the maximum likelihood estimates for the j-th sample set. The test statistic D^j was computed with $\hat{q}^j, \hat{\beta}^j, \hat{\eta}^j$ in place of $\hat{q}, \hat{\beta}, \hat{\eta}$. This yielded 1000 observations of Kolmogorov-Smirnov test statistic D. We counted the number of observations where D^j exceeds D_0 and computed the p-value as this number divided by 1000. For the five cases, we determined the goodness-of-fit of the NHPP q-Weibull model, the modified Kolmogorov-Smirnov statistic D_0 and the p-value are shown in the Table 4-8. We can observe that mostly the cases have low KS test statistic and high p-value, thus the NHPP q-Weibull model can fit the data sets.

Table 4-8: The goodness-of-fit results for five cases

Case	NHPP estimated parameters				
	$\hat{q}$	$\hat{\beta}$	$\hat{\eta}$	D_0	p-value
1	0.5576	0.2447	0.2630	0.1286	0.600
2	0.5212	0.5974	1.8523	0.1539	0.638
3	0.7548	0.7532	1.9073	0.1150	0.648
4	0.8199	1.1574	1.7714	0.0827	0.883
5	0.5801	0.2387	0.1326	0.1878	0.167

4.6 *Confidence Intervals*

We developed the asymptotic confidence intervals for estimated parameters of the NHPP q-Weibull and Clayton copula models. According to the asymptotic properties of the maximum likelihood estimators, for the NHPP q-Weibull model parameters $\hat{q}, \hat{\beta}, \hat{\eta}$, the related covariance matrix associated with the ML estimators can be estimated by the inverse of the observed information matrix $I\left(\hat{q}, \hat{\beta}, \hat{\eta} \middle| t\right)$, the negative of the second derivation of the log-likelihood function in Equation (4-19) evaluated at the point estimates $\hat{q}, \hat{\beta}, \hat{\eta}$ given data t [66][67].

$$\widehat{var}\left(\hat{q},\hat{\beta},\hat{\eta}\middle|t\right) = I^{-1}\left(\hat{q},\hat{\beta},\hat{\eta}\middle|t\right)$$

$$= -\begin{bmatrix} \frac{\partial^2 L\left(\hat{q},\hat{\beta},\hat{\eta}\middle|t\right)}{\partial\hat{q}^2} & \frac{\partial^2 L\left(\hat{q},\hat{\beta},\hat{\eta}\middle|t\right)}{\partial\hat{q}\partial\hat{\beta}} & \frac{\partial^2 L\left(\hat{q},\hat{\beta},\hat{\eta}\middle|t\right)}{\partial\hat{q}\partial\hat{\eta}} \\ \frac{\partial^2 L\left(\hat{q},\hat{\beta},\hat{\eta}\middle|t\right)}{\partial\hat{\beta}\partial\hat{q}} & \frac{\partial^2 L\left(\hat{q},\hat{\beta},\hat{\eta}\middle|t\right)}{\partial\hat{\beta}^2} & \frac{\partial^2 L\left(\hat{q},\hat{\beta},\hat{\eta}\middle|t\right)}{\partial\hat{\beta}\partial\hat{\eta}} \\ \frac{\partial^2 L\left(\hat{q},\hat{\beta},\hat{\eta}\middle|t\right)}{\partial\hat{\eta}\partial\hat{q}} & \frac{\partial^2 L\left(\hat{q},\hat{\beta},\hat{\eta}\middle|t\right)}{\partial\hat{\eta}\partial\hat{\beta}} & \frac{\partial^2 L\left(\hat{q},\hat{\beta},\hat{\eta}\middle|t\right)}{\partial\hat{\eta}^2} \end{bmatrix}^{-1} \tag{4-35}$$

This information matrix $I\left(\hat{q},\hat{\beta},\hat{\eta}\middle|t\right)$ can be computed using 'hessian' function in MATLAB Symbolic Math Toolbox. The details can be found on the GitHub.

The asymptotic confidence intervals with $(1-\gamma)\cdot 100\%$ of confidence for $\hat{q},\hat{\beta},\hat{\eta}$ are given by, respectively:

$$CI[\hat{q},(1-\gamma)\cdot 100\%] = \left[\hat{q} + z_{\frac{\gamma}{2}}\sqrt{\widehat{var}_{11}};\ \hat{q} + z_{1-\frac{\gamma}{2}}\sqrt{\widehat{var}_{11}}\right] \tag{4-36}$$

$$CI\left[\hat{\beta},(1-\gamma)\cdot 100\%\right] = \left[\hat{\beta} + z_{\frac{\gamma}{2}}\sqrt{\widehat{var}_{22}};\ \hat{\beta} + z_{1-\frac{\gamma}{2}}\sqrt{\widehat{var}_{22}}\right] \tag{4-37}$$

$$CI[\hat{\eta},(1-\gamma)\cdot 100\%] = \left[\hat{\eta} + z_{\frac{\gamma}{2}}\sqrt{\widehat{var}_{33}};\ \hat{\eta} + z_{1-\frac{\gamma}{2}}\sqrt{\widehat{var}_{33}}\right] \tag{4-38}$$

Where $z_{\frac{\gamma}{2}}, z_{1-\frac{\gamma}{2}}$ are the $\frac{\gamma}{2}$ and $1-\frac{\gamma}{2}$ quantiles of the standard normal distribution and $\widehat{var}_{11}, \widehat{var}_{22}, \dots, \widehat{var}_{33}$ are the diagonal elements of the covariance matrix associated with the maximum likelihood estimators $\hat{q},\hat{\beta},\hat{\eta}$.

Similarly, the covariance matrix associated with the Clayton copula model can be obtained by taking the negative of the second derivation of the log-likelihood function in Equation (4-29).

4.7 *Application to Machine Failure Data*

To illustrate the proposed model, we applied it to the failure data of a load-haul-dump (LHD) machine (see Figure 4-11). The LHD machine is modeled as a series system with six subsystems: engine (E), hydraulics (H), transmission (Tr), brakes (B), tires and wheels (T), and others (O) (including body, cabin, and chassis). The reliability block diagram of the LHD machine is shown in Figure 4-12. The example is a data set of 44 failure times (in hours) for LHD A machine given by Kumar et al. [7]; these are shown in Table 4-9 with the abbreviation of the subsystem that failed at that time. The data appear to have a bathtub-shaped failure intensity. For failure data with non-monotonic intensity, commonly used distributions such as the Weibull distribution are usually inappropriate. Pulcini [8] proposed a Superposed-PLP (S-PLP) model that superpositions two independent power law processes to fit this data set, which confirmed the bathtub shape of the failure intensity. We used the q-Weibull distribution to analyze this failure data.

Figure 4-11: An LHD machine [7]

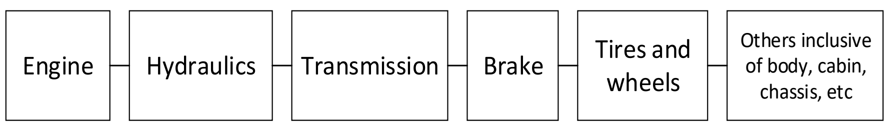

Figure 4-12: A reliability block diagram of an LHD machine [7]

Table 4-9: Failure times of LHD A machine [7]. The letters refer to the subsystems that failed: E=Engine; H=Hydraulics; Tr=Transmission; B=Brake; T=Tires and wheels; O=Others inclusive of body, cabin, and chassis.

16 (E)	39 (B)	71 (B)	95 (B)	98 (T)	110 (B)	114 (B)	226 (E)	294 (E)
344 (Tr)	555 (E)	599 (T)	757 (E)	822 (Tr)	963 (E)	1077 (B)	1167 (E)	1202 (B)
1257 (O)	1317 (Tr)	1345 (E)	1372 (E)	1402 (B)	1536 (E)	1625 (B)	1643 (B)	1675 (E)
1726 (H)	1736 (H)	1772 (E)	1796 (Tr)	1799 (E)	1814 (O)	1868 (H)	1894 (O)	1970 (E)
2042 (T)	2044 (H)	2094 (B)	2127 (T)	2291 (E)	2295 (T)	2299 (T)	2317 (T)	

We assumed that the LHD machine had minimal repair, and we applied the proposed three models to analyze this data set. For the Clayton copula models, we specified all the subsystems' failure times follow Weibull distribution with a hazard rate $h_i(x) = \frac{\alpha_i}{\lambda_i}\left(\frac{x}{\lambda_i}\right)^{\alpha_i - 1}$, for $i = 1,2, \dots, 6$, where α_i is shape parameter and λ_i is a scale parameter for the *i*-th subsystem.

For comparison, instead of q-Weibull as underlying intensity function, we also considered the NHPP model with other two intensity functions, one was a Weibull distribution with the intensity function $h(t) = \frac{\alpha}{\lambda}\left(\frac{t}{\lambda}\right)^{\alpha-1}$ and another one was an S-PLP model with the intensity function $h(t) = \frac{\beta_1}{\alpha_1}\left(\frac{t}{\alpha_1}\right)^{\beta_1-1} + \frac{\beta_2}{\alpha_2}\left(\frac{t}{\alpha_2}\right)^{\beta_2-1}$ [8]. Moreover, we considered the independent models, in which the subsystems are independent with each other. The independent models are special cases of the Clayton copula models with $\theta = 0$. *Independent model 1* represents the independent model known the component cause for the system failure, and *independent model 2* represents the independent model unknown the component cause for the system failure.

All the parameters were estimated by optimizing the ML problems using the adaptive hybrid artificial bee colony algorithm [64]. Table 4-10 compares the performance of these models. The proposed NHPP q-Weibull model yielded the smallest Akaike information criterion (AIC) and Bayesian information criterion (BIC) values. Using the modified Kolmogorov-Smirnov (KS) goodness-of-fit test approach described in Section 4.5, we determined the goodness-of-fit of these models, the modified Kolmogorov-Smirnov statistic D_0 and the p-value are also shown in Table 4-10. Both the Clayton copula model 1 with known component cause and the proposed NHPP q-Weibull model have high p-value of 0.972 and 0.971, respectively, which indicates these two models can fit the data very well. The NHPP Weibull model has the lowest p-value of 0.03. Besides, the NHPP q-Weibull model with the shape parameters q and β shows that LHD machine has a bathtub-shaped failure intensity, which has also been observed by Pulcini [8]. Moreover, the parameter q in the PDF of q-Weibull distribution is estimated as $\hat{q} = 0.9495$, the corresponding approximation to the Clayton copula parameter $\hat{\theta} = \frac{1}{2-\hat{q}} - 1 = -0.0481$. The results from the Clayton copula model 1 with information regarding exact component cause show that the dependence among the subsystems exists with degree characterized by $\hat{\theta} = -0.0245$. This dependence can be approximated by the q-Weibull distribution.

Table 4-10: Comparison of NHPP and Clayton copula models.

AIC = Akaike information criterion; BIC = Bayesian information criterion.

Models	Parameters	Log-likelihood	AIC	BIC	D_0	p-value
NHPP q-Weibull	$\hat{q} = 0.9495$ $\hat{\beta} = 0.5652$ $\hat{\eta} = 15.1495$	-207.9311	421.8622	427.2148	0.0764	0.971
NHPP S-PLP	$\hat{\alpha}_1 = 11.89$ $\hat{\beta}_1 = 0.603$ $\hat{\alpha}_2 = 912$ $\hat{\beta}_2 = 3.211$	-207.4867	422.9735	430.1102	0.0742	0.578
NHPP Weibull	$\hat{\alpha} = 0.9257$ $\hat{\lambda} = 40.8725$	-210.3067	424.6134	428.1818	0.1746	0.030

Clayton Copula Model 1	$\hat{\alpha}_1 = 0.7722$ $\hat{\lambda}_1 = 114.6580$ $\hat{\alpha}_2 = 0.4395$ $\hat{\lambda}_2 = 20.3106$ $\hat{\alpha}_3 = 3.6608$ $\hat{\lambda}_3 = 1937.33$ $\hat{\alpha}_4 = 0.8612$ $\hat{\lambda}_4 = 770.2656$ $\hat{\alpha}_5 = 1.1892$ $\hat{\lambda}_5 = 687.9873$ $\hat{\alpha}_6 = 2.3211$ $\hat{\lambda}_6 = 1913.7023$ $\hat{\theta} = -0.0245$	-271.8093	569.6186	592.8131	0.0752	0.972
Clayton Copula Model 2	$\hat{\alpha}_1 = 1.1823$ $\hat{\lambda}_1 = 104.0686$ $\hat{\alpha}_2 = 0.5896$ $\hat{\lambda}_2 = 9.2704$ $\hat{\alpha}_3 = 7.3851$ $\hat{\lambda}_3 = 1395.8181$ $\hat{\alpha}_4 = 2.2130$ $\hat{\lambda}_4 = 6638.3318$ $\hat{\alpha}_5 = 1.3512$ $\hat{\lambda}_5 = 2810258.52$ $\hat{\alpha}_6 = 33.0909$ $\hat{\lambda}_6 = 72870.6967$ $\hat{\theta} = 16.8148$	-202.7734	431.5468	454.7413	0.0698	0.788
Independent Model 1	$\hat{\alpha}_1 = 0.9071$ $\hat{\lambda}_1 = 126.32$ $\hat{\alpha}_2 = 0.5142$ $\hat{\lambda}_2 = 26.3187$ $\hat{\alpha}_3 = 4.3294$ $\hat{\lambda}_3 = 1682.14$ $\hat{\alpha}_4 = 1.0629$ $\hat{\lambda}_4 = 628.7710$ $\hat{\alpha}_5 = 1.4752$ $\hat{\lambda}_5 = 619.5290$ $\hat{\alpha}_6 = 2.8359$ $\hat{\lambda}_6 = 1572.83$	-272.1280	568.2560	589.6663	0.1052	0.729

Independent Model 2	$\hat{\alpha}_1 = 0.6082$ $\hat{\lambda}_1 = 161.5754$ $\hat{\alpha}_2 = 0.6082$ $\hat{\lambda}_2 = 54.3404$ $\hat{\alpha}_3 = 156.2540$ $\hat{\lambda}_3 = 2302.1496$ $\hat{\alpha}_4 = 0.6082$ $\hat{\lambda}_4 = 75.0654$ $\hat{\alpha}_5 = 2.6416$ $\hat{\lambda}_5 = 788.0360$ $\hat{\alpha}_6 = 0.6082$ $\hat{\lambda}_6 = 1998.8373$	-213.2592	450.5185	471.9288	0.0825	0.350

Note: in the Clayton Copula Models and Independent Models (special cases of Clayton Copula Models with $\theta = 0$), the estimators $\hat{\alpha}_{\mathrm{i}}$, $\hat{\lambda}_{\mathrm{i}}$ ($j = 1, 2, \ldots, 6$) represent shape parameter and scale parameter for subsystems: Engine, Brake, Hydraulics, Transmission, Tire, and Other.

The cumulative intensity functions and intensity functions for the above models are compared in Figure 4-13 and Figure 4-14, respectively. From Figure 4-13, with the exception of the NHPP Weibull model and the independent model with known component cause, one can observe that the models fit this data relatively well. Moreover, Figure 4-14 gives several interesting observations. Firstly, only the NHPP Weibull model shows monotonically decreasing intensity function, whereas the other models show a bathtub-shaped intensity function. Secondly, the proposed NHPP q-Weibull model (which has the advantage of fewer parameters to be estimated) is comparable with the Clayton copula model 1, which needs more information regarding the exact component causing the system failure. Thirdly, both the Clayton copula model 2 with unknown component cause and the independent model 2 with an unknown component cause have a rapid increase at the end of the failure intensity function curve, while the Clayton copula model 2 with unknown component cause also has a jump around time 1200. This result occurs due to data overfitting because while these models have 12 or 13 parameters, the sample size of 44 is relatively small. As shown in Table 4-10, some of the estimated parameters are unreasonably large. In comparison, provided the information about which component caused the system to fail, one can observe that the Clayton copula model 1's failure intensity curve is smoother.

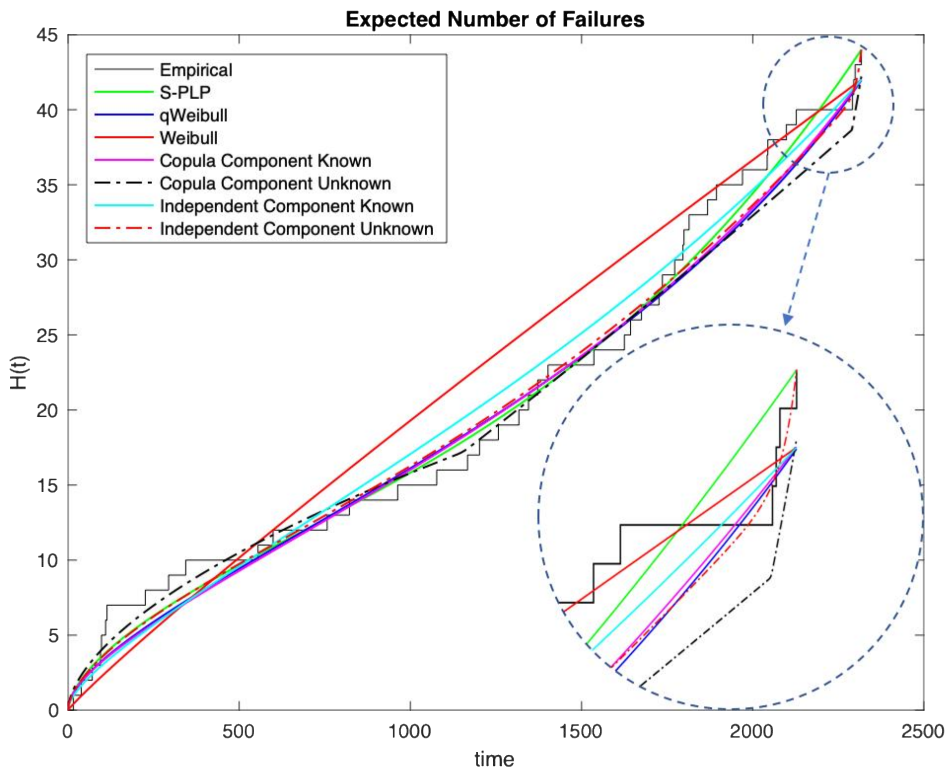

Figure 4-13: Comparison of cumulative intensity for LHD machine failures

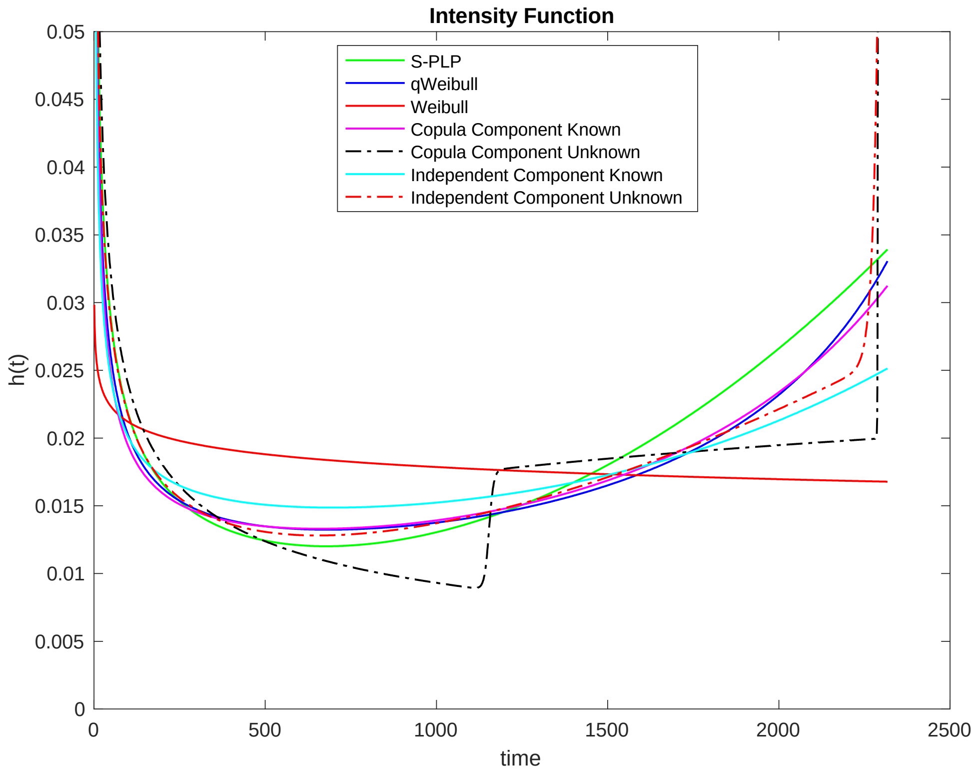

Figure 4-14: Comparison of intensity functions for LHD machine failures

Results from this application example show that the proposed NHPP with q-Weibull distribution could model the series system comprising dependent subsystems characterized by Clayton copula. The shape parameter q in the q-Weibull distribution is connected with the Clayton copula parameter θ, which measures the degree of dependence among the subsystems.

We also provide the asymptotic confidence intervals for estimated parameters of the NHPP q-Weibull model and the Clayton copula model 1 in the application example according to the approach proposed in Section 4.6. The confidence intervals are shown in Table 4-11 and Table 4-12. Note that the negative lower bound of scale parameter η may be due to the small sample size. Compared to the Clayton copula model, the proposed NHPP q-Weibull model has fewer parameters with smaller uncertainty.

Table 4-11: Asymptotic confidence intervals for NHPP q-Weibull model

Parameters	90% Confidence Intervals
$\hat{q} = 0.9495$	[0.9082, 0.9908]
$\hat{\beta} = 0.5652$	[0.2671, 0.8633]
$\hat{\eta} = 15.1495$	[-15.7541, 46.0531]

Table 4-12: Asymptotic confidence intervals for Clayton copula and subsystems

Subsystems	Parameters	90% Confidence Intervals
Engine	$\hat{k} = 0.7722$ $\hat{\alpha} = 114.6580$	[0.3205, 1.2240] [-72.5527, 301.8687]
Brake	$\hat{k} = 0.4395$ $\hat{\alpha} = 20.3106$	[0.1615, 0.7175] [-41.7100, 82.3313]
Hydraulics	$\hat{k} = 3.6608$ $\hat{\alpha} = 1937.3287$	[0.0373, 7.2843] [1105.2781, 2769.3794]
Transmission	$\hat{k} = 0.8612$ $\hat{\alpha} = 770.2656$	[-0.0025, 1.7249] [-461.8302, 2002.3614]
Tire	$\hat{k} = 1.1892$ $\hat{\alpha} = 687.9873$	[0.2250, 2.1534] [-78.5869, 1454.5615]
Other	$\hat{k} = 2.3211$ $\hat{\alpha} = 1913.7023$	[-0.3307, 4.9730] [649.1040, 3178.3005]
System Copula	$\hat{\theta} = -0.0245$	[-0.0695,0.0206]

4.8 Summary

In this chapter, we have shown that the q-Weibull distribution can model a series system with dependent component failure times that are described by Clayton copula and that the parameter q can approximate the Clayton copula parameter θ, which measures the degree of dependence. We have also proposed the NHPP with q-Weibull as the underlying time to first failure (TTFF) distribution model as an approximation to the minimal repair process of a series system composed of multiple dependent components characterized by Clayton copula. The maximum likelihood (ML) method was developed to estimate the model parameters, and asymptotic confidence intervals based on ML asymptotic theory were also developed.

Simulation experiments were conducted to validate the proposed NHPP q-Weibull model and showed that parameter q could approximate the parameter θ in Clayton copula, for both systems with monotonic and non-monotonic failure intensity functions. Estimating the parameters of the q-Weibull model does not require information about which components failed, which is necessary for accurately estimating the parameters of the Clayton model. In the simulation, we developed a sampling method for conditional failure times of dependent subsystems modeled by Clayton copula. A modified Kolmogorov-Smirnov (KS) goodness-of-fit test statistic and p-value were used to determine the goodness-of-fit of the proposed NHPP q-Weibull model.

The proposed model and parameter estimation procedure have been successfully applied to a real failure data set of a load-haul-dump (LHD) machine characterized by a bathtub-shaped intensity function. The results have shown that the proposed NHPP q-Weibull model has the advantage of fewer parameters with smaller uncertainty when used as an approximation to the Clayton copula approach, which in turn needs more information on the assumption for the underlying distributions of components and the exact component cause of system failure. The goodness-of-fit test results also have agreed that the proposed NHPP q-Weibull model outperformed other commonly used minimal repair process models, including NHPP Weibull and NHPP S-PLP, and the independent models (special cases of the Clayton copula model with $\theta = 0$).

Chapter 5: Modeling Dependent Parallel Systems

5.1 Overview

This chapter considers the problems of modeling a parallel system with dependent component failure times and estimating the model's parameters from failure time data. We propose a q-Fréchet distribution, which can be used to approximate the distribution of the failure time of a parallel system with dependent component failure times that are modeled as a Clayton copula. Similar to the q-Weibull distribution, the parameter q in q-Fréchet distribution is an approximation to the parameter θ in Clayton copula, which measures the degree of dependence among the components. The maximum likelihood method is used for the model parameters estimation. A simulation study is conducted to evaluate q-Fréchet approximation. This chapter also shows that the q-Fréchet distribution can model the parallel system with common cause dependence. The proposed q-Fréchet distribution is applied to a data set of 18 two-motor parallel system's failure times.

5.2 q-Fréchet Distribution

Fréchet distribution, also known as inverse Weibull distribution, is the type II generalized extreme value (GEV) distribution, which is the limit distribution of properly normalized maxima of a sequence of independent and identically distributed random variables.

The cumulative distribution function (CDF) of Fréchet distribution is as follows:

$$F_f(t) = exp\left(-\left(\frac{t}{\sigma}\right)^{-\xi}\right), \tag{5-1}$$

where $\xi > 0$ is shape parameter and $\sigma > 0$ is scale parameter, and support is $t > 0$.

Similar to the q-type generalization of Weibull distribution, by the substitution of the exponential function by a q-exponential [30], we develop a q-Fréchet distribution with the CDF as follows:

$$F_{qf}(t) = exp_q\left(-\left(\frac{t}{\sigma}\right)^{-\xi}\right), \tag{5-2}$$

where the q-Exponential function $\exp_q(x)$ is defined as:

$$exp_q(x) = \begin{cases} (1 + (1-q)x)^{\frac{1}{1-q}}, & \text{if } 1 + (1-q)x > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{5-3}$$

Therefore, the q-Fréchet CDF can be rewritten as:

$$F_{qf}(t) = \left[1 - (1-q)\left(\frac{t}{\sigma}\right)^{-\xi}\right]^{\frac{1}{1-q}}. \tag{5-4}$$

The probability density function (PDF) of the q-Fréchet distribution is as follows:

$$f_{qf}(t) = \frac{\xi}{\sigma}\left(\frac{t}{\sigma}\right)^{-\xi-1}\left[1 - (1-q)\left(\frac{t}{\sigma}\right)^{-\xi}\right]^{\frac{q}{1-q}}, \tag{5-5}$$

where $q > 0$ and $\xi > 0$ are shape parameters, and $\sigma > 0$ is a scale parameter.

In the limit $q \to 1$, $F_{qf}(t)$ reduces to the Fréchet CDF $F_f(t)$.

Then, the hazard rate function is defined as:

$$h_{qf}(t) = \frac{f_{qf}(t)}{R_{qf}(t)} = \frac{\frac{\xi}{\sigma}\left(\frac{t}{\sigma}\right)^{-\xi-1}\left[1-(1-q)\left(\frac{t}{\sigma}\right)^{-\xi}\right]^{\frac{q}{1-q}}}{1-\left[1-(1-q)\left(\frac{t}{\sigma}\right)^{-\xi}\right]^{\frac{1}{1-q}}}. \quad (5\text{-}6)$$

Equation (5-6) can represent different types of hazard rate functions according to the values of the shape parameters. Figure 5-1 illustrates the different behaviors of $h_{qf}(t)$ for $\sigma = 5$ and specific values of the shape parameters q and ξ. Table 5-1 shows a comparison between q-Weibull distribution and q-Fréchet distribution.

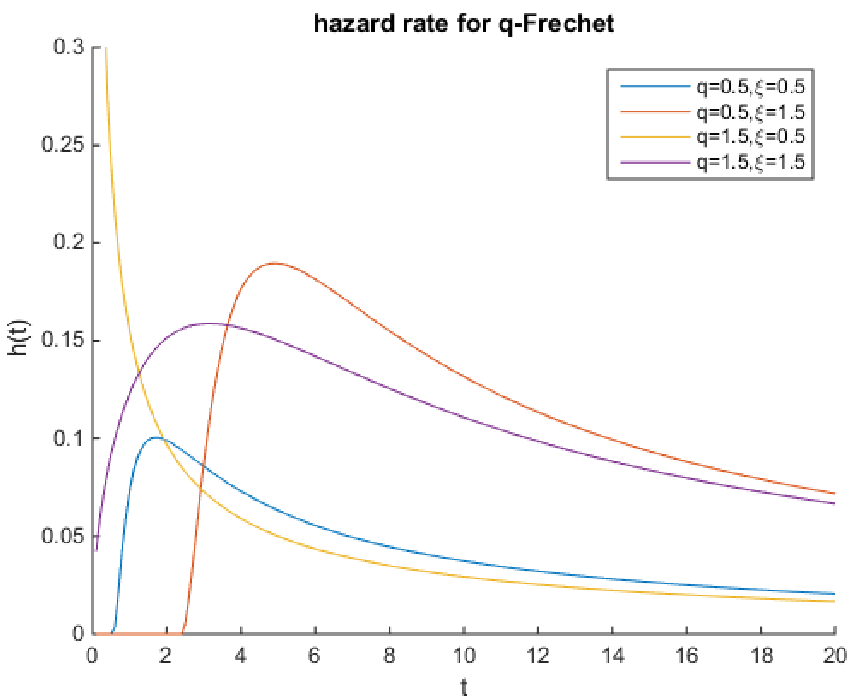

Figure 5-1: Behaviors of the q-Fréchet hazard rate function for $\sigma = 5$ and different values of the shape parameters q and ξ.

Table 5-1: Comparison between q-Weibull distribution and q-Fréchet distribution.

	q-Weibull	q-Fréchet
$R(t)\ or\ F(t)$	$R(t) = exp_q\left(-\left(\frac{t}{\eta}\right)^{\beta}\right) = \left[1-(1-q)\left(\frac{t}{\eta}\right)^{\beta}\right]^{\frac{1}{1-q}}$	$F(t) = exp_q\left(-\left(\frac{t}{\sigma}\right)^{-\xi}\right) = \left[1-(1-q)\left(\frac{t}{\sigma}\right)^{-\xi}\right]^{\frac{1}{1-q}}$
$f(t)$	$f(t) = \frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1}\left[1-(1-q)\left(\frac{t}{\eta}\right)^{\beta}\right]^{\frac{q}{1-q}}$	$f(t) = \frac{\xi}{\sigma}\left(\frac{t}{\sigma}\right)^{-\xi-1}\left[1-(1-q)\left(\frac{t}{\sigma}\right)^{-\xi}\right]^{\frac{q}{1-q}}$
Parameters	Shape parameters: q and β Scale parameter: η $q > 0$, $\beta > 0$ and $\eta > 0$	Shape parameters: q and ξ Scale parameter: σ $q > 0$, $\xi > 0$ and $\sigma > 0$
Base model	Weibull $R(t) = exp\left(-\left(\frac{t}{\eta}\right)^{\beta}\right)$	Fréchet $F(t) = exp\left(-\left(\frac{t}{\sigma}\right)^{-\xi}\right)$
Applicable system structure	Series $t = \min\{x_1, x_2, \dots, x_d\}$	Parallel $t = \max\{x_1, x_2, \dots, x_d\}$
Dependence model	Clayton survival copula $R(x_1, x_2, \dots, x_d) = \hat{C}\big(R_1(x_1), R_2(x_2), \dots, R_d(x_d)\big) = \left[R_1^{-\theta}(x_1) + R_2^{-\theta}(x_2) + \cdots + R_d^{-\theta}(x_d) - d + 1\right]^{-\frac{1}{\theta}}$	Clayton copula $F(x_1, x_2, \dots, x_d) = C\big(F_1(x_1), F_2(x_2), \dots, F_d(x_d)\big) = \left[F_1^{-\theta}(x_1) + F_2^{-\theta}(x_2) + \cdots + F_d^{-\theta}(x_d) - d + 1\right]^{-\frac{1}{\theta}}$
Assumption for component	$R_i(t) = 1 - \left(\frac{t}{\lambda}\right)^{\alpha} + o\left(\left(\frac{t}{\lambda}\right)^{\alpha}\right)$ $\frac{t}{\lambda} \to 0$	$F_i(t) = 1 - \left(\frac{t}{\lambda}\right)^{-\alpha} + o\left(\left(\frac{t}{\lambda}\right)^{-\alpha}\right)$ $\frac{t}{\lambda} \to \infty$

5.3 *Maximum Likelihood Estimation for Parameters*

The parameters of the q-Fréchet distribution are estimated via the maximum likelihood estimation method. Let $\underline{t} = (t_1, t_2, \dots, t_n)$ be an n-dimensional vector of observed failure times $t_i, i = 1, \dots, n$, independently drawn from a q-Fréchet distribution. The likelihood function is given by:

$$L(\underline{t}|\sigma, \xi, q) = \prod_{i=1}^{n} f_{qf}(t_i)$$
$$= \prod_{i=1}^{n} \frac{\xi}{\sigma}\left(\frac{t}{\sigma}\right)^{-\xi-1}\left[1-(1-q)\left(\frac{t}{\sigma}\right)^{-\xi}\right]^{\frac{q}{1-q}}. \tag{5-7}$$

The log-likelihood function is as follows:

$$\mathcal{L}(\underline{t}|\eta, \beta, q) = nln(\xi) + n\xi \ln(\sigma) + (-\xi - 1)\sum_{i=1}^{n} \ln(t_i) +$$
$$\frac{q}{1-q}\sum_{i=1}^{n} \ln\left[1-(1-q)\left(\frac{t_i}{\sigma}\right)^{-\xi}\right]. \tag{5-8}$$

5.4 *Modeling System Failure Time*

This section considers the time to failure of a parallel system with dependent component failure times described by Clayton copula and shows that a q-Fréchet distribution can approximate the time to failure distribution.

Herein, we use the notation $o(x)$ [58] to denote a function of x that satisfies the following property: $\lim_{x\to 0} \frac{o(x)}{x} = 0$.

5.4.1 Clayton Copula Model

Consider a system with d components in parallel. The dependence of these d failure times can be described by a Clayton copula. Let the random vector $(X_1, X_2, \dots, X_d)$ represents the lifetimes of the d components. Let $F_i(x_i) = \Pr(X_i < x_i), i = 1, \dots, d$ be the marginal distribution function. Let $C(F_1(x_1), F_2(x_2), \dots, F_d(x_d))$ be a Clayton copula. Assume the joint distribution function of the vector $(X_1, X_2, \dots, X_d)$ can be modeled as the Clayton copula:

$$\begin{aligned} \mathrm{P}\{X_1 < x_1, X_2 < x_2, \dots, X_d < x_d\} &= C\big(F_1(x_1), F_2(x_2), \dots, F_d(x_d)\big) \\ &= \big[F_1^{-\theta}(x_1) + F_2^{-\theta}(x_2) + \cdots \\ &\quad + F_d^{-\theta}(x_d) - d + 1\big]^{-\frac{1}{\theta}}. \end{aligned} \tag{5-9}$$

where $\theta \in [-1, \infty)\backslash\{0\}$.

Because the components are in parallel, the system failure time is the maximum of all the components' failure times: $t = \max\{X_1, X_2, \dots, X_d\}$. That is, the system fails at time *t* when all the components failed at time *t*. Thus, from Equation (5-9), the CDF of the parallel system at time *t* is given as follows:

$$\begin{aligned} F_s(t) &= P(max\{X_1, X_2, \dots, X_d\} < t) \\ &= P(X_1 < t, X_2 < t, \dots, X_d < t) \\ &= C\big(F_1(t), F_2(t), \dots, F_d(t)\big) \\ &= \big[F_1^{-\theta}(t) + F_2^{-\theta}(t) + \cdots + F_d^{-\theta}(t) - d + 1\big]^{-\frac{1}{\theta}}. \end{aligned} \tag{5-10}$$

5.4.2 q-Fréchet Approximation

Consider a parallel system with identical components. Now, suppose that the CDF for a component i can be expressed as follows:

$$F_i(t) = 1 - \left(\frac{t}{\lambda}\right)^{-\alpha} + o\left(\left(\frac{t}{\lambda}\right)^{-\alpha}\right), \text{ as } \frac{t}{\lambda} \to \infty \tag{5-11}$$

This leads to the following expression:

$$F_i^{-\theta}(t) = 1 + \theta\left(\frac{t}{\lambda}\right)^{-\alpha} + o\left(\left(\frac{t}{\lambda}\right)^{-\alpha}\right), \text{ as } \frac{t}{\lambda} \to \infty \tag{5-12}$$

After substituting Equation (5-12) into Equation (5-10), the system CDF can be expressed as follows:

$$\begin{aligned} F_s(t) &= \left[d \cdot \left(1 + \theta\left(\frac{t}{\lambda}\right)^{-\alpha} + o\left(\left(\frac{t}{\lambda}\right)^{-\alpha}\right)\right) - d + 1\right]^{-\frac{1}{\theta}} \\ &= \left[d \cdot \left(\theta\left(\frac{t}{\lambda}\right)^{-\alpha} + o\left(\left(\frac{t}{\lambda}\right)^{-\alpha}\right)\right) + 1\right]^{-\frac{1}{\theta}} \\ &\approx \left[1 + \theta d\left(\frac{t}{\lambda}\right)^{-\alpha}\right]^{-\frac{1}{\theta}}. \end{aligned} \tag{5-13}$$

Now, set $\theta = -(1-q)$, $\alpha = \xi$, and $d^{\frac{1}{\alpha}}\lambda = \sigma$. Substituting these into Equation (5-13) yields the following:

$$F_s(t) \approx \left[1 - (1-q)\left(\frac{t}{\sigma}\right)^{-\xi}\right]^{\frac{1}{1-q}}. \tag{5-14}$$

This is the CDF for a q-Fréchet distribution, as shown in Equation (5-4). Thus, the dependent parallel system time-to-failure is approximately distributed as q-Fréchet distribution.

5.5 *Simulation Experiments*

To evaluate the accuracy of the a-Fréchet model approximating the lifetime distribution of the dependent parallel system presented in Section 5.4. We conducted simulation experiments of multiple parallel systems with dependent component failure times described by Clayton copula. In particular, the experiments were designed to show how well the q-Fréchet model could estimate the parallel system's reliability function. The simulated systems included those with decreasing hazard rates and those with unimodal hazard rates. Section 5.5.1 describes the process for sampling failure times. Section 5.5.2 presents the simulated systems and the simulation results.

5.5.1 Data Generating

Let $t_1, t_2, \dots, t_n$ ($t_1 < t_2 < \cdots < t_n$) represent the system's failure times, which refer to the corresponding components $j_1, j_2, \dots, j_n$ of the system. The system failure time t_i is determined by $t_i = \max\{t_1^i, t_2^i, \dots, t_d^i\}$, where $t_1^i, t_2^i, \dots, t_d^i$ represent the components' failure times.

We develop a data sampling method based on the sampling method for Clayton copula [62], to generate components' failure times $t_1^i, t_2^i, \dots, t_d^i$, then the system failure time is determined by the maximum of the components' failure times $t_i = \max\{t_1^i, t_2^i, \dots, t_d^i\}$ for $i = 1,2,\dots,n$. The pseudo-code of the parallel system's failure times data generating algorithm is given in Figure 5-2.

01: **For** $i = 1,2,\dots,n$

02: Generate a variate V with distribution function gamma distribution $\mathrm{Ga}(\frac{1}{\theta})$

03: Generate independent uniform variates $X_1, X_2, \dots, X_d$

04: Compute $(U_1, \dots, U_d) = \left(\psi\left(-\frac{\ln(X_1)}{V}\right), \dots, \psi\left(-\frac{\ln(X_d)}{V}\right)\right)$, where $\psi(t) = (1+t)^{-1/\theta}$

05: Generate $(t_1^i, t_2^i, \dots, t_d^i)$ by taking the inverse of $F(t) = e^{-\left(\frac{t}{\sigma}\right)^{-\xi}}$

06: $t_i = \max(t_1^i, t_2^i, \dots, t_d^i)$

07: **End**

08: Return $(t_1, t_2, \dots, t_n)$

Figure 5-2: Pseudo-code of the parallel system's failure times data generating algorithm

5.5.2 Simulated Systems

In this section, simulation experiments were conducted to verify that a parallel system of identical components from Fréchet distribution with dependence described by Clayton copula approximately follows a q-Fréchet distribution. The algorithms were implemented in MATLAB. In these experiments, the Clayton copula parameter θ satisfies $\theta \in [-1, 0)$, and parameter q in q-Fréchet can approximate parameter θ in Clayton copula in the equivalent form with $q = 1 + \theta$. In this simulation study, suppose a parallel system composed of d components with dependence. The components' failure times follow Fréchet distributions with the CDF $F_i(x) = exp\left[-(x/\sigma_i)^{-\xi_i}\right]$, which satisfies Equation (5-

11). In these experiments, set the scale parameter $\sigma_i = 5$, for all the components $i = 1, 2, \dots, d$. The ML estimates for parameters of the q- Fréchet distribution were obtained by maximizing the log-likelihood function in Equation (5-8) via an adaptive hybrid artificial bee colony algorithm [64]. Table 5-2 shows the parameters of the simulated systems from which we generated samples and the maximum likelihood estimated (MLE) parameters. For each simulated case, we ran 20 replications, and each replication had $d = 50$ components and $n = 200$ failures. The results in Table 5-2 suggest that the parameter q in the q-Fréchet distribution can approximate the Clayton copula parameter θ in an equivalent form with $q = 1 + \theta$.

Table 5-2: Simulation settings for systems and MLE parameters

Parameters to generate data			MLE parameters		
Clayton copula	Component shape	Component scale	System shape	System shape	System scale
θ	ξ_i	σ_i	$\hat{q}$ $(1+\theta)$	$\hat{\xi}$	$\hat{\sigma}$
0.2	1	5	1.12	0.94	242.48
0.5	1	5	1.37	0.94	249.74
0.8	1	5	1.66	0.99	245.80
0.2	2	5	1.22	2.03	35.24
0.5	2	5	1.54	2.09	36.45
0.8	2	5	1.79	2.05	36.59

5.6 *Application to a Two-Motor System Failure Data*

In this section, the q-Fréchet distribution is applied to the failure data of a parallel system with two motors. In the parallel system, the system fails when both motors fail. This is a parallel model taking into consideration the dependence between the failure times of the two motors. The data were published and analyzed in Reliability Edge Home [68]. The data in Table 5-3 shows the time to failure for 18 such systems, and a graphical representation of the data is given in Figure 5-3. This data set has been analyzed by [69][70][71].

Table 5-3: Time to failure (in days) for two motors [68]. The letters refer to the motors that failed: A=Motor A; B=Motor B.

System	First failure	Second failure	System	First failure	Second failure
System 1	65 (B)	102 (A)	System 2	84 (A)	148 (B)
System 3	88 (A)	202 (B)	System 4	121 (B)	156 (A)
System 5	123 (B)	148 (A)	System 6	139 (A)	150 (B)
System 7	156 (B)	245 (A)	System 8	172 (B)	235 (A)
System 9	192 (B)	220 (A)	System 10	207 (A)	214 (B)
System 11	212 (B)	250 (A)	System 12	212 (A)	220 (B)
System 13	213 (A)	265 (B)	System 14	220 (A)	275 (B)
System 15	243 (A)	300 (B)	System 16	248 (B)	300 (A)
System 17	257 (A)	330 (B)	System 18	263 (A)	350 (B)

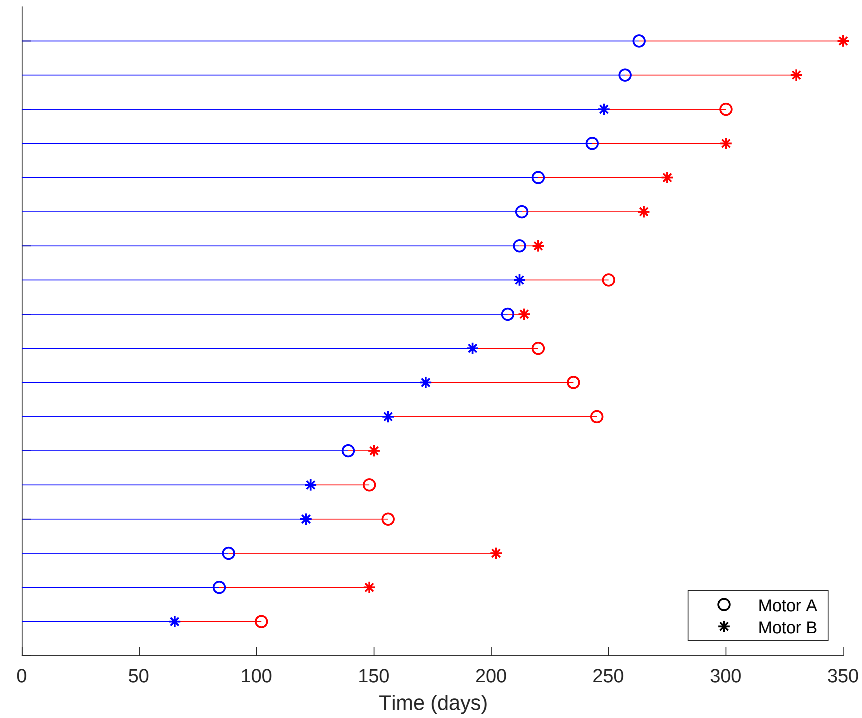

Figure 5-3: Times to failure (in days) for two motors.

Consider the second failure time is the system's failure time; we apply the q-Fréchet distribution to fit the system's failure times data. The ML estimated parameters are shown in Table 5-4.

Table 5-4: MLE parameters of q-Fréchet distribution

q-Fréchet	$\hat{q} = 4.5413, \hat{\xi} = 10.8367, \hat{\sigma} = 259.3707,\ \mathcal{L} = -101.4542$

The cumulative distribution function (CDF) and hazard rate function for the fitted q-Fréchet distribution are shown in Figure 5-4 and Figure 5-5, respectively. It shows that the model fits this data well, and the q-Fréchet hazard rate shows unimodal.

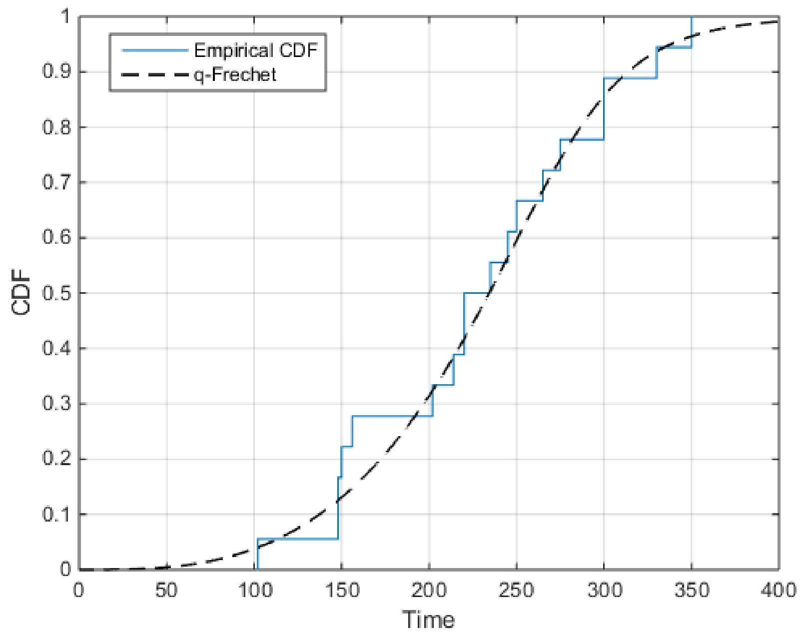

Figure 5-4: The empirical and fitted cumulative distribution function

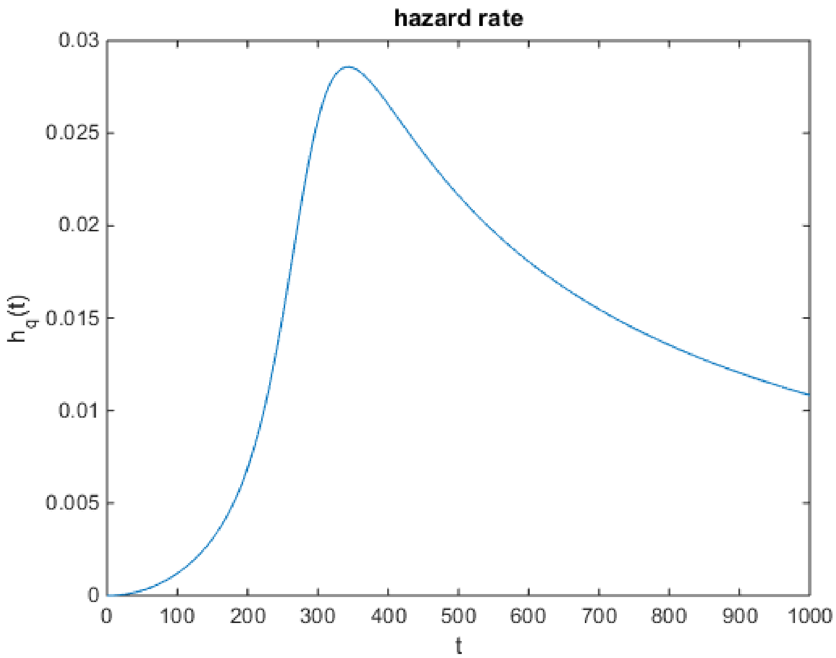

Figure 5-5: Hazard rate function for the two-motor system.

5.7 *q-Fréchet from Environmental Common Cause Failure*

In this section, we show that the dependence of a parallel system could result from the environmental common cause failure. The q-Fréchet distribution can model a parallel system with common cause dependence.

5.7.1 q-Fréchet Model

When a system is deployed in a random environment, the reliability function will be affected by that environment. The environment has a common effect on all the components. For example, all of the components are vulnerable to the temperature condition, with higher temperature, the components have higher hazard rates and vice versa. The lifetime distribution function will change accordingly. Assume the lifetime of component i in lab environment follows a Fréchet distribution, i.e., the CDF is:

$$F_i(t) = e^{-\left(\frac{t}{\sigma_i}\right)^{-\xi_i}} \tag{5-15}$$

Suppose the field environment where the component operates will have a random effect on the component's lifetime distribution function, as below:

$$F_{i,z} = \left(F_i(t)\right)^z = e^{-z\left(\frac{t}{\sigma_i}\right)^{-\xi_i}} \tag{5-16}$$

Where z represents the random effect from the environment, following a gamma distribution $f(z) = \frac{b^a}{\Gamma(a)} z^{a-1} e^{-bz}$. Notice that all the components in a given environment have the same z value. We chose the gamma distribution for two reasons: 1) gamma distribution is commonly used to describe the latent environment effect, and 2) using gamma distribution will lead to a closed-form solution as below.

At a given environment, assume these components are conditionally independent, let the random vector $(X_1, X_2, \dots, X_d)$ represents the lifetimes of the d components, then the conditional joint CDF of d components at a specific environment is determined as:

$$\begin{aligned} & P(X_1 < t_1, X_2 < t_2, \dots, X_d < t_d | z) \\ &= F_{1,z}(t_1) \cdot F_{2,z}(t_2) \cdot \dots \cdot F_{d,z}(t_d) \\ &= e^{-z\left(\frac{t_1}{\sigma_1}\right)^{-\xi_1}} \cdot e^{-z\left(\frac{t_2}{\sigma_2}\right)^{-\xi_2}} \dots \cdot e^{-z\left(\frac{t_d}{\sigma_d}\right)^{-\xi_d}} \\ &= e^{-z * \sum_{i=1}^{d}\left(\frac{t_i}{\sigma_i}\right)^{-\xi_i}} \end{aligned} \tag{5-17}$$

If the components are in parallel, then the systems' failure time is:

$$t = max\,\{X_1, X_2, \dots, X_d\} \tag{5-18}$$

Then the lifetime distribution function of the parallel system with *d* components at a given environment is:

$$\begin{aligned} F_s(t) &= F_s(max\{X_1, X_2, \dots, X_d\} < t) \\ &= P(X_1 < t, X_2 < t, \dots, X_d < t) \end{aligned} \tag{5-19}$$

Considering all the environmental effects, the expected lifetime distribution function of the parallel system is:

$$\begin{aligned} F_s(t) &= \int_0^\infty P(X_1 < t, X_2 < t, \dots, X_d < t|z) f(z) dz \\ &= \int_0^\infty e^{-z\cdot\sum_{i=1}^{d}\left(\frac{t}{\sigma_i}\right)^{-\xi_i}} \frac{b^a}{\Gamma(a)} z^{a-1} e^{-bz} dz \\ &= \frac{b^a}{\Gamma(a)} \int_0^\infty z^{a-1} e^{-\left(b+\sum_{i=1}^{d}\left(\frac{t}{\sigma_i}\right)^{-\xi_i}\right)z} dz \\ &= \frac{b^a}{\Gamma(a)} \cdot \frac{\Gamma(a)}{\left(b + \sum_{i=1}^{d}\left(\frac{t}{\sigma_i}\right)^{-\xi_i}\right)^a} \\ &= \left(1 + \sum_{i=1}^{d} \frac{1}{b}\left(\frac{t}{\sigma_i}\right)^{-\xi_i}\right)^{-a} \end{aligned} \tag{5-20}$$

When all components are identical, the above system's lifetime distribution function is a q-Fréchet distribution function as:

$$F_s(t) = \left(1 + \frac{d}{b}\left(\frac{t}{\sigma_i}\right)^{-\xi_i}\right)^{-a} \tag{5-21}$$

5.7.2 Clayton Copula from the Environmental Common Cause Failure

This subsection clarifies the relationship between the system's and components' lifetime distribution functions. Similar to the derivation of the system's lifetime distribution, the lifetime distribution function of a single component i is determined as:

$$F_i(t) = \left(1 + \frac{1}{b}\left(\frac{t}{\sigma_i}\right)^{-\xi_i}\right)^{-a}, \tag{5-22}$$

which is a q-Fréchet distribution.

Then, the system's lifetime distribution function can be rewritten as:

$$\begin{aligned} F_s(t) &= \left(1 + \sum_{i=1}^{d} \frac{1}{b}\left(\frac{t}{\sigma_i}\right)^{-\xi_i}\right)^{-a} \\ &= \left(1 + \sum_{i=1}^{d} \left(1 + \frac{1}{b}\left(\frac{t}{\sigma_i}\right)^{-\xi_i} - 1\right)\right)^{-a} \\ &= \left(1 + \sum_{i=1}^{d} \left(\left(F_i(t)\right)^{-\frac{1}{a}} - 1\right)\right)^{-a} \\ &= \left(1 + \sum_{i=1}^{d} \left(\left(F_i(t)\right)^{-\frac{1}{a}}\right) - d\right)^{-a} \\ &= \left(\sum_{i=1}^{d} \left(F_i(t)\right)^{-\frac{1}{a}} - (d-1)\right)^{-a} \end{aligned} \tag{5-23}$$

which means the system's lifetime distribution function can be described as a function of the components' lifetime distribution functions. This relationship is exactly the Clayton copula.

This provides one justification of our assumption in section 5.4 using the Clayton copula to model the dependence among components. The common cause failure from randomized environment would lead to the Clayton copula dependence among components.

5.7.3 Simulation Validation

We performed a simulation study to verify that q-Fréchet distribution can model a parallel system with common cause dependence when all components' hazard rates are affected by a common randomized environmental effect. For a parallel system with $d = 2$ identical components, each component follows the Fréchet distribution $F_i(t) = exp\left(-\left(\frac{t}{\sigma_i}\right)^{-\xi_i}\right)$, with parameter $\sigma_i = 1, \xi_i = 2$ $(i = 1,2)$. The environmental effect follows a gamma distribution $f(z) = \frac{b^a}{\Gamma(a)} z^{a-1} e^{-bz}$ with $a = 1, b = 1$. Then, considering the environmental effect, the component's lifetime distribution is $F_{i,z} = \left(F_i(t)\right)^z = e^{-z\left(\frac{t}{\sigma_i}\right)^{-\xi_i}}$. We generated *N*=1000 failure times for the parallel system using the algorithm below:

$N = 1000, a = 1, b = 1, d = 2, \sigma_i = 1, \xi_i = 2$, samples = []
For *j* = 1, …, *N*
    $z = \text{gamma\_random}(a, b)$
    For *i* = 1,…,d
        u=uniform_random()
        $t_i = \sigma_i \left(\frac{z}{-ln(u)}\right)^{\frac{1}{\xi_i}}$
    End
    $t_{system} = \max\ (t_i)$
    samples$[j] = t_{system}$

```
        End
        Return samples
```

Figure 5-6: Pseudo-code of the failure times data generating algorithm for a parallel system with common cause dependence.

Then, we used a q-Fréchet distribution to fit the sampled parallel system's failure times data. The fitted CDF was compared with the empirical CDF, as shown in the following Figure 5-7. The figure shows that the two lines are perfectly aligned. This validates our theory that the common cause failure results from a randomized environment leads to a q-Fréchet distributed lifetime.

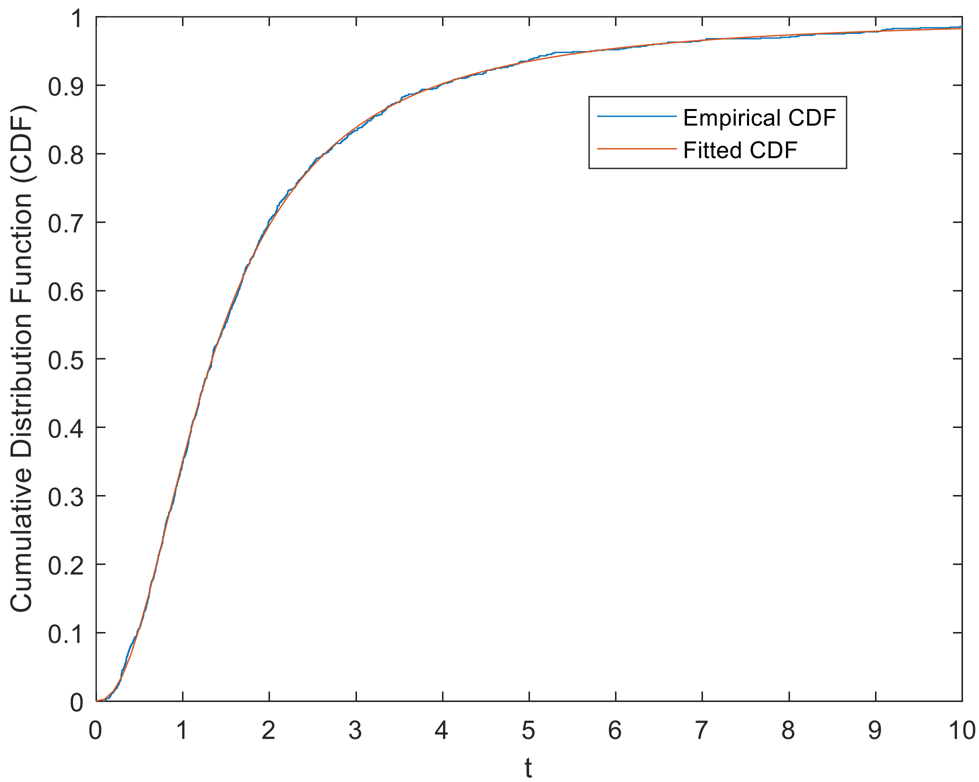

Figure 5-7: Comparison of the empirical and fitted CDFs of the system's lifetime.

We also used the above algorithm to simulate the lifetime of the two-motor parallel system example. For a parallel system with identical components, the lifetime distribution of the system is $F_s(t) = \left(1 + \frac{d}{b}\left(\frac{t}{\sigma_i}\right)^{-\xi_i}\right)^{-a}$ as shown in Equation (5-21), where $d = 2$. Meanwhile, the lifetime distribution of the system has the standard q-Fréchet representation $F_s(t) = \left[1 - (1-q)\left(\frac{t}{\sigma}\right)^{-\xi}\right]^{\frac{1}{1-q}}$ as shown in Equation (5-4), whose parameters were estimated by ML estimation, as $q = 4.5413, \xi = 10.8382, \sigma = 259.4715$ as shown in Table 5-4. Comparing the two representations, we can have the shape parameter of the environmental effect $a = \frac{1}{q-1} = 0.2824$. Notice that, in the simulation, we take the scale parameter of the environment effect $b = a$, so that the expected environmental effect is $E(z) = \int_0^{\infty} zf(z)dz = b/a = 1$. Lastly, from $1 + \frac{d}{b}\left(\frac{t}{\sigma_i}\right)^{-\xi_i} = 1 - (1-q)\left(\frac{t}{\sigma}\right)^{-\xi}$, we can have components' parameters $\sigma_i = \sigma\left(\frac{b(q-1)}{d}\right)^{1/\xi} = 243.3967, \xi_i = \xi = 10.8382$.

Using the algorithm in Figure 5-6, setting these parameters $a = 0.2824, b = 0.2824, d = 2, \sigma_i = 243.3967, \xi_i = 10.8382$, we sampled N=1000 lifetime pairs of motor A and B, plotted in the following Figure 5-8, it shows that the common cause failures between the two motors, i.e., the two motors fail relatively close to each other.

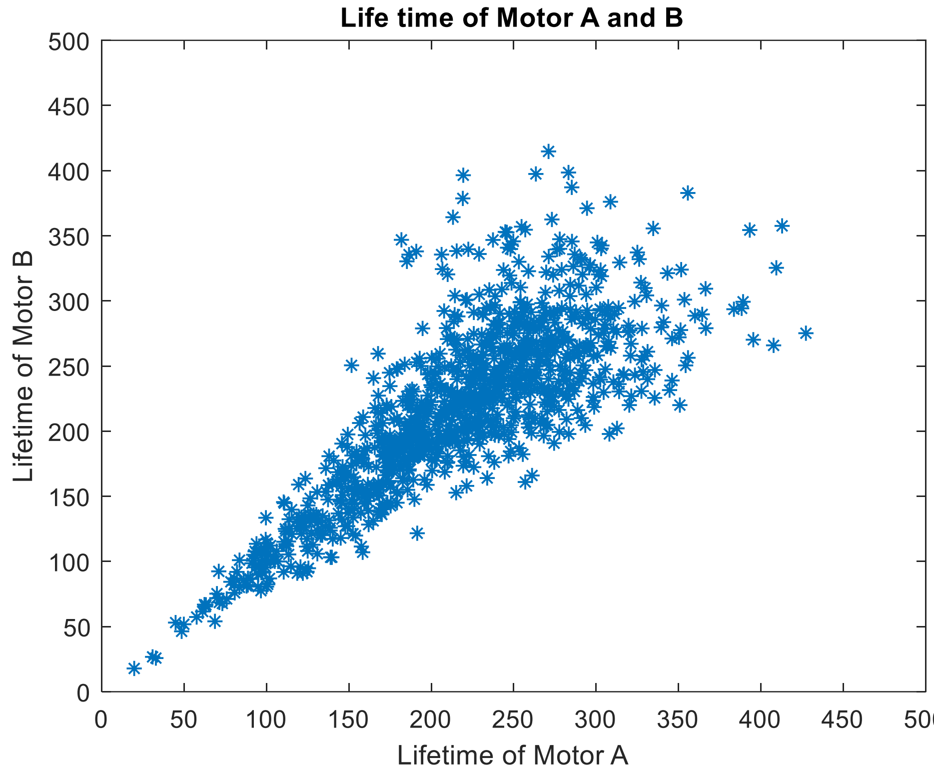

Figure 5-8: Simulated lifetime pairs of motor A and B.

The system's lifetime is the maximum value of the two motors' lifetimes. Hence, we have the 1000 system's lifetimes. Figure 5-9 shows the well alignment between the empirical CDF from the 1000 system's lifetimes and the predicted CDF. Note that the predicted CDF is not fitted from the 1000 sampled lifetimes but predicted from the original data with 18 samples.

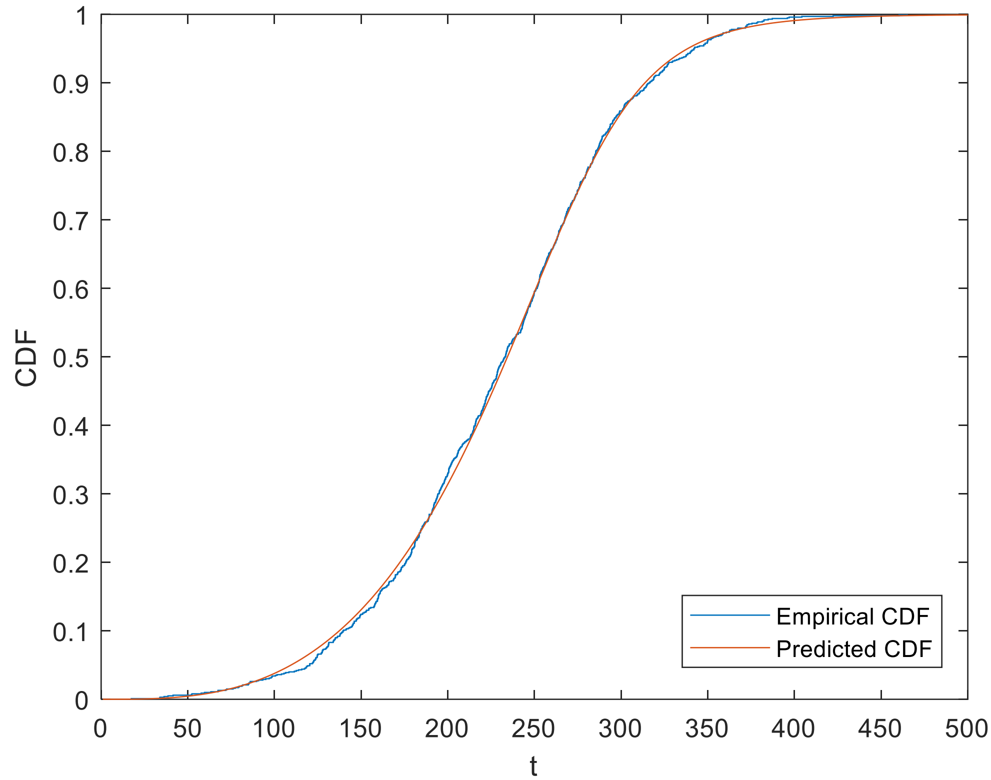

Figure 5-9: Comparison of simulated empirical and predicted CDFs.

5.8 *Summary*

In this chapter, a q-Fréchet distribution was proposed to approximate the distribution of the failure time of a parallel system with dependent component failure times that are modeled as a Clayton copula. We implemented the maximum likelihood (ML) method for estimating the model's parameters from failure time data. Simulation experiments were conducted to evaluate the q-Fréchet approximation and showed that parameter q could approximate the parameter θ in Clayton copula. One example of dependence was illustrated as common cause dependence when all components' hazard rates were affected by a common randomized environmental effect. We have shown that the q-Fréchet distribution can model the parallel system with common cause dependence. The proposed q-Fréchet distribution was successfully applied to a data set of 18 two-motor parallel systems' failure times.

Chapter 6: Conclusions, Contributions and Recommendations for Future Research

6.1 Conclusions

This dissertation has demonstrated that the q-Weibull distribution is a promising alternative distribution for reliability modeling and constitutes another alternative distribution model for the reliability analyst. In this research, q-Weibull distribution has been successfully applied to fit failure times data and to model the reliability of systems, including dependent series systems and dependent parallel systems.

The q-Weibull distribution is flexible and useful in the context of reliability engineering as it allows for the modeling and analysis of various behaviors of the hazard rate - monotonically decreasing, monotonically increasing, constant, unimodal, and bathtub-shaped - with a single set of parameters. The maximum likelihood (ML) method was developed to estimate the q-Weibull distribution parameters. The ML estimates of the q-Weibull parameters were accurate and precise with small bias and MSE. Intervals estimates for the q-Weibull parameters were provided, including asymptotic intervals based on the ML asymptotic theory, parametric and non-parametric bootstrapped confidence intervals. The proposed method for the ML constrained q-Weibull problem was also applied to an example involving failure data characterized by a bathtub-shaped hazard rate function.

In terms of system reliability, this dissertation has shown that the q-Weibull distribution can model a series system with dependent component failure times that are described by Clayton copula and that the parameter q can approximate the Clayton copula parameter θ, which measures the degree of dependence. The NHPP with q-Weibull as the underlying time to first failure (TTFF) distribution model was proposed as an approximation to the minimal repair process of a series system composed of multiple dependent components. The maximum likelihood (ML) method was developed to estimate the model parameters, and asymptotic confidence intervals based on ML asymptotic theory were also developed. Estimating the parameters of the q-Weibull model does not require information about which components failed, which is necessary for accurately estimating the parameters of the Clayton model. Simulation experiments were conducted to validate the proposed NHPP q-Weibull model and showed that parameter q could approximate the parameter θ in Clayton copula, for both systems with monotonic and non-monotonic failure intensity functions. In the simulation, we developed a sampling method for conditional failure times of dependent subsystems modeled by Clayton copula. The proposed model and parameter estimation procedure have been successfully applied to a real failure data set of a load-haul-dump (LHD) machine characterized by a bathtub-shaped intensity function. The results have shown that the proposed NHPP q-Weibull model has the advantage of fewer parameters with smaller uncertainty when used as an approximation to the Clayton copula approach, which in turn needs more information on the assumption for the underlying distributions of components and the exact component cause of system failure. A modified Kolmogorov-Smirnov (KS) goodness-of-fit test statistic and p-value were used to determine the goodness-of-fit of the proposed NHPP q-Weibull model. The goodness-of-fit test results also have agreed that the proposed NHPP q-Weibull model outperformed other commonly used minimal repair process models, including NHPP Weibull and NHPP S-PLP, and the independent models.

Besides modeling series systems, this dissertation proposed a q-Fréchet distribution, dual distribution to q-Weibull distribution, to approximate the distribution for the failure time of a parallel system with dependent component failure times that are modeled as a Clayton copula. Similar to q-Weibull distribution, the parameter q in q-Fréchet distribution is an approximation to the parameter θ in Clayton copula, which measures the degree of dependence among the components. A simulation study was conducted to evaluate q-Fréchet approximation. We also have shown that the q-Fréchet distribution could model the parallel system with common cause dependence. The proposed q-Fréchet model was applied to a data set of 18 two-motor parallel systems' failure times. The data appeared to have a unimodal failure intensity.

In this research, all the models' parameters were estimated by the maximum likelihood (ML) method. However, the intricate likelihood functions imposed significant numerical difficulties in estimating its parameters, which has limited the number of applications of q-Weibull distribution so far. Such a difficulty can explain the limited number of applications based on the q-Weibull model given that parameter estimation, and data fitting are crucial steps for reliability analyses. In this research, an adaptive hybrid artificial bee colony (AHABC) algorithm has been proposed to solve the ML problem, which combines the global exploration of ABC and the local exploitation of the Nelder-Mead simplex search.

The exploitation ability of Nelder-Mead improves the local search performance of ABC. Numerical results showed that the proposed AHABC algorithm efficiently finds the optimal solution for the q-Weibull ML problem, comprising different behaviors of the hazard rate and sample sizes. The results also showed that the proposed AHABC outperformed both ABC and similar algorithms in terms of accuracy and convergence speed in the context of the maximum likelihood problem for the q-Weibull distribution.

To conclude, the proposed AHABC for parameter estimation showed that the q-Weibull is a promising alternative distribution for reliability modeling of failure times data and series and parallel systems composed of multiple dependent components.

6.2 Contributions

The major contribution of this research to state-of-the-art is the fundamental understanding of q-Weibull distribution in modeling lifetime data and the dependence among components. Specifically, the dissertation has the following state-of-the-art contributions:

- Analytically derived that the q-Weibull distribution approximates the distribution of failure time of a series system with dependent component failure times that are described by Clayton copula and that the parameter q could approximate the Clayton copula parameter θ, which measures the degree of dependence;
- A novel NHPP with q-Weibull as the underlying time to first failure (TTFF) distribution model was proposed as an approximation to the minimal repair process of a series system composed of multiple dependent components. The

maximum likelihood (ML) method was developed to estimate the model parameters, and asymptotic confidence intervals based on ML asymptotic theory were also developed.

- A new q-Fréchet distribution was proposed that could approximate the distribution of the failure time of a parallel system with dependent component failure times that are modeled as a Clayton copula.
- The environmental common cause failure was revealed as one example of the dependence modeled by the Clayton copula.

Besides the theoretical contributions, this dissertation also contributes to the engineering practice of using q-Weibull distribution. The contributions of this research to reliability engineering practice can be summarized as follows:

- A complete approach was developed for the reliability lifetime data fitting by q-Weibull distribution; A new algorithm AHABC was proposed to solve q-Weibull distribution ML problem;
- A modified Kolmogorov-Smirnov (KS) goodness-of-fit test statistic was developed to determine the goodness-of-fit of the proposed NHPP q-Weibull model;
- Two powerful lifetime distributions: q-Weibull distribution and q-Fréchet distribution were provided for fitting lifetimes data as alternative distributions to other commonly used ones.
- Multiple simulation algorithms, including the conditional failure time data sampling for a series system with dependence described by Clayton copula, the

failure time data sampling for a parallel system with dependence described by Clayton copula, and the failure time data sampling for a parallel system with common cause dependence were presented to generate dependent failure time data.

6.3 Recommendations for Future Research

This research considered the value of using the q-Weibull distribution in solving typical problems in reliability engineering: the lifetime data fitting and modeling of series and parallel systems with dependence. Although we have demonstrated the successful applications of q-Weibull in these problems, there are some limitations of this research and some future research are recommended as follows:

- The introduction of additional generalizations, like the use of linear or nonlinear transformation of time, use of multiple distributions, the time dependence of parameters, etc., as it was done with Weibull, will further enhance flexibility and accuracy of the q-Weibull model;
- This research proposes to use a q-Weibull distribution to model a dependent series system, and to use a q-Fréchet distribution to model a dependent parallel system, both series ($k = n$) and parallel systems ($k = 1$) are special cases of the k-out-n system, more complex systems such as k-out-n systems, series-parallel systems should be studied;

- A wider range of reliability problems can incorporate the q-Weibull model, such as stress-strength analysis [72], optimal preventive maintenance policies [73][74], optimal system design [75], competitive risks [76].

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