Avoiding Internal Gaps with Heterogeneous Circle Coverings via Optimal Power Diagrams ⋆

Andrew C. Frommer [*] **Yancy Diaz-Mercado** [*]

[*] *Department of Mechanical Engineering, University of Maryland, College Park, MD 20742 USA (e-mail: afromm88@umd.edu, yancy@umd.edu).*

Abstract: In this work, we present a strategy for distributing a collection of heterogeneous circles over a convex domain such that there are no gaps between circles. We find optimal power diagram weights to partition the domain and repeatedly update the location of the circles to ensure there are no gaps between circles. Results presented demonstrate the algorithm's effectiveness and comparisons are provided with two other naive coverage algorithms. We show an improvement in coverage over naive Voronoi diagram coverage and demonstrate no internal gaps for feasible configurations.

Keywords: Multi Agent and Networked Systems, Circle Covering, Coverage Control, Optimization, Nonlinear Control

1. INTRODUCTION

Circle packing is a problem that deals with fitting as many circles as possible into a space without overlap. While circle packing problem has attracted much research attention, its dual problem, circle covering, has received less study despite numerous applications, particularly those related to the distribution of resources. Circle covering is a problem in which circles are positioned so that they provide a covering of a domain. This problem is considerably challenging, and few optimal solutions have been found (Bánhelyi et al. (2015)).

Applications of circle covering include the distribution of radio transmission stations so that an entire country is covered, as demonstrated by Bánhelyi et al. (2015). Circle covering has also been used to study sensor placement to ensure that all areas of a region are measured, as in Boualem et al. (2022). Similarly, Briskorn and Dienstknecht (2020) use circle covering to strategically place cranes within a construction site. We approach the problem from the perspective of designing a strategy to place pursuers in a pursuit-evasion game. Frommer et al. (2023) presents a strategy for a team of slow pursuers to capture a fast evader in a three-dimensional reach avoid game by using coverage control on a partitioning plane. However, there is no consideration for the feasibility of the pursuer team in guarding their assigned location. This paper proposes a method of positioning heterogeneous circles to provide a covering for a domain while avoiding internal gaps.

There are several variants of the standard covering problem when the objective is to find the maximum radius that N unit circles can cover Zahn (1962). Additionally, some works focus on minimizing the covering circles' area or how to cover non-circular domains. Kershner (1939) finds a bound on the number of circles of a certain radius to cover a bounded region. Bánhelyi et al. (2015) use interval arithmetic to find the optimal radii to cover an arbitrary polygon with circles when the circles have fixed centers. Rocha et al. (2014) seek to cover a polygon with an arbitrary number of heterogeneous circles on the medial axis of the polygonal domain. This means that the circles can only be placed in a small subset of the domain, greatly reducing the complexity of the problem. Additionally, both the circle radii and the positions are adjusted to find a solution. Das et al. (2006) use Voronoi diagrams and Lloyd's algorithm to place an a priori assigned number of circles in a convex region and find a minimum radius such that the set of circles covers the domain. Briskorn and Dienstknecht (2020) seek to cover polygonal sites within a domain using circles of discrete radii. They show that this problem reduces to a classic set cover problem. Birgin et al. (2022) seek to cover an arbitrary region (e.g., disconnected, non-convex) with identical balls of minimum radii. Thus far there is little work that seeks to solve the circle covering problem on an arbitrary domain, with non-uniform circles, placed in non-stationary locations.

This paper presents a strategy to position a series of circles with fixed heterogeneous radii within a convex domain such that there are no gaps between circles. This strategy is similar to the strategy presented by Das et al. (2006), using Lloyd's algorithm to update the position of the circles. Rather than using standard Voronoi diagrams to partition the space, we instead use power diagrams. We present a strategy to choose the power weights using optimization that constrains the power vertices so they are contained in the intersection of circles, leading to the absence of internal gaps.

The structure of the paper is as follows. We present a formal definition of the circle covering problem in Section 2. In Section 3, we first provide a sufficient condition to achieve a covering using the properties of tessellations as well as a relaxed condition to guarantee no gaps between circles. We then include a brief summary of power diagrams and their properties. To solve the weight selection problem, we provide an analytical expression for the location of a power diagram vertex. This allows the problem of choosing power weights to then be formulated into an optimization problem, whose structure is discussed. Lloyd's algorithm is then used to repeatedly update the circles' positions. In Section 4 we provide a demonstration of the algorithm positioning circles to cover a domain, and we compare this covering algorithm to standard coverage control

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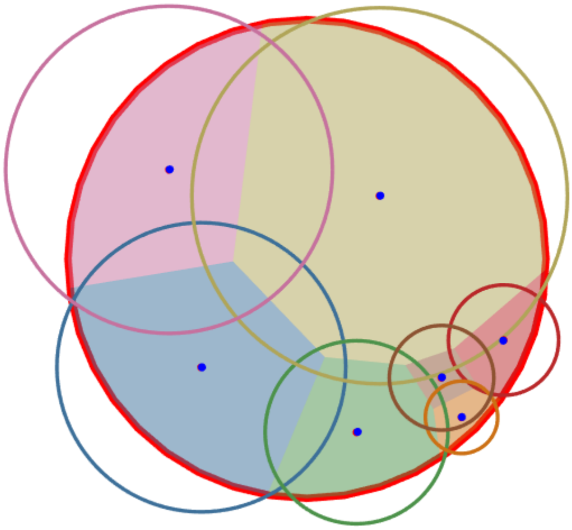


Fig. 1. An example of a series of heterogeneous circles placed using the proposed algorithm. Each vertex of the power diagram tesseallation is inside its cell's respective circle, ensuring no gaps between circles

which uses unweighted Voronoi tessellations. Concluding remarks are provided in Section 5.

2. PROBLEM STATEMENT

Problem 1. Consider a collection of N closed circles $\mathcal{C} = \{C_1, \ldots, C_N\}$ with centers $\{\mathbf{c}_1 \ldots \mathbf{c}_N\}$ with fixed radii $\mathbf{r} = \{r_1 \ldots r_N\}$. Formally,

$$\mathcal{C}_i = \left\{\mathbf{x} \in \mathbb{R}^2 \mid \|\mathbf{x} - \mathbf{c}_i\| \leq r_i\right\} \tag{1}$$

Also consider a convex set $\mathcal{D} \in \mathbb{R}^2$. We wish to find a scalable strategy that positions the elements of $\mathcal{C}$ such that the union of circles covers the domain

$$\bigcup_{i=1}^{N} \mathcal{C}_i \supseteq \mathcal{D}$$

Problem 2. Consider a tessellation $\mathcal{T}$ of a closed domain $\mathcal{D}$ with N convex polygonal cells $\mathcal{T} = \{P_1, \ldots, P_N\}$. Denote the vertices of P_i by $\{v_{i,1}, \ldots, v_{i,m_i}\}$. Given a set of circles $\mathcal{C} = \{C_1, \ldots, C_N\}$ with fixed heterogeneous radii, position the circles such that for $\forall\ i$ and $\forall\ j$, $v_{i,j} \in C_i$

Problem 3. Solve Problem 2 with the relaxation that tessellation vertices on the boundary of $\mathcal{D}$, $v_{i,j} \in \partial\mathcal{D}$ are not necessarily contained in C_i.

In this paper, we will focus on solving 3 as a crucial stepping stone to solving Problems 1 and 2.

3. METHODS

In this section, we propose a solution to place circles of arbitrary radius in a domain without gaps between them. We first prove a sufficient condition for covering. We then introduce power diagrams as a means to design tessellations conducive to covering. We then derive analytical expressions for the location of the vertices of power diagram cells. Using this relationship, we formulate an optimization problem using the locations of the vertices as constraints. Finally, we introduce an algorithm to repeatedly solve this optimization problem and update the location of the circles.

3.1 A Sufficient Condition on tessellation Vertices

Lemma 4. If every vertex of a convex polygon is an element of a circle, then the polygon is a subset of the circle.

Proof. Consider a convex polygon P with M vertices $\{v_1, \ldots, v_M\} \in C$ where C is a circle. Consider any two vertices, v_i and v_j. By the convexity of the circle C, the segment $\overline{v_i, v_j} \subset C$. Since every segment connecting two vertices is contained in C, then the $P \subset C$ because the exterior of P is comprised solely of segments connecting vertices.

Lemma 5. If every cell in tessellation $\mathcal{T}$, $P_i \subset C_i$ then the union of circles $\bigcup_i^N C_i$ covers $\mathcal{D}$

Proof. By the definition of a tessellation, $\mathcal{T}$ covers $\mathcal{D}$ or equivalently

$$\bigcup_{i=1}^{N} P_i \supseteq \mathcal{D}$$

Since $\forall i$, $P_i \subset C_i$, then

$$\mathcal{D} \subseteq \bigcup_{i=1}^{N} P_i \subset \bigcup_{i=1}^{N} C_i$$

□

Theorem 6. If for every vertex v_i of tessellation $\mathcal{T}$

$$v_i \in \bigcap_{k \in \mathcal{N}_i} C_k$$

where $\mathcal{N}_i$ is the list of polygons $P_k \in \mathcal{T}$ such that $v_i \in P_k$, and C_k is a circle such that $P_k \subset C_k$ then

$$\mathcal{D} \subset \bigcup_{k=1}^{N} C_k$$

covers the domain $\mathcal{D}$

Proof. Follows from Lemma 4 and Lemma 5. □

Remark 7. Satisfying Theorem 6 is equivalent to solving Problem 2

3.2 Power Diagrams

A Voronoi diagram is a tessellation of space such that each cell represents the set of points closest to a generator point under the Euclidean distance. We utilize a variation of the Voronoi diagram called the power diagram. The power diagram is also a tessellation of space; however, the distance metric is the squared Euclidean distance minus a weight. Consider a set of points $\{\mathbf{p}_1 \ldots \mathbf{p}_n\}$ with associated weights $\mathbf{w} = \{w_1 \ldots w_n\}$. For a generator point $\mathbf{p}_i \in \mathbb{R}^d$ and weight w_i power cell is given by

$$P_i = \left\{\mathbf{x} \in \mathbb{R}^d \mid \|\mathbf{p_i} - \mathbf{x}\|^2 - w_i < \left\|\mathbf{p}_j - \mathbf{x}\right\|^2 - w_j,\ \forall j \neq i\right\} \tag{2}$$

Power diagrams have the useful property that all cells are convex polygons Aurenhammer (1987). If we can design a series of weights and position the center of the circles such that the vertices meet the conditions of Theorem 6, then this will solve Problem 1.

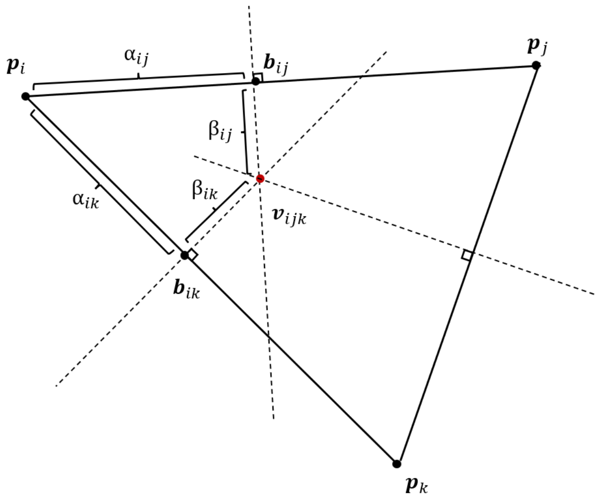


Fig. 2. The geometry of the location of a particular power diagram vertex $\mathbf{v}_{ijk}$. The dotted lines indicate the location boundaries between power cells. The distance from agent i to the boundaries is denoted α_{ij} and α_{ik}. The location of that intersection is $\mathbf{b}_{ij}$ and $\mathbf{b}_{ik}$ The distance from each $\mathbf{b}$ to the vertex $\mathbf{v}_{ijk}$ is β_{ij} and β_{ik} respectively.

3.3 Location of a Vertex

The first step in satisfying the conditions of Theorem 6 is to find an expression for the location of the power diagram's vertices as a function of the agents' positions and weights. Two agents are said to be adjacent (or neighbors) if their individual power diagrams share an edge. Whenever two adjacent agents share a common neighbor, a power diagram vertex will exist for the triple of agents. The vertex will be found at the intersection. This relationship is shown in Fig. 2.

Let us define some quantities related to the geometry of the triple of agents $\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k$. Let $\mathbf{t}_{ij}$ be a unit vector in the direction $\mathbf{p}_j - \mathbf{p}_i$

$$\mathbf{t}_{ij} = \frac{\mathbf{p}_j - \mathbf{p}_i}{\left\|\mathbf{p}_j - \mathbf{p}_i\right\|} \tag{3}$$

Let $\mathbf{n}_{ij}$ be a unit vector orthogonal to $\mathbf{t}_{ij}$

$$\mathbf{n}_{ij} = \mathbf{S}\mathbf{t}_{ij} \tag{4}$$

where $\mathbf{S}$ is a rotation matrix that rotates the unit vector by 90 degrees counterclockwise

$$\mathbf{S} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{5}$$

Let $\mathbf{b}_{ij}$ be the location of the intersection of the segment connecting $\mathbf{p}_j - \mathbf{p}_i$, and the boundary of P_i, as shown in Fig. 2. Using the definition of the power cell (2) we define α_{ij} as the distance from $\mathbf{p}_i$ to $\mathbf{b}_{ij}$

$$\alpha_{ij} = \frac{\left\|\mathbf{p_j} - \mathbf{p_i}\right\|^2 + w_i - w_j}{2\left\|\mathbf{p_j} - \mathbf{p_i}\right\|} \tag{6}$$

Lemma 8. Let β_{ij} be the distance from $\mathbf{b}_{ij}$ to the vertex $\mathbf{v}_{ijk}$. Then β_{ij} and β_{ik} are given by

$$\begin{aligned} \beta_{ij} &= \frac{1}{\sin\theta}\left(\alpha_{ik} - \alpha_{ij}\cos\theta\right) \\ \beta_{ik} &= \frac{1}{\sin\theta}\left(\alpha_{ij} - \alpha_{ik}\cos\theta\right) \end{aligned} \tag{7}$$

where we denote θ_{jik} as the angle $\angle \mathbf{p}_j\mathbf{p}_i\mathbf{p}_k$.

Proof. We can observe that

$$\mathbf{v}_{ijk} = \mathbf{b}_{ij} + \beta_{ij}\mathbf{n}_{ij} = \mathbf{b}_{ik} + \beta_{ik}\mathbf{n}_{ik} \tag{8}$$

such that

$$\beta_{ij}\mathbf{n}_{ij} - \beta_{ik}\mathbf{n}_{ik} = \mathbf{b}_{ik} - \mathbf{b}_{ij} \tag{9}$$

or equivalently

$$\begin{bmatrix}\mathbf{n}_{ij} & -\mathbf{n}_{ik}\end{bmatrix}\begin{bmatrix}\beta_{ij} \\ \beta_{ik}\end{bmatrix} = \mathbf{b}_{ik} - \mathbf{b}_{ij} \tag{10}$$

Then we can find β_{ij} and β_{ik} as

$$\begin{bmatrix}\beta_{ij} \\ \beta_{ik}\end{bmatrix} = \begin{bmatrix}\mathbf{n}_{ij} & -\mathbf{n}_{ik}\end{bmatrix}^{-1}\left(\mathbf{b_{ik}} - \mathbf{b_{ij}}\right) \tag{11}$$

Note that

$$\begin{bmatrix}\mathbf{n}_{ij} & -\mathbf{n}_{ik}\end{bmatrix}^{-1} = \frac{1}{\mathbf{t}_{ij}^\top \mathbf{S}^\top \mathbf{t}_{ik}}\begin{bmatrix}\mathbf{t}_{ik}^\top \\ \mathbf{t}_{ij}^\top\end{bmatrix} \tag{12}$$

By substituting $\mathbf{b}_{ij} = \mathbf{p}_i + \alpha_{ij}\mathbf{v}_{ij}$ we find that

$$\begin{bmatrix}\beta_{ij} \\ \beta_{ik}\end{bmatrix} = \frac{1}{\mathbf{t}_{ij}^\top \mathbf{S}^\top{}_{ik}}\begin{bmatrix}\mathbf{t}_{ik}^\top \\ \mathbf{t}_{ij}^\top\end{bmatrix}\left[\alpha_{ik}\mathbf{t}_{ik} - \alpha_{ij}\mathbf{t}_{ij}\right] \tag{13}$$

Multiplying out yields

$$\beta_{ij} = \frac{1}{\mathbf{t}_{ij}^\top \mathbf{S}\mathbf{t}_{ik}}\left(\alpha_{ik} - \alpha_{ij}\mathbf{t}_{ij}\cdot\mathbf{t}_{ik}\right) \tag{14}$$

$$\beta_{ik} = \frac{1}{\mathbf{t}_{ij}^\top \mathbf{S}\mathbf{t}_{ik}}\left(\alpha_{ik} - \alpha_{ik}\mathbf{t}_{ij}\cdot\mathbf{t}_{ik}\right) \tag{15}$$

Note that $\mathbf{t}_{ik}\cdot\mathbf{t}_{ij} = \cos\theta_{jik}$ since $\mathbf{t}_{ij}$, $\mathbf{t}_{ik}$ are both unit vectors. Similarly, $\mathbf{t}_{ij}^\top\mathbf{S}^\top\mathbf{t}_{ik} = \sin\theta_{jik}$ Thus the final formulation for both β's is the result. □

Substituting the result of Lemma 8 into (8) provides an analytical expression for $\mathbf{v}_{ijk}$

3.4 Quadratic Form

Recall that our goal is to specify $\mathbf{v}_{ijk} \in C_i \bigcap C_j \bigcap C_k$. Focusing on $\mathbf{v}_{ijk} \in C_i$ we now show an equivalent inequality as a function of $\mathbf{p}_i$, $\mathbf{p}_j$, $\mathbf{p}_k$, and r_i. First, define the following variables

$$\begin{aligned} d_{ij} &= \left\|\mathbf{p}_j - \mathbf{p}_i\right\| \\ d_{ik} &= \left\|\mathbf{p}_k - \mathbf{p}_i\right\| \\ a &= \frac{1}{4d_{ij}^2} \\ b &= \frac{1}{4d_{ik}^2} \\ c &= -\frac{\cos\theta_{jik}}{2d_{ij}d_{ik}} \end{aligned} \tag{16}$$

Theorem 9. The expression $\mathbf{v}_{ijk} \in C_i$ is equivalent to

$$\mathbf{w}_{ijk}{}^\top\mathbf{H}_{ijk}\mathbf{w}_{ijk} + \mathbf{k}_{ijk}\mathbf{w}_{ijk}{}^\top + D_{ijk} - \left(r_i\sin\theta_{jik}\right)^2 \leq 0 \tag{17}$$

where

$$\mathbf{H}_{ijk} = \begin{bmatrix} a+b+c & -(2a+c)/2 & -(2b+c)/2 \\ -(2a+c)/2 & a & c/2 \\ -(2b+c)/2 & c/2 & b \end{bmatrix} \tag{18}$$

$$\mathbf{k} = \begin{bmatrix} 1 - \frac{\cos\theta}{2}\left(\frac{d_{ij}}{d_{ik}} + \frac{d_{ik}}{d_{ij}}\right) \\ \frac{d_{ik}\cos\theta}{2d_{ij}} - 1/2 \\ \frac{d_{ij}\cos\theta}{2d_{ik}} - 1/2 \end{bmatrix} \tag{19}$$

$$D_{ijk} = ad_{ij}^4 + bd_{ik}^4 + cd_{ij}^2 d_{ik}^2 \tag{20}$$

$$\mathbf{w}_{ijk} = \left[w_i,\ w_j,\ w_k\right]^\top \tag{21}$$

Proof. The expression $\mathbf{v}_{ijk} \in C_i$ is equivalent to

$$\left\|\mathbf{v}_{ijk} - \mathbf{p}_i\right\| \leq r_i \tag{22}$$

Since $\mathbf{t}_{ij}$ and $\mathbf{n}_{ij}$ are orthogonal we can re-write (22) as

$$\alpha_{ij}^2 + \beta_{ij}^2 \leq r_i^2 \tag{23}$$

Since both α_{ij} and β_{ij} have a $\sin^2\left(\theta_{jik}\right)$ in the denominator it will be more convenient to write this as

$$\sin^2(\theta_{jik})\left(\alpha_{ij}^2 + \beta_{ij}^2\right) \leq \left(r_i \sin(\theta_{jik})\right)^2 \tag{24}$$

Then when we expand $\alpha_{ij}^2 + \beta_{ij}^2$ we get the following expression

$$\begin{aligned}
&\sin^2\left(\theta_{jik}\right)\left(\alpha_{ij}^2 + \beta_{ij}^2\right) \\
&= a\left(d_{ij}^4 + 2d_{ij^2} w_i - 2d_{ij}^2 w_j - 2w_i w_j + w_i^2 + w_j^2\right) \\
&+ b\left(d_{ik}^4 + 2d_i k^2 w_i - 2d_{ik}^2 w_k - 2w_i w_k + w_i^2 + w_k^2\right) \\
&+ c(d_{ij}^2 d_{ik}^2 + d_{ij}^2 w_i - d_{ij}^2 w_k + d_{ik}^2 w_i \\
&+ w_i^2 - w_i w_k - d_{ik}^2 w_j - w_i w_j + w_j w_k)
\end{aligned} \tag{25}$$

We can collect terms as follows

$$\begin{aligned}
&\sin^2\theta_{jik}\left(\alpha_{ij}^2 + \beta_{ij}^2\right) \\
&= \left(a d_{ij}^4 + b d_{ik}^4 + c d_{ij}^2 d_{ik}^2\right) \\
&+ \left(2a d_{ij}^2 + 2b d_{ik}^2 + c\left(d_{ij}^2 + d_{ik}^2\right)\right) w_i \\
&+ \left(-2a d_{ij}^2 - c d_{ik}^2\right) w_j + \left(-2b d_{ik}^2 - c d_{ij}^2\right) w_k \\
&+ (-2a - c) w_i w_j + (-2b - c) w_i w_k + (c) w_j w_k \\
&+ (a + b + c) w_i^2 + (a) w_j^2 + (b) w_k^2
\end{aligned} \tag{26}$$

The matrix equation $\mathbf{w}_{ijk}{}^\top \mathbf{H}_{ijk} \mathbf{w}_{ijk} + \mathbf{k}_{ijk} \mathbf{w}_{ijk}{}^\top + D_{ijk}$ is quadratic with respect to $\mathbf{w}_{ijk}$. Thus the expression for $\mathbf{v}_{ijk} \in C_i$ is equivalent to the statement of the proof. □

We now have an explicit quadratic inequality constraint on the weights of a three agent triple $\mathbf{w}_{ijk}$ that when satisfied guarantees that power vertex $\mathbf{v}_{ijk}$ is inside a circle centered at $\mathbf{p}_i$ with radius r_i

$$\mathbf{w}_{ijk}{}^\top \mathbf{H} \mathbf{w}_{ijk} + \mathbf{k} \mathbf{w}_{ijk}{}^\top + D_{ijk} - \left(r_i \sin\theta_{jik}\right)^2 \leq 0 \tag{27}$$

3.5 Optimization Formulation

We now seek to find a feasible set of weights $\mathbf{w} = \{w_1, \ldots, w_N\}$ such that for every i, j, k who share a vertex, equation (17) is satisfied. One methodology to accomplish this is through the framework of optimization. We already have constraints that define the feasible domain of $\mathbf{w}$, let us now choose an objective function. We choose to minimize the norm squared of the weight vector. We formally write the optimization problem as

$$\begin{aligned}
&\min_{\mathbf{w}} \|\mathbf{w}\|^2 \\
&\text{s.t. } \mathbf{w}_{ijk}{}^\top \mathbf{H}_{ijk} \mathbf{w}_{ijk} + \mathbf{k}_{ijk} \mathbf{w}_{ijk}{}^\top + D_{ijk} - \left(r_i \sin\theta_{jik}\right)^2 \leq 0 \\
&\qquad \forall i, j, k \text{ where } i, j, k \text{ are neighbors in a power diagram}
\end{aligned} \tag{28}$$

As both the objective function and the constraints are quadratic, this is a quadratically constrained quadratic program (QCQP). QCQP problems, in general, are known to be NP-hard, as described in d'Aspremont Alexandre and Boyd (2003). However, convex QCQP can be reformulated as a second-order cone program (SOCP), which can be readily solved using interior point methods, shown in Alizadeh and Goldfarb (2003).

Theorem 10. $\mathbf{H}_{ijk}$ is always positive semidefinite with at least one zero eigenvalue

Proof. We can observe that $[1\ 1\ 1]^\top$ is an eigenvector of the zero eigenvalue. It can be shown that the remaining two roots of the characteristic polynomial are given by

$$\lambda_{2,3} = a + b + c/2 \pm \sqrt{a^2 - ab + ac + b^2 + bc + c^2} \tag{29}$$

Because $\mathbf{H}_{ijk}$ is symmetric, all the eigenvalues are real. Then the discriminant of equation (29) and its root are nonnegative. Thus, to show that all the eigenvalues are nonnegative it is sufficient to lower bound the minor solution by zero.

$$a + b + c/2 - \sqrt{a^2 - ab + ac + b^2 + bc + c^2} \geq 0 \tag{30}$$

$$\begin{aligned}
a + b + c/2 &\geq \sqrt{a^2 - ab + ac + b^2 + bc + c^2} \\
a^2 + b^2 + c^2/4 + 2ab + bc + ac &\geq a^2 - ab + ac + b^2 + bc + c^2 \\
4ab &\geq c^2
\end{aligned}$$

Substituting a, b, c

$$\cos^2\theta_{jik} \leq 1 \tag{31}$$

Thus, for all configurations of $\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k$, the constraint matrix $\mathbf{H}_{ijk}$ is positive semi-definite. □

Corollary 11. Showing that $\mathbf{H}$ is positive semidefinite confirms that the optimization problem (28) is convex.

Remark 12. $\lambda_2 = 0$ only when $\theta_{jik} = 0$ or π, that is, when $\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k$ are colinear.

3.6 Coverage via Lloyd's

Now that for a collection of points, $\{\mathbf{p}_1, \ldots \mathbf{p}_N\}$, we can find a feasible set of weights $\mathbf{w}$ such that the vertices of the power diagram generated by the points and the weights are located inside circles centered at the points with radii $\mathbf{r}$. We would like to position the generator points to cover the domain $\mathcal{D}$. Coverage control is a mechanism for distributing resources over a domain. Lloyd's algorithm allows us to reposition points in a domain via the following control law

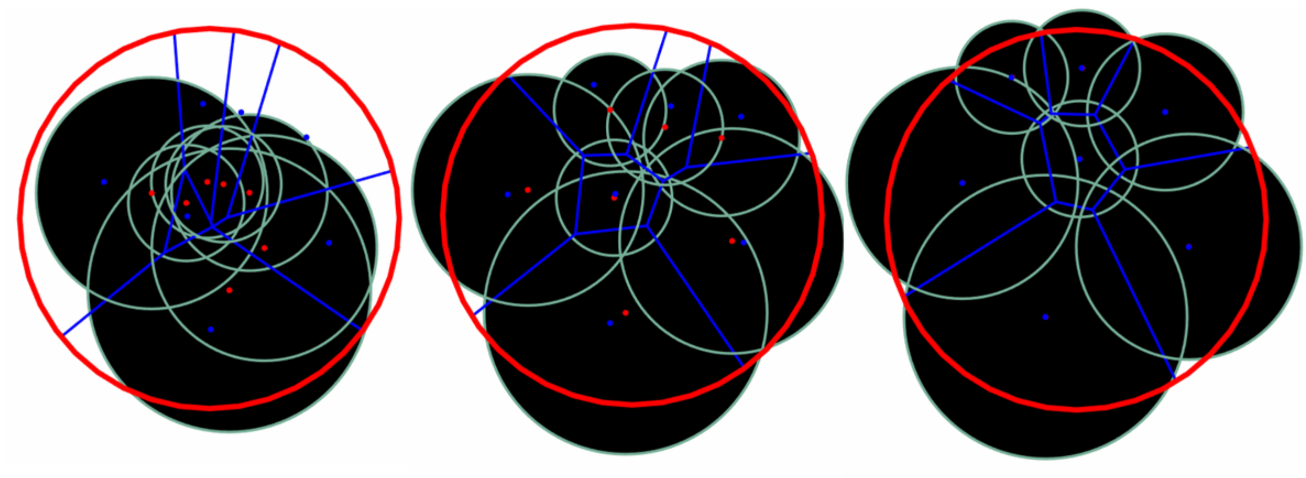
$$\mathbf{u}_i = \kappa\left(\mathbf{c}_{Pi}(\mathbf{p}) - \mathbf{p}_i\right) \tag{32}$$

$$\dot{\mathbf{p}}_i = \mathbf{u}_i \tag{33}$$

where $\mathbf{c}_{Pi}$ is the centroid of the state-dependent power cell P_i, κ is a positive control gain and $\mathbf{u}_i$ is a control signal. Lloyd's algorithm has been shown to converge to a centroidal Voronoi tessellation (CVT), a necessary condition for optimal distribution, for static domains when the cells are standard Voronoi cells as shown in Cortes et al. (2004). We will use Lloyd's algorithm in concert with the framework of power diagrams. Rather than using Voronoi cells, we run Lloyd's algorithm using the power cells as in Arslan and Koditschek (2016). The complete process is written in Algorithm 1.

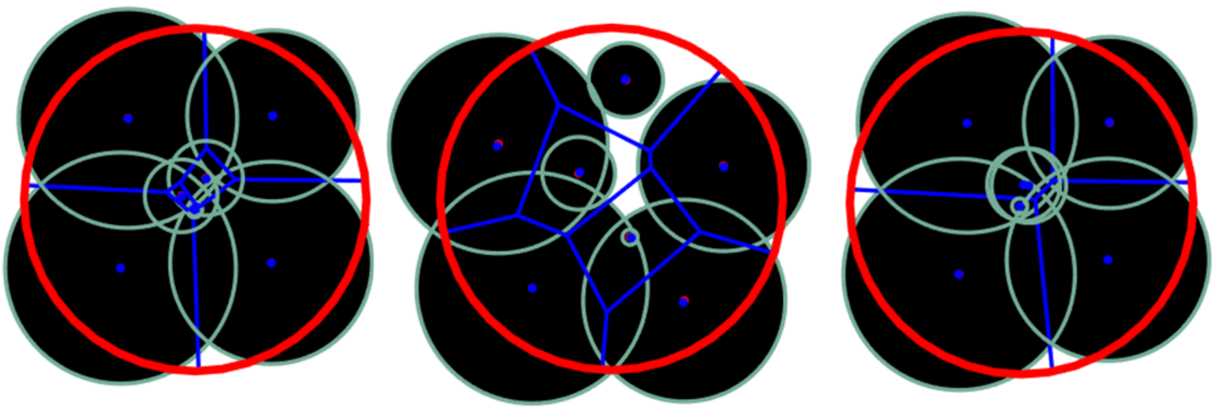
Algorithm 1 Update rule for circle locations

Initialize $\{\mathbf{p}_1 \ldots \mathbf{p}_N\} \in \mathcal{D}$
while $\exists\, i$ s.t. $\|\mathbf{u}_i\| \geq \varepsilon$ **do**
  $\forall i, j, k \in \{1 \ldots\}$ set $\mathbf{H}_{ijk}$, $\mathbf{k}_{ijk}$, D_{ijk} by Theorem 9
  Set $\mathbf{w}$ by Equation (28)
  Set $\{P_1 \ldots P_N\}$ using Equation (2)
  Set $\{\mathbf{c}_{P1} \ldots \mathbf{c}_{PN}\}$ as the centroids of $\{P_1 \ldots P_N\}$
  for $i \in \{1, \ldots, N\}$ **do**
    $\mathbf{u}_i \leftarrow \kappa(\mathbf{c}_{Pi} - \mathbf{p}_i)$
    $\mathbf{p}_i \leftarrow \mathbf{p}_i + \delta t\, \mathbf{u}_i$
  end for
end while



Initial | Intermediate | Final

Fig. 3. Three snapshots of the proposed strategy show the initialization, mobilization, and convergence of the circles. The domain is the large red circle, the small red dots are the locations of the centers of the circles, and the blue lines are the power diagram cells with the blue dots denoting their centroids. The circles are filled in black to easily visualize any gaps between circles and their borders are colored teal for contrast.



Optimized Power Diagram | Voronoi Diagram | Naive Power Diagram

Fig. 4. Comparison of presented coverage strategy, Voronoi diagram coverage, and a naive power diagram. Because the Voronoi coverage has no notion of circle size there are significant gaps between agents. Both power diagram methods fill the domain without gaps between agents, but there is more overlap in the naive approach, leading to under-utilization of resources.

4. RESULTS

We now present results that demonstrate the presented coverage strategy and compare it to two other tessellations, Voronoi diagrams and naively weighted power diagrams. Fig. 3 shows three snapshots of the proposed strategy converging. Observe that the vertices of the tessellation are always located in the intersection of the neighboring circles, and the circles spread out to fill the domain while leaving no gaps. It is also observable that the circles do not entirely cover the domain. This is because there is no constraint on the power vertices that occur at the domain boundaries. In Fig. 4 we compare the proposed power diagram-based strategy and two other strategies. One uses an unweighted Voronoi diagram to position the circles, and the other uses a power diagram with fixed weights equal to the radii of each circle, which we term naive. In the unweighted Voronoi example, there are clear gaps between the circles of the Voronoi diagram strategy. These gaps form because there is no notion of the radii of the circles. The naive weighted power diagram strategy also fills the domain well. However there it can

Table 1. Mean Percentage Coverage over Circular Domain and Convex Hull of Circle Centers over 80 Coverage Simulations

Tessellation Method	Optimized Power Diagram	Voronoi Diagram
Percent of Domain Covered	98.42	93.64
Percent of Hull Covered	98.63	96.41

converge to a solution with gaps between neighboring circles. We also empirically compared the ability of both the proposed weighted power diagram and the Voronoi diagram strategy. In each method, the initial positions and radii were set identically and ran until Lloyd's algorithm converged. To evaluate the performance, we computed two metrics for each strategy; the percent of the domain covered, and the percent of the convex hull of the circles' centers covered. The results of the empirical comparison are shown in Table 1.

5. CONCLUSION

In this work, we present a method of tackling the circle-covering problem which formulates an optimization problem to find appropriate power diagram weights, and places circles to cover the domain using coverage control over a power diagram. We showed sufficient conditions for covering, derived inequality constraints to satisfy such conditions, and provided a mechanism to continually update the circles' locations. We demonstrated through numerical simulation, the proposed strategy's improvement over standard Voronoi coverage, and naive power diagrams, particularly in the case where the radii are non-uniform. Future work includes adding the boundary vertices to the inequality constraints to solve Problem 2. We would also like to find conditions on when a feasible solution to equation (28) exists and show the convergence properties of Algorithm 1. One of the motivations for this work is to provide a strategy to place pursuers in a 3D pursuit-evasion game. To that end, future work includes incorporating state-dependent radii based on the relative position of the pursuer and evader.

REFERENCES

Alizadeh, F. and Goldfarb, D. (2003). Second-order cone programming. *Mathematical programming*, 95(1), 3–51.

Arslan, O. and Koditschek, D.E. (2016). Voronoi-based coverage control of heterogeneous disk-shaped robots. In *2016 IEEE International Conference on Robotics and Automation (ICRA)*, 4259–4266. IEEE.

Aurenhammer, F. (1987). Power diagrams: properties, algorithms and applications. *SIAM journal on computing*, 16(1), 78–96.

Bánhelyi, B., Palatinus, E., and Lévai, B.L. (2015). Optimal circle covering problems and their applications. *Central European Journal of Operations Research*, 23(4), 815–832.

Birgin, E.G., Laurain, A., Massambone, R., and Santana, A.G. (2022). A shape-newton approach to the problem of covering with identical balls. *SIAM Journal on Scientific Computing*, 44(2), A798–A824.

Boualem, A., De Runz, C., and Ayaida, M. (2022). Partial paving strategy: application to optimize the area coverage problem in mobile wireless sensor networks. *Journal of Wireless Mobile Networks, Ubiquitous Computing, and Dependable Applications*, 13(2), 1–22.

Briskorn, D. and Dienstknecht, M. (2020). Covering polygons with discs: The problem of crane selection and location on construction sites. *Omega*, 97, 102114.

Cortes, J., Martinez, S., Karatas, T., and Bullo, F. (2004). Coverage control for mobile sensing networks. *IEEE Transactions on robotics and Automation*, 20(2), 243–255.

Das, G.K., Das, S., Nandy, S.C., and Sinha, B.P. (2006). Efficient algorithm for placing a given number of base stations to cover a convex region. *Journal of Parallel and Distributed Computing*, 66(11), 1353–1358.

d'Aspremont Alexandre and Boyd, S. (2003). Lecture notes on relaxations and randomized methods for nonconvex qcqps.

Frommer, A.C., Abate, M., Rivera-Ortiz, P.M., and Diaz-Mercado, Y. (2023). A capture strategy for multi-pursuer coordination against a fast evader in 3d reach-avoid games. *IFAC-PapersOnLine*, 56(3), 217–222.

Kershner, R. (1939). The number of circles covering a set. *American Journal of mathematics*, 61(3), 665–671.

Rocha, P., Rodrigues, R., Gomes, A.M., Toledo, F.M., and Andretta, M. (2014). Circle covering representation for nesting problems with continuous rotations. *IFAC Proceedings Volumes*, 47(3), 5235–5240.

Zahn, Jr., C. (1962). Black box maximization of circular coverage. *Journal of Research of the National Bureau of Standards, Section B: Mathematics and Mathematical Physics*, 66B(4), 181–216.