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Chromatic Polynomials of Large Maps

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CHROMATIC POLYNOMIALS OF LARGE MAPS

by

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INTRODUCTION

The classical unsolved problem in the coloring of maps is to determine whether four colors are sufficient to color any map on a sphere (1). The problem has stimulated more general investigations such as the study of certain chromatic polynomials,[1] each of which gives the exact number of ways in which an associated map can be colored in any number of colors (2-9). This approach has led to a conjecture of Birkhoff and Lewis which is a strong form of the four-color conjecture (5, pp. 411-413). Their conjecture is based on the results of an actual calculation of the chromatic polynomials of certain significant maps which are regular in the sense of Birkhoff (1). The results of their calculation are given in a table of chromatic polynomials of maps having seventeen or fewer regions.

It is of particular interest to determine whether this conjecture holds in the case of a regular map which has the additional topological property that no two of its pentagons are contiguous. The simplest such map is the truncated icosahedron which contains thirty-two regions, twelve mutually isolated pentagons and twenty hexagons.

The chromatic polynomial of the truncated icosahedron

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[1] Cf. reference 5 of the bibliography for fundamental principles and definitions underlying the treatment of chromatic polynomials.

could conceivably be determined by extending the table calculated by Birkhoff and Lewis until it included the polynomial of the truncated icosahedron, but the magnitude of such an undertaking would be impractically large. The amount of calculation could be considerably reduced by employing the following modified form of this approach.

A set of maps containing the truncated icosahedron can be selected by successively adding regions to a submap corresponding to one of its hemispheres. By calculating the polynomials of these maps in turn, it is possible to build up a sequence which will eventually yield the chromatic polynomial of the truncated icosahedron. This method was attempted in 1948 by D. I. Rudebeck, now Mrs. J. L. Vanderslice, who found the polynomials of a number of maps of this type having twenty-one or fewer regions (7).

Over the intervening years this program has been continued by Mrs. Vanderslice and Professor Dick Wick Hall. They have obtained the chromatic polynomials of several hundred additional maps containing twenty-five or fewer regions. The results of these calculations are as yet unpublished. They indicate that the continuation of the attack along these lines, by extending this set all the way until it finally reaches the truncated icosahedron, would require a stupendous calculation.

The approach of the present paper differs from previous methods in several respects. For example, it begins with the complete truncated icosahedron itself and constructs a sequence of maps of decreasing size. The chromatic polynomial of the truncated icosahedron can then be expressed in terms of the polynomials of a certain set of maps each one of which belongs to this sequence, and contains no more than, say, 26 regions. Such a set will be denoted by T(26). The further reduction of each map of such a set is often a major undertaking in itself, sometimes involving months of calculation. Thus the reduction of the set T(26) by even a single map would materially decrease the total amount of calculation required. Even this approach, while more direct than the method of Rudebock, would normally be expected to entail a calculation of astronomical proportions.

The present approach also differs from previous methods in the manner of constructing the sequence. Using the present method, it has been found possible to construct a set T(26) which contains 35 maps. It has been estimated that, using previous methods, the size of such a set would be measured not in terms of dozens or scores of maps but in terms of hundreds of maps, and that the time required to reduce such a set would be measured in years. The present method thus has greatly reduced the total amount of calculation involved in determining the chromatic polynomial of the truncated icosahedron.

The methods of the present analysis actually are applied to a general class, H(n), of large maps, consisting of all those maps of n regions which are homeomorphic with any map having a proper nine-ring such that the inside of the ring contains a central hexagon surrounded by three pentagons and three hexagons which lie alternately around its boundary, and such that the inside and outside of the ring are absolutely invariant under a group of transformations which is isomorphic with the symmetric group of degree three. These methods make it possible to express the chromatic polynomial of any map belonging to this class in terms of the chromatic polynomials of a certain set of thirty-six of its submaps, each of which contains no more than $n-6$ regions. The truncated icosahedron is a member of this general class of large maps. Further reductions possible in this special case yield an expression for the chromatic polynomial of the truncated icosahedron in terms of the chromatic polynomials of a set thirty-five of its submaps, each of which contains no more than twenty-six regions. This expression, combined with known chromatic polynomials, brings within range the determination of the chromatic polynomial of the truncated icosahedron itself.

The details of the application of the present method to the class $M(n)$ and the truncated icosahedron are given in the next chapter. The rationale of the method and certain features of its present application are discussed in the last chapter.

CHAPTER I

REDUCTION SEQUENCES AND CHROMATIC POLYNOMIAL REDUCTION FORMULAS FOR MAPS OF THE CLASS M(n)

1. Conventions for Specifying Maps

The proper nine-ring which enters the definition of the class M(n) is illustrated in Figure 1.

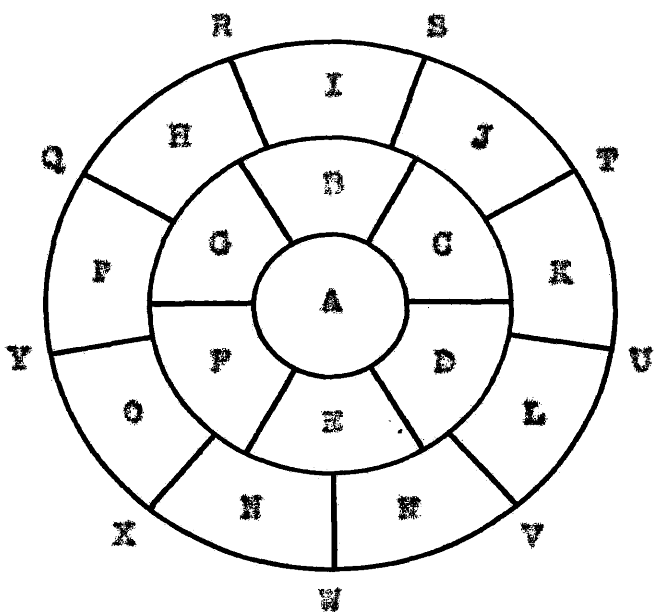

Figure 1

The regions H,I,J,K,L,M,N,O, and P, form the ring, and the regions A,B,C,D,E,F, and G, lie in the inside of the ring. The labels Q,R,S,T,U,V,W,X, and Y, are associated not with regions but with those vertices of the nine ring regions which appear in Figure 1 and belong to the outside of the ring. Each of these will be called a _ring vertex_.[1] The curve consisting of these vertices and those sides shown in Figure 1 which lie entirely in the outside of the ring will be called the _ring boundary_. In the following, for convenience, the term _nine-ring interior_ will refer to the portion of the map consisting of the inside of the ring, and the remaining portions of the ring regions and their boundaries. The same term will be applied to the corresponding portion of any submap of a map of the class H(n). The complement of the nine-ring interior in a map of the class H(n) or in any of its submaps will be called the _nine-ring exterior_. Where no confusion can result, a nine-ring interior will be referred to simply as a map.

The _group of symmetries of the class H(n)_ can be specified by means of the following permutations, written as the product of disjoint cycles, of three of the nine ring vertex labels indicated in Figure 1:

(Q)(T)(W)	(Q)(TW)
(QTW)	(T)(WQ)
(QWT)	(W)(QT) .

The original nine-ring interior represented in Figure 1 is seen to be invariant under each of the rigid motions of this group. The nine-ring exterior of each map of the class H(n) has a homomorph which is absolutely invariant

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[1] Terms arising in the present discussion are listed in the index.

under this group of rotations and reflections, but otherwise is arbitrary.

The simplest map of the class $M(n)$ is actually shown completely in Figure 1. Its nine-ring exterior is simply the 9-gon bounded by the ring boundary. Figure 1 represents only a portion of the more complex maps of the class $M(n)$. In such cases, the representation is completed by adding the remaining boundaries and vertices, and regarding additional points of the ring boundary as vertices. In the truncated icosahedron, for example, the ring boundary separates two equivalent hemispheres and contains two corresponding sets of nine vertices, either of which can be regarded as a set of ring vertices. This map appears in Figure 2.

Maps which are regular in the sense of Birkhoff and Lewis and contain fewer than ten regions can be specified completely by means of a symbol of the form: $(n;n_4,n_5,\ldots,n_j)$, where n denotes the total number of regions in the map and n_i denotes the number of i-sided regions in the map, for $i = 4,5,\ldots,j$ (5, Chapter II). In the following, this symbol will be referred to as a <u>region symbol</u>, and the term regular will be used exclusively in the sense of Birkhoff and Lewis.

In more complex maps, contacts between a given region and those contiguous with it can be denoted by a symbol such as (A-BCDEFG), which specifies the contacts of the region, A, in Figures 1 and 2. In this symbol, letters

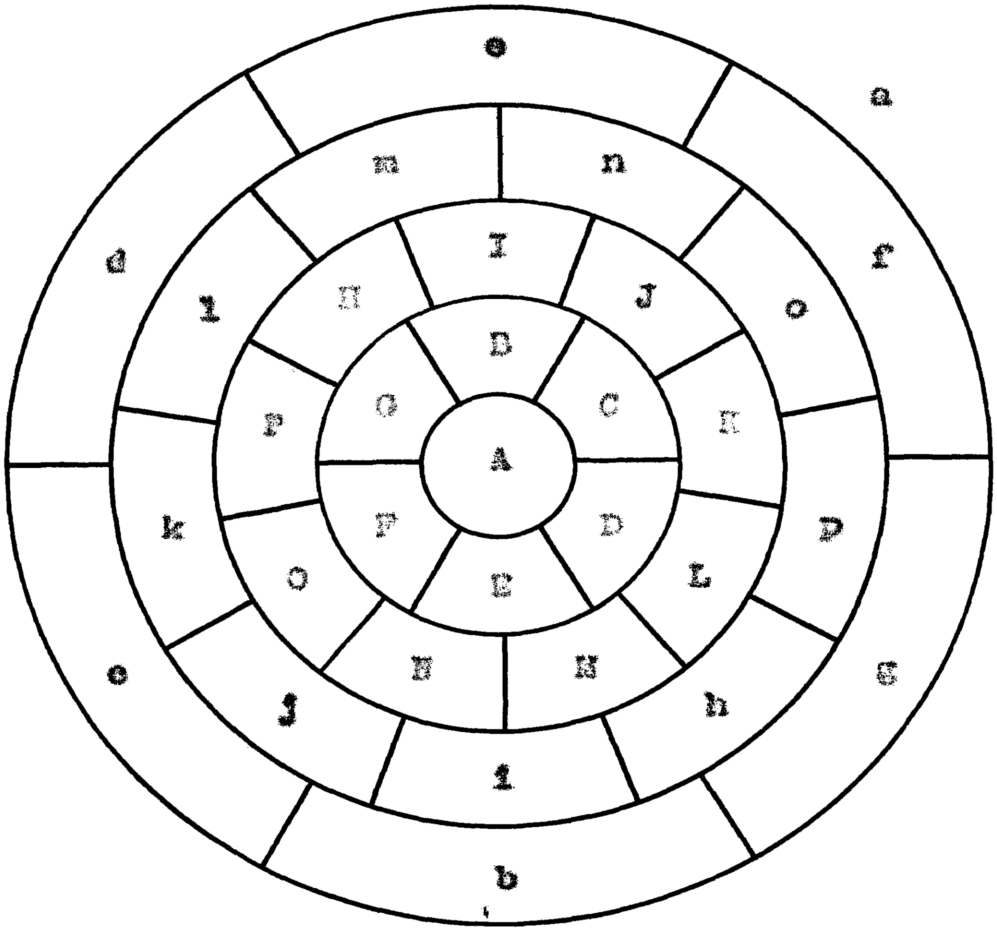

Figure 2

denoting regions contiguous with A are given in the order in which they are situated in the map. Any map can be complete-ly specified by writing such a symbol for each of the regions, to form a contact symbol for the entire map.

When the modifications of the map involve only bound-ries lying entirely inside a ring such as HIJKLMN of Figure 1, it is possible to condense the contact symbol and at the same time to simplify the process of recon-structing the map from it. If, in a map, those ring regions possess the labels and all the contacts among themselves and with the regions of the outside of the ring which exist in the original map corresponding to Figure 1, all of these contacts will be specified by the symbol Z. This same convention will apply in the case of the ring BCDEFG. Contacts among the regions of the nine-ring exterior are the same in each of the submaps as they are in the original map hence this portion of the map need be specified only once, as it is for the truncated icosahedron in Figure 2, for example. There is then no necessity for specifying the nine-ring exterior explicitly in each of the contact symbols. Thus, in this sense, the complete contact symbol for any map of the class M(n) can be written in terms of Figure 1 as follows:

(Z;Z)(A-BCDEFG),

where the first Z denotes the ring HIJKLMNOP, and the second, the ring BCDEFG.

When a ring cannot be specified by this convention, the symbol Z is replaced by a list of the labels of the remaining <u>ring regions</u>, that is, those whose labels and contacts with the regions of the outside of the ring are the same as in the original map. If two such ring regions possess, in the submap, the contact which they had in the original map, the labels are adjacent in the list. Otherwise they are not. Thus the contact symbols for the maps of Figures 3 and 5 are:

(;[illegible],[illegible])(A-[illegible]), and ([illegible];[illegible]/[illegible],[illegible])(A-[illegible]),
respectively.

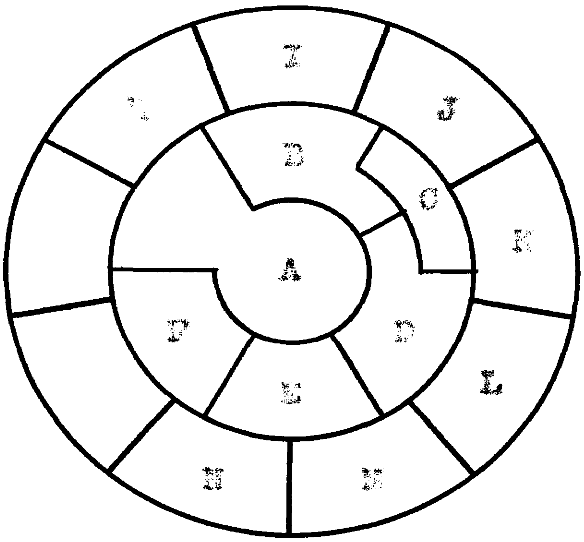

Figure 3

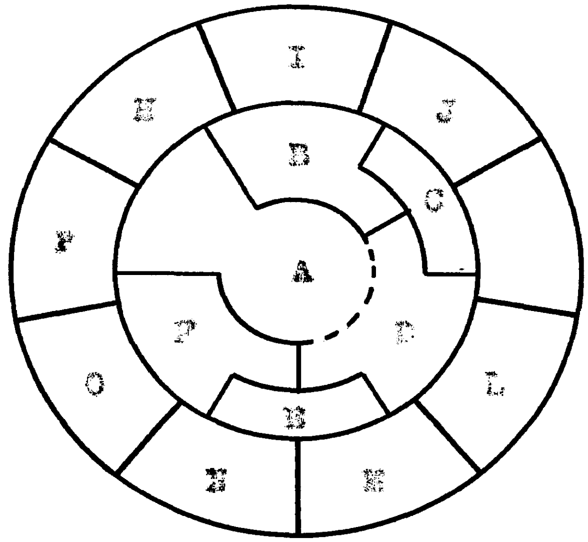

Figure 4

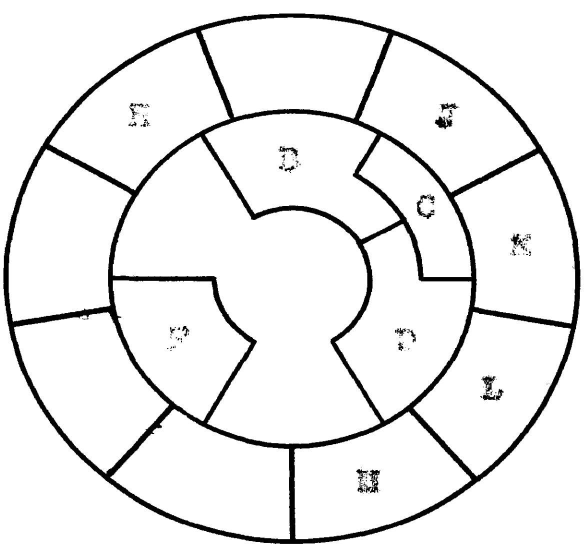

Figure 5

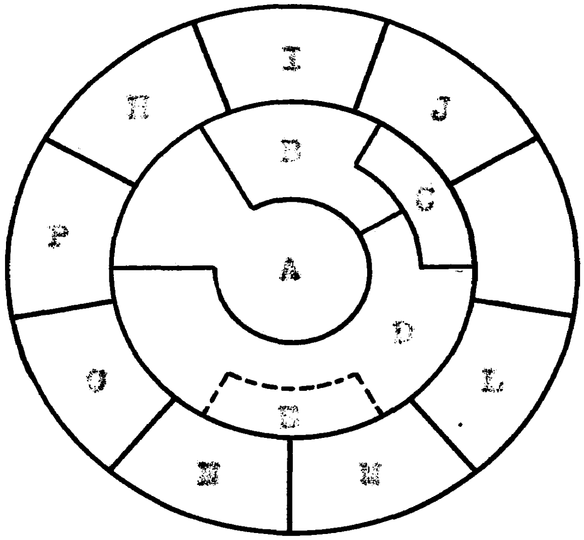

Figure 6

The additional contact between the two ring regions B and D in each of these two maps is specified by the notation B-D which completes the second ring contact symbol in each case. Thus, the contact symbol for the partial map of Figure 6, where the broken boundary between the regions A and D is understood to be present in the map, is (Z;BCDEF,D-EF)(A-CDFHE). The same conventions apply to the nine-ring HIJKLMNOP. Contacts between ring vertices and regions of the nine-ring interior on whose boundaries they lie will be indicated in the same manner employed for ordinary region contacts, i.e. implicitly for ring regions, explicitly otherwise. These contacts can actually be regarded as ordinary region contacts in the case of the truncated icosahedron, since there exists a one-one correspondence between the ring vertices and the regions of the nine-ring exterior on whose boundaries they lie.

2. Symmetric Equivalence

The group of the class H(n) finds application in the reduction of a map of this class. It is of value in establishing chromatical equivalences among submaps of such a map by means of certain properties of the corresponding nine-ring interiors.

Two such configurations are topologically equivalent if there exists a one-one correspondence between the regions and contacts indicated by their contact symbols. Two cases can be distinguished according as this correspondence does or does not imply one of the permutations of the group of symmetries of the class M(n). In the former case, the two nine-ring interiors will be said to be symmetrically equivalent, otherwise they will be said to be symmetrically distinct.

Since each map of the class M(n) has a homeomorph whose nine-ring exterior is absolutely invariant under the transformations of this group, the symmetric equivalence of two nine-ring interiors implies the topological equivalence of the corresponding complete maps. The converse is not necessarily true, however. Two complete maps can be homeomorphic if the corresponding nine-ring interiors are topologically equivalent, but symmetrically distinct. This is true, for example, in the case of the simplest map of the class M(n) which is shown in its entirety in Figure 1. Two complete maps can also be topologically equivalent if the corresponding nine-ring interiors are topologically distinct. This case actually occurs in the present reduction sequences in an example discussed in Section 4 of this chapter.

3. Conventions for Specifying Reduction Sequences

The chromatic polynomial of a map can be determined with the aid of the following formulas which are discussed by Birkhoff and Lewis (5, Chapter II):

(1) $$Q_n(u) = Q_n^e(u) - Q_{n-1}(u) + Q_{n-1}^e(u),$$

(2) $$Q_n(u) = (u+1)Q_{n-1}(u),$$

(3) $$Q_n(u) = uQ_{n-1}(u),$$

(4) $$Q_n(u) = (u-1)Q_{n-1}(u) + uQ_{n-2}(u) + Q_{n-2}^e(u),$$

(5) $$Q_n(u) = (u-2)Q_{n-1}^1(u) + (u-1)\left[Q_{n-2}^2(u) + Q_{n-2}^5(u)\right] + Q_{n-2}^1(u) + Q_{n-2}^3(u) + Q_{n-2}^4(u).$$

In the present discussion, for convenience, formulas (1) through (5) will be called <u>n-gon reduction formulas</u> for $n = 1,\ldots,5$, respectively. In particular, formula (1) will also be referred to as <u>the fundamental principle</u>, and formulas (4) and (5) will also be called the quadrilateral and pentagon reduction formulas, respectively. The application of a reduction formula to a map will be called a <u>reduction</u>, and an ordered set of reductions will be referred to as a <u>reduction sequence</u>.

In the present reduction sequences, the pentagon reduction formula is applied only to maps which have a homeomorph which is symmetric about an axis of symmetry of the pentagon. Formula (5) can then be written in the following simplified form:

(5a) $$Q_n(u) = (u-2)Q_{n-1}^1(u) + 2(u-1)Q_{n-2}^5(u) + Q_{n-2}^1(u) + 2Q_{n-2}^4(u).$$

The formulas actually used in the present reduction sequences are (1) through (4), and (5a). In the following, these will be referred to as the formulas (1) through (5a).

The relationship between the terms of a reduction formula will be indicated in a general way by referring to the left member as the antecedent, and each of the terms of the right member as a consequent. More specific information will be given by associating a consequent symbol with each of the twelve consequent terms of the formulas (1) through (5a) in the following way. The digits 1,2, and 3, will denote, respectively, the three consequents of the fundamental principle written in formula (1). The digits 4,5, and 6, will denote, respectively, the three consequents of the quadrilateral reduction formula (4). The digits 7,8,9, and 0, will denote, respectively, the four consequents of the pentagon reduction formula, (5a). The consequents of formulas (2) and (3) will be denoted by a superscript bar, and a prime, respectively.

Combinations of these consequent symbols formed in the manner indicated in the following examples will be called sequence symbols. For example, it will be seen in the next section that the map of Figure 3 has the sequence symbol 12 in the present reduction sequences.

The specification of each consequent in terms of its antecedent in a particular application of one of the formulas (1) through (5a) can be completed by means of additional letters as follows. The fundamental principle, formula (1), involves modifications to a portion of the antecedent obtained by isolating one of its sides. This side is denoted by the pair of letters assigned to the regions having contact across it. Thus, for example, the <u>reduction sequence symbol</u>:

(6) 12AB123 .

completely specifies the three consequents of the map 12 of Figure 3 which are obtained by applying the fundamental principle to the boundary AB. These three consequent maps, which have the sequence symbols 121, 122, and 123, are shown in Figures 4, 5, and 6, respectively. Here the broken boundaries are understood to exist in the maps of Figures 4 and 6.

The reduction of the 3-gon, E, in the latter map can be specified in the reduction sequence symbol (6) by writing it in the form:

(7) 12AB123'E .

If, in addition, the maps 122 and 123 had each contained a 2-gon labeled C, the reduction of these would be specified by writing the reduction sequence symbol as follows: 12AB12$\overset{2}{3}$'CE where the order of the last two letters would denote the order in which the corresponding regions are reduced, and the sequence symbol 12$\overset{2}{3}$' would denote the map so obtained.

The erasure of sides to obtain the consequents 2 and 3 such as those of Figures 5 and 6, is always accompanied by the elimination of the region label nearest the end of the alphabet. When one side of a region is erased, as in the case of formulas (2) and (3), and the consequents 4 and 7 of formulas (4) and (6a), respectively, the label of the region being reduced is eliminated. When two sides are erased as in the case of the consequents 5,6,8,9, and 0, the label of the region being reduced is the one retained.

The remaining conventions associated with the quadrilateral reduction formula can be illustrated by the following example. Suppose that the 4-sided region C of the map having the sequence symbol 123' is to be reduced in such a way that the consequent of formula (4) denoted by the symbol 4 is obtained by erasing the side CD in Figure 6. A reduction sequence symbol completely specifying this application of formula (4) is:

(8) 123'CD456 .

In this symbol, 123' denotes the antecedent; C denotes the region being reduced; CD denotes the side which is erased to form the consequent denoted by the symbol 4; and, recalling the significance of the terms of formula (4), the symbols 5 and 6 necessarily denote the consequents obtained by erasing the pairs of sides CD and CJ, and CD and CK, respectively. The complete sequence symbols of these three consequents are, then, 123'4, 123'5 and 123'6, respectively.

In similar fashion the reduction of the pentagon A in Figure 4, where the broken side is understood to be present in the map, is indicated by the reduction sequence symbol:

(9) 121AD7890 .

The four consequent maps specified by this symbol necessarily are: 1217 obtained by erasing the broken side AD, and 1218, 1219, and 1210, obtained by erasing the pairs of sides AD and AF, AB and AF, and AF and AH, respectively.

Reference to Figure 6 shows that precisely the same reduction is specified by either of the alternative symbols 123'CD456 and 123'CJ456. A combination of the two symbols might be preferable aesthetically, however for the sake of conciseness, only one of these simpler alternatives will be employed in each such case.

The reduction sequence symbols such as (6) through (9) specify each consequent unambiguously in terms of its antecedent. The sequence symbols serve as convenient identifications for the maps of the reduction sequences.

In addition, the one-one correspondence existing between each of the five consequent symbol components of a sequence symbol such as 123'4 and the appropriate terms of formulas (1), (3), and (4) makes it possible to infer the relative number of regions in the map. In this case, for example, the consequent 123'4 contains four regions fewer than the original map.

In the present reduction sequences the map 121 is symmetrically equivalent to the map 114, hence the two maps are elements of a single symmetric equivalence class. The first sequence symbol of each such symmetric equivalence class to arise in the reduction sequences will be called the <u>representative</u> of its class, and will serve to denote the class. In this case, the map 121 belongs to the class having representative sequence symbol 114. Also, the map 123' belongs to the symmetric equivalence class whose representative is 13". This information is conveyed by writing the symbol (7) in the form

(10) 12*1*23'*S(114,13") .

Here the stars denote the consequents which are symmetrically equivalent, respectively, to the maps whose representative sequence symbols appear in the parentheses.

4. Reduction Sequences For Maps of the Class M(n) and For the Truncated Icosahedron

The application of the methods of the present analysis yields an expression for the chromatic polynomial of each map of n regions belonging to the class, M(n), in terms of the chromatic polynomials of a certain set of thirty-six of its submaps, each of which contains no more than n-6 regions. The corresponding set of thirty-six consequents of the original nine-ring interior indicated in Figure 1 is denoted by the symbol S(n-6). In the special case of the truncated icosahedron, the set of complete submaps corresponding to S(n-6) is further reduced to yield the set T(26), which contains only thirty-five submaps.

The reduction sequences leading to the sets S(n-6) and T(26) can be written in various ways, depending upon the particular combinations of reduction techniques which are used. The present sequences employ only the formulas (1) through (6a) and hence are specified entirely in terms of the notation based upon these formulas. The reduction sequence symbols leading to the sets S(n-6) and T(26) are listed in Table I. They are arranged in seven sequences, the first of which consists of the reductions in which each of the consequents contains the complete nine-ring Z, i. e. the ring HIJKLMNOP of Figure 1. This sequence begins with the application of the fundamental principle to the boundary AC of the original nine-ring interior shown in Figure 1. This application, specified by the reduction sequence symbol:

$$AC123'C \; ,$$

yields the three consequents having the sequence symbols 1,2, and 3'. This reduction is entirely analogous to the one discussed in connection with the symbol given in (7) and illustrated in Figures 3 through 6. In similar fashion, the fundamental principle then is applied in turn to to boundary AC in each of the maps 1,2, and 3'. The first application yields the maps 11,12, and 13". The reduction of the map 12 is the one actually illustrated in Figures 3 through 6 and discussed in connection with the symbols (6), (7), and (10). The latter symbol appears as the sixth element of the first reduction sequence of Table I.

The map 13" is symmetrically equivalent to 123' of Figure 6. The reduction of 123' discussed in connection with (8) was actually applied to 13". The symbol corresponding to (8) but written in terms of 13" is:

(11) 13"CD456 .

where the regions labeled C and D in the map arising as 13" in the reduction sequences correspond to the regions labeled C and D of the map which arises as 123' and appears in Figure 6. It can be seen from that Figure and the discussion accompanying (8) that the consequents 13"5 and 13"6 of (11) no longer contain the complete nine-ring Z. Since the first reduction sequence includes only the reductions in which all the consequents contain this nine-ring, this sequence does not include (11).

The second reduction sequence of Table I begins with this symbol and comprises all the reductions of 13" and its submaps which were employed in constructing T(n-6) and T(86). For example, the symbol given in (11) yields three consequents, the first of which is further reduced as follows:

(12) 13"4B34<u>5'</u>D<u>6</u> .

to yield the submaps 13"44, 13"45', and 13"46. A glance at the last two sequence symbols shows that these maps contain n-7 and n-6 regions, respectively, hence they lie in S(n-6) and T(26). This fact is denoted by underlining the consequent symbols $\underline{6'}$ and $\underline{6}$ in (12). In this case, a single reduction of each of the remaining maps of (11) and (12), namely 13"5, 13"6, and 13"44, yields submaps all of which belong to the sets S(n-6) and T(26). Three of these are symmetrically equivalent to maps already in the reduction sequences, hence they are not regarded as distinct elements of these sets.

Similarly, the reduction of the map 121 of Figure 4 which was discussed in connection with the symbol (9) was actually applied to the symmetrically equivalent map having the representative sequence symbol 114. The reduction of the pentagon in the latter map is specified by the symbol

(13) 114:E7=B9=0(115,13") ,

in which the labels E and B denote the regions corresponding to those labeled A and D, respectively, in the analogous symbol (9) and in Figure 4. It is seen that two of the consequents in (13) are symmetrically equivalent to maps already in the reduction sequences. The symbol (13) specifies the first reduction of the fourth sequence in Table I.

The map 114 is analogous to 13" in that it arises in the first reduction sequence, is not in S(n-6) or T(26), and the initial reduction of it yields consequents which do not contain the complete nine-ring E. A reduction sequence is devoted to each of the six maps of this type which arise in the first sequence. The map 3'5' arising in the fourth reduction of that sequence actually belongs to S(n-6) and T(86). Each of the remaining maps in the first reduction sequence either is further reduced in that sequence, or is symmetrically equivalent to a map which has already occurred. Within each of the reduction sequences the symbols appear in the order in which the corresponding antecedents arise.

The fact that the complete map is specified in the case of the truncated icosahedron makes possible further reductions. In thirteen of the nine-ring interiors of the set S(n-6), two adjacent ring vertices are end points of a single boundary line which is not part of the ring boundary. Such a boundary line is a side of a 5-gon in each complete submap of every map whose original ring boundary contains exactly eighteen vertices disposed as in Figure 2. The situation of each of these _ring 5-gons_ relative to the ring hijklmnop, of Figure 2, is analogous to the situation of the 5-gon, E, relative to the ring HIJKLMNOP in Figure 6. The map 12856', as well as eleven maps of the set T(86), are consequents of the reduction of ring 5-gons. The reduction of each of these ring 5-gons is indicated in the sequence symbols and the reduction sequence symbols and the letters denoting these regions are underlined both in the latter symbols and in the contact symbols given in Section 6.

The nine-ring interiors 23'25 and 12886' are topologically distinct and hence symmetrically distinct, yet in the case of the truncated icosahedron the corresponding complete maps are topologically equivalent. This homeomorphism can be established with the aid of the contact symbols of Section 6 by observing, for example, that the contiguous 7-gons D and E of 23'26 correspond respectively to the contiguous 7-gons A and B of 12236'. Thus the submap 122366 of 12236 belongs to S(n-6) but not to T(26), hence the reduction of the ring 3-gons is not specified in the reduction sequence symbol in this case.

Table I

The Reduction Sequences Yielding the Sets S(n-6) and T(26).

AC123'C

1AC123"AC

2AC1=23'C(12)

3'AC1'=A2=3"ABD(13",23')

11AE456'=E(13")

12AE1=23'=E(114,13")

23'AE1=23"=AE(13",3'3")

122BD1=2'=C3(22,23'2)

13"CD456

13"4ED45'D6

13"5EC4'=D5'D46(13"45')

13"6EC456=(13"56)

13"44DB4=56'L(3'3")

22EA7=6'DG=0(1223,23'2)

223'EE456'I

220BA456'I

2204AK123

22041FA4=56'O(13"64)

114EB7=69=0(115,13")

1140CE45'F6

1140DB45'F6

11406FB4=5'EG'O(13"5)

11406CE4'D5'D14

11404CD4=6=6=(2204,22042,22043)

11406CE4=5'D6(22042)

23'GDA456'L

23'24AJ12=2(13"64)

23'241AE1=2=3'=A

(13"44,13"46,13"45')

1223DA456'=I(23'25)

12234FA4=56'O(13"44)

12235DB4=56'L(122345)

12236DA4=6=6(122346,122356)

115AE12=3=(1140,1140)

1151AE12=3=(1146,13")

11511CD456

115114AF12'=C3=(23'24,12234)

115115GF4=5'A6=(22041,22042)

115116DE1=2=3'=D

(11404,11406,11405')

1151141GD7=6'=F9=0=

(12234,226'4,23'24,2204) .

5. Chromatic Polynomial Reduction Formulas For Maps of the Class M(n) and For the Truncated Icosahedron

The reduction formula which expresses the chromatic polynomial $P_n(x)$, of each map of the class M(n) in terms of the chromatic polynomials of the submaps corresponding to the elements of the set S(n-6) is given on pages 26 and 27. Here $P_n(x)$ denotes the number of ways that a map P_n of n regions can be colored using some or all of x given colors. The polynomials of the submaps are also denoted by the corresponding representative sequence symbols which appear in parentheses following the usual polynomial symbols. The submaps themselves are specified by means of the contact symbols given in the next section. The coefficients of the reduction formula are written in powers of u, where $u \equiv x - 3$.

The special form of this formula which applies to the truncated icosahedron is given on pages 28 and 29. The representative sequence symbols in this formula denote the submaps belonging to the set T(26). These maps are also specified by means of the contact symbols of the next section.

$$
\begin{aligned}
P_n(x) = {} & (u^6-6u^5+24u^4-57u^3+70u^2-36u+16)P_{n-6}(x)\,(5'\bar{5}'') \\
& +(u^4-3u^3+6u^2-10u+6)P_{n-6}(x)\,(13''46) \\
& +(2u^5-7u^4+3u^3+34u^2-62u+30)P_{n-6}(x)\,(13''64) \\
& +(2u^5-12u^4+30u^3-44u^2+12u)P_{n-6}(x)\,(203'4) \\
& +(6u)P_{n-6}(x)\,(2205) \\
& +(6)P_{n-6}(x)\,(2206) \\
& +(6u^2-30u+30)P_{n-6}(x)\,(22046) \\
& +(u^4-12u^2+32u-18)P_{n-6}(x)\,(22045) \\
& +(-6u^2)P_{n-6}(x)\,(11405') \\
& +(u^4-u^3-4u^2+6u)P_{n-6}(x)\,(11406') \\
& +(-6u^2+6u)P_{n-6}(x)\,(114045) \\
& +(-6u+6)P_{n-6}(x)\,(114046) \\
& +(3u)P_{n-6}(x)\,35'55) \\
& +(3)P_{n-6}(x)\,(33'56) \\
& +(-u^4+6u^3-15u^2+19u-9)P_{n-6}(x)\,(25'345) \\
& +(u^5-6u^4+21u^3-30u^2+14u)P_{n-6}(x)\,(122345) \\
& +(u^4-6u^3+21u^2-30u+14)P_{n-6}(x)\,(122346) \\
& +(5u^5-12u^4+21u^3-6u^2-3u)P_{n-7}(x)\,(13''45') \; +
\end{aligned}
$$

$+(3u^3-9u^2+9u-3)P_{n-7}(x)(13''56)$

$+(3u^3-6u^2+3u)\ P_{n-7}(x)(13''65)$

$+(u^6-6u^5+16u^4-30u^3+40u^2-16u)P_{n-7}(x)(13''445)$

$+(u^5-6u^4+16u^3-30u^2+40u-16)P_{n-7}(x)(13''446)$

$+(6u^3-6u^2)P_{n-7}(x)(222'5)$

$+(6u^2-6u)P_{n-7}(x)(222'6)$

$+(2u^5-6u^4+22u^2-24u)P_{n-7}(x)(220415)$

$+(2u^4-6u^3+22u-24)P_{n-7}(x)(220416)$

$+(-6u^2+6u)P_{n-7}(x)(114064')$

$+(-6)P_{n-7}(x)(114066)$

$+(2u-6)P_{n-7}(x)(114066)$

$+(3u^3-4u^2)P_{n-7}(x)(122355)$

$+(6u^2-6u)P_{n-7}(x)(122356)$

$+(3u-4)P_{n-7}(x)(122366)$

$+(u^6-4u^4+6u^3)P_{n-7}(x)(1151155')$

$+(3u^5-9u^4+3u^3+9u^2-6u)P_{n-8}(x)(13''5\bar{5})$

$+(-6u^2-6u)P_{n-8}(x)(11406\bar{5})$

$+(2u^3-6u^2)P_{n-8}(x)(114065')$.

$$P_{36}(x) = (u^6-6u^5+24u^4-57u^3+78u^2-5[illegible]u+16)P_{26}(x)(5'3'')$$

$$+(u^4-3u^3+6u^2-10u+6)P_{26}(x)(23''46)$$

$$+(2u^5-7u^4+6u^3+24u^2-[illegible]u+30)P_{26}(x)(23''64)$$

$$+(2u^5-10u^4+20u^3-34u^2+10u)P_{26}(x)(2220'4)$$

$$+(6u)P_{26}(x)(2205$$

$$+(6u^2-20u+30)P_{26}(x)(22042)$$

$$+(u^4-12u^2+32u-18)P_{26}(x)(22048)$$

$$+(-6u^3)P_{26}(x)(11408')$$

$$+(u^4-u^3-4u^2+6u)P_{26}(x)(11403')$$

$$+(6u^2-u)P_{26}(x)(23'25)$$

$$+(-u^4+6u^3-12u^2+19u-9)P_{26}(x)(23'243)$$

$$+(u^5-6u^4+21u^3-30u^2+14u)P_{26}(x)(122348)$$

$$+(6u^5-10u^4+21u^3-6u^2-2u)P_{26}(x)(23''46')$$

$$+(3u^3-6u^2+6u-3)P_{26}(x)(23''56)$$

$$+(3u^3-6u^2+3u)P_{26}(x)(23''65)$$

$$+(u^6-6u^5+10u^4-32u^3+40u^2-16u)P_{26}(x)(23''445)$$

$$+(6u^3-6u^2)P_{26}(x)(223'3)$$

$$+(6u)P_{26}(x)(2206') +$$

$+(2u^5-6u^4+22u^2-24u)P_{25}(x)(220416)$

$+(-6u^3+6u^2)P_{25}(x)(114648')$

$+(-6u^2+6u)P_{25}(x)(114646')$

$+(-6u^2+6u)P_{25}(x)(114664')$

$+(-6)P_{25}(x)(114666)$

$+(2u-6)P_{25}(x)(114666)$

$+(3u)P_{25}(x)(25'26')$

$+(u^5-6u^4+10u^3-22u^2+10u)P_{25}(x)(122248')$

$+(3u^3-4u^2)P_{25}(x)(122856)$

$+(u^5-4u^4+6u^3)P_{25}(x)(1151156')$

$+(u^6-6u^5+10u^4-22u^3+40u^2-10u)P_{24}(x)(15''446')$

$+(6u^3-6u^2)P_{24}(x)(226'6')$

$+(2u^5-6u^4+22u^2-24u)P_{24}(x)(220416')$

$+(6u^3-6u^2)P_{24}(x)(124066')$

$+(3u^3-4u^2)P_{24}(x)(122256')$

$+(3u^6-6u^5+3u^4+6u^3-6u^2)P_{23}(x)(13''5\overline{6}')$

$+(-6u^3-6u^2)P_{23}(x)(11406\overline{5}')$.

6. Maps Arising in the Reduction Sequences Yielding the Sets S(n=6) and T(26)

The representative of each symmetric equivalence class arising in the reduction sequences is specified in Table II by means of the symbols defined in the first section of this chapter. The maps are identified by means of the representative sequence symbols. Following each of these is a region symbol for the corresponding consequent of the truncated icosahedron, and a contact symbol. The first set of symbols refers to the original configurations shown in Figures 1 and 2. The remaining sets are given in the order in which the representative sequence symbols arise in the reduction sequences of Section 4.

Sequence symbols corresponding to the maps of the sets S(n=6) and T(26) are underlined. Labels of ring 5-gons appear, underlined, in the contact symbols. The sequence symbols and region symbols, however, refer to the consequents of the truncated icosahedron in which these 5-gons are reduced, since the maps in this form are regular. For the same reason, these conventions are also applied to 10CCC'6', even though this map is not actually in T(26).

The symmetries of the consequents in the sets S(n=6) and T(26) can be described in terms of the subgroups of the group of symmetries of the class E(n). All of the consequents are, of course, invariant under the subgroup consisting of the identity alone, and one of the consequents, (3'$\overline{3}$"), is invariant under the whole group. In addition, certain consequents are invariant under proper subgroups. The consequent (114066) is invariant under the normal subgroup, which can be interpreted as the group of rotations of the class M(n). This subgroup is isomorphic with the alternating group on the three labels Q, T, and V. Each of the consequents: (13"5$\overline{5}$), (13"5$\overline{5}$'), (13"66), (13"445), (13"446), (13"446'), (11405'), (23'55), (23'66), (23'66'), (100344), (102300), is invariant under one of the three remaining proper subgroups. Each of these conjugate subgroups, isomorphic with the symmetric group of degree two and order two, can be interpreted as a group of reflections about an axis of symmetry of a map of the class M(n).

The information given in Table II can be derived from Table I, but the form in which it is presented in Table II is more convenient for reconstructing the maps. Numerous checks exist, both within a given table, and among the various sources. Thus the number of regions can be read directly not only from each of the different types of symbols in the previous paragraph and in Tables I and II, but also, for example, from the polynomials of the preceding section. All of the information which is given in various special and convenient forms in these sources can be obtained with the aid of Table I.

Table II

Maps Arising In the Reduction Sequences

Yielding the Sets S(n-6) and T(26).

(32;0,12,20)(Z;Z)(A-BCDEFG)

(1)(32;1,12,17,2)(Z;Z,B-D)(A-BDEFG)

(2)(31;0,13,17,1)(Z;DEFGH)(A-DEFGHJK)

(3')(30;1,13,15,0,0,1)(Z;EFG)(A-EFGH)(B-AGHIJKLMN)

(11)(32;3,10,16,2,1)(Z;Z,B-DF)(A-BDEF)

(12)(31;1,11,18,1)(Z;BCDEF,B-D)(A-BCDEFH)

(13")(29;2,11,15,0,0,1)(Z;CDE)(B-CDENOPHIJ)

(23)(30;1,12,16,0,1)(Z;DEF/B)(A-DEFGHIJK)

(23')(29;0,14,14,0,1)(Z;DE)(A-DEBJK)(B-ABNOPHIJ)

(3'3")(26;0,15,10,0,0,1)(Z)(B-HIJKLMNOP)

(114)(31;2,10,18,0,1)(Z;Z,B-DEF)

(115)(30;2,11,16,0,0,1)(Z;CD/FG)(A-CDEFGHIJ)

(122)(30;2,9,19,1)(Z;BCD/F,B-D)(A-BCDEFFH)

(23'2)(28;1,11,15,1)(Z;D)(A-DHEBJK)(B-ANOPHIJ)

(1225)(28;3,9,18,0,0,1)(Z;B/D/F)(A-BJKDEEFFH)

(13"4)(30;1,13,13,0,0,1)(Z;DE)(B-DENOPHIJK)

(13"5)(27;3,11,12,0,0,0,0,1)(LMNOPHIJ;B)(D-LMNO)

(C-MNOPHIJKBID)

(13"6)(27;2,11,12,1,1)(HIMNOPHI;B)(B-MNOPHIC)

(C-HIJKLMST)

(13"44)(27;1,13,12,0,0,1)(Z;D)(B-DMNOPHIJK)

(13"45')(26;2,12,10,0,0,0,1)(NOPHIJKL)(K-NOPHIJKLVW)

(13"46)(26;1,12,11,2)(OPHIJKLM)(B-OPHIJKM)(K-OPHIJMWX)

Table II (continued)

Maps Arising in the Reduction Sequences

Yielding the Sets S(m-6) and T(96).

(13"66') (23;2,12,8,0,0,0,1) (NOPHIJ/L) (N-VWNOPHIJTUL)

(L-NUV)

(13"56) (23;2,11,11,0,0,1) (OPHIJ/LM) (N-LMWXOC) (C-LMOPHIJTU)

(13"64) (26;1,12,12,0,1) (KLMNOPHI) (D-NOPHIC) (C-KLMWNIST)

(13"65) (25;2,12,10,0,0,0,1) (OPHI/KLM) (B-OPHIE)

(E-KLMWXOPIST)

(13"445) (25;2,12,10,0,0,0,1) (NNOPHIJK) (D-NNOPHIJNUV)

(13"446') (24;0,14,6,2) (NOPHIJ/L) (B-NOPHIJD)

(D-VWNHJTUL) (L-NUV)

(226') (27;3,10,13,0,0,0,1) (NOPHIJKL;B/F) (B-NPPHBJKLVW)

(220) (23;3,9,14,1,1) (OPHIJKLM;B) (A-PHBJKEF) (F-OPAE)

(E-OPAKLMWX)

(226'4) (26;2,12,11,0,0,0,1) (NOPHIJKL;F) (B-NPPHIJKLVW)

(226'5) (26;4,9,11,0,0,0,0,1) (NOPH/JKL;F) (B-NPPHB JKLVW)

(226'6') (24;2,11,9,1,1) (KL/NOP/I;F) (B-KLVWNST) (B-STHEPQHI)

(I-NRS)

(2204) (27;2,11,12,1,1) (OPHIJKLM) (A-PHIJKEF) (F-OPAE)

(E-OPAKLMWX)

(2205) (26;4,8,12,0,2) (JKLM/OPH) (F-OPBE) (B-PHRS JKEF)

(E-KLMWXOPB)

(2206') (23;2,11,10,1,1) (KLM/OP/I) (F-OPAE) (E-KLMWXOPA)

(A-KEFPB) (B-STHAPQRI) (I-NRS)

(22041) (27;2,11,13,0,0,1) (OPHIJKLM) (A-PHIJEF) (F-OPAE)

(E-OPAJKLMWX)

Table II (continued)

Maps Arising In the Reduction Sequences

Yielding the Sets S(n=6) and T(26).

(220422)(26;3,10,11,1,0,1)(OPHIJ/LM)(F=OPAE)(A=PHIJTULMF)

(E=OPAIMNX)

(22043)(26;5,11,11,0,0,0,0,1)(OPHI/HLM)(A=PHIJF)(F=OPAE)

(E=OPAISTKLMNX)

(220412)(25;2,10,11,2)(PHIJKLM)(E=JKLMWXF)(F=PHIJKXY)

(220410')(24;2,12,0,0,0,0,1)(HIJKLM/D)(A=HIJF)

(F=YQHAJKLMNXO)(Q=PKY)

(1142)(29;3,11,14,0,0,0,0,1)(NOPHIJKL;FG/C)(D=CKLE)

(E=NFGHIJCDLVW)

(1140)(29;3,10,14,1,0,1)(OPHIJKLM;GBC)(F=OPGBE)

(E=OFBCKLMNX)

(11404)(28;3,11,13,0,0,0,0,1)(NOPHIJKL;F/C)(D=CKLE)

(E=NFPHIJCDLVW)

(11405')(26;5,8,12,0,0,0,0,0,1)(HIJKL/NO;C)(D=CKLE)

(G=HIJCDLVWNOYQ)

(11406)(27;3,10,12,1,0,1)(IJKL/NOP;C)(D=CKLE)

(E=IJCDLVWNO)(G=NOPQRIE)

(11404)(28;3,10,13,1,0,1)(OPHIJKLM;C)(B=PHIJCEF)(F=OPBE)

(E=OFBCKLMNX)

(11405')(26;4,8,12,0,2)(HIJKLM/O;C)(E=CKLMNXOG)

(G=HIJCEOYQ)(O=GEXY)

(11406)(27;5,10,12,1,0,1)(IJKLM/OP;C)(B=IJCBG)(G=IJBOPQE)

(E=OGBCKLMNX)

Table II (continued)

Maps Arising In the Reduction Sequences

Yielding the Sets S(n=6) and T(26).

(114845')(25;4,9,11,0,0,0,0,1)(FHIJKL/N;C)(D-CKLF)

(F-XYFHIJCDLVWN)(N-FWX)

(114846')(25;2,11,11,0,0,1)(HIJKL/O;C)(D-CKLE)

(E-HIJCDLVWF)(F-VQHENXO)(O-FXY)

(114864')(25;2,11,10,1,1)(IJKL/NOP)(E-IJKLVWNO)

(G-IENOPQR)

(114865')(23;2,11,9,1,1)(NOP/IJ/L)(G-NOPQRIC)

(C-VWEGIJTUL)(L-CUV)

(114866)(26;2,10,11,3)(NOP/I/KL)(G-ISTKLN)(E-ICLVWND)

(G-IENOPQR)

(114065')(24;4,8,11,0,0,0,1)(NEH/OP/T)(G-OPQRIC)

(C-OGISTKLNWX)

(114066)(25;3,9,10,3)(IJ/KN/OP)(C-IJTULNW)(E-LMNKOGC)

(G-OPQRICE)

(23'24)(27;0,13,13,1)(Z)(A-JKLMNB)(B-NOPHIJA)

(23'25)(26;2,10,12,2)(MNOPHIJK)(B-NOPHIJD)(D-MNBJKUV)

(23'26')(25;1,12,10,2)(NOPHIJ/L)(B-NOPHIJA)(A-NBJD)

(D-VWBAJTUL)(L-DUV)

(23'241)(27;0,14,12,0,1)(Z)(A-KLMNB)(B-NOPHIJKA)

(23'243)(26;2,12,11,0,0,0,1)(LMNOPHIJ)(A-LMNB)

(B-LANOPHIJKU)

(12234)(26;2,11,14,0,0,1)(Z;D/F)(A-DMNFHIJK)

(12235)(27;4,8,14,0,0,0,1)(JKLMNOPH;D/F)(B-JKDMNFHTB)

Table II (continued)

Maps Arising in the Reduction Sequences

Yielding the Sets S(n=6) and T(26).

(1[illegible]6')(26;2,10,12,2)([illegible]/I;D/F)(A-[illegible])

(D-[illegible]I)(I-[illegible])

(122345)(26;3,10,12,0,0,0,1)([illegible];D)(F-[illegible])

(122340')(25;1,12,10,2)([illegible]/D;D)(A-[illegible])

(F-[illegible])(D-[illegible])

(122355)(25;5,7,12,0,0,0,0,1)([illegible]/JK;F)(D-[illegible])

(122256')(24;3,9,10,1,1)([illegible]/J/L;F)(B-[illegible])

(D-[illegible]L)(L-[illegible])

(12250'6')(23;1,12,8,2)([illegible]/I/L;F)(A-[illegible])(B-[illegible]I)

(D-[illegible]L)(I-[illegible])(L-[illegible])

(1151)(30;2,11,15,1,1)(Z;FG/C)(D-[illegible])(A-[illegible])

(11511)(30;2,10,16,2)(Z;FG/C)(A-[illegible])(D-[illegible])

(115114)(29;1,11,16,1)(Z;FG)(A-[illegible])(D-[illegible])

(115115)(28;2,11,14,0,0,1)([illegible];FG)(A-[illegible])

(C-[illegible])

(115116)(28;2,11,14,0,0,1)([illegible];FG)

(D-[illegible])(C-[illegible])

(11[illegible]1141)(29;0,14,14,0,1)(Z;FG)(A-[illegible])(D-[illegible])

(1151153')(26;3,9,11,1,1)([illegible]/I)(C-[illegible])(C-[illegible])

CHAPTER II

A METHOD FOR DETERMINING CHROMATIC POLYNOMIALS OF LARGE MAPS

The present method for determining chromatic polynomials of large maps was applied to the class $M(n)$ in the preceding chapter. The principal features of the method are discussed in this chapter. The concepts underlying the method are treated in the first section. Certain applications of these concepts are considered in the next section. Finally, various aspects of previous methods and of the present method are compared in the last section.

1. Relationships Among Elements of Reduction Sequences

Chromatic Differences. In previous methods, advantage is taken of the fortuitous occurrence of chromatical equivalence among submaps arising in a reduction sequence. It is possible to generalize the notion of chromatical equivalence in ways which are particularly fruitful from the standpoint of the efficiency of the reduction sequences. The discussion of the first part of this section is included to indicate in a general way some of the topological ideas underlying these generalizations.

Suppose that the reduction formulas are applied, not generally throughout the entire map, but only in a relatively small portion of it, say the interior of an n-ring which contains a relatively small number of regions of the original map. The repeated application of reduction formulas to this portion of the map and to the corresponding portions of its submaps will yield a number of submaps. These are not isolated instances of maps which are chromatically unrelated, but rather, they are fairly closely related. Some may actually be chromatically equivalent. Many of those which are not may nevertheless be not very different from one another chromatically speaking. This suggests that the simple dichotomy which characterizes maps as being either chromatically equivalent or chromatically distinct may be too gross in this situation.

These intuitive ideas can be formulated in terms of precise concepts in more than one way. For example, two maps will be called <u>fundamentally similar</u> if they are chromatically equivalent or if they can be obtained from one another by means of a single application of the fundamental principle. Maps of this type which arise whenever the fundamental principle is applied are not of primary interest in connection with the efficiency of the reduction sequences. In such cases the corresponding relation will be thought of as an <u>improper</u> fundamental similarity.

Fundamental similarity, whether proper or improper, is a reflexive, symmetric binary relation. It is not necessarily transitive, however, since a given map, a, can be fundamentally similar to each of two different maps, b, and c, through modifications of different boundaries in a. In such a case, the maps b and c will not, in general, be fundamentally similar.

Reflexive, Symmetric Relationships Among Maps and Bases in Reduction Sequences. Reflexive, symmetric binary relations among maps arising in reduction sequences exist not only in connection with the fundamental principle, but also in connection with the remaining n-gon reduction formulas. Denote the number of terms or maps in the particular n-gon reduction formula under consideration by t(n). Then if t(n)-k of these maps are contained as a subset in a given set M, this subset will be called an n,k-associate set of maps, for n = 1, 2, ... , and k = 0, 1, ... , g, where g = max (1, t(n)-2). Each of the two maps of every pair it contains will be called an improper or a proper n,k-associate of the other according as the subset did or did not arise as a result of the explicit application of that particular n-gon reduction formula. Unless otherwise specified, n,k-associate maps and sets will be understood to be proper. Also, each map of M will be called an n,k-associate of every map chromatically equivalent to it, where n and k have the ranges given above.

Thus the relationship of n,k-association is both reflexive and symmetric on M. It is not necessarily transitive however, since the type of situation described in the special case of fundamental similarity is general.

The index, k, will be referred to as the association index of the corresponding set of maps. When k vanishes, the maps form a linearly dependent set which will be referred to as an n-dependent set. When k has the value unity the set will be referred to as an *n-associate* set.

Thus, for special values of n and k there are various synonyms. For example, a 4,0-associate set of maps is quadrilaterally dependent, two 1,2-associate maps are fundamentally similar, and so forth. Where no confusion can result, n,k-associate sets will be referred to simply as *associate sets*.

The relationship of *n-association* between maps m_i and m_j will be denoted symbolically as follows:

$$m_i \; A_n \; m_j \, , \qquad i, j = 1, 2, \ldots , t(n)-1.$$

In the case of such an n-associate set, one of the maps corresponding to a term of the formula lies outside the given set M. An *n-extension*, M', of the given set, M, is formed by adding this map to M. This map and the set M will be referred to as *n-complements* of each other in the extension. Each of the maps of the n-dependent set in the extension is the n-complement of an *n-associate basis* of the extension. Thus, corresponding to each pair of n-associate maps:

$$p \; A_n \; q \, ,$$

is a pair of n-associate bases:

$$P \; A_n \; Q \, ,$$

where A_n denotes the relation of n-association, and P and Q denote the bases containing the maps p and q, respectively. M itself is a basis for the extension in the sense that each map of M' can be expressed as a linear combination of the maps in M. Actually there exist $t(n)$ such associate bases and a duality between relationships which they satisfy and the corresponding relationships among their complements, the $t(n)$ n-associate maps in the extension. Thus the following definitions:

$$(14) \qquad M_i = M' - m_j, \qquad i, j = 1, 2, \ldots, t(n),$$

and relations:

$$M_i \, A_n \, M_j, \qquad i, j = 1, 2, \ldots, t(n),$$

hold if and only if, in M',

$$m_i \, \bar{A}_n \, m_j,$$

for $\qquad i, j = 1, 2, \ldots, t(n).$

This duality between n-associate maps and bases thus implies that the relation of n-association between bases is both reflexive and symmetric, but not necessarily transitive, on a set of bases of a set of maps of the form $(M')'$. Here $(M')'$ is an extension of M', but not necessarily an extension of M. The set of all maps lying in extensions of M of the form $M', (M')', \ldots$, where each extension in these sequences may be formed with respect to any value of n, will be denoted by $\bar{M}$.

<u>Relations on General Sets.</u> Consider a set, M, of elements and a set, R, of binary relations which are defined, reflexive and symmetric on M. Two elements of M will be said

to possess a *join sequence* if and only if they belong to a finite sequence of elements of H in which each pair of elements which are adjacent in the sequence are related under one of the relations of the set R. Define an additional binary relation, S, on the set H, as follows. Two elements of H stand in the relation S if and only if there exists a join sequence for that pair of elements. In symbols,

$$a \; S \; b,$$

if and only if there exists a join sequence ,

$$a \; R_1 \; a_1 \; R_2 \; \ldots \; R_n \; b,$$

where the elements $a, a_1, \ldots, b$, are in H, and the relations $R_1, R_2, \ldots, R_n$, are in R.

Theorem 1. The binary relation S is an equivalence relation on the set H.

Proof. The reflexivity and symmetry of the relations R_i imply that the relation S is reflexive and symmetric, the latter property following from the fact that the order of a join sequence can be inverted in view of the symmetry of the R_i. Given any three elements of H, the existence of join sequences for two of the pairs of elements implies the existence of a join sequence for the third pair, hence S is also transitive on H.

The relation S will be referred to as *R-equivalence*.

Define a binary relation, E, on the set M as follows. Two elements a and b of M stand in the relation E if and only if, for every m in M and for every R in B, the following relations hold:

$$a \, R \, m \, ,$$

if and only if

$$b \, R \, m \, .$$

Theorem 2. The binary relation E is an equivalence relation on M.

The proof of the theorem follows directly from the definition.

The relation E will be referred to as _B-equivalence_.

It follows from the definitions that $a \, E \, b$ implies $a \, R \, b$, for every R in B, and hence that $a \, E \, b$ implies $a \, S \, b$.

Each equivalence class obtained through a partition of M with respect to the relation S can be regarded as a space, H, the points of which are the equivalence classes, a^*, for each a in H, obtained through the partition of each such class H with respect to the relation E.

Define the length of a join sequence to be the number of relations which it contains. For each pair of S-equivalent elements a and b in M, denote the length of a join sequence having minimum length by $d(a,b)$. Such a join sequence will be called _minimal_. The function, d, satisfies the triangle inequality since minimal join sequences for two of the pairs of an arbitrary set of three elements of an S-equivalence class correspond to a join sequence, not necessarily minimal, for the third pair of elements. It follows from the definition of E that, for f in a^* and g in b^*, $d(f,g) = d(a,b)$, and hence that it is possible to associate a number, $d'(a^*,b^*)$, with each pair of S-equivalence classes as follows:

$$d'(a^*,b^*) = d(a,b).$$

For each H define a function, $d^*(a^*,b^*)$, for every pair of points a^* and b^* in H as follows.
Let

$$d^*(a^*,b^*) = 0,$$

if and only if

$$a^* = b^*,$$

otherwise, let

$$d^*(a^*,b^*) = d'(a^*,b^*) \quad .$$

Theorem 3. Each H is a metric space with metric $d^*(a^*,b^*)$, for a^*, b^* in H.

The proof follows directly from the definitions, and the properties of d' and d.

The theorems of this section were included for completeness. Their application in the context of the preceding material will now be discussed.

<u>Equivalence Relationships Among Maps and Bases in Reduction Sequences.</u> Corresponding to the set of binary reflexive, symmetric relations of n,b-association is an equivalence relation which is a special case of S-equivalence. The binary reflexive, symmetric relations of n-association, A_n, among maps and bases form a subset which is of special interest, particularly in connection with the determination of chromatic polynomials of large maps. The S-equivalence defined in terms of this set will be called <u>sequence equivalence</u>, and will be denoted, as before, by the symbol, S. Thus two maps will be said to be <u>sequence equivalent</u> with respect to a given set $\overline{M}$ if and only if they are elements of a finite sequence of maps belonging to $\overline{M}$ such that each pair of maps adjacent in the sequence are related under one of the binary relations of n-association which are defined, reflexive and symmetric on $\overline{M}$. The corresponding sequence equivalence relation exists for bases. The relations of <u>n-equivalence</u> for each $n = 1, 2, \ldots$, are special cases of the relation of sequence equivalence for which all the reflexive, symmetric binary relations of at least one join sequence are the same relation, A_n, for each $n = 1, 2, \ldots$, respectively.

The role of the equivalence relation E can be played by symmetric, absolute, topological, or chromatical equivalence. Each of the first three implies the last and, in turn, chromatical equivalence implies sequence equivalence. In each case, however, the converse does not necessarily follow. The metric defined as in connection with theorem 3 for classes of maps with respect to sequence equivalence and chromatic equivalence will be called <u>chromatic distance</u>. It affords a quantitative measure of the chromatic differences between maps which are chromatically distinct but sequence equivalent. The corresponding function exists for sequence equivalent bases.

Maps or bases which differ by a unit of chromatic distance are n-associates. Two n-associate bases contain the same number of maps, namely one fewer than their common extension. Hence two sequence equivalent bases also contain the same number of maps. This property can be of value in determining chromatic polynomials of large maps. Its practical significance follows from the fact that while each of these various sequence equivalent bases initially contain the same number of maps, certain of them may be potentially more efficient than others. For example, it is sometimes possible to replace a map of the given set M by a sequence equivalent map having a certain symmetry property, or a smaller number of regions.

3. Applications of Associate Sets and Sequence Equivalence

<u>Applications of Associate Sets.</u> The present reduction sequences contain numerous examples of the application of n,k-association and sequence equivalence to the problem of determining chromatic polynomials of large maps.

An example of n,k-association occurs in connection with the maps 114 and 13'. These maps are symmetrically equivalent, respectively, to the maps 121 and 123 shown in Figures 4 and 6. The maps 12, 114, and 13' thus form a fundamentally associate set in which 12 plays the role of the antecedent in formula (1), and 114 and 13' correspond to the consequent terms denoted by the symbols 1 and 3, respectively. These relationships can be indicated schematically both by means of the following list:

(15) 12—114(1)
  13'(3)

and by writing the fundamental principle in a form in which only the coefficients and the sequence symbols are explicit:

(16) (12) = (114) - (122) + (13')

The n,k-associate sets given in Table III are found among the maps arising in the present reduction sequences. They are arranged according to the values of n and k and written either as in (15) or in the modified form of (15) discussed below. The sets corresponding to particular values of n and k appear in the order in which the antecedents are reduced in the reduction sequences.

The associate sets for which n = 2 and n = 3 are not listed separately for the following reasons. When a map contains a 3-gon, say, it corresponds to the antecedent of formula (3), hence n = 3. If the consequent is not symmetrically equivalent to a map already in the sequences, then k = 1, and the 3,1-associate set consists of the single map containing the 3-gon. Such a map can always be identified by the prime in its sequence symbol, since the 3-gon is always reduced. If the consequent is symmetrically equivalent to a map which has already arisen, then k = 0. In this event, the antecedent itself may or may not be symmetrically equivalent to a map already in the sequences. Examples of these two cases occur, respectively, in connection with the set (15) and the set: 1223-23'25(6'). Strictly speaking, the latter set lies in the 3-extension containing the map 2225, say, where it has the form: 1223-2325(6). In sets corresponding to either of these two cases, for convenience, the reduction of the 3-gons is denoted in a list such as (15). There, for example, the last line takes the form: 13"(3').

The set 1223-23'25(6') actually refers only to the consequents of the truncated icosahedron, while the 4,1-associate set having 12236 as its antecedent is employed only in the reduction sequences leading to the set T(n-6). Each of the remaining associate sets refers to the reduction sequences leading both to T(n-6) and to T(36).

The significance of the association index, k, from the standpoint of the efficiency of the reduction sequences lies in the fact that only k of the consequents of a map such as 12 of the set (15) require further reduction. In that case, k = 1, the single map remaining to be reduced being 122. Chief interest naturally centers upon the associate sets for which k is small. Those for which k = 0 are actually n-dependent sets. Examples of these appear in Table III.

Table III

Associate Sets in the Reduction Sequences

Yielding the Sets S(n-6) and T(26).

Fundamental k-associate sets

k = 0	k = 1	k = 2
23'241-13"44(1)	5'-13"(1')	2-12(1)
13"46(2)	23'(2)	
13"45'(3')		23'24-13"64(2)
	12-114(1)	
115116-11404(1)	13"(3')	
11406(2)		
11405'(3')	23'-13"(1)	
	5'$\bar{3}$"(3")	
	122-22(1)	
	23'2(2')	
	116-1146(2)	
	1140(3)	
	1151-1140(2)	
	13"(3)	
	115114-23'24(3')	
	12334(3)	

Table XII (continued)

Associate Sets in the Reduction Sequences

Yielding the Sets S(n-6) and T(26).

4.k-associate sets

k = 0	k = 1	k = 2
11404-2204(4)	12236-122346(4)	11-13"(6')
22042(5)	122356(5)	
22043(6)		13"5-13"45'(4')
	115116-22041(4)	
	22042(6)	13"6-13"56(6)
		13"44-3'$\bar{3}$"(4)
		22041-13"64(4)
		11404-13"5(4)
		11406-22042(4)
		1223-23'26(6')
		12234-13"44(4)
		12236-122346(4)

5.k-associate sets

k = 0	k = 2
1151141-12234(7)	22-1223(7)
223'4(8')	23'2(9)
23'24(9)	
2204(0)	114-116(7)
	13"(9) .

<u>Applications of Sequence Equivalence.</u> A sequence equivalence can be constructed whenever the association index has the value zero or unity. In the present context, for example, the efficiency of the 2-gon and 3-gon reduction formulas can be regarded as stemming from the fact that $k \leqq 1$ invariably holds for these formulas, and hence that a map containing a 2-gon or a 3-gon is always sequence equivalent to the simpler map obtained by reducing this region.

Sequence equivalences of a less straightforward type can be obtained in connection with each of the sets of Table III for which $k \leqq 1$. For example, the map 12 of the set (15) is sequence equivalent to 114 and 13'. More important, from the standpoint of the efficiency of the reduction sequences is the fact that, in the extension, the map 12 shown in Figure 3 is also sequence equivalent to the simpler map 122 of Figure 3 in which the boundary AF of Figure 3 has been erased. This relationship can be regarded either as a fundamental association:

$$(17) \qquad 12 \; A_1 \; 122 \; .$$

or as the sequence equivalence implied thereby:

$$(18) \qquad 12 \approx 122 \; .$$

Several of the concepts of the previous section can be illustrated in connection with these relations. Denote by M the set of maps which existed in the reduction sequences prior to the application of the fundamental principle which is indicated in formula (16). Then the set

$$N' = N + (122) \quad ,$$

is a fundamental extension of N. Two of the fundamentally associate bases of this extension are:

$$N = N' - (122) \quad ,$$

and

$$N_1 = N' - (12) \quad .$$

Each of the last two equations is a special case of one of the relations of (14). Each of the relations (16) through (18) can be regarded as exchanging the unreduced map 12 for the simpler unreduced map 122, and correspondingly, exchanging the basis N for its simpler fundamental associate N_1. The maps 12 and 122 of Figures 3 and 5 are also seen to be fundamentally similar, and to be separated by a unit of chromatic distance.

A particularly interesting example of a sequence equivalence arises in connection with the reduction of the map 115. In the presence of the set consisting of all of the remaining maps arising in the reduction sequences, this map is sequence equivalent to the much simpler map 1151156'. The sequence equivalence:

$$(19) \qquad 115 \equiv 1151156' \quad ,$$

can be established by means of the join sequence:

115 A_1 1151 A_1 11511 A_4 115115 A_4 1151155 A_3 1151155' .

Consider the initial fundamental association of this join sequence:

(20) 115 A_1 1151 .

It is based upon the existence of the fundamentally associate set:

(21) 115-1140(2)
1140(3) .

In the reduction sequences, which is exploited by applying the fundamental principle to 115. This map is symmetrically equivalent to the one obtained from Figure 4 by erasing the broken side, AD, and denoting the resulting 9-sided region by D. The application of the fundamental principle to the boundary, DF, of this region yields the map 1151 of Figure 7,

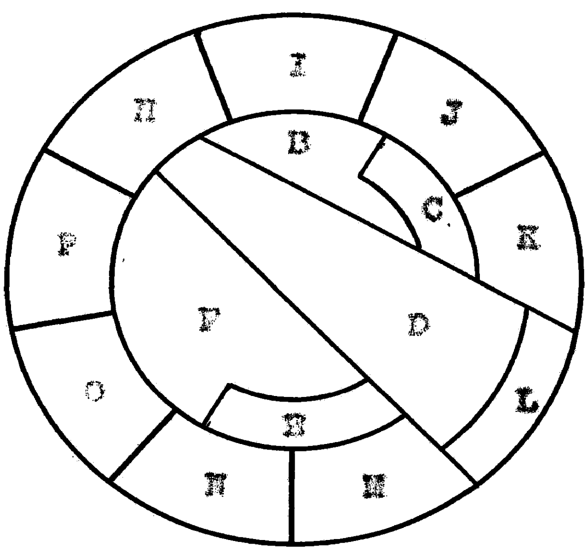

Figure 7

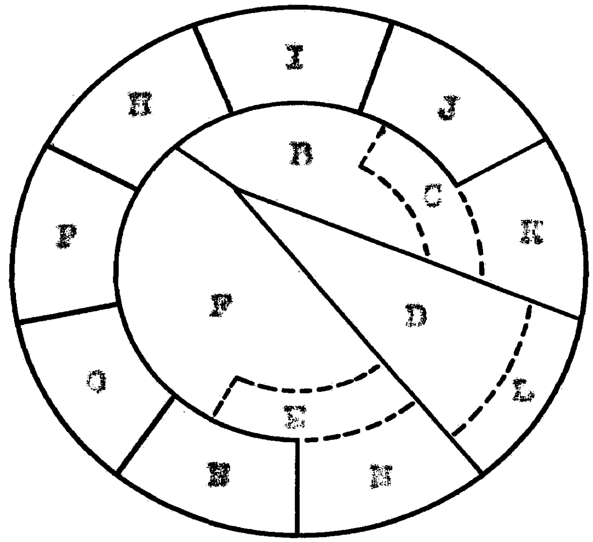

Figure 8

and the maps 1152 and 1153 which are symmetrically equivalent, respectively, to 1140 and 1140 of the associate set (21).

The relation (20) and the remaining association relationships forming the join sequence establishing the sequence equivalence (19) can be indicated schematically, in the manner of (16), as follows:

fundamental association: $115\ A_1\ 1151$.

(22) $(115) = (1151) - (1140) + (1140')$;

fundamental association: $1151\ A_1\ 11511$.

(23) $(1151) = (11511) - (1140) + (13'')$;

4-association: $11511\ A_4\ 115115$.

(24) $(11511) = (u-1)(115114) + u(115115) + (115116)$;

4-association: $115115\ A_4\ 1151156$.

(25) $(115115) = (u-1)(22041) + u(1151156) + (22042)$;

3-association: $1151156\ A_3\ 1151156'$.

(26) $(1151156) = u(1151156')$.

In the case of third m-association, (24), all four of the maps entering the corresponding linear relation actually lie not in W itself, but in its extension set $\overline{W}$. The fundamental extensions containing 11511 are defined by (22) and (23). The fundamental extensions containing 115114 and 115116 are defined by the following relations:

fundamental extension containing 115114:

(27) $\qquad$ (115114) = (1151141) − (1151142) + (12234) ;

fundamental extension containing 115116:

(28) $\qquad$ (115116) = (11404) − (11406) + (11405) .

In turn, (27) involves two additional extensions:

5-extension containing 1151141:

$$(1151141) = (u-2)(12234) + 2(u-1)(11511418)$$
$$+ (23'24) + 2(2204) \; ;$$

3-extension containing 1151142:

$$(1151142) = u(23'24) \; ;$$

the first of which involves the:

3-extension containing 11511418:

$$(11511418) = u(223'4) \; .$$

In the presence of these seven extensions, the relation (24) also defines the extension containing 115115. The map 11311 of Figure 8 in which all the broken sides are understood to be present in the map, is obtained from 1151 of Figure 7 by applying the fundamental principle (23) to the boundary DH of the 8-sided region D. The maps 115115 and 1151155 are obtained through the quadrilateral reduction formulas (24) and (25) by erasing the pairs of broken sides EF and FM, and CB and CK, respectively in Figure 8. In the resulting map the region D now has only three sides one of which, DL say, can be erased in accordance with the 3-gon reduction formula (26) to yield the final map in the join sequence, 1151155'. This is the map of Figure 8 in which it is understood that each of the five broken sides is obliterated.

In terms of the reduction sequences used to determine chromatic polynomials of maps of the class M(n), the existence of this sequence equivalence means that each submap which has n-8 regions and contains the configuration 115 can be exchanged for the much simpler submap containing the configuration 1151155', and having only n-7 regions.

8. Techniques for Determining Chromatic Polynomials of Large Maps

It is of interest to compare the present method with previous methods in terms of the preceding material. With the aid of formulas such as (1) through (8), the chromatic polynomial of a given map can be expressed in terms of the polynomials of maps having a smaller number of regions. The process can be repeated until it yields a set of submaps, all of whose polynomials either are directly available from previous work, or can be written by inspection. The construction of such a reduction sequence can be begun with the original map, or with simpler maps whose polynomials are known, or at some convenient intermediate stage. In either of the last two methods a sequence leading to the given map can be outlined *a priori* in terms of maps containing progressively larger numbers of regions. This is the approach of Rudsbock (7). In the first method, the selection of the sequence leading from the original map to simpler maps can be made _a posteriori_ in accordance with the objective of increasing the efficiency of the process. The present method for determining chromatic polynomials of large maps uses this approach. It begins with the original map and expresses its polynomial in terms of those of maps having smaller numbers of regions. The advantage of this approach has become especially significant due to the fact that more efficient methods have been developed for decreasing the number of maps entering the reduction sequences.

In previous methods, the use of formulas such as (1) through (5) was governed by considerations of the following type. Formulas (3) through (5), for example, are seen to be in order of their efficiency when appraised according to the number of new maps which they introduce, and according to the relative number of regions which these new maps contain. The fundamental principle (1) is not only least efficient in this respect but is actually seen to involve a retrogression in this sense, since it exchanges a single map of n regions for a set of three maps, one of which also has n regions.

Thus, in order to increase the efficiency of the reduction sequences it is apparently desirable to proceed by reducing all the 2-gons and 3-gons to obtain regular maps and then, in turn, to reduce the 4-gons, the 5-gons and so forth. Such a method was employed by Birkhoff and Lewis (5, pp. 376-378), and by Rudebock (7).

In the present method, the process is conceived not as one in which individual maps are successively reduced, but rather the reduction sequences are constructed by considering the set as a whole. The set of maps at hand at any given stage is actually regarded as a basis for possible extensions which can contain n,k-associate sets and sequence equivalent maps. Efficient reduction sequences can be selected by comparing different combinations of these sets and equivalences.

The two methods can be contrasted in connection with the reduction of the map 115. This is the map of Figure 4 in which the broken boundary is understood to be erased. It is seen to contain two symmetrically located quadrilaterals C and E, each of which can be reduced by obliterating the side in common with the region D so as to yield the additional quadrilaterals B and F, respectively. Thus, this map happens to be particularly well suited to the previous method for reducing a regular map. Under that method the reduction of these quadrilaterals is apparently the most efficient procedure, while the use of the fundamental principle would appear to be the least efficient approach. In contrast, however, the join sequence underlying (19) actually begins with two applications of the fundamental principle, namely (68) and (85). Each of these, however, takes advantage of the presence of a fundamentally associate set in the reduction sequences. The sequence equivalence involves two additional applications of the fundamental principle in connection with the fundamental extensions which are established by (27) and (28). The efficacy of this strategy is illustrated by the fact that the use of the previous method to reduce the number of regions in 115 by five results in an increase of seven in the number of maps in the sequence. By means of the present method, which takes advantage of a sequence equivalence, the same reduction was accomplished without any increase whatsoever in the number of maps.

This example and the discussions of the previous section illustrate the types of sequence equivalence which can be discovered by exploring and comparing various possible reduction sequences.

A noteworthy feature of the present method is the fact that, in it, the fundamental principle plays a much more important role than it does in the previous methods. This is evident both from the illustration of sequence equivalence and in the array of associate sets given in Table III. The array contains a preponderance of maps in fundamental k-associate sets, particularly in the categories giving rise to sequence equivalence, that is, those for which $k \leqq 1$.

Once the reduction sequences have been determined, it remains actually to express the chromatic polynomial of the original map in terms of the polynomials of the submaps. This is done by compounding the coefficients in sequences involving formulas such as (1) through (5). This process, too can be carried out in more than one way. The order indicated by the reduction formulas themselves is not necessarily the most efficient. This is especially true when the reduction sequences contain large numbers of associate sets. In such cases it is often preferable to partition the entire set of maps arising in the reduction sequences with respect to chromatic equivalence or symmetric equivalence, and to order these equivalence classes according to the number of regions which their maps contain. Within these sets the equivalence classes can be arranged according to the order in which their representative sequence symbols arise in the reduction sequences. The chromatic polynomial coefficients corresponding to these equivalence classes can then be determined in the order established in this way.

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ABSTRACT

Joseph W. Siry, M. S. 1947 (B. S. Rutgers University)
Title of thesis: Chromatic Polynomials of Large Maps
Thesis directed by Professor Dick Wick Hall
Major: Mathematics
Minors: Electrical Engineering, Physics
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Birkhoff and Lewis have stated a strong form of the four-color conjecture in terms of conditions on chromatic polynomials and have established, by actually determining the polynomials involved, that these conditions are satisfied for certain proper maps which are regular in the sense of Birkhoff and contain seventeen or fewer regions. It is of particular interest to determine whether their conjecture holds in the case of a regular map which possesses the additional topological property that no two of its pentagons are contiguous. The simplest such map is the truncated icosahedron which contains thirty-two regions. In 1940 a program to determine the chromatic polynomial of this map was begun by B. I. Tudebook, now Mrs. J. L. Vanderslice. Over the intervening years this program has been continued by Mrs. Vanderslice and Professor Dick Wick Hall. They have obtained the chromatic polynomials of several hundred maps having twenty-five or fewer regions.

Further progress has been greatly facilitated by the method of the present paper for determining chromatic polynomials of large maps. The method is based upon an analysis of the structure of sets of maps by means of a certain class of reflexive, symmetric binary relations and a related class

ABSTRACT (continued)

of equivalence relations. When applied to a class of maps each having a certain proper nine-ring and a homeomorph invariant under a transformation group isomorphic with the symmetric group of degree three, the method yields an expression for the chromatic polynomial of each map of n regions belonging to the class in terms of the chromatic polynomials of a set of thirty-six of its submaps, each of which contains no more than n-6 regions. This expression, combined with known chromatic polynomials, brings within range the determination of the chromatic polynomial of the truncated icosahedron itself.

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