ABSTRACT

Title of dissertation: CRITICAL THRESHOLDS IN EULERIAN DYNAMICS

Dongming Wei
Doctor of Philosophy, 2007

Dissertation directed by: Professor Eitan Tadmor
Center for Scientific Computation
And Mathematical Modeling
Department of Mathematics
Institute for Physical Science & Technology

In this thesis, we study the critical regularity phenomena in Eulerian dynamics, $u_t + u \cdot \nabla u = F(u, Du, \cdots)$, here F represents a general force acting on the flow and by regularity we seek to obtain a large set of sub-critical initial data.

We analyze three prototype models, ranging from the one-dimensional Euler-Poisson equations to two-dimensional system of Burgers equations to three-dimensional, four-dimensional and even higher-dimensional restricted Euler systems.

We begin with the one-dimensional Euler-Poisson equations, where F is the Poisson forcing term together with the usual γ-law pressure. We prove that global regularity of the Euler-Poisson equations with $\gamma \geq 1$ depends on whether or not the initial configuration crosses an intrinsic critical threshold.

Next, we discuss multi-dimensional examples.

The first multi-dimensional example that we focus our attention on is the two dimensional pressureless flow, where $F = \epsilon \Delta u$. Our analysis shows that there is a

uniform BV bound of the solutions u^ϵ. Moreover, if the initial velocity gradient ∇u_0 does not have negative eigenvalues, then its vanishing viscosity limit is the smooth solution of the corresponding equations of the inviscid fluid flow.

The second multi-dimensional example we discuss here is the restricted Euler dynamics, where $\nabla F = \frac{1}{n}\mathrm{tr}(\nabla u)^2 I_{n\times n}$. Our analysis shows that for the three-dimensional case, the finite-time breakdown of the restricted Euler system is generic, and for the four-dimensional case, there is a surprising global existence for sub-critical initial data. Further analysis extends the above result to the general n-dimensional $(n > 4)$ restricted Euler system.

CRITICAL THRESHOLDS IN EULERIAN DYNAMICS

by

Dongming Wei

Dissertation submitted to the Faculty of the Graduate School of the
University of Maryland, College Park in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
2007

Advisory Committee:
Professor Eitan Tadmor, Chair/Advisor
Professor Stuart Antman
Professor Jian-Guo Liu
Professor Matei Machedon
Professor Athanasios Tzavaras

DEDICATION

This dissertation is dedicated to my parents and my wife Huilin.

ACKNOWLEDGEMENTS

First and foremost, I would like to express my deepest appreciation to my advisor, Professor Eitan Tadmor, for his support, patience and encouragement throughout my graduate studies. Eitan is always a great source of knowledge and inspiration for me. From him, I not only learned how to do good mathematical research but also how to organize and write good mathematical papers.

I thank Professors Stuart Antman, Jian-Guo Liu, Matei Machedon, and Athanasios Tzavaras for agreeing to serve on my dissertation committee. I also thank Professors Bob Pego, Georg Dolzmann, Bo Li and Konstantina Trivisa for their valuable teaching.

I would like to express my gratitude to my friends and colleagues in the PhD program for their help and friendship. In particular, I thank Ning Jiang, Zhiwei Chen, Weiran Sun, Bin Cheng, Jie Liu, Linbao Zhang and Weigang Zhong, for discussing with me and teaching me various kind of mathematics.

I would like to thank my parents and my brother for their emotional support and encouragement.

Last but not the least, I want to thank my beautiful wife Huilin just for everything.

Table of Contents

1 Introduction 1
1.1 Euler dynamics – three prototype models . . . 1
1.1.1 Euler-Poisson systems . . . 2
1.1.2 Viscous fluids . . . 4
1.1.3 Restricted Euler/Navier-Stokes equations . . . 6
1.2 Spectral dynamics of the velocity gradient field . . . 11
1.3 Outline of the thesis . . . 13

2 Critical thresholds of the one-dimensional Euler-Poisson system 15
2.1 Euler-Poisson system and related models . . . 15
2.2 Isentropic gas dynamics . . . 18
2.3 Pure Euler-Poisson systems . . . 22
2.4 Euler-Poisson equations with γ-law pressure . . . 24
2.4.1 Riemann invariants . . . 25
2.4.1.1 The Euler-Poisson equations with γ-law pressure: $\gamma > 1$ . . . 25
2.4.1.2 The isothermal case $\gamma = 1$ . . . 27
2.4.2 Global smooth solutions for sub-critical initial data . . . 28
2.4.2.1 Critical Threshold for isothermal case: $\gamma = 1$ . . . 29
2.4.2.2 Critical threshold for $\gamma > 1$ . . . 32
2.4.3 Finite-time breakdown for super-critical initial date . . . 36

3 Vanishing viscosity limit of the system of Burgers equations 39
3.1 Irrotational viscous fluids . . . 39
3.2 Two-dimensional general viscous flow . . . 45
3.2.1 Spectral dynamics and BV bound . . . 45
3.2.2 Vanishing viscosity limit of smooth solutions . . . 50
3.2.3 Open questions . . . 57

4 Critical thresholds for restricted Euler dynamics 58
4.1 Restricted Euler equations and spectral dynamics . . . 58
4.2 Trace dynamics . . . 59
4.3 Three-dimensional critical thresholds . . . 61
4.4 Four-dimensional critical thresholds . . . 64
4.5 n-dimensional critical thresholds . . . 73

Bibliography 82

Chapter 1

Introduction

1.1 Euler dynamics – three prototype models

In a general class of Eulerian flows, the velocity field, u, is governed by the Newtonian law,

$$u_t + u \cdot \nabla u = F(u, Du, \cdots), \qquad u : \mathbb{R}^n \to \mathbb{R}^n, \tag{1.1}$$

where F represents a general force acting on the flow. Different physical models are dictated by different forcing.

We mainly focus our attention on the question of time regularity of equation (1.1), which is of fundamental importance from both mathematical and physical points of view. There is a considerable effort that has been and is being devoted to this issue for Euler equations of both compressible and incompressible fluids. See [BKM84], [Si85],[CFM96],[Li96],[Gr98] and [T01] for a partial reference list. In particular, the possible phenomena of finite-time breakdown for three-dimensional incompressible flows signifies the onset of turbulence in higher Reynolds number flows.

In this thesis, we study three prototype models. The first is the one-dimensional Euler-Poisson equations. The second is the two-dimensional viscous dusty medium model. And the third is the restricted model of Euler equations.

1.1.1 Euler-Poisson systems

We consider the one-dimensional Euler-Poisson equations driven by both pressure and Poisson forcing,

$$\begin{cases} \rho_t + (\rho u)_x = 0\,, \\ (\rho u)_t + (\rho u^2)_x = -p(\rho)_x - k\rho\varphi_x, \quad k > 0\,, \end{cases} \tag{1.2}$$

where the pressure $p = p(\rho)$ is given by the usual γ-law, $p(\rho) = A\rho^\gamma$ with $\gamma \geq 1$, and $\varphi = \varphi(\rho)$ is the potential, which is dictated by the (one-dimensional) Poisson equation, $\varphi_{xx} = -\rho$.

These equations govern different phenomena, ranging from the largest scale of, e.g., the evolution gravitational collapse in stars, to applications in the smallest scale of e.g., semi-conductors. There is a considerable amount of literature available on the local and global behavior of the Euler-Poisson and related problems. Consult [Ma86] for local existence in the small H^s-neighborhood of a steady state of self-gravitating stars, [CW96] for global existence of weak solutions with geometrical symmetry, [Gu98] for global existence for three-dimensional irrotational flow, [MN95] for isentropic case, and [JR00] [PRV95] for isothermal case. Consult [Pe90] [MP90] [Si85] [En96] [WC98], [BW98] and in particular, [En96], for non-existence results and singularity formation. The question of global smoothness vs. finite-time breakdown was studied in a recent series of works of Engelberg, Liu and Tadmor, in terms of a critical threshold phenomena for one-dimensional "pressure-less" Euler-Poisson equations, [ELT01] and two-dimensional restricted Euler-Poisson equations, [LT02, LT03].

There are two systems closely related to the Euler-Poisson system (1.2). The first is the 2×2 system of isentropic gas dynamics, which corresponds to $k = 0$ in (1.2). Following [La64], one can show that finite-time breakdown for this system is generic. The other is the one-dimensional "pressure-less" Euler-Poisson equations, which corresponding to $A = 0$. It was shown in [ELT01] that for this pure Euler-Poisson system, there is a "large set" of initial configurations which yield global smooth solutions. Thus, it turns out that the pressure is a destabilizing term and the Poisson forcing is a stabilizing term.

The natural question that arises in the present context of full Euler-Poisson equations (1.2) is whether the pressure enforces a generic finite-time breakdown or, whether the presence of Poisson forcing preserves global regularity for a "large set" of initial configurations. We gave our answer in [TW06]. More precisely, we answer this question of "competition" between pressure and Poisson forcing, proving that the Euler-Poisson equations (1.2) with $\gamma \geq 1$ admit global smooth solutions for a "large set" of sub-critical initial data such that

$$u_{0x}(x) > -K_0\sqrt{\rho_0(x)} + \sqrt{A\gamma}\frac{|\rho_{0x}(x)|}{\rho_0(x)^{\frac{3-\gamma}{2}}}, \qquad \gamma \geq 1. \tag{1.3}$$

Here, K_0 is a constant depending on k, γ and the initial data. In the particular (and important) case of isothermal equations, $\gamma = 1$, we have $K_0 = \sqrt{2k}$ and (1.3) amounts to a sharp critical threshold,

$$u_{0x}(x) \geq -\sqrt{2k\rho_0(x)} + \sqrt{A}\frac{|\rho_{0x}(x)|}{\rho_0(x)}, \qquad \gamma = 1. \tag{1.4}$$

The inequalities (1.3), (1.4) quantify the competition between the destabilizing pressure effects, as the range of sub-critical initial configurations shrinks with the growth of the amplitude of the pressure, A, while the stabilizing effect of the Poisson forcing increases the sub-critical range with a growing k. In particular, (1.4) with $A = 0$ recovers the pressure-free critical threshold, with $k = A = 0$ recovers the inviscid Burgers.

Formation of singularities and global regularity of (1.2) were addressed earlier by Engelberg in [En96]. His results show finite-time breakdown if $u_{0x}(x) - \sqrt{A\gamma}|\rho_{0x}(x)|\rho_0(x)^{\frac{\gamma-3}{2}}$ is "...sufficiently negative at some point". Our contribution here is to *quantify* the critical threshold behind this asymptotic statement. To fully appreciate this quantified threshold, we turn to the converse statement in [En96, Theorem 2]: it asserts the global regularity of (1.2) for a class of initial data such that $u_{0x}(x) - \sqrt{A\gamma}|\rho_{0x}(x)|\rho_0(x)^{\frac{\gamma-3}{2}} > 0$. It is a "non-generic" class in the sense of requiring both Riemann invariants (for details of Riemann invariants, see section 2.2) at $t = 0$ to be *globally* increasing. In fact, by (1.3) one has a *negative* threshold, $-K_0\sqrt{\rho_0}$, implying the existence of a "large" class of sub-critical initial data with global regularity.

1.1.2 Viscous fluids

Next, we consider the viscosity forces, $F := \epsilon\Delta u$, which lead to the so-called viscous dusty medium model,

$$\partial_t u + u \cdot \nabla u = \epsilon\Delta u\,, \qquad u : \mathbb{R}^n \to \mathbb{R}^n\,, \tag{1.5}$$

where $\epsilon > 0$ is a viscosity amplitude. Other suggested names for this system are Burgers system [ES00], Hopf system, Riemann equation (for $n = 1$).

We also consider the equation of the corresponding inviscid fluid

$$\begin{cases} u_t + u \cdot \nabla u = 0\,, \\ u(x,0) = u_0(x)\,. \end{cases} \tag{1.6}$$

These systems were proposed by Zeldovitch [ZE70] as a model describing the evolution of the rarefied gas of non-interacting particles.

It seems rather natural to expect that solutions of (1.6) can be obtained as limits of solutions of (1.5). This question has been positively answered when the initial data are irrotational (i.e., $u_0 = \nabla\phi$), in the sense of viscosity solutions for Hamilton-Jacobi equations. In [LT02], still considering the irrotational two-dimensional model, Liu and Tadmor provided a novel approach at the level of the velocity field. The essential tool they used is spectral dynamics.

There are other approaches to the solutions of the pressureless gas dynamics system. For example, people consider the Euler equations of compressible fluids in the whole space $\mathbb{R}^d$,

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho u) = 0\,, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \epsilon\nabla p(\rho) = 0\,, \end{cases} \tag{1.7}$$

where the pressure p has the following form

$$p(\rho) = \rho^\gamma\,, \qquad 1 < \gamma \leq 1 + \frac{2}{d}\,,$$

and the pressureless gas dynamics system in $\mathbb{R}^d$, which corresponding to $\epsilon = 0$,

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho u) = 0\,, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) = 0\,. \end{cases} \tag{1.8}$$

Consult [Gr98], [Se97] for global smooth solutions to the multidimensional Euler equations of compressible fluids. Consult [GS00], [Go04] and [Co06] for convergence of smooth solutions of (1.7) towards smooth solutions of (1.8). Also consult [CL04] for the phenomena of concentration and cavitation of the vanishing pressure limit of solutions of the full Euler equations for nonisentropic compressible fluids with a scaled pressure.

In this thesis, we consider equations of the two-dimensional pressureless gas dynamics systems – (1.5) and (1.6), subject to general initial data. Based on analyzing the spectra of ∇u, we obtain an uniform BV bound of solutions u^ϵ, see [TW07]. In the particular case in which ∇u_0 does not have negative eigenvalues, the limiting flow is shown to be the smooth solution of the equations (1.6) – the corresponding inviscid fluid flow, see [TW07].

1.1.3 Restricted Euler/Navier-Stokes equations

For the forcing involving viscosity and pressure, we meet the well known Navier-Stokes equations of incompressible fluid flow in n space dimensions, which can be expressed as the system of $n+1$ equations,

$$\partial_t u + u \cdot \nabla u = \nu \Delta u - \nabla p\,, \quad u : \mathbb{R}^n \to \mathbb{R}^n\,, \quad t > 0\,, \tag{1.9a}$$

$$\nabla \cdot u = 0\,, \tag{1.9b}$$

$$u(x,0) = u_0(x)\,. \tag{1.9c}$$

Here $\nu > 0$ is the kinematic viscosity. When the coefficient ν vanishes in (1.9a), we have the Euler equations for incompressible fluid flow. We will only discuss fluid flows occupying the whole space so that the important effects of boundary layers are ignored. In most applications, ν is an extremely small quantity, typically ranging from 10^{-6} to 10^{-3} in turbulent flows. Thus one can anticipate that the behavior of solutions of the Euler equations with $\nu = 0$ (which corresponds to the inviscid fluid flows) is rather important in describing solutions of the Navier-Stokes equations when ν is small.

Differentiating of the Euler equation of incompressible fluid flow with respect to x , we obtain the equation satisfied by the local velocity gradient tensor $M := \nabla u$

$$\partial_t M + (u \cdot \nabla) M + M^2 = -(\nabla \otimes \nabla) p\,. \tag{1.10}$$

Taking the trace of M and noting $\mathrm{tr} M = \nabla \cdot u = 0$, we find

$$\mathrm{tr} M^2 = -\Delta p\,. \tag{1.11}$$

This gives $p = -\Delta^{-1}(trM^2)$. The right-hand side in (1.10) therefore amounts to the $n \times n$ time-dependent matrix

$$(\nabla \otimes \nabla)\Delta^{-1}(\mathrm{tr} M^2) = R[\mathrm{tr} M^2]\,.$$

Here $R[w]$ denotes the $n \times n$ matrix whose entries are given by $(R[w])_{ij} := R_i R_j(w)$ where R_j denote the Riesz transforms, $R_j = -(-\Delta)^{-1/2}\partial_j$, i.e.,

$$[\widehat{R_j(w)}](\xi) = -i\frac{\xi_j}{|\xi|}\hat{w}(\xi) \qquad \text{for} \qquad 1 \le j \le n\,.$$

This yields the equivalent formulation of the Euler equations which reads

$$\partial_t M + (u \cdot \nabla) M + M^2 = R[\mathrm{tr} M^2]\,, \tag{1.12}$$

subject to the trace-free initial data

$$M(\cdot,0)=M_0\,,\qquad \mathrm{tr}M_0=0\,.$$

Note that the invariance of incompressibility is already taken account in (1.12) since $\partial_t\mathrm{tr}M=0$ and hence $\mathrm{tr}M=\mathrm{tr}M_0=0$. It is the global term in the above equations, $R[\mathrm{tr}M^2]$, which makes the problem rather intricate to solve, both analytically and numerically. Various simplifications to this pressure Hessian were sought, see, e.g. [CA92],[C86],[CPB99] and [V82].

We now turn to discuss the restricted Euler dynamics proposed in [V82] as a localized alternative of the full Euler equation (1.12). By the definition of the operator R, one has

$$R[\mathrm{tr}M^2]=\nabla\otimes\nabla\Delta^{-1}[\mathrm{tr}M^2]=\nabla\otimes\nabla\int_{\mathbb{R}^n}K(x-y)\mathrm{tr}M^2(y)dy\,,$$

where the kernel $K(\cdot)$ is given by

$$K(x)=\begin{cases}\dfrac{1}{2\pi}\,, & n=2\,,\\[2mm] \dfrac{1}{(2-n)\omega_n|x|^{n-2}}\,, & n>2\,,\end{cases}$$

with ω_n denoting the surface area of the unit sphere in n-dimensions. A direct computation yields

$$\partial_i\partial_jK*\mathrm{tr}M^2=\frac{\mathrm{tr}M^2}{n}\delta_{ij}+\int_{\mathbb{R}^n}\frac{|x-y|^2\delta_{ij}-n(x_i-y_i)(x_j-y_j)}{\omega_n|x-y|^{n+2}}\mathrm{tr}M^2(y)dy\,.$$

This shows that the local part of the global term $R[\mathrm{tr}M^2]$ is $\dfrac{\mathrm{tr}M^2}{n}I_{n\times n}$. We thus use this local term, $\dfrac{\mathrm{tr}M^2}{n}I_{n\times n}$, to approximate the pressure Hessian. The corresponding local gradient field then evolves according to the following restricted Euler model

$$\partial_tM+(u\cdot\nabla)M+M^2=\frac{\mathrm{tr}M^2}{n}I_{n\times n}\,. \tag{1.13}$$

This is a matrix Ricatti equation which is responsible for the formation of singularities at finite time, while the local source on the right provides certain balancing effect. We observe that as in the global model, the incompressibility is still maintained in this localized model, since $\mathrm{tr}M^2 = \mathrm{tr}\left[\frac{\mathrm{tr}M^2}{n} I_{n\times n}\right]$ implies $\partial_t \mathrm{tr}M = 0$. As a local approximation of the pressure Hessian, the above model, the so-called restricted Euler dynamics, has caught great attention since it was first introduced in [V82], because it can be used to understand the local topology of the Euler dynamics and to capture certain statistical features of the physical flow.

In this thesis, we identify and compare the critical thresholds in three-dimensional and four-dimensional cases, respectively, see [LTW07]. To do so, we consider a bounded, divergence-free, smooth vector field $u : \mathbb{R}^n \times [0, T] \to \mathbb{R}^n$. Let $x = x(\alpha, t)$ denote an orbit associated to the flow by

$$\frac{dx}{dt} = u(x,t), \qquad 0 < t < T, \qquad x(\alpha, 0) = \alpha \in \mathbb{R}^n. \tag{1.14}$$

Then along this orbit, the velocity gradient tensor of the restricted Euler equations (1.13) satisfies

$$\dot{M} + M^2 = \frac{\mathrm{tr}M^2}{n} I_{n\times n}, \qquad \dot{\{\}} := \frac{d}{dt} = \partial_t + u \cdot \nabla_x. \tag{1.15}$$

By the spectral dynamics lemma 3.1 in [LT02], (we will go through the details of this lemma in sec 1.2), the corresponding eigenvalues satisfy

$$\dot{\lambda}_i + \lambda_i^2 = \frac{1}{n}\sum_{j=1}^{n} \lambda_j^2, \qquad i = 1, \ldots, n. \tag{1.16}$$

This is a closed system, which serves as a simple approximation for the evolution of the velocity gradient field.

For arbitrary $n \geq 3$, one can use the spectral dynamics of M to show a finite-time breakdown of (1.16), which generalizes the previous result of [V82]. More precisely, in [LT02], the finite-time breakdown was established after a set of $\left[\frac{n}{2}\right]+1$ global invariants in terms of the eigenvalues of M were identified, and moreover, the precise topology of the flow at the breakdown time was also studied.

The question we are interested in is: how generic is the finite-time breakdown of the restricted Euler equations (1.16)? Is there a critical threshold such that finite-time breakdown occurs only when the initial configuration crosses such critical threshold, while sub-critical initial data yield global smooth solutions?

It turns out that the finite-time breakdown for the three-dimensional restricted Euler dynamics is generic, while for the four-dimensional restricted Euler dynamics, there is a surprising global existence for sub-critical initial data. More precisely, we describe critical thresholds for both 3-dimensional and 4-dimensional Euler dynamics in the following two theorems.

Theorem 1.1.1 *Solutions to* (1.16) *with* $n = 3$ *remain bounded for all time if and only if the initial data* $\Lambda_0 := (\lambda_1, \lambda_2, \lambda_3)$ *lie in the following set*

$$r\{(-1,-1,2),(-1,2,-1),(2,-1,-1)\}, \qquad \forall\, r \geq 0.$$

Theorem 1.1.2 *Solutions to* (1.16) *with* $n = 4$ *remain bounded for all time if and only if the initial data* $\Lambda_0 := (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ *lie in the following set*

$$\Lambda_0 \in \Omega \cap \left\{\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0\right\},$$

where

$$\Omega := \Omega_1 \cup \Omega_2 \cup \Omega_3\,,$$

$$\Omega_1 := \Big\{ \big(\lambda_1, \lambda_2, \lambda_3, \lambda_4\big) \ \Big| \ \ \lambda_{1,2,3,4} \in \mathbb{R} \quad \textit{and} \quad \lambda_{i_1} = \lambda_{i_2} \le \lambda_{i_3} \le \lambda_{i_4}, \\ \textit{here } (i_1, i_2, i_3, i_4) \textit{ is a permutation of } (1,2,3,4) \Big\},$$

$$\Omega_2 := \Big\{ \big(\lambda_1, \lambda_2, \lambda_3, \lambda_4\big) \ \Big| \ \ \lambda_{i_1} = \lambda_{i_2} \in \mathbb{R} \quad \textit{and} \quad \lambda_{i_3,i_4} \in \mathbb{C}/\ \mathbb{R}\,, \\ \textit{here } (i_1, i_2, i_3, i_4) \textit{ is a permutation of } (1,2,3,4) \Big\},$$

and

$$\Omega_3 := \Big\{ \big(\lambda_1, \lambda_2, \lambda_3, \lambda_4\big) \ \Big| \ \ \lambda_{1,2,3,4} \in \mathbb{C}/\ \mathbb{R} \Big\}.$$

For arbitrary $n > 4$, we extend the above results, obtaining sub-critical initial configurations which yield global smooth solutions, see [W07].

1.2 Spectral dynamics of the velocity gradient field

In this section, we introduce the basic lemma of the spectral dynamics of the velocity gradient field, [LT02]. This lemma is the main tool when dealing with multi-dimensional Euler dynamics.

Let us recall the Euler dynamics

$$u_t + u \cdot \nabla u = F(u, Du, \cdots)\,. \tag{1.17}$$

Differentiation of the above equation with respect to x yields

$$\partial_t M + (u \cdot \nabla) M + M^2 = \nabla F\,, \tag{1.18}$$

where $M := \nabla u$. To quantify all the entries of the velocity gradient tensor M is usually difficult. Instead, people analyze suitable linear combinations such as divergence and vorticity. Here we focus on the special role played by the eigenvalues of M, which depend on the entries of M in a strong nonlinear way.

Lemma 1.2.1 *(Spectral dynamics,[LT02]). Consider the general dynamical system (1.18) associated with the arbitrary velocity field u and forcing F. Let $\lambda(M)$ be a (possibly complex) eigenvalue of M associated with the corresponding left and right eigenvectors l and r, where l and r be normalized so that $lr = 1$. Then the dynamics of $\lambda(M)$ is governed by the corresponding Ricatti-like equation*

$$\partial_t \lambda + u \cdot \nabla \lambda + \lambda^2 = \langle l, \nabla F r \rangle .$$

Proof. Let the left and right eigenvectors of M associated with λ be l and r, normalized so that $lr = 1$. Then

$$Mr = \lambda r, \qquad lM = \lambda l .$$

Differentiation of the first relation with respect to t gives

$$\partial_t M r + M \partial_t r = \partial_t \lambda r + \lambda \partial_t r .$$

Multiply l on the left of this equation to obtain

$$l \partial_t M r + \lambda l \partial_t r = \partial_t \lambda + \lambda l \partial_t r ,$$

whence

$$l \partial_t M r = \partial_t \lambda .$$

Similarly differentiation of the relation $Mr = \lambda r$ with respect to x_j leads to

$$\partial_j M r + M \partial_j r = \partial_j \lambda r + \lambda \partial_j r.$$

Multiply this equation on the left by lu_j with $lr = 1$ to get

$$l u_j \partial_j M r = u_j l \partial_j \lambda r = u_j \partial_j \lambda .$$

Therefore

$$lu \cdot \nabla M r = u \cdot \nabla \lambda$$

A combination of the above facts together with $lM^2 r = \lambda^2$ gives

$$\partial_t \lambda + u \cdot \nabla \lambda + \lambda^2 = \langle l, \nabla F r \rangle \, .$$

This completes the proof. $\square$

1.3 Outline of the thesis

In Chapter 2, we present the results of the one-dimensional Euler-Poisson equations. First, we review the 2×2 system of isentropic gas dynamics, its Riemann invariants and its generic finite-time breakdown. Next, we review the pure Euler-Poisson equations, and its critical threshold. Then, we discuss the full Euler-Poisson equations driven by both pressure and Poisson forcing term. We give the critical threshold for the isothermal case and sub-critical initial data for $\gamma > 1$. At last, we show examples of finite-time breakdown of the full Euler-Poisson equations.

In Chapter 3, we present results of the two-dimensional system of Burgers equations. We begin with the irrotational initial data, showing that the vanishing viscosity limit is the solution of the equation of the corresponding inviscid fluid. Then we move to general initial data, and we obtain an uniform BV-bound of the viscous solutions. Finally, we prove that if ∇u_0 does not have negative eigenvalues, then the equation of the inviscid fluid admits a smooth solution, and this smooth solution is the the vanishing viscosity limit of the viscous fluids.

In the last Chapter, we deal with the restricted Euler dynamics. We compare the critical thresholds for three-dimensional and four-dimensional cases, showing that the finite-time breakdown is generic for three-dimensional restricted Euler dynamics and there is a surprising global existence for sub-critical initial data of four-dimensional restricted Euler dynamics. And finally, we extend our results to the general n-dimensional case.

Chapter 2

Critical thresholds of the one-dimensional Euler-Poisson system

2.1 Euler-Poisson system and related models

It is well known that the systems of Euler equations for compressible flows can and will breakdown at a finite time even if the initial data are smooth. A prototype example for such systems is provided by the 2×2 system of isentropic gas dynamics

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ (\rho u)_t + (\rho u^2)_x = -p_x \,, \end{cases} \tag{2.1}$$

where the pressure $p = p(\rho)$ is given by the usual γ-law, $p(\rho) = A\rho^{\gamma}$. By using the method introduced in [La64] to deal with pairs of conservation laws, it can be shown that (2.1) will lose the C^1-smoothness due to the appearance of shock discontinuities unless its two Riemann invariants are nondecreasing — we will go through the details in section 2.2. Thus, the finite-time breakdown of (2.1) is generic in the sense that it holds for all but a "small set" of initial data.

On the other hand, if we replace the pressure by Poisson forcing, then we arrive at the system of Euler-Poisson equations

$$\begin{cases} \rho_t + (\rho u)_x = 0 \,, \\ (\rho u)_t + (\rho u^2)_x = -k\rho\varphi_x \qquad k > 0 \,, \end{cases} \tag{2.2}$$

subject to initial data $(u_0 \,, \rho_0)$. Here $\varphi = \varphi(\rho)$ is the potential, which is dictated by the (one-dimensional) Poisson equation, $\varphi_{xx} = -\rho$. In this case, there is a "large set" of initial configurations which yield global smooth solutions. More precisely, [ELT01] have shown that (2.2) admits a global smooth solution if and only if

$$u_{0x}(x) > -\sqrt{2k\rho_0(x)}\,. \tag{2.3}$$

Thus, following the terminology of [LT02], the curve $u_{0x} + \sqrt{2k\rho_0} = 0$ is a "critical threshold" in configuration space which separates between initial configurations leading to finite-time breakdown and a "large set" of sub-critical initial configurations which yield global smooth solutions. In particular, (2.3) allows negative velocity gradients (depending on the local amplitude of the density), which otherwise are excluded in the case of Burgers equations of inviscid fluids, corresponding to $k = 0$.

In this Chapter we turn our attention to the full Euler-Poisson equations driven by both pressure and Poisson forcing,

$$\begin{cases} \rho_t + (\rho u)_x = 0\,, \\ (\rho u)_t + (\rho u^2)_x = -p(\rho)_x - k\rho\varphi_x, \quad k > 0, \\ -\varphi_{xx} = \rho\,. \end{cases} \tag{2.4}$$

The natural question that arises in the present context of full Euler-Poisson equations (2.4) is whether the pressure enforces a generic finite-time breakdown or, whether the presence of Poisson forcing preserves global regularity for a "large set" of initial configurations. We answer this question of "competition" between pressure and Poisson forcing, proving that the Euler-Poisson equations (2.4) with $\gamma \geq 1$ admit global smooth solutions for a "large set" of sub-critical initial data such that

$$u_{0x}(x) > -K_0\sqrt{\rho_0(x)} + \sqrt{A\gamma}\frac{|\rho_{0x}(x)|}{\rho_0(x)^{\frac{3-\gamma}{2}}}, \qquad \gamma \geq 1. \tag{2.5}$$

Here, K_0 is a constant depending on k, γ and the initial data. In the particular (and important) case of isothermal equations, $\gamma = 1$, we have $K_0 = \sqrt{2k}$ and (2.5) amounts to a sharp critical threshold,

$$u_{0x}(x) \geq -\sqrt{2k\rho_0(x)} + \sqrt{A}\frac{|\rho_{0x}(x)|}{\rho_0(x)}, \qquad \gamma = 1. \tag{2.6}$$

The inequalities (2.5),(2.6) quantify the competition between the destabilizing pressure effects, as the range of sub-critical initial configurations shrinks with the growth of the amplitude of the pressure, A, while the stabilizing effect of the Poisson forcing increases the sub-critical range with a growing k. In particular, (2.6) with $A = 0$ recovers the pressure-free critical threshold (2.3).

This chapter is organized as follows. In section 2.2, we review the 2×2 system of isentropic gas dynamics, its Riemann invariants and its generic finite-time breakdown. In section 2.3, we review the pure Euler-Poisson system and its critical threshold. In section 2.4, we derive the critical threshold of the full Euler-Poisson system. We begin by reformulating the system (2.4) with its Riemann invariants as a preparation for the analysis carried out in subsections 2.4.2 and 2.4.3. In subsection 2.4.2, we prove our main results, providing sufficient conditions for "large sets" of sub-critical initial configurations which yield global smooth solution. In section 2.4.3, we give examples of finite-time breakdown for super-critical initial data. Combining our results in sections 2.4.2 and 2.4.3, we confirm the existence of a critical threshold phenomena for the full Euler-Poisson equations (2.4).

2.2 Isentropic gas dynamics

In this section, we use Lax's method ([La64]), to show that the finite-time breakdown of isentropic gas dynamics

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ (\rho u)_t + (\rho u^2)_x = -p_x\,, \qquad p(\rho) = A\rho^\gamma\,, \end{cases} \tag{2.7}$$

is generic.

Note that system (2.7) consists of two conservation laws. And the most significant difference between systems of two conservation laws and systems consisting of more than two is the existence of Riemann invariants for the first class.

To find the Riemann invariants of (2.7), we rewrite the system as

$$\begin{pmatrix} \rho \\ u \end{pmatrix}_t + J \begin{pmatrix} \rho \\ u \end{pmatrix}_x = \begin{pmatrix} 0 \\ -k\varphi_x \end{pmatrix}, \tag{2.8}$$

where the Jacobian $J := \begin{pmatrix} u & \rho \\ A\gamma\rho^{\gamma-2} & u \end{pmatrix}$ has two different eigenvalues

$$\lambda := u - \sqrt{A\gamma}\rho^{\frac{\gamma-1}{2}} < \mu := u + \sqrt{A\gamma}\rho^{\frac{\gamma-1}{2}}\,.$$

The corresponding left eigenvectors are

$$l_\lambda = \Big(-\sqrt{A\gamma}\rho^{\frac{\gamma-3}{2}},\, 1 \Big)\,,$$

and

$$l_\mu = \Big(\sqrt{A\gamma}\rho^{\frac{\gamma-3}{2}},\, 1 \Big)\,.$$

Thus

$$\Big(-\sqrt{A\gamma}\rho^{\frac{\gamma-3}{2}}\rho_t + u_t \Big) + \lambda \Big(-\sqrt{A\gamma}\rho^{\frac{\gamma-3}{2}}\rho_x + u_x \Big) = 0\,, \tag{2.9a}$$

and

$$\left(\sqrt{A\gamma}\rho^{\frac{\gamma-3}{2}}\rho_t + u_t\right) + \mu\left(\sqrt{A\gamma}\rho^{\frac{\gamma-3}{2}}\rho_x + u_x\right) = 0\,. \tag{2.9b}$$

Let R and S be functions of ρ and u which take the following forms

$$R := u - \frac{2\sqrt{A\gamma}}{\gamma-1}\rho^{\frac{\gamma-1}{2}}\,, \qquad S := u + \frac{2\sqrt{A\gamma}}{\gamma-1}\rho^{\frac{\gamma-1}{2}}\,, \qquad \text{for} \qquad \gamma > 1\,,$$

and

$$R := u - \sqrt{A}\ln\rho\,, \qquad S := u + \sqrt{A}\ln\rho\,, \qquad \text{for}\gamma = 1\,.$$

Then

$$\operatorname{grad} R = \left(-\sqrt{A\gamma}\rho^{\frac{\gamma-3}{2}},\, 1\right)^T,$$

and

$$\operatorname{grad} S = \left(\sqrt{A\gamma}\rho^{\frac{\gamma-3}{2}},\, 1\right)^T.$$

It follows (2.9a) and (2.9b) that

$$R_t + \lambda R_x = 0\,, \tag{2.10a}$$

and

$$S_t + \mu S_x = 0\,. \tag{2.10b}$$

In this chapter, here and below, $\{\}^{\backprime} := \partial_t + \lambda\partial_x$ and $\{\}' := \partial_t + \mu\partial_x$ denote differentiation along the λ and μ particle paths,

$$\Gamma_\lambda := \left\{(x,t)\,\middle|\,\dot{x}(t) = \lambda\big(\rho(x,t), u(x,t)\big)\right\},$$

$$\Gamma_\mu := \left\{(x,t)\,\middle|\,\dot{x}(t) = \mu\big(\rho(x,t), u(x,t)\big)\right\}.$$

Then (2.10a) and (2.10b) can be rewritten as

$$R^{\backprime} = 0\,, \tag{2.11a}$$

and

$$S' = 0\,. \tag{2.11b}$$

Thus, as functions of x and t, R is constant along the λ particle path, and S is constant along the μ particle path. This is the reason why R and S are called *Riemann invariants.*

If we set $r := R_x$, $s := S_x$ then upon differentiation of (2.32) we get

$$r_t + \lambda r_x + \lambda_S rs + \lambda_R r^2 = 0\,, \tag{2.12a}$$

$$s_t + \mu s_x + \mu_S s^2 + \mu_R rs = 0\,. \tag{2.12b}$$

I.e.,

$$r^{\backprime} + \lambda_S rs + \lambda_R r^2 = 0\,, \tag{2.13a}$$

$$s' + \mu_S s^2 + \mu_R rs = 0\,. \tag{2.13b}$$

Next, we observe that $\lambda = \frac{R+S}{2} - \frac{\gamma-1}{4}(S-R)$ and $\mu = \frac{R+S}{2} + \frac{\gamma-1}{4}(S-R)$. Hence, expressed in terms of $\theta := \dfrac{\gamma-1}{2}$, we have for $\gamma \geq 1$,

$$\lambda_R = \mu_S = \frac{1+\theta}{2} \quad \text{and} \quad \lambda_S = \mu_R = \frac{1-\theta}{2}, \qquad \theta := \frac{\gamma-1}{2} \geq 0,$$

and the pair of equations (2.13) is recast into the form

$$r^{\backprime} + \frac{1+\theta}{2}r^2 + \frac{1-\theta}{2}rs = 0, \tag{2.14a}$$

$$s' + \frac{1+\theta}{2}s^2 + \frac{1-\theta}{2}rs = 0\,. \tag{2.14b}$$

It follows from (2.10b) that

$$s = S_x = \frac{S^{\backprime}}{\lambda - \mu} = -\frac{S^{\backprime}}{\theta(S-R)}\,. \tag{2.15}$$

Substituting (2.15) into (2.14a) we obtain

$$r^{\grave{}} + \frac{1+\theta}{2} r^2 - r \frac{1-\theta}{2\theta} \frac{S^{\grave{}}}{S-R} = 0 \,. \tag{2.16}$$

Let $h := -\dfrac{1-\theta}{2\theta} \ln(S-R)$. Then h satisfies

$$h_S = -\frac{1-\theta}{2\theta} \frac{1}{S-R} \,.$$

We know that along the λ particle path, $R^{\grave{}} = 0$; thus

$$h^{\grave{}} = h_R R^{\grave{}} + h_S S^{\grave{}} = -\frac{1-\theta}{2\theta} \frac{1}{S-R} S^{\grave{}} \,.$$

Substituting this into (2.16) we obtain

$$r^{\grave{}} + \frac{1+\theta}{2} r^2 + h^{\grave{}} r = 0 \,. \tag{2.17}$$

Multiplying (2.17) by e^h, and using the abbreviations

$$q := e^h r, \qquad w = \frac{1+\theta}{2} e^{-h} \,, \tag{2.18}$$

we rewrite the resulting equation as

$$q^{\grave{}} + w q^2 = 0 \,. \tag{2.19}$$

An explicit formula for q can be found:

$$q(t) = \frac{q_0}{1 + q_0 W(t)} \,, \tag{2.20}$$

where $q_0 = q(0)$ and

$$K(t) = \int_0^t w dt \,,$$

with the integration along the λ particle path.

Suppose the initial values of R and S are bounded. Then R and S stay between the same bounds for all time, since R and S are constant along characteristics. The quantity $\frac{1+\theta}{2}e^{-h}$ has then a positive lower bound w_0. Thus $W(t)$ satisfies

$$W(t) \geq w_0 t \qquad \text{for all} \qquad t \geq 0\,. \tag{2.21}$$

Substituting (2.21) into (2.20), we conclude that if $q_0 \geq 0$, then $q(t)$ stays bounded, and if $q_0 < 0$, then $q(t)$ becomes unbounded after a finite time. A similar result holds for the other variable S. To summarize, we have the following result.

Theorem 2.2.1 *Let* (ρ, u) *be a solution of* (2.7) *whose initial values are bounded. Then the solution remains in* C^1 *if and only if* $R_{0x} \geq 0$ *and* $S_{0x} \geq 0$.

2.3 Pure Euler-Poisson systems

In this section, we reiterate the results in [ELT01], illustrating that there is a "large set" of initial configurations which yield global smooth, C^1-solutions for the pure Euler-Poisson system

$$\rho_t + (\rho u)_x = 0\,, \tag{2.22a}$$

$$(\rho u)_t + (\rho u^2)_x = -k\rho\varphi_x \qquad k > 0\,, \tag{2.22b}$$

$$\varphi_{xx} = -\rho\,. \tag{2.22c}$$

Set $d := u_x(x,t)$, then differentiation of (2.22a) and (2.22b) yields

$$\dot{\rho} + \rho d = 0\,, \tag{2.23a}$$

$$\dot{d} + d^2 = k\rho\,, \qquad \dot{\{\}} := \partial_t + u\partial_x\,. \tag{2.23b}$$

Multiply (2.23b) by $\dfrac{1}{\rho}$, (2.23a) by $\dfrac{d}{\rho^2}$, and take the difference. This gives

$$\dot{\left(\frac{d}{\rho}\right)} = \frac{\rho\dot{d} - d\dot{\rho}}{\rho^2} = k\,, \tag{2.24}$$

and upon integration one gets

$$\frac{d}{\rho} = \beta(t) := kt + \frac{d_0}{\rho_0}\,. \tag{2.25}$$

Substituting (2.25) into (2.23) we obtain

$$\dot{\rho} + \beta(t)\rho^2 = 0\,, \tag{2.26a}$$

$$\dot{d} + d^2 = \frac{k}{\beta(t)}d\,. \tag{2.26b}$$

We then find explicit formulas for ρ and d:

$$\rho(t) = \frac{\rho_0}{1 + \rho_0 \int_0^t \beta(\tau)d\tau}\,, \tag{2.27a}$$

$$d(t) = \frac{d_0 e^{\int_0^t k/\beta(\tau)d\tau}}{1 + d_0 \int_0^t e^{\int_0^\tau k/\beta(s)ds} d\tau}\,. \tag{2.27b}$$

Direct calculation shows that

$$1 + \rho_0 \int_0^t \beta(\tau)d\tau = 1 + d_0 \int_0^t e^{\int_0^\tau k/\beta(s)ds} d\tau = 1 + d_0 t + \frac{k}{2}\rho_0 t^2\,.$$

Thus, the boundedness or not of ρ and d depends on whether $1 + d_0 t + \dfrac{k}{2}\rho_0 t^2$ ever takes on the value 0. It follows that if $d_0 > -\sqrt{2k\rho_0}$ then global regularity for ρ and d is ensured. To summarize, we obtain the following result, (Theorem 2.2, [ELT01]):

Theorem 2.3.1 *The system of Euler-Poisson equations* (2.22a)-(2.22c) *admits a global smooth solution if and only if*

$$u_{0x}(\alpha) > -\sqrt{2k\rho_0(\alpha)}\,, \qquad \forall\, \alpha \in \mathbb{R}\,. \tag{2.28}$$

In this case the solution of (2.22a)-(2.22c) *is given by*

$$\rho(x(\alpha,t),t) = \frac{\rho_0}{\Gamma(\alpha,t)}\,, \qquad u_x(x(\alpha,t),t) = \frac{u_{0x}+k\rho_0 t}{\Gamma(\alpha,t)}\,,$$

$$\Gamma(\alpha,t) := 1 + u_{0x}t + \frac{k\rho_0 t^2}{2}\,,$$

so that $\rho \sim t^{-2}$ *and* $u_x \sim t^{-1}$ *as long as* $\rho_0 \neq 0$ *. If condition* (2.28) *fails, then the solution breaks down at the finite time,* t_c *, where* $\Gamma(\alpha,t_c)=0$ *.*

Thus, following the terminology of [LT02], the curve $u_{0x}+\sqrt{2k\rho_0}=0$ is a "critical threshold" in configuration space which separates between initial configurations leading to finite-time breakdown and a "large set" of sub-critical initial configurations which yield global smooth solutions. In particular, (2.28) allows negative velocity gradients (depending on the local amplitude of the density), which otherwise are excluded in the case of Burgers equations of inviscid fluids, corresponding to $k = 0$.

2.4 Euler-Poisson equations with γ-law pressure

In this section, we use a variant of Lax's method to find subcritical initial data of the full Euler-Poisson equation

$$\begin{cases} \rho_t + (\rho u)_x = 0\,, \\ (\rho u)_t + (\rho u^2)_x = -p(\rho)_x - k\rho\varphi_x, \quad k>0, \\ -\varphi_{xx} = \rho\,, \end{cases} \tag{2.29}$$

where the pressure $p = p(\rho)$ is given by the usual γ-law, $p(\rho) = A\rho^\gamma$ with $\gamma \geq 1$.

2.4.1 Riemann invariants

In this subsection, we reformulate system (2.29) with its Riemann invariants as a preparation for the analysis carried out in following subsections.

2.4.1.1 The Euler-Poisson equations with γ-law pressure: $\gamma > 1$

We begin by rewriting the Euler-Poisson equations (2.4) as a first order quasi-linear system

$$\begin{pmatrix} \rho \\ u \end{pmatrix}_t + J \begin{pmatrix} \rho \\ u \end{pmatrix}_x = \begin{pmatrix} 0 \\ -k\varphi_x \end{pmatrix}. \tag{2.30}$$

As showed in section 2.2, the Jacobian $J := \begin{pmatrix} u & \rho \\ A\gamma\rho^{\gamma-2} & u \end{pmatrix}$ has two different eigenvalues

$$\lambda := u - \sqrt{A\gamma}\rho^{\frac{\gamma-1}{2}} < \mu := u + \sqrt{A\gamma}\rho^{\frac{\gamma-1}{2}}\,.$$

And let R and S denote the Riemann invariants of the corresponding Euler system (2.1)

$$R := u - \frac{2\sqrt{A\gamma}}{\gamma-1}\rho^{\frac{\gamma-1}{2}} \quad \text{and} \quad S := u + \frac{2\sqrt{A\gamma}}{\gamma-1}\rho^{\frac{\gamma-1}{2}}\,. \tag{2.31}$$

They satisfy the coupled system of equations,

$$R_t + \lambda R_x = -k\varphi_x\,, \tag{2.32a}$$

$$S_t + \mu S_x = -k\varphi_x\,, \tag{2.32b}$$

coupled through the Poisson equation $-\phi_{xx} = \rho$. Here we redo the process we did in section 2.2. Let us set $r := R_x$, $s := S_x$ then upon differentiation of (2.32) we get

$$r_t + \lambda r_x + \lambda_S rs + \lambda_R r^2 = k\rho\,, \tag{2.33a}$$

$$s_t + \mu s_x + \mu_S s^2 + \mu_R rs = k\rho\,. \tag{2.33b}$$

Next, we observe that $\lambda = \frac{R+S}{2} - \frac{\gamma-1}{4}(S-R)$ and $\mu = \frac{R+S}{2} + \frac{\gamma-1}{4}(S-R)$. Hence, expressed in terms of $\theta := \dfrac{\gamma-1}{2}$, we have for $\gamma \geq 1$,

$$\lambda_R = \mu_S = \frac{1+\theta}{2} \quad \text{and} \quad \lambda_S = \mu_R = \frac{1-\theta}{2}, \qquad \theta := \frac{\gamma-1}{2} \geq 0,$$

and the pair of equations (2.33) is recast into the form

$$r^{\grave{}} + \frac{1+\theta}{2}r^2 + \frac{1-\theta}{2}rs = k\rho, \tag{2.34a}$$

$$s' + \frac{1+\theta}{2}s^2 + \frac{1-\theta}{2}rs = k\rho\,. \tag{2.34b}$$

Here $\{\}^{\grave{}} := \partial_t + \lambda\partial_x$ and $\{\}' := \partial_t + \mu\partial_x$ denote differentiation along the λ and μ particle paths,

$$\Gamma_\lambda := \big\{(x,t)\,\big|\,\dot{x}(t) = \lambda\big(\rho(x,t), u(x,t)\big)\big\}, \quad \Gamma_\mu := \big\{(x,t)\,\big|\,\dot{x}(t) = \mu\big(\rho(x,t), u(x,t)\big)\big\}.$$

To continue, we rewrite the equation for ρ as

$$(\rho_t + \lambda\rho_x) + \frac{\mu-\lambda}{2}\rho_x + \rho\frac{s+r}{2} = 0\,. \tag{2.35}$$

Since $s - r = S_x - R_x = 2\sqrt{A\gamma}\rho^{\frac{\gamma-3}{2}}\rho_x$, we can express $\dfrac{\mu-\lambda}{2}\rho_x = \sqrt{A\gamma}\rho^{\frac{\gamma-1}{2}}\rho_x = \rho\dfrac{s-r}{2}$, so that the ρ equation (2.35) can be written along the λ particle path as $\rho^{\grave{}} + \rho s = 0$. Similarly, it can be written along the μ particle path as $\rho' + \rho r = 0$. Assembling the above equations, we arrive at the following system governing r, s

and ρ,

$$\left\{\begin{aligned} r' + \frac{1+\theta}{2} r^2 + \frac{1-\theta}{2} rs &= k\rho\,, \\ \rho' + \rho s &= 0\,, \end{aligned}\right. \tag{2.36a}$$

and

$$\left\{\begin{aligned} s' + \frac{1+\theta}{2} s^2 + \frac{1-\theta}{2} rs &= k\rho\,, \\ \rho' + \rho r &= 0. \end{aligned}\right. \tag{2.36b}$$

Finally, we use the integration factors $1/\sqrt{\rho}$ and $r/2\rho\sqrt{\rho}$ in the first and second equations of each pair in (2.36), to conclude that

$$\left(\frac{r}{\sqrt{\rho}}\right)' + \frac{1+\theta}{2}\frac{r^2}{\sqrt{\rho}} - \frac{\theta}{2}\frac{rs}{\sqrt{\rho}} = k\sqrt{\rho}, \tag{2.37a}$$

$$\left(\frac{s}{\sqrt{\rho}}\right)' + \frac{1+\theta}{2}\frac{s^2}{\sqrt{\rho}} - \frac{\theta}{2}\frac{rs}{\sqrt{\rho}} = k\sqrt{\rho}. \tag{2.37b}$$

2.4.1.2 The isothermal case $\gamma = 1$

In this case, the two eigenvalues are $\lambda = u - \sqrt{A} < \mu = u + \sqrt{A}$ with the corresponding Riemann invariants $R = u - \sqrt{A}\ln\rho$ and $S = u + \sqrt{A}\ln\rho$. Their derivatives, r and s, satisfy the pair of equations, corresponding to (2.37a), (2.37b) with $\theta = (\gamma - 1)/2 = 0$,

$$\left(\frac{r}{\sqrt{\rho}}\right)' + \frac{1}{2}\frac{r^2}{\sqrt{\rho}} = k\sqrt{\rho}, \tag{2.38a}$$

$$\left(\frac{s}{\sqrt{\rho}}\right)' + \frac{1}{2}\frac{s^2}{\sqrt{\rho}} = k\sqrt{\rho}. \tag{2.38b}$$

2.4.2 Global smooth solutions for sub-critical initial data

For the pressureless Euler-Poisson equations (2.2), the evolution of u_x and ρ could be traced backwards along the same particle path to their initial data at $t = 0$. The scenario becomes more complicated with the additional pressure term, due to the coupling of r and s along *different* particle paths which are traced back to different neighborhoods of the initial line $t = 0$. This is the main obstacle in finding the sharp critical threshold of the full Euler-Poisson system (2.4). To this end, we will seek invariant regions for the coupled system, governing the Riemann invariants. We begin this section with the following lemma.

Lemma 2.4.1 *Let the total charge* $E_0 := \int_{-\infty}^{\infty} \rho_0(x)dx < \infty$. *Then* $\rho(x,t)$ *and* $u(x,t)$ *remain uniformly bounded for all* $t > 0$.

Proof. Under the given condition, we can set (e.g., [ELT01, p. 116])

$$\varphi_x(x,t) = \frac{1}{2}\bigg(\int_{-\infty}^{x} \rho(\xi,t)d\xi - \int_{x}^{\infty} \rho(\xi,t)d\xi \bigg),$$

which satisfies $-E_0 \le \varphi_x(x,t) \le E_0$, for all $t \ge 0$ and $x \in \mathbb{R}$. Recall the transport equations (2.32a), (2.32b) which govern the Riemann invariants along different characteristics $R' + k\varphi_x = S' + k\varphi_x = 0$. Since φ_x is bounded, these transport equations tell us that R and S remain uniformly bounded with at most a linear growth in time. Indeed, for all $M \gg 1$ we have

$$\sup_{|x| \le M} \{|R(x,t)|, |S(x,t)|\} \le C_0 + kE_0 t, \quad C_0 := \sup_{|x| \le M + u_\infty t} \{|R_0(x)|, |S_0(x)|\}. \quad (2.39)$$

Take the sum and difference of S and R to find that $u(x,t)$ and $\rho(x,t)$ in (2.31) remain bounded,

$$u_\infty := \sup_{|x|\le M} |u(x,t)| \le C_0 + kE_0 t, \tag{2.40a}$$

$$\sup_{|x|\le M} \rho(x,t) \le Const. \begin{cases} (C_0 + kE_0 t)^{\frac{2}{\gamma-1}}, & \gamma > 1, \\ \exp(kE_0 t), & \gamma = 1. \end{cases} \tag{2.40b}$$

□

We note in passing that the time growth asserted in (2.40) is probably not sharp; the estimate can be improved after taking into account the *uniform bounds* of $R_x/\sqrt{\rho}$ and $S_x/\sqrt{\rho}$ discussed in theorems 2.4.1 and 2.4.2 below.

2.4.2.1 Critical Threshold for isothermal case: $\gamma = 1$

We begin with the isothermal case, $\gamma = 1$, which plays an important role in various applications. Compared with the general case (2.37), the isothermal case becomes simpler due to the fact that $\theta = 0$ decouples the dependence on r and s through the mixed term θrs, which disappears from left-hand side of (2.38). Here we prove the following sharp characterization of the critical threshold phenomena.

Theorem 2.4.1 *Consider the isothermal Euler-Poisson system* (2.4) *with pressure forcing* $p(\rho) = A\rho$, *and subject to initial data* $(u_0, \rho_0 > 0)$ *with finite total charge,* $E_0 = \int_{-\infty}^{\infty} \rho_0(x)dx < \infty$. *The system admits a global smooth,* C^1*-solution if and only if*

$$u_{0x}(x) \ge -\sqrt{2k\rho_0(x)} + \sqrt{A}\frac{|\rho_{0x}(x)|}{\rho_0(x)}, \quad \forall x \in \mathbb{R}. \tag{2.41}$$

Remark 2.4.1 *Expressed in terms of the Riemann invariants specified in* §2.4.1.2, $u_x \pm \sqrt{A}\rho_x/\rho$, *theorem* 2.4.1 *states that the isothermal Euler-Poisson equations admit* *global smooth solutions for sub-critical initial conditions,*

$$s_0 \geq -\sqrt{2k\rho_0} \quad \text{and} \quad r_0 \geq -\sqrt{2k\rho_0}. \tag{2.42}$$

Proof. We define $X := \dfrac{r}{\sqrt{\rho}}$ and $Y := \dfrac{s}{\sqrt{\rho}}$. Equations (2.38a), (2.38b) then read

$$X' = \frac{\sqrt{\rho}}{2}(2k - X^2), \tag{2.43a}$$

$$Y' = \frac{\sqrt{\rho}}{2}(2k - Y^2). \tag{2.43b}$$

It follows that

$$X' \begin{cases} > 0\,, & X \in (-\sqrt{2k}, \sqrt{2k}\,)\,, \\ = 0\,, & |X| = \sqrt{2k}\,, \\ < 0\,, & |X| > \sqrt{2k}, \end{cases}$$

and similarly,

$$Y' \begin{cases} > 0\,, & Y \in (-\sqrt{2k}, \sqrt{2k}\,)\,, \\ = 0\,, & |Y| = \sqrt{2k}\,, \\ < 0\,, & |Y| > \sqrt{2k}\,. \end{cases}$$

Thus, starting with (2.42), $X_0, Y_0 \geq -\sqrt{2k}$, we find that X and Y remain bounded within the invariant region $[-\sqrt{2k}, \sqrt{2k}]$, and they are decreasing outside this interval. We conclude that

$$X(\cdot, t), Y(\cdot, t) \leq \max\left\{\sqrt{2k}, X_0(\cdot), Y_0(\cdot)\right\}.$$

Lemma 2.4.1 tells us that ρ is bounded. The boundedness of X , Y and ρ imply that $r = X\sqrt{\rho}$ and $s = Y\sqrt{\rho}$ remain bounded for all $t < \infty$, and hence the Euler-Poisson system (2.4) admits a global smooth C^1-solution.

Conversely, suppose that there exists $X_0 = X(x_0) < -\sqrt{2k}$. We will show that this value will evolve along $\Gamma_\lambda(x_0, 0)$ such that $X(\cdot, t)$ will tend to $-\infty$ at a finite time. To this end, assume that Y is well behaved, i.e., $Y_0(\cdot) \geq -\sqrt{2k}$ so that $Y(\cdot, t) \leq Y_1 := \max\left\{Y_0(\cdot), \sqrt{2k}\right\}$ for all t's (otherwise, the finite-time blow up of Y can be argued along the same lines). It follows that $s = Y\sqrt{\rho} \leq Y_1\sqrt{\rho}$ and inserting this into $\rho' = -\rho s$, we find $\rho' \geq -Y_1\rho^{3/2}$. This yields the lower-bound

$$\rho \geq \left(\frac{2}{Y_1 t + 2/\sqrt{\rho_0}}\right)^2,$$

which together with (2.43a), implies that $X(\cdot, t)$ satisfies the following Ricatti equation along the Γ_λ-path,

$$X' \leq -\frac{X_1}{Y_1 t + 2/\sqrt{\rho_0}} X^2, \qquad X_1 := (X_0^2 - 2k)/X_0^2 > 0. \tag{2.44}$$

Integration of (2.44) yields

$$X(\cdot, t) \leq \frac{Y_1}{X_1 \ln\left(1 + \sqrt{\rho_0} Y_1 t/2\right) + Y_1 X_0} \tag{2.45}$$

Thus, starting with $X_0 < -\sqrt{2k} < 0$ we find that there exists a finite critical time $t_c > 0$ such that $X(t \uparrow t_c)$ tends to $-\infty$. $\square$

The critical threshold condition (2.41) reflects the competition between the Poisson forcing and the pressure. It yields global smooth solutions for a "large set" of initial configurations allowing negative velocity gradients. In the particular case that there is no pressure, $A = 0$, (2.41) is reduced to the critical threshold condition of the "pressureless" Euler-Poisson equations $u_{0x} > -\sqrt{2k\rho_0(x)}$ of [ELT01].

2.4.2.2 Critical threshold for $\gamma > 1$

The equations for the Riemann invariants (2.37a), (2.37b) are coupled through the mixed term, $\theta rs/2$. We note in passing that it is possible to get rid of this mixed term when integrating (2.36a), (2.36b) with the integration factors $\rho^{(\gamma-3)/4}$, and $r\rho^{(\gamma-7)/4}(3-\gamma)/4$ in the first and second equations in each pair, yielding

$$\left(r\rho^{\frac{\theta-1}{2}}\right)' + \frac{1+\theta}{2} r^2 \rho^{\frac{\theta-1}{2}} = k\rho^{\frac{1+\theta}{2}},$$
$$\left(s\rho^{\frac{\theta-1}{2}}\right)' + \frac{1+\theta}{2} s^2 \rho^{\frac{\theta-1}{2}} = k\rho^{\frac{1+\theta}{2}}.$$

Nevertheless, it will prove useful to use the same integration factors, $1/\sqrt{\rho}$ and $r/2\rho\sqrt{\rho}$ which led to (2.37). The main task is to identify the invariant region associated with (2.37), corresponding to the isothermal invariant region $[-\sqrt{2k}, \sqrt{2k}]$ discussed in theorem 2.4.1.

Theorem 2.4.2 *Consider the Euler-Poisson system* (2.4) *with γ-law pressure $p(\rho) = A\rho^\gamma$, $\gamma > 1$, subject to initial data $(u_0, \rho_0 > 0)$ with finite total charge, $E_0 = \int_{-\infty}^{\infty} \rho_0(x)dx < \infty$. Then, there exists a constant $K_0 > 0$ depending on k, γ and the initial conditions (specified in* (2.47b) *below), such that the Euler-Poisson equations* (2.4) *admit a global smooth, C^1-solution if,*

$$u_{0x}(x) \geq -K_0\sqrt{\rho_0(x)} + \sqrt{A\gamma}\frac{|\rho_{0x}(x)|}{\rho_0(x)^{\frac{3-\gamma}{2}}}. \tag{2.46}$$

Before we turn to the proof of this theorem, several remarks are in order.

Remark 2.4.2 *Expressed in terms of the Riemann invariants, $r = u_x - \sqrt{A\gamma}\rho_{0x}/\rho_0^{(3-\gamma)/2}$*

and $s = u_x + \sqrt{A\gamma}\rho_{0x}/\rho_0^{(3-\gamma)/2}$*, the critical threshold* (2.46) *reads*

$$\frac{r_0(x)}{\sqrt{\rho_0(x)}}, \frac{s_0(x)}{\sqrt{\rho_0(x)}} \geq -K_0. \tag{2.47a}$$

The constant K_0 *is given by*

$$K_0 = \frac{-\theta M_0 + \sqrt{\theta^2 M_0^2 + 8k(1+\theta)}}{2(1+\theta)}, \qquad M_0 = \max_x \left\{ \sqrt{2k}, \frac{r_0(x)}{\sqrt{\rho_0(x)}}, \frac{s_0(x)}{\sqrt{\rho_0(x)}} \right\}. \tag{2.47b}$$

We mention two simplifications which are summarized in the following two corollaries. We first observe that if the initial configurations satisfy the *upper-bound* $r_0(x), s_0(x) \leq \sqrt{2k\rho_0(x)}$, then (2.47b) yields $M_0 = \sqrt{2k}$, hence $K_0 = \dfrac{\sqrt{2k}}{1+\theta}$, and theorem 2.4.2 implies the following.

Corollary 2.4.1 *Consider the Euler-Poisson system* (2.4) *with* γ*-law pressure* $p(\rho) = A\rho^\gamma$*,* $\gamma > 1$*, subject to initial data* $(u_0, \rho_0 > 0)$ *with finite total charge,* $E_0 = \int_{-\infty}^{\infty} \rho_0(x)dx < \infty$*. Then, the Euler-Poisson equations* (2.4) *admit a global smooth,* C^1*-solution if for all* $x \in \mathbf{R}$*,*

$$|u_{0x}(x)| \leq \sqrt{2k\rho_0(x)} - \sqrt{A\gamma}\frac{|\rho_{0x}(x)|}{\rho_0(x)^{\frac{3-\gamma}{2}}}. \tag{2.48}$$

The next result follows from the trivial inequality

$$-K_0 \leq \frac{\theta M_0 - \big(\theta M_0 + \sqrt{8k(1+\theta)}\big)/\sqrt{2}}{2(1+\theta)}.$$

Corollary 2.4.2 *Consider the Euler-Poisson system* (2.4) *with a* γ*-law pressure* $p(\rho) = A\rho^\gamma$*,* $\gamma > 1$*, subject to initial data* $(u_0, \rho_0 > 0)$ *with finite total charge,* $E_0 = \int_{-\infty}^{\infty} \rho_0(x)dx < \infty$*. Then, the Euler-Poisson equations* (2.4) *admit a global smooth,* C^1*-solution, if for all* $x \in \mathbf{R}$*,*

$$u_{0x}(x) \geq -\sqrt{\frac{2k\rho_0(x)}{\gamma+1} + \sqrt{A\gamma}\frac{|\rho_{0x}(x)|}{\rho_0(x)^{\frac{3-\gamma}{2}}} +}$$
$$\Big(1-\frac{1}{\sqrt{2}}\Big)\frac{\gamma-1}{2(\gamma+1)}\max_x\left\{\sqrt{2k\rho_0(x)}, u_{0x}(x)+\sqrt{A\gamma}\frac{|\rho_{0x}(x)|}{\rho_0(x)^{\frac{3-\gamma}{2}}}\right\}. \tag{2.49}$$

Remark 2.4.3 *We observe that as in the isothermal case, the critical threshold in its various versions* (2.46), (2.47), (2.48) *and* (2.49)*, allow a "large set" of initial configurations with negative velocity gradient, due to the competition between the stabilizing Poisson forcing $k\rho\phi(\rho)_x$ and the destabilizing pressure $A(\rho^\gamma)_x$. In the extreme case that Poisson forcing is missing, (where $k = 0$), the breakdown of the system is generic unless u_{0x} is* positive enough *(so that $r_0, s_0 > 0$). In the other extreme of a "pressureless" Euler-Poisson system, $A = 0, \gamma = 1$, the critical thresholds* (2.46), (2.48) *are reduced to $u_{0x}(x) > -\sqrt{2k\rho_0(x)}$, which coincides with the "pressureless" critical threshold* (2.3) *found in [ELT01].*

Proof. Expressed in terms of $X := \dfrac{r}{\sqrt{\rho}}$ and $Y := \dfrac{s}{\sqrt{\rho}}$, equations (2.37) read

$$X` = \sqrt{\rho}\Big(k - \frac{1+\theta}{2}X^2 + \frac{\theta}{2}XY\Big), \tag{2.50a}$$

$$Y' = \sqrt{\rho}\Big(k - \frac{1+\theta}{2}Y^2 + \frac{\theta}{2}XY\Big). \tag{2.50b}$$

We seek an invariant region of the form $[-K_0, M_0]$, with $K_0, M_0 > 0$ yet to be determined. We begin by noticing that if $X, Y \leq M$ then[1] $X_+Y \leq M^2$, and

[1] We let $Z_+ = \max\{X, 0\}$ and $Z_- = \min\{Z, 0\}$ denote the positive and negative part of Z.

recalling that $\theta \geq 0$, (2.50) then yield

$$
\begin{aligned}
X` &\leq \sqrt{\rho}\left(k - \frac{1+\theta}{2}X^2 + \frac{\theta}{2}M^2\right), \quad X > 0, \\
Y' &\leq \sqrt{\rho}\left(k - \frac{1+\theta}{2}Y^2 + \frac{\theta}{2}M^2\right), \quad Y > 0.
\end{aligned}
$$

This in turn implies that

$$
X \text{ and } Y \text{ are decreasing if } X, Y > C_+, \quad C_+ = C_+(M) := \sqrt{\frac{2k + \theta M^2}{1+\theta}}. \tag{2.51}
$$

The solution of $C_+(M) = M$ yields $M = \sqrt{2k}$. Thus, X and Y are decreasing whenever $X, Y > M = \sqrt{2k}$, and we end up with the upper-bound

$$
X(\cdot, t), Y(\cdot, t) \leq M_0, \qquad M_0 := \max_x \left\{\sqrt{2k}, X_0(x), Y_0(x)\right\}. \tag{2.52}
$$

In a similar manner, we study the lower bound of the invariant region. By (2.52) and (2.50) yield

$$
X` \geq \sqrt{\rho}\left(k - \frac{1+\theta}{2}X^2 + \frac{\theta}{2}M_0 X\right), \quad X < 0, \tag{2.53a}
$$

$$
Y' \geq \sqrt{\rho}\left(k - \frac{1+\theta}{2}Y^2 + \frac{\theta}{2}M_0 Y\right), \quad Y < 0, \tag{2.53b}
$$

which in turn, imply that

$$
X \text{ and } Y \text{ are increasing if } 0 \geq X, Y > -K_0, \tag{2.54a}
$$

where K_0 is the smallest root of the quadratics on the right of (2.53),

$$
K_0 := \frac{-\theta M_0 + \sqrt{\theta^2 M_0^2 + 8k(1+\theta)}}{2(1+\theta)}. \tag{2.54b}
$$

The critical threshold condition (2.46) tells us that at $t = 0$, $X_0, Y_0 \geq -K_0$ and (2.54a) implies that $X(\cdot, t)$ and $Y(\cdot, t)$ remain above the same lower-bound, (2.46).

As before, the bounds of X, Y and ρ imply that $r = X\sqrt{\rho}$ and $s = Y\sqrt{\rho}$ remain bounded, and hence the Euler-Poisson system (2.4) a global smooth, C^1-solution.

$\square$

2.4.3 Finite-time breakdown for super-critical initial date

Consider the Euler-Poisson system (2.4) with a γ-law pressure, $\gamma \geq 1$, and subject to initial data such that $r_0(x), s_0(x) \leq \sqrt{2k}$. Then, according to corollary 2.4.1, the following critical threshold is sufficient for the existence of global smooth solutions,

$$u_{0x}(x) \geq -\sqrt{2k\rho_0(x)} + \sqrt{A\gamma}\frac{|\rho_{0x}(x)|}{\rho_0(x)^{\frac{3-\gamma}{2}}}.$$

In this section we show that this critical threshold is also *necessary* for global regularity.

Theorem 2.4.3 *Consider the Euler-Poisson system* (2.4) *with a* γ*-law pressure* $p(\rho) = A\rho^\gamma$, $\gamma \geq 1$, *subject to initial data* $(u_0, \rho_0 > 0)$. *The system loses the* C^1*-smoothness if there exists an* $x \in \mathbb{R}$ *such that*

$$u_{0x}(x) < -\sqrt{2k\rho_0(x)} + \sqrt{A\gamma}\frac{|\rho_{0x}(x)|}{\rho_0(x)^{\frac{3-\gamma}{2}}}. \tag{2.55}$$

Remark 2.4.4 *Expressed in terms of the Riemann invariants,* $r = u_x - \sqrt{A\gamma}\rho_{0x}/\rho_0^{(3-\gamma)/2}$ *and* $s = u_x + \sqrt{A\gamma}\rho_{0x}/\rho_0^{(3-\gamma)/2}$, *the condition* (2.55) *reads*

$$\exists\, x \in \mathbb{R} \quad \text{s.t.} \quad \mathrm{r_0(x)} < -\sqrt{2\mathrm{k}\rho_0(\mathrm{x})}\,, \quad \text{or} \quad \mathrm{s_0(x)} < -\sqrt{2\mathrm{k}\rho_0(\mathrm{x})}\,. \tag{2.56}$$

The lack of smoothness in this case was shown in theorem 2.4.1 *for* $\gamma = 1$ *and is extended for* $\gamma > 1$ *below.*

Proof. Recall equations (2.50) for $X := \dfrac{r}{\sqrt{\rho}}$ and $Y := \dfrac{s}{\sqrt{\rho}}$

$$X^{\backprime} = \sqrt{\rho}\left(k - \frac{1+\theta}{2}X^2 + \frac{\theta}{2}XY\right), \tag{2.57a}$$

$$Y' = \sqrt{\rho}\left(k - \frac{1+\theta}{2}Y^2 + \frac{\theta}{2}XY\right). \tag{2.57b}$$

In the proof of theorem 2.4.2, we have shown that X and Y have an upper bound

$$X(\cdot,t), Y(\cdot,t) \le M_0, \qquad M_0 := \max_x \left\{\sqrt{2k}, X_0(x), Y_0(x)\right\}. \tag{2.58}$$

Suppose that there exists $X_0 = X(x_0) < -\sqrt{2k}$. We will show that this value will evolve along $\Gamma_\lambda(x_0, 0)$ such that $X(\cdot, t)$ will tend to $-\infty$ at a finite time. To this end, assume that Y is well behaved, i.e., $Y_0(\cdot) \ge -\sqrt{2k}$ so that $Y(\cdot,t) \le M_0$ for all t's (otherwise, the finite time blow up of Y can be argued along the same lines). It follows that along $\Gamma_\lambda(x_0, 0)$

$$X^{\backprime} = \sqrt{\rho}\left(k - \frac{1+\theta}{2}X^2 + \frac{\theta}{2}XY\right) < \sqrt{\rho}\left(k - \frac{1}{2}X^2\right). \tag{2.59}$$

Following exactly what we have done in the proof of theorem 2.4.1, we obtain the inequality

$$X(\cdot,t) \le \frac{M_0}{X_1 \ln\left(1 + \sqrt{\rho_0}M_0 t/2\right) + M_0 X_0}, \tag{2.60}$$

where $X_1 := (X_0^2 - 2k)/X_0^2 > 0$. Thus, starting with $X_0 < -\sqrt{2k} < 0$ it follows that there exists a finite critical time $t_c > 0$ such that $X(t \uparrow t_c)$ tends to $-\infty$. $\square$

We conclude with an example for a finite-time breakdown.

Example: Suppose at $t = 0$, $u_0(x) = 0$ and

$$\rho_0(x) = \begin{cases} 1, & x < 0, \\ 1 - \dfrac{x}{2\epsilon}, & 0 \le x \le \epsilon, \\ \frac{1}{2}, & x > \epsilon. \end{cases}$$

Thus

$$s_0(x) = \begin{cases} -\sqrt{A\gamma}\Big(1 - \dfrac{x}{2\epsilon}\Big)/2\epsilon\,, & 0 < x < \epsilon\,, \\ 0\,, & \text{elsewhere}\,. \end{cases}$$

If we choose ϵ small enough, then $s_0(x) < -\sqrt{2k\rho_0(x)}$ for $0 < x < \epsilon$. According to theorem 2.4.3, the system (2.4) will break down at a finite time. This example shows that even if the fluid is near rest at $t = 0$, the pressure itself could still lead to a finite-time breakdown.

Chapter 3

Vanishing viscosity limit of the system of Burgers equations

In this chapter, we move to the multi-dimensional cases. We consider the viscous dusty medium flow, $u := u^\epsilon$, governed by:

$$\partial_t u + u \cdot \nabla u = \epsilon \Delta u \,, \qquad u : \mathbb{R}^n \to \mathbb{R}^n \,, \tag{3.1}$$

where $\epsilon > 0$ is a viscosity amplitude.

We also consider the corresponding equation of the inviscid fluid

$$\begin{cases} u_t + u \cdot \nabla u = 0 \,, \\ u(x, 0) = u_0(x) \,. \end{cases} \tag{3.2}$$

Our ultimate goal is to establish the convergence of solutions of (3.1) towards solutions of (3.2).

In order to simplify the analysis, we shall consider the same initial data for (3.1) and (3.2), which is compactly supported.

3.1 Irrotational viscous fluids

In this section, we mimic the method used in [LT02] to study the n-dimensional irrotational dusty medium model, and identify its vanishing viscosity limit.

Differentiating of equation (3.1) with respect to x, we obtain that the velocity gradient tensor $M := \nabla u$ satisfies

$$\partial_t M + u \cdot \nabla M + M^2 = \epsilon \Delta M \,. \tag{3.3}$$

It follows that if the initial velocity is irrotational, $\nabla \times u_0 = 0$, i.e. if $M_0 := \nabla u_0$ is symmetric, then the flow remains irrotational, $\nabla \times u = 0$, i.e. $M := \nabla u$ is symmetric.

Apply the spectral dynamics Lemma, one can obtain the following result (Lemma 5.1, [LT02], p444-445).

Lemma 3.1.1 *Assume that the flow is irrotational* $\nabla \times u_0 = 0$. *Then the real eigenvalues* $\lambda = \lambda(\nabla u)$ *satisfy*

$$\partial_t \lambda + u \cdot \nabla \lambda + \lambda^2 = \epsilon \Delta \lambda + Q_\lambda.$$

Here Q_λ *satisfies the constraint*

$$a(\lambda_{\min} - \lambda) \leq Q_\lambda \leq a(\lambda_{\max} - \lambda), \qquad \lambda_{\max} := \max \lambda(\nabla u), \qquad \lambda_{\min} := \min \lambda(\nabla u),$$

where a *is given by*

$$a := 2\epsilon \sum_k \partial_k r^\perp \partial_k r > 0$$

and r *is the right eigenvector of* ∇u *associated with* λ.

Proof. Let l and r be the normalized left and right eigenvectors of M associated with the eigenvalue λ. Then

$$\partial_t \lambda + u \cdot \nabla \lambda + \lambda^2 = \epsilon l \Delta M r.$$

Observe that M is symmetric because $\nabla \times u = 0$, and consequently λ are all real quantities. Differentiation of $lM = \lambda l$ with respect to x twice gives

$$\Delta l M + 2\nabla l \nabla M + l \Delta M = \Delta \lambda l + 2\nabla \lambda \nabla l + \lambda \Delta l,$$

which upon multiplication against r on the right leads to

$$l\Delta M r = \Delta\lambda + 2[(\nabla\lambda\nabla l)r - (\nabla l\nabla M)r]\,.$$

Here the differentiation operators apply component wise, e.g., $\nabla l\nabla M = \sum_{k=1}^n \partial_k l \partial_k M\,.$ On the other hand it follows from $Mr = \lambda r$ that

$$\nabla M r = \nabla\lambda r + \lambda\nabla r - M\nabla r\,.$$

This gives

$$(\nabla l\nabla M)r = \nabla l\nabla\lambda r + \lambda\nabla l\nabla r - \nabla l M\nabla r\,.$$

A combination of the above facts yields

$$Q_\lambda = 2\epsilon\Big[-\lambda\sum_{k=1}^n \partial_k l\partial_k r + \sum_{k=1}^n \partial_k l M\partial_k r\Big]\,.$$

Since the flow is irrotational we have $M^\top = M$ and $l = r^\top$, with superscript $^\top$ denoting the transpose. The second term in Q_λ is then bounded by

$$\lambda_{\min}\sum_{k=1}^n \partial_k r^\top\partial_k r \leq \sum_{k=1}^n \partial_k r^\top M\partial_k r \leq \lambda_{\max}\sum_{k=1}^n \partial_k r^\top\partial_k r\,,$$

which completes the proof. $\square$

It follows that the largest eigenvalue $\lambda_{\max}$ satisfies

$$\partial_t\lambda_{\max} + u\cdot\nabla\lambda_{\max} + \lambda_{\max}^2 \leq \epsilon\Delta\lambda_{\max}\,,$$

and by the comparison principle we obtain

$$\lambda_{\max} \leq \frac{1}{\lambda_{\max}(0)^{-1} + t} \leq \frac{1}{t}\,.$$

This bound enable us to establish the following.

Lemma 3.1.2 *Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, be real eigenvalues of the velocity gradient field ∇u^ϵ in* (3.1). *If $d(0) := \sum_{i=1}^{n} \lambda_i(0) \in L^1(\mathbb{R}^2)$, then*

$$||d(t)||_{L^1(\mathbb{R}^2)} \leq \textit{Const.}$$

Proof. The one-sided upper bound for $\lambda_{\max}$ implies that the positive part of the divergence, $(\sum_{i=1}^{n} u_{ix_i})_+ = (\sum_{i=1}^{n} \lambda_i)_+$ is bounded. We observe that λ_i are essentially supported on a finite domain in the sense of their exponential decay outside a finite region of propagation, and hence $\int_{\mathbb{R}^2}(\sum_{i=1}^{n} u_{ix_i})_+ \leq$ Const. This, combined with $\int_{\mathbb{R}^2} \sum_{i=1}^{n} u_{ix_i} = 0$, yields that $\sum_{i=1}^{n} u_{ix_i} = \sum_{i=1}^{n} \lambda_i \in L^1(\mathbb{R}^2)$. This completes the proof. □

Lemma 3.1.3 *Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, be real eigenvalues of the velocity gradient field ∇u^ϵ in* (3.1). *If $\eta(0) := (\lambda_1 - \lambda_n)(0) \in L^1(\mathbb{R}^2)$, then*

$$||\eta(t)||_{L^1(\mathbb{R}^2)} \leq \textit{Const.}$$

Proof. It follows from Lemma 3.1.1 that η satisfies

$$\eta' + (\lambda_1 + \lambda_n)\eta \leq \nu\Delta\eta \, .$$

We rewrite this equation as

$$\eta' + \eta d \leq \nu\Delta\eta + (\lambda_2 + \cdots + \lambda_{n-1})\eta \, .$$

Since $\eta \geq 0$ and $\lambda_1 < C$, spatial integration gives

$$\frac{d}{dt}||\eta(t)||_{L^1} \leq C||\eta(t)||_{L^1} \, .$$

Applying the Gronwall inequality, we find that $||\eta(t)||_{L^1}$ is bounded. □

Equipped with these two lemmas, and following the exactly same strategy of [LT02], we can identify the vanishing viscosity limit of (3.1).

We reiterate the procedure here briefly.

Lemma 3.1.4 *(BV bound). Consider the dusty medium equation* (3.1) *with compactly supported irrotational initial data* $u_0^\epsilon = u^\epsilon(x,0)$ *such that* $||u_0^\epsilon||_{BV(\mathbb{R}^2)}$ *is bounded. Then the corresponding velocity,* u^ϵ *, satisfies*

$$||u^\epsilon(\cdot,t)\,||_{BV(\mathbb{R}^2)} \leq Const.$$

Moreover, for $t_1, t_2 \geq 0$ *, we also have*

$$||u^\epsilon(x,t_2) - u^\epsilon(x,t_1)\,||_{L^1(\mathbb{R}^2)} \leq Const.|t_2 - t_1|^{1/3}\,. \tag{3.4}$$

Proof. Since $(\lambda_1 - \lambda_n)(t) \in L^1(\mathbb{R}^2)$, and

$$0 \leq (\lambda_i - \lambda_j) \leq (\lambda_1 - \lambda_n)\,, \qquad \forall\, 1 \leq i < j \leq n\,,$$

one has

$$(\lambda_i - \lambda_j)(t) \in L^1(\mathbb{R}^2)\,, \qquad \text{for all} \qquad 1 \leq i < j \leq n\,.$$

This combined with $d(t) \in L^1(\mathbb{R}^2)$ yields

$$\lambda_i \in L^1(\mathbb{R}^2)\,, \qquad i = 1, \cdots, n\,. \tag{3.5}$$

Since ∇u is symmetric, (3.5) gives

$$\int_{\mathbb{R}^2} ||\nabla u^\epsilon|| dx = \int_{\mathbb{R}^2} ||diag(\lambda_1, \cdots, \lambda_n)|| dx < \infty\,,$$

with the usual matrix norm, $||\cdot||$, defined as $||M|| =: \sup_{||\xi||=1} |M\xi|$. Thus the BV bound follows.

To estimate the modulus of continuity in time, we multiply (3.1) by a smooth test function $\psi \in C_0^\infty$ and use the spatial BV bound to obtain

$$\left|\psi(x)(u(x,t_2) - u(x,t_1))dx\right| \le \text{Const.}(t_2 - t_1)(|\psi|_\infty + |\Delta\psi|)\,.$$

This inequality and the BV estimate combined with Kružkov's interpolation theorem ([K70], p.233) yield (3.4). □

For irrotational flow, $\nabla \times u = 0$, one has $u \cdot \nabla u = \nabla(|u|^2/2)$, and the reduced equation of the inviscid fluid (3.2) can be recast into the conservative form

$$\partial_t u + \nabla\Big(\frac{|u|^2}{2}\Big) = 0\,. \tag{3.6}$$

The irrotational property of both viscous and inviscid fluids suggests that there exists a potential ϕ such that $u = \nabla\phi$, where ϕ solves the Hamilton-Jacobi equation

$$\phi_t + \frac{1}{2}|\nabla\phi|^2 = 0\,, \qquad \phi(x,0) = \phi_0\,. \tag{3.7}$$

There is an unique continuous solution to (3.7), which is expressed via the Hopf-Lax formulation, ([Ev98], p.560). Then [LT02] make

Definition 3.1.1 *([LT02], p.447) A measurable function u is called a weak solution of the equation* (3.2) *if $u = \nabla\phi$ with the potential ϕ being the unique weak solution of the Eikonal equation* (3.7)*.*

With this definition of a weak solution, we summarize our results by stating the following theorem (see [LT02], p.447 for detailed proof.)

Theorem 3.1.1 *(Vanishing viscosity limit). Consider the dusty medium equation* (3.1) *with compactly supported irrotational initial data $u(\cdot,0) \in L^\infty(\mathbb{R}^2)$, then the* *local velocity u^ϵ converges to the unique weak solution of* (3.2)*, i.e., we have*

$$u^\epsilon(x,t) \to u(x,t) \qquad in \qquad L^\infty([0,T];L^1(\mathbb{R}^2))\,, \tag{3.8}$$

where $u = \nabla\phi$ is the viscosity solution of the Eikonal equation (3.7)*.*

3.2 Two-dimensional general viscous flow

In this section, we discuss the two-dimensional system of Burgers equations (3.1) and (3.2) subject to general initial data.

3.2.1 Spectral dynamics and BV bound

For general initial data, we decompose M as $M = S + A$, where S is the symmetric part of M, and A is the skew-symmetric part of M, i.e.,

$$A = \begin{pmatrix} 0 & \dfrac{u_{1x_2} - u_{2x_1}}{2} \\ \dfrac{u_{2x_1} - u_{1x_2}}{2} & 0 \end{pmatrix},$$

$$S = \begin{pmatrix} u_{1x_1} & \dfrac{u_{1x_2} + u_{2x_1}}{2} \\ \dfrac{u_{1x_2} + u_{2x_1}}{2} & u_{2x_2} \end{pmatrix}.$$

Then A and S satisfy the coupled system of equations

$$\partial_t A + u\cdot\nabla A + AS + SA = \epsilon\Delta A\,, \tag{3.9a}$$

$$\partial_t S + u\cdot\nabla S + S^2 + A^2 = \epsilon\Delta S\,. \tag{3.9b}$$

Let us denote $a = \dfrac{u_{1x_2} + u_{2x_1}}{2}$. Noting that

$$AS + SA = \begin{pmatrix} 0 & a(u_{1x_1} + u_{2x_2}) \\ -a(u_{1x_1} + u_{2x_2}) & 0 \end{pmatrix} =: \begin{pmatrix} 0 & ad \\ -ad & 0 \end{pmatrix},$$

we find

$$\partial_t a + u \cdot \nabla a + ad = \epsilon \Delta a \,. \tag{3.10}$$

Since S is symmetric, it has two real eigenvalues, $\lambda \leq \mu$. Regarding the evolution of λ and μ, we state the following lemma:

Lemma 3.2.1 *The two real eigenvalues $\lambda \leq \mu$ of S satisfy*

$$\partial_t \lambda + u \cdot \nabla \lambda + \lambda^2 = \epsilon \Delta \lambda + \epsilon Q_\lambda + a^2, \tag{3.11a}$$

$$\partial_t \mu + u \cdot \nabla \mu + \mu^2 = \epsilon \Delta \mu + \epsilon Q_\mu + a^2. \tag{3.11b}$$

Here Q_λ and Q_μ satisfy the constraint

$$Q_\lambda = -Q_\mu \geq 0 \,.$$

Proof. First, we apply the spectral dynamics lemma ([LT02], p.442) to (3.9b). And note that $r^T A^2 r = q^T A^2 q = -a^2$, we obtain

$$\partial_t \lambda + u \cdot \nabla \lambda + \lambda^2 - a^2 = r^T \Delta S r \,, \tag{3.12a}$$

$$\partial_t \mu + u \cdot \nabla \mu + \mu^2 - a^2 = q^T \Delta S q \,, \tag{3.12b}$$

here r and q are the right eigenvectors of S associated with λ and μ respectively, normalized so that $r^T r = 1$ and $q^T q = 1$.

Next, we differentiate $r^T S = \lambda r^T$ with respect to x twice, and it yields

$$\Delta r^T S + 2\nabla r^T \nabla S + r^T \Delta S = \Delta \lambda r^T + 2\nabla \lambda \nabla r^T + \lambda \Delta r^T \,,$$

which upon multiplication against r on the right leads to

$$r^T \Delta S r = \Delta \lambda + 2[(\nabla \lambda \nabla r^T) r - (\nabla r^T \nabla S) r] \,.$$

Here the differentiation operators apply component wise, e.g., $\nabla r^T \nabla S = \sum_{k=1}^{2} \partial_k r^T \partial_k S$.

On the other hand, it follows from $Sr = \lambda r$ that

$$\nabla S r = \nabla \lambda r + \lambda \nabla r - S \nabla r \,.$$

This gives

$$(\nabla r^T \nabla S) r = \nabla r^T \nabla \lambda r + \lambda \nabla r^T \nabla r - \nabla r^T S \nabla r \,.$$

A combination of the above facts yields

$$Q_\lambda = 2\Big[-\lambda \sum_{k=1}^{2} \partial_k r^T \partial_k r + \sum_{k=1}^{2} \partial_k r^T S \partial_k r \Big] \,.$$

Similarly, we obtain

$$Q_\mu = 2\Big[-\mu \sum_{k=1}^{2} \partial_k q^T \partial_k q + \sum_{k=1}^{2} \partial_k q^T S \partial_k q \Big] \,.$$

Since S is symmetric, then $q = r^\perp$. Decompose $\partial_k r$ as $\partial_k r := \alpha_k r + \beta_k q$, then $\partial_k q = \alpha_k q - \beta_k r$. Thus

$$Q_\lambda = 2(\mu - \lambda) \sum_{k=1}^{2} \beta_k^2 \geq 0 \,,$$

and

$$Q_\mu = 2(\lambda - \mu) \sum_{k=1}^{2} \beta_k^2 = -Q_\lambda \,.$$

This completes the proof. $\square$

Setting $\eta := \mu - \lambda$, following Lemma 3.2.1, one has

$$\partial_t \eta + u \cdot \nabla \eta + \eta d \leq \epsilon \Delta \eta \,. \tag{3.13}$$

Equations (3.10) and (3.13) yield

Lemma 3.2.2 *If $a(0), \eta(0) \in L^1(\mathbb{R}^2)$, then*

$$||a(t)||_{L^1(\mathbb{R}^2)} \le ||a(0)||_{L^1(\mathbb{R}^2)} ,$$

and

$$||\eta(t)||_{L^1(\mathbb{R}^2)} \le ||\eta(0)||_{L^1(\mathbb{R}^2)} .$$

Proof. Noting that $\nabla \cdot u = \lambda + \mu$, thus

$$\partial_t a + \nabla \cdot (au) = \epsilon \Delta a ,$$

$$\partial_t \eta + \nabla \cdot (\eta u) \le \epsilon \Delta \eta .$$

Spatial integration gives the L^1 estimate for a and η as asserted. □

Taking the sum of (3.11a) and (3.11b), we obtain

$$\partial_t d + u \cdot \nabla d + \frac{d^2 + \eta^2}{2} = \epsilon \Delta d + 2a^2 . \tag{3.14}$$

This yields

Lemma 3.2.3 *If $\sup_x a(x,0) < \infty$ and $\sup_x d(x,0) < \infty$, then $\sup_x d(x,t) < \infty$ for all $t \ge 0$.*

Proof. For $d > 0$, $2a * (3.10) + d * (3.14)$ yields

$$\begin{aligned} \partial_t(a^2 + \frac{d^2}{2}) + u \cdot \nabla(a^2 + \frac{d^2}{2}) &= \epsilon(2a\Delta a + d\Delta d) - d\frac{d^2+\eta^2}{2} \\ &= \epsilon\Delta(a^2 + \frac{d^2}{2}) - \epsilon(2|\nabla a|^2 + |\nabla d|^2) - d\frac{d^2+\eta^2}{2} \end{aligned} \tag{3.15}$$

By the maximum principle, we obtain $a^2 + \frac{d^2}{2} \le \max_{d_0 > 0}(a_0^2 + \frac{d_0^2}{2})$. □

Equipped with all of the above Lemmas, we can establish the BV bound of u^ϵ.

Theorem 3.2.1 *Consider the dusty medium equation* (3.1) *with compactly supported initial data* $u_0^\epsilon = u^\epsilon(x,0)$ *such that* $||u_0^\epsilon||_{BV(\mathbb{R}^2)}$ *is bounded. Then the corresponding velocity,* u^ϵ*, satisfies*

$$||u^\epsilon(\cdot,t)\,||_{BV(\mathbb{R}^2)} \leq \mathit{Const.}$$

Proof. Since $d = u_{1x_1} + u_{2x_2}$ is essentially supported on a finite domain in the sense of its exponential decay outside a finite region of propagation, and d has an upper bound, we find that

$$\int_{\mathbb{R}^2} (u_{1x_1} + u_{2x_2})_+ \leq \text{Const.}$$

This, combined with $\int_{\mathbb{R}^2}(u_{1x_1} + u_{2x_2}) = 0$, yields that

$$u_{1x_1} + u_{2x_2} \in L^1(\mathbb{R}^2)\,. \tag{3.16}$$

Direct calculation shows that

$$\lambda = \frac{u_{1x_1} + u_{2x_2} - \sqrt{(u_{1x_1} - u_{2x_2})^2 + (u_{1x_2} + u_{2x_1})^2}}{2}\,,$$

and

$$\mu = \frac{u_{1x_1} + u_{2x_2} + \sqrt{(u_{1x_1} - u_{2x_2})^2 + (u_{1x_2} + u_{2x_1})^2}}{2}\,.$$

Since $\eta = \mu - \lambda \in L^1(\mathbb{R}^2)$, we obtain

$$u_{1x_1} - u_{2x_2}\,, u_{1x_2} + u_{2x_1} \in L^1(\mathbb{R}^2)\,. \tag{3.17}$$

Combine (3.16), (3.17) with

$$a = \frac{u_{1x_2} - u_{2x_1}}{2} \in L^1(\mathbb{R}^2)\,, \tag{3.18}$$

we conclude

$$\nabla u \in L^1(\mathbb{R}^2)\,. \tag{3.19}$$

Then the BV bound of u^ϵ follows. $\square$

3.2.2 Vanishing viscosity limit of smooth solutions

First, we show in the following lemma that if the initial velocity gradient tensor does not have negative eigenvalues, then equation (3.2) admits a global smooth solution.

Lemma 3.2.4 *Consider equation* (3.2) *with compactly supported initial data* $u_0 = u(x,0)$ *subject to the constraint that* ∇u_0 *has no negative eigenvalues. Then* $\forall\, T > 0$ *, there exists* $C > 0$ *, such that* $||\nabla u(x, t \leq T)|| < C$ *, with the usual matrix norm,* $||\cdot||$ *, defined as* $||M|| =: sup_{||\xi||=1}|M\xi|$ *.*

Proof. In this chapter, here and below, we denote $\{\,\}' := \partial_t + u \cdot \nabla$. Let χ and ω be the two eigenvalues of ∇u. Then χ and ω satisfy

$$\chi' + \chi^2 = 0\,, \tag{3.20a}$$

$$\omega' + \omega^2 = 0\,. \tag{3.20b}$$

Solving (3.20a), we obtain

$$\chi(t) = \frac{1}{t + \chi(0)^{-1}}\,. \tag{3.21}$$

If $\chi(0)$ is a positive number, then $\chi(t)$ decreases to 0. If $\chi(0) := c + pi$ is a complex number, ($c, p \in \mathbb{R}$, and without loss of generality, we can assume $p > 0$), then

$$\begin{aligned}
\chi(t) &= \frac{1}{t + (c+pi)^{-1}} \\
&= \frac{c^2+p^2}{(c^2+p^2)t + c - pi} \\
&= \frac{(c^2+p^2)[(c^2+p^2)t+c]}{[(c^2+p^2)t+c]^2+p^2} + \frac{(c^2+p^2)p}{[(c^2+p^2)t+c]^2+p^2} i \,.
\end{aligned}$$

Set $h(t) = (c^2+p^2)t + c$, then the real and imagine parts of $\chi(t)$ are

$$R_\chi(h) = \frac{(c^2+p^2)h}{h^2+p^2}\,, \quad \text{and} \quad I_\chi(h) = \frac{(c^2+p^2)p}{h^2+p^2}\,.$$

Differentiation of $R_\chi(h)$ with respect to h yields

$$\begin{aligned}
\frac{dR_\chi}{dh} &= \frac{c^2+p^2}{h^2+p^2} - \frac{(c^2+p^2)2h^2}{(h^2+p^2)^2} \\
&= \frac{(c^2+p^2)(p^2-h^2)}{h^2+p^2}\,.
\end{aligned}$$

It follows that $R_\chi(h)$ is decreasing on $(-\infty, -p)$, increasing on $(-p, p)$, and decreasing to 0 on (p, ∞).

We show that χ and ω will remain bounded. For every fixed $T > 0$, we decompose $\mathbb{R}^2$ as

$$\mathbb{R}^2 := \bigcup_{j=1}^{6} \Omega_j\,,$$

where

$$\Omega_1 := \{x | \chi(x,0) \geq 0\}\,,$$

$$\Omega_2 := \{x | R_\chi(x,0) \geq I_\chi(x,0) > 0\}\,,$$

$$\Omega_3 := \{x | I_\chi(x,0) > R_\chi(x,0) \geq 0\}\,,$$

$$\Omega_4 := \{x | R_\chi(x,0) \leq -\frac{1}{3T}\}\,,$$

$$\Omega_5 := \{x| -\frac{1}{3T} < R_\chi(x,0) < 0\,, \text{and } I_\chi(x,0) \geq -R_\chi\}\,,$$
$$\Omega_6 := \{x| -\frac{1}{3T} < R_\chi(x,0) < 0\,, \text{and } I_\chi(x,0) < -R_\chi\}\,.$$

Tracing $\chi(x,t)$ along the characteristic line, we obtain the following bounds subject to different sets:

(I) $x \in \Omega_1:\ 0 \leq \chi(t) \leq \chi(0)\,.$

(II) $x \in \Omega_2:\ 0 < R_\chi(t) < R_\chi(0)$ and $0 < I_\chi(t) < I_\chi(0)\,.$

(III) $x \in \Omega_3:\ 0 < R_\chi(t) < \dfrac{R_\chi(0)^2 + I_\chi(0)^2}{2I_\chi(0)^2} I_\chi(0) < I_\chi(0)$ and $0 < I_\chi(t) < I_\chi(0)\,.$

(IV) $x \in \Omega_4:$ Since $\chi(\cdot,0) \notin \mathbb{R}^-$, then there must exists $\delta > 0$ such that $I_\chi(x,0) > \delta$ for all $x \in \Omega_4$, otherwise χ will be negative somewhere. Then

$$R_\chi(t) \geq -\frac{R_\chi(0)^2 + I_\chi(0)^2}{2I_\chi(0)} \geq -\frac{\max_x |\chi(x,0)|}{2\delta}\,,$$

and

$$0 \leq I_\chi(t) \leq \frac{R_\chi(0)^2 + I_\chi(0)^2}{I_\chi(0)} \leq \frac{\max_x |\chi(x,0)|}{\delta}\,.$$

(V) $x \in \Omega_5:\ R_\chi(t) > R_\chi(0)$, and $0 < I_\chi(t) \leq \dfrac{R_\chi(0)^2 + I_\chi(0)^2}{I_\chi(0)} < 2I_\chi(0)\,.$

(VI) $x \in \Omega_6:\ R_\chi(t) = \dfrac{(R_\chi(0)^2 + I_\chi(0)^2)h(t)}{h(t)^2 + I_\chi(0)^2}$, here $h(t) = (R_\chi(0)^2 + I_\chi(0)^2)t + R_\chi(0)\,.$ Combine this with the facts that $R_\chi(0) > -\dfrac{1}{3T}$ and $I_\chi(0) < -R_\chi(0)$, we obtain $h(t) < \frac{1}{3}R_\chi(0)$ for all $0 < t \leq T$. Thus $R_\chi(t) > \dfrac{(R_\chi(0)^2 + I_\chi(0)^2)R_\chi(0)}{\frac{1}{9}R_\chi(0)^2 + I_\chi(0)^2} > 18R_\chi(0)\,.$ Similarly, we can obtain that $I_\chi(t) < 18I_\chi(0)\,.$

Combine all of the above results, we obtain an uniform bound of χ which depends on the initial data and T. Same result hold for ω. Therefore $d = \chi + \omega = u_{1x_1} + u_{2x_2}$ is uniformly bounded.

The boundedness of d and the structure of equation (1.6) enable us to control every single term of M. We state the details below.

Differentiating equation (3.2), we obtain the equations of $u_{kx_i}(k, i = 1, 2)$:

$$u'_{1x_1} + u^2_{1x_1} + u_{1x_2}u_{2x_1} = 0\,, \tag{3.22a}$$

$$u'_{1x_2} + u_{1x_1}u_{1x_2} + u_{1x_2}u_{2x_2} = u'_{1x_2} + u_{1x_2}d = 0\,, \tag{3.22b}$$

$$u'_{2x_1} + u_{1x_1}u_{2x_1} + u_{2x_1}u_{2x_2} = u'_{2x_1} + u_{2x_1}d = 0\,, \tag{3.22c}$$

$$u'_{2x_2} + u^2_{2x_2} + u_{1x_2}u_{2x_1} = 0\,. \tag{3.22d}$$

It follows from equation (3.22b) that along the characteristic line,

$$u_{1x_2}(t) = u_{1x_2}(0)e^{-\int_0^t d}. \tag{3.23}$$

Thus u_{1x_2} is bounded. Similarly, (3.22c) implies that u_{2x_1} is bounded. Take the difference of (3.22a) and (3.22d), we obtain

$$(u_{1x_1} - u_{2x_2})' + (u_{1x_1} - u_{2x_2})(u_{1x_1} + u_{2x_2}) = (u_{1x_1} - u_{2x_2})' + (u_{1x_1} - u_{2x_2})d = 0\,, \tag{3.24}$$

which implies the boundedness of $u_{1x_1} - u_{2x_2}$. Combine this with the boundedness of $d = u_{1x_1} + u_{2x_2}$, we have the control of u_{1x_1} and u_{2x_2}. This complete the proof. □

We now turn to the main theorem of this section.

Theorem 3.2.2 *We consider the dusty medium equation* (3.1) *with compactly supported initial data* $u_0 \in C^2(\mathbb{R}^2)$ *subject to the constraint that* ∇u_0 *has no negative eigenvalues. Then, the local velocity* u^ϵ *converges to the smooth solution* $\bar{u}$ *of* (3.2), *i.e., we have*

$$u^\epsilon(x,t) \to \bar{u}(x,t) \qquad in \qquad L^\infty\big([0,T]; L^1(\mathbb{R}^2)\big). \tag{3.25}$$

Proof. For every fixed $T > 0$, we have shown that $\nabla u(x,t)$ is uniformly bounded for all x and $0 \le t \le T$. Set

$$L := \inf_{x, 0 \le t \le T} \Big\{ \text{the smaller eigenvalue of } S(\nabla u(x,t)) =: \lambda\Big(S(\nabla u(x,t))\Big)\Big\} - 1 \,,$$

here $S(\nabla u)$ is the symmetric part of ∇u. According to (3.11a)

$$\partial_t \lambda + u \cdot \nabla \lambda + \lambda^2 = \epsilon \Delta \lambda + \epsilon Q_\lambda + a^2 \ge \epsilon \Delta \lambda \,, \tag{3.26}$$

and by the comparison principle we obtain

$$\lambda(x,t) \ge \frac{1}{t + L^{-1}} \,.$$

Let $T_1 := -\dfrac{1}{2L}$, then

$$\lambda \ge 2L \,, \quad \forall \epsilon > 0 \,, t \le T_1 \,, x \,.$$

Thus $d = \lambda + \mu \ge 4L$, for all x and $0 \le t \le T_1$. Recall the equations of a, η and μ

$$a' + ad = \epsilon \Delta a \,, \tag{3.27a}$$

$$\eta' + \eta d \le \epsilon \Delta \eta \,, \tag{3.27b}$$

$$\mu' + \mu^2 = \epsilon \Delta \mu + \epsilon Q_\mu + a^2 \le \epsilon \Delta \mu + a^2 \,. \tag{3.27c}$$

By the comparison principle, we obtain

$$|a(x,t)| \le \max_x |a(x,0)| e^{-LT_1} =: U_a \,, \qquad \forall\, x \,, \forall\, 0 \le t \le T_1 \,,$$

$$\eta(x,t) \le \max_x \eta(x,0) e^{-LT_1} =: U_\eta \,, \qquad \forall\, x \,, \forall\, 0 \le t \le T_1 \,,$$

and

$$\mu(x,t) \le \max_x \mu(x,0) + U_a^2 T_1 =: U_\mu \,, \qquad \forall\, x \,, \forall\, 0 \le t \le T_1 \,.$$

The upper bound of μ yields $d \le 2U_\mu$, for all x and $0 \le t \le T_1$. The boundedness of λ and μ implies every element of S is bounded, i.e., u_{1x_1} , u_{2x_2} and $\dfrac{u_{1x_2}+u_{2x_1}}{2}$ are uniformly bounded for all x and $0 \le t \le T_1$. The above facts combined with the boundedness of $a = \dfrac{u_{1x_2}-u_{2x_1}}{2}$ yield that u_{1x_2} and u_{2x_1} are uniformly bounded for all x and $0 \le t \le T_1$.

Since all the first order derivatives of u are uniformly bounded, we have the control of all the second order derivatives of u . The detailed explanation is given below. We differentiate

$$\partial_t M + u \cdot \nabla M + M^2 = \epsilon \Delta M \,,$$

with respect to x , obtaining the equations of all the second order derivatives of u . For example, the equation of $u_{1x_1x_1}$ is

$$u'_{1x_1x_1} = \epsilon \Delta u_{1x_1x_1} - (3u_{1x_1}u_{1x_1x_1} + 2u_{2x_1}u_{1x_1x_2} + u_{1x_2}u_{2x_1x_1}) \,. \tag{3.28}$$

Set $z(t) := \sup\limits_x \{ |u_{kx_ix_j}(x,t)| \,, k, i, j = 1, 2 \}$. Then

$$\frac{dz}{dt} \le 6C_{T_1} z \,, \tag{3.29}$$

which yields

$$z(t) \le z(0) e^{6T_1 C_{T_1}} \,. \tag{3.30}$$

Thus, u , Du and D^2u are all uniformly bounded. We observe that they are essentially supported on a finite domain in the sense of their exponential decay outside a finite region of propagation, and hence $u \in W^{2,p}(\mathbb{R}^2)$, here $1 \le p \le \infty$. By Rellich's compactness theorem, u^ϵ is precompact in $W^{1,q}(\mathbb{R}^2)$, $1 \le q \le \infty$. We multiply (1.5) by a smooth test function $\psi \in C_0^\infty$ to obtain

$$\left| \int_{\mathbb{R}^2} \psi(x)(u^\epsilon(x,t_2) - u^\epsilon(x,t_1))dx \right| \leq Const.(t_2 - t_1)(|\psi|_\infty + |\Delta\psi|) \,. \tag{3.31}$$

This inequality and $Du^\epsilon \in L^1$ combined with Kružkov's interpolation theorem (p.233,[K70]) yield

$$||u^\epsilon(x,t_2) - u^\epsilon(x,t_1)||_{L^1(\mathbb{R}^2)} \leq Const.|t_2 - t_1|^{1/3} \,. \tag{3.32}$$

Then, by the Cantor diagonalization process of passing to further subsequence if necessary, we obtain

$$u^\epsilon(x,t) \to \bar{u}(x,t) \qquad in \qquad L^\infty([0,T_1]; L^1(\mathbb{R}^2)) \,. \tag{3.33}$$

This combined with D^2u^ϵ are uniformly bounded yield $\bar{u} \in C^1$. Multiply (1.5) by $\varphi(x,t) \in C_0^\infty$, integrate by parts, and pass to the limit, we find $\bar{u}$ is the smooth solution of (1.6). Furthermore, due to the boundedness of u^ϵ, $\bar{u}$, D^2u^ϵ and $D^2\bar{u}$, there exists $\theta_1 > 0$, such that when $\epsilon < \theta_1$, then

$$||u^\epsilon(x,T_1) - \bar{u}(x,T_1)||_{C^1(\mathbb{R}^2)} < \theta_2 \,,$$

here $\theta_2 > 0$ is small enough to ensure that

$$\lambda\Big(S\Big(\nabla u^\epsilon(x, t \leq T_1)\Big)\Big) > L + \frac{1}{2} \,.$$

Thus, for u^ϵ with $\epsilon < \theta_1$, we can repeat the above proof to obtain a further subsequence which converges to $\bar{u}$ on $[0, 2T_1]$. By doing this $\left[\frac{T+T_1}{T_1}\right]$ times, we obtain a subsequence $u^\epsilon \to \bar{u}$ on $[0,T]$. By a general contradiction proof, we know that $u^\epsilon \to \bar{u}$ is true for every subsequence u^ϵ. $\square$

3.2.3 Open questions

For general initial data, we have obtained an uniform BV bound of u^ϵ. Due to this BV bound, there exists a subsequence of u^ϵ converges to a function u. The open problems are: What is the dynamic satisfied by this limit u? Is u a weak solution of the inviscid equation (3.2)? And how do we define the weak solution of (3.2)?

Chapter 4

Critical thresholds for restricted Euler dynamics

4.1 Restricted Euler equations and spectral dynamics

In this chapter, we discuss the restricted Eulerian dynamics

$$\partial_t M + u \cdot \nabla M + M^2 = \frac{\mathrm{tr} M^2}{n} I_{n\times n}\,, \qquad M = \nabla u\,, \tag{4.1}$$

By the spectral dynamics lemma, the corresponding eigenvalues of M satisfy

$$\dot{\lambda}_i + \lambda^2 = \frac{1}{n}\sum_{j=1}^{n} \lambda_j^2\,, \qquad i = 1, \cdots, n\,, \qquad \dot{\{\}} := \frac{d}{dt} = \partial_t + u \cdot \nabla_x\,. \tag{4.2}$$

We say that $\Lambda_0 \in \mathbb{R}^n$ is sub-critical if there exists a global smooth solution in time of (4.2), $\Lambda(t) = (\lambda_1(t), \lambda_2(t), \cdots, \lambda_n(t))$ subject to initial conditions, $\Lambda(0) = (\lambda_1(0), \lambda_2(0), \cdots, \lambda_n(0))$. A first observation rests on the obvious symmetries of (4.2).

Lemma 4.1.1 *If Λ is sub-critical then so is $r\Lambda$, $\forall\, r > 0$. Moreover, every permutation of Λ is sub-critical.*

It follows that the set of sub-critical initial data consist of rays, and therefore, it is enough to consider the projection of this set on the unit sphere.

4.2 Trace dynamics

This section is devoted to an alternative formulation of the spectral dynamics in terms of real quantities $m_k := \sum_{j=1}^{n} \lambda_j^k$, $k = 1, \cdots, n$. This is motivated by the trace dynamics originally studied in [V82] for $n = 3$.

Here we seek an extension for the general n-dimensional setting, which is summarized in the following.

Lemma 4.2.1 *[LT02] Consider the n-dimensional restricted Euler system* (4.2) *subject to the incompressibility condition $m_1 := \sum_{j=1}^{n} \lambda_j = 0$. Then the traces m_k for $k = 2, \cdots, n$ satisfy a closed dynamical system, which governs the local topology of the restricted flow.*

Proof. Based on the spectral dynamics, the evolution equation for each eigenvalue λ_i can be written as

$$\frac{d}{dt}\lambda_i + \lambda_i^2 = \frac{1}{n} m_2 \,, \qquad i = 1, \cdots, n \,.$$

By multiplying $k\lambda_i^{k-1}$ and summation over i we obtain

$$\frac{d}{dt} m_k + k m_{k+1} = \frac{1}{n} m_2 m_{k-1} \,, \qquad i = 2, \cdots, n \,.$$

Note that $m_1 = 0$, we obtain

$$\frac{d}{dt} m_2 + 2m_3 = 0 \,, \tag{4.3a}$$

$$\frac{d}{dt} m_3 + 3m_4 = \frac{3}{n} m_2^2 \,, \tag{4.3b}$$

$$\cdots$$

$$\frac{d}{dt}m_n + n m_{n+1} = m_2 m_{n-1}\,. \tag{4.3c}$$

To close the system, it remains to express m_{n+1} in terms of $(m_1, \cdots, m_n)$. To this end we utilize the characteristic polynomial

$$\lambda_j^n + q_1\lambda_j^{n-1} + \cdots + q_{n-1}\lambda_j + q_n = 0\,, \tag{4.4}$$

expressed in terms of the characteristic coefficients

$$q_1 = -m_1 = 0\,, \quad q_2 = -\frac{1}{2}m_2\,, \quad q_3 = -\frac{1}{3}m_3\,, \quad q_4 = -\frac{1}{4}m_4 + \frac{1}{8}m_2^2\,, \quad \cdots$$

Note that q's can be expressed in terms of $(m_1, \cdots, m_n)$. Using (4.4) one may reduce m_{n+1} in (4.3c) to lower-order products. In facts, $\sum_{j=1}^n(\lambda_j \times (4.3c)_j)$ gives

$$m_{n+1} + q_2 m_{n-1} + \cdots + q_{n-1}m_2 = 0\,. \tag{4.5}$$

Substituting this into (4.3c) yields the closed system we sought for. □

In order to demonstrate the above procedure we now turn to consider two examples, whose critical thresholds will be studied in subsequent sections.

Example 1. (3-dimensional case $n = 3$, see [V82],[CA92])

In the three-dimensional case one has

$$q_1 = 0\,, \qquad q_2 = -\frac{1}{2}m_2\,, \qquad q_3 = \prod_{j=1}^n \lambda_j = -\frac{1}{3}m_3\,,$$

hence

$$\lambda_i^3 - \frac{1}{2}m_2\lambda_i - \frac{1}{3}m_3 = 0\,, \qquad i = 1, 2, 3.$$

Multiplying by λ_i and taking the summation over i we find

$$m_4 = \frac{1}{2}m_2^2\,.$$

Thus a closed system is obtained,

$$\frac{d}{dt}m_2 + 2m_3 = 0\,, \tag{4.6a}$$

$$\frac{d}{dt}m_3 + \frac{1}{2}m_2^2 = 0\,. \tag{4.6b}$$

Example 2. (4-dimensional case $n = 4$)

In the four-dimensional case one has

$$q_1 = 0\,, \qquad q_2 = -\frac{1}{2}m_2\,, \qquad q_3 = -\frac{1}{3}m_3\,, \qquad q_4 = -\frac{1}{4}m_4 + \frac{1}{8}m_2^2\,.$$

Hence

$$\lambda_i^4 - \frac{1}{2}m_2\lambda_i^2 - \frac{1}{3}m_3\lambda_i - \frac{1}{4}m_4 + \frac{1}{8}m_2^2 = 0\,, \qquad i = 1, 2, 3, 4.$$

Multiplying by λ_i and taking the summation over i we obtain

$$m_5 = \frac{1}{2}m_2m_3 + \frac{1}{3}m_3m_2 = \frac{5}{6}m_2m_3\,.$$

Therefore the resulting closed system becomes

$$\frac{d}{dt}m_2 + 2m_3 = 0\,, \tag{4.7a}$$

$$\frac{d}{dt}m_3 + 3m_4 = \frac{3}{4}m_2^2\,, \tag{4.7b}$$

$$\frac{d}{dt}m_4 = -\frac{7}{3}m_3m_2\,. \tag{4.7c}$$

4.3 Three-dimensional critical thresholds

This section is devoted to the study of three-dimensional restricted models in terms of the critical thresholds.

It follows from (4.6) that

$$\frac{d}{dt}(6m_3^2 - m_2^3) = 12m_3\dot{m}_3 - 3m_2^2\dot{m}_2 = -6m_3m_2^2 + 6m_2^2m_3 = 0\,,$$

which yields a global invariant

$$6m_3^2 - m_2^3 = \text{Const.} \tag{4.8}$$

We consider the phase plane (m_2, m_3), except for the separatrix $6m_3^2 = m_2^3$, all other solutions would not approach the origin. The phase plane is divided into two parts by this separatrix. The nonlinearity ensures that trajectories which do not pass the origin must lead to infinity at a finite time. In fact for initial data from the region $\{(m_2, m_3)\,, \quad m_2 > \sqrt[3]{6}m_3^{2/3}\}$, the corresponding trajectories will remain in this region since system (4.6) is autonomous. Therefore (4.6b) leads to

$$\dot{m}_3 < -\frac{1}{2}\sqrt[3]{36}\, m_3^{4/3}\,. \tag{4.9}$$

Since $\dot{m}_3 = -\frac{1}{2}m_2^2$, $m_3(t)$ is always decreasing in time. Even for positive $m_3(0)$, there exists a finite time T^* such that $m_3(T^*) < 0$. The integration of (4.9) gives

$$m_3(t) < \left[\frac{1}{6}\sqrt[3]{36}(t - T^*) + m_3(T^*)^{-1/3}\right]^{-3}. \tag{4.10}$$

This shows that $m_3(t) \to -\infty$ when t approaches a time before

$$T^* + \sqrt[3]{6}(-m_3(T^*))^{-1/3}\,.$$

Finite-time breakdown can be similarly justified for initial data lying in the region $\{(m_2, m_3)\,, \quad m_2 < \sqrt[3]{6}m_3^{2/3}\}$. These facts enable us to conclude the following:

Theorem 4.3.1 *Consider the system* (4.6) *with initial data* $\big(m_2(0), m_3(0)\big)$. *The global bounded solution exists if and only if the initial data lie on the curve*

$$\Big\{(m_2, m_3) \,\big|\quad m_3 = \frac{1}{\sqrt{6}} m_2^{3/2}\Big\}.$$

We now turn to interpret this condition in terms of the eigenvalues. Set $\Lambda = (\lambda_1, \lambda_2, \lambda_3)$, the above critical stable set can be written as

$$\Omega = \{\Lambda \,|\quad \sum_{k=1}^{3} \lambda_k^3 = \frac{1}{\sqrt{6}} \Big(\sum_{k=1}^{3} \lambda_k^2\Big)^{3/2} \}$$

The homogeneity of the above constraint in terms of eigenvalues implies that if $\Lambda \in \Omega$, then $\epsilon\Lambda \in \Omega$, $\forall\, \epsilon > 0$.

Without loss of generality, we consider the restriction of Ω onto a ball $\sum_{k=1}^{3} \lambda_k^2 = r^2$, denoted by $\Omega(r)$. There are two cases to be considered:

The initial eigenvalues contain complex component, say $\Lambda_0 = (a - bi, a + bi, c)$ for real $a, b, c \in \mathbb{R}$. The restricted set $\Omega(\sqrt{6})$ is determined by

$$c + 2a = 0, \qquad 2a^2 - 2b^2 + c^2 = 6, \qquad 2a(a^2 - 3b^2) + c^3 = \frac{r^3}{\sqrt{6}} = 6\,.$$

Eliminating c we have

$$6a^2 - 2b^2 = 6\,, \qquad -6a(a^2 + b^2) = 6 \;\Rightarrow\; 4a^3 - 3a + 1 = 0\,,$$

which has real roots $a \in \{-1, 0.5, 0.5\}$, from which no real $b \neq 0$ can be found.

The only possible scenario is the real eigenvalues $\Lambda = (a, b, c) \in \mathbb{R}^3$. Restricted again on $\Omega(\sqrt{6})$ we have

$$a + b + c = 0, \quad a^2 + b^2 + c^2 = 6, \quad a^3 + b^3 + c^3 = \frac{r^3}{\sqrt{6}} = 6\,.$$

Eliminating a, b we have $c^3 - 3c - 2 = 0$ with real roots $c \in \{2, -1, -1\}$. The symmetric property implies that a, b also lie in the set $\{2, -1, -1\}$. In short, one has

$$\Omega(\sqrt{6}) = \big\{\Lambda \,\big|\quad (-1,-1,2),\, (-1,2,-1),\, (2,-1,-1)\big\}\,.$$

This leads to:

Theorem 4.3.2 *Solutions to* (4.2) *with* $n = 3$ *remain bounded for all time if and only if the initial data* $\Lambda_0 := (\lambda_1, \lambda_2, \lambda_3)$ *lie in the following set*

$$r\big\{(-1,-1,2),\, (-1,2,-1),\, (2,-1,-1)\big\}, \qquad \forall\, r \geq 0\,.$$

Thus, restricted to one orthant of the unit sphere we find that the three-dimensional restricted Euler equations admit only one sub-critical point. In this sense, the finite-time breakdown of three-dimensional restricted Euler equations is generic. This result was already obtained in [LT02] by spectral dynamics analysis. What we have presented is an alternative, equivalent argument which paves the way for the analysis carried out in next section.

4.4 Four-dimensional critical thresholds

In four-dimensional setting, the trace dynamics is governed by $m_1 = 0$ and

$$\frac{d}{dt}m_2 + 2m_3 = 0\,, \tag{4.11a}$$

$$\frac{d}{dt}m_3 + 3m_4 = \frac{3}{4}m_2^2\,, \tag{4.11b}$$

$$\frac{d}{dt}m_4 = -\frac{7}{3}m_3 m_2\,. \tag{4.11c}$$

Combine (4.11a) and (4.11c) we obtain

$$\frac{d}{dt}\big(m_4 - \frac{7}{12}m_2^2\big) = 0\,,$$

which gives a global invariant

$$m_4 = \frac{7}{12}m_2^2 + C_1, \qquad C_1 := m_4(0) - \frac{7}{12}m_2^2(0)\,. \tag{4.12}$$

Substituting this into (4.11b) we find

$$\frac{d}{dt}m_3 = -m_2^2 - 3C_1\,.$$

We then have a closed system for (m_2, m_3)

$$\frac{d}{dt}m_2 = -2m_3\,, \tag{4.13a}$$

$$\frac{d}{dt}m_3 = -m_2^2 - 3C_1\,. \tag{4.13b}$$

In order to ensure global bounded solution it is necessary to assume $C_1 < 0$, i.e.,

$$m_4(0) < \frac{7}{12}m_2^2(0)\,. \tag{4.14}$$

Set $-3C_1 = l^2$ with $l > 0$, thus system (4.15) is written as

$$\frac{d}{dt}m_2 = -2m_3, \tag{4.15a}$$

$$\frac{d}{dt}m_3 = -m_2^2 + l^2, \tag{4.15b}$$

with moving parameter l determined by the initial data through

$$m_4(0) = \frac{7}{12}m_2^2(0) - \frac{l^2}{3}\,. \tag{4.16}$$

This system has two critical points $(-l, 0)$ and $(l, 0)$; it is easy to verify that as equilibrium points, $(-l, 0)$ is a spiral point and $(l, 0)$ is a saddle point for the corresponding linearized system.

This structure suggests that part of separatrixes of this system may serve as the critical threshold. Note that

$$\frac{d}{dt}\big(3m_3^2 - m_2^3 + 3l^2 m_2\big) = 6m_3\frac{d}{dt}m_3 - 3m_2^2\frac{d}{dt}m_2 + 3l^2\frac{d}{dt}m_2 = 0\,.$$

Thus the second global invariant is

$$3m_3^2 - m_2^3 + 3l^2 m_2 = C_2\,. \tag{4.17}$$

The two separatrixes passing $(l, 0)$ are obtained by taking $C_2 = 2l^3$, i.e.,

$$3m_3^2 = m_2^3 - 3l^2 m_2 + 2l^3 = (m_2 + 2l)(m_2 - l)^2\,. \tag{4.18}$$

In the phase plane (m_2, m_3) , this consists of a closed curve for $-2l \le m_2 \le l$ and two open branches for $m_2 > l$. The phase plane analysis suggests that the global bounded solution exists if and only if the initial data satisfy

$$m_4(0) < \frac{7}{12}m_2^2(0)\,, \tag{4.19}$$

and

$$(m_2, m_3)(0) \in \Gamma\,, \tag{4.20}$$

where

$$\begin{aligned}\Gamma := &\left\{(m_2, m_3)\Big| \quad |m_3| \le \frac{l - m_2}{\sqrt{3}}\sqrt{m_2 + 2l}\,, \qquad -2l \le m_2 \le l\right\}\\ &\qquad \bigcup \left\{(m_2, m_3)\Big| \quad m_3 = \frac{m_2 - l}{\sqrt{3}}\sqrt{m_2 + 2l}\,, \qquad m_2 > l\right\}\end{aligned}$$

To interpret this condition in terms of the eigenvalues, we state the following theorem.

Theorem 4.4.1 *The solutions to* (4.2) *with* $n = 4$ *remain bounded for all time if and only if the initial data* $\Lambda_0 := (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ *lie in the following set*

$$\Lambda_0 \in \Omega \cap \Big\{\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0\Big\}\,,$$

where

$$\Omega := \Omega_1 \cup \Omega_2 \cup \Omega_3\,,$$

$$\begin{aligned}\Omega_1 := \Big\{\big(\lambda_1, \lambda_2, \lambda_3, \lambda_4\big) \;\Big|\; & \lambda_{1,2,3,4} \in \mathbb{R} \quad \textit{and} \quad \lambda_{i_1} = \lambda_{i_2} \le \lambda_{i_3} \le \lambda_{i_4}, \\ & \textit{here } (i_1, i_2, i_3, i_4) \textit{ is a permutation of } (1,2,3,4)\Big\}\,,\end{aligned}$$

$$\begin{aligned}\Omega_2 := \Big\{\big(\lambda_1, \lambda_2, \lambda_3, \lambda_4\big) \;\Big|\; & \lambda_{i_1} = \lambda_{i_2} \in \mathbb{R} \quad \textit{and} \quad \lambda_{i_3,i_4} \in \mathbb{C}/\ \mathbb{R}\,, \\ & \textit{here } (i_1, i_2, i_3, i_4) \textit{ is a permutation of } (1,2,3,4)\Big\}\,,\end{aligned}$$

and

$$\Omega_3 := \Big\{\big(\lambda_1, \lambda_2, \lambda_3, \lambda_4\big) \;\Big|\; \lambda_{1,2,3,4} \in \mathbb{C}/\ \mathbb{R}\Big\}\,.$$

Proof. In terms of traces m_k, we have showed that solutions to (4.2) with $n = 4$ remain bounded for all time if and only if the initial data (m_2, m_3, m_4) lie in the following set

$$S \cap \Gamma\,, \tag{4.21}$$

where

$$S := \Big\{\Lambda \;|\quad C_1 := m_4 - \frac{7}{12} m_2^2 \le 0\Big\}\,,$$

and

$$\begin{aligned}\Gamma := & \Big\{(m_2, m_3)\Big| \quad |m_3| \le \frac{l - m_2}{\sqrt{3}}\sqrt{m_2 + 2l}\,, \qquad -2l \le m_2 \le l\Big\} \\ & \bigcup \Big\{(m_2, m_3)\Big| \quad m_3 = \frac{m_2 - l}{\sqrt{3}}\sqrt{m_2 + 2l}\,, \qquad m_2 > l\Big\}\end{aligned}$$

We decompose Γ as

$$\Gamma := \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \,,$$

where

$$\Gamma_1 := \Big\{ (m_2, m_3) \;\Big|\; \quad |m_3| < \frac{l - m_2}{\sqrt{3}} \sqrt{m_2 + 2l}, \quad -2l \le m_2 < l \Big\} \,,$$

$$\Gamma_2 := \Big\{ (m_2, m_3) \;\Big|\; \quad |m_3| = \frac{l - m_2}{\sqrt{3}} \sqrt{m_2 + 2l}, \quad -2l \le m_2 < l \Big\} \,,$$

$$\Gamma_3 := \Big\{ (m_2, m_3) \;\Big|\; \quad m_3 = \frac{m_2 - l}{\sqrt{3}} \sqrt{m_2 + 2l}, \quad m_2 \ge l \Big\} \,,$$

and

$$l^2 = -3C_1, \qquad \text{with} \qquad l > 0 \,.$$

The meaning of this decomposition will be clear later.

We now turn to interpret condition (4.21) in terms of the eigenvalues $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$. we consider three cases.

Case I: all the eigenvalues are real.

Suppose the four eigenvalues are

$$\lambda_1 = a \,, \qquad \lambda_2 = b \,, \qquad \lambda_3 = c \,, \qquad \text{and} \qquad \lambda_4 = -(a + b + c) \,, \tag{4.22}$$

where $a, b, c \in \mathbb{R}$. Suppose $C_1 < 0$, then

$$m_2^2 - l^2 = m_2^2 + 3C_1 = 3m_4 - \frac{3}{4} m_2^2 = 3\Big(m_4 - \frac{1}{4} m_2^2 \Big).$$

By applying the general inequality $(\lambda^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2)^2 \le 4(\lambda_1^4 + \lambda_2^4 + \lambda_3^4 + \lambda_4^2)$, we obtain that $m_4 - \dfrac{1}{4} m_2^2 \ge 0$, for all $t > 0$. Thus $m_2^2 - l^2 \ge 0$ is always true. This implies that under the constraint that all the eigenvalues are real, $S \cap \Gamma = S \cap \Gamma_3$.

Because of the homogeneity, we can assume the four real eigenvalues are $-1+s, -1+w, -1$ and $3-s-w$. We then perform the following computation. Since

$$m_3 = \frac{m_2 - l}{\sqrt{3}} \sqrt{m_2 + 2l}\,, \quad \text{i.e.,} \quad 3m_3^2 = (m_2 - l)^2 (m_2 + 2l)$$

then

$$3m_3^2 = m_2^3 - 3m_2 l^2 + 2l^3\,. \tag{4.23}$$

Rewrite this equation as

$$3m_3^2 - m_2^3 + 3m_2(-3C_1) = 2l^3\,,$$

i.e.,

$$3m_3^2 - m_2^3 - 9m_2(m_4 - \frac{7}{12}m_2^2) = 2l^3\,. \tag{4.24}$$

Square both sides of (4.24), we obtain

$$\Big(3m_3^2 - m_2^3 - 9m_2(m_4 - \frac{7}{12}m_2^2)\Big)^2 = 4l^6\,.$$

Replace l^6 by $(-3C_1)^3 = 27(m_4 - \frac{7}{12}m_2^2)^3$, we obtain

$$0 = \Big(3m_3^2 - m_2^3 - 9m_2(m_4 - \frac{7}{12}m_2^2)\Big)^2 + 108\Big(m_4 - \frac{7}{12}m_2^2\Big)^3 =: p\,.$$

Direct calculation shows that

$$p = -27s^2w^2(s-w)^2(2s+w-4)^2(s+2w-4)^2(s+w-4)^2.$$

Thus $p = 0$ if and only if either $s = 0$, or $w = 0$, or $s - w = 0$, or $2s + w - 4 = 0$, or $s + 2w - 4 = 0$, or $s + w - 4 = 0$. They are all the same subject to homogeneity and permutation. Actually, it turns out that the four eigenvalues must be in the form $r(-1+s, -1, -1, 3-s)$, $\forall\, r \geq 0$, with arbitrary permutation. We claim that the range for s is $[0,4]$. When $0 \le s \le 4$, it's easy to check that $\Lambda \in S \cap \Gamma_3$. When $s > 4$ or $s < 0$, we rewrite $(-1+s,-1,-1,3-s)$ as $(1+\tilde{s},-1,-1,1-\tilde{s})$, where $\tilde{s} := s-2$, with $\tilde{s} < -2$ or $\tilde{s} > 2$. Then we find

$$l = \tilde{s}^2 - 4, \qquad m_2 = 4 + 2\tilde{s}^2, \qquad m_3 = 6\tilde{s}^2 .$$

Let

$$p_1 := \frac{m_2 - l}{\sqrt{3}} \sqrt{m_2 + 2l} = \frac{\tilde{s}^2 + 8}{\sqrt{3}} \sqrt{4\tilde{s}^2 - 4} = \frac{2(\tilde{s}^2 + 8)}{\sqrt{3}} \sqrt{\tilde{s}^2 - 1} .$$

Thus

$$p_1^2 - m_3^2 = \frac{3}{4}(\tilde{s}^2 - 4)^3 > 0 ,$$

which means

$$r(-1+s,-1,-1,3-s) \notin S \cap \Gamma_3 , \quad \text{for} \quad s \notin [0,4] .$$

So here we conclude: if all the eigenvalues are real, then $\Lambda_0 \in S \cap \Gamma$, if and only if Λ_0 takes the form $r(-1+s,-1,-1,3-s)$ with arbitrary permutation, where $0 \le s \le 4$ and $r \ge 0$. And more precisely, in this case, $\Lambda_0 \in S \cap \Gamma_3$.

Case II: a pair of complex eigenvalues and two real eigenvalues.

Let us suppose the four eigenvalues are

$$\lambda_1 = a + bi , \qquad \lambda_2 = a - bi , \qquad \lambda_3 = -a + c , \qquad \text{and} \qquad \lambda_4 = -a - c , \quad (4.25)$$

where $a, b, c \in \mathbb{R}$ and $b \neq 0$. Direct calculation yields

$$m_2 = 4a^2 - 2b^2 + 2c^2 ,$$

$$m_3 = -6a(b^2 + c^2) ,$$

$$m_4 = 4a^4 + 2b^4 + 2c^4 - 12a^2(b^2 - c^2)\,,$$

$$l^2 = \big(4a^2 + b^2 - c^2\big)^2 - 12b^2c^2\,,$$

and

$$m_2^2 - l^2 = 3\Big(m_4 - \frac{1}{4}m_2^2\Big) = 3\Big(b^2 + c^2\Big)^2 - 24a^2(b^2 - c^2)\,.$$

We show that

$$\Lambda_0 \in S \cap (\Gamma_2 \cup \Gamma_3)\,.$$

We need to consider the situation that $C_1 < 0$, and $-2l \leq m_2 \leq l$. If $4a^2+b^2-c^2 \geq 0$, then $l \leq 4a^2 + b^2 - c^2$. It follows that

$$\frac{l - m_2}{\sqrt{3}}\sqrt{m_2 + 2l} \leq \frac{3(b^2 - c^2)}{\sqrt{3}}\sqrt{12a^2} = 6|a(b^2 - c^2)| \leq |m_3|.$$

It becomes an equality if and only if $c = 0$. If $4a^2+b^2-c^2 < 0$, then $l < c^2-4a^2-b^2$, and $l - m_2 = b^2 - c^2 - 8a^2 < 0$. It's a contradiction to $m_2 < l$.

Thus if the four eigenvalues consist of two real and two non-real eigenvalues, then (m_2, m_3) must satisfies

$$m_3 = \frac{m_2 - l}{\sqrt{3}}\sqrt{m_2 + 2l}\,.$$

In an equivalent form, it is

$$3m_3^2 - m_2^3 - 9m_2\Big(m_4 - \frac{7}{12}m_2^2\Big) = 2l^3\,.$$

Square both sides of this equation and replace l^6 by $(-3C_1)^3 = 27(m_4 - \frac{7}{12}m_2^2)^3$, we obtain

$$p_2 := \Big(3m_3^2 - m_2^3 - 9m_2(m_4 - \frac{7}{12}m_2^2)\Big)^2 + 108\Big(m_4 - \frac{7}{12}m_2^2\Big)^3 = 0\,.$$

Direct calculation shows that

$$p_2 = 432b^2c^2\Big(b^2+(2a-c)^2\Big)^2\Big(b^2+(2a+c)^2\Big)^2 .$$

Thus $p=0$ if and only if $c=0$. We can verify that when $c=0$, we do have $C_1<0$ and $-2l \le m_2 \le l$. So here we conclude: if the eigenvalues consist of two real and non-real eigenvalues (4.25), then $\Lambda_0 \in S\cap\Gamma$ if and only if (a,b,c) takes the form $r(q,1,0)$, $\forall\, r\in\mathbb{R}$, and $\forall\, q\in\mathbb{R}$. More precisely, in the above form, $\Lambda_0\in S\cap\Gamma_2$.

Case III: two pairs of complex eigenvalues.

Let us suppose the four eigenvalues are

$$\lambda_1 = a+bi, \qquad \lambda_2 = a-bi, \qquad \lambda_3 = -a+ci, \qquad \lambda_4 = -a-ci\,, \tag{4.26}$$

where $a,b,c\in\mathbb{R}$, and $bc\neq 0$. We can calculate that

$$m_2 = 4a^2 - 2b^2 - 2c^2\,,$$

$$m_3 = 6a(c^2-b^2)\,,$$

$$m_4 = 4a^4 + 2b^4 + 2c^4 - 12a^2(b^2+c^2)\,,$$

$$m_2^2 - l^2 = 3\Big(m_4 - \frac{1}{4}m_2^2\Big) = 3\Big(b^2-c^2\Big)^2 - 24a^2\Big(b^2+c^2\Big)\,,$$

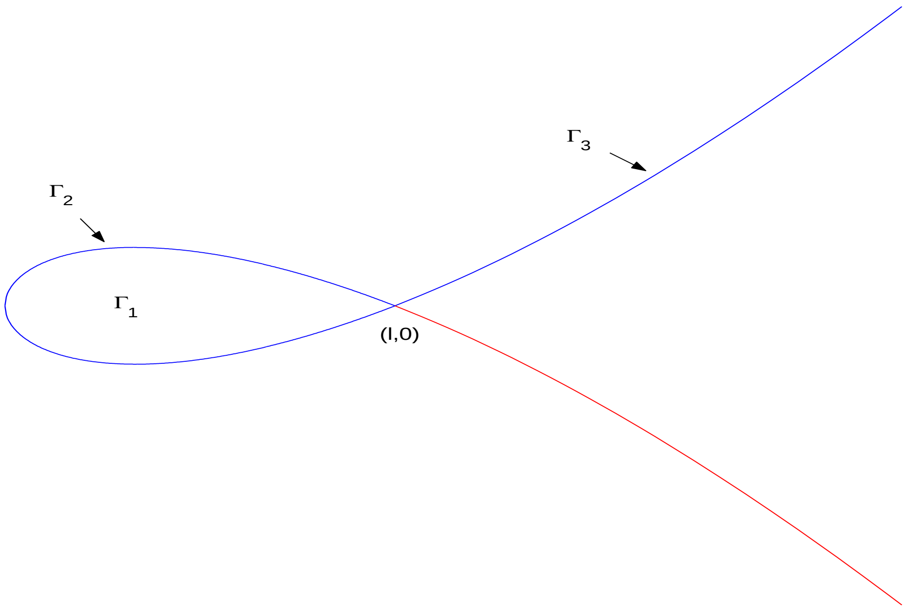
and

$$l^2 = \Big(4a^2+b^2+c^2\Big)^2 + 12b^2c^2\,.$$

It follows that $-2l < m_2 < l\,, \quad l > 4a^2+b^2+c^2\,, \quad$ and

$$\frac{l-m_2}{\sqrt{3}}\sqrt{m_2+2l} > \frac{3(b^2+c^2)}{\sqrt{3}}\sqrt{12a^2} = 6|a|(b^2+c^2) > |m_3|\,.$$

Thus $\forall (a,b,c)\in\mathbb{R}^3$ with $bc\neq 0$, $\Lambda_0\in S\cap\Gamma$, and solutions to (4.2) remain bounded for all time. More precisely, $\Lambda_0\in S\cap\Gamma_1$.



At last, we give the graph of Γ on the phase plane (m_2, m_3).

The meaning of each part of Γ were explained in Case I,II and III. This completes the proof. $\square$

Compared with the three-dimensional restricted Euler equations, for four-dimensional restricted Euler equations, here we found a surprising global existence for subcritical initial data.

4.5 n-dimensional critical thresholds

In this section, we partially extend our results to the general n-dimensional $(n > 4)$ restricted Euler equations.

For $n > 4$, we do not apply the trace dynamics. Instead, we deal with the eigenvalue system directly

$$\begin{aligned} &\frac{d}{dt}\lambda_i + \lambda_i^2 = \frac{1}{n}\sum_{j=1}^{n}\lambda_j^2\,, \qquad i = 1, \cdots, n\,, \\ &\sum_{i=1}^{n}\lambda_i(0) = 0\,. \end{aligned} \tag{4.27}$$

First, we prove that if all the initial λ's are real, then the finite-time breakdown of (4.27) is generic.

Theorem 4.5.1 *Reorder the initial data of* (4.27) *such that*

$$\lambda_1(0) = \lambda_2(0) = \cdots = \lambda_k(0) < \lambda_{k+1} \leq \cdots \leq \lambda_n(0)\,,$$

then λ's remain bounded if and only if $k \geq \dfrac{n}{2}$.

Proof. First we show that if

$$\lambda_1(0) < \lambda_2(0) \cdots \leq \lambda_n(0)\,,$$

then λ_1 will become unbounded in a finite time. We prove this by the contradiction method. Here we list some facts which are useful in our proof. Suppose all λ's remain bounded, then

Fact 1. $\lambda_i(t) < \lambda_j(t)\,, \quad \forall\, t \geq 0 \;$ and $\; i < j\,.$

Take the difference of the λ_i and λ_j equations, we obtain

$$\frac{d}{dt}(\lambda_i - \lambda_j) = -(\lambda_i^2 - \lambda_j^2)\,.$$

Divide the above equation by $(\lambda_i - \lambda_j)$, we find that

$$\frac{d}{dt}\Big(\ln(\lambda_i - \lambda_j)\Big) = -(\lambda_i + \lambda_j)\,,$$

which yields

$$\lambda_i(t) - \lambda_j(t) = \Big(\lambda_i(0) - \lambda_j(0)\Big) e^{-\displaystyle\int_0^t (\lambda_i + \lambda_j) s ds} < 0 .$$

Fact 2. If $\lambda_i(t_0) \geq 0$, then $\lambda_i(t) \geq 0$, $\forall t > t_0$.

Fact 3. If $\lambda_i(0) < \lambda_j(0) < 0$, then $(\lambda_i - \lambda_j)(t)$ is decreasing as long as $\lambda_j(t) \leq 0$.

Fact 4. If $\lambda_i(t_0) > \lambda_j(t_0) \geq 0$, then $(\lambda_i - \lambda_j)(t)$ is decreasing for $t > t_0$.

Let us suppose that

$$\lambda_1 < \lambda_2 < \cdots < \lambda_p < 0 \leq \lambda_{p+1} < \lambda_{p+2} < \cdots < \lambda_n , \qquad \forall\, t > 0 .$$

The equations for λ_1 and λ_2 are

$$\frac{d}{dt}\lambda_1 + \lambda_1^2 = \frac{1}{n}\sum_{j=1}^{n} \lambda_j^2 , \tag{4.28a}$$

$$\frac{d}{dt}\lambda_2 + \lambda_2^2 = \frac{1}{n}\sum_{j=1}^{n} \lambda_j^2 , \tag{4.28b}$$

Multiply (4.28a) by $1/\lambda_2$, (4.28b) by λ_1/λ_2^2, and take the difference, we obtain

$$\frac{d}{dt}\Big(\frac{\lambda_1}{\lambda_2}\Big) = \frac{1}{n}\Big(\frac{\lambda_2 - \lambda_1}{\lambda_2^2}\Big)\sum_{j=1}^{n} \lambda_j^2 + \Big(\frac{\lambda_1\lambda_2 - \lambda_1^2}{\lambda_2}\Big) > 0 , \tag{4.29}$$

which means $\dfrac{\lambda_1}{\lambda_2}$ is increasing. Applying Fact 3, we know there is an upper bound $C_{\lambda_1} < 0$ for $\lambda_1(t)$. Thus

$$\frac{\lambda_1}{\lambda_2}(t) \geq \frac{\lambda_1}{\lambda_2}(0) + Ct .$$

On the other hand,

$$\lambda_n(t) > \frac{\lambda_p(0) - \lambda_1(0)}{n - p} , \qquad \forall\, t > 0 .$$

By applying Fact 3, we find that $\lambda_n - \lambda_{p+1}$ goes to 0. Combine the above results, we obtain that when t is large enough,

$$\frac{d}{dt}\lambda_2 \geq \frac{1}{2}\left(\frac{1}{n}\lambda_1^2 + \frac{1}{n}\sum_{j=p+1}^{n}\left(\frac{\lambda_1}{n-p}\right)^2\right).$$

Thus λ_2 will be increasing, and eventually λ_2 will be greater than 0, which means $p = 1$. Then, we obtain that for a fixed $0 < \alpha < 1$, when t is large enough, λ_1 satisfies

$$\frac{d}{dt}\lambda_1 \leq \alpha\left(\frac{1}{n}\lambda_1^2 + \frac{1}{n}\sum_{j=2}^{n}\left(\frac{\lambda_1}{n-1}\right)^2 - \lambda_1^2\right) = \alpha\frac{2-n}{n-1}\lambda_1^2,$$

which implies λ_1 goes to $-\infty$ at a finite time.

Next, we consider the case

$$\lambda_1(0) = \lambda_2(0) = \cdots = \lambda_k(0) < \lambda_{k+1}(0) \leq \cdots \leq \lambda_n(0).$$

Following the same strategy, we can prove that when t is large enough, $\lambda_{k+1}(t)$ will be positive. So without loss of generality, we can consider

$$\lambda_1(0) = \lambda_2(0) = \cdots = \lambda_k(0) < 0 < \lambda_{k+1}(0) \leq \cdots \leq \lambda_n(0).$$

If $k < \frac{n}{2}$, then since $\sum_{j=1}^{n}\lambda_j = 0$, we obtain $\lambda_{k+1} < -\lambda_1$. Thus λ_{k+1} is increasing. If $\lambda_n \geq -\lambda_1$, then $\frac{d}{dt}\lambda_n < 0$. Combine that λ_{k+1} is increasing and λ_n is decreasing, we know λ_n will be less than $-\lambda_1$ eventually. By applying Fact 3, we obtain that for a fixed $0 < \alpha < 1$, when t large enough, λ_1 satisfies

$$\frac{d}{dt}\lambda_1 \leq \alpha\left(\frac{k}{n}\lambda_1^2 + \frac{1}{n}\sum_{j=k+1}^{n}\left(\frac{k\lambda_1}{n-k}\right)^2 - \lambda_1^2\right) = \alpha\frac{2k-n}{n-k}\lambda_1^2,$$

which implies λ_1 goes to $-\infty$ at a finite time. On the other hand, if $k \geq \frac{n}{2}$, then $\lambda_n > -\lambda_1$. Thus λ_n is decreasing all the time, which implies λ's approach 0 as t increases. This complete the proof. $\square$

According to Theorem 4.5.1, if the initial velocity is irrotational, i.e., if $M = \nabla u$ is symmetric, then finite-time breakdown of system (4.1) is generic. More precisely, it breaks down in such a way: the smallest eigenvalues go to $-\infty$ when t approaches the critical time, while all the other eigenvalues go to ∞ when t approaches the critical time.

Next, we prove that if all the eigenvalues are non-real, then solutions to (4.2) remain bounded for all time.

Theorem 4.5.2 *Suppose that initial data of* (4.2) *are non-real, i.e.,* $\Im(\lambda_i(0)) \neq 0\,, \forall\, i\,,$ *then solutions to system* (4.2) *remain bounded for all time.*

Proof. We perform singularity analysis to prove this theorem. For reader's convenience, we sketch the main steps briefly, and refer to [GH00] and references therein for more details of this method.

We assume a flow governed by the nonlinear ODE $w' = f(w)$ diverges at a finite t^*, then one can seek local solutions of the so-called *Psi-series* form

$$w = \omega \tau^p \left[1 + \sum_{j=1}^{\infty} a_j \tau^{j/q} \right],$$

where $\tau = (t^* - t)$, $p \in \mathbb{R}^n$ with at least one negative component, $q \in \mathbb{N}$, and a_j is a polynomial in $\log(t^* - t)$ of degree $N_j \leq j$. There are three steps to determine the above series:

Step 1: find the so-called balance pair, (ω, p), such that the dominant behavior, $\omega \tau^p$, is an exact solution of some truncated system $w' = \tilde{f}(w)$;

Step 2: computation of the resonances, which are given by the eigenvalues of the matrix $-\dfrac{\partial \tilde{f}(w)}{\partial w} - \mathrm{diag}(p)$;

Step 3: the last step of the singularity analysis consists of finding the explicit form for the different coefficients a_j by inserting the full series in the original system, $w' = f(w)$.

The singularity analysis asserts, if the system breaks down for some initial data, then there must exist a general solution in the *Psi-series* form with $\omega \in \mathbb{R}^n$. We use a corollary (Corollary 1, p443, [GH00]) to prove our theorem. More precisely, we show that there is no real balance pair (ω, p) such that $\omega\tau^p$ solves our system, thus the system does not have finite-time singularities.

Since all the eigenvalues are complex, the dimension number must be even. Suppose the dimension number is $2n$, and the eigenvalues are $a_k \pm ib_k$, where $1 \leq k \leq n$, $a_k, b_k \in \mathbb{R}$ and $b_k > 0$. Plug these into (4.2) we obtain

$$\frac{d}{dt}a_k + (a_k^2 - b_k^2) = \sum_{j=1}^{n}(a_j^2 - b_j^2)/n\,, \qquad k = 1, \cdots, n \tag{4.30a}$$

$$\frac{d}{dt}b_k + 2a_k b_k = 0\,, \qquad k = 1, \cdots, n \tag{4.30b}$$

Let us suppose the dominant behaviors of a_k and b_k have the form

$$a_k \sim \alpha_k \tau^{p_k}\,, \qquad b_k \sim \beta_k \tau^{q_k}\,.$$

Substituting this into (4.30), we find

$$-p_k\alpha_k\tau^{p_k-1} + (\alpha_k^2\tau^{2p_k} - \beta_k^2\tau^{2q_k}) = \frac{1}{n}\sum_{j=1}^{n}(\alpha_k^2\tau^{2p_k} - \beta_k^2\tau^{2q_k})\,, \qquad k = 1, \cdots, n \tag{4.31a}$$

$$-q_k\beta_k\tau^{q_k-1} + 2\alpha_k\beta_k\tau^{p_k+q_k} = 0\,, \qquad k = 1, \cdots, n \tag{4.31b}$$

It follows from (4.31b) that

$$p_k = -1\,, \qquad \text{and} \qquad \alpha_k = \frac{q_k}{2}\,, \qquad \forall\, k\,.$$

For q_k, there are three cases to be considered.

(1) If $\min_k q_k < -1$, then it follows from (4.31b) that $q_j = \min_k q_k$, $\forall j$. Thus $\alpha_k = \frac{q_k}{2}$ are all negative. This implies system (4.2) may diverge in such a way: all the real parts of the eigenvalues go to $-\infty$, and all the imagine parts go to ∞. This is a contradiction to the incompressibility condition, $\sum_{j=1}^n a_j = 0$.

(2) If $q_k \geq -1$ for all k, and there exists $q_j = -1$. We choose a m such that

$$\Re(\beta_m^2) = \max_{q_j=-1} \Re(\beta_j^2).$$

Then according to $(4.31a)_m$, the coefficients of the τ^{-2} terms satisfy

$$-\frac{1}{4} = \alpha_m + \alpha_m^2 = \sum_{j=1}^n \alpha_j^2 + (\beta_m^2 - \sum_{q_s=-1} \beta_s^2/n) > 0,$$

which is a contradiction.

(3) If $q_k > -1$ for all k. Then after dropping the lower order terms, we obtain that $\alpha_k = \frac{q_k}{2}$ satisfies

$$\frac{q_k}{2} + \frac{q_k^2}{4} = \sum_{j=1}^n \frac{q_j^2}{4n}.$$

According to the incompressibility condition, we have $\sum_{j=1}^n q_j = 0$. So there exists $q_r < 0$. Since

$$\frac{q_r}{2} + \frac{q_r^2}{4} > 0,$$

it yields that $q_r < -2$. It's a contradiction to $q_r > -1$.

Combine (1),(2) and (3), we complete the proof. □

At last, we prove that if (4.2) have both real and complex eigenvalues, the system will breakdown for initial data in an open set.

Theorem 4.5.3 *Suppose the initial eigenvalues of* (4.2) *have both real and complex numbers, then the system will break down at a finite time if the initial data are in an open set.*

Proof. Suppose the eigenvalues are $a_k \pm i b_k$ and c_l, where $1 \le k \le n$, $1 \le l \le m$, $a_k, b_k, c_l \in \mathbb{R}$. Suppose (4.2) breaks down at a critical time t^*, and the dominant behaviors of a_k, b_k and c_l have the form

$$a_k \sim \alpha_k \tau^{p_k}, \qquad b_k \sim \beta_k \tau^{q_k}, \qquad c_l \sim \gamma_l \tau^{r_k}, \qquad \tau = t^* - t.$$

Following the proof of Theorem 4.5.2, we obtain that $p_k = 1$ and $\alpha_k = \dfrac{q_k}{2}$ for all k. We then find a balance pair

$$\begin{cases} (p_1, p_2, \cdots, p_n) = (-1, -1, \cdots, -1) \\ (q_1, q_2, \cdots, q_n) = (\frac{2}{2n+m-2}, \frac{2}{2n+m-2}, \cdots, \frac{2}{2n+m-2}) \\ (r_1, r_2, \cdots, r_m) = (-1, -1, \cdots, -1) \\ (\alpha_1, \alpha_2, \cdots, \alpha_n) = (\frac{1}{2n+m-2}, \frac{1}{2n+m-2}, \cdots, \frac{1}{2n+m-2}) \\ \forall\, \beta_k \\ (\gamma_1, \gamma_2, \cdots, \gamma_m) = (\frac{1}{2n+m-2}, \frac{1}{2n+m-2}, \cdots, \frac{1-2n-m}{2n+m-2}, \frac{1}{2n+m-2}, \frac{1}{2n+m-2}) \end{cases}$$

According to (Theorem 1, p428, [GH00]), we know that for the initial data in an open set, (4.2) will break down in such a way: the smallest real eigenvalue goes to $-\infty$, all other real eigenvalues and real part of the complex eigenvalues go to ∞, and all the imagine parts of the complex eigenvalues go to 0. This completes the proof. $\square$

Remark 4.5.1 *Numerical experiments strongly suggest that if there is any real eigenvalue at the beginning, then* (4.2) *will break down at a finite time, unless there* *are at least* $\left[\frac{n+1}{2}\right]$ *real eigenvalues which equals each other, and they are less than all the other real eigenvalues. If this is true, then the sharp critical threshold for the* n*-dimensional restricted Euler is: solutions to* (4.2) *remain bounded for all time if and only if either* λ*'s are all non-real, or among the real eigenvalues there are at least* $\left[\frac{n+1}{2}\right]$ *minimums.*

Bibliography

[BKM84] J. T. Beale, T. Kato, A. Majda, Remarks on the breakdown of smooth solutions for the 3D Euler equation. Comm. Math. Phys. 94 (1984), 61–66.

[BW98] M. P. Brenner, T. P. Witelski, On Spherically Symmetric Gravitational Collapse, J. of Statistical Physics, 93 (1998), 863–900.

[CA92] B. J. Cantwell, Exact solution of a restricted Euler equation for the velocity gradient tensor, Phys. Fluids A, 4 (1992), 782–793.

[C86] P. Constantin, Note on loss of regularity for solutions of the 3-D incompressible Euler and related equations, Commun. Math. Phys., 104 (1986), 311–326.

[CFM96] P. Constantin, Ch. Fefferman, A. Majda, Geometric constraints on potentially singular solutions for the 3D Euler equation. Comm. Partial Diff. Equations 21 (1996), 559–571.

[CL04] G. -Q. Chen, H. Liu, Concentration and cavitation in the vanishing pressure limit of solutions to the Euler equations for nonisentropic fluids, Phys. D 189 (2004), 141–165.

[CPB99] M. Chertkov, A. Pumir, B. Shraiman, Lagrangian tetrad dynamics and the phenomenology of turbulence, Physics of fluids, 11 (1999), 2394–2410.

[Co06] J. Coulombel, From gas dynamics to pressureless gas dynamics, Proc. Amer. Math. Soc. 134 (2006), 683–688.

[CW96] G. -Q. Chen, D. Wang, Convergence of shock capturing scheme for the compressible Euler-Poisson equation, Comm. Math. Phys. 179 (1996), 333–364.

[En96] S. Engelberg, Formation of singularities in the Euler and Euler-Poisson equations, Physica D 98 (1996), 67–74.

[ELT01] S. Engelberg, H. Liu, E. Tadmor, Critical Thresholds in Euler-Poisson Equations, Indiana Univ. Math. J. 50 (2001), 109–157.

[ES00] W. E, Y. Sinai, New results in mathematical and statistical hydrodynamics. (Russian) Uspekhi Mat. Nauk 55 (2000), 25–58.

[Ev98] L. C. Evans, Partial Differential Equations, Rhode Island: AMS Providence, 1998.

[GH00] A. Goriely, C. Hyde, Necessary and sufficient conditions for finite time singularities in ordinary differential equations, J. Diff. Eqs. 161 (2000), 422-448.

[Go04] P. Godin, A class of global non-smooth axisymmetric solutions to the Euler equations of an isentropic perfect gas in 2 space dimensions. Hyperbolic problems: theory, numerics, applications, 549–555, Springer, Berlin, 2003.

[Gr98] M. Grassin, Global smooth solutions to Euler equations for a perfect gas, Indiana Univ. Math. J. 47 (1998), 1397–1432.

[GS00] T. Goudon, S. Junca, Vanishing pressure in gas dynamics equations, Z. Angew. Math. Phys. 51 (2000), 143–148.

[Gu98] Y. Guo, Smooth irrotational flows in the large to the Euler-Poisson system in $\mathbb{R}^{3+1}$, Comm. Math. Phys. 195 (1998), 249–265.

[JR00] S. Junca, M. Rascle, Relaxation of the isothermal Euler-Poisson system to the drift-diffusion equations, Quart. Appl. Math. 58 (2000), 511–521.

[K70] S. N. Kružkov, Generalized solutions to the Hamilton-Jacobi equations of Eikonal type, I. Mat. Sb. 27, (1975), 406–446 (in Russian).

[La64] P. D. Lax, Development of singularities of solutions of nonlinear hyperbolic partial differential equations, J. Math. Physics 5 (1964), 611–613.

[Li96] P. L. Lions, Mathematical Topics in Fluid Mechanics: Incompressible Models. Oxford Lecture Series in Mathematics and Its Applications, 3, Vols. 1-2, Oxford: Oxford Univ. Press, 1996.

[LT02] H. Liu, E. Tadmor, Spectral dynamics of the velocity gradient field in restricted flows, Comm. Math. Phys. 228 (2002), 435–466.

[LT03] H. Liu, E. Tadmor, Critical thresholds in 2D restricted Euler-Poisson equations, SIAM J. Appl. Math. 63 (2003), 889–1910.

[LTW07] H. Liu, E. Tadmor, D. Wei, Critical thresholds in 4D restricted Euler equations, In preparation (2007).

[Ma86] T. Makino, On a local existence theorem for the evolution of gaseous stars, In: Patterns and Waves(T. Nishida, M. Minura, H. Fujii, eds.), North-Holland/Kinokuniva, 1986, 459–479.

[MN95] P. Marcati, R. Natalini, Weak solutions to a hydrodynamic model for semiconductors and relaxation to the drift-diffusion equation, Arch. Rat. Mech. Anal. 129 (1995), 129–145.

[MP90] T. Makino, B. Perthame, Sur les solutions *à* symé*trie sph*é*rique de l'*é*quation d'Euler-Poisson pour l'*é*volution d'*é*toiles gazeuses, Japan J. Appl. Math. 7 (1990), 165–170.

[Pe90] B. Perthame, Nonexistence of global solutions to the Euler-Poisson equations for repulsive forces, Japan J. Appl. Math. 7 (1990), 363–367.

[PRV95] F. Poupaud, M. Rascle, J.-P. Vila, Global solutions to the isothermal Euler-Poisson system with arbitrarily large data, J. Diff. Eqs. 123 (1995), 93–121.

[Se97] D. Serre, Solutions classiques globales des équations d'Euler pour un fluide parfait compressible, Ann. Inst. Fourier (Grenoble), 47 (1997), 139–153.

[Si85] T. C. Sideris, Formation of singularities in three-dimensional compressible fluids, Comm. Math. Phys. 101 (1985), 475–485.

[T01] E. Tadmor, On a new scale of regularity spaces with applications to Euler's equations, Nonlinearity 14 (2001), 513–532.

[TW06] E. Tadmor, D. Wei, On the global regularity of sub-critical Euler-Poisson equations with pressure, Preprint (2006).

[TW07] E. Tadmor, D. Wei, The vanishing viscosity limit of the 2D system of Burgers equations, In preparation (2007).

[V82] P. Vieillefosse, Local interation between vorticity and shear in a perfect incompressible flow, J. Phys. (ParisP) 43 (1982), 837.

[WC98] D. Wang, G.-Q. Chen, Formation of singularities in compressible Euler-Poisson fluids with heat diffusion and damping relaxation, J. Diff. Eqs. 144 (1998), 44–65.

[W07] D. Wei, Critical thresholds in n-dimensional restricted Euler equations, In preparation (2007).

[ZE70] Ya. B. Zeldovitch, Gravitational instability: An approximate theory for large density perturbations, Astron. and Astrophys. 5 (1970), 84-89.