THESIS REPORT

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Modeling and Control of Dynamical Effects due to Impact on Flexible Structures

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Abstract

Title of Dissertation: MODELING AND CONTROL OF DYNAMICAL EFFECTS DUE TO IMPACT ON FLEXIBLE STRUCTURES

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In the first part of this dissertation, we consider modeling and approximation of impact dynamics on flexible structures. A nonlinear model is developed through Hertz law of impact in conjunction with the dynamic equation of the flexible structure. We have analyzed this nonlinear model and established the existence and uniqueness of solutions of the nonlinear equation. A numerical method is developed based on the contraction mapping principle. By utilizing the fact that impact interval is very short in general, one may approximate the transfer functions of the systems to which the impacting bodies belong by Taylor polynomials of low order. We have developed the first and second order approximations. The first order approximation yields a special function which can be used for analytical and computational purposes. The second order approximation leads to a two-parameter family of ordinary differential equations of which the solutions exhibit universal features of impact problems. Simulation results of various examples have demonstrated the usefulness of the developed numerical method and approximation methods.

The second part of this dissertation is devoted to control of impact dynamics. We have formulated and studied a control problem where a linear system is subjected to a series of impact forces. The impact forces are treated as disturbances to the system and modeled as finite duration events using the theory developed in part one. A reasonable control objective is to design a feedback controller to minimize the energy transferred from the disturbances to the controlled outputs in the L_2 norm sense. Under the assumption that the disturbance information is known a priori, a (sub)optimal control strategy is derived based on dynamic game theory. We have shown that, by taking advantage of the fact that the duration of each impact force is very short in general, we can derive a series of approximate solutions of the nonlinear problem. The higher order terms may be negligible for the disturbance attenuation problem in some applications. Hence, the approximation with the leading term renders a linear one. A (sub)optimal H_∞ controller is derived and a procedure to compute such a controller is given. The (sub)optimal solution is naturally associated with the existence of a stabilizing periodic solution of coupled Riccati equations. Hamiltonian theory is employed to analyze the coupled Riccati equations. Finally, we investigate the digital implementation of this control algorithm by using a sampled-data controller. We have shown that under a certain sampling condition, the controller structure could become simpler than the continuous version.

MODELING AND CONTROL OF DYNAMICAL EFFECTS DUE TO IMPACT ON FLEXIBLE STRUCTURES

by

Qifeng Wei

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Dedication

To my wife: Xumei Zhao

and to my parents

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Chapter 1

INTRODUCTION

This dissertation is composed of two parts. The first part (chapters 2, 3 and 4) is concerned with modeling and approximation of impact dynamics on flexible structures. Impact phenomena have interested scientists and engineers for a long time. Numerous attempts have been made to model accurately the dynamical effects of impact between two or more objects [1, 2]. Our motivation to study impact problems comes from force and constrained motion control in various robotic manipulations [1-8]. One popular example in these problems is to use a flexible robotic arm to catch a baseball, play tennis, snatch a stationary object from a table, hammer a nail into a board, or collect space debris while attached to a moving space vehicle. A physical contact between the manipulator and the object must occur before the desired force and motion can be applied to the object. To model the effects of these actions on a flexible arm, the impact dynamics must be examined. Salmatjidis [3] and Chapnik etc.[4] independently studied impact control of flexible manipulators. Both of them applied open-loop control strategies to cancel the impact forces due to the bandwidth limitation of actuators and sensors. The open-loop control scheme requires a precise model of impact force in order to achieve a good performance. Salmatjidis used the stereomechanical impact model [5] by assuming that the impact duration is negligible. Chapnik etc. used an impact model devised by Lee etc. [6], which accounts for the finite duration of impact. Experimental results from both studies showed that, during the contact period, although it is very short, impact dynamics can be a significant factor. If the impact phenomena are not modeled properly, or the manipulator is not controlled to diminish the effect of impact, and result could be the failure of the manipulator [7, 8, 9, 10]. Moreover, since high precision control of robotic manipulators has become increasingly important in a variety of industrial applications, e.g. laser beam technology, semiconductor wafer manufacturing etc., this requires paying extra attention to the usual dynamical effects as well as taking into consideration otherwise ignored features such as dynamical effects due to impact.

Consideration of displacement and use of Hertz law of impact at the region of contact seems to be the most successful approach [11]. When the contact involves a flexible structure Hertz law of impact leads to a nonlinear integral equation called the Hertz equation, which incorporates the effects of local elastic deformation at the region of contact [6]. This model has been widely applied to various impact situations [5, 12, 13], and experimental results obtained in [12, 13] well support the validation of this equation. However, there are several drawbacks associated with this nonlinear model.

1) Numerical methods: this nonlinear equation does not admit a closed form solution in general, hence numerical methods must be developed to solve this equation. There are some numerical methods such as Timoshenko's small-increment method and an energy method devised by Zener, Feshbach and Lee etc. More

efficient methods need to be developed due to the popular use of this model.

2) Mathematical analysis: although the above numerical methods are successfully used to solve the nonlinear equation, these methods were proposed without establishing their convergence or even existence of a unique solution. Our viewpoint is that theoretical analysis is necessary for both proving the validation of this equation and developing efficient numerical methods.

3) Computational complexity: in all existing numerical methods, each case, e.g. varying initial velocities, has to be numerically solved separately. Also, a fairly large computational burden has to be incurred for each numerical solution. One of the goals of this dissertation is to attempt to solve these problems.

Our methodology for modeling the impact dynamics is based on the Hertz law of impact. In chapter 3, we derive a nonlinear impact model through Hertz law of impact in conjunction with the dynamical equation of a flexible structure which involves impact. We establish the existence and uniqueness of solutions of the Hertz equation by applying the contraction mapping principle. The detailed proofs of both local and global solutions will be given. Chapter 4 addresses the problem of numerically solving the Hertz equation. Timoshenko's small-increment method [11] yields a solution within any degree of accuracy provided that sufficiently small time steps of integration are used. Hence it has become the basis for evaluating other approximation methods. Another useful method is the application of the energy method devised by Zener and Feshbach [14], and applied by Lee [6] to the central impact of a sphere on a simply supported beam. By combining the principle of conservation of energy during the impact, and momentum considerations for the impacting sphere, the resulting velocity of the impacting sphere and vibrational energy imparted to the beam are determined.

Impact force was assumed to be a sinusoidal function of time. This method is simple and fast, but not reliable, and we will give some numerical examples to illustrate this. In its place we will develop a numerical method using successive Picard iterations; $f_0, Pf_0, PPf_0, \cdots\cdots$, where the initial condition f_0 is obtained from the energy method, and P is a contraction operator. We show that this method is faster compared to the small-increment method, more accurate than the energy method, and its convergence is very fast.

Observe from numerical solutions that the impact period is very short in general. Therefore, one may approximate the transfer functions of the systems to which the impacting bodies belong by Taylor polynomials of low order. We explicitly carry out this computation in the cases of first and second order Taylor polynomial approximations of the transfer functions. We show that in the case of the first order approximation there is a universal ordinary differential equation that describes the impact behavior completely. Therefore, one can numerically solve this equation beforehand, save the results, and use it to predict the impact behavior with only a minimal computational burden. In the case of the second order approximation there is a two-parameter family of ordinary differential equations that govern the impact behavior. These approximation methods are validated via numerical examples.

The second part (chapter 5 and 6) is devoted to control aspects of impact dynamics. We formulate and study a control problem where a linear system is subjected to a series of impact forces. The impact forces are treated as the disturbances to the system and modeled as finite duration events using the theory developed in part one. Among the motivating factors is the need to study the control problems related to mechanical systems subject to impact forces, e.g. active control of the suspension system of a vehicle against irregularities of the road [15, 16], accurate pointing of guns, stabilization of an antenna on the space station subject to impact from space debris, or active damping of vibrations of flexible structures caused by impact forces [10, 17]. A reasonable control objective in all these problems is to design a stabilizing controller to minimize the energy transferred from the disturbances to the controlled outputs in the L_2 norm sense. This problem in turn can be studied in the framework of H_∞ control theory.

An important paradigm in control synthesis is the H_∞ control problem introduced by Zames [18]. In this formulation, the disturbances belong to a ball in a certain function space, and a quadratic cost function is minimized for the worst disturbance in this set. Various theories of H_∞ control problems for linear systems have been developed so far by many researchers (see e.g. [19, 20, 21, 22, 23, 24]). Solutions to these problems can be obtained via frequency domain methods, or recently developed state-space methods.

Recently, several researchers have attempted to extend H_∞ control results to nonlinear systems. Ball and Helton [25, 26], from a viewpoint of operator theory, have discussed H_∞ control theory of nonlinear systems for the first time. Basar and Isidori [27, 28] study the connection between the H_∞ control theory of nonlinear systems and differential game theory. In this setting, one is naturally led to a nonlinear partial differential equation known as the *Isaacs equation*. A straightforward application of the theory (see e.g., [29]) to the case of a plant modeled by an affine nonlinear system shows that once a solution of the appropriate *Isaacs equation* is found, a (full-information) feedback law providing disturbance attenuation (in the L_2 gain sense) can be computed right away. Van der Schaft [30, 31] analyzed the relation of the L_2 gain between nonlinear systems and their linearization, and gave a sufficient condition for the existence of smooth H_∞ state feedback. In addition, using the dissipative system theory [32, 33], Van der Schaft has discussed the relation between the L_2 gain and the Hamilton-Jacobi equation in [34]. However, his approach requires the equilibrium point of a suitable Hamilton system to be hyperbolic.

The H_∞ control of nonlinear systems is still largely open. One problem is the output feedback control. Van der Schaft derived an output feedback controller for a certain class of nonlinear systems [35], which however has the severe drawback of being infinite-dimensional. Much more effort has been put into obtaining finite-dimensional controllers in general (see, e.g. [36]), but without decisive answers thus far. Another main issue for the applicability of nonlinear H_∞ control is its computational complexity, one of the essential requirements in all of the above approaches is either to solve a Hamilton-Jacobi equation or a set of completed Hamilton-Jacobi inequalities [34, 35].

The disturbance attenuation of impact forces is a nonlinear control problem due to the nonlinearity of the impact model, which is different from the problems discussed in [36, 34, 35]. It is not affine in disturbance input, and the linearization around the equilibrium does not exist. Hence this nonlinear problem cannot be solved by the above proposed methods, but some analysis can be carried over to this problem. Our approach is based on the dynamic game theory [27, 29]. In this setting, the control problem naturally becomes a minimax optimization problem of a cost function $L(u, v)$ with two players, where u is the control and v is the disturbance. Roughly speaking, the player u tries to minimize $L(u, v)$ in U, while the player v tries to maximize $L(u, v)$ in V simultaneously, where U, V are the constraint sets of u and v respectively. [27, 29] show that the (sub)optimal disturbance attenuation problem has a solution for a given $\gamma > 0$ if the minimax optimization problem admits a saddle point. We show that, due to nature of impact dynamics, the saddle point may not exist in problems involving impacts. Motivated by the dynamic game approach, if the information of the disturbance v is assumed to be known a priori (e.g. sensors can predict the impact velocity before impact), one can seek an optimal strategy by using this information such that a certain attenuation level γ is achieved. A procedure is given to compute the optimal strategy $u(v)$, and the optimal attenuation level γ^*. Finally, by taking advantage of the fact that the duration of each impact force is very short in general, we derive a series of approximate models for the original nonlinear system. In many cases it turns out that the higher order terms are negligible. Hence a special linearization of the nonlinear system can be obtained by using the leading term as an approximation. Thus, in these cases the nonlinear problem can be approximated by a linear one. In chapter 6, motivated by the linear impulsive model obtained in chapter 5, we formulate a linear H_∞ control problem. In the infinite horizon case, a state-space solution for the state-feedback controller design problem is derived. The (sub)optimal H_∞ controller is naturally associated with the existence of stabilizing periodic solution of a coupled Riccati equations (one is a differential Riccati equation, one is a difference Riccati equation). Hamiltonian theory is employed to analyze the coupled Riccati equations. Finally, we investigate the digital implementation of this control algorithm by using a sampled-data controller. We show that under a certain sampling condition, the control problem can be converted into a standard discrete-time H_∞ problem. Then, the state-space solution is derived through dynamic game theory. The structure of the sampled-data controller could be simpler than the continuous version. An impact example is given and discussed.

In the final chapter, we summarize the main results obtained in this dissertation, and identify some interesting problems for future research.

Chapter 2

A REVIEW OF PREVIOUS WORK

In this chapter, we will review some previous work on modeling and control of impact dynamics.

2.1 Impact Modeling

Impact phenomena have interested scientists and engineers for a long time. Numerous attempts have been made to accurately model dynamical effects of impact between two or more objects. The first formulation of rigid-body impact theory is due to Galileo, which the focus is on the impulse-momentum law for rigid bodies and involves a minimum of mathematical difficulties. For perfectly elastic impact of two bodies, the law of conservation of energy provides the second relation required to uniquely determine the final velocities of the objects. Newton later introduced a coefficient of restitution e to account for the degree of plasticity of the collision and energy loss during impact. This modified impact law was widely used in many impact problems [1, 2, 15, 16]. However, it is incapable of describing the transient stresses, forces, or deformations produced, and is limited to a specification of the initial and terminal velocity states of the objects and the applied linear or angular impulse. The theory fails to account for local deformations at the contact point and further assumes that a negligible fraction of the initial kinetic energy of the system is transformed into vibrational energy of the colliding bodies. The last hypothesis has been found to be reasonably valid for the collision of two spheres, but not for collisions involving flexible structures (beams, rods, plates etc).

The first satisfactory analysis of the stresses caused by the impact of two elastic bodies is due to Hertz, who viewed the contact of two bodies as an equivalent problem in electrostatics. A solution was obtained in the form of a potential which described the stresses and deformations near the contact point as a function of the geometrical and elastic properties of the bodies.

When impact involves a flexible structure, consideration of displacement and use of Hertz's law of impact at the region of contact seems to be the most successful approach [11]. Hertz law of impact leads to a nonlinear integral equation called the Hertz equation, which incorporates the effects of local elastic deformation at the region of contact [6]. This model has been widely applied to various impact situations [5, 12, 13], and the experimental results obtained in [12, 13] well support the validity of this equation. Unfortunately, this nonlinear equation does not admit a closed form solution, hence numerical methods must be developed to solve this equation. Timoshenko [11] developed the so-called small-increment method to give numerical solutions, and it has become the basis for evaluating other approximation methods. Some other approximation methods also give very satisfactory results [6, 38]. One of them is the application of the energy method devised by Zener and Feshbach [14], and applied by Lee [6] to central impact of a sphere on a simply supported beam.

2.1.1 Stereomechanical impact

The classical theory of impact, called stereomechanics, is based primarily on the impulse-momentum law for rigid bodies and involves a minimum of mathematical difficulties in its formulation. For perfectly elastic impact of two bodies, the law of conservation of mechanical energy provides the second relation required to uniquely determine the final velocities of the objects. When the impact produces a permanent deformation this relation is replaced by the introduction of a coefficient of restitution e for the process. This coefficient purports to describe the degree of plasticity of the impact, and is usually defined as the ratio of final to initial relative velocity components of the striking objects in the direction normal to the contact surfaces. This impact theory has been used in various applications, e.g central impact of two rigid bodies. Assume the two rigid bodies of masses m_1 and m_2 respectively make initial contact with each other at a point on the line connecting their centers of gravity with velocities v_{10} and v_{20} respectively. By the use of the moment conservation law, the velocities of the two bodies after impact are given by [5],

$$\begin{aligned} v_{1f} &= v_{10} - (1+e)\frac{m_2(v_{10}-v_{20})}{m_1+m_2} \qquad (2.1.1) \\ v_{2f} &= v_{20} + (1+e)\frac{m_1(v_{10}-v_{20})}{m_1+m_2} \\ &= v_{20} + K_2(v_{10}-v_{20}), \qquad (2.1.2) \end{aligned}$$

where e is the coefficient of restitution and assumed to be a constant. For a given e and initial velocities, the final velocities of contacting objects can be determined immediately from these equations.

However, there are some limitations associated with this method, especially for impact involving a flexible structure. It is incapable of describing the transient stresses, forces, or deformations produced, and the method further assumes that a negligible fraction of the initial kinetic energy of the system is transformed into vibrational energy of the colliding bodies.

2.1.2 Hertz law of impact

The first satisfactory analysis of the stresses at the contact of two elastic bodies is due to Hertz, who viewed the contact of two bodies as an equivalent problem in electrostatics. A solution was obtained in the form of a potential which described the stresses and deformations near the contact point as a function of the geometrical and elastic properties of the bodies. The Hertz law of impact is,

$$\alpha(t) = K[f(t)]^{2/3}, \tag{2.1.3}$$

where $\alpha(t)$ is the relative approach, i.e. the difference between the displacements of the bodies, and K is the Hertz constant [5, 39],

$$K = \frac{4}{3}\{q/(Q_1 + Q_2\sqrt{A+B})\}, \tag{2.1.4}$$

where q, A and B are constants which depend on the local geometry of the region of contact, for a sphere of radius R_1 and plane surface contact case, $A = B = \frac{1}{2R_1}$ and $q = \pi^{1/3}$, and,

$$Q_1 = (1-\mu_1^2)/E_1\pi, \qquad Q_2 = (1-\mu_2^2)/E_2\pi, \tag{2.1.5}$$

where, μ_i and E_i are the Poisson ratio and Young's modulus for the two bodies respectively.

This impact law, although both static and elastic in nature, has been widely applied to various impact situations and the experimental results obtained well support the validity of this equation.

2.1.3 Central impact of a sphere on a simply supported beam

Timoshenko was first to use the Hertz law of impact to model the impact dynamics on a flexible beam. He formulated and solved numerically the problem of central impact of a sphere on a simply supported beam [11, 5]. The impact problem can be formulated as follows; a beam is struck transversely by a spherical mass m with initial moving velocity v_0. We further assume that the Hertz law of impact is valid i.e.

$$\alpha = K[f(t)]^{2/3}, \tag{2.1.6}$$

where α is the relative approach of the striking body, $f(t)$ is the contact force.

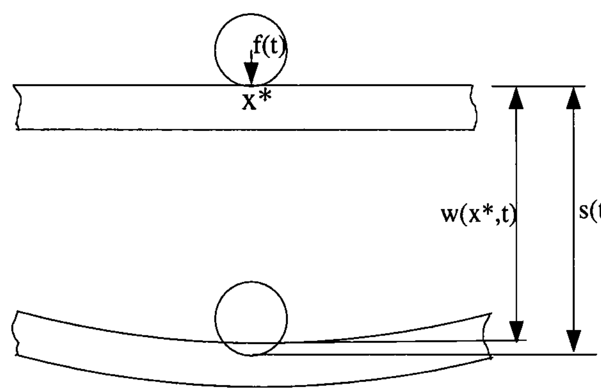

Figure 2.1: A sketch of the displacement

The relative approach is the difference between the displacement of the beam and the contacting body, measured from the instant of initial contact. Hence

$$\alpha = s(t) - w(x^*, t), \tag{2.1.7}$$

where $w(x^*, t)$ is deflection of the beam at the point of contact x^*, $s(t)$ is the displacement of the ball under of the contact force $f(t)$, and is given by,

$$s(t) = v_0 t - \frac{1}{m}\int_0^t f(\tau)(t - \tau)d\tau. \tag{2.1.8}$$

From the equations (2.1.6)-(2.1.8) we obtain the nonlinear integral equation,

$$K[f(t)]^{2/3} = v_0 t - \frac{1}{m}\int_0^t f(\tau)(t-\tau)d\tau - w(x^*, t). \tag{2.1.9}$$

The forced vibrations produced in the beam by the varying force $f(\tau)$ at the point of contact are expressed in terms of the normal modes of vibration. In the case of symmetrical vibration of a simply supported beam, the normalized characteristic deflections [11] are $\sqrt{2/l sin(n\pi x/l)}$ for $n = 1, 3, 5$, etc., where x is the coordinate of position along the beam of length l. The corresponding frequencies of the natural modes of vibration are w_n, related by the equation $w_n = n^2 w_1$.

The central deflection $w(x^*, t)$ due to the forced vibrations is given by

$$w(x^*, t) = \sum_{n=1,3,5}^{\infty} \frac{2}{ql}\int_0^t f(\tau)\frac{sin w_n(t-\tau)}{w_n}d\tau \tag{2.1.10}$$

where q is the constant mass per unit length of the beam. $ql/2$, denoted by M, is called the reduced mass and in this particular case is equal to half the mass of the beam. Substituting equation (2.1.10) in equation (2.1.9) gives the integral equation for the impact force from which the motion can be determined, that is

$$K[f(t)]^{2/3} = v_0 t - \frac{1}{m}\int_0^t f(\tau)(t-\tau)d\tau - \sum_{n=1,3,5}^{\infty} \frac{1}{M}\int_0^t f(\tau)\frac{sin w_n(t-\tau)}{w_n}d\tau. \tag{2.1.11}$$

The nonlinear term $K[f(t)]^{2/3}$, precludes any closed form solution for (2.1.11).

2.2 Numerical Methods

There are some numerical methods available in the existing literatures [6, 38, 40]. Timoshenko's small-increment method [11] is usually reliable, but could turn out to be time consuming. Another useful method is the application of the energy method devised by Zener and Feshbach [14], and applied by Lee [6] to central impact of a sphere on a simply supported beam. This method is simple and fast, but not reliable as we will illustrate via some numerical examples.

2.2.1 Small-increment method

In this section we will illustrate Timoshenko's small-increment method (see [11] for details). Recall that the Hertz equation is given by

$$K[f(t)]^{2/3} = v_0 t - \frac{1}{m}\int_0^t f(\tau)(t-\tau)d\tau - \sum_{i=1,3,5}^{\infty}\frac{1}{M}\int_0^t f(\tau)\frac{sinw_i(t-\tau)}{w_i}d\tau \quad (2.2.1)$$

Small-increment method divides the time interval $[0,t)$ into n sub-intervals with increments $\triangle\tau$, and the impact force is assumed to be constant during each sub-interval. Thus for the n^{th} time interval $t = n\tau$, equation (2.2.1) can be written as

$$\begin{aligned} K[f_n]^{2/3} &= v_0 n\triangle\tau - \frac{(\triangle\tau)^2}{m}\sum_{j=1}^{n} D_{n-j+1}f_j - \\ &\frac{1}{M}\sum_{j=1}^{n} f_j \sum_{i=1,3,5}^{\infty}\frac{cosw_i(n-j)\triangle\tau - cosw_i(n-j+1)\triangle\tau}{w_i^2} \end{aligned} \quad (2.2.2)$$

where $\triangle\tau$ is conveniently chosen as some small fraction of the fundamental period of vibration of the beam. The quantity $\sum_{j=1}^{n} D_{n-j+1}f_j$ arises from the double integration term and, for a linear continuous approximation of the force-time curve may be expressed as

$$\begin{aligned}\sum_{j=1}^{n} D_{n-j+1} f_j &= 2[(n-1)f_1 + (n-2)(f_2 - f_1) + \cdots \\ &\quad (n-j)(f_j - f_{j-1} + f_{j-2} - f_{j-3} + \cdots \mp f_1) + \cdots + \\ &\quad (f_{n-1} - f_{n-2} + f_{n-3} - \cdots \pm f_1)] + \\ &\quad 1/3(f_n - f_{n-1} + f_{n-2} - \cdots \pm f_1) \end{aligned} \tag{2.2.3}$$

The accuracy of the results and the labor of computation depend upon the proper evaluation of the term $\sum_{i=1}^{\infty} \frac{cosw_i(n-j)\Delta\tau - cosw_i(n-j+1)\Delta\tau}{w_i^2}$ being directly proportional to the number of modes included and the fineness of the time interval chosen. Procedural techniques for the step-wise approximation are given in reference[41]. Although this method is very lengthy, it is considered in the literature as the standard numerical method for solving the Hertz equation.

2.2.2 A solution by approximating impact force

Lorenertz [42] developed a concise method of solution by assuming that the duration of impact is small compared with the period of the fundamental mode of vibration of the beam and also that only fundamental model is excited. The first assumption is justified in most impact problems and, moreover, is easily checked, but the decision where or not it is permissible to neglect the effect of the higher modes of vibration of the beam is more difficulty. The latter assumption is equivalent to replacing the beam by the one degree-freedom system of a mass on a spring [6].

The method of solution is to assume a sinusoidal contact variation $f(t) = Psin(\lambda t)$ which enables the integrals in equation (2.2.1) to be evaluated, thus transforming it into an algebraic equation. Values of P and λ are required to satisfy this equation for all t during the impact. Since $f(t) = Psin(\lambda t)$ cannot be negative, contact occurs for $0 < \lambda t < \pi$, so that the duration of contact is $T_D = \pi/\lambda(sec)$. Substituting the sinusoidal variation for $f(t)$ and and evaluating the integral, the equation (2.2.1) becomes

$$K[Psin\lambda t]^{2/3} = v_0 t - \frac{P}{m}(\frac{t}{\lambda} - \frac{sin\lambda t}{\lambda^2}) - \frac{P}{M}\frac{\lambda sinw_1 t - w_1 sin\lambda t}{w_1(\lambda^2 - w_1^2)}. \quad (2.2.4)$$

Equation (2.2.4) is still too complicated for direct solution. Some simplifications can be made by writing $w_1 t$ for $sinw_1 t$, since $w_1 t$ remains small. In addition $[sin^{2/3}\lambda t]$ can be approximated by $Asin\lambda t$; where A is chosen so that the new contact force variation, which makes $\alpha = KP^{2/3}Asin\lambda t$, produces the same total change of momentum in the striking mass as did $Psin\lambda t$. This condition expressed mathematically becomes

$$\int_0^\pi sin\theta d\theta = A\int_0^\pi sin^{2/3}\theta d\theta$$

which gives $A = 1.093$, inserting these simplifications, Equation (2.2.4) becomes

$$KP^{2/3}A\ sin\lambda t = v_0 t - \frac{P}{m}(\frac{t}{\lambda} - \frac{sin\lambda t}{\lambda^2}) - \frac{P}{M}\frac{\lambda t - sin\lambda t}{\lambda^2 - w_1^2}. \quad (2.2.5)$$

Equating the coefficients of $sin\lambda t$ and t on each side gives two simultaneous equations for the determination of P and λ,

$$\begin{aligned} 0 &= v_0 - \frac{P}{m\lambda} - \frac{P}{M}\frac{\lambda}{\lambda^2 - w_1^2}, \\ KP^{2/3}A &= \frac{P}{m\lambda^2} + \frac{P}{M}\frac{1}{\lambda^2 - w_1^2}. \end{aligned} \quad (2.2.6)$$

After eliminating P we obtain

$$\lambda^5 = v_0\frac{[(m+M)\lambda^2 - Mw_1^2]^2}{A^3K^3m^2M^2(\lambda^2 - w_1^2)^2}. \quad (2.2.7)$$

Then P is determined from the first of the simultaneous equations [6] which gives

$$P = v_0 \frac{mM\lambda(\lambda^2 - w_1^2)}{(m+M)\lambda^2 - Mw_1^2}. \tag{2.2.8}$$

Numerical results obtained in [6] showed that this method is fairly good in some cases, but may cause a large error in other cases. This indicated that, the error in the solution lies not in the mathematical approximations adopted but in the physical assumption that only the fundamental mode of the beam affects the impact process. Hence some criterion must be found which will give conditions justifying such an assumption.

Among the other approximation methods found in the literature is the energy method devised by Zener and Feshbach [14], and applied by Lee [6] to central impact of a sphere on a simply supported beam.

2.2.3 Energy method

An important contribution to the understanding of impact characteristics was made by Zener and Feshbach [14] in their considerations of energy transfer. The procedure in the application of the principle of conservation of energy is to assume a contact force variation in terms of which the energy imparted to the beam can be represented. The vibrational energy absorbed by each mode is composed of the sum of kinetic energy and strain energy of bending at a particular time. Recall that the deflection of the beam can be written as $w(x^*,t) = \sum_{n=1}^{\infty} C_n(t)W_n(x^*)$ by using the Green's function method, where $C_n(t)$ is modal coordinate and is governed by the following differential equation,

$$\ddot{C}_n(t) + w_n^2 C_n(t) = W_n(x^*)f(t), \qquad n = 1, 2, 3 \cdots$$

It is easy to show that the energy for the n^{th} mode during impact is equal to

$$\begin{aligned} \triangle E_n &= \frac{1}{2}[\dot{C}_n^2(t) + w_n^2 C_n^2(t)] \\ &= \frac{1}{2} W_n(x^*)^2 | \int_0^T f(\tau) e^{iw_n \tau} d\tau |^2, \end{aligned} \tag{2.2.9}$$

where the T is the time when contact loses.

Errors resulting from ignorance of the contact-force variation $f(\tau)$ can be largely eliminated by expressing $f(\tau)$ in the form of a normalized force. The rebound velocity of the impactor can be expressed in the form ev_0 by introducing a restitution coefficient e. The total change in the momentum of the mass produced by the impact force is then

$$mv_0(1+e) = \int_0^T f(\tau) d\tau. \tag{2.2.10}$$

Now, defining the normalized impact force $F(\tau)$ by the relation $f(\tau) = mv_0(1+e)F(\tau)$, the momentum equation becomes

$$\int_0^T F(\tau) d\tau = 1. \tag{2.2.11}$$

Substitute for $f(\tau)$, equation (2.2.9) becomes

$$\triangle E_n = \frac{1}{2} m^2 v_0^2 (1+e)^2 W_n(x^*)^2 | \int_0^T F(\tau) e^{iw_n \tau} d\tau |^2. \tag{2.2.12}$$

The total vibrational energy of the beam is equal to the sum of the energies in the separate modes of vibrations, i.e., the total vibration energy $\triangle E = \sum_{n=1}^{\infty} \triangle E_n$. Let us define a function R as

$$R = m \sum_{n=1}^{\infty} W_n(x^*)^2 | \int_0^T F(\tau) e^{iw_n \tau} d\tau |^2. \tag{2.2.13}$$

The total vibrational energy $\triangle E$ now reduces to

$$\triangle E = \sum_{n=1}^{\infty} \triangle E_n = \frac{1}{2} m v_0^2 (1+e)^2 R = E(1+e)^2 R. \tag{2.2.14}$$

where E represents the initial kinetic energy of the impactor. Since energy is conserved, $\triangle E$ is also the loss of energy of the impactor, giving the equation

$$\triangle E = E(1 - e^2). \tag{2.2.15}$$

Now $\triangle E/E$ can be eliminated from equations giving the coefficient e in terms of the function R, or

$$e = \frac{1-R}{1+R}. \tag{2.2.16}$$

If we assume that an expression for $F(\tau)$ satisfies the condition (2.2.11), then R and e can be determined. However, the $F(\tau)$ is generally unknown, hence an approximation must be selected. In order to avoid heavy computational burden to evaluate function R, a simple force function can be chosen. One example given in [6] is of the form

$$\begin{aligned} F(t) &= \frac{T_0}{2\pi} sin(\pi t/T_0) \\ &= \frac{1}{2\lambda} sin\lambda t, \qquad \forall t \in [0, T_0]. \end{aligned} \tag{2.2.17}$$

where, T_0 is the estimated impact duration by assuming two rigid body impact, calculated from [5]. Hence, the impact force is given by

$$f(t) = K_0 sin\lambda t. \tag{2.2.18}$$

where the K_0 is a constant.

2.3 Control of Impact Forces

As we have pointed out in chapter 1, the force and constrained motion control in various robotic manipulations is the main motivation to this research. In general, there are three modes of operation so that a robot can complete a task: motion in free space, impact and constrained motion. Obviously the manipulator must change from one mode to the other readily. Usually switching from the constrained space to free space presents no problems. However, the opposite switch has the significant problem of impact forces. The impact forces can be very large, and can drive an otherwise stable controller into instability. Typically, it is the force control strategy that must deal with this transient phenomenon. However, the natural elasticity of impacting bodies (demonstrated by the nonlinear impact model), can cause the manipulator to rebound from the environment. Thus, the manipulator is once again unconstrained. This phenomenon can cause oscillatory behavior [43]. Obviously it is the goal of any controller to pass this transitory period successfully, and have the manipulator stably exerting forces on the environment in the end. The controller must, therefore, pass through the impact phase by attempting to maintain contact with the environment until all of the energy of the impact has been absorbed. Most early research treated the impact as a transient and did not account for the impact effect, which resulted in performance tradeoffs. This problem has drawn more attention recently since high precision control of robotic manipulators has become increasingly important in a variety of industrial applications, e.g. laser beam technology, semiconductor wafer manufacturing etc., thus requiring extra attention to the usual dynamical effects as well as taking into consideration otherwise ignored features such as dynamical effects due to impact. Below we describe some the impact control schemes that have been developed in the past few years.

2.3.1 Open-loop control strategy

One useful method that has been used for controlling impact dynamics is open-loop control schemes. The motivation for this strategy stems from the observation that feedback signals may obscure effects due to impact because of the bandwidth limitations of actuators and sensors [4]. Salmatjidis [3] and Chapnik

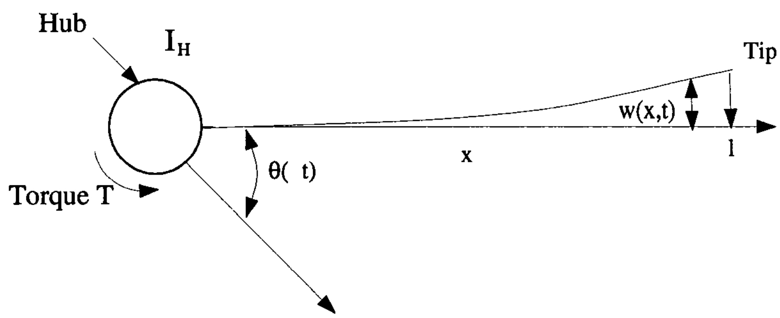

Figure 2.2: A single-link flexible robot

etc.[4] independently studied impact and force control problem of a single-link flexible robot manipulator shown in Figure 2.2, where a flexible robot with a flexible beam is mounted on a moving base driven by a torque $T(t)$. Suppose the flexible robot is transversely collided with an impactor with mass m and velocity v_0. If we just consider the transverse deflection of the beam and ignore the rotary inertia effects, by the extended Hamilton principle[44], the dynamic equation of the beam is

$$\rho\frac{\partial^2 y}{\partial t^2} + EI\frac{\partial y^4}{\partial x^4} = \bar{f}(x^*, t), \qquad 0 < x < l, \tag{2.3.1}$$

where $y(x,t) = w(x,t) + x\theta(t)$, $w(x,t)$ is the deflection of the beam, and $\theta(t)$ is the rotational angel of the hub, and $\bar{f}(x^*,t)$ is the distributed load. Equation (2.3.1) describes an infinite dimensional system. Typical finite dimensional model approximations are used for the controller design.

Finite Element Model:

The elastic deflection $w(x,t)$ is to be approximated using finite element techniques [44]. Assembling the element mass and stiffness matrices leads to the semi-discretized equations of motion for the flexible arm systems,

$$M\ddot{Q} + C\dot{Q} + KQ = F \tag{2.3.2}$$

where M, C, and K are the global mass, damping, and stiffness matrices, respectively, F is the external force vector (applied torque is specified in the first entry of this vector), and F contains the unknown reaction (impact) forces at each element. $Q = [\theta, w_1, \dot{w}_1, w_2, \dot{w}_2 \cdots\cdots, w_N, \dot{w}_N]^T$ for some finite integer $N > 0$.

Simulation:

In order to calculate the time-domain response of the beam from the equation (2.3.2), the Newmark-β integration scheme [45] has been implemented, and yields the vector $\ddot{Q}, \dot{Q}$, and Q from equation after every integration time step. In order to solve the equation completely, a model of impact force must be given.

Impact Model and Control:

Salmatjidis used a stereomechanical impact model by assuming that the impact duration is negligible. The impulsive force is defined as

$$\hat{f} = \lim_{\Delta t \to 0} \int_{t_0}^{t_0+\Delta t} f(t)dt \tag{2.3.3}$$

Plugging the impulsive force model into the dynamic equation (2.3.2) and taking the limit process, the problem is left to determine to the amplitude of $\hat{f}$. Based on this impact model the authors realized that a contact problem is actually a problem with holonomic constraints. The primary kinematic axiom of a contact problem is, that the structures involved do not penetrate each other. Hence the kinematic constraint provides an extra equation necessarily to determine the impact force $\hat{f}$. An open-loop control algorithm is proposed, which operates the three modes of the flexible robot as follows;

1). The controller monitors contact through the FSR (force sensitive resister) output, while in free space.

2). When contact is detected, an open-loop control scheme is implemented. The desired displacement and velocity profiles that the system should follow are specified by using the simplified dynamic model derived through kinematic (holonomic) constraints which hold when contact is established, then the required torque is computed.

3). Once the transient phenomenon is over and contact is established, the force regulator is turned on (PD control), making the last minor corrections.

Experimental results showed that this control scheme is better than convention force control methods without accounting for the impact forces. As pointed out by the authors, this method is still in a quite primitive state.

Chapnik etc. [4] proposed another open-loop approach. An impact model (energy method) developed by [6, 5] is used for their analysis of impact force, where the finite duration of impact is accounted for

$$f(t) = K_0 sin\lambda t. \tag{2.3.4}$$

The control strategy is based on a computed torque scheme proposed by Bayo [46]. As Bayo has shown, only one degree of freedom needs be specified from equation (2.3.2) in order to solve for the required torque. Once this variable has been specified, by an equation in time domain, the other degrees of freedom can be calculated. It was decided that, in accordance with Bayo, the variable specified would be the global tip acceleration, $\ddot{y}_N$. This allows the specification to include the final resting place of the beam, and to force minimum tip vibration in the global sense. Once the required variable is specified, Fast Fourier Transforms are used to calculated the required torque.

The weakness of the open-loop control schemes is obvious. Without any feedback the performance of the control is determined by the accuracy of the system and impact models. Modeling errors may result in even greater impact forces than without control.

2.3.2 Proportional explicit force control

The another successful method is proposed by Volpe [9, 10], an experimental approach devised to solve impact control problems. By using an explicit force control and properly tuning the feedback gain, the author showed that impact dynamics can be compensated satisfactorily. To better understand this control scheme, let us consider a generic force-based explicit force controller shown in Figure 2.3, where G is the plant, H is the controller, and R is the feedfor-

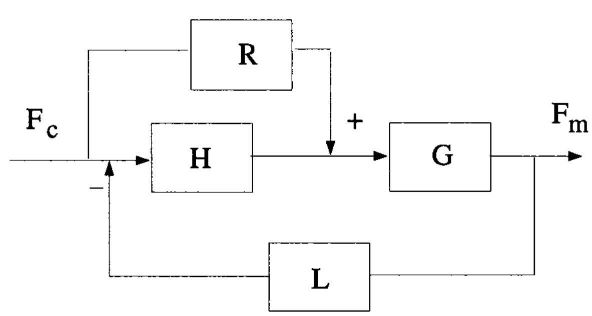

Figure 2.3: Explicit force control block diagram

ward transfer function, and L is a force feedback filter. The plant G may be represented by the fourth order model or the reduced second order model [43]. The controller H is usually some subset of PID control (e.g. P, PI, PD). In

[9], authors implemented a force-based explicit force controller with proportional gain and extra feedback for reaction force compensation. The plant G has be expanded into its components. It has been shown that the system is equivalent to one shown in Figure 2.4. An analysis using the root locus method showed

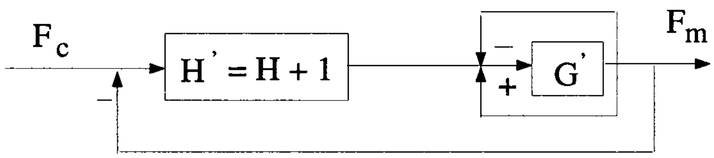

Figure 2.4: Impact control block diagram

that the stability of the closed loop system is guaranteed for $H' \geq 0$, i.e. negative proportional force control gains down to -1 are stable. The experimental results showed that $H \rightarrow -1$ is desirable for impact control. An interpretation in [10] goes as as follows;

By using this control scheme, the controller does not utilize the force error signal, since $H' = H + 1 \approx 0$. However, the reaction force of the impact is directly negated by a feedback signal. Viewed this way, the impact controller does not bounce, because the oscillations in the commanded force and those in the experienced force are equal and opposite. Thus the surface is at a node of two interfering pressure waves. No net force means no net acceleration. Any initial oscillation is damped out by natural and active damping. The drawback of this control algorithm is that the best feedback gain for impact dynamics is very poor for tracking purposes which results in performance tradeoff.

2.3.3 Other approaches

There are some other approaches aiming to control impact forces.

A) **Passive Compliance and Damping**

One proposed method of dealing with the impact problem is to use passive compliance, either on the end effector or in the environment. Some researchers have proposed the use of soft force sensors [47]. These methods appear to provide stable impact in two ways. First, the material used naturally provides passive damping that helps to absorb some of the energy of impact. Second, the compliance of the material effectively lessens the stiffness of the system composed of the material and the environment. There are problems with the passive compliance: it may not be modified without physical replacement of the material and it also limits the effective stiffness of the manipulator during position control etc.

B) **Active Damping**

Another method is to employ maximal damping during the impact phase [48]. The goal of this strategy is to damp out the oscillations caused by the transition. While this may be successful for soft environments, stiff environment may present some problems, since the feedback signal seems obscure the effect of impact forces [4].

Chapter 3

MODELING OF IMPACT ON FLEXIBLE STRUCTURES

In this chapter, we will provide a systematic way to model the impact forces on flexible structures via use of Hertz law of impact. For the sake of simplicity, we only consider a flexible structure subject to impact forces occurring from contact with an elastic body. Here a flexible structure means both finite-dimensional and infinite-dimensional systems (a lumped-parameter structure and a distributed-parameter structure). We restrict attention to the problem of modeling impact dynamics, existence of solutions to the model, and application examples. Numerical aspects will be addressed in chapter 3.

In section 2 of this chapter an impact is formulated and the Hertz equation is derived through the Hertz law of impact. In section 3 we will discuss some basic properties of the Green's function associated with the Euler Bernoulli beam equation. This equation is used to describe the motion of the beam. In section 4 we establish the existence and uniqueness of solutions of the Hertz equation by applying the contraction mapping principle. We will discuss two examples in section 5.

3.1 Formulation of the Impact Problem

3.1.1 Impact on a lumped-parameter structure

We first study the impact between an elastic body and a lumped-parameter structure. For our purpose, the impact problem can be formulated as follows; A lumped-parameter structure is described in Figure 2.1, where m_i is the mass of the i_{th} rigid body and k_i represents the stiffness $(i = 1, 2, \cdots, n-1, n)$ of a spring between bodies i and $i+1$. The words "rigid body" certainly need to be clarified here, the meaning of "rigid body" is macroscopic. For the impact problem, "rigid body" is still assumed to be locally deformable. Suppose that the system is struck by a mass m having a spherical surface at the point of contact with initial moving velocity v_0. We further assume that the Hertz law of impact is valid , i.e,

$$\alpha = K[f(t)]^{2/3}, \tag{3.1.1}$$

where, $\alpha(t)$, the relative approach, is the difference between the displacements of the first rigid body and the contacting body, measured from the instant of initial contact. Hence,

$$\alpha = s(t) - x_1(t), \tag{3.1.2}$$

where $x_1(t)$ is the displacement of the first rigid body of the system, $s(t)$ is the displacement of the ball under the contact force $f(t)$, and is given by,

$$s(t) = v_0 t - \frac{1}{m}\int_0^t f(\tau)(t - \tau)d\tau. \tag{3.1.3}$$

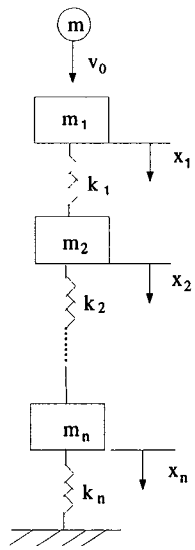

Figure 3.1: Impact on a lumped-parameter structure

From equations (3.1.1), (3.1.2) and (3.1.3), we obtain the following nonlinear integral equation,

$$K[f(t)]^{2/3} = v_0 t - \frac{1}{m}\int_0^t f(\tau)(t - \tau)d\tau - x_1(t). \qquad (3.1.4)$$

If we assume that the flexible structure is at rest just before impact, it is easy to show that the displacement $x_1(t)$ can be expressed as

$$x_1(t) = \int_0^t G_1(t-\tau)f(\tau)d\tau \qquad (3.1.5)$$

where the function $G_1(\cdot)$ is the Green's function of the system. Replacing $x_1(t)$ in equation (3.1.4) by (3.1.5), we get the nonlinear integral equation

$$K[f(t)]^{2/3} = v_0 t - \frac{1}{m}\int_0^t f(\tau)(t - \tau)d\tau - \int_0^t G_1(t-\tau)f(\tau)d\tau. \quad (3.1.6)$$

We note here that in general it is impossible to solve (3.1.6) analytically.

3.1.2 Impact on a flexible beam

We now study the impact between an elastic body and a flexible beam with various boundary conditions. Suppose a beam is struck transversely by a mass m having a spherical surface at the point of contact with initial moving velocity v_0 (see Figure 2.2). In the same way as for the lumped-parameter structure in

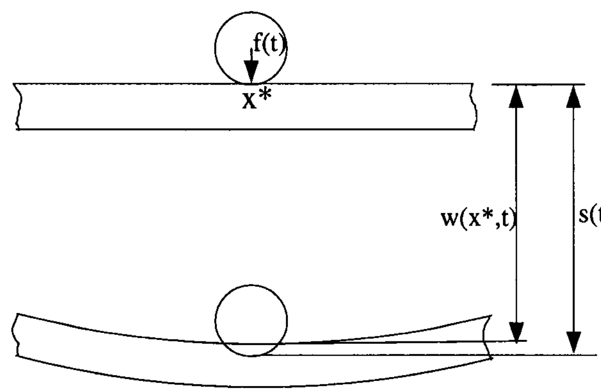

Figure 3.2: Impact on a flexible beam

section 2.2.1, we end up with the nonlinear integral equation

$$K[f(t)]^{2/3} = v_0 t - \frac{1}{m}\int_0^t f(\tau)(t - \tau)d\tau - w(x^*, t). \tag{3.1.7}$$

where $w(x^*, t)$ is the deflection of the beam at the point of contact x^*. Before we can carry out further analysis, it is necessary to represent the deflection of the beam at the point of contact $w(x^*, t)$.

3.2 Deflection of the Beam and Green's Function

If we restrict attention to transverse vibrations only and assume that the beam is long and slender, the transverse shear and torsional effects can be neglected, and the dynamics of the beam can be described by the Euler Bernoulli beam equation. The deflection of the beam is in turn obtained by solving the following partial differential equation

$$\rho \frac{\partial^2 w}{\partial t^2} + EI \triangle w = \bar{f}(x,t) \qquad 0 < x < l, \tag{3.2.1}$$

in which $\triangle = \frac{\partial^4}{\partial x^4}$, l is the length of the beam, ρ is the mass density and EI is the bending stiffness (here ρ and EI are assumed constants), and $\bar{f}(x,t)$ is the distributed load. The deflection of the beam is uniquely determined when equation (3.2.1) is solved under the appropriate initial and boundary conditions. If we assume that the beam is at rest just before impact, the initial conditions are

$$w(x,0) \;=\; \dot{w}(x,0) \;=\; 0. \tag{3.2.2}$$

Various boundary conditions of interest can be described as

$$B_i w(x,t) \;=\; 0, \qquad i \;=\; 1,2, \qquad x \;=\; 0,l, \tag{3.2.3}$$

where B_i is a linear homogeneous differential operator of maximum order 3.

The concentrated load $f(t)$ is obtained as a limiting process of a uniformly distributed load $\bar{f}(x,t)$ over a small range 2δ of the beam. Thus, by letting $\bar{f}(x,t) \to \infty$ while $\delta \to 0$, the contact force $f(t)$ is obtained as

$$f(t) \;=\; \lim_{\delta \to 0, \bar{f} \to \infty} \int_{x^*-\delta}^{x^*+\delta} \bar{f}(x,t)dx \tag{3.2.4}$$

One of the most popular methods for analyzing linear partial differential equations is the Green's function method [49]. With the aid of a Green's function, the solution of a certain class of PDE can be expressed as an integral.

Definition 3.2.1 *A function $G(x, \zeta; t) \in L^2[0, l]$ is called a Green's function of (3.2.1)-(3.2.3), if it satisfies the following conditions:*

i) As a function of the argument x, it satisfies the homogeneous differential equation , i.e. $\bar{f} = 0$, everywhere except at $x = \zeta$ where it may have a singularity.

ii) As a function of the argument t, it satisfies the homogeneous differential equation everywhere except at $t = 0$ where it may have a singularity.

iii) As a function of the argument x , $G(x, \zeta; t)$ satisfies the boundary condition (3.2.3).

iv) It satisfies the initial conditions

$$G(x, \zeta; 0^+) = 0 \qquad \textit{and} \qquad \frac{\partial G(x, \zeta; 0^+)}{\partial t} = \delta(x, \zeta)/\rho. \tag{3.2.5}$$

Note that (i) and (ii) above lead to

$$\{EI \bigtriangleup + \rho \frac{\partial^2}{\partial t^2}\} G(x, \zeta; t) = \delta(x, \zeta)\delta(t). \tag{3.2.6}$$

If the PDE (3.2.1)-(3.2.3) is self-adjoint (which depends on the nature of the boundary conditions). Cantilevered and simply supported beams result in self-adjoint PDE's. $G(x, \zeta; t)$ is symmetric with respect to x and ζ, and can be expressed in terms of an eigenfunction expansion. Therefore, in this case it can be shown that the Green's function for the PDE (3.2.1)-(3.2.3) can be expressed in the form (we refer the reader to [50] for details)

$$G(x, \zeta; t) = \sum_{k=1}^{\infty} W_k(x) W_k(\zeta) \frac{sin w_k t}{w_k} H(t), \tag{3.2.7}$$

where $H(t)$ is the unit step function, $\{W_k(x)\}_{k=1}^{\infty}$ is an orthonormal basis of eigenfunctions and $\{w_k\}_{k=1}^{\infty}$ are the corresponding eigenvalues. The following theorem is standard in the theory of partial differential equations.

Theorem 3.2.1 *[50] A representation of the solution to the PDE (3.2.1)-(3.2.3) in terms of the Green's function $G(x, \zeta; t)$ is*

$$w(x,t) = \int_0^t \int_0^l G(x, \zeta; t-\tau) \bar{f}(\zeta, \tau) d\zeta d\tau \tag{3.2.8}$$

□

For the impact problem, because the contact can be treated as a point contact, the contact force has the special form (3.2.4). Hence equation (3.2.8) can be further simplified as

$$w(x,t) = \int_0^t G(x, x^*; t-\tau) f(\tau) d\tau. \tag{3.2.9}$$

For simplicity, we write $G(x^*; t)$ instead of $G(x^*, x^*; t)$ in the rest of the dissertation. From equations (3.1.7) and (3.2.9), the Hertz equation becomes

$$K[f(t)]^{2/3} = v_0 t - \frac{1}{m} \int_0^t f(\tau)(t-\tau) d\tau - \int_0^t G(x^*; t-\tau) f(\tau) d\tau. \tag{3.2.10}$$

Remark 3.2.1 *For the impact problems in section 3.2.1 and 3.2.2, we have derived the equations of impact dynamics (3.1.6) and (3.2.10) by means of the Hertz law of impact. Note that both equations have the same structure; the only difference is in the nature of their Green's functions. For a finite-dimensional system, the Green's function is much simpler in general, we only use equation (3.2.10) for analysis.*

3.3 Analysis of the Hertz Equation

We should point out that there already exist some equations similar to equation (3.2.10) to model impact dynamics on flexible structures [5, 6]. Timoshenko [11] derived one for a simply supported beam. Though these equations haven't been analyzed in any great detail in the existing literature, some numerical methods for solving these equations have been presented in some detail. Our view is that some theoretical analysis is necessary for both proving the validity of this equation and developing efficient numerical methods. T contraction mapping technique is employed here to show that a unique solution exists for the Hertz equation. Before invoking the contraction mapping theorem some rearrangements are necessary. Let

$$L(t) \;=\; t \;+\; mG(x^*;t). \qquad \forall t \geq 0 \tag{3.3.1}$$

Equation (3.2.10) can be rewritten as

$$f(t)^{2/3} \;=\; v_0' t \;-\; \frac{1}{m'}\int_0^t f(\tau)L(t-\tau)d\tau, \tag{3.3.2}$$

where $v_0' \;=\; v_0/K; \qquad m' \;=\; mK,$

$$\begin{aligned} f(t) &= [v_0' t - \frac{1}{m'}\int_0^t f(\tau)L(t-\tau)d\tau]^{3/2} && (3.3.3)\\ &= [v_0' t - \frac{1}{m'}\int_0^t f(\tau)(t-\tau)d\tau - \frac{1}{K}\int_0^t f(\tau)G(x^*;t-\tau)d\tau]^{3/2}. && (3.3.4)\end{aligned}$$

Note that v_0' is assumed to be positive always. Both equations (3.3.3) and (3.3.4) will be used in the following analysis. The following result is well known (see, e.g., [51]).

Theorem 3.3.1 *(Contraction Mapping Theorem) Let X be a Banach space, and B a closed subset of X. Let P: $B \to B$ be an operator satisfying the following condition: $\exists\ \rho < 1$ such that*

$$||Px - Py|| \;\leq\; \rho||x-y||, \qquad \forall x, y \in B.$$

Then

a) P has exactly one fixed point in B (denoted by x^*).

b) For any $x_0 \in B$, the sequence $\{x_n\}_0^\infty$ defined by

$$x_{n+1} = Px_n, \qquad n \geq 0$$

converges to x^*. Moreover,

$$||x_n - x^*|| \leq \frac{\rho^n}{1-\rho}||Px_0 - x_0||.$$

We will use this theorem as the main tool to show that equation (3.3.3) has a unique solution by constructing a contraction operator P on an appropriate closed subset B of a Banach space.

3.3.1 Local existence and uniqueness

We shall first establish some conditions under which (3.3.3) has exactly one solution over every finite interval $[0, \delta]$ for δ sufficiently small, i.e., conditions for local existence and uniqueness. We shall then obtain stronger conditions for global existence and uniqueness, i.e., conditions under which (3.3.3) has exactly one solution over $[0, T]$ for some finite T. Hypotheses in both theorems are satisfied by finite dimensional systems as well as simply supported and cantilevered beams.

Theorem 3.3.2 *Suppose that the Green's function $G(x^*; t)$ is uniformly bounded over $x^* \in [0, l]$ and $t \in [0, T]$ for some $T > 0$. Then there exists a small enough $\delta > 0$ such that (3.3.3) has a unique continuous solution for $t \in [0, \delta]$.*

Proof : *Let $M > 0$ be such that*

$$|G(x^*; t)| \leq M \qquad \forall t \geq 0 \quad \text{and} \quad \forall x^* \in [0, l].$$

Let $N > 0$ be a sufficiently large constant. Let $\delta > 0$ be small enough such that

(i) $\delta \leq \frac{N^{2/3}}{v_0' + MN/K}$;

(ii) $\delta \leq \frac{2v_0'}{[1/m' + M/K]}$;

(iii) $2(\delta^2/m' + M\delta/K)\sqrt{(v_0' + MN/K)\delta} < 1$.

Let the Banach space be $C[0,\delta]$, *the space of continuous real valued functions from* $[0,\delta]$, *endowed with the sup norm, i.e.* $||f||_\infty = \sup_{t\in[0,\delta]} |f(t)|$. *Let us define the mapping* $P : C[0,\delta] \to C[0,\delta]$ *by*

$$Pf(t) = [v_0't - \frac{1}{m'}\int_0^t f(\tau)L(t - \tau)d\tau]^{3/2}, \quad \forall t \in [0,\delta]. \quad (3.3.5)$$

The domain of P *is defined by*

$$\begin{aligned} B[0,\delta] = \quad & \{f(\cdot) \in C[0,\delta]; N \geq f(t) \geq 0; \\ & v_0't - \frac{1}{m'}\int_0^t f(\tau)L(t - \tau)d\tau \geq 0 \quad \forall t \in [0,\delta]\}. \quad (3.3.6) \end{aligned}$$

Obviously, $B[0,\delta]$ *is a closed subset of the Banach space of continuous functions on* $[0,\delta]$.

The rest of the proof is divided into two parts: first, we show that P *maps* $B[0,\delta]$ *into itself. Then we show that* P *is a contraction mapping on* $B[0,\delta]$.

a) $Pf \geq 0 \quad \forall f \in B[0,\delta]$ *by definition. Define mapping* $F : B[0,\delta] \to B[0,\delta]$ *by*

$$Ff(t) = [v_0't - \frac{1}{m'}\int_0^t f(\tau)L(t-\tau)d\tau]$$

Now $f \in B[0,\delta]$ *implies* $\frac{1}{m'}\int_0^t f(\tau)(t - \tau)d\tau \geq 0$. *Hence,*

$$\begin{aligned} |Ff(t)| &= |[v_0't - \frac{1}{m'}\int_0^t f(\tau)(t-\tau)d\tau - \frac{1}{K}\int_0^t f(\tau)G(x^*;t-\tau)d\tau]| \\ &\leq v_0't + \frac{1}{K}\int_0^t |f(\tau)||G(x^*;t-\tau)|d\tau \\ &\leq (v_0' + MN/K)t, \quad t \in [0,\delta]. \end{aligned}$$

$Ff(t) \geq 0$ *by definition. Therefore,*

$$Ff(t) \leq (v_0' + MN/K)t$$

implies $\quad Pf(t) \leq [(v_0' + MN/K)t]^{3/2} \leq t, \quad \forall t \in [0,\delta]$.

$$
\begin{aligned}
FPf(t) &= v_0't - \frac{1}{m'}\int_0^t Pf(\tau)(t-\tau)d\tau - \frac{1}{K}\int_0^t Pf(\tau)G(x^*;t-\tau)d\tau \\
&\geq v_0't - \frac{1}{m'}\int_0^t (t-\tau)\tau d\tau - \frac{1}{K}\int_0^t \tau M d\tau \\
&\geq v_0't - \frac{1}{m'}\frac{t^2}{2} - \frac{t^2 M}{2K} \\
&\geq 0 \qquad\qquad \forall t \in [0,\delta].
\end{aligned}
$$

Thus, we have shown that $PB[0,\delta] \subset B[0,\delta]$.

b) $\forall f_1, f_2 \in B[0,\delta]$,

$$
\begin{aligned}
Pf_1(t) - Pf_2(t) &= [v_0't - \frac{1}{m'}\int_0^t f_1(\tau)L(t-\tau)d\tau]^{3/2} \\
&\quad -[v_0't - \frac{1}{m'}\int_0^t f_2(\tau)L(t-\tau)d\tau]^{3/2}. \\
\textit{Let} \qquad x(t) &= [v_0't - \frac{1}{m'}\int_0^t f_1(\tau)L(t-\tau)d\tau]^{1/2}. \\
y(t) &= [v_0't - \frac{1}{m'}\int_0^t f_2(\tau)G(t-\tau)d\tau]^{1/2}.
\end{aligned}
$$

Since $f_1, f_2 \in B[0,\delta]$ *imply* $x(t), y(t)$ *are well defined.*

$$
\begin{aligned}
Pf_1(t) - Pf_2(t) &= x^3(t) - y^3(t) \qquad\qquad \forall t \in [0,\delta] \\
|Pf_1(t) - Pf_2(t)| &\leq |x^2(t) - y^2(t)||x(t) + y(t)| \\
x^2(t) - y^2(t) &= v_0't - \frac{1}{m'}\int_0^t f_1(\tau)L(t-\tau)d\tau \\
&\quad -(v_0't - \frac{1}{m'}\int_0^t f_2(\tau)L(t-\tau)d\tau) \\
&= \frac{1}{m'}\int_0^t (f_2(\tau) - f_1(\tau))L(t-\tau)d\tau. \\
|x^2(t) - y^2(t)| &= |\frac{1}{m'}\int_0^t (f_2(\tau) - f_1(\tau))L(t-\tau)d\tau| \\
&\leq \frac{1}{m'}\int_0^t |f_2(\tau) - f_1(\tau)|(t-\tau)d\tau \\
&\quad +\frac{1}{K}\int_0^t |f_2(\tau) - f_1(\tau)||G(x^*;t-\tau)|d\tau \\
&\leq (t^2/m' + Mt/K)||f_2 - f_1||_\infty \\
&\leq (\delta^2/m' + M\delta/K)||f_2 - f_1||_\infty.
\end{aligned}
$$

$$
\begin{aligned}
|x(t)+y(t)| &\leq |x(t)|+|y(t)| \leq 2\sqrt{v_0't+MNt/K} \\
&\leq 2\sqrt{(v_0'+MN/K)\delta}. \\
|Pf_1(t)-Pf_2(t)| &\leq |x^2(t)-y^2(t)||x(t)+y(t)| \\
&\leq 2(\delta^2/m'+M\delta/K)\sqrt{(v_0'+MN/K)\delta}||f_2(\cdot)-f_1(\cdot)||_\infty \\
&\leq \rho||f_2-f_1||_\infty \qquad \forall t\in[0,\delta].
\end{aligned}
$$

where $\rho = 2(\delta^2/m'+M\delta/K)\sqrt{(v_0'+MN/K)\delta}$, *and by the property(iii) of* δ, $\rho < 1$. *Therefore,*

$$
\begin{aligned}
||Pf_1-Pf_2||_\infty &= \sup_{t\in[0,\delta]} |Pf_1(t)-Pf_2(t)| \\
&\leq \rho||f_2-f_1||_\infty.
\end{aligned}
$$

so that P *is a contraction mapping on* $B[0,\delta]$.

Finally, using theorem 2.4.1, it follows that the mapping P *has a unique fixed point in* $B[0,\delta]$. *It is clear that* f *is a solution of the equation(3.3.3) over* $[0,\delta]$ *if and only if* $Pf = f$, *i.e.* f *is a fixed point of* P *over* $B[0,\delta]$. *This completes our proof.* □

The above theorem shows that a unique solution exists over $t \in [0,\delta]$ for some small δ. Our interest is to find the impact force variation during the entire contact period. The following theorem will establish this global result.

3.3.2 Global existence and uniqueness

Theorem 3.3.3 *Suppose that equation (3.3.3) has a local unique solution over* $[0,\delta]$ *for some sufficiently small* δ. *If* $f(\delta) > 0$, *then* $\exists\epsilon > 0$, *such that equation(3.3.3) has a unique continuous solution on* $[0,\delta+\epsilon]$.

Proof : *The argument is similar to the proof of the local version. We will only carry out the details of a crucial step here.*

Let $g : [0, \delta] \to R$ be the unique solution of (3.3.3) established in the proof of theorem 2.4.2. Let N be a positive number larger than $||g||_\infty$. Let $\epsilon > 0$ be a small positive constant. Let

$$\begin{aligned} D[0,\delta+\epsilon] &= \{f \in C[0,\delta+\epsilon]; f\mid_{[0,\delta]} = g; \\ &\quad v_0't - \frac{1}{m'}\int_0^t f(\tau)L(t-\tau)d\tau \geq 0, \quad \forall t \in [0,\delta+\epsilon]\}. \quad (3.3.7) \end{aligned}$$

Clearly D is a closed subset of $(C[0,\delta+\epsilon], ||\cdot||_\infty)$.

Let $F : D[0,\delta+\epsilon] \to C[0,\delta+\epsilon]$ be

$$Ff(t) \;=\; v_0't \;-\; \frac{1}{m'}\int_0^t f(\tau)L(t-\tau)d\tau,$$

and, let $P \;=\; F^{3/2}$.

We will show that for small enough ϵ, $P(D) \subset D$, and P is a contraction mapping, thus establishing the theorem. Note that it follows easily as in the proof of theorem 2.4.2 that

$$|Pf(t)| \;\leq\; N \qquad \forall t \in [0,\delta+\epsilon].$$

The crucial step is to show that $FPf(t) \geq 0 \quad \forall t \in [0,\delta+\epsilon], \quad \forall f \in D$. Now,

$FPf(t) \;=\; f(t) \quad \forall t \leq \delta$ since $f\mid_{[0,\delta]}$ satisfies (3.3.3). For $0 \leq t \leq \epsilon$,

$$\begin{aligned} FPf(t+\delta) &= v_0'\delta - \frac{1}{m'}\int_0^\delta Pf(\tau)(t+\delta-\tau)d\tau \\ &\quad -\frac{1}{K}\int_0^\delta Pf(\tau)G(x^*;t+\delta-\tau)d\tau \\ &\quad +v_0't - \frac{1}{m'}\int_0^t Pf(\tau+\delta)L(t-\tau)d\tau \\ &= v_0'\delta - \frac{1}{m'}\int_0^\delta Pf(\tau)(\delta-\tau)d\tau - \frac{1}{K}\int_0^\delta Pf(\tau)G(x^*;\delta-\tau)d\tau \end{aligned}$$

$$
\begin{aligned}
& -\frac{1}{K}\int_0^\delta f(\tau)(G(x^*,t+\delta-\tau)-G(x^*;\delta-\tau))d\tau \\
& +v_0't-\frac{t}{m'}\int_0^\delta f(\tau)d\tau-\frac{1}{m'}\int_0^t Pf(\tau+\delta)L(t-\tau)d\tau \\
= \; & [f^{2/3}(\delta)-\frac{t}{m'}\int_0^\delta f(\tau)d\tau-\bar{G}(x^*;t)+\bar{G}(x^*;0) \\
& +v_0't-\frac{1}{m'}\int_0^t Pf(\tau+\delta)L(t-\tau)d\tau]
\end{aligned}
$$

where $\bar{G}(x^*;t) = \frac{1}{K}\int_0^\delta f(\tau)G(x^*;t+\delta-\tau)d\tau$*. By the continuity of the Green's function, for small enough* ϵ*,*

$$
|\bar{G}(x^*,t)-\bar{G}(x^*,0)| \;\leq\; \frac{1}{2}f^{2/3}(\delta); \qquad \forall t\in[0,\epsilon]
$$

For such ϵ*,*

$$
FPf(t+\delta) \;\geq\; 0 \qquad \forall t\in[0,\epsilon].
$$

Hence, we have shown that $PD[0,\delta+\epsilon]\subset D[0,\delta+\epsilon]$*.*

It is easy to show that P *is a contraction mapping. The existence of a unique solution on* $[0,\delta+\epsilon]$ *follows at once now.* □

Remark 3.3.1 *The global version has the following physical interpretation. The condition* $f(t)>0$ *means that the contact is in progress at* $t\geq 0$*; finally,* $f(T)=0$ *means that the objects are just about to cease to be in contact, i.e.,* T *is the impact duration.*

3.4 Two Applications

3.4.1 A single-link flexible robot

High precision control of robotic manipulators is becoming increasingly important in a variety of applications, e.g., laser beam technology, semiconductor wafer manufacturing etc. This requires paying extra attention to the usual dynamical effects as well as taking into consideration otherwise ignored features such as dynamical effects due to impact. Here we consider a flexible robot with a flexible beam mounted on a moving base driven by a torque $T(t)$. Suppose the flexible robot transversely collides with an impactor with mass m and velocity v_0. If we just consider the transverse deflection of the beam and ignore the rotary inertia effects, by the extended Hamilton principle [44], the dynamic equation of the beam is

$$\rho\frac{\partial^2 y}{\partial t^2} + EI\frac{\partial y^4}{\partial x^4} = \bar{f}(x^*, t), \qquad 0 < x < l. \tag{3.4.1}$$

where $y(x,t) = w(x,t) + x\theta(t)$, $w(x,t)$ is the deflection of the beam, and $\theta(t)$

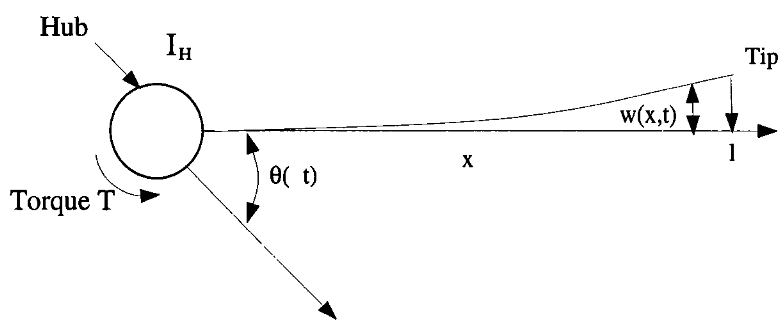

Figure 3.3: A single-link flexible robot

is the rotation angel of the hub and the $\bar{f}(x^*, t)$ is the distributed load. For the impact problem, the impact force has the general form

$$f(t) = \lim_{\delta\to 0, \bar{f}\to\infty} \int_{x^*-\delta}^{x^*+\delta} \bar{f}(x,t)dx \tag{3.4.2}$$

where x^* is the contact point of the beam. The boundary conditions are

$$\begin{aligned} EI\frac{\partial y^2}{\partial x^2} - I_H\frac{\partial y^3}{\partial t^2\partial x^1} + T(t) &= 0, \;\; y(x,t) = 0, \;\; at \;\; x = 0 \qquad (3.4.3) \\ EI\frac{\partial y^2}{\partial x^2} = EI\frac{\partial y^3}{\partial x^3} &= 0, \;\; at \;\; x = l. \end{aligned}$$

where I_H is the inertia of the base and $T(t)$ is the torque applied to the hub.

In order to apply the Green's method as described in section 3.3 to obtain the deflection of the flexible robot, we first need to solve the eigenvalue problem of equation (3.4.1)-(3.4.3). The general methods can be found in [52]. Posbergh [53] applied spectral operator theory to carry out some rigorous analysis. For the sake of brevity some useful results are listed here (see [53] for details).

The dynamic equations are first put into the infinite-dimensional state-space form

$$\frac{d}{dt}s(t) = As(t) + Bu(t),$$

where $s(t)$ are state variables, $u(t)$ are inputs. A and B are two abstract operators defined on some Hilbert space H.

$R1$: The operator A is a skew adjoint operator.

$R2$: The spectrum of operator A is discrete and the characteristic equation is

$$\frac{I_H\beta^3}{\rho}(1 + cos(\beta l)cosh(\beta l)) - sin(\beta l)cosh(\beta l) + sinh(\beta l)cos(\beta l) = 0 \quad (3.4.4)$$

where the eigenvalues λ are given by $\lambda^2 = -\frac{EI}{\rho}\beta^4$.

In general, there are a countably infinite number of β which satisfy this equation, denoted by $\beta_n,\ n = 1, 2, \cdots$, giving rise to a discrete set of eigenvalues.

Remark 3.4.1 *If $I_H = 0$ (the hinged-free case), the equation(3.4.4) reduces to*

$$sin(\beta_n l)cosh(\beta_n l) = sinh(\beta_n l)cos(\beta_n l).$$

Remark 3.4.2 *If $I_H \to \infty$ (the cantilevered case), the equation(3.4.4) reduces to*

$$cos(\beta_n l)cosh(\beta_n l) = -1.$$

$R3$: The associated eigenfunctions are orthonormal and have the following properties

$$\begin{aligned} W_n''''(x) &= \frac{\rho}{EI} w_n^2 W_n(x), \qquad && 0 < x < l, \qquad (3.4.5) \\ W_n''(0) &= -\frac{I_H}{EI} w_n^2 W_n'(0), && x = 0. \\ < W_n, W_m > &= \delta_{n,m} \end{aligned}$$

where the inner product defined on H is given by

$$< f, g >= \int_0^l \rho f g dx + I_H \frac{df(0)}{dx} \frac{dg(0)}{dx}.$$

Next, we derive the Green's function for equations (3.4.1)-(3.4.3). Since our interest is to model the impact dynamics, we will set $T(t) = 0$ for all t.

Because $\{W_n\}_{n=1}^{\infty}$ are orthonormal and form a complete basis of the Hilbert space H, the solution of equation (3.4.1) can be expressed in the form [44]

$$y(x,t) = \sum_{k=0}^{\infty} W_k(x) C_k(t), \qquad (3.4.6)$$

where $C_k(t)$ is the k^{th} modal coordinate. From equations (3.4.1) and (3.4.5),

$$\begin{aligned} \ddot{C}_k(t) + w_k^2 C_k(t) &= W_k(l) f(t), \\ (I_B + I_H)\ddot{\theta}(t) &= 0. \qquad (3.4.7) \end{aligned}$$

If we assume that the flexible robot is at rest before impact, the corresponding initial conditions are zero, and the solution $y(x,t)$ can be expressed as

$$y(x,t) = \int_0^t G(x, x^*; t - \tau) f(\tau) d\tau \qquad (3.4.8)$$

where the Green's function G is given by

$$G(x, \zeta; t) = \sum_{k=1}^{\infty} W_k(x) W_k(\zeta) . \frac{sin w_k t}{w_k}$$

In order to apply theorems 2.4.2 and 2.4.3, we then need to show that the Green's function $G(x, \zeta; t)$ is uniformly bounded. Observe that from $R2$ we get two boundary cases (hinged-free and cantilevered-beam case) as the inertia of the base I_H approaches two extreme cases (0 and ∞), it is shown in Appendix A.2 that the Green's functions are uniformly bounded in the two cases. It is possible to carry out similar analysis for the other cases, but we will not do so here.

3.4.2 A smart actuator

Piezoelectric and magnetostrictive actuators are gaining increasing attention for their potential use as positioners, motors, and vibration suppressors [54, 55]. They are anticipated to play a prominent role in high precision manufacturing devices for optical instruments such as lasers and cameras, and high precision positioning in semiconductor chip manufacturing chip etc. They are being applied already in high precision optical telescopes, cameras etc. [56, 57].

An actuator using piezoelectric and magnetostrictive materials is being designed and fabricated by R. Venkataraman [58] as a part of a project on smart structures technology with applications to rotorcraft systems at the University of Maryland. A description of the actuator is shown in Figure 2.4, where a piezoelectric stack and two magnetostrictive rods are electrically connected and controlled by a single sinusoidal voltage signal $v(t)$. During the electrical half cycle the piezoelectric stack expands and clamps onto a disk, while the magnetostrictive rod pushes the clamp. The combined motion results in the rotation of the disk. During the second half cycle the piezoelectric stack releases its hold on the disk and the magnetostrictive actuators return to their initial positions.

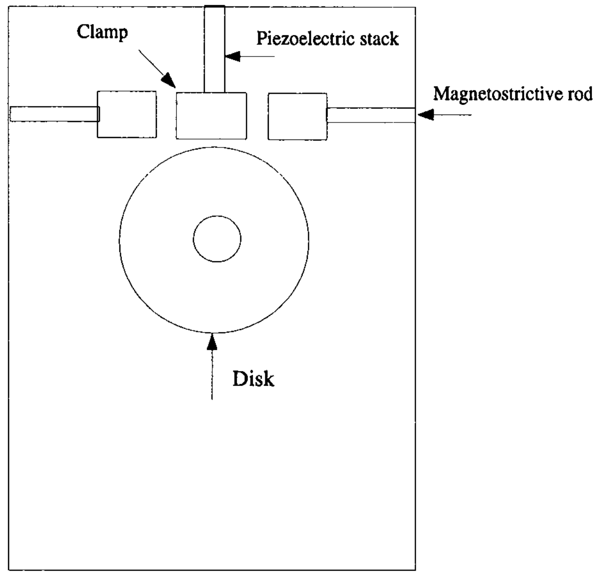

Figure 3.4: A sketch of smart actuator

The speed rotation of the disk is regulated by the frequency of the power supply. Phase angle relationships of the piezoelectric stack and magnetostrictive rods are given in Figure 2.5. During the expansion of the piezoelectric stack, it collides with the disk and an impact occurs. Proper modeling of such impact dynamics is important because:

1). The impact force may become so large that it will shatter the materials. The impact model described earlier in the chapter can be used to select materials (clamp and disk) as well as to determine a proper impact velocity.

2). The impact duration is non-zero and finite, and this fact turns out to be very important for design considerations. In simulations it has been found that the average impact duration could be as high as 50 times the time step in adaptive-step size numerical simulations which ignore this phenomena.

The disk and the clamp system is modeled as a spring-mass system as shown in Figure 3.6. We assume that the Hertz law of impact is valid,

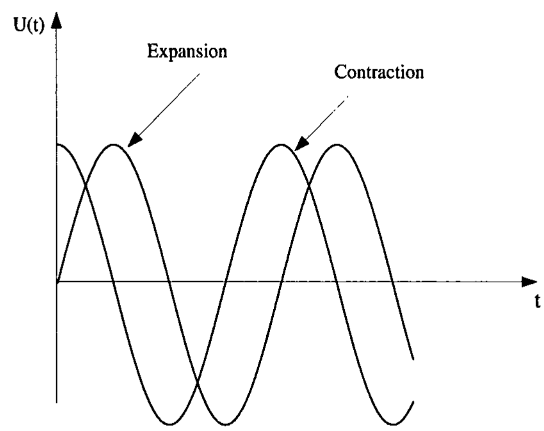

Figure 3.5: The relationship of expansion and contraction

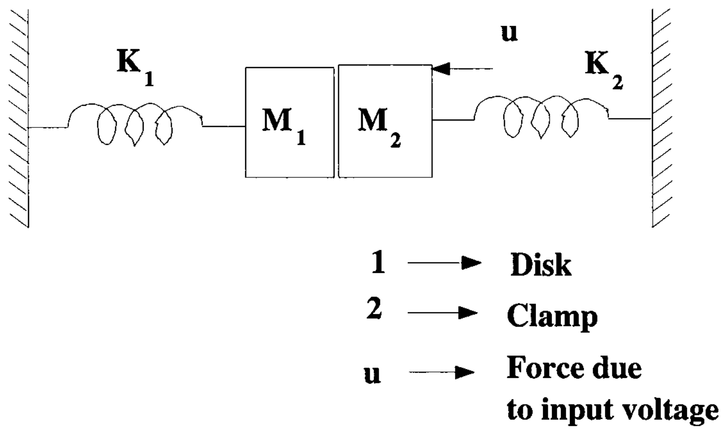

Figure 3.6: Disk and clamp system.

$$K\left[f\left(t\right)\right]^{2/3} = s\left(t\right) - w\left(t\right), \tag{3.4.9}$$

where $w\left(t\right)$ is the displacement of the disk at the point of contact, $s\left(t\right)$ is the displacement of the clamp under the contact force $f\left(t\right)$, and is given by,

$$s\left(t\right) = \frac{v_{clamp}\, sin\left(\omega_{n\,clamp} t\right)}{\omega_{n\,clamp}} + \frac{C_1 V sin\left(\omega_{n\,clamp} t\right)}{m_{clamp}\omega_{n\,clamp}}$$

$$-\frac{1}{\omega_{n\,clamp}\, m_{clamp}} \int_0^t f\left(\tau\right)\, sin\left[w\left(t-\tau\right)\right]\, d\tau \tag{3.4.10}$$

where

$$C_1 = \frac{d_e\, A_e\, N_e}{s^e\, l_e}$$

and $w\left(t\right)$ is given by

$$w(t) = \frac{v_{disk}\, sin(\omega_{disk} t)}{\omega_{disk}} + \frac{1}{\omega_{disk}\, m_{disk}} \int_0^t f(\tau)\, h[w(t-\tau)]\, d\tau \tag{3.4.11}$$

$$\omega_{disk} = \omega_{n\,disk}\sqrt{1-\xi^2} \tag{3.4.12}$$

$$h\left(t\right) = e^{-\xi \omega_{n\,disk} t}\, sin\left(\omega_{disk}\, t\right) \tag{3.4.13}$$

where

$$\omega_{n\,clamp} = \sqrt{\frac{k_{clamp}}{m_{clamp}}} \tag{3.4.14}$$

$$\omega_{n\,disk} = \sqrt{\frac{k_{disk}}{m_{disk}}} \tag{3.4.15}$$

$$\xi = \frac{damp_{disk}}{2\, m_{disk}} \tag{3.4.16}$$

and v_{clamp}, v_{disk} are the initial velocities of the clamp and the disk. Equations (3.4.9) through (3.4.11) give,

$$f(t) = \left(\frac{1}{K}\left(a(t) - \frac{1}{\omega_{n\,clamp}\, m_{clamp}} \int_0^t f(\tau)\, sin[w(t-\tau)]\, d\tau\right.\right.$$

$$\left.\left.-\frac{1}{\omega_{disk}\, m_{disk}} \int_0^t f(\tau)\, h[w(t-\tau)]\, d\tau\right)\right)^{3/2} \tag{3.4.17}$$

where,

$$a(t) = \frac{v_{clamp}\, sin\,(\omega_{n\,clamp}t)}{\omega_{n\,clamp}} + \frac{C_1 V sin\,(\omega_{n\,clamp}t)}{m_{clamp}\omega_{n\,clamp}} - \frac{v_{disk}\, sin(\omega_{disk}t)}{\omega_{disk}}. \quad (3.4.18)$$

The disk and clamp system is similar to the lumped-parameter structure we discussed in section 2.2.1. Without much difficulty, we can show that the Green's function of this system satisfies the conditions of theorem 2.4.2 and 2.4.3. Hence, the Hertz equation has a unique solution and the impact model is valid. The numerical solutions of the impact model for the actuator will be given in the next chapter. For simplicity, this dissertation has dealt solely with the case of beam. Extensions of the method to multiple space dimensions e.g. plates, are feasible.

Chapter 4

NUMERICAL AND APPROXIMATION METHODS

In chapter three we used the Hertz law of impact to derive an impact model when impact involves a flexible structure, and we also established the existence and uniqueness of solutions of the Hertz equation. Unfortunately, this impact model is nonlinear and does not admit a closed form solution. Hence, computational aspects must be considered.

Inspired by the contraction mapping theorem, we develop a numerical method using the successive Picard iterations: $f_0, Pf_0, PPf_0, \cdots\cdots$, where the initial condition f_0 is obtained from the energy method, and P is a contraction operator. Our experience is that this method is faster compared to the small-increment method, more accurate than the energy method, and its convergence is very fast.

Although the methods (small-increment method and energy method etc.) could provide the numerical solutions to the Hertz equation, these methods share some drawbacks, the main one is that parameter variations e.g. varying initial velocities, demand a whole new computation of the numerical solution. Also, a fairly large computational burden has to be incurred for each numerical solution.

We propose a method to alleviate this problem. Let us first observe from our experience that the impact period is very short in general. Therefore, one may approximate the transfer functions of the systems to which the impacting bodies belong by Taylor polynomials of low order. We explicitly carry out this computation in the cases of first and second order Taylor polynomial approximations. We show that in the case of the first order approximation, there is a universal ordinary differential equation that describes the impact behavior completely in the sense that parameter variations only require proper rescaling of functions. Therefore, one can numerically solve this equation beforehand, save the results, and can use it to predict the impact behavior with only a minimal computational burden. In the case of the second order approximation, there is a two parameters family of ordinary differential equations that govern the impact behavior.

4.1 A Numerical Method

The energy method is simple and fast, although less accurate than the small-increment method. Hence, the results obtained by the energy method can be used as a good initial approximation.

Inspired by the contraction mapping theorem, we develop a numerical method using the successive Picard iterations, $f_0, Pf_0, PPf_0, \cdots\cdots$, where the initial condition f_0 is obtained from the energy method, and P is the contraction operator defined by,

$$Pf(t) \quad = \quad [v_0' t \; - \; \frac{1}{m'} \int_0^t f(\tau) L(t \; - \; \tau) d\tau]^{3/2}, \qquad \forall t \in [0, T_0].$$

Recall that the function $L(t)$ is given by

$$L(t) \;=\; t + mG(x^*; t).$$

The successive approximations to a solution of (3.2.10) are given by

$$f_n(t) \;=\; [v_0' t \;-\; \frac{1}{m'}\int_0^t f_{n-1}(\tau)L(t \;-\; \tau)d\tau]^{3/2}, \quad n = 1, 2, \cdots \tag{4.1.1}$$

We take the initial condition f_0 to be the one given by (2.2.18). Closed form solution of the first iteration is,

$$\begin{aligned} f_1(t) &= [v_0' t \;-\; \frac{1}{m'}\int_0^t f_0(\tau)L(t \;-\; \tau)d\tau]^{3/2}, \qquad \forall t \in [0, T_0]. \\ &= [v_0' t - \frac{K_0}{m'}(\frac{t}{\lambda} - \frac{1}{\lambda^2} sin\lambda t) \\ &\quad -K_0 \sum_{k=1}^{\infty} W_k^2(x^*) \frac{\lambda sin w_k t - w_k sin\lambda t}{w_k(\lambda^2 - w_k^2)}]^{3/2}. \end{aligned} \tag{4.1.2}$$

It turns out that even this first approximation yields reasonable results in some cases (see e.g. Figure 4.4).

4.2 Numerical Examples

We consider some impact examples involving a flexible structure; a simply supported beam on which a spherical body impacts at the center. Solution of the Hertz equation depends on several parameters. The following two examples are used to show how the impact force is affected by the approach velocity v_0 and material properties, and these results give us some "feel" for the impact problem. The model parameters used in the numerical simulations are given in Table 3.1.

Remark 4.2.1 *The purpose of the first example is to address the question of how the impact force is affected by the approach velocity. We use different approach velocities while keeping the material properties fixed. Force profiles are plotted in Figure 4.1.*

m_1: mass of the impactor (steel ball)	0.2 kg
v_0: velocity of impactor	0.1 m/s
l : length of the beam	1.0 m
h: thickness of the beam	0.02 m
b: width of the beam	0.06 m

Table 4.1: Model parameters

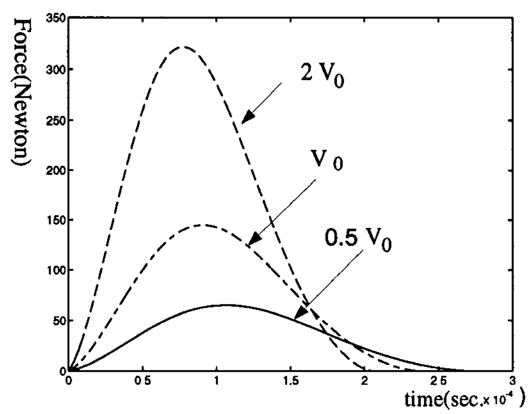

Figure 4.1: Impact forces by different velocities; material: aluminum

For a given pair of impacting objects the impact forces and impact durations are determined by the approach velocity v_0. *Smaller* $v_0's$ *result in smaller impact forces and longer impact durations, while the larger* $v_0's$ *result in greater impact forces and shorter impact durations. The approach velocity* v_0 *is usually controllable in robotics applications. Hence, it can be effectively exploited in the control design [59].*

Remark 4.2.2 *The second example illustrates how the impact force is affected by material properties. We use three different materials typically used in robotics. The approach velocity is fixed and the impact force profiles are plotted. The fol-*

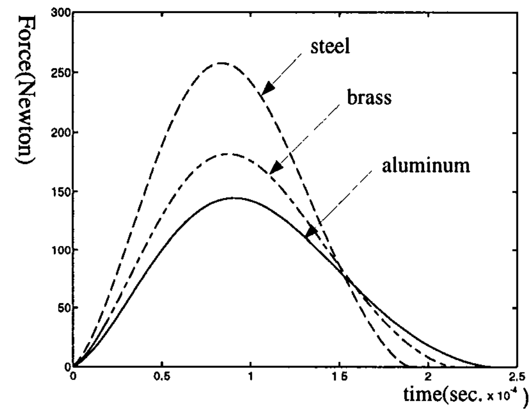

Figure 4.2: Impact forces using different materials; velocity: $v_0 = 0.1m/s$

lowing conclusions can be drawn from Figure 4.2. For a given approach velocity v_0*, the force magnitudes and impact durations are determined by material properties of both objects. The softer material (aluminum) results in smaller impact force and longer impact duration while the harder one (steel) results in greater impact force and shorter impact duration. This relationship leads to the concept of passive compliance. When the approach velocity cannot be controlled to be arbitrarily small, the passive compliance can be introduced by the use of soft contact surfaces (such as soft force sensors), thereby reducing the impact force greatly [47].*

4.2.1 A simply supported beam

We compare the three numerical methods discussed before. We consider a simply supported beam case for which parameters are given in Table 3.1. Impact occurs at the center of the beam. Figure 4.3 shows the solutions obtained from the energy method, the small-increment method and three iterations using the solution obtained from the energy method as the initial data. We note that the

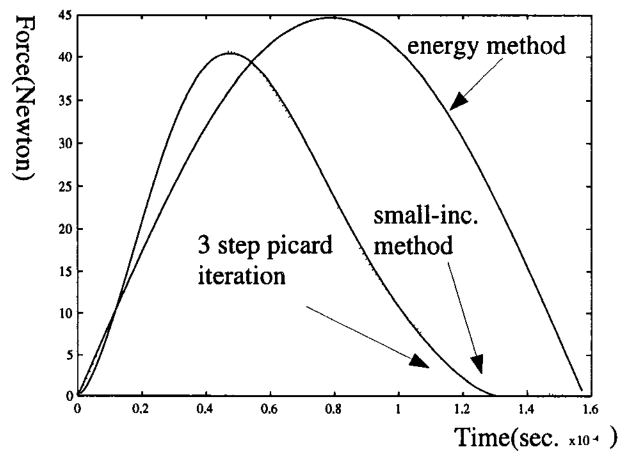

Figure 4.3: Central impact on a simply supported beam

energy method yields a large error. In order to apply the Picard iteration method to this case we need first to show that the Green's function associated with this case is uniformly bounded. Details are given in Appendix A.1. The result obtained from three iterations is very good, and the computational complexity is just 1/3 that of the small-increment method.

4.2.2 A cantilevered beam

In this example, we consider a cantilevered beam and the system parameters are the same as in Table 3.1. Impact occurs at the tip of the beam. We see that fairly large errors occurred by using the energy method. On the other hand, the Piccard iteration method gives a good closed form result even after just one iteration. Figure 4.4 also shows the fast convergence of this algorithm. Again, the Green's function for this case is uniformly bounded as proven in Appendix A.2.

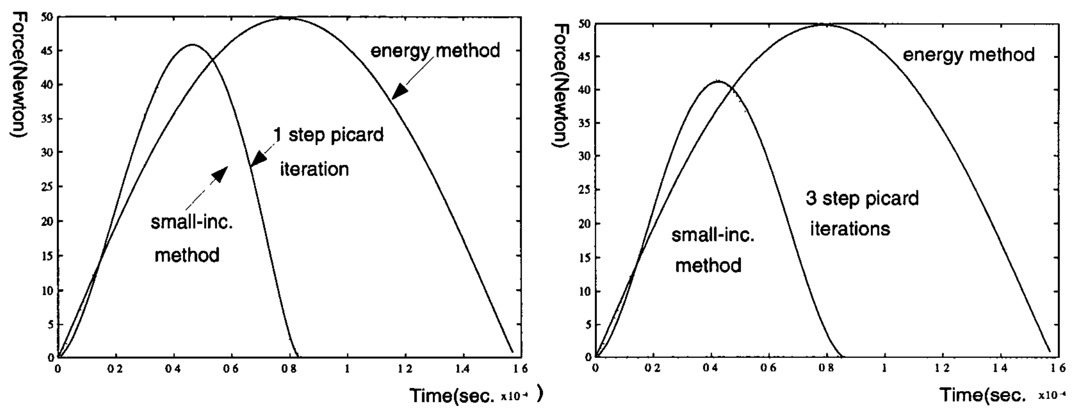

Figure 4.4: Tip impact on a cantilevered beam

	Material	*Mass kg*	*Radius m*	*Velocity m/s*
Body 1	*PhosphorBronze*	0.0114	∞	0.03
Body 2	*Steel*	1.044	0.05	−0.01

Table 4.2: Impact parameters of motor

4.2.3 A smart actuator

We now revist the example of a smart actuator discussed in chapter 2. A typical impact force profile is given in Figure 4.5, which is useful in the design procedures such as selecting materials etc. Venkataraman [58] has done extensively numerical simulations, and found the results by small-increment method and 2-steps Piccard iteration method are essentially same but the computation of the latter is just 1/4 of the former method.

4.3 Approximation Methods

Numerical data obtained from the impact model provides useful information as illustrated above. These numerical methods share some drawbacks, one of

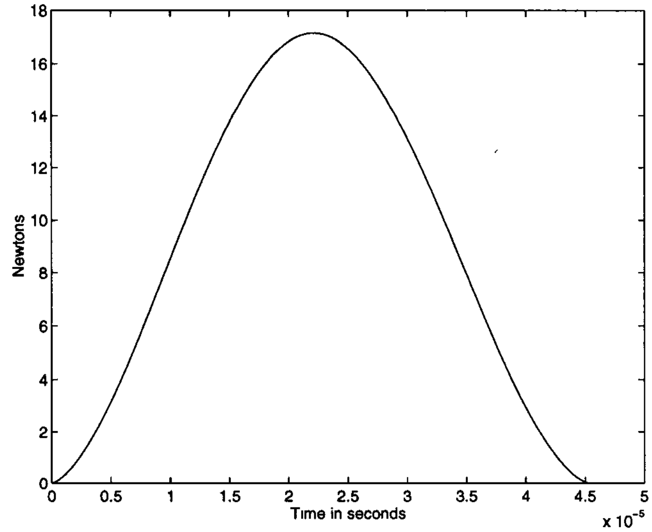

Figure 4.5: Impact force profile

which is that each situation, e.g., resulting from varying initial velocities, has to be numerically solved separately. Also, a fairly large computational burden is incurred for each numerical solution.

Our objective in this section is to show that it is possible to take advantage of the fact that the impact period is very short in general. Therefore, one may approximate the transfer functions of the systems to which the impacting bodies belong by Taylor polynomials of low order. We carry out this computation explicitly in the cases of first and second order Taylor polynomial approximations. We show that in the case of the first order approximation there is a universal ordinary differential equation that describes the impact behavior completely. Therefore, one can solve this equation numerically beforehand, save the results, and use them to predict the impact behavior with only a minimal computational burden. In the case of the second order approximation, there is a two-parameter family of ordinary differential equations that govern the impact behavior.

In order to motivate the problem, for the sake of simplicity, let us consider two flexible bodies in linear motion as shown in Figure 4.6. This could be a simplified model for a robot and a work-piece [43, 60], where m_i is the mass of the i^{th} body, c_i and k_i are the associated viscous frictional coefficient and stiffness respectively, and $u(t)$ is the control input (force). Let $f(t)$ denote the impact force between the bodies. The dynamical equations for the system during the

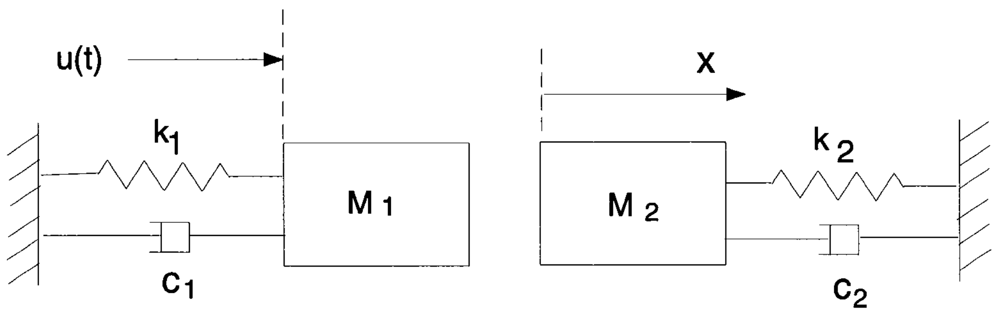

Figure 4.6: A model of robot and workpiece

impact are:

$$\begin{aligned} m_1\ddot{x}_1(t) + c_1\dot{x}_1(t) + k_1 x_1(t) &= u(t) - f(t), \\ m_2\ddot{x}_2(t) + c_2\dot{x}_2(t) + k_2 x_2(t) &= f(t). \end{aligned} \tag{4.3.1}$$

Without loss of generality, we assume that the approach velocity of the first body just before contact is $v_0 \geq 0$ and that the second body is initially at rest. The Hertz law of impact is assumed valid, i.e.

$$\alpha(t) = K[f(t)]^{2/3}. \tag{4.3.2}$$

Let us assume that the external forces acting on the bodies are negligible in comparison to the impact force. We can write the displacements of the bodies as

$$x_1(t) = h_0(t) - \int_0^t h_1(t-\tau)f(\tau)d\tau,$$

$$x_2(t) = \int_0^t h_2(t-\tau)f(\tau)d\tau. \quad (4.3.3)$$

where $h_0(t)$ is a function of initial condition and $h_0(t) = v_0 m_1 h_1(t)$. The Green's functions h_i are given by

$$h_i(t) = \frac{\omega_i}{k_i\sqrt{1-\zeta_i^2}} e^{-\zeta_i\omega_i t} sin(\omega_i\sqrt{1-\zeta_i^2}\ t), \quad (4.3.4)$$

where $i = 1, 2$ and ω_i and ζ_i are defined by

$$\omega_i^2 = \frac{k_i}{m_i}, \qquad \frac{c_i}{m_i} = 2\zeta_i\omega_i.$$

Hence, the relative approach can be written as

$$\begin{aligned} \alpha(t) &= h_0(t) - \int_0^t h_1(t-\tau)f(\tau)d\tau - \int_0^t h_2(t-\tau)f(\tau)d\tau, \\ &= h_0(t) - \int_0^t \bar{h}(t-\tau)f(\tau)d\tau, \end{aligned} \quad (4.3.5)$$

where $\bar{h}(t) := h_1(t) + h_2(t)$.

Equations (4.3.2) - (4.3.5) lead to,

$$K[f(t)]^{2/3} = h_0(t) - \int_0^t \bar{h}(t-\tau)f(\tau)d\tau \quad (4.3.6)$$

If the Green's function $h(t)$ belongs to $C^\infty[0,T]$ (this condition is satisfied in many applications), then $h(t)$ can be expanded in the form of a Taylor series:

$$h(t) = k_1 t + \frac{1}{2!}k_2 t^2 + \frac{1}{3!}k_3 t^3 + \cdots\cdots, \quad (4.3.7)$$

where the coefficients k_i are determined by $h(t)$ and are finite. Since the impact duration is short, one can frequently approximate $h(t)$ by its first or second order Taylor polynomials very well.

4.3.1 First-order approximation

The solution of nonlinear equation (4.3.6) can be greatly simplified if we use the first-order approximation. Somewhat surprisingly, equation(4.3.6) renders a universal ordinary differential equation (UODE) in this case. The first-order approximations of functions $h_0(t)$ and $\bar{h}(t)$ are $v_0 t$ and $k_1 t$ respectively. By plugging these approximations into equation (4.3.6) we obtain

$$K[f(t)]^{2/3} = v_0 t - \int_0^t k_1(t-\tau) f(\tau) d\tau. \tag{4.3.8}$$

Because the relative approach $\alpha(t) = K f^{2/3}(t)$, replacing $f(t)$ by $\alpha(t)$ in equation (4.3.8) gives

$$\alpha(t) = v_0 t - \int_0^t k_1 (\frac{1}{K})^{3/2} (t-\tau) \alpha^{3/2}(\tau) d\tau. \tag{4.3.9}$$

Differentiating equation (4.3.9) twice leads to,

$$\frac{d^2\alpha(t)}{dt^2} = -k_1 (\frac{1}{K})^{3/2} \alpha^{3/2}(t). \tag{4.3.10}$$

By our assumptions, the initial conditions are $\alpha(0) = 0$, and $\dot{\alpha}(0) = v_0$. Equation (4.3.10) can be integrated once to obtain

$$(\frac{d\alpha(t)}{dt})^2 - v_0^2 = -\frac{4}{5} k_1 (\frac{1}{K})^{3/2} \alpha^{5/2}(t). \tag{4.3.11}$$

From equation (4.3.11), the maximum relative approach, α_{max}, can be immediately obtained by letting $\dot{\alpha}(t) = 0$, and the maximum impact force is easily obtained from (4.3.2):

$$\begin{aligned} \alpha_{max} &= [\frac{5}{4} \frac{K^{3/2} v_0^2}{k_1}]^{2/5}, \\ f_{max} &= [\frac{5}{4} \frac{v_0^2}{k_1 K}]^{3/5}. \end{aligned} \tag{4.3.12}$$

Next, we show that equation (4.3.11) can be put into a universal form by scaling both the time and magnitude of $\alpha(t)$. Let us introduce two new variables β and τ by

$$\beta(\tau) = \lambda\alpha(t), \qquad t = \mu\tau, \tag{4.3.13}$$

where β and τ will play the roles of the dependent and the independent variables, λ and μ are two constants to be determined shortly. It is easy to show that $\frac{d\beta(\tau)}{d\tau} = \lambda\mu\frac{d\alpha(t)}{dt}$. Replacing α and t by the expressions in (4.3.13) we obtain from (4.3.11)

$$(\frac{d\beta(\tau)}{d\tau})^2 - \lambda^2\mu^2 v_0^2 = -\frac{4}{5}\mu^2 k_1(\frac{1}{K})^{3/2}\frac{1}{\sqrt{\lambda}}\beta(\tau)^{5/2}. \tag{4.3.14}$$

Let us now choose λ and μ such that

$$\lambda^2\mu^2 v_0^2 = 1, \qquad \frac{4}{5}\mu^2 k_1(\frac{1}{K})^{3/2}/\sqrt{\lambda} = 1. \tag{4.3.15}$$

Equation (4.3.14) now takes the universal form

$$\frac{d\beta(\tau)}{d\tau} = \sqrt{1 - \beta(\tau)^{5/2}}, \tag{4.3.16}$$

subject to the initial condition $\beta(0) = 0$. This completes the construction. □

The universal ordinary differential equation (4.3.16) has a unique solution with the given initial condition, and can be solved numerically. The result is plotted in Figure 4.7. Once the solution of this equation is obtained, any impact problem can be solved immediately by using the following relation. From equation (4.3.15), we can solve for λ and μ respectively:

$$\lambda = [(\frac{4}{5})^2\frac{k_1^2}{v_0^4 K^3}]^{1/5}, \qquad \mu = [(\frac{5}{4})^2\frac{K^3}{v_0 k_1^2}]^{1/5}. \tag{4.3.17}$$

Let τ_{max} be the time before the solution of (4.3.16) crosses the time axis once again. Then the impact duration T for the original problem is

$$T = \mu\tau_{max} = [(\frac{5}{4})^2 \frac{K^3}{v_0 k_1^2}]^{1/5} \tau_{max}. \tag{4.3.18}$$

The impact force can be obtained from

$$f(t) = (\frac{1}{K\lambda})^{3/2} \beta(t/\mu)^{3/2} = (\frac{5}{4}\frac{v_0^2}{k_1 K})^{3/5} \beta(t/\mu)^{3/2}. \tag{4.3.19}$$

By scaling back the time from τ to t, we can recover the actual force $f(t)$. Let

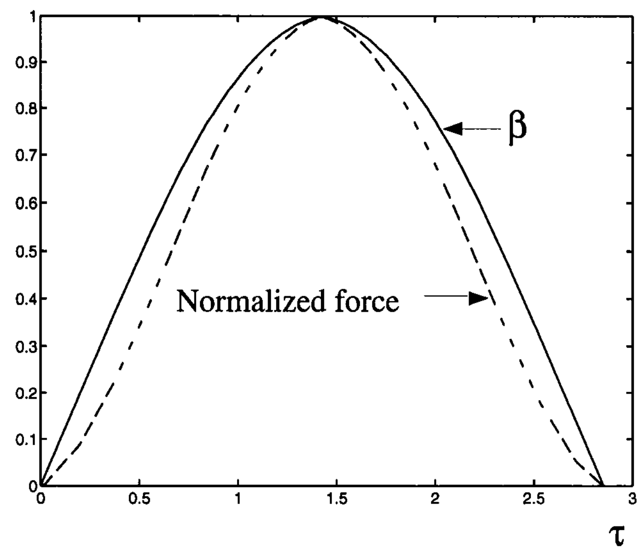

Figure 4.7: Numerical solution of UODE

us apply this method to the system shown in Figure 4.6 for which the model and impact parameters are given in Table 3.2.

Case A: The two impacting bodies are assumed to be made of steel. Figure 4.8 shows that the approximate solution is practically indistinguishable from the exact solution (numerically computed with the small increment method).

Remark 4.3.1 *The first-order approximation has an interesting physical interpretation. Note that* $k_1 = (1/m_1 + 1/m_2)$. *This means that we treat the two*

m_1:	2.0 kg
c_1 :	20.0 $N.s/m$
k_1:	10^4 N/m
m_2:	$0.5kg$
c_2:	15 $N.s/m$
k_2:	10^4 N/m
v_0:	0.1 m/s

Table 4.3: Model parameters of flexible bodies

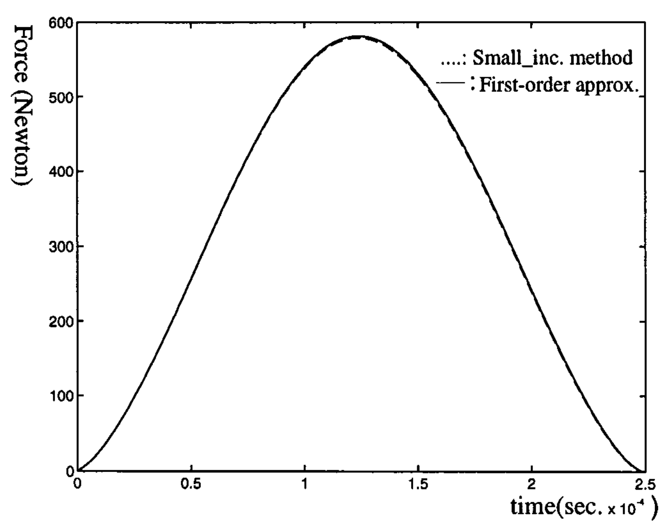

Figure 4.8: Solutions by 1_{st} order approximation and small-incre. methods: Case A

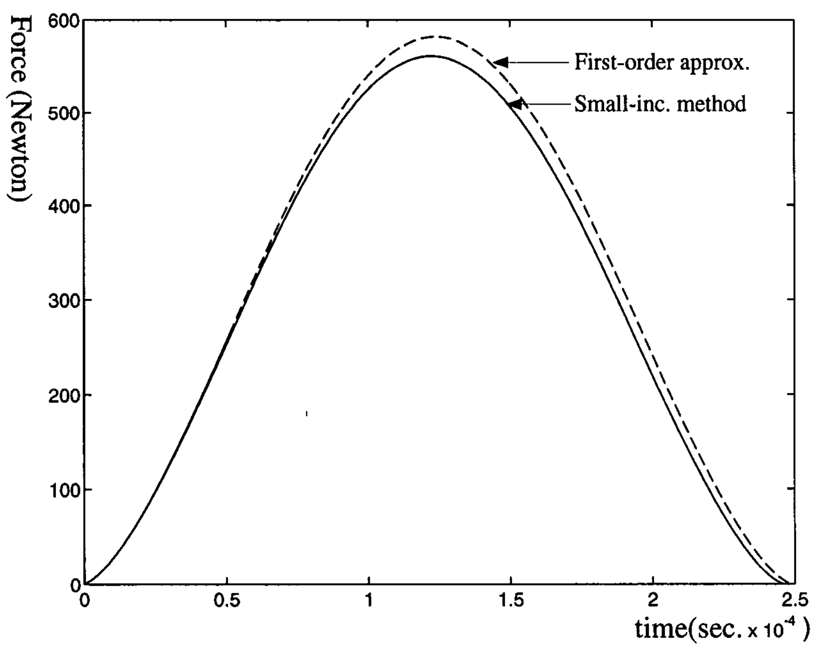

Figure 4.9: Solutions by 1_{st} order approximation and small-incre. methods: Case B

flexible bodies as if they were two single rigid bodies in free space. When the impact duration is relatively large (low-speed impact, softer materials contacting etc.), the effects of other factors such as damping cannot be ignored. The first-order approximation will introduce some errors.

Case B: For the purpose of comparison, we increased the damping coefficients c_i by a factor of 10. Results are shown in Figure 4.9. The solution obtained from the first-order approximation now begins to deviate from the exact solution. This motivates us to consider the development of a second order approximation.

4.3.2 Second-order approximation

The second-order approximation of the function $\bar{h}(t)$ is $k_1 t + \frac{1}{2}k_2 t^2$, where $k_1 \neq 0$, $k_2 \neq 0$, and the approximation of $h_0(t)$ is $v_0 t + \frac{1}{2}k_0 t^2$. We substitute these approximations into the nonlinear equation (4.3.6):

$$K[f(t)]^{2/3} = v_0 t + \frac{1}{2}k_0 t^2 - \int_0^t [k_1(t-\tau) + \frac{1}{2}k_2(t-\tau)^2] f(\tau) d\tau. \quad (4.3.20)$$

Using the relation (4.3.2), replace $f(\cdot)$ by $\alpha(\cdot)$,

$$\alpha(t) = v_0 t + \frac{1}{2}k_0 t^2 - \int_0^t (\frac{1}{K})^{3/2} [k_1(t-\tau) + \frac{1}{2}k_2(t-\tau)^2] \alpha^{3/2}(\tau) d\tau.$$

Hence we obtain

$$\frac{d^3\alpha(t)}{dt^3} = -\frac{3}{2}k_1' \sqrt{\alpha(t)} \frac{d\alpha(t)}{dt} - k_2' \alpha(t)^{3/2}, \quad (4.3.21)$$

where, $k_1' = k_1(\frac{1}{K})^{3/2}$, *and* $k_2' = k_2(\frac{1}{K})^{3/2}$.

Let us introduce two new variables:

$$\beta(\tau) = \lambda\alpha(t), \qquad t = \mu\tau. \quad (4.3.22)$$

From equation (4.3.21) we obtain,

$$\begin{aligned}
\frac{d^3\beta(\tau)}{d\tau^3} &= -\mu^3\lambda[\frac{3}{2}k_1'\sqrt{\beta(\tau)/\lambda}\frac{1}{\lambda\mu}\frac{d\beta(\tau)}{d\tau} + k_2'(\beta(\tau)/\lambda)^{3/2}] \\
&= -\frac{3}{2}k_1'\frac{\mu^2}{\sqrt{\lambda}}\sqrt{\beta(\tau)}\frac{d\beta(\tau)}{d\tau} - k_2'\frac{\mu^3}{\sqrt{\lambda}}\beta(\tau)^{3/2}. \qquad (4.3.23)
\end{aligned}$$

Let us first assume that $k_2 > 0$, and choose the parameters λ and μ such that

$$\frac{3}{2}k_1'\frac{\mu^2}{\sqrt{\lambda}} = 1; \qquad k_2'\frac{\mu^3}{\sqrt{\lambda}} = 1. \qquad (4.3.24)$$

Equation (4.3.23) now becomes a universal differential equation

$$\frac{d^3\beta(\tau)}{d\tau^3} = -\sqrt{\beta(\tau)}\frac{d\beta(\tau)}{d\tau} - \beta(\tau)^{3/2}, \qquad (4.3.25)$$

with the initial conditions given by

$$\beta(0) = 0; \qquad \frac{d\beta(\tau)}{d\tau}|_{\tau=0} = \lambda\mu v_0 = \gamma_1; \qquad \frac{d^2\beta(\tau)}{d\tau^2}|_{\tau=0} = \lambda\mu^2 k_0 = \gamma_2. \, (4.3.26)$$

Hence, we have a universal ordinary differential equation (4.3.25), and a two-parameter family of initial data, parameterized by γ_1 and γ_2.

Equation (4.3.24) can be solved for μ and λ respectively:

$$\begin{aligned}
\mu = \frac{3}{2}\frac{k_1'}{k_2'} = \frac{3}{2}\frac{k_1}{k_2}; &\qquad \lambda = (\frac{3}{2})^3\frac{k_1'^4}{k_2'^2}. \\
\gamma_1 = (\frac{3}{2})^4\frac{k_1'^5}{k_2'^3}v_0; &\qquad \gamma_2 = (\frac{3}{2})^5\frac{k_1'^6}{k_2'^4}k_0. \qquad (4.3.27)
\end{aligned}$$

Remark 4.3.2 *When $k_2 < 0$, let $k_2'\frac{\mu^3}{\sqrt{\lambda}} = -1$. By the same procedure, we obtain the universal differential equation*

$$\frac{d^3\beta(\tau)}{d\tau^3} = -\sqrt{\beta(\tau)}\frac{d\beta(\tau)}{d\tau} + \beta(\tau)^{3/2}, \qquad (4.3.28)$$

with the same initial conditions (4.3.26), where the parameters are,

$$\begin{aligned}
\mu = -\frac{3}{2}\frac{k_1'}{k_2'} = -\frac{3}{2}\frac{k_1}{k_2}; &\qquad \lambda = (\frac{3}{2})^3\frac{k_1'^4}{k_2'^2}, \\
\gamma_1 = -(\frac{3}{2})^4\frac{k_1'^5}{k_2'^3}v_0, &\qquad \gamma_2 = -(\frac{3}{2})^4\frac{k_1'^6}{k_2'^4}k_0. \qquad (4.3.29)
\end{aligned}$$

Once again, the universal differential equation (4.3.25) with two-parameter γ_1 and γ_2 can be solved numerically. A table can be created to store the results with different γ_1 and γ_2, and any impact problem can be immediately solved by using data from the table. Obviously, higher-order approximations may increase the accuracy, but only at the expense of computational load. Let us apply the second-order approximation to the above examples.

Case C: For the purpose of comparison, we increased the damping coefficients c_i by a factor of 50 . It is clear that $k_2 = -(c_1/m_1^2 + c_2/m_2^2) < 0$; For the given model parameters, a table is generated with two variables γ_1 and γ_2. It is seen in Figure 4.10 that the result obtained by using the second order approximation agrees with true results to a high degree of accuracy.

In summary, an approximation method has been developed for the analysis of

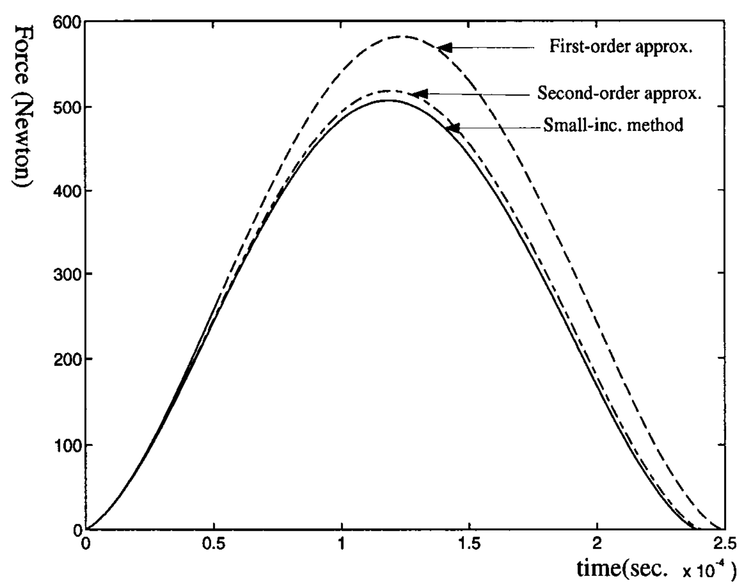

Figure 4.10: Solutions by 1_{st} and 2_{nd} order approximation and small-incre. methods

dynamics of flexible bodies undergoing impact. The first-order approximation yields a special function which can be used for analytical and computational purposes. This approximation seems to give results which are very close to the exact solutions. A second order approximation was developed as well, and it leads to a two parameter family of ordinary differential equations of which the solutions contain universal features of impact problems.

Chapter 5

NONLINEAR OPTIMAL CONTROL OF IMPACT FORCES

In this chapter, we study some control issues of impact dynamics. We will study a control problem where a linear system is subjected to a series of impact forces. The impact forces are treated as disturbances to the system and modeled as finite duration events using the theory developed in chapters 3 and 4. Among the motivating factors is the need to study the control problems related to mechanical systems subject to impact forces, e.g., active control of the suspension system of a vehicle against irregularities of the road, stabilization of an antenna on the space station subject to impact from space debris, or active damping of vibrations of flexible structures caused by impact forces [10, 17]. A reasonable control objective in all these problems is to design a stabilizing controller to minimize the effects from the disturbances to the controlled outputs in some sense. This problem in turn can be studied in the framework of H_∞ control theory.

An important paradigm in control synthesis is the H_∞ control problem introduced by Zames [18]. In this formulation the disturbances belong to a ball in a certain function space, and a quadratic cost function is minimized for the worst disturbance in this set. Various theories of H_∞ control problems for linear systems have been developed so far by many researchers [19, 20, 21, 22, 23]. Solutions to these problems can be obtained via frequency domain methods, or recently developed state-space methods.

Recently, several researchers have attempted to extend the H_∞ control theory to the case of nonlinear systems. Ball and Helton [25, 26], from a viewpoint of operator theory, discussed H_∞ control theory of nonlinear systems for the first time. Basar and Isidori [27, 28] have connected the H_∞ control theory of nonlinear systems with dynamic game theory. In this setting, one is naturally led to a nonlinear partial differential equation known as the *Isaacs* equation. A straightforward application of the theory (see e.g., [29]) to the case of a plant modeled by an affine nonlinear system shows that once a solution of the appropriate *Isaacs* equation is found, a (full-information) feedback law providing disturbance attenuation (in the sense of the L_2 gain) can be computed right away. Van der Schaft [30, 31] analyzed the relation of the L_2 gain between nonlinear systems and their linearization, and gave a sufficient condition for the existence of smooth H_∞ state feedback. In addition, using the dissipative system theory [32, 33], Van der Schaft has discussed the relation between the L_2 gain and the Hamilton-Jacobi equation in [34].

The disturbance attenuation of impact forces is a nonlinear control problem due to the nonlinearity of the impact model, which is different from the problems discussed in [36, 34, 35]. It is not affine in the disturbance input, and the linearization around the equilibrium does not exist. Hence this nonlinear problem cannot be solved by the above proposed methods, but some analysis can be carried over to this problem. Our approach is based on dynamic game theory [27, 29]. In this setting, the control problem naturally becomes a minimax optimization problem of a cost function $L(u, v)$ with two players, where u is the control and v is the disturbance. Roughly speaking, the player u tries to minimize $L(u, v)$ over U, while the player v tries to maximize $L(u, v)$ in V simultaneously, where U, V are the constraint sets of u and v respectively. [27, 29] show that the (sub)optimal disturbance attenuation problem has a solution for a given $\gamma > 0$ if the minimax optimization problem admits a saddle point. We show that, due to the nature of impact dynamics, the saddle point may not exist in this control problem. Motivated by the dynamic game approach, if the information of the disturbance v is assumed to be known a priori (e.g., sensors can predict the impact velocity before impact), one can seek an optimal strategy by using this information such that a certain attenuation level γ is achieved. A procedure is given to compute the optimal strategy $u(v)$, and the optimal attenuation level γ^*. Finally, by taking advantage of the fact that the duration of each impact force is usually very short, we derive a series of approximation models for the original nonlinear system. We show that the higher order terms are negligible, hence a special linearization of the nonlinear system can be obtained by using the leading term as an approximation. Thus the nonlinear problem can be approximated by a linear one. The complete analysis and the solution of this linear control problem are discussed in detail in the next chapter.

5.1 Dynamic equation

Let l_2 denote the Hilbert space of square-summable complex-valued sequences $\{x_k, k \geq 0\}$ with the norm defined by

$$||x||_{l_2} := (\sum_{k=0}^{\infty} |x_k|^2)^{1/2}. \tag{5.1.1}$$

Let L_2 denotes the Hilbert space of all complex-valued Lebesque measurable functions $x(t)$ defined on $[0, \infty)$ with the property that $|x|^2$ is Lebesque integrable. The norm is defined by

$$||x||_{L_2} := (\int_0^{\infty} ||x(t)||^2 dt)^{1/2}. \tag{5.1.2}$$

For simplicity, we consider a single-input and single-output (SISO) linear system subject to a series of impact forces, occurring at constant time intervals $kT_0, k = 0, 1, 2, \cdots\cdots, T_0 > 0$. The analysis and theories developed here can be easily extended to MIMO systems and some nonlinear systems. We further assume that a state-space form of the SISO system is given by

$$\dot{x}(t) = Ax(t) + B_1 u(t) + B_2 \sum_{k=0}^{N} f(v_k, t - kT_0) \tag{5.1.3}$$

where $x \in R^n$ are state variables, A, B_1, B_2 are constant matrices with appropriate dimensions, $N \geq 0$ is a finite integer, $and f(v_k, t - kT_0)$ is the impact force generated by the k^{th} impact, which is parameterized by the variable v_k. We assume that $|v_k| \leq M, k = 1, 2, \cdots\cdots N$ for some finite number $M > 0$. The function $f(v_k, t - kT_0)$ is a causal nonlinear function derived from the Hertz law of impact, which is given by

$$\begin{aligned} K[f(v_k, t - kT_0)]^{2/3} &= v_k(t - kT_0) - \int_{kT_0}^{t} G(t - \tau) f(v_k, \tau), \quad t \geq kT_0 \\ f(v_k, t - kT_0) &= 0, \quad \forall t < kT_0. \end{aligned} \tag{5.1.4}$$

where $G(t)$ is the Green's function of the system and is assumed to be known. Since there is no closed-form solution to the nonlinear model (5.1.4) in general, and various numerical methods to solve this nonlinear equation are developed in chapter 3. It is easy to show that if $v \in l_2$, then impact forces $\sum_{k=0}^{N} f(v_k, t - kT_0) \in L_2$.

5.2 Formulation of the control problem

In this subsection, we formulate a nonlinear optimal control problem by specifying a control objective and stating the general assumptions.

5.2.1 The L_2 gain of a nonlinear system

Let us consider the system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 u(t) + B_2 \sum_{k=0}^{N} f(v_k, t - kT_0) \\ z(t) &= Cx(t) + Du(t) \end{aligned} \tag{5.2.1}$$

where $z(t)$ is the controlled output vector, and C, D are constant matrices of appropriate dimension. If we treat the impact forces as disturbances to the system (5.2.1), a nonlinear H_∞ control problem for disturbance attenuation can be formulated. A word concerning terminology is certainly in order here, because the H_∞ norm is defined as a norm on transfer matrices and hence does not directly generalize to nonlinear systems. However, when translated to the time-domain, the H_∞ norm is none other than the L_2-induced operator norm from the input-output viewpoint. This latter norm is extended to the nonlinear setting by specifying that the gain should not exceed a prespecified limit.

Definition 5.2.1 *Let $\gamma > 0$. System (5.2.1) is said to have L_2 gain less than or equal to γ if*

$$\int_0^T ||z(t)||^2 \leq \gamma^2 \sum_{k=0}^{N} v_k^2 \tag{5.2.2}$$

for all $T \geq 0$ and all $\sum_{k=0}^{N} v_k^2 < \infty$, subject to the initial condition $x(0) = 0$.

Now let $\gamma > 0$ be a fixed number. Then in the standard nonlinear H_∞ (sub) optimal control problem (for disturbance attenuation level γ) one seeks a feedback controller u such that the closed-loop system has L_2 gain less than or equal to γ, i.e., such that inequality (5.2.2) holds. In a dynamic game setting, the cost function is defined as

$$L_\gamma(u, v) = \int_0^T ||z(t)||^2 - \gamma^2 \sum_{k=0}^{N} v_k^2. \tag{5.2.3}$$

The requirement that the L_2 gain does not exceed to γ is equivalent to existence of a saddle point (u^*, v^*) for the dynamic game

$$\begin{aligned} L_\gamma(u^*, v) &\leq L_\gamma(u^*, v^*) \leq L_\gamma(u, v^*), \\ \textit{and} \quad L_\gamma(u^*, v^*) &\leq 0. \end{aligned} \tag{5.2.4}$$

5.3 Main results

Before we study the nonlinear control problem, it is necessary to make some simplifications. Since the impact force $f(v_k, t - kT_0), k = 0, 1, 2, \cdots\cdots, N$ has no closed-form expression in general, it is impossible to analyze and solve the control problem (5.2.1)-(5.2.2) explicitly.

5.3.1 An approximation method

In chapter 4, we developed first-order and second-order approximations for impact force functions. The first-order approximation is governed by a universal differential equation and it essentially captures all features of impact dynamics. We use the first-order approximation to replace $f(v_k, t - kT_0)$. To be explicit, let us take $k = 0$. Now $f(v_0, t)$ is given by the equation

$$f(v_0, t) = (1/K)^{3/2}\alpha^{3/2}(v_0, t) \tag{5.3.1}$$

where $K > 0$ is a constant which depends on the local geometry of the region of contact and the material properties of the contacting bodies. In general $K << 1$, and

$$\alpha(v_0, t) = \frac{1}{\lambda}\beta(t/\tau), \tag{5.3.2}$$

where λ and μ are given by

$$\lambda = [(\frac{4}{5})^2\frac{k_1^2}{v_0^4K^3}]^{1/5}, \qquad \mu = [(\frac{5}{4})^2\frac{K^3}{v_0k_1^2}]^{1/5}. \tag{5.3.3}$$

In (5.3.3), k_1 is a constant and $\beta(\tau)$ is governed by the universal differential equation

$$\frac{d\beta(\tau)}{d\tau} = \sqrt{1 - \beta(\tau)^{5/2}}, \quad \beta(0) = 0. \tag{5.3.4}$$

From equations (5.3.2)-(5.3.3), we have

$$\alpha(v_0, t) = \frac{1}{\lambda}\beta(t/\mu) = \frac{1}{b}v_0^{4/5}\beta(bv_0^{1/5}t), \tag{5.3.5}$$

where the constant b is given by

$$b = [(\frac{4}{5})^2\frac{k_1^2}{K^3}]^{1/5}. \tag{5.3.6}$$

Since $K << 1$, it follows that $b >> 1$. Here the constant b can be thought of as an indicator of the time scale of impact dynamics. By plugging $\alpha(v_0, t)$ into equation (5.3.1), we finally arrive at

$$f(v_0, t) = (1/K)^{3/2}\alpha^{3/2}(v_0, t) = av_0^{6/5}r(bv_0^{1/5}t) \qquad (5.3.7)$$

where $r(\cdot) = \beta^{3/2}(\cdot)$ is called the normalized force. Numerical solutions of $\beta(\tau)$ and $r(\tau)$ are given in Figure 5.1. By using the first-order approximation, the

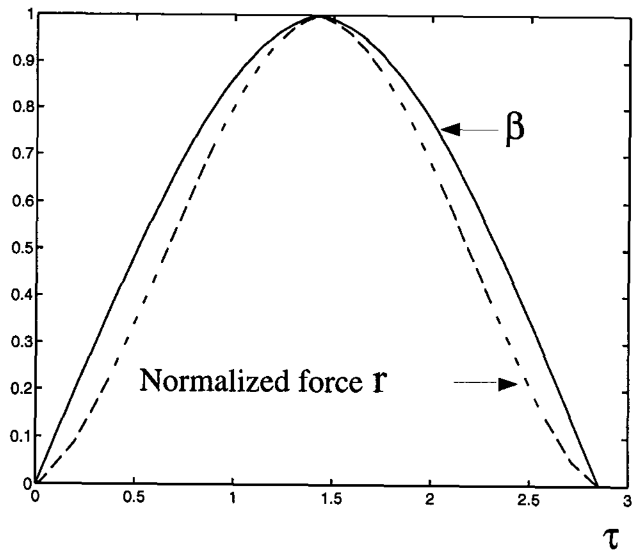

Figure 5.1: Numerical solutions of $\beta(\tau)$ and $r(\tau)$

nonlinear system (5.2.1) becomes

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1u(t) + B_2\sum_{k=0}^{N} av_k^{6/5}r(bv_k^{1/5}(t - kT_0)) \\ z(t) &= Cx(t) + Du(t). \end{aligned} \qquad (5.3.8)$$

The equation (5.3.8) has a simpler form than (5.2.1), but it is still too complicated to analyze. Next, we show that it is possible to take advantage of the fact that the impact period is very short in general. Therefore, one may further approximate the nonlinear system (5.3.8) by Taylor polynomials of low order. In order to motivate our approach, let us consider a simple example. For a scalar linear system subjected to a single impact with velocity v_0, we obtain

$$\dot{x}(t) = a_1 x(t) + b_1 v_0^{6/5} r(b v_0^{1/5} t), \quad x(0) = 0. \tag{5.3.9}$$

where a_1, b_1 are constants. $x(t)$ can be solved explicitly to obtain,

$$x(t) = e^{a_1 t} \int_0^t e^{-a_1 \tau} b_1 v_0^{6/5} r(b v_0^{1/5} \tau) d\tau, \quad t \geq 0. \tag{5.3.10}$$

Let us perform time-scaling by defining

$$b v_0^{1/5} \tau = \sigma. \tag{5.3.11}$$

Then $x(t)$ in (5.3.10) becomes

$$\begin{aligned} x(t) &= e^{a_1 t} \int_0^{b v_0^{1/5} t} e^{-\frac{a_1}{b v_0^{1/5}} \sigma} b_1 v_0^{6/5} \frac{1}{b v_0^{1/5}} r(\sigma) d\sigma \\ &= e^{a_1 t} \frac{b_1}{b} v_0 \int_0^{b v_0^{1/5} t} e^{-\frac{a_1}{b v_0^{1/5}} \sigma} r(\sigma) d\sigma. \end{aligned} \tag{5.3.12}$$

Expanding $e^{-\frac{a_1}{b v_0^{1/5}} \sigma}$ in the form of a Taylor series, we have

$$e^{-\frac{a_1}{b v_0^{1/5}} \sigma} = 1 - \frac{a_1}{b v_0^{1/5}} \sigma + \frac{1}{2!} \frac{a_1^2}{(b v_0^{1/5})^2} \sigma^2 + \cdots\cdots, \tag{5.3.13}$$

where a_1 is the time constant for the system (5.3.9). A series of approximate models is given as

$$\begin{aligned} x(t) &= e^{a_1 t} \frac{b_1}{b} v_0 \int_0^{b v_0^{1/5} t} r(\sigma) [1 - \frac{a_1}{b v_0^{1/5}} \sigma + \cdots\cdots] d\sigma \\ &= e^{a_1 t} \bar{b}_1 v_0 \theta_1 (b v_0^{1/5} t) - e^{a_1 t} \bar{b}_2 v_0^{4/5} \theta_2 (b v_0^{1/5} t) + \cdots\cdots, \end{aligned} \tag{5.3.14}$$

where $\theta_1(t)$ and $\theta_2(t)$ are given by

$$\theta_1(t) = \int_0^t r(\sigma) d\sigma, \quad \theta_2(t) = \int_0^t \sigma r(\sigma) d\sigma \tag{5.3.15}$$

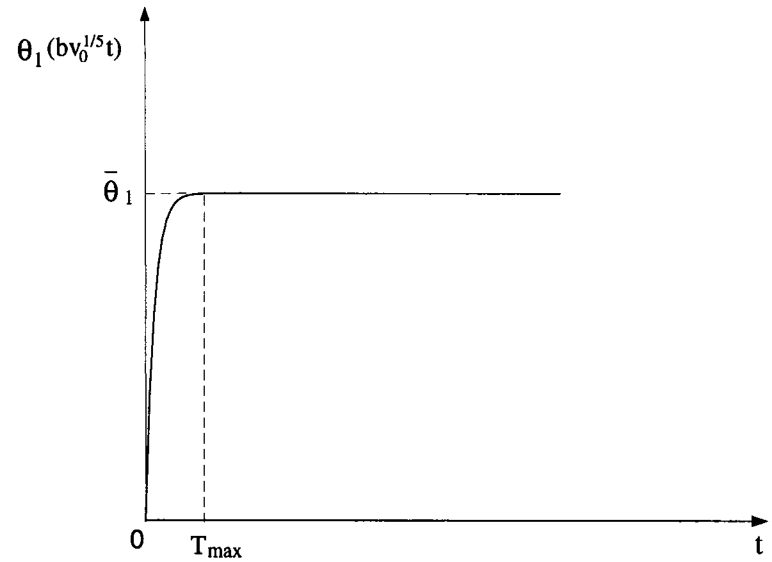

Figure 5.2: Plot of nonlinear function $\theta_1(t)$

Since impact dynamics is much faster than system dynamics in general, we have $\frac{a_1}{b} << 1$. Thus we can often approximate $x(t)$ by low order terms very well. Let τ_{max} be the time before the solution of equation (5.3.4) crosses the time axis once again (see Figure 5.1) and $T_{max} = \tau_{max}/(bv_0^{1/5})$, define the constants $\bar{\theta}_1$ $\bar{\theta}_2$ as

$$\bar{\theta}_1 = \int_0^{\tau_{max}} r(\sigma)d\sigma, \quad \bar{\theta}_2 = \int_0^{\tau_{max}} \sigma r(\sigma)d\sigma \tag{5.3.16}$$

A plot of $\theta_1(bv_0^{1/5}t)$ is given in Figure 5.2.

Remark 5.3.1 *Since* $0 \leq T_{max} << 1$, *the slope of* $\theta_1(bv_0^{1/5}t)$ *is very sharp. Notice that* $v_0\theta_1(bv_0^{1/5}t) \equiv 0, \forall t \geq 0$ *if* $v_0 = 0$. *We can approximate the function* $v_0\theta_1(bv_0^{1/5}t)$ *by* $v_0\bar{\theta}_1$ *very well.*

5.3.2 A nonlinear control problem

We now consider the following infinite horizon nonlinear problem with zero initial condition:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 u(t) + B_2 \sum_{k=0}^{N} a v_k^{6/5} r(b v_k^{1/5}(t - kT_0)) \\ z(t) &= Cx(t) + Du(t), \end{aligned} \tag{5.3.17}$$

where the cost function

$$J_\gamma(u, v) = \int_0^\infty ||z||_2^2 dt - \gamma^2 ||v||_{l_2}^2 \tag{5.3.18}$$

for a given $\gamma > 0$. For simplicity, let $w(t) = \sum_{k=0}^{N} a v_k^{6/5} r(b v_k^{1/5}(t - kT_0))$.

In view of the definition of L_2 gain, for a given $\gamma > 0$, we want to find a feedback control u such that $J_\gamma(u, v) \leq 0, \ \forall v \in V$. It is easy to show [27, 29] that this condition is equivalent to the fact that a dynamic game admits the following saddle point,

$$\begin{aligned} L_\gamma(u^*, v) &\leq L_\gamma(u^*, v^*) \leq L_\gamma(u, v^*), \\ \textit{and} \quad L_\gamma(u^*, v^*) &\leq 0. \end{aligned} \tag{5.3.19}$$

In this control problem, the saddle point (u^*, v^*) of the dynamic game may not exist due to the nonlinearity of impact dynamics, hence it cannot be solved by the standard approaches [27, 29]. Motivated by the condition (5.3.19), we consider a related optimal control problem.

Impact control problem:

Let us assume that the disturbance v is known a priori. We want to use this information to design an optimal strategy $u(v)$ such that a certain disturbance attenuation level $\gamma > 0$ can be achieved, i.e., the following inequality holds for all $v \in V$,

$$L_\gamma(u(v), v) \leq 0. \tag{5.3.20}$$

A sample problem of this type is to stabilize an antenna on a space station subject to impact from space debris, where we assume that the debris coming within a certain distance is measurable and are modeled as a sequence of events. Towards this end, we consider the following max-min optimization problem

$$\max_{v \in V} \min_{u \in U} J_\gamma(u, v) \tag{5.3.21}$$

under the constraint (5.3.17).

Remark 5.3.2 *The max-min optimization problem is much easier to solve than the minimax problem. For a given v, the minimization problem with respect to u can be solved by the standard optimal control theory. The cost function $J_\gamma(u(v), v)$ can be explicitly evaluated by using the above approximation method. The cost function $J_\gamma(u(v), v)$ is a nonlinear function of v. Since $v \in V$ and V is a compact set, the maximization problem with respect to v always has a solution.*

We assume that the state variables are available and the disturbance v is known a priori. The following standard assumptions are made:

A1) (A, B_1) and (A, B_2) are stabilizable, and (C, A) is detectable.

A2) $D'[C \;\; D] = [0 \;\; I]$.

Theorem 5.3.1 *Let the pair $(\hat{u}, \hat{v})$ be a solution of the max-min problem (5.3.21) under the constraint (5.3.17). If the cost function $J_\gamma(\hat{u}, \hat{v}) \leq 0$ for a given $\gamma > 0$, then the inequality (5.3.20) holds for all $v \in V$ and the optimal strategy is given by the solution of the minimization problem* $\min_u J_\gamma(u, v)$.

Proof: *Let $u(v)$ denote the solution of the minimization problem of $J_\gamma(u,v)$, so that v becomes the only variable of the function $J_\gamma(u(v),v)$. Let $\hat{v}$ be the solution of the maximization problem of $J_\gamma(u(v),v)$ under constraint V. Then by the assumption, we have the following inequality*

$$J_\gamma(u(v),v) \leq J_\gamma(\hat{u},\hat{v}) \leq 0 \tag{5.3.22}$$

where $\hat{u} = u(\hat{v})$. □

Now we discuss the problem of computing the optimal control strategy $u(v)$. For a given v, the Hamiltanion of (5.3.17)-(5.3.18) is given by

$$H(x,u,p) = \frac{1}{2}(x'C'Cx + u'u) + p'(Ax + B_1u + B_2w). \tag{5.3.23}$$

The minimum principle now yields

$$\frac{\partial H}{\partial u} = 0, \Longrightarrow \; u^* = -B_1'p \tag{5.3.24}$$

The state and adjoint equations are given by

$$\begin{aligned} \dot{x} &= \frac{\partial H(x,u^*,p)}{\partial p} = Ax - B_1B_1'p + B_2w \\ \dot{p} &= -\frac{\partial H(x,u^*,p)}{\partial x} = -A'p - C'Cx \end{aligned} \tag{5.3.25}$$

$$\begin{aligned} d/dt \begin{bmatrix} x \\ p \end{bmatrix} &= \begin{bmatrix} A & -B_1B_1' \\ -C_1'C_1 & -A' \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} + \begin{bmatrix} B_2 \\ 0 \end{bmatrix} w \\ &= H \begin{bmatrix} x \\ p \end{bmatrix} + Pw, \; t \geq 0 \end{aligned} \tag{5.3.26}$$

Note for $t \geq (N+1)T_0, w(t) \equiv 0$, and we have the $2n \times 2n$ Hamiltanion system

$$d/dt \begin{bmatrix} x \\ p \end{bmatrix} = \begin{bmatrix} A & -B_1B_1' \\ -C_1'C_1 & -A' \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix}, \; t \geq (N+1)T_0 \tag{5.3.27}$$

The algebraic Riccati equation associated with (5.3.27) is

$$A'Q + QA - Q'B_1B_1'Q + C'C = 0. \tag{5.3.28}$$

The Riccati equation admits a unique solution $Q = Q' \geq 0$ by assumption $A1)$ and the LQG theory. It is easy to show that $p((N+1)T_0) = Qx((N+1)T_0)$. The linear system (5.3.26) can be solved in backward time with initial condition $[x((N+1)T_0) \;\; p((N+1)T_0)]'$,

$$\begin{aligned}\begin{bmatrix} x(t) \\ p(t) \end{bmatrix} &= e^{H(t-(N+1)T_0)} \begin{bmatrix} x((N+1)T_0) \\ p((N+1)T_0) \end{bmatrix} \\ &\quad + e^{Ht} \int_{(N+1)T_0}^{t} e^{-H\tau} Pw(\tau)d\tau, \quad 0 \leq t \leq (N+1)T_0. \end{aligned} \tag{5.3.29}$$

We then apply the approximation method developed in section 4.3.1 to simplify expression (5.3.29). A Taylor expansion of the function $e^{-\frac{1}{bv_k^{1/5}}H\tau}$ is given as

$$e^{-\frac{1}{bv_k^{1/5}}H\tau} = I - \frac{1}{bv_k^{1/5}}H\tau + \frac{1}{2!}\frac{1}{(bv_k^{1/5})^2}H^2\tau^2 + \cdots\cdots \tag{5.3.30}$$

The following integral can be written as a series,

$$\begin{aligned}\int_0^t e^{-H\tau}Pw(\tau)d\tau &= \sum_{k=0}^{N}(Pb_1v_k\theta_1(bv_k^{1/5}(t-kT_0)) \\ &\quad - HPb_2v_k^{4/5}\theta_2(bv_k^{1/5}(t-kT_0)) + \cdots\cdots), \end{aligned} \tag{5.3.31}$$

and

$$\int_0^{(N+1)T_0} e^{-H\tau}Pw(\tau)d\tau = \sum_{k=0}^{N}(Pb_1v_k\bar{\theta}_1 - HPb_2v_k^{4/5}\bar{\theta}_2 + \cdots\cdots), \tag{5.3.32}$$

Note that

$$\int_{(N+1)T_0}^{t} e^{-H\tau}Pw(\tau)d\tau = \int_0^t e^{-H\tau}Pw(\tau)d\tau - \int_0^{(N+1)T_0} e^{-H\tau}Pw(\tau)d\tau \tag{5.3.33}$$

We can explicitly express $x((N+1)T_0)$ as a function of v by using equations (5.3.29), (5.3.32) and the initial condition $x(0) = 0$. Once an expression for

$x((N+1)T_0)$ is obtained, $x(t)$ and $p(t)$ can be expressed as functions of v during the interval $[0, (N+1)T_0]$. For simplicity, let

$$x(t) = f_1(t, v), \; p(t) = f_2(t, v), \; 0 \leq t \leq (N+1)T_0. \tag{5.3.34}$$

where f_1 and f_2 are two nonlinear functions of v in general. Thus the optimal strategy $u(v)$ is given by

$$u(t) = -B_1' p(t) = \begin{cases} -B_1' f_2(t, v), & 0 \leq t \leq (N+1)T_0 \\ \\ -B_1' Q x(t), & t > (N+1)T_0 \end{cases} \tag{5.3.35}$$

By *LQG* theory, the control strategy $u(t)$ asymptotically stabilizes the closed-loop system.

Once the expressions for $p(t), x(t)$, and $x((N+1)T_0)$ over the interval $[0, (N+1)T_0]$ are obtained , the cost function of $\int_0^\infty ||z||_2^2 dt$ can be easily evaluated as

$$\int_0^\infty ||z||_2^2 dt = \int_0^{(N+1)T_0} ||Cx||_2^2 + ||u||_2^2 dt + \int_{(N+1)T_0}^\infty ||Cx||_2^2 + ||u||_2^2 dt \tag{5.3.36}$$

Let us denote the first integral as $\int_0^{(N+1)T_0} ||Cx||_2^2 + ||u||_2^2 dt = g_1(v) \geq 0$, and from the *LQG* theory the second integral can be expressed as $x((N+1)T_0)' Q x((N+1)T_0) = g_2(v) \geq 0$. Now the total cost function becomes

$$J_\gamma(v, u(v)) = g_1(v) + g_2(v) - \gamma^2 ||v||_{l_2}^2. \tag{5.3.37}$$

The maximization problem $\max_v J_\gamma(v, u(v))$ can be solved for a given γ under constraint V. Observe that the function $J_\gamma(\hat{v}, \hat{u})$ decreases monotonically as γ increases, hence we can always find a γ such that $J_\gamma(\hat{v}, \hat{u}) \leq 0$. The minimal γ^* such that $J_\gamma(\hat{v}, \hat{u}) \leq 0$ is called the optimal disturbance attenuation level.

In order to illustrate the procedure, let us take a scalar linear system subjected to a single impact:

$$\dot{x}(t) = 2x(t) + u(t) + av_0^{6/5}r(bv_0^{1/5}t), \tag{5.3.38}$$

where v_0 is a variable $v_0 \in [0,\ 0.1]$. The cost function is defined as

$$J_\gamma(u, v_0) = \int_0^\infty (x+u)^2 dt - \gamma^2 v_0^2. \tag{5.3.39}$$

Now we follow the above procedures to compute the optimal strategy $u(v)$ and optimal attenuation level γ^*, we have

$$\begin{aligned}
H(x,u,p) &= \frac{1}{2}(x+u)^2 + p[2x(t) + u(t) + av_0^{6/5}r(bv_0^{1/5}t)] \\
\frac{\partial H}{\partial u} &= 0 \Longrightarrow u^* = -p - x \\
H(x,u^*,p) &= \frac{1}{2}p^2 + p[x(t) - p + av_0^{6/5}r(bv_0^{1/5}t)] \\
\dot{x} &= \frac{\partial H}{\partial p} = x - p + av_0^{6/5}r(bv_0^{1/5}t), \\
\dot{p} &= -\frac{\partial H}{\partial x} = -p.
\end{aligned} \tag{5.3.40}$$

Without loss of any generality, let $T_0 = 1$. It is easy to obtain $p(1) = 2x(1)$ from equations (5.3.40). Thus

$$\begin{aligned}
p(t) &= 2e^{-(t-1)}x(1), \\
x(t) &= e^{1-t}x(1) + \int_1^t e^{t-\tau}av_0^{6/5}r(bv_0^{1/5}\tau), 0 \le t \le 1.
\end{aligned} \tag{5.3.41}$$

Since $x(0) = 0$, we have

$$x(1) = e^{-1}\int_0^1 e^{-\tau}av_0^{6/5}r(bv_0^{1/5}\tau). \tag{5.3.42}$$

First-order and second-order approximations are considered here for the purpose of comparison. The first-order approximation yields

$$\begin{aligned}
x_1(1) &= e^{-1}\frac{a}{b}\bar{\theta}_1 v_0, \\
p_1(t) &= 2e^{-t}\frac{a}{b}\bar{\theta}_1 v_0.
\end{aligned} \tag{5.3.43}$$

	a	b
steel	9918.555	7934.8442
aluminum	308.8	247.044
rubber	20.7873	16.63

Table 5.1: Impact parameters

and the total cost function is

$$J_\gamma^1(u, v_0) = \int_0^\infty (x+u)^2 dt - \gamma^2 v_0^2 = 2\frac{a^2}{b^2}\bar{\theta}_1^2 v_0^2 - \gamma^2 v_0^2. \tag{5.3.44}$$

The second-order approximation yields

$$\begin{aligned} x_2(1) &= e^{-1}(\frac{a}{b}\bar{\theta}_1 v_0 - \frac{a}{b^2}\bar{\theta}_2 v_0^{4/5} \\ p_2(t) &= 2e^{-t}(\frac{a}{b}\bar{\theta}_1 v_0 - \frac{a}{b^2}\bar{\theta}_2 v_0^{4/5}) \\ J_\gamma^2(u, v_0) &= 2(\frac{a}{b}\bar{\theta}_1 v_0 - \frac{a}{b^2}\bar{\theta}_2 v_0^{4/5})^2 - \gamma^2 v_0^2. \end{aligned} \tag{5.3.45}$$

The parameters a, b are determined by the local geometry of the region of contact and the material properties of contacting bodies. In order to show that the approximation is valid for a large range of applications, we select three materials: steel, aluminum and rubber. They are representatives of a hard, medium and soft material. The parameters are given in Table 5.1.

Case1: Steel.

$$J_\gamma^1(u, v_0) = 9.68v^2 - \gamma^2 v_0^2 \tag{5.3.46}$$

$$J_\gamma^2(u, v_0) = 9.68v^2 - 1.6 \times 10^{-4} v^{9/5} + 3.2 \times 10^{-7} v^{8/5} - \gamma^2 v_0^2 \tag{5.3.47}$$

Let γ^* be defined as the optimal attenuation level. The equation (5.3.46) represents the leading term approximation and equation (5.3.47) represents the second-order approximation. The optimal attenuation levels of (5.3.46) and

(5.3.47) are computed as $\gamma_1^* = 3.111, \gamma_2^* = 3.111$ respectively.

Case2: Aluminum.

$$J_\gamma^1(u, v_0) = 9.68v^2 - \gamma^2 v_0^2 \tag{5.3.48}$$

$$J_\gamma^2(u, v_0) = 9.68v^2 - 0.117v^{9/5} + 3.3 \times 10^{-4} v^{8/5} - \gamma^2 v_0^2 \tag{5.3.49}$$

The optimal attenuation levels of (5.3.48) and (5.3.49) are computed as $\gamma_1^* = 3.111, \gamma_2^* = 3.08$ respectively.

Case3: Rubber.

$$J_\gamma^1(u, v_0) = 9.68v^2 - \gamma^2 v_0^2 \tag{5.3.50}$$

$$J_\gamma^2(u, v_0) = 9.68v^2 - 1.78v^{9/5} + 0.054v^{8/5} - \gamma^2 v_0^2 \tag{5.3.51}$$

The optimal attenuation levels of (5.3.50) and (5.3.51) are $\gamma_1^* = 3.111, \gamma_2^* = 2.641$ respectively.

Remark 5.3.3 *For the hard and medium impact objects, the accuracy of the leading term approximation is almost the same as for the second-order approximation, and the accuracy of the second-order approximation is improved by approximately* 15% *for soft impact materials. Hence the following conclusion can be drawn: For impact problems involving hard and medium materials which consists of a large part of the applications we discussed before, the leading term of the Taylor series provides a very good approximation. From the remark 4.3.1, this approximation basically renders a linear system. For impact problems involving soft materials and low velocity, we need to keep the higher order term in order to obtain better approximations and need to solve the nonlinear problem directly.*

In the cases where the leading term is sufficient, the impact control problem is

Definition 6.3.2 *[62] An eigenvalue λ of $\Psi(T,0)$ is said to be $(A(t), B(t))$ uncontrollable if there exists an n-dimensional vector $\eta \neq 0$ such that*

$$\Psi(T,0)'\eta = \lambda\eta \tag{6.3.5}$$

$$B'\Psi^{-1}(t,0)'\eta = 0, \quad \forall t \in [0,T]. \tag{6.3.6}$$

Remark 6.3.3 *The pair $(A(t), B(t))$ is said to be stabilizable if and only if all the eigenvalues λ of $\Psi(T,0)$ such that $|\lambda| \geq 1$ are $(A(t), B(t))$ controllable.*

Let $\Phi(t,\tau)$ defined in (6.2.11) be partitioned into four $n \times n$ matrices as follows,

$$\Phi(t,\tau) = \begin{bmatrix} \Phi_{11}(t,\tau) & \Phi_{12}(t,\tau) \\ \Phi_{21}(t,\tau) & \Phi_{22}(t,\tau) \end{bmatrix}. \tag{6.3.7}$$

Obviously $\Phi(t,\tau)$ has the property for all t and τ,

$$\Phi(t+T, \tau+T) = \Phi(t,\tau). \tag{6.3.8}$$

Lemma 6.3.1 *The eigenvalues of $\Phi(t+T,t)$ are independent of t.*

Proof: *Without loss generality, we assume $t \in (0,T)$. By definition,*

$$\begin{aligned} \Phi(t+T,t) &= \Phi_H(t+T,T^+)F\Phi_H(T,t) \\ &= \Phi_H(t,0^+)F\Phi_H(T,t) \\ &= \Phi_H^{-1}(T,t)\Phi_H(T,0^+)F\Phi_H(T,t) \\ &= \Phi_H^{-1}(T,t)\Phi(T,0)\Phi_H(T,t). \end{aligned}$$

Thus, the eigenvalues of $\Phi(t+T,t)$ are independent of t. □

The hybrid state transition matrix $\Phi(t+T,t)$ displays same properties familiar with periodic systems. Hence we may expect that there exists a periodic solution of equations (6.2.1)-(6.2.2). For simplicity, we will assume that the matrix

$\Phi(T,0)$ has distinct eigenvalues. Let Λ be the corresponding diagonal Jordan form such that

$$S^{-1}(0)\Phi(T,0)S(0) \;=\; \Lambda, \tag{6.3.9}$$

where the matrix $S(0)$ is composed of eigenvectors of $\Phi(T,0)$. It should be noted that the derivation of results can be applied to the case of multiple eigenvalues with minor modifications. Now partition $S(0)$ into four $n \times n$ matrices

$$S(0) \;=\; \begin{bmatrix} Y(0) & V(0) \\ X(0) & U(0) \end{bmatrix} \tag{6.3.10}$$

and partition Λ similarly. Then equation (6.3.9) can be written as,

$$\begin{bmatrix} \Phi_{11}(T,0) & \Phi_{12}(T,0) \\ \Phi_{21}(T,0) & \Phi_{22}(T,0) \end{bmatrix}\begin{bmatrix} Y(0) & V(0) \\ X(0) & U(0) \end{bmatrix}=\begin{bmatrix} Y(0) & V(0) \\ X(0) & U(0) \end{bmatrix}\begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix} \tag{6.3.11}$$

where Λ_1 is an $n \times n$ diagonal matrix,

$$\Lambda_1 \;=\; diag\{\lambda_1, \cdots\cdots, \lambda_n\} \tag{6.3.12}$$

and $\{\lambda_1, \cdots\cdots, \lambda_n\}$ are n of the 2n eigenvalues of $\Phi(T,0)$. Let us define a $2n \times 2n$ matrix $S(t)$ for $t \geq 0$ by

$$S(t) \;=\; \Phi(t,0)S(0). \tag{6.3.13}$$

Again, partition $S(t)$ as follows,

$$S(t) \;=\; \begin{bmatrix} Y(t) & V(t) \\ X(t) & U(t) \end{bmatrix} \tag{6.3.14}$$

Lemma 6.3.2 *The equation $\Phi(t+T,t)S(t) = S(t)\Lambda$ holds for all $t \geq 0$.*

Proof:

$$\begin{aligned}
\Phi(t+T,t)S(t) &= \Phi(t+T,T)\Phi(T,t)S(t) \\
&= \Phi(t,0)\Phi(T,t)\Phi(t,0)S(0) \\
&= \Phi(t,0)\Phi(T,0)S(0) \\
&= \Phi(t,0)S(0)\Lambda \\
&= S(t)\Lambda
\end{aligned}$$

Thus,

$$\Phi(t+T,t)S(t) = S(t)\Lambda, \quad \forall t \geq 0. \tag{6.3.15}$$

□

It is easy to show that the following equation also holds,

$$\begin{aligned}
\frac{dS(t)}{dt} &= HS(t), \quad t \neq iT, \\
S(t^{+}) &= FS(t), \quad t = iT, i = 0, 1, 2, \cdots\cdots,
\end{aligned} \tag{6.3.16}$$

Theorem 6.3.1 *Let λ be an eigenvalue of $\Phi(t,\tau)$ and $\begin{bmatrix} y \\ x \end{bmatrix}$ be the corresponding eigenvector. Then λ^{-1} is also an eigenvalue of $\Phi(t,\tau)$ and $\begin{bmatrix} -x \\ y \end{bmatrix}$ is the eigenvector of $\Phi(t,\tau)'$ associated with λ^{-1}.*

Proof: *Define a $2n \times 2n$ symplectic matrix*

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \tag{6.3.17}$$

Since H is a Hamiltonian matrix and F is a symplectic matrix, we have the following equations,

$$H'J + JH = 0,$$

$$F'JF \quad = \quad J. \tag{6.3.18}$$

On the other hand,

$$\frac{\partial}{\partial t}[\Phi(t,\tau)'J\Phi(t,\tau)] = \Phi(t,\tau)'[H'J + JH]\Phi(t,\tau) = 0. \tag{6.3.19}$$

The possible initial conditions are

$$\begin{aligned} \Phi(\tau,\tau)'J\Phi(\tau,\tau) &= J, \quad \tau \neq iT, \\ \Phi(\tau^+,\tau)'J\Phi(\tau^+,\tau) &= F'JF = J, \quad \tau = iT \end{aligned} \tag{6.3.20}$$

Hence,

$$\Phi(t,\tau)'J\Phi(t,\tau) = J, \quad \forall t,\tau \tag{6.3.21}$$

Equation (6.3.21)shows that $\Phi(t,\tau)$ is non-singular for all t and τ implying that none of the eigenvalues of $\Phi(t,\tau)$ vanishes. Let,

$$\Phi(t,\tau)\begin{bmatrix} y \\ x \end{bmatrix} = \lambda \begin{bmatrix} y \\ x \end{bmatrix} \tag{6.3.22}$$

then, from equation (6.3.21), we have

$$\Phi^{-1}(t,\tau)'\begin{bmatrix} -x \\ y \end{bmatrix} = J\Phi(t,\tau)\begin{bmatrix} y \\ x \end{bmatrix} = \lambda J\begin{bmatrix} y \\ x \end{bmatrix} = \lambda\begin{bmatrix} -x \\ y \end{bmatrix} \tag{6.3.23}$$

Hence,

$$\Phi(t,\tau)'\begin{bmatrix} -x \\ y \end{bmatrix} = \lambda^{-1}\begin{bmatrix} -x \\ y \end{bmatrix}$$

Therefore, if λ is an eigenvalue of $\Phi(t,\tau)$, then so is λ^{-1}. $\square$

6.3.2 Parameterization of periodic solutions with jump

The following theorem gives a parameterization of all periodic solutions of equations (6.2.1)-(6.2.2) in terms of $X(t)$ and $Y(t)$ defined in equation (6.3.14).

Theorem 6.3.2 *Suppose $Y(t)$ defined in equation (6.3.14) is non-singular for all $t \in [0, T]$. Then $P(t)$ given by*

$$P(t) \;=\; X(t)Y(t)^{-1}, \quad t \geq 0 \tag{6.3.24}$$

is a periodic solution of equation (6.2.1) and (6.2.2).

Proof*:*

$$\begin{aligned} S(t+T) &= \Phi(t+T,0)S(0) \\ &= \Phi(t+T,t)\Phi(t,0)S(0) \\ &= S(t)\Lambda. \end{aligned}$$

Therefore,

$$\begin{bmatrix} Y(t+T) \\ X(t+T) \end{bmatrix} = \begin{bmatrix} Y(t) \\ X(t) \end{bmatrix} \Lambda_1, \quad t \geq 0. \tag{6.3.25}$$

$$\begin{aligned} \dot{P}(t) &= \dot{X}(t)Y(t)^{-1} - X(t)Y(t)^{-1}\dot{Y}(t)Y(t)^{-1} \\ &= [-C_1'C_1Y(t) - A'X(t)]Y(t)^{-1} - X(t)Y(t)^{-1} \\ &\quad [AY(t) - B_1B_1'X(t)]Y(t)^{-1} \\ &= -A'P(t) - P(t)A + P(t)B_1B_1'P(t) - C_1'C_1 \\ P(0^+) &= X(0^+)Y(0^+)^{-1} \\ &= X(0)[Y(0) - \gamma^{-2}B_2B_2'X(0)]^{-1} \\ &= P(0)[I - \gamma^{-2}B_2B_2'P(0)]^{-1} \end{aligned} \tag{6.3.26}$$

It is easy to show that $P(0^+)$ satisfies equation (6.2.2). Thus $P(t)$ satisfies equations (6.2.1) and (6.2.2). It is periodic because from (6.3.25),

$$P(t+T) = X(t+T)Y(t+T)^{-1} = X(t)\Lambda_1[Y(t)\Lambda_1]^{-1} = P(t). \qquad (6.3.27)$$

$\square$

6.3.3 Analysis of periodic solutions with jumps

The following Lemmas(4.3)-(4.6) characterize the properties of a solution $P(t)$ that depend on the choice of n eigenvalues $\lambda_1, \cdots\cdots \lambda_n$ for Λ_1. Let

$$\begin{aligned} \Omega(t) &= Y(t)^* X(t), && (6.3.28) \\ \hat{\Omega}(t) &= Y(t)' X(t), && (6.3.29) \end{aligned}$$

where $*$ is used to denote the complex conjugate transpose.

Lemma 6.3.3 *If $\lambda_i^* \neq \lambda_j^{-1}$ for all $i, j, 1 \leq i, j \leq n$, then $\Omega(t)$ is Hermitian and $\hat{\Omega}(t)$ is symmetric.*

Proof: *Let $x_i(t)$ and $y_i(t)$ be the i^{th} columns of $X(t)$ and $Y(t)$ respectively, and also let*

$$z_i(t) = \begin{bmatrix} y_i(t) \\ x_i(t) \end{bmatrix}$$

The $ij-$element $\omega_{ij}(t)$ of $\Omega(t)$ is then described as

$$\omega_{ij}(t) = y_i^*(t) x_j(t) \qquad (6.3.30)$$

and hence

$$\omega_{ij}(t) - \omega_{ji}^*(t) = z_i^*(t) J z_j(t) \qquad (6.3.31)$$

Noting that $\lambda_i^* - \lambda_j^{-1} \neq 0$, equations (6.3.15) and (6.3.21) leads to,

$$\begin{aligned}
\omega_{ij}(t) - \omega_{ji}^*(t) &= (\lambda_i^* - \lambda_j^{-1})^{-1}[\lambda_i^* z_i^*(t) J z_j(t) - \lambda_j^{-1} z_i^*(t) J z_j(t)] \\
&= (\lambda_i^* - \lambda_j^{-1})^{-1} z_i^*(t)[\Phi(t+T,t)'J - J\Phi^{-1}(t+T,t)] z_j(t) \\
&= 0. \qquad (6.3.32)
\end{aligned}$$

The proof that $\hat{\Omega}(t)$ is symmetric is analogous to above. □

Lemma 6.3.4 *Assume that $|Y(t)| \neq 0$ for all $t \in [0,T]$. If $\lambda_i^* \neq \lambda_j^{-1}$ and $\lambda_i \neq \lambda_j^{-1}$ for all $i, j, 1 \leq i, j \leq n$, then $P(t)$ defined by equation (6.3.24) is real.*

Proof: *Under the assumptions, $P(t)$ can be rewritten as,*

$$\begin{aligned}
P(t) &= X(t)Y(t)^{-1} \\
&= (Y(t)^{-1})^* \Omega(t) Y(t)^{-1} \\
&= (Y(t)^{-1})' \hat{\Omega}(t) Y(t)^{-1}. \qquad (6.3.33)
\end{aligned}$$

By Lemma (6.3.3), $\Omega(t)$ and $\hat{\Omega}(t)$ are Hermitian and symmetric respectively. Equation (6.3.33) then shows that $P(t)$ is Hermitian as well as symmetric. Thus $P(t)$ is a real matrix. □

Lemma 6.3.5 *Suppose that (A, B_1) is controllable and (C_1, A) is observable. If $|\lambda_i| < 1$ for all $i = 1, 2, \cdots\cdots n$, then $\Omega(t)$ is positive definite.*

Proof: *Let $k = 0, 1, 2, \cdots\cdots$*

$$U(k) = \begin{bmatrix} Y(t) \\ X(t) \end{bmatrix} \Lambda_1^k \qquad (6.3.34)$$

then from equation (6.3.15)

$$U(k+1) = \Phi(t+T,t)U(k). \qquad (6.3.35)$$

Let us define a $2n \times 2n$ matrix V as,

$$V = \begin{bmatrix} 0 & 0 \\ -I & 0 \end{bmatrix}. \tag{6.3.36}$$

$\Omega(t)$ can then be expressed as

$$\Omega(t) = -U^*(0)VU(0). \tag{6.3.37}$$

Since $|\lambda_i| < 1$ for all $i = 1, 2, \cdots\cdots n$, then $U(k) \to 0$ as $k \to \infty$. Therefore,

$$\begin{aligned} \Omega(t) &= \sum_{k=0}^{\infty} U^*(k+1)VU(k+1) - U^*(k)VU(k) \\ &= \sum_{k=0}^{\infty} U^*(k)[\Phi(t+T,t)'V\Phi(t+T,t) - V]U(k). \end{aligned} \tag{6.3.38}$$

Define a matrix $M(t,\tau)$ by,

$$M(t,\tau) = \Phi(t,\tau)'V\Phi(t,\tau) - V. \tag{6.3.39}$$

Then

$$\begin{aligned} \frac{\partial M(t,\tau)}{\partial t} &= \Phi(t,\tau)'[H'V + VH]\Phi(t,\tau) \\ &= \Phi(t,\tau)'N'N\Phi(t,\tau) \end{aligned} \tag{6.3.40}$$

where

$$N = \begin{bmatrix} C_1 & 0 \\ 0 & B_1' \end{bmatrix}. \tag{6.3.41}$$

Since

$$M(\tau,\tau) = 0, \quad \tau \neq iT, \tag{6.3.42}$$

$$M(\tau^+,\tau) = F'VF - V = \begin{bmatrix} 0 & 0 \\ 0 & \gamma^{-2}B_2B_2' \end{bmatrix} \geq 0, \tau = iT. \tag{6.3.43}$$

Thus,

$$M(t,\tau) = \int_\tau^t \Phi(s,\tau)'N'N\Phi(s,\tau)ds, \quad \tau \neq iT, \tag{6.3.44}$$

$$M(t,\tau) = \int_\tau^t \Phi(s,\tau)'N'N\Phi(s,\tau)ds + M(\tau^+,\tau), \ \tau = iT. \tag{6.3.45}$$

Since (A, B_1) is controllable and (C_1, A) is observable, it is easy to show that $M(t,\tau) > 0$ for all $t \geq \tau$. Thus from equations (6.3.38) and (6.3.39), it can be concluded that $\Omega(t) > 0$. □

Theorem 6.3.3 *Assume that for all eigenvalues of $\Phi(T, 0)$, $|\lambda_i| \neq 1$, $i = 1, 2 \cdots\cdots 2n$ and that $|Y(t)| \neq 0$ for all $t \in [0, T]$. If n eigenvalues in Λ_1 are chosen such that $|\lambda_i| < 1, i = 1, 2 \cdots\cdots n$, then a periodic solution $P(t)$ given by equation (6.3.24) is real, symmetric and positive definite.*

Proof*: The assumption $|\lambda_i| < 1$ for all $i = 1, 2 \cdots\cdots n$ guarantees that $\lambda_i^* \neq \lambda_j^{-1}$ and $\lambda_i \neq \lambda_j^{-1}$ for all $i, j, 1 \leq i, j \leq n$. Therefore $P(t)$ is real and symmetric by Lemmas (6.3.3) and (6.3.4) respectively. That $P(t)$ is positive definite follows from Lemma (6.3.5) and equation (6.3.34).* □

6.3.4 The periodic Lyapunov equation

Our approach for the analysis of the closed-loop system (6.2.12) is based on the Lyapunov method. As is well-known, the Lyapunov equation plays an important role in the analysis of Riccati equations and the associated closed-loop systems [63, 69, 70]. Here we extend some useful results on the periodic Lyapunov equation to our particular problem, namely, to the periodic Lyapunov equation with jumps.

Let $P(t)$ be a solution of the coupled Riccati equations (6.2.1)-(6.2.2),

$$\dot{P}(t) = -A'P(t) - P(t)A + P(t)B_1B_1'P(t) - C_1'C_1, \ t \neq iT \tag{6.3.46}$$

$$P(iT^+) = P(iT) + (B_2'P(iT))'(\gamma^2 I - B_2'P(iT)B_2)^{-1}B_2'P(iT). \quad (6.3.47)$$

It can be rewritten as a Lyapunov-type equation,

$$\dot{P}(t) = -A_c(t)'P(t) - P(t)A_c(t) - H(t)'H(t), \ t \neq iT \quad (6.3.48)$$

$$P(iT^+) = P(iT) + Q_{tmp}(iT). \quad (6.3.49)$$

where $A_c(t), H(t)$ and $Q_{tmp}(iT)$ are given in (6.2.17). Therefore, we can obtain the following results from [69, 71]. If $P(\cdot)$ is a symmetric periodic solution of the Riccati equation (6.3.46)-(6.3.47), then $P(\cdot)$ is also a solution of the Lyapunov equation (6.3.48)-(6.3.49). The structural properties of the pair $(H(t), A_c(t))$ can be related to the ones of the pair (C_1, A) by means of the following Lemma [71, 72].

Lemma 6.3.6 *The pair (C_1, A) is observable if and only if, for any $T-$periodic matrix $P(\cdot)$, the pair $(H(t), A_c(t))$ is observable.* □

The solution of equations (6.3.48)-(6.3.49) can be also given by the celebrated formula [65]. Let P_0 be the initial condition of equation (6.3.48) at time t_0, the solution of the equation (6.3.48) is given by

$$P(t) = \Phi_{A_c}'(t_o, t)P_0\Phi_{A_c}(t_0, t) - \int_{t_0}^{t} \Phi_{A_c}'(\tau, t)H(\tau)'H(\tau)\Phi_{A_c}'(\tau, t)d\tau. \quad (6.3.50)$$

Without loss of generality, we assume $t_0 = 0$, then $P(T)$ is given by

$$P(T) = \Phi_{A_c}'(T, 0)^{-1}P_0\Phi_{A_c}(T, 0)^{-1} - \int_0^T \Phi_{A_c}'(\tau, T)H(\tau)'H(\tau)\Phi_{A_c}'(\tau, T)d\tau. \quad (6.3.51)$$

Since we are looking for the periodic solutions of equations (6.3.48)-(6.3.49), the periodic generator P_0 must satisfy the following algebraic Lyapunov equation

$$P_0 = \Phi_{A_c}'(T, 0)P_0\Phi_{A_c}(T, 0) + W(T) \quad (6.3.52)$$

expressed as

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B_1 u(t) + B_2 \sum_{k=0}^{N} v_k \delta(t - kT_0) \\ z(t) &= Cx(t) + Du(t)\end{aligned} \tag{5.3.52}$$

where $\delta(t)$ is the standard Dirac delta distribution. The control objective is given by

$$||z(t)||_{L_2}^2 \leq \gamma^2 ||v||_{l_2}^2. \tag{5.3.53}$$

Remark 5.3.4 *The linear model has a physical interpretation. Since each impact duration is very short, we can ignore the finite duration and treat the impact as an impulsive event. This linear model has been used in various impact problems.*

We should point out that the control problem (5.3.52)-(5.3.53) is also different from the standard linear ones discussed in [20, 22]. The disturbance $v \in l_2$ is a discrete-time signal, while the controlled output $z(t)$ are continuous signals. The standard approaches [20, 22] cannot be applied to this problem directly. We will analyze and solve this special linear control problem in the next chapter.

Chapter 6

H_∞ CONTROL FOR IMPULSIVE DISTURBANCES

In chapter 5, we studied the nonlinear control of impact forces when the impact forces are modeled as finite duration events. We have shown that, in certain cases, by taking advantage of the fact that the impact duration is very short, we can approximate the system dynamics undergoing impact by low order terms of the Taylor polynomials. For the disturbance attenuation problem the leading term provides a very good approximation. A linear model can be obtained if we treat the finite event by an impulsive event. Thus, the nonlinear control problem can be solved via a linear one. In this chapter, we analyze and study the disturbance attenuation problem for the corresponding linear system. This control problem has some special features; the disturbance input $v \in l_2$ is a discrete-time signal, while the controlled output z is a continuous signal. It cannot be solved by the existing H_∞ approaches [19, 20, 24, 21, 22, 23]. We are motivated by the approaches [20, 24] for a standard H_∞ control problem, where a state-space solution of H_∞ control is closely related to the solutions of the Riccati equations. If full state-feedback is available, we knew that a controller exists if and only if the unique solution of the associated algebraic Riccati equation is positive definite. In addition, a formula for the state-feedback gain matrix was given in terms of the solution of the Riccati equation. In this control problem, instead of the one algebraic Riccati equation involved in problems [20, 24], we will have two coupled Riccati equations in order to account for the impulses in the state-feedback case. Hamiltonian theory is used to analyze the coupled Riccati equations, and necessary and sufficient conditions are obtained for the existence of a unique solution. Because the state variables may experience jumps due to the effects of impulsive forces, an extended Lyapunov lemma is derived to prove the stability of the closed-loop system.

6.1 Formulation of the Control Problem

We consider a finite-dimensional linear system

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} v \\ u \end{bmatrix} \tag{6.1.1}$$

where $z(t), y(t)$ are the controlled and measured output vectors, respectively. They are piecewise-continuous signals. There are two kinds of input signals, u, the control vector which is assumed to be piecewise-continuous, and, v, the impulsive disturbance vector which is assumed to be in the l_2 space. $L_{ij}, i, j = 1, 2$ are linear operators mapping from $\{v, u\}$ to $\{z, y\}$. This is a hybrid system because it contains both continuous time and discrete time components. In view of the results obtained in chapter 5 we assume that the system (6.1.1) admits a state space realization of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 u(t) + \sum_{i=0}^{\infty} B_2 v(i)\delta(t - iT), \\ z(t) &= C_1 x(t) + D_1 u(t), \\ y(t) &= C_2 x(t). \end{aligned} \tag{6.1.2}$$

It is convenient to use the state space equations (6.1.2) in the form

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + B_1 u(t), \quad t \neq iT, \\
x(t^+) &= x(t) + B_2 v(i), \quad t = iT, \quad i = 0, 1, 2, \cdots\cdots, \\
z(t) &= C_1 x(t) + D_1 u(t), \\
y(t) &= C_2 x(t).
\end{aligned} \tag{6.1.3}$$

where $x(t^+)$ denote the values of state variables immediately after a jump. The state variables $x(t)$ are right continuous and may be left discontinuous due to the possible jump.

The control problem here is to design a stabilizing controller to attenuate the effects of the impulsive disturbance v on the controlled output z. Let us define K as the set of all causal, finite-dimensional linear stabilizing controllers.

We will now introduce a minimax performance measure which is motivated by [27]. For $k \in K$ define a performance measure

$$J(k) = \sup_{v \in l_2, v \neq 0} \left(\frac{||z||_{L_2}^2}{||v||_{l_2}^2} \right). \tag{6.1.4}$$

$J(k)$ can also be viewed as the induced operator norm from $l_2 \to L_2$.

We want to find a controller $k \in K$ to minimize the worst case performance measure $J(k)$. Specifically, we solve the following (sub)optimal problem. Given $\gamma > 0$, find a $k \in K$ such that following inequality holds:

$$J(k) \; < \; \gamma^2 \tag{6.1.5}$$

Equivalently, we require that

$$||z||_{L_2}^2 - \gamma^2 ||v||_{l_2}^2 \; < \; 0, \tag{6.1.6}$$

for all possible $v \in l_2$, under the constraints of the system equations (6.1.3). If such a controller k exists, we call it a $\gamma-$level disturbance attenuation controller.

6.2 Main Results

We start with a standard H_∞ control problem treated in [20, 24] where we consider a continuous LTI system. It is well known that a state-space solution to the H_∞ control problem is closely related to Riccati equations. If full state-feedback is available, a controller exists if and only if the unique solution of the associated algebraic Riccati equation is positive definite. In addition, a formula for the state-feedback gain matrix was given in terms of the solution of the Riccati equation. Similar results can be obtained for discrete-time systems [22] and time-varying systems [61].

It should be noted that the control problem defined in (6.1.6) is different from the standard H_∞ problems treated in [20, 22, 24, 61]. Since the system (6.1.3) is a hybrid system which contains both continuous and discrete components. The control problem cannot be solved by the formulas obtained in [20, 24, 22, 61]. However, the essential ideas can be carried over to analyze the control problem (6.1.6). For state-feedback control, instead of the one algebraic Riccati equation involved in problems [20, 24], we will have two coupled Riccati equations given by

$$\begin{aligned} \dot{K}(t) &= -A'K(t) - K(t)A + K(t)B_1B_1'K(t) - C_1'C_1, \ t \neq iT \quad (6.2.1) \\ K(iT^+) &= K(iT) + (B_2'K(iT))'(\gamma^2 I - B_2'K(iT)B_2)^{-1}B_2'K(iT) \quad (6.2.2) \end{aligned}$$

where $\gamma > 0$ is a given real number, the $n \times n$ matrix-valued function $K(t)$ is right continuous and may be left discontinuous, $K(iT^+)$ represents the value of $K(t)$ after the i^{th} jump, while $K(iT)$ represents the value of $K(t)$ just before the i^{th} jump, i.e., $K(iT) := \lim_{\epsilon>0,\epsilon\to 0} K(iT-\epsilon)$ assuming that the limit exists.

The main results of this chapter are stated as Theorems 6.2.1- 6.2.3. We assume that all state variables are measured, and consider the infinite horizon problem with zero initial state. First the following assumptions are made.

i) (A, B_1) is controllable, (C_1, A) is observable.

ii) $D_1'[C_1 \quad D_1] = [0 \quad I]$.

Assumption i) is standard in the quadratic regulation of a linear system. It can be relaxed by the assumption that (A, B_1) is stabilizable and (C_1, A) is detectable. Assumption ii) is made here just for the sake of simplicity. Relaxing this assumption will only complicate the formulas, but an analysis can be carried out along lines similar to what appears below. We will state the main theorems first and defer the proofs to the following sections.

Theorem 6.2.1 *Consider the hybrid system described by (6.1.3). Let $\gamma > 0$ be given. Suppose that the assumptions i) and ii) hold. Then there exists a controller $k \in K$ such that $J(k) < \gamma^2$ if there exists a unique stabilizing positive definite periodic solution $P(t)$ of the coupled Riccati equations (6.2.1)-(6.2.2). Moreover, if this condition is satisfied, one such stabilizing state-feedback controller is given by*

$$u(t) = -B_1'P(t)x(t), \quad \forall t \geq 0. \tag{6.2.3}$$

□

Remark 6.2.1 *As $\gamma \to \infty$, the coupled Riccati equations (6.2.1)-(6.2.2) degenerate into the continuous-time Riccati equation*

$$\dot{K}(t) = -A'K(t) - K(t)A + K(t)B_1B_1'K(t) - C_1'C_1. \tag{6.2.4}$$

This Riccati equation will yield a unique positive definite constant matrix solution under assumption i). It is well known that this unique solution internally *stabilizes the associated closed-loop system from LQG theory. Thus, the H_∞ problem degenerates into a LQG problem as $\gamma \to \infty$.*

Via the Hamiltonian theory, the solution of a Riccati equation can be obtained by solving a suitable set of linear differential equations. The Hamiltonian matrix associated with the continuous time Riccati equation (6.2.1) is given by

$$H = \begin{bmatrix} A & -B_1 B_1' \\ -C_1' C_1 & -A' \end{bmatrix}. \tag{6.2.5}$$

As usual let us consider a $2n_{th}$ order differential equation

$$\dot{X}(t) = HX(t), \quad t \neq iT. \tag{6.2.6}$$

The state transition matrix associated with H is

$$X(t) = \Phi_H(t,\tau)X(\tau), \quad t \geq \tau, \quad and \quad t,\tau \neq iT, \tag{6.2.7}$$

where $\Phi_H(t,\tau)$ has the following properties,

$$\frac{\partial \Phi_H(t,\tau)}{\partial t} = H\Phi_H(t,\tau), \quad \Phi_H(\tau,\tau) = I. \tag{6.2.8}$$

The symplectic matrix associated with the difference Riccati equation (6.2.2) is given by

$$F = \begin{bmatrix} I & -\gamma^{-2} B_2 B_2' \\ 0 & I \end{bmatrix}. \tag{6.2.9}$$

Let us define a $2n_{th}$ order difference (jump) equation

$$X(t^+) = FX(t), \quad t = iT. \tag{6.2.10}$$

The combined equations (6.2.6) and (6.2.10) are a hybrid system. For $\tau = iT, i = 0, 1, \cdots, \cdots$, and $\tau \leq t \leq (i+1)T$, we have,

$$\begin{aligned} X(t) &= \Phi_H(t,\tau)FX(\tau), \\ &= \Phi(t,\tau)X(\tau). \end{aligned} \tag{6.2.11}$$

We will show later that this state transition matrix $\Phi(t,\tau)$ of the hybrid system displays the same properties as for a periodic system.

Now, let $P(t)$ be a periodic solution of equations (6.2.1)-(6.2.2). The associated closed-loop system is given by

$$\dot{x}(t) = [A - B_1 B_1' P(t)]x(t). \tag{6.2.12}$$

Definition 6.2.1 *A periodic solution $P(t)$ of (6.2.1)-(6.2.2) is called a stabilizing solution if the closed-loop system (6.2.12) is asymptotically stable.*

Let us consider the following hybrid system first:

$$\begin{aligned} \dot{x}(t) &= [A - B_1 B_1' P(t)]x(t), \\ &= A_c(t), \quad t \neq iT, \end{aligned} \tag{6.2.13}$$

$$\begin{aligned} x(t^+) &= [I - \gamma^{-2} B_2 B_2' P(t)]x(t), \\ &= F_c(t), \quad t = iT. \end{aligned} \tag{6.2.14}$$

We will show that if (6.2.12) is asymptotically stable, then the hybrid system asymptotically stable also.

Theorem 6.2.2 *Let $\gamma > 0$ be given. Suppose that the assumptions i) and ii) hold. Then a necessary and sufficient condition for the existence of a unique positive definite periodic solution $P(t)$ to equations (6.2.1) - (6.2.2), such that the hybrid system (6.2.13)-(6.2.14) is asymptotically stable, is that no eigenvalue of $\Phi(T,0)$ lies on the unit circle.* □

The next theorem gives necessary and sufficient conditions that the unique positive definite periodic solution stabilizes the associated closed-loop system.

Theorem 6.2.3 *Suppose that the assumptions i) and ii) hold. Let $P(t)$ be the unique positive definite periodic solution of (6.2.1) and (6.2.2). Then $P(t)$ is a stabilizing solution if and only if the inequality*

$$W_c(T) - \Phi'_{A_c}(T,0)Q_{tmp}(T)\Phi_{A_c}(T,0) > 0. \tag{6.2.15}$$

holds, where

$$W_c(T) = \int_0^T \Phi_{A_c}(\tau,0)'H(\tau)'H(\tau)\Phi_{A_c}(\tau,0)d\tau \tag{6.2.16}$$

and $\Phi_{A_c}(t,\tau)$ is state transition matrix of $A_c(t)$, and where the matrices $A_c(t)$, $H(t)$ and $Q_{tmp}(T)$ are defined by

$$\begin{aligned} A_c(t) &= A - B_1B_1'P(t), \\ H(t)'H(t) &= C_1'C_1 + P(t)B_1B_1'P(t), \\ Q_{tmp}(T) &= (B_2'P(T))'(\gamma^2 I - B_2'P(T)B_2)^{-1}B_2'P(T). \end{aligned} \tag{6.2.17}$$

□

6.3 Background and Technical Lemmas

In order to prove Theorems (6.2.1)-(6.2.3), we need to develop theory and technical machinery to analyze the coupled Riccati equations (6.2.1)-(6.2.2). Specifically, 1) we will give conditions for existence of periodic solutions of equations (6.2.1)-(6.2.2), 2) If such conditions are satisfied, we will parameterize all stabilizing periodic solutions. The technical machinery developed here is based on the standard tools for the analysis of periodic systems [62, 63, 64]. We will show that equations (6.2.1)-(6.2.2) display same properties similar to a standard periodic system, the main difference being that we need to take care of the jumps in the state which actually result in the periodic solutions.

6.3.1 Periodic system and stability analysis

Consider the following linear periodic system

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) \end{aligned} \tag{6.3.1}$$

where $x(t) \in R^n, y(t) \in R^m, u(t) \in R^p$ are state, output and input vector variables respectively, $A(\cdot), B(\cdot), C(\cdot)$ are assumed to be periodic matrices of period T, i.e,

$$A(t+T) = A(t), \;\; B(t+T) = B(t), \;\; C(t+T) = C(t), \;\; \forall t \geq 0. \tag{6.3.2}$$

The state transition matrix of system (6.3.1) is denoted by $\Psi(t, t_0)$. The matrix $\Psi(t+T, t)$ is called the monodromy matrix at time t. Since $\Psi(t+T, t) = \Psi(t, \tau)\Psi(\tau + T, \tau)\Psi(t, \tau)^{-1}$, the eigenvalues of $\Psi(t+T, t)$ are independent of t. they are also called the characteristic multipliers [65, 66, 67].

Remark 6.3.1 *[67, 68] The system (6.3.1) is asymptotically stable if and only if the eigenvalues of $\Psi(T, 0)$ are all inside the unit circle.*

Definition 6.3.1 *[62] An eigenvalue λ of $\Psi(T, 0)$ is said to be $(C(t), A(t))$ unobservable if there exists an n-dimensional vector $\xi \neq 0$ such that*

$$\Psi(T, 0)\xi = \lambda\xi \tag{6.3.3}$$

$$C\Psi(t, 0)\xi = 0, \;\; \forall t \in [0, T]. \tag{6.3.4}$$

Remark 6.3.2 *The pair $(C(t), A(t))$ is said to be detectable if and only if all eigenvalues λ of $\Psi(T, 0)$ such that $|\lambda| \geq 1$ are $(C(t), A(t))$ observable.*

at the sampling instances are given by

$$\begin{aligned} x(k+1) &= Fx(k) + G_1 u(k) + G_2 v(k) \\ y(k) &= C_2 x(k) \end{aligned} \tag{6.5.2}$$

where $F = e^{AT}$, $G_1 = \int_0^T e^{A(T-\tau)} B_1 d\tau$. $G_2 = e^{AT} B_2$. The controlled output is continuous and is given by

$$z(kT+t) = C_1 e^{At} x(kT) + \int_0^t C_1 e^{A(t-\tau)} B_1 d\tau u(kT) + C_1 e^{At} B_2 v(kT)$$
$$\forall t \in [kT, kT+T]. \tag{6.5.3}$$

It is well-known if (A, B_i) is controllable and (C_i, A) is observable, for a generic choice of the sampling period T so are the pairs (F, G_i) and (C_i, F) [74]. We assume that the sampling period T satisfies generically condition in the rest of this chapter.

Let $\tilde{K}$ be the set of all discrete-time LTI stabilizing controllers. The H_∞ control problem we address here is that for given $\gamma > 0$, find a $\tilde{k} \in \tilde{K}$ such that the following inequality holds

$$J_\gamma(\tilde{K}) = ||z||_{L_2}^2 - \gamma^2 ||v||_{l_2}^2 \leq 0. \tag{6.5.4}$$

for all $v \in l_2$, and zero initial conditions. It is known that a sampled-data control system in general is time-varying [76]. H_∞ control design problem for sampled-data control systems has been addressed by many researchers recently (see [23, 77, 75] for details). Our interest here is to adapt the theory to the case of impulsive disturbances attenuation. We will show that by using a sampled-data controller, the control problem can be converted into a LTI discrete-time H_∞ control problem.

The term $||z||_{L_2}^2$ during $[kT,(k+1)T]$ can be rewritten as

$$\int_0^T z'(kT+t)z(kT+t)dt = x'(k)M_1x(k) + 2u'(k)P_1'x(k) + u'(k)M_2u(k) + 2v'(k)P_2'x(k) + v'(k)M_3v(k) + 2v'(k)P_3'u(k). \quad (6.5.5)$$

where M_i are weighting matrix associated with state $x(k)$, control $u(k)$ and disturbance $v(k)$, and P_i are cross-terms respectively. They are given by,

$$\begin{aligned}
M_1 &= \int_0^T e^{A't}C_1'C_1e^{At}dt, \quad P_1 = \int_0^T\int_0^t e^{A't}C_1'C_1e^{A(t-\tau)}B_1 d\tau dt \\
M_2 &= \int_0^T\int_0^t B_1'e^{A'(t-\tau)}C_1'C_1e^{A(t-\tau)}B_1 d\tau dt, \ P_2 = \int_0^T e^{A't}C_1'C_1e^{At}B_2dt \\
M_3 &= \int_0^T B_2'e^{A't}C_1'C_1e^{At}B_2dt, \ P_3 = \int_0^T\int_0^t B_1'e^{A't}C_1'C_1e^{A(t-\tau)}B_2 d\tau dt. \quad (6.5.6)
\end{aligned}$$

Hence the square of the controlled output's norm is in turn given by

$$\begin{aligned}
||z||_{L_2}^2 &= \int_0^\infty z'(t)z(t)dt = \sum_{k=0}^{\infty}\int_k^{(k+1)T} z'(t)z(t)dt \\
&= \sum_{k=0}^{\infty}\int_0^T z(kT+t)'z(kT+t)dt \\
&= \sum_{k=0}^{\infty} x'(k)M_1x(k) + 2u'(k)P_1'x(k) + u'(k)M_2u(k) + \\
&\quad 2v'(k)P_2'x(k) + v'(k)M_3v(k) + 2v'(k)P_3'u(k). \quad (6.5.7)
\end{aligned}$$

The sampled-data control problem is now converted into the following discrete-time H_∞ control problem, where the system is given by,

$$\begin{aligned}
x(k+1) &= Fx(k) + G_1u(k) + G_2v(k) \\
y(k) &= C_2x(k), \quad k = 0,1,2,\cdots \quad (6.5.8)
\end{aligned}$$

and the control objective is to find a $\tilde{k} \in \tilde{K}$ such that

$$J_\gamma(\tilde{K}) = ||z||_{L_2}^2 - \gamma^2||v||_{l_2}^2 \leq 0 \quad (6.5.9)$$

where $||z||^2_{L_2}$ is given by (6.5.7).

Among different state-space approaches to this class of H_∞ control problems, the one that uses the framework of dynamic game theory seems to be the most natural. This is so because the original H_∞-optimal control problem (in its equivalent state-space formation) is in fact a minimax optimization problem, and hence a zero-sum game [27], where the controller can be viewed as the minimizing player and disturbance as the maximizing player. We will use this approach to derive the (sub)optimal solution to the problem (6.5.8)-(6.5.9).

6.5.2 Full state-feedback control

Consider a standard discrete-time system given by

$$\begin{aligned} x(k+1) &= Ax(k) + B_1 u(k) + B_2 v(k) \\ y(k) &= C_2 x(k). \end{aligned} \tag{6.5.10}$$

The performance measure is given by

$$J_\gamma(\tilde{K}) = \sum_{k=0}^{\infty} x'(k)Qx(k) + u'(k)u(k) - \gamma^2 v'(k)v(k) \leq 0 \tag{6.5.11}$$

for some specified $\gamma > 0$ and $Q \geq 0$, and $v \in l_2$.

First we consider the case when the full state variables ($C_2 = I$) are available. We make the following standard assumptions:

A1) The pairs (A, B_1) and (A, B_2) are controllable.

A2) $(A, Q^{1/2})$ is observable.

Let us define an algebraic Riccati equation,

$$M = A'(M^{-1} + B_1 B_1' - \gamma^{-2} B_2 B_2')^{-1} A + Q. \qquad (R1)$$

Theorem 6.5.1 *[27] Consider discrete-time system described by (6.5.10), let $\gamma > 0$ be given. Suppose that the assumptions $A1) - A2)$ hold. Then there exists a controller $\tilde{k} \in \tilde{K}$ such that inequality (6.5.11) hold if and only if*

1) (R1) admits a positive definite solution M_+.

2) $\gamma^2 I - B_2' M_+ B_2 > 0$.

Moreover, when these conditions hold, one such stabilizing feedback controller is given by

$$\begin{aligned} u(k) &= -B_1' M_+ \Delta^{-1} A x(k), \qquad (6.5.12)\\ v(k) &= -\gamma^{-2} B_2' M_+ \Delta^{-1} A x(k),\\ \Delta : &= I + (B_1 B_1' - \gamma^{-2} B_2 B_2') M_+. \quad k = 1, 2, 3, \cdots\cdots \end{aligned}$$

and the closed-loop matrix $(I - B_1 B_1' M_+ \Delta^{-1})A$ is Hurwitz. □

Now we are ready to present our main results. Let us first make the following assumptions:

$\bar{A}1)$ $\gamma^2 I - M_3 > 0$,

$\bar{A}2)$ The pairs (A, B_1), (A, B_2) are controllable and (C_1, A) is observable,

$\bar{A}3)$ B_1 is full rank,

$\bar{A}4)$ $\bar{Q} \geq 0$ and is defined by

$$\bar{Q} = M_1 + P_2 \bar{M_3}^{-1} P_2' - (P_1' + P_3 \bar{M_3}^{-1} P_2')' \bar{M_2}^{-1} (P_1' + P_3 \bar{M_3}^{-1} P_2') \quad (6.5.13)$$

where the $\bar{M_2}$ and $\bar{M_3}$ are given by

$$\bar{M_3} = \gamma^2 I - M_3, \qquad \bar{M_2} = M_2 + P_3 \bar{M_3}^{-1} P_3'. \qquad (6.5.14)$$

and M_i and P_i are defined by (6.5.6).

Theorem 6.5.2 *Consider the sampled-data system described by (6.5.1). Let $\gamma > 0$ be given. Then there exists a controller $\tilde{k} \in \tilde{K}$ such that (6.5.9) hold if and only if following conditions hold:*

1') The Riccati equation (R1) associated with $(\bar{A}, \bar{B}_1, \bar{B}_2, \bar{Q})$ has a positive definite solution $\bar{M}_+$,

2') $\gamma^2 I - \bar{B}_2{}'\bar{M}_+\bar{B}_2 > 0$.

When these conditions hold, one such stabilizing controller is given by

$$\begin{aligned} \bar{u}(k) &= -\bar{B}_1{}'\bar{M}_+\Delta^{-1}\bar{A}x(k), \qquad (6.5.15)\\ \bar{v}(k) &= -\gamma^{-2}\bar{B}_2{}'\bar{M}_+\Delta^{-1}\bar{A}x(k), \\ \Delta: &= I + (\bar{B}_1\bar{B}_1{}' - \gamma^{-2}\bar{B}_2\bar{B}_2{}')\bar{M}_+. \quad k = 1, 2, 3, \cdots\cdots \end{aligned}$$

where the $\bar{A}, \bar{B}_1, \bar{B}_2$ are defined by,

$$\begin{aligned} \bar{A} &= F + G_1\bar{M}_3{}^{-1}P_2' - (\gamma G_1\bar{M}_3{}^{-1}P_3')\bar{M}_2{}^{-1}(P_1' + P_3\bar{M}_3{}^{-1}P_2') \\ \bar{B}_1 &= (G_1 + \gamma G_2\bar{M}_3{}^{-1}P_3')\bar{M}_2{}^{-1/2} \\ \bar{B}_2 &= \gamma G_2\bar{M}_3{}^{-1/2} \qquad (6.5.16) \end{aligned}$$

Proof: *In order to apply theorem (6.5.1) to this problem, we first need to convert the problem (6.5.8)-(6.5.9) into the standard problem described by (6.5.10)-(6.5.11), then we show that the assumptions of theorem (6.5.1) are satisfied. Since the pairs $(A, B_1)$and(A, B_2) are controllable and (C_1, A) is observable by assumption $\bar{A}2$), hence the discrete system $(F, G_1), (F, G_2)$ are also controllable and (C_1, F) is observable by the assumption of that generic condition of the sampling period T is satisfied.*

Next, we carry out linear transformations. By doing so, we first define a new *variable* $\bar{v}(k)$ *as*

$$\bar{v}(k) = \frac{1}{\gamma}\bar{M_3}^{1/2}[v(k) - \bar{M_3}^{-1}(P_2'x(k) + P_3'u(k))], \quad k = 0, 1, 2, \cdots \tag{6.5.17}$$

The $\bar{M_3}$ *is invertible by assumption* $\bar{A}1$*). The system (6.5.8) and the performance measure (6.5.9) become*

$$x(k+1) = (A + G_2\bar{M_3}^{-1}P_2')x(k) + (G_1 + G_2\bar{M_3}^{-1}P_3')u(k) + \gamma G_2\bar{M_3}^{-1/2}\bar{v}(k)$$

$$\begin{aligned} J_\gamma(\tilde{K}) &= \sum_{k=0}^{\infty} x'(k)(M_1 + P_2\bar{M_3}^{-1}P_2')x(k) + 2u(k)'(P_1' + P_3\bar{M_3}^{-1}P_2')x(k) + \\ & \quad u'(k)(M_2 + P_3\bar{M_3}^{-1}P_3')u(k) - \gamma^2\bar{v}(k)'\bar{v}(k) \end{aligned}$$

Since (C_1, A) *is observable, and* B_1 *is full rank by assumption* $\bar{A}3$*), it is easy to show that the matrix* M_2 *is positive definite. The* $\bar{M_3}^{-1} > 0$ *follows from* $\bar{M_3} > 0$*. This implies that* $P_3\bar{M_3}^{-1}P_3' \geq 0$*. Hence,* $\bar{M_2} = M_2 + P_3\bar{M_3}^{-1}P_3' > 0$ *is positive definite. Define a new variable* $\bar{u}(k)$ *as*

$$\bar{u}(k) = \bar{M_2}^{1/2}[u(k) + \bar{M_2}^{-1}(P_1' + P_3\bar{M_3}^{-1}P_2')]x(k) \tag{6.5.18}$$

and $\bar{Q} = M_1 + P_2\bar{M_3}^{-1}P_2' - (P_1' + P_3\bar{M_3}^{-1}P_2')'\bar{M_2}^{-1}(P_1' + P_3\bar{M_3}^{-1}P_2')$*. After some simplifications, we finally arrive at*

$$x(k+1) \;=\; \bar{A}x(k) + \bar{B}_1\bar{v}(k) + \bar{B}_2\bar{u}(k) \tag{6.5.19}$$

$$J_\gamma(\tilde{K}) = \sum_{k=0}^{\infty} x(k)'\bar{Q}x(k) + \bar{u}(k)'\bar{u}(k) - \gamma^2\bar{v}(k)'\bar{v}(k) \tag{6.5.20}$$

Since the linear transformations do not affect the controllability and observability, it follows that $(\bar{A}, \bar{B}_1)$*and*$(\bar{A}, \bar{B}_2)$ *are controllable.* $\bar{Q}$ *is positive semi-definite by its structure and assumption* $\bar{A}4$*). Since* (C_1, A) *is observable, it is straightforward to show that* $(\bar{A}, \bar{Q}^{1/2})$ *is also observable. Hence, the assumptions* $A1$*) and* $A2$*) of theorem (6.5.1) are satisfied, and the proof is completed by invoking the theorem (6.5.1) to the problem (6.5.19)-(6.5.20).* □

The feedback controller is expressed by in the new variables $\bar{u}$ and $\bar{v}$. By changing back the variables, we get

$$\begin{aligned} u(k) &= -\bar{M}_2^{-1/2}[\bar{B}_1'\bar{M}_+\Delta^{-1}\bar{A} - \bar{M}_2^{-1/2}(P_1' + P_3\bar{M}_3^{-1}P_2')]x(k), \quad (6.5.21) \\ v(k) &= -\bar{M}_3^{-1/2}[\gamma^{-1}\bar{B}_2'\bar{M}_+\Delta^{-1}\bar{A} - \bar{M}_3^{-1/2}P_2']x(k) + \bar{M}_3^{-1}P_3'u(k). \end{aligned}$$

In most practical control applications, only partial state variables are available, we will derive the solution for output-feedback problem in next section.

6.5.3 Output-feedback control

When the full state variables are not available, we need to construct a observer-based feedback controller. The following theorem gives us the standard results. We first assume the following assumptions hold.

A1) The pairs(A, B_1), (A, B_2) are controllable and (C_2, A) is observable.

A2) Q is positive semi-definite.

Let us define two algebraic Riccati equations as follows,

$$\begin{aligned} M &= A'(M^{-1} + B_1B_1' - \gamma^{-2}B_2B_2')^{-1}A + Q \qquad (R1) \\ N &= A(N^{-1} + C_2'C_2 - \gamma^{-2}Q)^{-1}A' + B_2B_2' \qquad (R2) \end{aligned}$$

Theorem 6.5.3 *[27] Consider discrete-time system described by (6.5.10), let $\gamma > 0$ be given. Suppose that the assumptions A1) − A2) hold. Then there exists a controller $\tilde{k} \in \tilde{K}$ such that inequality (6.5.11) hold if and only if*

1) (R1) and (R2) have positive definite solutions M_+ and N_+.

2) $\rho(M_+N_+) < \gamma^2$.

Moreover, when these conditions hold, one such stabilizing feedback controller is given by

$$\tilde{x}(k+1) = A\tilde{x}(k) + A({N_+}^{-1} + C_2'C_2 - \gamma^{-2}Q)^{-1}[\gamma^{-2}Q\tilde{x}(k)$$

$$+C_2'(y(k) - C_2\tilde{x}(k))] + B_1u(k) \tag{6.5.22}$$

$$u(k) \;=\; -B_1'({M_+}^{-1} + B_1B_1' - \gamma^{-2}B_2B_2')^{-1}A(I - \gamma^2N_+M_+)^{-1}\tilde{x}(k),$$

where the $\tilde{x}$ is an observer. □

Now we solve the problem (6.5.8)-(6.5.9) by applying theorem (6.5.3). Let us first make the following assumptions:

$\bar{A}1)$ $\gamma^2 I - M_3 > 0$,

$\bar{A}2)$ The pairs (A, B_1) is controllable and (C_1, A) is observable, so are (A, B_2) and (C_2, A) respectively,

$\bar{A}3)$ B_1 is full rank,

$\bar{A}4)$ $\bar{Q} \geq 0$ and is defined by

$$\bar{Q} = M_1 + P_2\bar{M_3}^{-1}P_2' - (P_1' + P_3\bar{M_3}^{-1}P_2')'\bar{M_2}^{-1}(P_1' + P_3\bar{M_3}^{-1}P_2') \tag{6.5.23}$$

where the $\bar{M_2}$ and $\bar{M_3}$ are given by

$$\bar{M_3} = \gamma^2 I - M_3, \qquad \bar{M_2} = M_2 + P_3\bar{M_3}^{-1}P_3'. \tag{6.5.24}$$

and M_i and P_i are defined by (6.5.6).

Theorem 6.5.4 *Consider the sampled-data system described by (6.5.1). Let $\gamma > 0$ be given. Then there exists a controller $\tilde{k} \in \tilde{K}$ such that (6.5.11) hold if and only if following conditions hold:*

1') The two Riccati equations (R1) and (R2) associated with $(\bar{A}, \bar{B}_1, \bar{B}_2, \bar{C}_2, \bar{Q})$ have positive definite solutions $\bar{M}$ and $\bar{N}$.

2') $\rho(\bar{M}\bar{N}) < \gamma^2$.

When these conditions hold, one such stabilizing controller is given by

$$\tilde{x}(k+1) \;=\; \bar{A}\tilde{x}(k) + \bar{A}(\bar{N}^{-1} + \bar{C_2}'\bar{C_2} - \gamma^{-2}\bar{Q})^{-1}[\gamma^{-2}\bar{Q}\tilde{x}(k)$$

$$
\begin{aligned}
& \quad + \bar{C_2}'(y(k) - \bar{C_2}\tilde{x}(k))] + \bar{B}_1\bar{u}(k) && (6.5.25) \\
\bar{u}(k) &= -\bar{B_1}'(\bar{M}^{-1} + \bar{B}_1\bar{B_1}' - \gamma^{-2}\bar{B}_2\bar{B_2}')^{-1}\bar{A}(I - \gamma^2\bar{N}\bar{M})^{-1}\tilde{x}(k)
\end{aligned}
$$

where the $(\bar{A}, \bar{B}_1, \bar{B}_2, \bar{C}_2)$ *are defined by,*

$$
\begin{aligned}
\bar{A} &= F + G_2\bar{M_3}^{-1}P_2' - (\gamma G_2\bar{M_3}^{-1}P_3')\bar{M_2}^{-1}(P_1' + P_3\bar{M_3}^{-1}P_2') \\
\bar{B}_1 &= (G_1 + \gamma G_2\bar{M_3}^{-1}P_3')\bar{M_2}^{-1/2} \\
\bar{B}_2 &= \gamma G_2\bar{M_3}^{-1/2} \\
\bar{C}_2 &= C_2. && (6.5.26)
\end{aligned}
$$

Proof: *The proof is analogous to the proof of theorem (6.5.2) and is omitted here.* □

The output-feedback controller (6.5.25) is expressed by in the new variable $\bar{u}(k)$, and the actual control $u(k)$ can be obtained by changing back the variable (6.5.18)

$$
\begin{aligned}
\tilde{x}(k+1) &= \bar{A}\tilde{x}(k) + \bar{A}(\bar{N}^{-1} + \bar{C_2}'\bar{C}_2 - \gamma^{-2}\bar{Q})^{-1}[\gamma^{-2}\bar{Q}\tilde{x}(k) + \bar{C_2}'(y(k) \\
&\quad - \bar{C}_2\tilde{x}(k))] + \bar{B}_1\bar{M_2}^{-1/2}(P_1' + P_3\bar{M_3}^{-1}P_2')\tilde{x}(k) + \bar{B}_1\bar{M_2}^{1/2}u(k) \\
u(k) &= -\bar{M_2}^{-1/2}\bar{B_1}'(\bar{M}^{-1} + \bar{B}_1\bar{B_1}' - \gamma^{-2}\bar{B}_2\bar{B_2}')^{-1}\bar{A}(I - \gamma^2\bar{N}\bar{M})^{-1}\tilde{x}(k) \\
&\quad - \bar{M_2}^{-1}(P_1' + P_3\bar{M_3}^{-1}P_2')\tilde{x}(k). && (6.5.27)
\end{aligned}
$$

To recap, let us list the steps in the design procedures for the output feedback problem:

step 1: Calculate the matrices M_i and P_i according to (6.5.6),

step 2: Give $\gamma > 0$, check if the conditions $\bar{A}1$) and $\bar{A}4$) are satisfied, if not, increase γ until these conditions are satisfied,

step 3: Calculate matrices $\bar{A}, \bar{B}_1, \bar{B}_2, \bar{C}_2$ according to (6.5.26), step 4: Solve two algebraic Riccati equations $(R1), (R2)$ for $\bar{M}, \bar{N}$ associated with $(\bar{A}, \bar{U}_1, \bar{B}_2, \bar{C}_2, \bar{Q})$, check if conditions $1'$ *and* $2'$ are satisfied, if not, increase γ, go back to step 3.

The state-feedback is a special case, the design procedure is similar to output feedback.

6.5.4 Example

Our motivation for developing the H_∞ optimal control for impulsive disturbances is from various impact control problems. Let us consider the following impact control example shown in Figure 6.2. The system represents a flexible body with mass m_2, and the k, c represents its stiffness and damping coefficient respectively, and u is the control input. The system is subjected to a series of impulsive disturbances (impact forces). When the impact occurs, a significant portion of the kinetic energy of the impactor will be transferred to the flexible body, causing vibrations. Hence, a reasonable control objective is to minimize the energy transferred to the system by the impact forces. This control problem can be viewed as a disturbance attenuation problem. Suppose the displacement of the mass is available. The state space representation of the system with impulsive disturbances is given by,

$$
\begin{aligned}
\dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ -k/m_2 & -c/m_2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1/m_2 \end{bmatrix} u(t) \\
&\quad + \sum_{k=0}^{\infty} \begin{bmatrix} 0 \\ b_2/m_2 \end{bmatrix} v(k)\delta(t-kT) \\
&= Ax(t) + B_1 u(t) + \sum_{k=0}^{\infty} B_2 v(k)\delta(t-kT)
\end{aligned}
$$

where the $W(T)$ is given by

$$W(T) = W_c(T) - \Phi'_{A_c}(T,0)Q_{tmp}(T)\Phi_{A_c}(T,0). \tag{6.3.53}$$

Proposition 6.3.1 *[70] The Lyapunov equations (6.3.48)-(6.3.49) admit a unique positive periodic solution if and only if the algebraic Lyapunov equation (6.3.52) admits a positive definite solution P_0.* □

Proposition 6.3.2 *[68] Suppose that the $A_c(t)$ is asymptotically stable. Then the necessary condition that algebraic Lyapunov equation (6.3.52) admits a positive definite solution is that $W(T)$ defined in (6.3.53) is positive definite.* □

6.4 Proofs of Theorem 6.2.1- 6.2.3

Now, we are ready to prove the main results of this chapter. First we prove the Theorem (6.2.2), then we prove the Theorem (6.2.3). Finally, we consider the Theorem (6.2.1).

6.4.1 Proof of theorem 6.2.2

Recall the hybrid system defined in (6.2.13)-(6.2.14),

$$\begin{aligned} \dot{x}(t) &= [A - B_1 B_1' P(t)]x(t), \\ &= A_c(t), \quad t \neq iT, \qquad (6.4.1) \\ x(t^+) &= [I - \gamma^{-2} B_2 B_2' P(t)]x(t), \\ &= F_c(t), \quad t = iT. \qquad (6.4.2) \end{aligned}$$

Denote the state transition matrix of the hybrid system (6.4.1) -(6.4.2) by $\Phi_c(t,\tau)$. It has the following properties,

$$\frac{\partial \Phi_c(t,\tau)}{\partial t} = A_c(t)\Phi_c(t,\tau), \tag{6.4.3}$$

$$\Phi_c(\tau^+, \tau) = F_c(\tau). \tag{6.4.4}$$

Theorem 6.4.1 *Let $P(t)$ in equation(6.3.24) be substituted into equation (6.4.1) and (6.4.2). Also let $\lambda_i, i = 1, 2, \cdots\cdots n$ be n eigenvalues of $\Phi(T, 0)$ used to form Λ_1 in equation(6.3.12), then $\Phi_c(t, 0)$ is given by*

$$\Phi_c(t, 0) = Y(t)Y(0)^{-1} \tag{6.4.5}$$

and the eigenvalues of $\Phi_c(T, 0)$ are $\lambda_i, i = 1, 2, \cdots\cdots n$.

Proof: *Using equation (6.3.15), we have*

$$\begin{aligned}\dot{\Phi}_c(t, 0) &= \dot{Y}(t)Y(0)^{-1} \\ &= [AY(t) - B_1 B_1' X(t)]Y(0)^{-1} \\ &= [A - B_1 B_1' P(t)]\Phi_c(t, 0) \\ &= A_c(t)\Phi_c(t, 0).\end{aligned} \tag{6.4.6}$$

$$\begin{aligned}\Phi_c(0^+, 0) &= Y(0^+)Y(0)^{-1} \\ &= [Y(0) - \gamma^{-2} B_2 B_2' X(0)]Y(0)^{-1} \\ &= [I - \gamma^{-2} B_2 B_2' P(0)] \\ &= F_c(0).\end{aligned} \tag{6.4.7}$$

From equation (6.3.25),

$$\begin{aligned}Y(T) &= Y(0)\Lambda_1 \\ \Phi_c(T, 0) &= Y(T)Y(0)^{-1} = Y(0)\Lambda_1 Y(0)^{-1}.\end{aligned} \tag{6.4.8}$$

Thus, eigenvalues of $\Phi_c(T, 0)$ are $\lambda_i, i = 1, 2, \cdots\cdots n$. □

The following Lemmas relate the positive definite periodic solution of the coupled Riccati equations and the structure properties of systems (controllability and observability).

Lemma 6.4.1 *Assume (C_1, A) is observable and no eigenvalue of $\Phi(T, 0)$ lies on the unit circle. Then there exists a positive definite periodic solution $P(t)$ to equations (6.2.1) - (6.2.2) such that the hybrid system (6.4.1)-(6.4.2) is asymptotically stable if (A, B_1) is controllable.*

Proof*: Since only solutions $P(t)$ of equations (6.2.1) and (6.2.2) that lead to asymptotically stable hybrid systems are under consideration, n eigenvalues in Λ_1 must be chosen so that $|\lambda_i| < 1$ for $i = 1, 2, \cdots\cdots, n$ according to Theorem (6.4.1). Such a choice is possible under the current hypothesis and Theorem (6.3.1). From Theorem (6.3.3), the resulting $P(t)$ clearly satisfies $P(t) > 0$. The non-existence of $P(t) > 0$ then arises only when the condition that $|Y(t)| \neq 0$ for all $t \in [0, T]$ is violated. In the sequel, we will show that (A, B_1)-controllability implies $|Y(t)| \neq 0$ for all $t \in [0, T]$. Suppose to the contrary that for some $t = s, s \in [0, T], Y(t)$ is singular. Let $b \neq 0$ be an n-dimensional vector belonging to the null space of $Y(s)$, i,e.*

$$Y(s)b \;=\; 0 \tag{6.4.9}$$

Also for $k = 0, 1, \cdots\cdots$. Let $\alpha(k)$ and $\beta(k)$ be n-dimensional vectors such that

$$\begin{bmatrix} \alpha(k) \\ \beta(k) \end{bmatrix} = \begin{bmatrix} Y(s)\Lambda_1^k b \\ X(s)\Lambda_1^k b \end{bmatrix} \tag{6.4.10}$$

Define a vector $u(k)$ as

$$u(k) \;=\; \begin{bmatrix} \alpha(k) \\ \beta(k) \end{bmatrix} \tag{6.4.11}$$

From equation (6.3.15), we have

$$u(k+1) \;=\; \Phi(S+T, S)u(k) \tag{6.4.12}$$

Let us define a scalar $W(k)$ *as*

$$W(k) \;=\; \beta^*(k)\alpha(k) - \beta^*(k+1)\alpha(k+1) \tag{6.4.13}$$

Since $\alpha(0) = Y(s)b = 0$, *and* $\alpha(k), \beta(k) \to 0$ *as* $k \to \infty$,

$$\sum_{k=0}^{\infty} W(k) \;=\; 0 \tag{6.4.14}$$

On the other hand, from equation (6.3.39)

$$\begin{aligned} W(k) &= u^*(k+1)Vu(k+1) - u^*(k)Vu(k) \\ &= u^*(k)[\Phi(s+T,s)'V\Phi(s+T,s) - V]u(k) \\ &= u^*(k)M(s+T,s)u(k) \end{aligned} \tag{6.4.15}$$

Since $M(s+T,s) \geq 0$ *from equations (6.3.43)-(6.3.45), hence yielding* $W(k) \geq 0$. *Equation (6.4.14) then implies that* $W(k) = 0$ *for all* $k = 0, 1, 2, \cdots\cdots$. *Therefore,* $M(s+T,s)$ *is not positive definite, hence* (A, B_1) *is not controllable, a contradiction.* □

Lemma 6.4.2 *Assume that no eigenvalue of* $\Phi(T,0)$ *lies on the unit circle and* (A, B_1) *is controllable. Then there exists a unique positive definite periodic solution* $P(t)$ *to equations (6.2.1)-(6.2.2) if* (C_1, A) *is observable.*

Proof: *Under the assumptions, the existence of at least one solution* $P(t) \geq 0$ *is guaranteed by Lemma (6.4.1). Such a solution* $P(t)$ *is obviously associated with the asymptotically stable hybrid system (6.4.1) and (6.4.2). In the following, we shall show that the additional condition* (C_1, A)*-observability, necessarily implies the asymptotic stability of the hybrid system (6.4.1) and (6.4.2). The uniqueness of the solution can be established by this way. Assume to the contrary that the hybrid system (6.4.1) and (6.4.2) is unstable. It then follows from Theorem*

(6.4.1) that Λ_1 in equation (6.3.12) has at least one eigenvalue λ of $\Phi(T,0)$ such that $|\lambda| > 1$. Since the order of eigenvalues in Λ_1 is irrelevant to solution, it can be assumed that $\lambda_1 = \lambda$ without loss of generality. Denote by $\hat{P}(t)$ the solution of equations (6.2.1) and (6.2.2) corresponding to this set of eigenvalues.

Let x_1 and y_1 be n-dimensional vectors that constitute the eigvector corresponding to λ_1, namely,

$$\Phi(T,0)\begin{bmatrix} y_1 \\ x_1 \end{bmatrix} = \lambda \begin{bmatrix} y_1 \\ x_1 \end{bmatrix} \tag{6.4.16}$$

Also for $k = 0, 1, 2 \cdots\cdots$. Let $\gamma(k)$ and $\delta(k)$ be n-dimensional vectors such that

$$\begin{bmatrix} \gamma(k) \\ \delta(k) \end{bmatrix} = \begin{bmatrix} y_1 \\ x_1 \end{bmatrix} \lambda_1^{-k} \tag{6.4.17}$$

Furthermore, Let $v(k) = \begin{bmatrix} \gamma(k) \\ \delta(k) \end{bmatrix}$, then from equations (6.4.16) and (6.4.17), we obtain

$$v(k) = \Phi(T,0)v(k+1) \tag{6.4.18}$$

Define a scalar $\hat{w}(k)$ by

$$\hat{w}(k) = \delta^*(k+1)\gamma(k+1) - \delta^*(k)\gamma(k) \tag{6.4.19}$$

since $\gamma(k), \delta(k) \to 0$ as $k \to \infty$, we have

$$\sum_{k=0}^{\infty} \hat{w}(k) = -x_1^* y_1. \tag{6.4.20}$$

On the other hand, similar steps used to derive equation (6.3.38) lead to

$$\hat{w}(k) \geq 0, \quad k = 0, 1, 2, \cdots\cdots \tag{6.4.21}$$

We consider two cases:

(1) $t \neq iT$, differentiate $V(t)$ along the trajectory of system (6.2.12),

$$\begin{aligned} \dot{V}(t) &= \dot{x}(t)'P(t)x(t) + x(t)'\dot{P}(t)x(t) + x(t)'P(t)\dot{x}(t) \\ &= x(t)'(\dot{P}(t) + A_c(t)'P(t) + P(t)A_c(t))x(t) \\ &= -x(t)'H(t)'H(t)x(t) \qquad (6.4.24) \\ &< 0. \end{aligned}$$

Hence inequality holds for all $t_1, t_2 \neq iT$.

(2) $t = iT, i = 0, 1, 2, \cdots\cdots$. In the rest of proof, we use i to replace iT for the sake of brevity. Since $\dot{V}(t) = -x(t)'H(t)'H(t)x(t) < 0$,

$$\begin{aligned} V(i) &= V((i-1)^+) - \int_{i-1}^{i} x(\tau)'H(\tau)'H(\tau)x(\tau)d\tau \\ V(i^+) &= x(i)'P(i^+)x(i) \\ &= x(i)'P(i)x(i) + x(i)'Q_{tmp}(i)x(i) \\ &= V(i) + x(i)'Q_{tmp}(i)x(i). \qquad (6.4.25) \end{aligned}$$

Hence,

$$V(i^+) - V((i-1)^+) = x(i)'Q_{tmp}(i)x(i) - \int_{i-1}^{i} x(\tau)'H(\tau)'H(\tau)x(\tau)d\tau. \qquad (6.4.26)$$

From equation (6.2.12), we can explicitly solve $x(t)$ starting at arbitrary $x(i-1)$,

$$x(t) = \Phi_{A_c}(t, i-1)x(i-1), \ \forall t \in [(i-1), \ i]. \qquad (6.4.27)$$

Replace $x(t)$ in equation (6.4.26) by (6.4.27),

$$\begin{aligned} & V(i^+) - V((i-1)^+) \\ = \ & x(i-1)'[\Phi'_{A_c}(i, i-1)Q_{tmp}(i)\Phi_{A_c}(i, i-1) \\ & - \int_{i-1}^{i} \Phi'_{A_c}(\tau, i-1)H(\tau)'H(\tau)\Phi_{A_c}(\tau, i-1)d\tau]x(i-1). \qquad (6.4.28) \end{aligned}$$

Since $A_c(t), H(t)'H(t)$ and $Q_{tmp}(i)$ are periodic, $\Phi_{A_c}(i, i-1) = \Phi_{A_c}(T, 0)$, and $Q_{tmp}(T) = Q_{tmp}(T)$ for all $i = 0, 1, 2, \cdots\cdots$. The equation (6.4.28) can be further simplified as

$$
\begin{aligned}
V(i^+) - V((i-1)^+) &= x(i-1)'[\Phi'_{A_c}(T,0)Q_{tmp}(T)\Phi_{A_c}(T,0) \\
&\quad - \int_0^T \Phi'_{A_c}(\tau,0)H(\tau)'H(\tau)\Phi_{A_c}(\tau,0)d\tau]x(i-1) \\
&< 0, \ \textit{by inequality } (6.2.15). \qquad (6.4.29)
\end{aligned}
$$

(Necessity):

Since the closed-loop system $A_c(t)$ is asymptotically stable and $P(t)$ is positive definite, the inequality (6.2.15) follows immediately from Propositions 4.3.1 and 4.3.2. □

6.4.3 Proof of theorem 6.2.1

We use the standard completion of squares method while accounting for the possible jumps. Differentiating $x'(t)P(t)x(t)$ along the trajectory of system (6.1.3), we obtain

$$
\begin{aligned}
\frac{d}{dt}x'Px &= \dot{x}'(t)P(t)x(t) + x'(t)\dot{P}(t)x(t) + x'(t)P(t)\dot{x}(t) \\
&= x'(t)[A'P(t) + \dot{P}(t) + P(t)A]x(t) + 2 < u, B_1'Px > . \qquad (6.4.30)
\end{aligned}
$$

Replace term $[A'P(t) + \dot{P}(t) + P(t)A]$ by equation (6.2.1), and use assumption *ii*),

$$
\begin{aligned}
\frac{d}{dt}x'Px &= -x'(t)C_1'C_1x(t) + x'(t)P(t)B_1B_1'P(t)x(t) + 2 < u, B_1'Px > \\
&= -||z||^2 + ||u + B_1'Px||^2. \qquad (6.4.31)
\end{aligned}
$$

Integrating equation (6.4.31) from $[iT, (i+1)T]$ for some i, the right-hand side (RHS) of equation (6.4.31) becomes

$$RHS = -||z||^2_{[iT,(i+1)T]} + ||u + B_1'Px||^2_{[iT,(i+1)T]}. \tag{6.4.32}$$

where the left-hand side (LHS) of equation (6.4.30) becomes

$$\begin{aligned} LHS &= x'(iT^+)P(iT^+)x(iT^+) - x'(iT)P(iT)x(iT) \\ &= [x(iT) + B_2v(i)]'P(iT^+)[x(iT) + B_2v(i)] \\ &\quad -x'(iT)P(iT)x(iT) \\ &= v'(i)B_2'P(iT^+)B_2v(i) + 2 < v(i), B_2'P(iT^+)x(iT) > \\ &\quad +x(iT)[P(iT^+) - P(iT)]x(iT) \\ &= -\gamma^2||v(i)||^2 + ||(\gamma^2 I + B_2P(iT^+)B_2)^{1/2} \\ &\quad \times[v(i) + (\gamma^2 I + B_2'P(iT^+)B_2)^{-1}B_2'P(iT^+)x(iT)]||^2. \end{aligned} \tag{6.4.33}$$

Since the closed-loop system is stable, hence $x(t) \to 0$ as $t \to \infty$. Assume the initial condition $x(0) = 0$. Integrate above equation from 0 to ∞, we obtain the following equation,

$$\begin{aligned} & -\gamma^2||v||^2_{l_2} + \sum_{i=0}^{\infty} ||(\gamma^2 I + B_2P(iT^+)B_2)^{1/2} \\ & \times[v(i) + (\gamma^2 I + B_2'P(iT^+)B_2)^{-1}B_2'P(iT^+)x(iT)]||^2 \\ = & -||z||^2_{L_2} + ||u + B_1'Px||^2_{L_2}. \end{aligned} \tag{6.4.34}$$

Taking a feedback controller $u(t) = -B_1'P(t)x(t)$ and since the summation term is always nonnegative, we have the following inequality

$$||z||^2_{L_2} < \gamma^2||v||^2_{l_2}. \tag{6.4.35}$$

It is easy to see from (6.4.34) that the worst case impulsive disturbances is

$$v(i) = -(\gamma^2 I + B_2'P(kT^+)B_2)^{-1}B_2'P(iT^+)x(iT). \tag{6.4.36}$$

□

The procedure to compute such a feedback controller is outlined as follows:

1). Given $\gamma > 0$, check if the matrix $\Phi(T, 0)$ has no eigenvalues on the unit circle. If not, increase γ until the condition is satisfied.

2). Compute the periodic solution $P(t)$ according to equations (6.3.9), (6.3.13) and (6.3.24).

3). Check if the inequality (6.2.15) is satisfied. If not, go back to step 1) and increase γ until this condition is satisfied.

4). Compute the feedback controller $u(t)$ according to equation (6.2.3).

6.5 Implementation By A Digital Controller

In this section we will address the optimal H_∞ control for impulsive disturbances by using a sampled-data controller. This consideration is partially motivated by the need for digital implementation of any controller. In chapter 4, we have deduced that the (sub) optimal solution and the controller is time-varying in general. When using a sampled-data controller we are able to convert the control problem into a discrete-time H_∞ control problem under a proper sampling condition, hence the control structure is simpler than the continuous one.

6.5.1 A sampled-data controller

To facilitate exposition, the problem we consider here is a simple case which captures all the essential features of the general problem. The linear system is

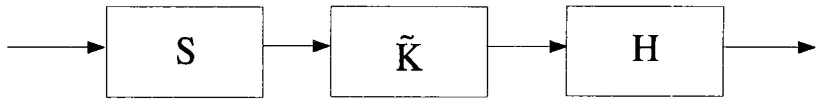

Figure 6.1: A sampled-data controller

given by,

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + B_1 u(t) + \sum_{k=0}^{\infty} B_2 v \delta(t - kT_1) \\
z(t) &= C_1 x(t) + D_1 u(t) \\
y(t) &= C_2 x(t)
\end{aligned}
\tag{6.5.1}
$$

We shall study the problem of controlling (6.5.1) using a sampled-data controller $\tilde{k}$, and $\tilde{k}$ has the form of Figure 6.1, where the S is sampling operator with sampling period T_2, $\tilde{k}$ is a discrete-time LTI system. H is the hold operator (zero-order) [73, 74, 75]. Suppose that the sampling period is T_2. The structure of this system with sampled-data controller is similar to so called multirate sampling systems. [76] has shown that given two sampling periods T_1 and T_2, if the ratio of two periods is a rational number $T_1/T_2 = N_1/N_2$, where N_1 and N_2 have no common factor. Then there exists a smallest integer N and a real number T such that $T_1 = TN_1/N, T_2 = TN_2/N$, and $N = N_1 N_2$. If the samples are synchronized, it follows that the control and measurement signals will be constant over sampling periods of length T/N. Sampling with that period gives a discrete-time system that is periodic with period T. The system can then be described as time-invariant discrete-time system if the values of system variables are considered only at integer multiples of T. The ordinary discrete-time theory and state-space form can then be applied. In our problem, we assume that $T_1/T_2 = n$, where $n \geq 1$ is an integer, hence the above assumptions are satisfied. For simplicity, we assume $T_1 = T_2 = T$ in this chapter. The state and observation

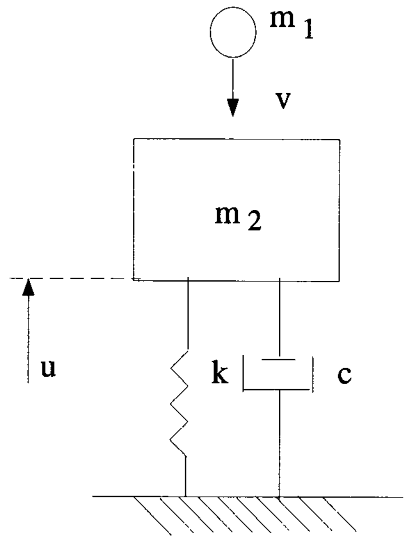

Figure 6.2: System diagram

$$\begin{aligned} z(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x(t) &= C_1 x(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) &= C_2 x(t) \end{aligned} \quad (6.5.28)$$

Let us apply the state-space formula to solve this control problem. We consider the output-feedback case. Without loss of generality, we choose k, c, m_2, and b_1 such that the matrices are given as,

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, B_1 = B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ C_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}, C_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}. (6.5.29)$$

It is easy to check that assumptions $\bar{A}2$) and $\bar{A}3$) are satisfied. Let us follow the design procedures.

Step 1: Let sampling period $T = 1$, the M_i and P_i are given by,

$$M_1 = \begin{bmatrix} 0.177113 & -0.0540768 \\ -0.0540768 & 0.147066 \end{bmatrix}, \quad P_1 = \begin{bmatrix} -0.0885563 \\ 0.0270384 \end{bmatrix},$$

$$M_2 = 0.127396, \quad P_2 = \begin{bmatrix} -0.0540768 \\ 0.147066 \end{bmatrix},$$

$$M_3 = 0.147066, \quad P_3 = 0.0270384. \tag{6.5.30}$$

Step 2: Let $\gamma = 1$, $\gamma^2 I - M_3 = 0.852934 > 0$, and $\bar{Q}$ is given by

$$\bar{Q} = \begin{bmatrix} 0.117005 & -0.0410887 \\ -0.0410887 & 0.164588 \end{bmatrix} > 0,$$

Step 3: Calculate $\bar{A}, \bar{B}_1, \bar{B}_2, \bar{C}_2$ according to (6.5.16).

Step 4: Solve two algebraic Riccati equations $(R1)$ and $(R2)$, the solutions are

$$\bar{M} = \begin{bmatrix} 0.2840 & -0.0086 \\ -0.0086 & 0.1789 \end{bmatrix}, \quad \bar{N} = \begin{bmatrix} 0.0405 & -0.0007 \\ -0.0007 & 1.1872 \end{bmatrix} \tag{6.5.31}$$

It is easy to show that both $\bar{M}$ and $\bar{N}$ are positive definite and $\rho(\bar{N}\bar{M}) < 1$. Hence, the discrete-time observer $\tilde{x}(k)$ and suboptimal controller $u(k)$ are given by,

$$\begin{aligned} \tilde{x}(k+1) &= \begin{bmatrix} 0.456713 & 0.302102 \\ -0.696961 & 0.142489 \end{bmatrix} \tilde{x}(k) + \begin{bmatrix} 0.145365 \\ 0.0319954 \end{bmatrix} \\ &\quad (y(k) - \begin{bmatrix} 1 & 1 \end{bmatrix} \tilde{x}(k)) + \begin{bmatrix} 0.199788 \\ 0.264245 \end{bmatrix} u(k), \\ u(k) &= \begin{bmatrix} 0.684444 & -0.382274 \end{bmatrix} \tilde{x}(t), \quad k = 0, 1, 2, \cdots \end{aligned} \tag{6.5.32}$$

6.6 Conclusion

In this chapter, we described a complete state-space solution to the (sub)optimal control problem of a class of linear systems subject to impulsive disturbances. The state feedback controller can be computed in terms of the unique positive definite periodic solution of a coupled Riccati equations. Finally, we implemented this control algorithm by using a sampled-data controller, we showed that the structure of the sampled-data controller could be simpler that continuous version under a proper choice of the sampling period.

Chapter 7

CONCLUSIONS AND FUTURE WORK

In this dissertation, we have derived a general nonlinear model for impact dynamics on flexible structures based on Hertz law of impact. A mathematical analysis of this model has been carried out to prove the existence of a unique solution. A numerical method has been developed based on the contraction mapping principle. By taking advantage of the fact that the impact period is usually very short we have developed a series of approximation solutions. The first order approximation yields a special function which can be used for analytical and computational purposes. The second order approximation leads to a two-parameter family of ordinary differential equations of which the solutions contain universal features of impact problems. Simulation results of various examples have demonstrated the usefulness of the developed numerical method and approximate methods. We believe this nonlinear model and its approximation solutions are useful in the design and analysis of various engineering problems involving impact. This methodology has already been used for the analysis and design of a smart motor and a walking robot at the University of Maryland at college park. Besides the applications discussed in the introduction to this dissertation, the potential areas could be found in biomechanical engineering such as the need to study of human jumping, foot landing etc.

We have systematically investigated the control of impact forces by using a nonlinear model for impact developed in the first part of this dissertation, where the impact forces are treated as disturbances to the system. A nonlinear optimal control problem is formulated under the assumption that the information of impact forces is known a priori. Optimal control strategies were derived via the use of dynamic game theory. A more interesting problem is to derive optimal solutions where sensors information on impacting velocities is not available a priori. This requirement leads to a nonlinear H_∞ control problem which needs further investigation. Again, by taking advantage of the fact that impact period is very short in general we have developed a series of approximate solutions for the nonlinear optimal control problem. We have shown that the higher order terms are negligible in some cases, hence the approximation leads to a linear problem. A linear H_∞ control problem is formulated and a state-space solution has been obtained. The solution is naturally associated with the existence of a stabilizing periodic solution of a coupled Riccati equations. Hamiltonian theory is employed to analyze the coupled Riccati equations. Finally, we investigate the digital implementation of this control algorithm by using a sampled-data controller. We show that under a certain sampling condition, the controller structure could become simpler than the continuous one. These control methods are proposed and analyzed on the theoretical base. It would be more interesting and challenging to implement and test these algorithms on some real world problems. One application is force and constrained motion control of a flexible robot. An experimental apparatus in the ISL laboratory at the University of

Maryland could be a good test bed. Another application is to investigate the active control of the suspension system of a vehicle [16, 15].

Appendix A

PROOFS OF CHAPTER 3

A .1 A simply supported beam

For the free vibration, the deflection of a beam is governed by the PDE,

$$\rho\frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial x^4} = 0 \qquad 0 < x < l, \tag{A.1}$$

and boundary conditions for the simply supported case are

$$\begin{aligned} w(0,t) &= w''(0,t) = 0, \\ w(l,t) &= w''(l,t) = 0. \end{aligned} \tag{A.2}$$

The corresponding eigenvalue problem can be written as

$$W'''' = \beta W(x), \qquad (\frac{\partial^4 w}{\partial x^4} = W'''') \qquad 0 < x < l, \tag{A.3}$$

where β is an eigenvalue and the $W(x) \in L^2[0,l]$ is the corresponding eigenfunction. The boundary conditions are

$$\begin{aligned} W(0) &= W''(0) = 0, \\ W(l) &= W''(l) = 0. \end{aligned} \tag{A.4}$$

The general solution can be expressed as

$$W(x) = c_1 sin\beta x + c_2 cos\beta x + c_3 sinh\beta x + c_4 cosh\beta x, \tag{A.5}$$

where the c_1 to c_4 are constants to be determined by boundary conditions (A.4). In order to satisfy boundary conditions (A.4) we get,

$$sin\beta l \;=\; 0. \tag{A.6}$$

The solution of equation(A.6) is $\beta_k l \;=\; k\pi, k = 1, 2, \cdots\cdots\infty$

The normal eigenfunctions are

$$W_k(x) \;=\; \sqrt{2} sin\beta_k x \qquad k = 1, 2, \cdots\cdots\infty \tag{A.7}$$

Lemma A .1 $\forall x, \zeta \in [0, l]$ *and* $\forall t > 0$, *the Green's function is uniformly bounded.*

Proof: *It is easily checked that (A.3), (A.4) is self -adjoint. Hence, its Green's function can be expressed as*

$$G(x, \zeta; t) \;=\; \sum_{k=1}^{\infty} W_k(x) W_k(\zeta) \frac{sinw_k t}{w_k} H(t). \tag{A.8}$$

Since,

$$\begin{aligned}
W_k(x) &= \sqrt{2} sin\beta_k x \qquad \forall x \in [0, l] \\
|W_k(x)W_k(\zeta)| &= 2|sin\beta_k x sin\beta_k \zeta| \leq 2 \\
|G(x, \zeta; t)| &= |\sum_{k=1}^{\infty} W_k(x) W_k(\zeta) \frac{sinw_k t}{w_k} H(t)| \\
&\leq \sum_{k=1}^{\infty} |W_k(x) W_k(\zeta)| |\frac{sinw_k t}{w_k} H(t)| \\
&\leq \sum_{k=1}^{\infty} 2 \frac{|sinw_k t|}{w_k} \\
&\leq 2 \sum_{k=1}^{\infty} \frac{1}{w_k} \qquad (w_k^2 = \beta_k^4 \rho / EI)
\end{aligned}$$

converges, $\exists M > 0$*, such that* $G(x, \zeta; t) \leq M$ *,* $\qquad \forall x, \zeta \in [0, l]$.

A .2 A cantilevered beam

The PDE is same as in equation (A.1), and the boundary conditions for the cantilevered beam are

$$\begin{aligned} w(0,t) &= w^{'}(0,t) = 0, \\ w(l,t)^{''} &= w^{'''}(l,t) = 0. \end{aligned} \tag{A.9}$$

The approach to solve the eigenvalue problem is the same as in A.1. Again, we write the general solution and plug in the boundary conditions (A.9) into this equation. After simplification, we get the beam characteristic equation,

$$cos\beta l cosh\beta l = -1. \tag{A.10}$$

The orthonormal eigenfunctions are determined by the following equations,

$$\begin{aligned} W_k(x) &= \frac{1}{A_k}\bar{W}_(x) \qquad\qquad (A.11) \\ \text{where} \quad \bar{W}_k(x) &= \frac{cosh\beta_k x - cos\beta_k x}{cosh\beta_k l + cos\beta_k l} - \frac{sinh\beta_k x - sin\beta_k x}{sinh\beta_k l + sin\beta_k l} \\ \text{and} \quad A_k^2 &= \int_0^l \bar{W}_k^2(x)dx = \frac{cos^2\beta_k l}{sin^4\beta_k l} \qquad k = 1, 2, \cdots\cdots \infty. \end{aligned}$$

Lemma A .2 *The orthonormal eigenfunctions* $\{W_k(x)\}_{k=1}^{\infty}$ *are uniformly bounded.*

Proof*: There are infinitely many solutions to the characteristic equation (A.10),* $0 < \beta_1 l < \beta_2 l < \cdots\cdots < \infty$*, where* $\beta_1 l = 4.73$.

Note that $cosh\beta_k l > 0$*,* $cosh\beta_k x > 0$*,* $sinh\beta_k l > 0$*,* $sinh\beta_k x > 0$*.* $\forall x \in [0, l], \forall k$.

$$\bar{W}_k(x) = \frac{cosh\beta_k x - cos\beta_k x}{cosh\beta_k l + cos\beta_k l} - \frac{sinh\beta_k x - sin\beta_k x}{sinh\beta_k l + sin\beta_k l}$$

$$\begin{aligned} &= \frac{C_k(x) - D_k(x)}{(cosh\beta_k l + cos\beta_k l)(sinh\beta_k l + sin\beta_k l)} \\ where \quad C_k(x) &= (cosh\beta_k x - cos\beta_k x)(sinh\beta_k l + sin\beta_k l); \\ D_k(x) &= (sinh\beta_k x - sin\beta_k x)(cosh\beta_k l + cos\beta_k l). \end{aligned}$$

Now, we carry out the simplifications:

$$\begin{aligned} C_k(x) - D_k(x) &= cosh\beta_k x sinh\beta_k l - sinh\beta_k x cosh\beta_k l - cos\beta_k x sinh\beta_k l \\ &\quad + cosh\beta_k x sin\beta_k l + sin\beta_k x cosh\beta_k l - sinh\beta_k x cos\beta_k l \\ &\quad + sin\beta_k x cos\beta_k l - cos\beta_k x sin\beta_k l \\ &= sinh\beta_k(l-x) + sin\beta_k(x-l) - cos\beta_k x sinh\beta_k l \\ &\quad + cosh\beta_k x sin\beta_k l + sin\beta_k x cosh\beta_k l - sinh\beta_k x cos\beta_k l \end{aligned}$$

$$\begin{aligned} |C_k(x) - D_k(x)| &\leq sinh\beta_k(l-x) + |sin\beta_k(x-l)| + |cos\beta_k x| sinh\beta_k l \\ &\quad + cosh\beta_k x |sin\beta_k l| + |sin\beta_k x| cosh\beta_k l + sinh\beta_k x |cos\beta_k l| \\ &\leq sinh\beta_k(l-x) + 1 + sinh\beta_k l \\ &\quad + cosh\beta_k x + cosh\beta_k l + sinh\beta_k x \\ &\leq 1/2 e^{\beta_k(l-x)} + 1 + e^{\beta_k l} + e^{\beta_k x} = h_k(x) > 0; \end{aligned}$$

Since $\beta_k l > 4 \;\; \forall k$, *it follows that* $sinh\beta_k l > 2$, and $cosh\beta_k l > 2, \forall k$. *Hence,*

$$\begin{aligned} |\bar{W}_k| &\leq \frac{|C_k(x) - D_k(x)|}{|cosh\beta_k l + cos\beta_k l||sinh\beta_k l + sin\beta_k l|} \\ &\leq \frac{h_k(x)}{(cosh\beta_k l - 1)(sinh\beta_k l - 1)} \\ &\leq \frac{2h_k(x)}{cosh\beta_k l(sinh\beta_k l - 1)} \qquad since \quad cosh\beta_k l > 2; \\ &\leq \frac{4h_k(x)}{cosh\beta_k l sinh\beta_k l} \qquad since \quad sinh\beta_k l > 2; \end{aligned}$$

$$since\ cos\beta_k l cosh\beta_k l \;=\; -1 \;\; \Longrightarrow cos^2\beta_k l = \frac{1}{cosh^2\beta_k l} \;\; and \;\; cosh\beta_k l > 0;$$

$$A_k^2 = \frac{cos^2\beta_k l}{sin^4\beta_k l} \quad \Longrightarrow \frac{1}{A_k} = sin^2\beta_k l cosh\beta_k l \quad \Longrightarrow$$

$$\begin{aligned}
|W_k(x)| &= |\frac{1}{A_k}\bar{W}_k(x)| \\
&= sin^2\beta_k l cosh\beta_k l \frac{4h_k(x)}{cosh\beta_k l sinh\beta_k l} \\
&\leq \frac{4h_k(x)}{sinh\beta_k l} \\
&= \frac{8(e^{\beta_k(l-x)}/2 + e^{\beta_k l} + e^{\beta_k x} + 1)}{e^{\beta_k l} - e^{-\beta_k l}} \\
&= \frac{8(e^{-\beta_k x}/2 + e^{\beta_k(x-l)} + e^{-\beta_k l} + 1)}{1 - e^{-2\beta_k l}} \\
&\leq \frac{8(e^{-\beta_1 x}/2 + e^{\beta_1(x-l)} + e^{-\beta_1 l} + 1)}{1 - e^{-2\beta_1 l}}.
\end{aligned}$$

Hence, $\exists M_0 > 0$ *such that*

$$|W_k(x)| \leq M_0 \qquad \forall x \in [0, l]; k = 1, 2, \cdots\cdots.$$

For the cantilevered beam, the differential operator is also self-adjoint. Thus, the proof that the Green's function is uniformly bounded is similar to lemma A.1.

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