ABSTRACT

Title of dissertation: LOWER BOUNDS ON ESSENTIAL DIMENSION FOR CONGRUENCE COVERS OF MIXED SHIMURA VARIETIES

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We use the fixed point method to obtain general lower bounds for the essential dimension of congruence covers $\Gamma'\backslash\mathcal{X}^0 \to \Gamma\backslash\mathcal{X}^0$ of mixed Shimura varieties. Our main result states that $\mathrm{ed}_{\mathbb{C}}(\Gamma'\backslash\mathcal{X}^0 \to \Gamma\backslash\mathcal{X}^0; p)$ is bounded from below by the dimension of certain unipotent subgroups associated with the rational boundary components of the corresponding mixed Shimura datum. This generalizes theorems of Brosnan and Fakhruddin ([BF24, 33, 34]) to the case of an arbitrary mixed Shimura datum. We also discuss a few examples of lower bounds in the last section.

LOWER BOUNDS ON ESSENTIAL DIMENSION FOR CONGRUENCE COVERS OF MIXED SHIMURA VARIETIES

by

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To all the mysterious and interesting questions, and to the meaning of life:

We must know. We will know.

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1 Introduction

Let k be a ground field and K/k an extension field. Given a finite-dimensional étale K-algebra A, (following [Rei25]) we say that A/K descends to A'/K' if K'/k is a subfield of K and A' is a K'-algebra such that the canonical map $A' \otimes_{K'} K \to A$ is an isomorphism. We define the *essential dimension* of A/K to be

$$\mathrm{ed}_k(A/K) = \min_{A'/K'} \mathrm{tr.deg}_k K'.$$

More generally in the language of schemes, given a generically étale morphism $f : X \to Y$ of schemes of finite type over k with Y integral, we say f can be compressed to another morphism $f' : X' \to Y'$ of the same kind if

(1) there exists a dominant rational morphism $g : Y \dashrightarrow Y'$;

(2) $X \to Y$ and $Y \times_{Y'} X' \to Y$ are isomorphic over the generic point of Y.

In other words, we have a diagram

$$\begin{tikzcd} X \arrow[r, dashed, "g'"] \arrow[d, "f"] & X' \arrow[d, "f'"] \\ Y \arrow[r, dashed, "g"] & Y' \end{tikzcd}$$

and an open dense subscheme $V \subseteq Y$ on which g is defined and $f|_V$ is the pullback of f' via g. We define the essential dimension $\mathrm{ed}_k(X \xrightarrow{f} Y)$ as the minimum dimension of Y' for which such a compression exists. (Here we are basically using the definition given in [BF24].)

If p is a prime, define the *essential p-dimension* $\mathrm{ed}_k(X \xrightarrow{f} Y; p)$ to be the minimum of $\mathrm{ed}_k(Y'' \times_Y X \to Y'')$ for all generically finite morphisms $Y'' \to Y$ of degree prime to p with Y'' integral. (By convention, the essential dimension for $p = 0$ is the usual one without base change.) We also say that $X \to Y$ is incompressible (resp. p-incompressible) if $\mathrm{ed}_k(X \to Y)$ (resp. $\mathrm{ed}_k(X \to Y; p)$) equals $\dim Y$.

Suppose the above $X \xrightarrow{f} Y$ is a G-torsor with G a finite (constant) group scheme over k. Then f is finite étale. The essential dimension $\mathrm{ed}_k(G)$ (resp. essential p-dimension $\mathrm{ed}_k(G; p)$) of G is defined as the maximum of $\mathrm{ed}_k(X \to Y)$ (resp. the maximum of $\mathrm{ed}(X \to Y; p)$ for all G-torsors $X \to Y$) where we require all $X' \to Y'$ to be G-torsors and $g' : X \dashrightarrow X'$ to be G-equivariant.

The notion of essential dimension dates back to the time of Klein and Kronecker and is closely related to Hilbert's 13th problem (which involves another important invariant called the resolvent degree). It is first formally defined for finite groups by Buhler and Reichstein in [BR97] and then extended to an algebraic group G in [Rei00] and functors $\mathcal{F} : \mathit{Fields}_k \to \mathit{Sets}$ in [BF03]. Roughly speaking, the essential dimension of an algebraic group measures the complexity of its torsors. For a detailed survey, see [Mer13]. For more historical remarks, see for example [BR97] and [FKW21]. We will present a collection of results needed on essential dimension in Section 2.

Shimura varieties are objects of both algebro-geometric and arithmetic interest. Recall that a *Shimura datum* is a pair $(G, \mathcal{X})$. Here G is a connected reductive $\mathbb{Q}$-group and $\mathcal{X}$ is a $G(\mathbb{R})$-conjugacy class of morphisms $\mathbb{S} \to G_{\mathbb{R}}$ with $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)$ the Deligne torus. They are required to satisfy a few axioms which guarantee that $\mathcal{X}$ is a union of Hermitian symmetric domains parametrizing pure Hodge structures on

Lie G.

Let $G(\mathbb{Q})$ act diagonally on $\mathcal{X} \times G(\mathbb{A}_f)$ from the left and let an open compact subgroup $K_f \subseteq G(\mathbb{A}_f)$ act on $G(\mathbb{A}_f)$ via right multiplication (with $\mathbb{A}_f$ the ring of finite adeles), the double quotient

$$M^{K_f}(G,\mathcal{X})(\mathbb{C}) = G(\mathbb{Q})\backslash\mathcal{X} \times G(\mathbb{A}_f)/K_f$$

has a natural structure of a quasi-projective complex algebraic variety. If $\mathcal{X}^0$ is a connected component of $\mathcal{X}$ and $p_f \in G(\mathbb{A}_f)$, then $\Gamma \coloneqq \mathrm{Stab}_{G(\mathbb{Q})}(\mathcal{X}^0) \cap p_f K_f p_f^{-1}$ is an arithmetic subgroup of $G(\mathbb{Q})$ that acts on $\mathcal{X}^0$. The inclusion map $\mathcal{X}^0 \times \{p_f\} \hookrightarrow \mathcal{X} \times G(\mathbb{A}_f)$ induces an embedding $\Gamma\backslash\mathcal{X}^0 \hookrightarrow M^{K_f}(G,\mathcal{X})(\mathbb{C})$, realizing $\Gamma\backslash\mathcal{X}^0$ as a connected component. Such a $\Gamma\backslash\mathcal{X}^0$ is called a locally symmetric variety. If $K_f' \subseteq K_f \subseteq G(\mathbb{A}_f)$ are two open compact subgroups, we get a covering $\Gamma'\backslash\mathcal{X}^0 \to \Gamma\backslash\mathcal{X}^0$. When K_f' is a normal subgroup of K_f, the action of the finite group Γ/Γ' is transitive on the fiber.

Now it makes sense to study $\mathrm{ed}_{\mathbb{C}}(\Gamma'\backslash\mathcal{X}^0 \to \Gamma\backslash\mathcal{X}^0)$ or $\mathrm{ed}_{\mathbb{C}}(\Gamma'\backslash\mathcal{X}^0 \to \Gamma\backslash\mathcal{X}^0; p)$. Such results are first available in Farb, Kisin, and Wolfson's paper [FKW21], where they computed the essential dimension for congruence covers of many Shimura varieties using arithmetic methods. For example, let $\mathcal{A}_{g,N}$ be the moduli space of principally polarized abelian varieties with level-N structure. They proved that $\mathrm{ed}_{\mathbb{C}}(\mathcal{A}_{g,nN} \to \mathcal{A}_{g,N}; p) = \dim \mathcal{A}_g = \binom{g+1}{2}$, i.e. it is p-incompressible, where $g \geq 2$, $\gcd(n,N) = 1$, $p|n$. The paper also includes incompressibility results on the "principal p-level coverings" $\Gamma\backslash\mathcal{X} \to \Gamma'\backslash\mathcal{X}$. Their results are strengthened in many aspects in [FKW24] through non-vanishing characteristic classes in prismatic cohomology and toroidal compactifications. Fakhruddin and Saini also proved incompressibility of congruence covers for certain pure Shimura varieties of PEL type. Compare [FS22, 5,6].

On the other hand, in [BF24], Brosnan and Fakhruddin obtained lower bounds on the essential dimension of congruence covers of locally symmetric varieties by "the fixed point method". This leads to new incompressibility theorems while retrieving many of those in [FKW21] including the above one on $\mathcal{A}_{g,N}$. Roughly speaking, the p-rank of a finite abelian group H contained in Γ/Γ' bounds the essential p-dimension from below, provided that this H has a smooth fixed point in some H-equivariant (partial) compactification of $\Gamma'\backslash\mathcal{X}^0$. They proved the existence of smooth fixed points for some H-equivariant toroidal compactifications of the cover $\Gamma'\backslash\mathcal{X}^0 \to \Gamma\backslash\mathcal{X}^0$ and applied the fixed point method to get their results.

The approaches of [FKW21] and [BF24] complement each other. Each applies to a class of objects not accessible to the other. For a more refined comparison, we refer the reader to the introduction of [BF24] as well as some remarks in Section 5 thereof.

In this paper, we generalize Theorems 33 and 34 in [BF24] to a broader class of covers. Namely, we prove general lower bounds of essential dimension for congruence covers of mixed Shimura varieties with the fixed point method.

A *mixed Shimura datum* $(P, \mathcal{X})$ consists of a connected algebraic group P over $\mathbb{Q}$ and a finite covering $\mathcal{X}$ of a conjugacy class of morphisms $h_x : \mathbb{S}_{\mathbb{C}} \to P_{\mathbb{C}}$, subject to some axioms formulated by Pink in his thesis [Pin90]. The group P may be non-reductive with unipotent radical W. For a pure Shimura datum $(G, \mathcal{X})$, $\mathcal{X}$ parametrizes rational pure Hodge structures on $\operatorname{Lie} G$. Whereas, for a mixed Shimura datum $(P, \mathcal{X})$, each h_x induces a rational mixed Hodge structure of weight $\{0, -1, -2\}$

on $\mathrm{Lie}\, P_{\mathbb{C}}$ with $\mathrm{Lie}\, W$ corresponding to the subspace of weight ≤ -1. Further, there is a $\mathbb{Q}$-subgroup U central in W and normal in P whose Lie algebra corresponds to the subspace of weight -2. The group $P(\mathbb{R})\cdot U(\mathbb{C}) \subseteq P(\mathbb{C})$ acts transitively on $\mathcal{X}$. As in the pure case, the double quotient $P(\mathbb{Q})\backslash\mathcal{X} \times P(\mathbb{A}_f)/K_f$ has the structure of a quasi-projective algebraic variety.

Mixed Shimura data arise naturally. The rational boundary components of a pure/mixed Shimura datum are mixed Shimura data. (The definition of a rational boundary component will be given in 4.4.) Thus in this sense they are more natural objects to consider. Despite some notational sophistication, the general theory of mixed Shimura varieties resembles that of pure Shimura varieties. (Toroidal compactifications are also closely related to the Baily-Borel compactification. See for example Proposition 4.6.7.) The systematic treatment of the subject is [Pin90]. See also the introduction therein for more insightful remarks.

We illustrate our main theorem with the following example. Let $\mathcal{Y}_{g,d}$ be the universal family of principally polarized abelian varieties with full level-d structure. We have a canonical projection map $\mathcal{Y}_{g,d} \to \mathcal{A}_{g,d}$ onto the moduli space $\mathcal{A}_{g,d}$ mentioned above. In fact, $\mathcal{Y}_{g,d} \to \mathcal{A}_{g,d}$ is an abelian scheme for $d \geq 3$. ($\mathcal{Y}_{g,d}$ is an example of *Kuga varieties* and can be constructed explicitly using the vector space V of dimension $2g$ on which Sp_{2g} acts. A Kuga datum $(P, \mathcal{Y})$ is a mixed Shimura datum with the group U trivial.) For a prime p not dividing d, we get a canonical cover $\varphi : \mathcal{Y}_{g,pd} \to \mathcal{Y}_{g,d}$ which is the composition $\mathcal{Y}_{g,pd} \xrightarrow{p} \mathcal{Y}_{g,pd} \to \mathcal{Y}_{g,d}$. Here the first map is multiplication by p on each fiber and the second map the pullback of $\mathcal{A}_{g,pd} \to \mathcal{A}_{g,d}$

via $\mathcal{Y}_{g,d} \to \mathcal{A}_{g,d}$, as in the diagram.

$$\begin{tikzcd}
\mathcal{Y}_{n,pd} \arrow[r, "p"] \arrow[dr] & \mathcal{Y}_{n,pd} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pullback}"] & \mathcal{Y}_{n,d} \arrow[d] \\
& \mathcal{A}_{n,pd} \arrow[r] & \mathcal{A}_{n,d}
\end{tikzcd}$$

It follows from our main theorem that $\mathrm{ed}_{\mathbb{C}}(\mathcal{Y}_{g,pd} \xrightarrow{\varphi} \mathcal{Y}_{g,d}; p) = g + \binom{g+1}{2}$ for $p \nmid d \geq 3$. Thus the cover $\mathcal{Y}_{g,pd} \to \mathcal{Y}_{g,d}$ is p-incompressible. This may be viewed as a generalization of Brosnan's conjecture, which states that the multiplication-by-p map $A \xrightarrow{p} A$ is p-incompressible for an abelian variety A over a field of characteristic 0. For many interesting cases of this conjecture, the reader may consult [FS22], [FKW24, 2.3.5], and [KZ25]. See also Subsection 6.5 for more details on this example.

Finally, we can state our main theorem. For a mixed Shimura datum $(P, \mathcal{X})$, let $Z(P)$ be the center of P and denote by $\mathrm{Id}_{Z(P)(\mathbb{Q})}(\mathcal{X})$ the set $\{z \in Z(P)(\mathbb{Q}) \mid z|_{\mathcal{X}} = \mathrm{id}\}$. I.e. this is the set of all elements of $Z(P)(\mathbb{Q})$ that act trivially on $\mathcal{X}$.

Any rational boundary component $(P_1, \mathcal{X}_1)$ of $(P, \mathcal{X})$ has a corresponding $U_1 \trianglelefteq P_1$. In particular, as a unipotent group, it stabilizes any component $\mathcal{X}^0$ of $\mathcal{X}$. With open compact subgroups $K'_f \subseteq K_f \subseteq P(\mathbb{A}_f)$, we define the following groups:

$$\Gamma \coloneqq \mathrm{Stab}_{P(\mathbb{Q})}(\mathcal{X}^0) \cap \left(p_f K_f p_f^{-1}\right).$$

Γ is the stabilizer in $P(\mathbb{Q})$ of $\mathcal{X}^0 \times \{p_f K_f\} \subset \mathcal{X} \times P(\mathbb{A}_f)/K_f$.

$$\Gamma_Z \coloneqq \left(\mathrm{Id}_{Z(P)(\mathbb{Q})}(\mathcal{X})\right) \cap \left(p_f K_f p_f^{-1}\right).$$

Γ_Z is the kernel of the action of Γ on $\mathcal{X}^0$. In fact, as we shall see in Lemma 4.3.5, it is also equal to the stabilizer in Γ of any point $x \in \mathcal{X}^0$. Therefore, Γ/Γ_Z acts freely on $\mathcal{X}^0$.

$$\Gamma_{ZU_1} \coloneqq \left(\mathrm{Id}_{Z(P)(\mathbb{Q})}(\mathcal{X}) \times U_1(\mathbb{Q})\right) \cap \left(p_f K_f p_f^{-1}\right).$$

Γ_{ZU_1} is introduced mainly to define the group Γ_{U_1} below. It acts on $U_1(\mathbb{C})$ by translation through its image Γ_{U_1} under the projection $Z(P)\times U_1 \to U_1$. In Subsection 5.3, we will also define a similar group $\Omega_{Z_1U_1} \subset \Gamma_{ZU_1}$ that is closely related to the structure of a unipotent fiber (cf. [Pin90, 3.13] or Proposition 4.3.9).

$$\Gamma_{U_1} \coloneqq \frac{\Gamma_{ZU_1}}{\Gamma_Z} \subset \frac{\Gamma}{\Gamma_Z}.$$

Γ_{U_1} is the image of Γ_{ZU_1} under the projection $\mathrm{Id}_{Z(P)(\mathbb{Q})}(\mathcal{X}) \times U_1(\mathbb{Q}) \twoheadrightarrow U_1(\mathbb{Q})$. The groups Γ', Γ'_{ZU_1}, Γ'_Z and Γ'_{U_1} are defined similarly.

Theorem 1.1. *Let $(P, \mathcal{X})$ be a mixed Shimura datum and $\mathcal{X}^0 \subset \mathcal{X}$ any connected component. Suppose $(P_1, \mathcal{X}_1)$ is any rational boundary component of $(P, \mathcal{X})$ such that $\mathcal{X}^0 \subset \mathcal{X}^+_{(P_1,\mathcal{X}_1)}$ (as in Proposition 4.4.8). Given neat open compact subgroups $K'_f \trianglelefteq K_f \subseteq P(\mathbb{A}_f)$ and $p_f \in P(\mathbb{A}_f)$, define $\Gamma, \Gamma_Z, \Gamma_{U_1}$ as above. Let Σ be a K_f-admissible complete cone decomposition for $(P, \mathcal{X})$ that defines a smooth projective toroidal compactification $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ (as in Theorem 4.6.8) and denote by $\left(\Gamma\backslash\mathcal{X}^0\right)_\Sigma \subset M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ the connected component containing $\Gamma\backslash\mathcal{X}^0$. Then*

(1) The cover $\Gamma'\backslash\mathcal{X}^0 \to \Gamma\backslash\mathcal{X}^0$ is a $\dfrac{\Gamma/\Gamma_Z}{\Gamma'/\Gamma'_Z}$-torsor.

(2) Any $\dfrac{\Gamma/\Gamma_Z}{\Gamma'/\Gamma'_Z}$*-equivariant partial compactification* Y *containing* $\Gamma'\backslash\ \mathcal{X}^0$ *with a proper morphism* $Y \to \left(\Gamma\backslash\mathcal{X}^0\right)_\Sigma$ *extending* $\Gamma'\backslash\mathcal{X}^0 \to \Gamma\backslash\mathcal{X}^0$ *has a* $\Gamma_{U_1}/\Gamma'_{U_1}$*-fixed point.*

(3) For any prime $p > 0$,

$$\mathrm{ed}_{\mathbb{C}}(\Gamma'\backslash\mathcal{X}^0 \to \Gamma\backslash\mathcal{X}^0; p) \geq \mathrm{rank}\frac{\Gamma_{U_1}}{\Gamma'_{U_1}} \otimes \frac{\mathbb{Z}}{p\,\mathbb{Z}} (\leq \dim U_1).$$

(4) The lower bound can be attained: For any neat open compact K_f *and any prime* $p > 0$*, there exists a neat open compact subgroup* $K'_f \subseteq K_f$ *in* $P(\mathbb{A}_f)$ *such that* $\mathrm{rank}\dfrac{\Gamma_{U_1}}{\Gamma'_{U_1}} \otimes \dfrac{\mathbb{Z}}{p\,\mathbb{Z}} = \dim U_1$.

In particular, $\mathrm{ed}_{\mathbb{C}}(\Gamma'\backslash\mathcal{X}^0 \to \Gamma\backslash\mathcal{X}^0; p) \geq \dim U_1$.

Remark 1.1. The condition $\mathcal{X}^0 \subset \mathcal{X}^+_{(P_1,\mathcal{X}_1)}$ is not really a restriction. Indeed, given any $\mathcal{X}^0$ and any $\mathbb{Q}$-admissible parabolic subgroup $Q \subset P$ (to be defined in Subsection 4.4), there exists a $(P_1, \mathcal{X}_1)$ associated with Q such that $\mathcal{X}^0 \subset \mathcal{X}^+_{(P_1,\mathcal{X}_1)}$. As $\dim U_1$ only depends on Q, we get equally good lower bounds for every $\mathcal{X}^0$.

The terminology will be made clear in the subsequent sections. We will review the definition of a rational boundary component in 4.4 and the construction of toroidal compactifications in 4.6.

We conclude the introduction with an outline of the paper. In Section 2 we recall some elementary properties of and several important results on essential dimension. Section 3 contains the precise statement of the fixed point method and the theorem in [BF24] on the existence of fixed points, needed in the proof of Theorem 1.1. The delicate machinery of mixed Shimura varieties is introduced in Section 4, leading to the construction of their toroidal compactifications, as developed in [Pin90]. Finally, Section 5 completes the proof of Theorem 1.1 after the establishment of some technical lemmas. In Section 6, we present a few examples to illustrate our main theorem.

2 Essential dimension

In the introduction, we have defined $\mathrm{ed}_k(G)$ for G a finite group. In fact, essential dimension can be defined for any algebraic group G. We include the definition and some basic results below. Our main references for this section are [Mer13] and [BF24].

Let G be an algebraic group over a field k. A *G-scheme* is a scheme of finite type over k with a left G-action. A morphism of G-schemes $f : X \to Y$ is a *G-torsor* if

(1) Y is integral and G acts trivially on Y;

(2) f is faithfully flat;

(3) The morphism $G \times_k X \to X \times_Y X : (g, x) \mapsto (g \cdot x, x)$ is an isomorphism.

A G-scheme is generically free if there exists an open dense G-invariant subscheme U such that U is part of a G-torsor $U \to Y$. (In some literature, our generically free G-scheme is called primitive as G acts transitively on the set of irreducible components of X.) Following [Mer13], we use $k(X)^G$ to denote $k(Y)$. $k(X)^G$ does not depend on the choice of U.

Given a generically free G-scheme X, a *G-compression* is a G-equivariant dominant rational map $X \dashrightarrow X'$ to another generically free G-scheme X'. The essential dimension of X (resp. of G) is defined to be

$$\mathrm{ed}_k(X, G) \coloneqq \min_{X \dashrightarrow X'} \{\dim X' - \dim G\}$$

$$\mathrm{ed}_k(G) \coloneqq \max_{X} \{\mathrm{ed}_k(X, G)\}$$

Similarly, the essential p-dimension of X is defined to be

$$\mathrm{ed}_k(X, G; p) \coloneqq \min_{Y'' \dashrightarrow Y} \{\mathrm{ed}_k(Y'' \times_Y X, G)\}$$

$$\mathrm{ed}_k(G; p) \coloneqq \max_X \{\mathrm{ed}_k(X, G; p)\}$$

where the minimum is taken over all generically finite dominant rational maps $Y'' \dashrightarrow Y$ of degree prime to p.

A generically free G-scheme X is *incompressible* (resp. p-incompressible) if $\mathrm{ed}_k(X, G) = \dim X - \dim G$ (resp. $\mathrm{ed}_k(X, G; p) = \dim X - \dim G$). In other words, X cannot be compressed to a generically free G-scheme of lower dimension.

Proposition 2.1 ([BF03])**.**

$$\mathrm{ed}_k(X, G) = \mathrm{ed}_k(T_X \to \mathrm{Spec}\, k(X)^G)$$

$$\mathrm{ed}_k(G) = \max_{T \to \mathrm{Spec}\, K} \{\mathrm{ed}_k(T \to \mathrm{Spec}\, K)\}$$

Here $\mathrm{ed}_k(T \to \mathrm{Spec}\, K)$ *is the minimum of the transcendence degree of a field of definition of the* G*-torsor* $T \to \mathrm{Spec}\, K$*.*

Among all G-schemes, some have the "highest" essential dimension. This is taken into account by the notion of versality. A G-scheme X is *versal* if every open dense G-invariant subscheme $U \subseteq X$ satisfies the condition that every generically free G-scheme X' with $k(X')^G$ infinite admits a G-equivariant rational map $X' \dashrightarrow U$. Intuitively, this means whenever X can be compressed to some extent, X' can be compressed at least to the same extent "through X". Therefore, the versal generically free G-schemes should have the highest essential dimension. It is clear by definition that any open dense G-invariant subscheme is also versal. Further, if $X \dashrightarrow X'$ is a G-compression of generically free G-schemes with X versal, then X' is versal.

By analogy, there is a notion of p-versality. A G-scheme X is *p-versal* if, for every G-invariant dense open $U \subseteq X$ and every generically free G-scheme X' with $k(X')^G$ infinite, there exists a G-equivariant dominant rational map $X'' \dashrightarrow X'$ with $k(X'')^G \leftarrow k(X')^G$ finite of degree prime to p such that X'' admits a G-equivariant rational map to U. All the above observations carry over to p-versality.

In summary, we have

Proposition 2.2 ([Mer13, Proposition 3.11] and [BF24, Lemma 6]).

(1) If X is a versal generically free G-scheme, then

$$\mathrm{ed}_k(G) = \mathrm{ed}_k(X, G) = \min_{\text{versal } X'} \{\dim X' - \dim G\}$$

(2) If X is a p-versal generically free G-scheme, then

$$\mathrm{ed}_k(G; p) = \mathrm{ed}_k(X, G; p) = \min_{p\text{-versal } X'} \{\dim X' - \dim G\}$$

Versal G-schemes are not hard to find. For every k-vector space V, we say a linear representation $G \to \mathrm{GL}_V$ is generically free if V as a G-scheme is so. In particular, if G is a finite group, a linear representation is generically free if and only if it is faithful. Then it can be shown, for example, that every linear G-representation V is G-versal (cf. [Mer13, Proposition 3.10]).

Consider an algebraic subgroup $H \subseteq G$. Since every generically free G-scheme is a generically free H-scheme and every G-compression is also an H-compression, we have $\mathrm{ed}_k(X, G; p) + \dim G \geq \mathrm{ed}_k(X, H; p) + \dim H$ for all $p \geq 0$.

Proposition 2.3 ([BRV10, Lemma 2.2])**.** *For every $p \geq 0$,*

$$\mathrm{ed}_k(G; p) + \dim G \geq \mathrm{ed}_k(H; p) + \dim H.$$

In particular, if $H \subseteq G$ are finite groups, then $\mathrm{ed}_k(H; p) \leq \mathrm{ed}_k(G; p)$ for all $p \geq 0$.

We conclude this section by listing a few results on the essential dimension of finite p-groups, which are the ones needed in this article.

Theorem 2.4.

(1) ([KM08, 4.1, 5.1, 5.2]) Let G be a finite p-group and k a field of characteristic not equal to p containing a primitive p-th root of unity. Then $\mathrm{ed}_k(G; p) = \mathrm{ed}_k(G)$ is equal to the least dimension of a faithful representation of G over k. Moreover

$$\mathrm{ed}_k(G_1 \times G_2) = \mathrm{ed}_k(G_1) + \mathrm{ed}_k(G_2),$$

$$\mathrm{ed}_k(\mathbb{Z}/p^n\mathbb{Z}) = [k(\xi_{p^n}) : k],$$

where ξ_{p^n} is a primitive p^n-th root of unity.

(2) If k is as above, in particular, $\mathrm{ed}_k\left((\mathbb{Z}/p\mathbb{Z})^d\right) = d$.

Remark 2.5. As indicated in the assumptions of the theorem, $\mathrm{ed}_k(G)$ or $\mathrm{ed}_k(G; p)$ depends on the ground field k.

Remark 2.6. The equality $\mathrm{ed}_k(G_1 \times G_2) = \mathrm{ed}_k(G_1) + \mathrm{ed}_k(G_2)$ is only for p-groups. In general, $\mathrm{ed}_k(G_1 \times G_2; p) \leq \mathrm{ed}_k(G_1; p) + \mathrm{ed}_k(G_2; p)$ for all $p \geq 0$ as a $G_1 \times G_2$-torsor is naturally the product of a G_1-torsor with a G_2-torsor.

3 The fixed point method

In this section, we briefly explain the fixed point method. Our starting point is the following theorem, which relates the existence of a G-fixed smooth k-point to p-versality.

Theorem 3.1 ([DR15, Corollary 8.6])**.** *Let G be a smooth affine algebraic group over a field k and X be a generically free G-variety that is geometrically integral. If G fixes a smooth point $x \in X(k)$, then X is p-versal.*

Remark 3.2. In practice, G will be a finite group and k will be the field of complex numbers.

Combining Theorem 2.4 and Theorem 3.1, while noting that the essential p-dimension is a birational invariant, we are led to

Theorem 3.3 ([BF24, Proposition 10])**.** *Let G be a finite group and X an irreducible generically free G-variety over a base field k of characteristic not equal to p containing a primitive p-th root of unity. Suppose G has a subgroup H such that H is a p-group and H has a smooth fixed point in X. Then*

(1) $\mathrm{ed}_k(X_0, G; p) \geq \mathrm{ed}_k(X_0, H; p) = \mathrm{ed}_k(H; p)$ *for any open dense G-invariant subvariety $X_0 \subseteq X$.*

(2) If $H = (\mathbb{Z}/p\mathbb{Z})^r$, then $\mathrm{ed}_k(X_0, G; p) \geq \mathrm{ed}_k(X_0, H; p) = r$.

To state the next theorem, we need a few definitions, which are also from [BF24]. We here assume k is an algebraically closed field.

Definition 3.4. A *toroidal singularity* is a pair (S, φ), involving the following data.

(1) T is a torus over k and T_σ is the torus embedding with respect to a strongly convex rational polyhedral cone $\sigma \subseteq Y_*(T)_\mathbb{R}$ with $\dim \sigma = \dim T$. ($Y_*(T)$ is the cocharacter group of T.)

(2) S is a scheme over k and $\varphi : S \xrightarrow{\sim} \operatorname{Spec} \widehat{\mathcal{O}}_{T_\sigma,x}$ is an isomorphism of S with the completion of the local ring of T_σ at its σ-stratum (which is a torus fixed point).

Remark 3.5. The definitions of a toroidal action and a toroidal map of toroidal singularities depend on the choice of φ above, as remarked in [BF24]. For more preliminaries on torus embeddings, see Section 4.5.

Given a (S, φ) as above, we define $S^\circ \subset S$ to be the inverse image of $T \subset T_\sigma$ under the morphism $S \xrightarrow{\sim} \operatorname{Spec} \widehat{\mathcal{O}}_{T_\sigma,x} \to T_\sigma$. (Thus we have a pullback diagram as below, where $\operatorname{Spec} \widehat{O}_{T_\sigma,x} \cap T$ is the preimage of T.) Each ray $\ell \leq \sigma$ defines a stratum of codimension 1, whose closure is a boundary divisor.

$$\begin{tikzcd}
S^\circ \arrow[r, "\sim"'] \arrow[d, tail] & \operatorname{Spec} \widehat{O}_{T_\sigma,x} \cap T \arrow[r] \arrow[d, tail] & T \arrow[d, tail] \\
S \arrow[r, "\varphi", "\sim"'] & \operatorname{Spec} \widehat{O}_{T_\sigma,x} \arrow[r] & T_\sigma
\end{tikzcd}$$

Example 3.6. Let $T = \mathbb{G}_m \times \mathbb{G}_m$. Then $Y_*(T) = \mathbb{Z} \oplus \mathbb{Z}$. We take $\sigma = \mathbb{R}_{\geq 0} \cdot e_1 + \mathbb{R}_{\geq 0} \cdot e_2$ for $e_1 : \mathbb{G}_m \rightarrowtail \mathbb{G}_m \times \{1\}$ and $e_2 : \mathbb{G}_m \rightarrowtail \{1\} \times \mathbb{G}_m$. Then $T_\sigma = \operatorname{Spec} \mathbb{C}[x, y] = \mathbb{A}^2_\mathbb{C}$ with x, y the dual generators of e_1, e_2 in $X^*(T)$, the character group of T. Then the σ-stratum is given by the ideal $I = (x, y)$. The pair $(S, \varphi) = (\operatorname{Spec} \mathbb{C}[[x, y]], \mathrm{id})$ is a toroidal singularity.

Moreover, since $T = \operatorname{Spec} \mathbb{C}[x, y]_{xy}$, we have $S^\circ = \mathbb{C}[[x, y]]_{xy}$, the Laurent series ring in two variables.

Theorem 3.7 ([BF24, Proposition 16])**.** *Let Y be an irreducible variety over k and let $f : X \to Y$ an irreducible finite étale Galois cover with Galois group G. Let S be a simplicial toroidal singularity and S° be the complement of the boundary divisor. Suppose there exists a morphism $g : S^\circ \to Y$ such that the image of the composite $\pi_1^{\mathrm{et}}(S^\circ, s) \xrightarrow{g_*} \pi_1^{\mathrm{et}}(Y, g(s)) \twoheadrightarrow G$ is a finite (abelian) group A of order not divisible by $\operatorname{char} k$ (for $s \in S^\circ$ any geometric point). Assume that g extends to a morphism $\bar{g} : S \to \overline{Y}$, where $\overline{Y}$ is a partial compactification of Y. Then*

(1) Any G-equivariant partial compactification $\overline{X} \supseteq X$ admitting a proper morphism $\overline{f} : \overline{X} \to \overline{Y}$ extending f has an A-fixed point.

(2) Any smooth proper variety X' with a G-action which is equivariantly birational to X has an A-fixed point.

Remark 3.8. As pointed out in [BF24, 17], when $k = \mathbb{C}$, it suffices to find holomorphic maps $g, \bar{g}$ on a sufficiently small open neighborhood of the torus fixed point $x \in T_\sigma$. Then we have the desired maps $S \to \overline{Y}$ and $S^\circ \to Y$ since we can complete the induced homomorphism between the analytic local rings.

Remark 3.9. Suppose our X is known to be smooth and we have such a G-equivariant proper $\overline{f} : \overline{X} \to \overline{Y}$. In order to apply Theorem 3.7, we need a smooth $\overline{X}'$. This can be achieved by applying equivariant resolution of singularities to $\overline{X}$. See for example [EV98].

4 Basics on mixed Shimura varieties

This section is mainly an exposition of the theory of mixed Shimura varieties. Our primary reference is [Pin90]. (Some lemmas are also deduced in Subsection 4.3.) The proof of our main theorem will make use of many elements in this section, especially those on rational boundary components and the construction of toroidal compactifications, as developed in Subsections 4.4 and 4.6. Hopefully, this section can provide the reader with more background on mixed Shimura varieties and help him navigate through [Pin90] on his own if necessary. Readers familiar with mixed Shimura varieties and eager to know the proof may move on to Section 5 directly.

4.1 Mixed Hodge strctures

We introduce the notion of mixed Hodge structures in this subsection.

Let V be a finite-dimensional $\mathbb{Q}$-vector space.

Definition 4.1.1. (1) A pure rational Hodge structure on V of weight $n \in \mathbb{Z}$ is a decomposition $V_{\mathbb{C}} = \oplus_{(p,q)\in\mathbb{Z}\oplus\mathbb{Z}} V^{p,q}$ such that $V^{p,q} = 0$ if $p + q \neq n$ and $\overline{V^{p,q}} = V^{q,p}$. The Hodge filtration of this pure Hodge structure is the decreasing nested sequence of subspaces $\{F^pV = \oplus_{p\leq p'} V^{p',q}\}_{p\in\mathbb{Z}}$. Note that $F^pV \cap \overline{F^qV} = V^{p,q}$. Thus the decomposition and the Hodge filtration determine each other.

(2) A mixed rational Hodge structure on V consists of an increasing, separated and exhaustive filtration of $\mathbb{Q}$-subspaces $\{W_nV\}_{n\in\mathbb{Z}}$ and a decreasing, separated and exhaustive filtration $\{F^pV\}_{p\in\mathbb{Z}}$ such that on each $Gr_nV := W_nV/W_{n-1}V$,

$\{F^pV\}_{p\in\mathbb{Z}}$ induces a pure Hodge structure of weight n.

Given a mixed rational Hodge structure on V, we define the Hodge numbers $h^{p,q}$ as $:= \dim_{\mathbb{C}}(Gr_nV)^{p,q}$. By definition, $h^{p,q} = h^{q,p}$. If $A \subseteq \mathbb{Z}\oplus\mathbb{Z}$ is a subset, a Hodge structure on V has type A if $h^{p,q} = 0,\ (p,q) \notin A$.

For example, a pure Hodge structure of type $\{(-1,0),(0,-1)\}$ is just a complex structure on V.

Morphisms of Hodge structures are $\mathbb{Q}$-linear maps $V \to V'$ preserving the weight and Hodge filtrations. With morphisms as arrows and Hodge-structured vector spaces as objects, mixed rational Hodge structures form an abelian category. Given rational mixed Hodge structures on V, V_1 and V_2, we have induced structures on $V_1 \oplus V_2$, $V_1 \otimes_{\mathbb{Q}} V_2$ and $V^\vee$ as follows

$$
\begin{aligned}
(V_1 \oplus V_2)^{p,q} &= V_1^{p,q} \oplus V_2^{p,q} \\
(V_1 \otimes_{\mathbb{Q}} V_2)^{p,q} &= \bigoplus_{\substack{(p_1,q_1)+(p_2,q_2)\\=(p,q)}} V_1^{p_1,q_1} \otimes_{\mathbb{Q}} V_2^{p_2,q_2} \\
(V^\vee)^{p,q} &= V^{-p,-q}
\end{aligned}
$$

Thus there are also induced Hodge structures on $\mathrm{Hom}(V_1, V_2) = V_1^\vee \otimes_{\mathbb{Q}} V_2$.

We say a mixed Hodge structure on V splits over $\mathbb{R}$ if there exists a decomposition $V_{\mathbb{C}} = \bigoplus_{p,q} V^{p,q}$ such that $W_nV = \oplus_{p+q\le n} V^{p,q}$, $F^pV = \oplus_{p\le p'} V^{p',q}$ and $\overline{V^{p,q}} = V^{q,p}$. Thus a pure Hodge structure splits over $\mathbb{R}$ by definition. Although not every mixed Hodge structure splits over $\mathbb{R}$, we have the following

Proposition 4.1.2 ([Pin90, 1.2]). *Let V be a finite-dimensional vector space equipped* *with a rational mixed Hodge structure.*

(1) *There exists a decomposition* $V_{\mathbb{C}} = \bigoplus_{p,q} V^{p,q}$ *such that* $W_n V = \bigoplus_{p+q\leq n} V^{p,q}$ *and* $F^p V = \bigoplus_{p\leq p'} V^{p',q}$.

(2) *There exists a unique decomposition as in (a) that satisfies the condition* $\overline{V^{p,q}} \equiv V^{q,p} \operatorname{mod}(\bigoplus_{p'<p,q'<q} V^{p',q'})$.

The above definitions are in the classical setting. Now we reformulate them in terms of representations of algebraic groups.

Let $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)$ be the Deligne torus. Then $\mathbb{S}_{\mathbb{C}} \xrightarrow{\sim} \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}}$ and we can choose the isomorphism so that $\mathbb{C}^{\times} = \mathbb{S}(\mathbb{R}) \to \mathbb{S}(\mathbb{C}) \xrightarrow{\sim} \mathbb{C}^{\times} \times \mathbb{C}^{\times} : z \mapsto (z, \bar{z})$. Complex conjugation acts on $\mathbb{S}(\mathbb{C})$ as $(z_1, z_2) \mapsto (\overline{z_2}, \overline{z_1})$. There are also the following frequently used morphisms of $\mathbb{S}$:

$$\omega : \mathbb{G}_{m,\mathbb{R}} \to \mathbb{S}, \quad z \mapsto (z^{-1}, z^{-1});$$

$$\iota : S^1 \to \mathbb{S}, \qquad z \mapsto (z, z^{-1});$$

$$\mu_0 : \mathbb{G}_{m,\mathbb{C}} \to \mathbb{S}_{\mathbb{C}},\ z \mapsto (z, 1);$$

$$N : \mathbb{S} \to \mathbb{G}_{m,\mathbb{R}}, \quad (z_1, z_2) \mapsto z_1 z_2.$$

(Note that a morphism over $\mathbb{R}$ is given in terms of its expression over $\mathbb{C}$. And one can easily check they are defined over $\mathbb{R}$.) By definition, we have a short exact sequence $1 \to \mathbb{G}_{m,\mathbb{R}} \xrightarrow{\omega} \mathbb{S} \xrightarrow{N} \mathbb{G}_{m,\mathbb{R}} \to 1$.

Since $\mathbb{S}(\mathbb{C}) = \mathbb{C}^{\times} \times \mathbb{C}^{\times}$, a linear representation $h : \mathbb{S}_{\mathbb{C}} \to GL_{V,\mathbb{C}}$ is equivalent to a decomposition $V_{\mathbb{C}} = \oplus_{p,q} V^{p,q}$, where $V^{p,q}$ is the eigenspace for the character $(z_1, z_2) \mapsto z_1^p z_2^q$ (and $GL_{V,\mathbb{C}}$ is the base change of GL_V to $\mathbb{C}$).

From now on, we will adopt the convention, to denote by $V^{p,q}$ the eigenspace for the character $(z_1, z_2) \mapsto z_1^{-p} z_2^{-q}$. Therefore through $h \circ \omega : \mathbb{G}_{m,\mathbb{C}} \to GL_{V,\mathbb{C}}$, the action on $V^{p,q}$ is through $t \mapsto t^{p+q}$. As before, $W_n^h V := \oplus_{p+q \leq n} V^{p,q}$ and $F_h^p V := \oplus_{p \leq p'} V^{p',q}$. The superscript or subscript h indicates the representation. We will often drop it if no confusion arises.

Suppose the morphism $h : \mathbb{S}_{\mathbb{C}} \to GL_{V,\mathbb{C}}$ defines a rational mixed Hodge structure on V. Since $\overline{V^{p,q}} = V^{q,p}$ if and only if h is defined over $\mathbb{R}$, the Hodge structure splits over $\mathbb{R}$ if and only if h is defined over $\mathbb{R}$. And if $h : \mathbb{S} \to GL_{V,\mathbb{R}}$ and $h' : \mathbb{S} \to GL_{V',\mathbb{R}}$ induce pure Hodge structures on V and V' respectively, a morphism of pure Hodge structures is a $\mathbb{Q}$-linear map $V \to V'$ that is $\mathbb{S}$-equivariant.

Now that we have the above viewpoint, let's make two more definitions. The *Tate Hodge structure* $\mathbb{Q}(n)$ is defined to be the $\mathbb{Q}$-subspace $(2\pi i)^n \cdot \mathbb{Q} \subset \mathbb{C}$ with the pure Hodge structure of type $(-n, -n)$ on $\mathbb{Q}_{\mathbb{C}} = \mathbb{C}$. Suppose $h : \mathbb{S} \to GL_{V,\mathbb{R}}$ defines a pure Hodge structure of weight n on V. A polarization of the pure Hodge structure on V is a morphism $\psi : V \otimes_{\mathbb{Q}} V \to \mathbb{Q}(-n)$ of Hodge structures (of weight $2n$!) such that the pairing

$$\begin{aligned}(2\pi i)^n (\psi_{\mathbb{R}})_{h(i)} : V_{\mathbb{R}} \times V_{\mathbb{R}} &\xrightarrow{\psi} \mathbb{R}(-n) \xrightarrow{(2\pi i)^n} \mathbb{R} \\ (u, v) &\longmapsto (2\pi i)^n \psi(u, h(i)v)\end{aligned}$$

is symmetric and positive definite. If $\psi : V \otimes_{\mathbb{Q}} V \to \mathbb{Q}(-n)$ is known to be a morphism of Hodge structures, then $(2\pi i)^n (\psi_{\mathbb{R}})_{h(i)}$ is symmetric if and only if (1) n is even and ψ is symmetric or (2) n is odd and ψ is alternating.

The next proposition provides some motivation for the definition of a mixed Shimura datum.

Proposition 4.1.3 ([Pin90, 1.4]). *Let P be a connected affine algebraic group over $\mathbb{Q}$ with unipotent radical W and $\pi : P \to P/W =: G$. Suppose $h : \mathbb{S}_{\mathbb{C}} \to P_{\mathbb{C}}$ is a morphism satisfying*

(1) $\pi \circ h : \mathbb{S}_{\mathbb{C}} \to G_{\mathbb{C}}$ is defined over $\mathbb{R}$;

(2) $\pi \circ h \circ \omega : \mathbb{G}_{m,\mathbb{R}} \to G_{\mathbb{R}}$ is a cocharacter defined over $\mathbb{Q}$ into the center of G;

(3) In the decomposition induced on $\mathrm{Lie}\, P$ via the adjoint representation of P, $W_{-1}^{\mathrm{Ad}_P \circ h}(\mathrm{Lie}\, P) = \mathrm{Lie}\, W$.

Then we have

(1) For every $\mathbb{Q}$-representation $\rho : P \to GL_V$, $\rho \circ h : \mathbb{S}_{\mathbb{C}} \to P_{\mathbb{C}} \to GL_{V,\mathbb{C}}$ induces a rational mixed Hodge structure on V;

(2) The weight filtration $\{W_n^{\rho\circ h} V\}_{n\in\mathbb{Z}}$ is P-invariant;

(3) For any $p \in P(\mathbb{R}) \cdot W(\mathbb{C})$, same conclusions as in (a) and (b) hold for the conjugate $\mathrm{int}(p) \circ h$. Further, $\mathrm{int}(p) \circ h$ induces the same weight filtration and Hodge numbers.

Corollary 4.1.4. *Let P, W and $\pi : P \to P/W = G$ be as above. Suppose $\lambda : \mathbb{G}_{m,\mathbb{C}} \to P_{\mathbb{C}}$ is a cocharacter such that $\pi \circ \lambda$ maps into $Z(G)_{\mathbb{C}}$ and is defined over $\mathbb{Q}$. Then for any $\mathbb{Q}$-representation $\rho : P \to GL_V$, the weight filtration $\{W_n^{\rho\circ\lambda} V\}_{n\in\mathbb{Z}}$ (defined by $\mathbb{G}_{m,\mathbb{C}} \xrightarrow{\lambda} P_{\mathbb{C}} \xrightarrow{\rho_{\mathbb{C}}} GL_{V,\mathbb{C}}$) is defined over $\mathbb{Q}$ and stable under P.*

Proof. This follows from the proof of Proposition 1.4 in [Pin90], as we can just argue the same way with respect to the weight filtration. $\square$

Remark 4.1.5. Actually, in the second condition of Proposition 4.1.3, we can replace $\mathbb{Q}$ with any subfield K between $\mathbb{Q}$ and $\mathbb{R}$ and the conclusion will become that every representation of $P_K \to GL_{V_K}$ induces a mixed Hodge structure on V_K with weight filtration consisting of K-subspaces.

A similar remark holds for Corollary 4.1.4 as well.

4.2 Mixed Shimura data

In this subsection, we introduce the notion of a mixed Shimura datum. The number of axioms may seem a bit overwhelming at first. However, they become lighter once the reader recognizes that they can be roughly divided into two parts: the ones concerning the mixed Hodge structure and the ones concerning the algebraic group.

Definition 4.2.1 (mixed Shimura datum, [Pin90, 2.1])**.** Let P be a connected affine algebraic group over $\mathbb{Q}$ with unipotent radical W and a subgroup $U \subset W$ that is normal in P. Denote by $\pi : P \to G := P/W$ and $\pi' : P \to P/U$ the canonical projections. A *mixed Shimura datum* is a pair $(P, \mathcal{X})$ equipped with a map $h : \mathcal{X} \to \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbb{C}})$ satisfying the following axioms.

(A1) $\mathcal{X}$ is a (left) $P(\mathbb{R}) \cdot U(\mathbb{C})$-homogeneous space and the map h is $P(\mathbb{R}) \cdot U(\mathbb{C})$-equivariant with finite fibers.

For some (and thus for all) $x \in \mathcal{X}$,

(A2) $\pi' \circ h_x : \mathbb{S}_{\mathbb{C}} \to P_{\mathbb{C}} \to (P/U)_{\mathbb{C}}$ is defined over $\mathbb{R}$.

(A3) $\pi \circ h_x \circ \omega : \mathbb{G}_{m,\mathbb{R}} \to G$ is a cocharacter into the center $Z(G)$ of G.

(A4) $\mathrm{Ad}_P \circ h_x : \mathbb{S}_{\mathbb{C}} \to GL_{\mathrm{Lie}\, P,\mathbb{C}}$ defines a rational mixed Hodge structure on $\mathrm{Lie}\, P$

of type

$$\{(-1,1),(0,0),(1,-1)\} \cup \{(-1,0),(0,-1)\} \cup \{(-1,-1)\}$$

(A5) The weight filtration on $\mathrm{Lie}\, P$ is as follows: $W_{-2}(\mathrm{Lie}\, P) = \mathrm{Lie}\, U$, $W_{-1}(\mathrm{Lie}\, P) = \mathrm{Lie}\, W$ and $W_0(\mathrm{Lie}\, P) = \mathrm{Lie}\, P$.

(A6) $\mathrm{int}(\pi \circ h_x(i))$ induces a Cartan involution on $G^{\mathrm{ad}}_{\mathbb{R}}$.

(A7) G^{ad} has no simple $\mathbb{Q}$-factors of compact type.

(A8) The connected center $Z(G)^0$ of G acts on U and on $V = W/U$ through a torus which is an almost direct product of a split $\mathbb{Q}$-torus with a $\mathbb{Q}$-torus of compact type.

Remark 4.2.2. A few clarifications on the definition:

(i) For a variety X over $\mathbb{Q}$ (not assumed to be irreducible), we say X is of compact type if and only if $X(\mathbb{R})$ is compact in the euclidean topology.

(ii) We will often denote the image of h by $\mathcal{H}$. Thus $\mathcal{H}$ is a $P(\mathbb{R}) \cdot U(\mathbb{C})$-conjugacy class of morphisms $\mathbb{S}_{\mathbb{C}} \to P_{\mathbb{C}}$.

(iii) There is a canonical $P(\mathbb{R}) \cdot U(\mathbb{C})$-invariant complex structure on $\mathcal{H}$ and $\mathcal{X}$. ([Pin90, 2.1(c)].).

(iv) The fourth axiom above implies Griffiths transversality. It guarantees that there is a variation of mixed Hodge structures on the image of $\mathcal{X}$ in a suitable Grassmannian. See [Pin90, 1.10].

(v) For the reasons to have axiom (8), we refer the reader to [Pin90, 1.13].

(vi) As we will see, U is central in W. Thus the statement of (8) makes sense.

(vii) If $W = 1$, $(P, \mathcal{X})$ is called a pure Shimura datum. The definition given in [Del79] is the special case that $W = 1$ and $\mathcal{X} \cong h(\mathcal{X})$. The definition here is slightly more general in that we allow finite coverings.

(viii) ([Pin90, 1.19]) For the quotient Shimura datum $(P, \mathcal{X})/Z(P)$ (defined in Proposition 4.2.7), stronger axioms hold:

- In Axiom (A3), $\pi \circ h_x \circ \omega : \mathbb{G}_{\mathbb{C}} \to G_{\mathbb{C}}$ is a cocharacter into $Z(G)$ that is defined over $\mathbb{Q}$.
- In Axiom (A8), the identity component $Z(G)^0$ is an almost direct product of a split $\mathbb{Q}$-torus with a $\mathbb{Q}$-torus of compact type.

Let $(P, \mathcal{X})$ and $(P', \mathcal{X}')$ be two mixed Shimura data. A morphism between them is a pair (φ, ξ), where $\varphi : P \to P'$ is a morphism of algebraic groups over $\mathbb{Q}$ and $\xi : \mathcal{X} \to \mathcal{X}'$ is a $P(\mathbb{R}) \cdot U(\mathbb{C})$-equivariant map. They are required to fit into the following commutative diagram. (The bottom map comes from $\mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{\mathbb{C}}) \to \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P'_{\mathbb{C}})$ induced by φ.)

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\xi} & \mathcal{X}' \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h'} \\ \mathcal{H} & \xrightarrow{\varphi\circ -} & \mathcal{H}' \end{array}$$

Here are a few basic examples.

Example 4.2.3. Let T be a torus over $\mathbb{Q}$. Then a mixed Shimura datum $(T, \mathcal{X})$ consists of a finite set $\mathcal{X}$ on which $T(\mathbb{R})$ acts transitively as well as a morphism $h : \mathbb{S} \to T_{\mathbb{R}}$. $\mathcal{X} \to \{h\}$ is the constant map.

Example 4.2.4. For a more specific example on torus, consider the group $\mathbb{G}_{m,\mathbb{Q}}$ together with the norm morphism $N : \mathbb{S} \to \mathbb{G}_{m,\mathbb{R}}, z \mapsto z\bar{z}$. Define $\mathcal{H}_0$ to be the set of isomorphisms $\mathbb{Z} \xrightarrow{\sim} 2\pi i\mathbb{Z}$. So $\mathcal{H}_0$ contains two points $n \mapsto \pm 2\pi in$. We let $\mathbb{G}_m(\mathbb{R}) = \mathbb{R}^\times$ act on $\mathcal{H}_0$ via $\mathbb{R}^\times \to \pi_0(\mathbb{R}^\times) = \{\pm 1\}$, where -1 acts on $\mathcal{H}_0$ by interchanging the two points. With the constant projection map $\mathcal{H}_0 \to \{N\}$, $(\mathbb{G}_{m,\mathbb{Q}}, \mathcal{H}_0)$ is a pure Shimura datum.

Example 4.2.5. Another classical example comes from the Siegel modular variety. Fix $\psi : V \times V \to \mathbb{Q}$, a non-degenerate alternating bilinear form on a $\mathbb{Q}$-vector space V of dimension $2n > 0$. Define CSp_V to be the $\mathbb{Q}$-subgroup of GL_V consisting of linear automorphisms that preserve ψ up to a scalar, i.e. $CSp_V(R) = \{g \in GL_V(R) \,|\, \psi(g \cdot v, g \cdot v') = \mu(g)\psi(v, v')$ for some $\mu(g) \in R^\times\}$ for all small $\mathbb{Q}$-algebras R. We get a scalar character $\mu : CSp_V \to \mathbb{G}_{m,\mathbb{Q}}$ and a short exact sequence $1 \to Sp_V \to CSp_V \xrightarrow{\mu} \mathbb{G}_{m,\mathbb{Q}} \to 1$, where $Sp_V \cong Sp_{2n}$.

Let $\mathcal{H}_V$ be the set of morphisms $h : \mathbb{S} \to GL_{V,\mathbb{R}}$ with the properties that

(1) h induces a Hodge structure of type $\{(-1, 0), (0, -1)\}$ on $V_\mathbb{C}$.

(2) $\psi_\mathbb{R}(h(i)u, h(i)v) = \psi_\mathbb{R}(u, v)$ for all $u, v \in V_\mathbb{R}$.

(This is equivalent to $\psi_\mathbb{R}(h(z)u, h(z)v) = z\bar{z}\,\psi_\mathbb{R}(u, v)$. Therefore, $\mu \circ h = N$ for all such h.)

(3) The (symmetric) bilinear form $(u, v) \mapsto \psi_\mathbb{R}(u, h(i)v)$ is either positive or negative definite.

By definition, every $h \in \mathcal{H}_V$ factors through CSp_V. It is well-known that CSp_V is a connected reductive algebraic group and $CSp_V(\mathbb{R})$ acts transitively on $\mathcal{H}_V$. ($Sp_V(\mathbb{R})$

is connected and acts transitively on either of the two connected components of $\mathcal{H}_V$.)

Then $(CSp_V, \mathcal{H}_V)$ is a pure Shimura datum. If $V = \mathbb{Q}^{2n}$ with ψ given by the matrix $\begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$, we will write $(CSp_{2n}, \mathcal{H}_{2n})$.

Example 4.2.6. There is a morphism $(CSp_V, \mathcal{H}_V) \to (\mathbb{G}_{m,\mathbb{Q}}, \mathcal{H}_0)$. For any $h \in \mathcal{H}_V$, there is a unique $\lambda \in \mathcal{H}_0$ such that $(u, v) \mapsto \lambda \circ \psi_{\mathbb{R}}(u, h(i)v)$ is a polarization of the Hodge structure on V defined by h. (If $\psi_{\mathbb{R}}(-, h(i) \cdot -)$ is positive definite, take $\lambda : 1 \mapsto 2\pi i$; if it is negative definite, take $\lambda : 1 \mapsto -2\pi i$.) This way we get a diagram

$$\begin{tikzcd} \mathcal{H}_V \arrow[r] \arrow[d, "\wr"] & \mathcal{H}_0 \arrow[d] \\ \mathcal{H}_V \arrow[r] & \{N\} \end{tikzcd}$$

compatible with the action of $CSp_V(\mathbb{R}) \to \mathbb{G}_{m,\mathbb{R}}(\mathbb{R})$.

Some consequences of the axioms are summarized below.

Proposition 4.2.7.

(1) Every connected component $\mathcal{X}^0 \subset \mathcal{X}$ is mapped isomorphically onto a connected component $\mathcal{H}^0 = h(\mathcal{X}^0) \subset \mathcal{H}$.

(2) For any morphism (φ, ξ) of mixed Shimura data as above, ξ is a holomorphic map. If both φ and ξ are injective (whence we call (φ, ξ) an embedding), then ξ is a closed embedding.

(3) We can define the product $(P, \mathcal{X}) \times (P', \mathcal{X}') = (P \times P', \mathcal{X} \times \mathcal{X}')$ with $h \times h' : \mathcal{X} \times \mathcal{X}' \to \mathcal{H} \times \mathcal{H}'$ having the desired universal property and projection maps.

(4) Let $N \trianglelefteq P$ be a normal $\mathbb{Q}$-subgroup. There exists a quotient mixed Shimura

datum $(P,\mathcal{X})/N = (P/N, \mathcal{X}/N)$ *with a projection morphism* $(\pi,\zeta):(P,\mathcal{X}) \longrightarrow (P,\mathcal{X})/N$*. It is defined up to isomorphism with the universal property that, if in a given morphism* $(\varphi,\xi):(P,\mathcal{X}) \to (P',\mathcal{X}')$*,* φ *factors through* P/N*, then* (φ,ξ) *factors uniquely through* $(P,\mathcal{X})/N$*. in other words, we have a diagram*

$$\begin{tikzcd}
 & (P,\mathcal{X}) \arrow[dl, "(\pi,\zeta)"'] \arrow[dr, "(\varphi,\xi)"] & \\
(P,\mathcal{X})/N \arrow[rr] & & (P',\mathcal{X}')
\end{tikzcd}$$

Let $P_1 = P/U$ *and* U_1 *be the image of* U *under* $\pi : P \to P_1$*. Fix one* $x \in \mathcal{X}$*. Then* $\mathcal{X}/N$ *is defined as* $\dfrac{P_1(\mathbb{R})\cdot U_1(\mathbb{C})}{\pi(\mathrm{Stab}_{P(\mathbb{R})\cdot U(\mathbb{C})}(x))}$ *with* $\mathcal{H}/N$ *the* $P_1(\mathbb{R})\cdot U_1(\mathbb{C})$*-conjugacy class of* $\pi \circ h_x$*.*

A mixed Shimura datum $(P,\mathcal{X})$ is called *irreducible* if P itself is the smallest $\mathbb{Q}$-normal subgroup in P containing the image of all h_x. (Again, it suffices to check this for some h_x.) Given an arbitrary $(P,\mathcal{X})$, Let P_1 be the smallest $\mathbb{Q}$-normal subgroup of P through which some h_x factors (P_1 must be connected!). Since $h_x(\mathbb{G}_{m,\mathbb{C}})$ acts trivially on P/P_1, we must have $W \subset P_1$ by (A5). Then it can be shown (cf. [Pin90, 2.13]) that the $P_1(\mathbb{R})\cdot U(\mathbb{C})$-orbit of $x \in \mathcal{X}$ is a union of connected components of $\mathcal{X}$. (We just have to show that $P_1(\mathbb{R})\cdot U(\mathbb{C})\cdot \mathrm{Stab}_{P(\mathbb{R})\cdot U(\mathbb{C})}(x)$ is a finite-index subgroup of $P(\mathbb{R})\cdot U(\mathbb{C})$.) Denote by $\mathcal{X}_1$ this orbit. Then $(P_1,\mathcal{X}_1) \to (P,\mathcal{X})$ is an embedding. Dividing $\mathcal{X}$ into a disjoint union of $P_1(\mathbb{R})\cdot U(\mathbb{C})$-orbits, we see that every mixed Shimura datum $(P,\mathcal{X})$ is canonically a disjoint union of irreducible ones.

Irreducible mixed Shimura data have the following nice property.

Proposition 4.2.8. *Let* $(P,\mathcal{X})$ *be an irreducible mixed Shimura datum. Then the* *representation $P \to GL_U$ factors through the scalar cocharacter $\mathbb{G}_{m,\mathbb{Q}} \rightarrowtail GL_U$. That is, P acts on U through scalars.*

We can also extract some useful information about the structure of the unipotent radical W. As $\operatorname{Lie} U$ and $\operatorname{Lie} W$ are of weight -2 and < 0, respectively, $[\operatorname{Lie} U, \operatorname{Lie} W] = 0$. Similarly, the derived group $W^{\mathrm{der}} \subset U$. It follows that U is central in W and V is abelian. Moreover, we have a well-defined commutator pairing

$$\begin{aligned}\Psi : & V \times V \longrightarrow U \\ & (v, v') \longmapsto [v, v']\end{aligned}$$

where (in terms of $\mathbb{Q}$-points) $[v, v'] = [w, w']$ for any $w, w' \in W(\mathbb{Q})$ lifting $v, v' \in V(\mathbb{Q})$. This pairing determines the group structure on W as follows: Since we are over characteristic 0, W is isomorphic to $U \times V$ as a scheme, and $(u, v) \cdot (u', v') :=$ $(u + u' + \frac{1}{2}\Psi(v, v'), v + v')$ defines the group structure on $U \times V$. Conversely, any such Ψ defines a unipotent extension $1 \to U \to W \to V \to 1$ with Ψ as its commutator pairing.

The last construction to be introduced is the unipotent extension. Let $(P, \mathcal{X})$ be a mixed Shimura datum. Suppose we have a short exact sequence $1 \to W_0 \to P_1 \to P \to 1$ with W_0 a unipotent group. We wish to extend $(P, \mathcal{X})$ to $(P_1, \mathcal{X}_1)$ such that $(P_1, \mathcal{X}_1)/W_0 \to (P, \mathcal{X})$ is an isomorphism.

Let V_0 be a subquotient of W_0 such that $\operatorname{Lie} V_0$ is irreducible as a P_1-representation. Then V_0 must be abelian as $\operatorname{Lie}\left(V_0^{\mathrm{der}}\right)$ is a P_1-invariant subspace (not equal to $\operatorname{Lie} V_0$

as V_0 is unipotent, hence solvable). Therefore, the action of P_1 factors through P and then further through G as $W \subset P$ must act trivially on V_0. With this observation we state

Proposition 4.2.9. *Let $(P, \mathcal{X})$, P_1, W_0 be as above. Suppose that for every irreducible subquotient* $\mathrm{Lie}\, V_0$ *of* $\mathrm{Lie}\, W_0$*,*

(1) The Hodge structure on $\mathrm{Lie}\, V_0$ *as a representation of G induced by $\pi \circ h_x$ is of type $\{(-1,-1)\}$ or $\{(-1,0),(0,-1)\}$.*

(2) The connected center of G acts on V_0 through a torus that is an almost direct product of a split $\mathbb{Q}$-torus with a $\mathbb{Q}$-torus of compact type.

Then

(1) the projection map $P_1 \to P$ extends to a morphism of mixed Shimura data $(P_1, \mathcal{X}_1) \to (P, \mathcal{X})$ such that $(P_1, \mathcal{X}_1)/W_0 \to (P, \mathcal{X})$ is an isomorphism. The $(P_1, \mathcal{X}_1)$ is defined up to canonical isomorphism.

(2) Given a morphism of mixed Shimura data $(\varphi, \xi) : (P', \mathcal{X}') \to (P, \mathcal{X})$, if $\varphi : P' \to P$ factors through P_1, then (φ, ξ) factors uniquely through $(P_1, \mathcal{X}_1)$.

Furthermore, for any unipotent extension $(P_1.\mathcal{X}_1) \to (P, \mathcal{X})$, the diagram

$$\begin{array}{ccc} \mathcal{X}_1 & \longrightarrow & \mathcal{X} \\ {\scriptstyle h_1}\downarrow & & \downarrow{\scriptstyle h} \\ \mathcal{H}_1 & \longrightarrow & \mathcal{H} \end{array}$$

is a pullback.

Now let $(\varphi, \xi) : (P_1.\mathcal{X}_1) \to (P, \mathcal{X})$ be a unipotent extension for $1 \to W_0 \to P_1 \to P \to 1$. By (A5), it is not hard to deduce that the fiber $\xi^{-1}(x)$ over a point $x \in \mathcal{X}$ is an orbit under $W_0(\mathbb{R})\cdot U_0(\mathbb{C})$ where $U_0 := W_0\cap U_1 \subset P_1$. Choose any $x_1 \in \xi^{-1}(x)$, then $\mathrm{Lie}\,(W_0(\mathbb{R})\cdot U_0(\mathbb{C})) = (\mathrm{Lie}\,W_0)_{\mathbb{R}} + (\mathrm{Lie}\,U_0)_{\mathbb{C}} \xrightarrow{\sim} \dfrac{(\mathrm{Lie}\,W_0)_{\mathbb{C}}}{F^0_{h_{x_1}}(\mathrm{Lie}\,W_0)_{\mathbb{C}}}$ for $\{F^p_{h_{x_1}}(\mathrm{Lie}\,W_0)_{\mathbb{C}}\}_p$ the Hodge filtration on $\mathrm{Lie}\,W_0$ induced by h_{x_1}. When the short exact sequence splits, it can be shown that $\mathcal{X}_1 \to \mathcal{X}$ is isomorphic to a holomorphic complex vector bundle over $\mathcal{X}$.

Since $1 \to W \to P \to G \to 1$ splits, we obtain a structural result on $\mathcal{X}$.

Proposition 4.2.10. *For a mixed Shimura datum $(P, \mathcal{X})$, any connected component $\mathcal{X}^0$ of $\mathcal{X}$ is a holomorphic vector bundle over a Hermitian symmetric domain $\mathcal{H}^0 \subset \mathcal{H}$.*

Finally, we list a few examples of unipotent extension which will be useful.

Example 4.2.11. Let $\mathbb{G}_{m,\mathbb{Q}}$ act on $U_0 = \mathbb{Q}$ through multiplication. Then under the norm morphism $N : \mathbb{S} \to \mathbb{G}_{m,\mathbb{Q}}$, the induced Hodge structure on $\mathrm{Lie}\,U_0$ is of type $\{(-1,-1)\}$. Let $(P_0, \mathcal{X}_0) \to (\mathbb{G}_{m,\mathbb{Q}}, \mathcal{H}_0)$ be the unipotent extension corresponding to $1 \to U_0 \to U_0 \rtimes \mathbb{G}_{m,\mathbb{Q}} =: P_0 \to \mathbb{G}_{m,\mathbb{Q}} \to 1$.

The Hodge structure on $\mathrm{Lie}\,P_0$ is of type $\{(0,0),(-1,-1)\}$. $(P_0, \mathcal{X}_0)$ may be the simplest "authentic" mixed Shimura datum.

Example 4.2.12. Let $V = \mathbb{Q}^{2n}$ and $\psi : V_{2n} \times V_{2n} \to U_{2n} = \mathbb{Q}$ the standard symplectic form on V_{2n}. Let $(CSp_{2n}, \mathcal{H}_{2n})$ be defined as in Example 4.2.5 and $1 \to U_{2n} \to W_{2n} \to V_{2n} \to 1$ the central extension with ψ as the commutator pairing. Let CSp_{2n} act on U_0 through the multiplier map $CSp_{2n} \to \mathbb{G}_{m,\mathbb{Q}}$. Then ψ becomes CSp_{2n}-equivariant. Through unipotent extensions we obtain the following morphisms of mixed Shimura data.

$$(W \rtimes CSp_{2n}, \mathcal{X}_{2n}) \to (V \rtimes CSp_{2n}, \mathcal{Y}_{2n}) \to (CSp_{2n}, \mathcal{H}_{2n})$$

We will denote the first mixed Shimura datum as $(P_{2n}, \mathcal{X}_{2n})$.

4.3 Mixed Shimura varieties

We now turn to our main objects of study. Mixed Shimura varieties have a definition akin to their pure counterparts and possess morphisms and functoriality formulated in a similar way. For the purpose of constructing their toroidal compactifications, it is necessary, however, to examine more closely the fibers of a unipotent extension. Due to their nature as more (mixed) general objects, the notations will get a bit more complicated as we move on. But most of them are adopted to avoid further confusion. This subsection runs parallel to Chapter 3 of [Pin90].

Definition 4.3.1 (mixed Shimura varieties)**.** Let $(P, \mathcal{X})$ be a mixed Shimura datum and $K_f \subset P(\mathbb{A}_f)$ an open compact subgroup. For the product space $\mathcal{X} \times P(\mathbb{A}_f)$, we let $P(\mathbb{Q})$ act from the left on both factors and K_f act from the right on $P(\mathbb{A}_f)$. The associated *mixed Shimura variety* is defined to be the double coset space

$$M^{K_f}(P, \mathcal{X})(\mathbb{C}) = P(\mathbb{Q})\backslash\mathcal{X} \times P(\mathbb{A}_f)/K_f.$$

We equip $M^{K_f}(P, \mathcal{X})(\mathbb{C})$ with the quotient topology of $\mathcal{X} \times P(\mathbb{A}_f)/K_f$ (with $P(\mathbb{A}_f)/K_f$ a discrete space). The set of connected components of $M^{K_f}(P, \mathcal{X})(\mathbb{C})$ is finite, as it is mapped onto by the finite set $\pi_0(P(\mathbb{Q})\backslash P(\mathbb{R}) \cdot U(\mathbb{C}) \times P(\mathbb{A}_f)/K_f) \overset{\sim}{\leftarrow} \pi_0(P(\mathbb{Q})\backslash P(\mathbb{R}) \times P(\mathbb{A}_f)/K_f) = \pi_0(P(\mathbb{Q})\backslash P(\mathbb{A})/K_f)$ (cf. [Pin90, 3.2]). Similarly, $\pi_0(P(\mathbb{Q})\backslash P(\mathbb{A}_f)/K_f) = P(\mathbb{Q})\backslash P(\mathbb{A}_f)/K_f$ is also mapped onto by $\pi_0(P(\mathbb{Q})\backslash P(\mathbb{A})/K_f)$, hence finite.

Since $\mathcal{X}$ has only finitely many connected components, for any component $\mathcal{X}^0$, the subgroup $\mathrm{Stab}_{P(\mathbb{Q})}(\mathcal{X}^0) \subset P(\mathbb{Q})$ is of finite index. Therefore $\mathrm{Stab}_{P(\mathbb{Q})}(\mathcal{X}^0)\backslash P(\mathbb{A}_f)/K_f$ is a finite set. Choose a set of representatives $\{p_f\} \subset P(\mathbb{A}_f)$ for it. It follows that the connected components of $M^{K_f}(P, \mathcal{X})(\mathbb{C})$ are precisely $\Gamma_{p_f}\backslash\mathcal{X}^0$ where $\Gamma_{p_f} = \mathrm{Stab}_{P(\mathbb{Q})}(\mathcal{X}^0) \cap p_f K_f p_f^{-1}$. So we have a decomposition

$$\bigsqcup_{p_f} \Gamma_{p_f}\backslash\mathcal{X}^0 \xrightarrow{\sim} M^{K_f}(P, \mathcal{X})(\mathbb{C})$$

The following proposition tells us that $M^{K_f}(P, \mathcal{X})(\mathbb{C})$ is reasonably good as a topological space.

Proposition 4.3.2 ([Pin90, 3.3])**.** *Fix $\mathcal{X}^0 \subset \mathcal{X}$, $p_f \in P(\mathbb{A}_f)$, and let Γ_{p_f} be defined as above.*

(1) Γ_{p_f} acts properly discontinuously on $\mathcal{X}^0$.

(2) $M^{K_f}(P, \mathcal{X})(\mathbb{C})$ inherits the structure of a normal complex space from the $P(\mathbb{R}) \cdot U(\mathbb{C})$-invariant complex structure on $\mathcal{X}$. Its singularities are at most quotient singularities by finite groups.

(3) If $K_f \subset P(\mathbb{A}_f)$ is a neat open compact subgroup, then $M^{K_f}(P, \mathcal{X})(\mathbb{C})$ is a complex manifold.

Remark 4.3.3.

(i) Γ_{p_f} is generally an infinite group. To say it acts properly discontinuously on some space Y means some finite quotient Γ_{p_f}/Γ_0 acts on Y properly discontinuously in the usual sense.

(ii) For the generalized notion of a neat open compact subgroup $K_f \subset P(\mathbb{A}_f)$, see (the last part of) Chapter 0 of [Pin90]. Generalized neatness retains the most commonly used properties. For example, it is inherited by subgroups, images, and preserved under automorphisms of P as well as inner automorphisms of $P(\mathbb{A}_f)$.

We define some morphisms of mixed Shimura varieties.

Let $(P, \mathcal{X})$ be a mixed Shimura datum and $K_f, K'_f \subset P(\mathbb{A}_f)$ be open compact subgroups. Suppose $p_f \in P(\mathbb{A}_f)$ satisfies $p_f^{-1} K_f p_f \subseteq K'_f$, then multiplying on the left by p_f induces a well-defined morphism

$$
\begin{aligned}
[\cdot p_f] = [\cdot p_f]_{K'_f, K_f} : M^{K_f}(P, \mathcal{X})(\mathbb{C}) &\longrightarrow M^{K'_f}(P, \mathcal{X})(\mathbb{C}) \\
[x, p'_f] &\longmapsto [x, p'_f \cdot p_f]
\end{aligned}
$$

of mixed Shimura varieties. $[\cdot p_f]$ is holomorphic, surjective with finite fibers (thus open and closed). If $p_f^{-1} K_f p_f = K'_f$ it is an isomorphism. This is the case, for example, if $K'_f = K_f$ and p_f normalizes K_f.

For another type of morphism, let $\varphi : (P, \mathcal{X}) \to (P', \mathcal{X}')$ be a morphism of mixed Shimura data (where we abuse the notation by writing (φ, ξ) as φ). If $K_f \subset P(\mathbb{A}_f)$, $K'_f \subset P'(\mathbb{A}_f)$ are open compact subgroups such that $\varphi(K_f) \subseteq K'_f$. then φ induces a well-defined morphism of mixed Shimura varieties

$$
\begin{aligned}
[\varphi] = [\varphi]_{K'_f, K_f} : M^{K_f}(P, \mathcal{X})(\mathbb{C}) &\longrightarrow M^{K'_f}(P', \mathcal{X}')(\mathbb{C}) \\
[x, p_f] &\longmapsto [\varphi(x), \varphi(p_f)]
\end{aligned}
$$

This again is holomorphic. If $\varphi : P \to P'$ is an isomorphism of groups and $\varphi(K_f) = K'_f$, $[\varphi]$ is an isomorphism.

Remark 4.3.4.

(i) For any $k_f \in K_f$, $[\cdot k_f]$ is the identity morphism $[\mathrm{id}] : M^{K_f}(P, \mathcal{X})(\mathbb{C}) \to M^{K_f}(P, \mathcal{X})(\mathbb{C})$.

(ii) If K'_f is a normal open compact subgroup of K_f, then K_f/K'_f is finite and acts on $M^{K'_f}(P, \mathcal{X})(\mathbb{C})$. From this perspective, $M^{K_f}(P, \mathcal{X})(\mathbb{C})$ becomes the quotient space under this finite group action. (However, as we shall see, this is in general not a faithful action.)

For more properties of mixed Shimura varieties, we refer the reader to 3.7-3.12 in Chapter 3 of [Pin90]. It ought to be mentioned that fiber products do not exist in general in the category of mixed Shimura varieties. They do exist for the pullback of a unipotent extension $(P_1, \mathcal{X}_1) \to (P, \mathcal{X})$ under an arbitrary morphism $\varphi : (P_2, \mathcal{X}_2) \to (P, \mathcal{X})$. Even this case only holds under a mild assumption on φ.

We next prove a technical "centralizer" lemma that will be crucial later on. Basically, it means that if an element of a neat arithmetic subgroup of $P(\mathbb{Q})$ commutes with the image of h_x for some $x \in \mathcal{X}$, then it lies in the center of P and thus commutes with every one of them.

Lemma 4.3.5. *Let $(P, \mathcal{X})$ be a mixed Shimura datum and $x \in \mathcal{X}$. Then for any neat arithmetic subgroup $K \subseteq P(\mathbb{Q})$, we have $K \cap \mathrm{Cent}_{P(\mathbb{Q})}(h_x) \subseteq Z(P)(\mathbb{Q})$.*

Proof. Let $K_0 \coloneqq K \cap \mathrm{Cent}_{P(\mathbb{Q})}(h_x)$. It suffices to show that the image of K_0 in $(P/Z(P))(\mathbb{Q})$ is trivial. Thus we may assume that the connected center $Z(G)^0$ of G

is an almost direct product of a split $\mathbb{Q}$-torus with a $\mathbb{Q}$-torus of compact type. Since $P/P^{\mathrm{der}} \xrightarrow{\sim} G/G^{\mathrm{der}}$ and $Z(G)^0$ projects onto this, P/P^{der} is also an almost direct product of a split $\mathbb{Q}$-torus with a $\mathbb{Q}$-torus of compact type. Any neat arithmetic subgroup in such a group is trivial. Therefore,

$$
\begin{aligned}
K_0 &\subseteq \mathrm{Cent}_{P^{\mathrm{der}}(\mathbb{Q})}(h_x) \\
&\subseteq \mathrm{Cent}_{P^{\mathrm{der}}(\mathbb{R})\cdot U(\mathbb{C})}(h_x) \\
&\underset{\sim}{\xrightarrow{\pi}} \mathrm{Cent}_{G^{\mathrm{der}}(\mathbb{R})}(\pi \circ h_x) \text{ (cf. [Pin90, Lemma 1.17])} \\
&\subseteq \mathrm{Cent}_{G^{\mathrm{der}}(\mathbb{R})}(\pi \circ h_x(i)) \\
&\subseteq \left\{ g \in G^{\mathrm{der}}(\mathbb{C}) \mid \mathrm{int}(\pi \circ h_x(i))(g) = \bar{g} \right\}
\end{aligned}
$$

But the last set is compact as $\mathrm{int}(\pi \circ h_x(i))$ induces a Cartan involution on G^{der}. As K_0 is discrete and torsion-free, it must be trivial. □

Corollary 4.3.6. *Let $(P, \mathcal{X})$ be a mixed Shimura datum and $x \in \mathcal{X}$. Let $K_f \subseteq P(\mathbb{A}_f)$ be a neat open compact subgroup. Then $\mathrm{Stab}_{P(\mathbb{Q})}(x) \cap K_f \subseteq \mathrm{Id}_{Z(P)(\mathbb{Q})}(\mathcal{X})$, where $\mathrm{Id}_{Z(P)(\mathbb{Q})}(\mathcal{X})$ is the subgroup of $Z(P)(\mathbb{Q})$ that acts trivially on $\mathcal{X}$.*

Proof. We have $\mathrm{Stab}_{P(\mathbb{Q})}(x) \subseteq \mathrm{Cent}_{P(\mathbb{Q})}(h_x)$. And if an element $z \in Z(P)(\mathbb{Q})$ fixes a point $x \in \mathcal{X}$, it fixes the whole of $\mathcal{X}$ as $P(\mathbb{R}) \cdot U(\mathbb{C})$ acts transitively on this space. □

The next proposition is also used in the study of unipotent fibers.

Proposition 4.3.7. *Let $(P, \mathcal{X})$ be a mixed Shimura datum with $G = P/W$. Denote by $\pi : (P, \mathcal{X}) \to (G, \mathcal{X}_G) = (P, \mathcal{X})/W$ the morphism constructed from the canonical projection. Suppose $K_f \subset G(\mathbb{A}_f)$ is a neat open compact subgroup. Then*

$\mathrm{Id}_{Z(G)}(\mathcal{X}_G) \cap K_f \subseteq \pi(\mathrm{Id}_{Z(P)(\mathbb{Q})}(\mathcal{X}))$. *In particular,*

$$\left(\pi|_{Z(P)}\right)^{-1}\left(\mathrm{Id}_{Z(G)}(\mathcal{X}_G) \cap K_f\right) \xrightarrow{\sim} \mathrm{Id}_{Z(G)}(\mathcal{X}_G) \cap K_f.$$

Proof. Choose a splitting $P = W \rtimes G$. As $A := \mathrm{Id}_{Z(G)}(\mathcal{X}_G) \cap K_f$ is a neat arithmetic subgroup of $Z(G)(\mathbb{Q})$ acting on U and V through an almost direct product of a split $\mathbb{Q}$-torus with a $\mathbb{Q}$-torus of compact type, A must act trivially on U and V and hence also on W. This means under the inclusion $G \rightarrowtail W \rtimes G$, A is mapped into the center of $W \rtimes G$. But A fixes points in $\mathcal{X}_G$ and hence must fix $\mathcal{X}$ as well. (By Proposition 4.2.7, the splitting of groups $G \rightarrowtail W \rtimes G \to G$ extends to a splitting of mixed Shimura data $(G, \mathcal{X}_G) \rightarrowtail (P, \mathcal{X}) \to (G, \mathcal{X}_G)$.) Therefore A maps into $\mathrm{Id}_{Z(P)(\mathbb{Q})}(\mathcal{X})$. □

Corollary 4.3.8. *Let $\varphi : (P_1, \mathcal{X}_1) \to (P, \mathcal{X})$ be any unipotent extension and $K_f \subset P(\mathbb{A}_f)$ any neat open compact subgroup. Then under the map $\varphi|_{Z(P_1)}\colon \mathrm{Id}_{Z(P_1)(\mathbb{Q})}(\mathcal{X}_1) \rightarrowtail \mathrm{Id}_{Z(P)(\mathbb{Q})}(\mathcal{X})$, we have*

$$\left(\varphi|_{Z(P_1)}\right)^{-1}\left(\mathrm{Id}_{Z(P)(\mathbb{Q})}(\mathcal{X}) \cap K_f\right) \xrightarrow{\sim} \mathrm{Id}_{Z(P)(\mathbb{Q})}(\mathcal{X}) \cap K_f.$$

Proof. Let $\pi : (P, \mathcal{X}) \to (G, \mathcal{X}_G) = (P, \mathcal{X})/W$ be as above and $K_f^G := \pi(K_f)$. Then we have the diagram as follows (with $\pi_1 = \pi \circ \varphi$).

$$\begin{tikzcd}
(\varphi)^{-1}\left(\mathrm{Id}_{Z(P)(\mathbb{Q})}(\mathcal{X})\cap K_f\right) \arrow[r, "\sim"] \arrow[d, hook] & \mathrm{Id}_{Z(P)}(\mathcal{X})\cap K_f \arrow[d, hook] \\
(\pi_1)^{-1}\left(\mathrm{Id}_{Z(G)(\mathbb{Q})}(\mathcal{X}_G)\cap K_f^G\right) \arrow[r, "\varphi", "\sim"'] \arrow[dr, "\sim", "\pi_1"'] & (\pi)^{-1}\left(\mathrm{Id}_{Z(G)(\mathbb{Q})}(\mathcal{X}_G)\cap K_f^G\right) \arrow[d, "\pi", "\wr"'] \\
& \mathrm{Id}_{Z(G)(\mathbb{Q})}(\mathcal{X}_G)\cap K_f^G
\end{tikzcd}$$

Since π_1 induces an isomorphism with φ and π injective, both φ and π are isomorphisms. The corollary follows immediately as $(\varphi)^{-1}\left(\mathrm{Id}_{Z(P)(\mathbb{Q})}(\mathcal{X})\cap K_f\right)$ is contained in $(\pi_1)^{-1}\left(\mathrm{Id}_{Z(G)(\mathbb{Q})}(\mathcal{X}_G)\cap K_f^G\right)$. $\square$

With the above results, we can analyze the structure of unipotent fibers in more detail. Let $(\varphi,\xi):(P',\mathcal{X}')\to(P,\mathcal{X})$ be a unipotent extension with kernel $W_0\subset P'$. $K'_f\subseteq P'(\mathbb{A}_f)$ is an open compact subgroup and define $K_f=\varphi(K'_f)\subseteq P(\mathbb{A}_f)$ to be its image. Fix $x'\in\mathcal{X}', p'_f\in P'(\mathbb{A}_f)$; let x, p_f be their images in $\mathcal{X}$, $P(\mathbb{A}_f)$, respectively.

As W_0 is unipotent, we have a short exact sequence $1\to W_0(\mathbb{Q})\to P'(\mathbb{Q})\to P(\mathbb{Q})\to 1$. Thus the preimage of the orbit $P(\mathbb{Q})\cdot(x,p_f)\cdot K_f\subset\mathcal{X}\times P(\mathbb{A}_f)$ is $P'(\mathbb{Q})\cdot\left(\xi^{-1}(x)\times(p'_f\cdot W_0(\mathbb{A}_f)K'_f)\right)$. Then the fiber over the point $[x,p_f]$ of the morphism $[\varphi]:M^{K'_f}(P',\mathcal{X}')(\mathbb{C})\to M^{K_f}(P,\mathcal{X})(\mathbb{C})$ is

$$\begin{aligned}
[\varphi]^{-1}([x,p_f]) &= P'(\mathbb{Q})\backslash\left(P'(\mathbb{Q})\cdot\left((\xi^{-1}(x))\times\frac{p'_f\cdot W_0(\mathbb{A}_f)K'_f}{K'_f}\right)\right)\\
&\xleftarrow{\sim}\mathrm{Stab}_{P'(\mathbb{Q})}(\mathbf{x})\backslash(\xi^{-1}(x))\times\frac{p'_f\cdot W_0(\mathbb{A}_f)K'_f}{K'_f}\\
&\xleftarrow{\sim}\mathrm{Stab}_{P'(\mathbb{Q})}(\mathbf{x})\backslash(\xi^{-1}(x))\times\frac{W_0(\mathbb{Q})p'_f\cdot K'_f}{K'_f}\\
&\xleftarrow{\sim}\mathrm{Stab}_{P'(\mathbb{Q})}(\mathbf{x})\cap p'_fK'_fp'^{-1}_f\backslash\left(\xi^{-1}(x)\times\frac{p'_f\cdot K'_f}{K'_f}\right)\\
&=\mathrm{Stab}_{P'(\mathbb{Q})}(\mathbf{x})\cap p'_fK'_fp'^{-1}_f\backslash\xi^{-1}(x)
\end{aligned}$$

(The arrows indicate the direction of natural inclusions.)

As $\xi^{-1}(x)$ is isomorphic to $W_0(\mathbb{R}) \cdot U_0(\mathbb{C})$ (with $U_0 = W_0 \cap U_1$), we see first that the fiber is connected.

From now on assume that K'_f is neat. Then $\mathrm{Stab}_{P'(\mathbb{Q})}(\mathbf{x}) \cap p'_f K'_f p'^{-1}_f$ projects into $\mathrm{Stab}_{P(\mathbb{Q})}(\mathbf{x}) \cap p_f K_f p_f^{-1}$, which is contained in $\mathrm{Id}_{Z(P)(\mathbb{Q})}(\mathcal{X}) \cap K_f$ by Lemma 4.3.5. Therefore, by the above Corollary 4.3.8,

$$\begin{aligned} & \mathrm{Stab}_{P'(\mathbb{Q})}(\mathbf{x}) \cap p'_f K'_f p'^{-1}_f \subseteq \mathrm{Id}_{Z(P')(\mathbb{Q})}(\mathcal{X}') \times W_0(\mathbb{Q}) \\ \implies & \mathrm{Stab}_{P'(\mathbb{Q})}(\mathbf{x}) \cap p'_f K'_f (p'^{-1}_f = \big(\mathrm{Id}_{Z(P')(\mathbb{Q})}(\mathcal{X}') \times W_0(\mathbb{Q})\big) \cap p'_f K'_f p'^{-1}_f \end{aligned}$$

Define

$$\Gamma' := \frac{\big(\mathrm{Id}_{Z(P')(\mathbb{Q})}(\mathcal{X}') \times W_0(\mathbb{Q})\big) \cap p'_f K'_f p'^{-1}_f}{\mathrm{Id}_{Z(P')(\mathbb{Q})}(\mathcal{X}') \cap p'_f K'_f p'^{-1}_f}.$$

This is the projection of $\big(\mathrm{Id}_{Z(P')(\mathbb{Q})}(\mathcal{X}') \times W_0(\mathbb{Q})\big) \cap p'_f K'_f p'^{-1}_f$ onto $W_0(\mathbb{Q})$ under $Z(P') \times W_0 \to W_0$. Hence the fiber is isomorphic to $\Gamma' \backslash W_0(\mathbb{R}) \cdot U_0(\mathbb{C})$. The situation can be illustrated with the diagram below.

$$\begin{tikzcd} W_0(\mathbb{R}) \cdot U_0(\mathbb{C}) \arrow[r, "\sim"] \arrow[d, two heads] & \xi^{-1}(x) \arrow[d, two heads] \\ \Gamma' \backslash W_0(\mathbb{R}) \cdot U_0(\mathbb{C}) \arrow[r, "\sim"] & \dfrac{\xi^{-1}(x)}{\big(\mathrm{Id}_{Z(P')(\mathbb{Q})}(\mathcal{X}') \times W_0(\mathbb{Q})\big) \cap p'_f K'_f p'^{-1}_f} \end{tikzcd}$$

The bottom isomorphism is defined as $[w] \mapsto [w \cdot x', p'_f]$. It is in general non-canonical. But it is so when the unipotent extension splits.

Proposition 4.3.9. *Let $(P', \mathcal{X}') \to (P, \mathcal{X})$ be a unipotent extension by W_0. Given $[x', p'_f] \in M^{K'_f}(P', \mathcal{X}')(\mathbb{C})$ and its image $[x, p_f] \in M^{K_f}(P, \mathcal{X})(\mathbb{C})$, the fiber over $[x, p_f]$*

is isomorphic to $\Gamma'\backslash W_0(\mathbb{R}) \cdot U_0(\mathbb{C})$ *via the map* $[w] \mapsto [w \cdot x', p'_f]$ *as above.*

Finally, there are certain conditions under which Γ' has a simpler description. We summarize them below.

Proposition 4.3.10. *In the following two situations,* $\Gamma' = W_0(\mathbb{Q}) \cap p'_f K'_f p'^{-1}_f$.

(1) $Z(P)^0$ *is an almost direct product of a split* $\mathbb{Q}$*-torus with a* $\mathbb{Q}$*-torus of compact type.*

(2) The unipotent extension $1 \to W_0 \to P' \to P \to 1$ *splits.* W_0 *is abelian and* $K'_f = K_f^{W_0} \ltimes K_f$ *with* $K_f^{W_0} \subset W_0(\mathbb{A}_f)$ *any (neat) open compact subgroup.*

Remark 4.3.11. We will mostly apply the above result to the unipotent extension $1 \to U \to P \to P/U \to 1$. Then $\Gamma \subset U(\mathbb{Q})$ is an arithmetic lattice and $\Gamma\backslash U(\mathbb{C})$ is an algebraic torus. Thus $M^{K_f}(P, \mathcal{X})(\mathbb{C}) \to M^{\pi_U(K_f)}(P/U, \mathcal{X}/U)(\mathbb{C})$ is locally an analytic $\mathbb{G}_{m,\mathbb{C}}^{\dim U}$-torsor. This will be revisited in the construction of toroidal compactifications.

4.4 Rational boundary components

The rational boundary components of a given mixed Shimura datum $(P, \mathcal{X})$ are mixed Shimura data obtained from certain $\mathbb{Q}$-parabolic subgroups of P. When $(P, \mathcal{X})$ is irreducible, it is itself an (improper) rational boundary component. Rational boundary components play a key role in the construction of toroidal compactifications. In this subsection, we present the process of defining a rational boundary component and summarize some of their properties which will be useful later on. Of special importance is the associated open convex cone $C(\mathcal{X}^0, P_1) \subset U_1(\mathbb{R})(-1)$.

The first step to define a rational boundary component is to obtain a new "more" mixed Hodge structure on $\operatorname{Lie} P$ from the old one defined by some h_x. In order to achieve that, we make a few definitions.

Let $H_0 \coloneqq \mathbb{S} \times_{\mathbb{G}_{m,\mathbb{R}}} GL_{2,\mathbb{R}}$ be the pullback along the two morphisms $N : \mathbb{S} \to \mathbb{G}_{m,\mathbb{R}}$ and $\det : GL_{2,\mathbb{R}} \to \mathbb{G}_{m,\mathbb{R}}$. H_0 is called the reference group. It fits into the following diagram.

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{S}^1 \arrow[r] \arrow[d, "\mathrm{id}"'] & H_0 = \mathbb{S} \times_{\mathbb{G}_{m,\mathbb{R}}} GL_{2,\mathbb{R}} \arrow[r, "\mathrm{pr}_{GL_2}"] \arrow[d, two heads, "\mathrm{pr}_{\mathbb{S}}"'] & GL_{2,\mathbb{R}} \arrow[r] \arrow[d, two heads, "\det"] & 1 \\
1 \arrow[r] & \mathbb{S}^1 \arrow[r] & \mathbb{S} \arrow[r, two heads, "N"] & \mathbb{G}_{m,\mathbb{R}} \arrow[r] & 1
\end{tikzcd}$$

Let $h_0 : \mathbb{S} \to H_0$ be the morphism with two components $\mathrm{id} : \mathbb{S} \to \mathbb{S}$ and $\mathbb{S} \to GL_{2,\mathbb{R}}, z = a + bi \mapsto \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. (Over $\mathbb{C}$, this is $(z_1, z_2) \mapsto \begin{bmatrix} \frac{z_1+z_2}{2} & -\frac{z_1-z_2}{2i} \\ \frac{z_1-z_2}{2i} & \frac{z_1+z_2}{2} \end{bmatrix}$). Also, define $h_\infty : \mathbb{S}_{\mathbb{C}} \to H_{0,\mathbb{C}}$ to be the morphism with two components $\mathrm{id}_{\mathbb{C}} : \mathbb{S}_{\mathbb{C}} \to \mathbb{S}_{\mathbb{C}}$ and $\mathbb{S}_{\mathbb{C}} \to GL_{2,\mathbb{C}}, (z_1, z_2) \mapsto \begin{bmatrix} z_1 z_2 & i(z_1 z_2 - 1) \\ 0 & 1 \end{bmatrix}$. It is readily checked that they are well-defined morphisms to H_0 and $H_{0,\mathbb{C}}$. Note that h_0 is a morphism over $\mathbb{R}$ whereas h_∞ is a morphism over $\mathbb{C}$.

We observe that $\mathrm{pr}_{GL_2} \circ h_0 : \mathbb{S} \to GL_{2,\mathbb{R}}$ induces on $M_0 = \mathbb{R}^2$ a Hodge structure of type $\{(-1, 0), (0, -1)\}$ and $\mathrm{pr}_{GL_2} \circ h_\infty : \mathbb{S}_{\mathbb{C}} \to GL_{2,\mathbb{C}}$ induces a Hodge structure of type $\{(-1, -1), (0, 0)\}$. The two Hodge structures have the same Hodge filtration. For M_0, the only non-trivial step in the Hodge filtration is $F^0 V = \mathbb{C} \cdot \begin{bmatrix} -i \\ 1 \end{bmatrix}$. The weight spaces are as follows.

$$(M_0)^{0,-1}_{\mathrm{pr}_{GL_2}\circ h_0} = \mathbb{C}\cdot \begin{bmatrix} -i \\ 1 \end{bmatrix},$$

$$(M_0)^{-1,0}_{\mathrm{pr}_{GL_2}\circ h_0} = \mathbb{C}\cdot \begin{bmatrix} i \\ 1 \end{bmatrix} \text{ (the complex conjugate of } (M_0)^{0,-1}_{\mathrm{pr}_{GL_2}\circ h_0}),$$

$$(M_0)^{-1,-1}_{\mathrm{pr}_{GL_2}\circ h_\infty} = \mathbb{C}\cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ (follows from the definition)},$$

$$(M_0)^{0,0}_{\mathrm{pr}_{GL_2}\circ h_\infty} = \mathbb{C}\cdot \begin{bmatrix} -i \\ 1 \end{bmatrix}.$$

We can view the Hodge structure induced by $\mathrm{pr}_{GL_2}\circ h_\infty$ as a modification of that induced by $\mathrm{pr}_{GL_2}\circ h_0$. If we require the $(-1,-1)$-weight space of $\mathrm{pr}_{GL_2}\circ h_\infty$ to be spanned by $(1,0)$ and that the two Hodge structures have the same Hodge filtration, then $\mathrm{pr}_{GL_2}\circ h_\infty$ is uniquely determined by h_0.

Let $B \subset GL_2$ be the Borel subgroup of invertible upper-triangular matrices and $B_0 \subset H_0$ be the preimage of B under pr_{GL_2}. Define the cocharacter $\lambda_0 : \mathbb{G}_{m,\mathbb{C}} \to H_0^{\mathrm{der}} \cap B_0$ as $z \mapsto \begin{bmatrix} z^{-1} & i(z^{-1}-z) \\ 0 & z \end{bmatrix}$. Thus $h_\infty \circ \omega = (h_0\circ\omega)\cdot\lambda_0$: $\begin{bmatrix} z^{-2} & i(z^{-2}-1) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} z^{-1} & 0 \\ 0 & z^{-1} \end{bmatrix}\begin{bmatrix} z^{-1} & i(z^{-1}-z) \\ 0 & z \end{bmatrix}$. (Note our convention on ω inverts z).

In the next proposition, $N_0 = \mathbb{R}^2$ is the representation of H_0 via $H_0 \stackrel{\mathrm{pr}_{\mathbb{S}}}{\to} \mathbb{S} \to GL_{2,\mathbb{R}}$ where the second arrow induces the standard complex structure on $\mathbb{R}^2 = \mathbb{C}$ (i.e.

$\mathrm{pr}_{GL_2} \circ h_0$). We view M_0 as a representation of H_0 via pr_{GL_2}.

Proposition 4.4.1 ([Pin90, 4.4])**.** *Let $(P, \mathcal{X})$ be a mixed Shimura datum, $x \in \mathcal{X}$ and $\alpha : H_{0,\mathbb{C}} \to P_{\mathbb{C}}$ a morphism such that $\pi \circ \alpha : H_{0,\mathbb{C}} \to G_{\mathbb{C}}$ is defined over $\mathbb{R}$. Suppose $\alpha \circ h_0 = h_x$, then as a representation of H_0 via $\pi \circ \alpha$,*

(1) $(\mathrm{Lie}\, U)_{\mathbb{R}}$ *is a direct sum of copies of $\wedge^2 M_0$. In particular, under $\pi \circ \alpha \circ h_\infty$, the induced Hodge structure on* $\mathrm{Lie}\, U$ *is of type* $\{(-1, -1)\}$.

(2) $(\mathrm{Lie}\, V)_{\mathbb{R}}$ *is a direct sum of copies of M_0 and N_0. In particular, under $\pi \circ \alpha \circ h_\infty$, the induced Hodge structure on* $\mathrm{Lie}\, V$ *is of type* $\{(0, 0), (-1, 0), (0, -1), (-1, -1)\}$.

(3) $(\mathrm{Lie}\, G)_{\mathbb{R}}$ *is a direct sum of copies of $(N_0)^\vee \otimes N_0$, $(N_0)^\vee \otimes M_0$, $(M_0)^\vee \otimes M_0$. In particular, under $\pi \circ \alpha \circ h_\infty$, the induced Hodge structure on* $\mathrm{Lie}\, G$ *is of type* $\{(1, 1), (0, 1), (1, 0), (-1, 1), (0, 0), (1, -1), (-1, 0), (0, -1), (-1, -1)\}$.

This shows that under $\omega \circ h_\infty$, the new Hodge structure becomes more mixed.

The second step in the construction of a rational boundary component involves a certain kind of parabolic subgroups of P. To streamline the statement of the result, we now briefly divert to introduce some general facts on algebraic groups.

Let P be an algebraic group over a field k. Then k-parabolic subgroups of P correspond bijectively with k-parabolic subgroups of G^{ad}. By the structure theorem of semisimple algebraic groups, $G^{\mathrm{ad}} = \prod_{1 \le i \le r} G_i$ with each G_i a k-simple adjoint group. A k-parabolic subgroup $Q \subseteq P$ is called k-admissible if it corresponds to $\prod_{1 \le i \le r} Q_i$, where Q_i is either maximal proper parabolic or equal to G_i itself.

Consider a reductive algebraic group G over a field k. Let Q be a k-parabolic subgroup of G and W_Q its unipotent radical. Choose a maximal k-split torus $T \subset$

Q. Then $\operatorname{Lie} Q$ is a direct sum of root spaces of T. Let $\beta_1, \ldots, \beta_r \in X^*(T)$ (the character group of T) be the roots that appear in $\operatorname{Lie} W_Q$ and $\beta_1^*, \ldots, \beta_r^* \in Y_*(T)_{\mathbb{Q}}$ (the cocharacter group of T tensored with $\mathbb{Q}$) the dual of the $\beta_i (1 \leq i \leq r)$. Define λ to be the unique generator of $(\mathbb{Q} \cdot (\beta_1^* + \cdots + \beta_r^*)) \cap Y_*(T)$ that is a $\mathbb{Q}_{<0}$-multiple of $\beta_1^* + \cdots + \beta_r^*$. The cocharacter λ only depends on Q and T.

It is well-known that $\operatorname{Lie} Q$ is the direct sum of root spaces $(\operatorname{Lie} Q)_\beta$ such that $\beta \circ \lambda \leq 0$. Moreover, if $\pi_Q : Q \to Q/W_Q$ is the canonical projection, then $\pi_Q \circ \lambda$ maps into the center of Q/W_Q. We say that λ *defines Q relative to T*. As maximal k-split tori of Q are conjugate under $Q(k)$, it can be checked that if $q \in Q(k)$, then $q \cdot \lambda = \operatorname{int}(q) \circ \lambda$ defines Q relative to $q \cdot T = qTq^{-1}$. Also, if $E \supset k$ is an extension field, then λ_E defines Q_E relative to any maximal E-split torus S containing T_E. Therefore, the field of definition of the $Q(E)$-conjugacy class of any such λ defined via a maximal E-split torus S is just k. In particular, if $Q \subseteq G$ is defined over $\mathbb{Q}$, then the $Q(\mathbb{C})$-conjugacy class of any such λ defined via a maximal $\mathbb{C}$-torus of $Q_{\mathbb{C}}$ has an element defined over $\mathbb{Q}$. And $\pi_Q \circ \lambda$ is defined over $\mathbb{Q}$.

We can now state the result relating $\mathbb{Q}$-admissible parabolics to H_0.

Proposition 4.4.2 ([Pin90, 4.6]). *Let $(P, \mathcal{X})$ be a mixed Shimura datum and $Q \subseteq P$ a $\mathbb{Q}$-parabolic subgroup of P. Let $\pi_U : P \to P/U$ be the projection. The following are equivalent.*

(1) Q is $\mathbb{Q}$-admissible.

(2) There exists an $x \in \mathcal{X}$ and a homomorphism $\alpha_x : H_{0,\mathbb{C}} \to P_{\mathbb{C}}$ satisfying the following conditions.

(i) $\pi_U \circ \alpha_x : H_{0,\mathbb{C}} \to (P/U)_{\mathbb{C}}$ *is defined over* $\mathbb{R}$*.*

(ii) $h_x = \alpha_x \circ h_0$*.*

(iii) $\pi \circ \alpha_x \circ h_\infty \circ \omega = \mu \cdot \lambda$*, where* $\mu = \pi \circ h_x \circ \omega$ *maps into* $Z(G)_{\mathbb{C}}$ *and* $\lambda = \pi \circ \alpha_x \circ \lambda_0$ *is a cocharacter defining the parabolic subgroup* $(Q/W)_{\mathbb{C}}$ *relative to some maximal* $\mathbb{C}$*-torus* $T \subset (Q/W)_{\mathbb{C}}$*. Moreover,* $\pi_Q \circ \lambda : \mathbb{G}_{m,\mathbb{C}} \to (Q/W)_{\mathbb{C}} \to (Q/W_Q)_{\mathbb{C}}$ *is defined over* $\mathbb{Q}$ *(and maps into the center of* Q/W_Q*).*

(3) For every $x \in \mathcal{X}$*, there exists a unique* $\alpha_x : H_{0,\mathbb{C}} \to P_{\mathbb{C}}$ *as above.*

Remark 4.4.3.

(i) Note that $\pi \circ \alpha_x \circ h_\infty \circ \omega = \pi \circ \alpha_x \circ ((h_0 \circ \omega) \cdot \lambda_0) = (\pi \circ \alpha_x \circ h_0 \circ \omega) \cdot (\pi \circ \alpha_x \circ \lambda_0) = (\pi \circ h_x \circ \omega) \cdot (\pi \circ \alpha_x \circ \lambda_0)$.

(ii) Combined with Proposition 4.4.1, this implies that $\alpha_x \circ h_\infty$ and $\alpha_x \circ h_0 \circ \omega = h_x \circ \omega$ both factors through $Q_{\mathbb{C}}$. Further, $(\operatorname{Lie} Q)_{\mathbb{C}}$ is the direct sum of all non-positive weight spaces under $\alpha_x \circ h_\infty \circ \omega$.

Example 4.4.4. Take $(GL_{2,\mathbb{Q}}, \mathcal{H}_2)$ be the Siegel modular Shimura datum defined in Example 4.2.5. And let $x \in \mathcal{H}_2$ be the point corresponding to $\operatorname{pr}_{GL_2} \circ h_0$ defined at the beginning of this subsection. Then with respect to the $\mathbb{Q}$-admissible parabolic subgroup $B \subset GL_{2,\mathbb{Q}}$ of upper-triangular matrices, α_x is just $\operatorname{pr}_{GL_2} : H_{0,\mathbb{R}} \to GL_{2,\mathbb{R}}$, which is already defined over $\mathbb{R}$ and consistent with the fact that $GL_{2,\mathbb{Q}}$ is reductive. It is left to the reader to verify that under $\alpha_x \circ h_\infty : (z_1, z_2) \mapsto \begin{bmatrix} z_1 z_2 & i(z_1 z_2 - 1) \\ 0 & 1 \end{bmatrix}$, the Lie algebra of $B \subset GL_{2,\mathbb{Q}}$ is the direct sum of subspaces of non-positive weights. (Actually, $(\operatorname{Lie} GL_{2,\mathbb{Q}})_{\mathbb{C}}$ is of type $\{(1,1),(0,0),(-1,-1)\}$.) Note that $\alpha_x \circ \lambda_0 : t \mapsto$

$\begin{bmatrix} t^{-1} & i(t^{-1}-t) \\ 0 & t \end{bmatrix}$ maps into $B_{\mathbb{C}}$ and is defined over $\mathbb{Q}$ modulo its unipotent radical $\left\{ \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \right\}$. It defines $B_{\mathbb{C}}$ relative to the maximal $\mathbb{C}$-torus

$$\left\{ \begin{bmatrix} x & i(x-y) \\ 0 & y \end{bmatrix} = \begin{bmatrix} 1 & -i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \begin{bmatrix} 1 & -i \\ 0 & 1 \end{bmatrix}^{-1} \middle| x, y \in \mathbb{C}^{\times} \right\}$$

Further, the smallest $\mathbb{Q}$-normal subgroup $P_1 \subset B$ containing the image of $\alpha_x \circ h_\infty$ consists of matrices $\left\{ \begin{bmatrix} * & * \\ 0 & 1 \end{bmatrix} \right\}$. Indeed, this is isomorphic to the group $\mathbb{U}_0 \rtimes \mathbb{G}_{m,\mathbb{Q}}$ as defined in Example 4.2.11, where $\mathbb{G}_{m,\mathbb{Q}}$ acts by multiplication on the first factor.

Let $(P, \mathcal{X})$ be a mixed Shimura datum and $Q \subseteq P$ a $\mathbb{Q}$-admissible parabolic subgroup. Each h_x now corresponds to an α_x. If $q \in Q(\mathbb{R}) \cdot U(\mathbb{C})$, then $\operatorname{int}(q) \circ \alpha_x = \alpha_{q \cdot x}$ by the uniqueness characterization above. Define $P_1 \trianglelefteq Q$ to be smallest $\mathbb{Q}$-normal subgroup through which $\alpha_x \circ h_\infty$ factors. By conjugacy, it follows that all $\alpha_x \circ h_\infty$ factor through P_1. We have seen that the assignment $\mathcal{X} \to \operatorname{Hom}(\mathbb{S}_{\mathbb{C}}, P_{1,\mathbb{C}}), x \mapsto \alpha_x \circ h_\infty$ is $Q(\mathbb{R}) \cdot U(\mathbb{C})$-equivariant.

Lemma 4.4.5 ([Pin90, 4.7-4.11])**.** *Let W_1 be the unipotent radical of P_1.*

(1) Each $\alpha_x \circ h_\infty$ induces a rational mixed Hodge structure on $\operatorname{Lie} Q$ *with $W_1 = \exp(W_{-1}(\operatorname{Lie} Q))$.*

(2) Define $U_1 := \exp(W_{-2}(\operatorname{Lie} Q))$. ($U_1$ is normal in Q.) The groups P_1, W_1, U_1, and the homomorphisms $\{\alpha_x \circ h_\infty\}_{x \in \mathcal{X}}$ satisfy axioms A(2)-A(8) in Definition

4.2.1. This set of data only depends on $(P, \mathcal{X})$ and Q.

(3) The assignment $\mathcal{X} \to \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{1,\mathbb{C}}), x \mapsto \alpha_x \circ h_\infty$ is $Q(\mathbb{R}) \cdot U(\mathbb{C})$-equivariant. In addition, it maps every connected component $\mathcal{X}^0 \subset \mathcal{X}$ to a $P_1(\mathbb{R})^0 \cdot U_1(\mathbb{C})$-orbit.

Remark 4.4.6.

(1) It follows from (1) that $W_Q = W_1 \cdot W$ and the weight decomposition on $\mathrm{Lie}\, W_Q$ induced by $\alpha_x \circ h_\infty$ is of type $\{(0,0), (-1,0), (0,-1), (-1,-1)\}$.

(2) By the proof of [Pin90, 4.7], $Q(\mathbb{R}) \cdot U(\mathbb{C})$ acts transitively on $\mathcal{X}$. Thus (3) says the map sends every $Q(\mathbb{R})^0 \cdot U(\mathbb{C})$-orbit to a $P_1(\mathbb{R})^0 \cdot U_1(\mathbb{C})$-orbit.

(3) We could have replaced our h_0 with any other morphism having the same $\mathrm{pr}_{\mathbb{S}} \circ h_0$ and such that $\mathrm{pr}_{GL_2} \circ h_0$ induces a Hodge structure of type $\{(-1,0), (0,-1)\}$ on M_0. (Recall that h_∞ is uniquely determined by h_0.) The resulting P_1, W_1, U_1 remains the same.

The map $\mathcal{X} \to \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{1,\mathbb{C}}), x \mapsto \alpha_x \circ h_\infty$ can map multiple $\mathcal{X}^0$'s to the same $P_1(\mathbb{R})^0 \cdot U_1(\mathbb{C})$-orbit. We wish to separate their targets apart so that distinct connected components have distinct images. More precisely, we consider the following map (in which we denote the connected component containing x as $[x]$ and let $P_1(\mathbb{R}) \cdot U_1(\mathbb{C})$ acts on $\pi_0(\mathcal{X})$ via $P_1(\mathbb{R})$).

$$\begin{aligned} \zeta : \mathcal{X} &\longrightarrow \pi_0(\mathcal{X}) \times \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{1,\mathbb{C}}) \\ x &\longmapsto ([x], \alpha_x \circ h_\infty) \end{aligned}$$

The image of ζ is contained in finitely many $P_1(\mathbb{R}) \cdot U_1(\mathbb{R})$-orbits and ζ by definition is injective on connected components. (After all, there are finitely many connected components in $\mathcal{X}$.) Let $\mathcal{X}_1$ be such an orbit (containing, say, $\zeta(x)$ for some $x \in \mathcal{X}$) and $\mathcal{X}^+_{(P_1,\mathcal{X}_1)} \coloneqq \zeta^{-1}(\mathcal{X}_1) \subseteq \mathcal{X}$ be its preimage. Then $\mathcal{X}^+_{(P_1,\mathcal{X}_1)} \to \mathcal{X}_1$ is bijective on connected components and $\mathrm{Stab}_{Q(\mathbb{R})}(\mathcal{X}_1) \cdot U(\mathbb{C})$-equivariant. The projection onto the second factor induces a $P_1(\mathbb{R}) \cdot U_1(\mathbb{C})$-equivariant map $\mathcal{X}_1 \to P_1(\mathbb{R}) \cdot U_1(\mathbb{C}) \cdot (\alpha_x \circ h_\infty)$ with finite fibers.

Definition 4.4.7 (rational boundary components)**.** Given mixed Shimura datum $(P, \mathcal{X})$ and $\mathbb{Q}$-admissible parabolic subgroup Q, a $(P_1, \mathcal{X}_1)$ thus constructed is called a *rational boundary component* of $(P, \mathcal{X})$.

With fixed $(P, \mathcal{X})$, Q and P_1, choosing an $\mathcal{X}_1$ amounts to choosing a $P_1(\mathbb{R})$-orbit in $\pi_0(\mathcal{X})$: the $P_1(\mathbb{R}) \cdot U_1(\mathbb{C})$-orbit generated by its image form such a $\mathcal{X}_1$. **Note that by construction, a rational boundary component is always irreducible.**

Let's collect some of the properties of $(P_1, \mathcal{X}_1)$.

Proposition 4.4.8 ([Pin90, 4.12, 4.13, 4.19])**.**

(1) $(P_1, \mathcal{X}_1)$ *is an irreducible mixed Shimura datum equipped with a map* $\mathcal{X}^+_{(P_1,\mathcal{X}_1)} \to \mathcal{X}_1$ *that is* $\mathrm{Stab}_{Q(\mathbb{R})}(\mathcal{X}_1) \cdot U(\mathbb{C})$*-equivariant and bijective on connected components.*

In particular, $\mathcal{X}^+_{(P_1,\mathcal{X}_1)} \to \mathcal{X}_1$ *is* $P_1(\mathbb{R}) \cdot Q(\mathbb{R})^0 \cdot U(\mathbb{C})$*-equivariant.*

(2) The map $\mathcal{X}^+_{(P_1,\mathcal{X}_1)} \to \mathcal{X}_1$ *is injective and holomorphic.*

(3) Given $x \in \mathcal{X}^+_{(P_1,\mathcal{X}_1)}$*, let* $x_1 \in \mathcal{X}_1$ *be its image. Suppose* $\rho : P \to GL_V$ *is a* $\mathbb{Q}$*-representation of* P*. Then the Hodge structures on* V *induced by* $\rho \circ h_x$ *and* $\rho \circ h_{x_1} = \rho \circ \alpha_x \circ h_\infty$ *have the same Hodge filtration.*

(4) Suppose $(P_2, \mathcal{X}_2)$ is a rational boundary component of $(P_1, \mathcal{X}_1)$ associated with a $\mathbb{Q}$-admissible parabolic subgroup $Q_{12} \subseteq P_1$. Then there exists a $\mathbb{Q}$-admissible parabolic subgroup $Q_2 \subseteq P$ such that $Q_{12} = Q_2 \cap P_1$ and $(P_2, \mathcal{X}_2)$ is the rational boundary component of $(P.\mathcal{X})$ associated with Q_2. Further, $\mathcal{X}^+_{(P_2,\mathcal{X}_2)}$ is contained in $\mathcal{X}^+_{(P_1,\mathcal{X}_1)}$ and mapped under $\mathcal{X}^+_{(P_1,\mathcal{X}_1)} \to \mathcal{X}_1$ into $(\mathcal{X}_1)^+_{(P_2,\mathcal{X}_2)}$. The composite map $\zeta^{\mathcal{X}_2}_{\mathcal{X}_1} \circ \zeta^{\mathcal{X}_1}_{\mathcal{X}} : \mathcal{X}^+_{(P_2,\mathcal{X}_2)} \to (\mathcal{X}_1)^+_{(P_2,\mathcal{X}_2)} \to \mathcal{X}_2$ is the map $\zeta^{\mathcal{X}_2}_{\mathcal{X}}$ defined for $(P_2, \mathcal{X}_2)$ as a rational boundary component of $(P, \mathcal{X})$.

Thus we have the following diagram.

$$
\begin{tikzcd}
\mathcal{X}^+_{(P_1,\mathcal{X}_1)} \arrow[r, tail, "\zeta^{\mathcal{X}_1}_{\mathcal{X}}"] & \mathcal{X}_1 & \\
\mathcal{X}^+_{(P_2,\mathcal{X}_2)} \arrow[u, tail] \arrow[r, tail] \arrow[d, equal] & (\mathcal{X}_1)^+_{(P_2,\mathcal{X}_2)} \arrow[u, tail] \arrow[r, tail, "\zeta^{\mathcal{X}_2}_{\mathcal{X}_1}"] & \mathcal{X}_2 \arrow[d, equal] \\
\mathcal{X}^+_{(P_2,\mathcal{X}_2)} \arrow[rr, "\zeta^{\mathcal{X}_2}_{\mathcal{X}}"] & & \mathcal{X}_2
\end{tikzcd}
$$

Loosely speaking, a rational boundary component of a rational boundary component is a rational boundary component. From now on, we will write $(P, \mathcal{X}) \dashrightarrow (P_1, \mathcal{X}_1) \dashrightarrow (P_2, \mathcal{X}_2)$ (or $(P, \mathcal{X}) \geq (P_1, \mathcal{X}_1) \geq (P_2, \mathcal{X}_2)$) to mean that $(P_1, \mathcal{X}_1)$ is a rational boundary component of $(P, \mathcal{X})$ and $(P_2, \mathcal{X}_2)$ is a rational boundary component of $(P_1, \mathcal{X}_1)$. We call this a chain of rational boundary components. (Note that $(P, \mathcal{X})$ itself may not be a real one if it is not irreducible.) By definition, we see that $U \subseteq U_1 \subseteq U_2$.

Example 4.4.9. Continuing Example 4.4.4, we've seen that for the rational boundary component $(P_1, \mathcal{X}_1)$ corresponding to the parabolic subgroup B, the group P_1 consists of matrices of the form $\begin{bmatrix} * & * \\ 0 & 1 \end{bmatrix}$. For $\mathcal{H}_2 \ni h_x : z = a + bi \mapsto \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, its image under

$$\mathcal{H}_2 \to \pi_0(\mathcal{H}_2) \times \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{1,\mathbb{C}}) = \{\pm 1\} \times \mathrm{Hom}(\mathbb{S}_{\mathbb{C}}, P_{1,\mathbb{C}})$$

is equal to

$$\mathcal{X}_1 \ni x_1 = (+1, h_{x_1} = \alpha_x \circ h_\infty : z_1 z_2 \mapsto \begin{bmatrix} z_1 z_2 & i(z_1 z_2 - 1) \\ 0 & 1 \end{bmatrix}).$$

Consider the adjoint representation of GL_2. The two Hodge structures induced by the two morphisms h_x and h_{x_1} both have Hodge filtration

$$\begin{aligned} F^1 (\mathrm{Lie}\, GL_2) &= \left\langle \begin{bmatrix} -i & 1 \\ 1 & i \end{bmatrix} \right\rangle, \\ F^0 (\mathrm{Lie}\, GL_2) &= \left\langle \begin{bmatrix} -i & 1 \\ 1 & i \end{bmatrix}, \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & i \end{bmatrix} \right\rangle, \\ F^{-1} (\mathrm{Lie}\, GL_2) &= (\mathrm{Lie}\, GL_2)_{\mathbb{C}}\,. \end{aligned}$$

We leave the details to the reader.

We now turn to the last significant ingredient. Again let $(P, \mathcal{X})$ be a mixed Shimura datum. Some $x_0 \in \mathcal{X}$ will have h_{x_0} already defined over $\mathbb{R}$. Since U is an abelian unipotent group over $\mathbb{Q}$, we may identify it with a $\mathbb{Q}$-vector space. Define

$U(\mathbb{R})(-1) \subset U(\mathbb{C})$ to be the subspace $(2\pi i)^{-1}U(\mathbb{R})$. Thus $U(\mathbb{C}) = U(\mathbb{R}) \times U(\mathbb{R})(-1)$. It follows that $P(\mathbb{R}) \cdot U(\mathbb{C}) = P(\mathbb{R}) \cdot U(\mathbb{R})(-1)$. Any other point $x \in \mathcal{X}$ is of the form $up \cdot x_0$ for some $up \in U(\mathbb{R})(-1) \cdot P(\mathbb{R})$. Therefore $u^{-1} \cdot x = p \cdot x_0$ is also defined over $\mathbb{R}$. Since $U(\mathbb{C})$ acts freely on $\mathcal{X}$, we have proved that for any point $x \in \mathcal{X}$, there exists a unique $u_x \in U(\mathbb{R})(-1)$ such that $h_{u_x^{-1} \cdot x} = u_x^{-1} \cdot h_x$ is defined over $\mathbb{R}$. The map

$$\mathrm{im}_{\mathcal{X}} : \mathcal{X} \longrightarrow U(\mathbb{R})(-1)$$
$$x \longmapsto u_x$$

is called the map of imaginary part. Let $P(\mathbb{R}) \cdot U(\mathbb{C}) = U(\mathbb{R})(-1) \rtimes P(\mathbb{R})$ act on $U(\mathbb{R})(-1) = \frac{U(\mathbb{R})(-1) \rtimes P(\mathbb{R})}{P(\mathbb{R})}$ via left multiplication. Thus $P(\mathbb{R})$ acts by conjugation and $U(\mathbb{R})(-1)$ acts by translation on itself. Then $\mathrm{im}_{\mathcal{X}}$ is $P(\mathbb{R}) \cdot U(\mathbb{C})$-equivariant. It is also compatible with a morphism of mixed Shimura data $(P, \mathcal{X}) \to (P', \mathcal{X}')$.

For a rational boundary component $(P_1, \mathcal{X}_1)$ of $(P, \mathcal{X})$ for the parabolic subgroup Q, we see that $\mathrm{im}_{\mathcal{X}_1}$ is $Q(\mathbb{R})^0 \cdot P_1(\mathbb{R}) \cdot U_1(\mathbb{C})$-equivariant.

The following proposition will be useful in the proof of our main theorem.

Proposition 4.4.10 ([Pin90, 4.15])**.** *Let $(P_1, \mathcal{X}_1)$ be a rational boundary component of $(P, \mathcal{X})$ with the map $\mathcal{X}^+_{(P_1, \mathcal{X}_1)} \to \mathcal{X}_1$ defined above. Let $\mathcal{X}^0$ be a connected component of $\mathcal{X}$ and $\mathcal{X}_1^0$ the corresponding connected component of $\mathcal{X}_1$.*

(1) $\mathcal{X}^+_{(P_1, \mathcal{X}_1)} \to \mathcal{X}_1$ is an open embedding.

(2) There exists an open convex cone $C(\mathcal{X}^0, P_1) \subset U_1(\mathbb{R})(-1)$ such that the image of $\mathcal{X}^0$ in $\mathcal{X}_1^0$ is the preimage of $C(\mathcal{X}^0, P_1)$ under $\mathrm{im}_{\mathcal{X}_1}|_{\mathcal{X}_1^0}$. Thus we have a

pullback diagram.

$$\begin{tikzcd} \mathcal{X}^0 \arrow[r, tail] \arrow[d, two heads] & \mathcal{X}_1^0 \arrow[d, two heads, "\mathrm{im}_{\mathcal{X}_1}|_{\mathcal{X}_1^0}"] \\ C(\mathcal{X}^0, P_1) \arrow[r, tail] & U_1(\mathbb{R})(-1) \end{tikzcd}$$

(3) $C(\mathcal{X}^0, P_1)$ *is an orbit under conjugation by* $Q(\mathbb{R})^0$ *and translation by* $U(\mathbb{R})(-1)$*. It is also invariant under translation by* $(U_1 \cap W)(\mathbb{R})(-1)$*.*

(4) Under $U_1(\mathbb{R})(-1) \to (U_1/(U_1 \cap W))(\mathbb{R})(-1)$*,* $C(\mathcal{X}^0, P_1)$ *projects onto a non-degenerate homogeneous self-adjoint open cone (in the sense of [AMRT10], Chapter II.)*

Proposition 4.4.11 ([Pin90, 4.21])**.** *Suppose we have a chain of rational boundary components* $(P, \mathcal{X}) \dashrightarrow (P_1, \mathcal{X}_1) \dashrightarrow (P_2, \mathcal{X}_2)$ *as above. Under the map* $\mathcal{X}^+_{(P_2,\mathcal{X}_2)} \to (\mathcal{X}_1)^+_{(P_2,\mathcal{X}_2)} \to (P_2, \mathcal{X}_2)$*, let* $\mathcal{X}^0 \rightarrowtail \mathcal{X}_1^0 \rightarrowtail \mathcal{X}_2^0$ *be a chain of corresponding connected components. We have* $U \subseteq U_1 \subseteq U_2$*.*

(1) $C(\mathcal{X}_1^0, P_2) = C(\mathcal{X}^0, P_2) + U_1(\mathbb{R})(-1)$*.*

(2) $C(\mathcal{X}^0, P_1)$ *is contained in the closure of* $C(\mathcal{X}^0, P_2)$*.*

(3) If $(P, \mathcal{X}) \dashrightarrow (P_1', \mathcal{X}_1') \dashrightarrow (P_2, \mathcal{X}_2)$ *is another chain of rational boundary components, then viewing* $C(\mathcal{X}^0, P_1)$ *and* $C(\mathcal{X}^0, P_1')$ *as subsets of* $U_2(\mathbb{R})(-1)$*, we have*

$$C(\mathcal{X}^0, P_1) \cap C(\mathcal{X}^0, P_1') \neq \varnothing \iff (P_1', \mathcal{X}_1') = (P_1, \mathcal{X}_1).$$

Remark 4.4.12. From (3) above, in particular, if $(P_1, \mathcal{X}_1) \dashrightarrow (P_2, \mathcal{X}_2)$ are distinct rational boundary components of $(P, \mathcal{X})$, then $C(\mathcal{X}^0, P_1) \cap C(\mathcal{X}^0, P_2) = \varnothing$.

The disjoint $C(\mathcal{X}^0, P_1)$ can be combined to define another cone. Let $(P_2, \mathcal{X}_2)$ be a rational boundary component of $(P, \mathcal{X})$. Fix $\mathcal{X}^0 \subset \mathcal{X}^+_{(P_2, \mathcal{X}_2)}$. Then for any chain $(P, \mathcal{X}) \dashrightarrow (P_1, \mathcal{X}_1) \dashrightarrow (P_2, \mathcal{X}_2)$, $\mathcal{X}^0 \subset \mathcal{X}^+_{(P_1, \mathcal{X}_1)}$ and thus $C(\mathcal{X}^0, P_1)$ makes sense. Define

$$C^*(\mathcal{X}^0.P_2) \coloneqq \bigsqcup_{(P_1, \mathcal{X}_1)} C(\mathcal{X}^0, P_1) \subseteq U_2(\mathbb{R})(-1),$$

where $(P_1, \mathcal{X}_1)$ runs through all rational boundary components between $(P, \mathcal{X})$ and $(P_2, \mathcal{X}_2)$.

Proposition 4.4.13 ([Pin90, 4.22])**.** *Fix a chain of rational boundary components $(P, \mathcal{X}) \dashrightarrow (P_1, \mathcal{X}_1) \dashrightarrow (P_2, \mathcal{X}_2)$ and $\mathcal{X}^0 \subset \mathcal{X}^+_{(P_2, \mathcal{X}_2)}$. Let $\mathcal{X}^0 \rightarrowtail \mathcal{X}^0_1 \rightarrowtail \mathcal{X}^0_2$ be a chain of corresponding connected components.*

(1) $C^*(\mathcal{X}^0, P_2)$ is a convex cone. (In general neither open nor closed.)

(2) $C^*(\mathcal{X}^0_1, P_2) = C^*(\mathcal{X}^0, P_2) + U_1(\mathbb{R})(-1)$.

(3) $C^*(\mathcal{X}^0, P_1) = C^*(\mathcal{X}^0, P_2) \cap (U_1(\mathbb{R})(-1)) = C^*(\mathcal{X}^0, P_2) \cap ((U_2 \cap W_1)(\mathbb{R})(-1))$.

Example 4.4.14. Resuming Example 4.4.9, $\mathcal{H}_2$ has two connected components $\mathcal{H}_2^\pm$ by definition, containing morphisms $h_x : \mathbb{S} \to GL_{2,\mathbb{R}}$ inducing positive definite (resp. negative definite) forms with respect to the alternating form $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Let's compute the cone $C(\mathcal{H}_2^\pm, P_1) \subseteq U_1(\mathbb{R})(-1) = \left\{ \begin{bmatrix} 1 & r \cdot (2\pi i)^{-1} \\ 0 & 1 \end{bmatrix} \middle| r \in \mathbb{R} \right\}$.

We know that for the h_x such that $h_x(i) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, its image in $\mathcal{X}_1$ is

$$\left(1, \alpha_x \circ h_\infty : (z_1, z_2) \mapsto \begin{bmatrix} z_1 z_2 & i(z_1 z_2 - 1) \\ 0 & 1 \end{bmatrix}\right).$$

But

$$\begin{bmatrix} z_1 z_2 & i(z_1 z_2 - 1) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 z_2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -i \\ 0 & 1 \end{bmatrix}^{-1}.$$

Thus $h_x \mapsto \alpha_x \circ h_\infty \overset{\mathrm{im}}{\mapsto} \begin{bmatrix} 1 & -i \\ 0 & 1 \end{bmatrix}$.

For the parabolic subgroup B, $B(\mathbb{R})^0 = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \middle| \, a, d > 0 \right\}$. By the $B(\mathbb{R})^0$-equivariance of the maps $\zeta_{\mathcal{H}_2}^{\mathcal{X}_1} : (\mathcal{H}_2)^+_{(P_1, \mathcal{X}_1)} \to \mathcal{X}_1$ and $\mathrm{im}_{\mathcal{X}_1} : \mathcal{X}_1 \twoheadrightarrow U_1(\mathbb{R})(-1)$, we see that if $q = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$, then the image of $q \cdot h_x$ under $\mathrm{im}_{\mathcal{X}_1} \circ \zeta_{\mathcal{H}_2}^{\mathcal{X}_1}$ is just

$$q \begin{bmatrix} 1 & -i \\ 0 & 1 \end{bmatrix} q^{-1} = \begin{bmatrix} 1 & (ad^{-1})(-i) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & (ad^{-1}) 2\pi (2\pi i)^{-1} \\ 0 & 1 \end{bmatrix}.$$

Hence $C(\mathcal{H}_2^+, P_1)$ is just the "positive" half axis while $C^*(\mathcal{H}_2^+, P_1)$, being the closure of $C(\mathcal{H}_2^+, P_1)$, is set of non-negative numbers.

Similarly, for the $h_{x'}$ such that $h_{x'}(i) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, since $h_{x'} = \mathrm{int}(b) \circ h_x$ where

$b = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, its image in $\mathcal{X}_1$ is $(-1, \alpha_{x'} \circ h_\infty)$, where the second factor is

$$\begin{aligned}
(z_1, z_2) \mapsto b &\begin{bmatrix} 1 & -i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 z_2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -i \\ 0 & 1 \end{bmatrix}^{-1} b^{-1} \\
&= b \begin{bmatrix} 1 & -i \\ 0 & 1 \end{bmatrix} b^{-1} b \begin{bmatrix} z_1 z_2 & 0 \\ 0 & 1 \end{bmatrix} b^{-1} b \begin{bmatrix} 1 & -i \\ 0 & 1 \end{bmatrix}^{-1} b^{-1} \\
&= \begin{bmatrix} 1 & i \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_1 z_2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -i \\ 0 & 1 \end{bmatrix}^{-1}.
\end{aligned}$$

Hence $\mathrm{im}_{\mathcal{X}_1}(\alpha_{x'} \circ h_\infty) = \begin{bmatrix} 1 & i \\ 0 & 1 \end{bmatrix}$ and the cone

$$C(\mathcal{H}_2^-, P_1) = \left\{ \begin{bmatrix} 1 & (ad^{-1})(i) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -(ad^{-1})2\pi(2\pi i)^{-1} \\ 0 & 1 \end{bmatrix} \middle| \, a, d > 0 \right\},$$

This is the "negative" half axis and $C^*(\mathcal{H}_2^-, P_1)$ is the set of non-positive numbers.

From above we see that the cone $C(\mathcal{X}^0, P_1)$ does depend on the connected component $\mathcal{X}^0$.

Definition 4.4.15. The *conical complex* of a mixed Shimura datum $(P, \mathcal{X})$ is defined as

$$\mathcal{C}(P, \mathcal{X}) := \bigsqcup_{\mathcal{X}^0 \subseteq \mathcal{X}} \left(\left(\bigsqcup_{\mathcal{X}^0 \subseteq \mathcal{X}^+_{(P_1, \mathcal{X}_1)}} C^*(\mathcal{X}^0, P_1) \right) \Big/ \sim \right),$$

where $\mathcal{X}^0 \subseteq \mathcal{X}$ runs over all connected components and the equivalence relation is generated by all (closed) embeddings $C^*(\mathcal{X}^0, P_1) \rightarrowtail C^*(\mathcal{X}^0, P_2)$ coming from chains $(P, \mathcal{X}) \dashrightarrow (P_1, \mathcal{X}_1) \dashrightarrow (P_2, \mathcal{X}_2)$.

Set-theoretically, $\mathcal{C}(P, \mathcal{X})$ is the disjoint union of all $C(\mathcal{X}^0, P_1)$. With the quotient topology, the image of each $C(\mathcal{X}^0, P_1)$ is a locally closed subset with closure the image of $C^*(\mathcal{X}^0, P_1)$. The canonical map $C^*(\mathcal{X}^0, P_1) \to \mathcal{C}(P, \mathcal{X})$ is a closed homeomorphism onto its image.

Just like the mixed Shimura variety $M^{K_f}(P, \mathcal{X})(\mathbb{C})$ is defined as the double quotient $P(\mathbb{Q})\backslash\mathcal{X} \times P(\mathbb{A}_f)/K_f$, the definition of toroidal compactifications will also use the product $\mathcal{C}(P, \mathcal{X}) \times P(\mathbb{A}_f)$ in some way. But before their construction, we still need some background on torus embeddings. This is the topic of the next section.

4.5 Torus embeddings

In this subsection, we briefly introduce the notion of torus embeddings and gather some propositions for later use. For a comprehensive reference, the reader can consult, e.g., [CLS11]. In [Pin90], the author also studies morphisms of torus embeddings as line bundles and an ampleness criterion for the construction of toroidal compactifications. We will mostly focus on the definitions and preliminary results.

Let V be a finite-dimensional $\mathbb{Q}$-vector space and $V^\vee$ its dual space. A subset σ of $V_\mathbb{R}$ is called a rational convex polyhedral cone if there exists a finite subset $\{I_i\}_{i\in I} \subset V^\vee$ such that $\sigma = \{v \in V_\mathbb{R} | \langle l_i, v\rangle \geq 0, \forall i \in I\}$. In other words, σ is the intersection of finitely many closed half-spaces defined over $\mathbb{Q}$. Equivalently, there exists a finite subset $\{v_i\}_{i\in I} \subset V$ such that $\sigma = \{v \in V_\mathbb{R} | v = \sum_{i\in I} x_i v_i, x_i \geq 0\}$.

Given such a σ defined by $\{l_i\}_{i\in I} \subset V^\vee$, a face τ of σ is a subset of the form $\sigma \cap_{i\in J} V(l_i)$, where $J \subseteq I$ and $V(l_i) = \{v \in V_\mathbb{R} | \langle l_i, v\rangle = 0\}$. So a face of σ is the zero locus of finitely many l_i on σ. A face of σ is also a rational convex polyhedral cone (by adding $\{-l_i\}_{i\in J}$ to the defining set). A proper face is a face not equal to σ. We write $\tau \le \sigma$ to mean τ is a face of σ. The interior σ^0 of σ is the complement of the union of all faces of σ. It is open in $\mathbb{R}\cdot\sigma$, but not open in $V_\mathbb{R}$ in general. A cone σ is *of top dimension* if $\mathbb{R}\cdot\sigma = V_\mathbb{R}$.

The *dual cone* of σ is the cone $\check{\sigma} \coloneqq \{l \in V_\mathbb{R}^\vee \,|\, l|_\sigma \ge 0\}$. The dual cone of a rational convex polyhedral cone is also a rational convex polyhedral cone and $\check{\check{\sigma}} = \sigma$. Moreover, $\tau \subseteq \sigma \iff \check{\tau} \supseteq \check{\sigma}$

Proposition 4.5.1. *$\{l_i\}_{i\in I}$ defines σ if and only if $\check{\sigma} = \sum_{i\in I} \mathbb{R}_{\ge 0}\cdot l_i$.*

Proof. Let $\sigma' = \sum_{i\in I} \mathbb{R}_{\ge 0}\cdot l_i$. Then $\{l_i\}_{i\in I} \subseteq \sigma' \subseteq \check{\sigma}$ and $(\{l_i\}_{i\in I})^\vee = \check{\sigma'} \supseteq \sigma$. Thus $\{l_i\}_{i\in I}$ defines $\sigma \iff (\{l_i\}_{i\in I})^\vee = \sigma \iff \check{\sigma'} = \sigma \iff \sigma' = \check{\sigma}$. $\square$

Proposition 4.5.2. *The following statements are equivalent for a rational convex polyhedral cone σ.*

(1) $\{0\}$ is a face of σ (in which case σ is said to be strongly convex*).*

(2) σ doesn't contain a non-trivial linear subspace of $V_\mathbb{R}$.

(3) $\check{\sigma}$ contains a basis of $V^\vee$.

Definition 4.5.3. Let $\Sigma \subseteq V_\mathbb{R}$ be a collection of rational convex polyhedral cones. Σ is called a partial convex polyhedral decomposition if the following three conditions hold.

(1) Every $\sigma \in \Sigma$ is strongly convex, i.e. $\{0\} \leq \sigma$.

(2) A face of every $\sigma \in \Sigma$ is still a cone in Σ.

(3) The intersection of two cones in Σ is a face of both cones.

(A collection of cones satisfying only the second and third conditions as above is also called a fan.) We say Σ is finite if it contains finitely many cones. Denote by $|\Sigma|$ the union of all cones in Σ. If $C \subset V_{\mathbb{R}}$ contains $|\Sigma|$, Σ is called a partial rational polyhedral decomposition of C. If $C = |\Sigma|$, then Σ is a complete rational polyhedral decomposition of C.

Next, let T be a split torus over a field of characteristic zero. We use $X^*(T)$ and $Y_*(T)$ to denote its character and cocharacter group, respectively. Then the affine ring of T is $k[X^*(T)]$. Let $\sigma \subseteq Y_*(T)_{\mathbb{R}}$ be a rational polyhedral convex cone. The *torus embedding* T_σ of T with respect to σ is defined to be $\operatorname{Spec} k[X^*(T) \cap \check{\sigma}]$. (In the rest of this article, we often use $X^*(T)_{\sigma \geq 0}$ to mean $X^*(T) \cap \check{\sigma}$.) The action of T on itself extends to T_σ. As $X^*(T) \cap \check{\sigma} \rightarrowtail X^*(T)$, we have a canonical T-equivariant map $T \to T_\sigma$. It is an open embedding if and only if σ is strongly convex. In this case, T_σ is a normal variety.

Example 4.5.4. Let $T = \mathbb{G}_{m,\mathbb{C}}^d$. Identify $Y_*(T)$ with $\mathbb{Z}^d$ and let $\sigma = \sum_{i=1}^d \mathbb{R}_{\geq 0} e_i$ where e_i is the i-th standard basis vector. Then $T \to T_\sigma$ is just $\mathbb{G}_{m,\mathbb{C}}^d \to \mathbb{A}_{\mathbb{C}}^d$.

A T-variety over k containing T as an open dense subvariety and for which the inclusion is T-equivariant is called a *toric variety.*

Theorem 4.5.5. *T_σ is a normal affine toric variety. And any normal affine toric variety is of the form T_σ for some strongly convex rational polyhedral cone $\sigma \subseteq Y_*(T)_{\mathbb{R}}$*

of top dimension.

Proof. Let $T'' \subseteq T$ be the subtorus such that $Y_*(T'')_{\mathbb{Q}}$ is the maximal $\mathbb{Q}$-linear subspace contained in σ and let $T \to T' = T/T''$ be the quotient map. Then σ is the preimage under $Y_*(T)_{\mathbb{R}} \to T_*(T')_{\mathbb{R}}$ of a strongly convex (rational!) polyhedral cone $\bar{\sigma}$ and $T_\sigma = T'_{\bar{\sigma}}$. The canonical map $T \to T_\sigma$ is just the composite map $T \to T' \to T'_{\bar{\sigma}}$. The rest follows from [CLS11, 1.3.5]. $\square$

A rational polyhedral convex cone $\sigma \subseteq Y_*(T)_{\mathbb{R}}$ is smooth with respect to the lattice $Y_*(T)$ if the monoid $Y_*(T) \cap \sigma$ is generated by a subset of a $\mathbb{Z}$-basis of $Y_*(T)$. The variety T_σ is smooth over k if and only if σ is smooth. A strongly convex cone is simplicial if its one-dimensional faces are linearly independent. If $f : T_1 \to T_2$ is a morphism of tori with convex cones $\sigma_1 \subseteq Y_*(T_1)_{\mathbb{R}}$, $\sigma_2 \subseteq Y_*(T_2)_{\mathbb{R}}$, then f extends to a (necessarily T_1-equivariant) diagram below if and only if $f_*(\sigma_1) \subseteq \sigma_2$.

$$\begin{tikzcd} T_1 \arrow[r, "f"] \arrow[d] & T_2 \arrow[d] \\ T_{1,\sigma_1} \arrow[r] & T_{2,\sigma_2} \end{tikzcd}$$

The action of T on T_σ has a unique closed orbit $T_\sigma^\sim$ defined by the ideal $I_\sigma = k[X^*(T)_{\sigma^0>0}]$. The affine ring of $T_\sigma^\sim$ is also isomorphic to $k[X^*(T)_{\sigma=0}]$. If σ is of top dimension, then $T_\sigma^\sim$ is just a point. We have a canonical surjective projection map $T_\sigma \to T_\sigma^\sim$ which fits into a diagram

$$\begin{tikzcd} T_\sigma^\sim \arrow[r, tail] \arrow[dr, "\sim"'] & T_\sigma \arrow[d, "\pi_\sigma"] \\ & T_\sigma^\sim \end{tikzcd}$$

The restriction $\pi_\sigma|_T : T \to T_\sigma$ makes $T_\sigma^\sim$ a quotient torus of T.

If $\tau \leq \sigma$ is a face, then $T_\tau \rightarrowtail T_\sigma$ is an open dense subvariety. The T-orbits in T_σ are just $\{T_\tau^\sim | \tau \leq \sigma\}$. Each $T_\tau^\sim$ is a locally closed subvariety. The minimal face corresponds to T itself. The closure of $T_\tau^\sim$ is the union of $T_\rho^\sim$ for all $\tau \leq \rho \leq \sigma$.

Now consider a partial rational polyhedral decomposition Σ of $Y_*(T)_\mathbb{R}$. As $\tau \leq \sigma \implies T_\tau \rightarrowtail T_\sigma$. It is not hard to see that the various T_σ for $\sigma \in \Sigma$ can be glued to form a variety T_Σ, equipped with open dense embeddings $T_\sigma \rightarrowtail T_\Sigma$ compatible with $T_\tau \rightarrowtail T_\sigma$. T_σ is the colimit of the diagram of morphisms $T_\tau \to T_\sigma$. T_σ is of finite type over k if and only if Σ is finite. It is smooth with $T_\sigma \backslash T$ a union of smooth divisors with normal crossings if and only if Σ consists of smooth cones and we say Σ is smooth in this case. T_σ is proper over k if and only if Σ is a finite and complete decomposition of $Y_*(T)_\mathbb{R}$.

Example 4.5.6. Let $T = \mathbb{G}_{m,\mathbb{C}}^2$. Identify $Y_*(T)$ with $\mathbb{Z} \times \mathbb{Z}$ and let Σ consist of all the cones spanned by $\pm e_1, \pm e_2$ where e_i is the i-th standard basis vector. In other words, the two-dimensional cones are the four quadrants. Then $T \to T_\Sigma$ is just $\mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}} \to \mathbb{P}^1_\mathbb{C} \times \mathbb{P}^1_\mathbb{C}$. (We view $\mathbb{P}^1_\mathbb{C}$ as $\mathbb{G}_{m,\mathbb{C}} \cup \{0\} \cup \{\infty\} = \mathbb{A}^1_\mathbb{C} \cup \{\infty\}$.)

The action of T extends to T_Σ. Again the T-orbits are in $1-1$ correspondence with the cones in Σ. Thus a T-invariant subset $S \subseteq T_\Sigma$ amounts to a subset $\Sigma_S \subseteq \Sigma$. S is open in T_Σ if and only if every face of a cone in Σ_S is still in it.

If $f : T_1 \to T_2$ is a morphism of tori and Σ_1, Σ_2 are partial rational polyhedral decompositions of $Y_*(T_1)_\mathbb{R}$, $Y_*(T_2)_\mathbb{R}$, respectively, then f extends to a (necessarily T_1-equivariant) morphism $T_{1,\Sigma_1} \to T_{2,\Sigma_2}$ if and only if for all $\sigma_1 \in \Sigma_1$, $f_*(\sigma_1) \subseteq \sigma_2$ for some $\sigma_2 \in \Sigma_2$. Therefore we have a similar diagram as below. A special case of this is that if Γ is a discrete group with a left action on T, then the Γ-action extends to T_Σ if and only if Σ is Γ-invariant, i.e. $\forall \gamma \in \Gamma, \forall \sigma \in \Sigma, \gamma_*(\sigma) \in \Sigma$.

$$\begin{tikzcd} T_1 \arrow[r, "f"] \arrow[d] & T_2 \arrow[d] \\ T_{1,\Sigma_1} \arrow[r] & T_{2,\Sigma_2} \end{tikzcd}$$

In the construction of toroidal compactifications, we will make use of the above result in the following situation: Let $f : T \to T'$ be an isogeny of split tori over k. Then under the isomorphism $Y_*(T)_{\mathbb{R}} \to T_*(T')_{\mathbb{R}}$, $\Sigma \subseteq Y_*(T)_{\mathbb{R}}$ is mapped to $\Sigma' \subseteq Y_*(T')_{\mathbb{R}}$. The kernel Γ of f acts on T and T' is just the quotient of T under this action. Further, the Γ-action extends to T_Σ. Under $T_\Sigma \to T'_{\Sigma'}$, $T'_{\Sigma'}$ becomes the scheme-theoretic quotient of the Γ-action.

Torus embeddings have relative versions. Let $X \to Y$ be a T-torsor over k and $\Sigma \subseteq Y_*(T)_{\mathbb{R}}$ a partial rational polyhedral decomposition. As the torsor is Zariski-locally trivial, it is isomorphic to $T \times V \to V$ for various open subsets $V \subseteq Y$. This gives a torus embedding $T \times V \rightarrowtail T_\Sigma \times V$ over V. Any two sections of X over V differ by a morphism $V \to T$. From this, it follows that the various torus embeddings can be glued to a global relative torus embedding $X \to X_\Sigma$ over Y, which is uniquely defined up to canonical isomorphism.

Relative torus embeddings have analogous functorial properties: Suppose $X_i \to Y_i$ is a T_i-torsor and $\Sigma_i \subseteq Y_*(T_i)_{\mathbb{R}}$ is a rational partial polyhedral decomposition, for $i = 1, 2$. Given a diagram as on the left below and a morphism $f : T_1 \to T_2$ such that

$f(\Sigma_1) \subseteq \Sigma_2$, the left diagram extends to a diagram as on the right T_1-equivariantly.

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d] & X_2 \arrow[d] \\ Y_1 \arrow[r] & Y_2 \end{tikzcd} \implies \begin{tikzcd} X_1 \arrow[r] \arrow[d, tail] & X_2 \arrow[d, tail] \\ X_{1,\Sigma_1} \arrow[r] \arrow[d] & X_{2,\Sigma_2} \arrow[d] \\ Y_1 \arrow[r] & Y_2 \end{tikzcd}$$

Finally, we introduce the "ord"-map. Let T be a $\mathbb{C}$-torus. Then the map ord_T is defined as follows:

$$\begin{aligned} \mathrm{ord}_T : \qquad T(\mathbb{C}) &\xleftarrow{\sim} Y_*(T) \otimes_{\mathbb{Z}} \mathbb{C}^\times \longrightarrow \quad Y_*(T) \otimes_{\mathbb{Z}} \mathbb{R}, \\ \prod_{j=1}^{d} \lambda_j(t_j) &\longleftarrow\!\shortmid \ \sum_{j=1}^{d} \lambda_j \otimes t_j \ \longmapsto \sum_{j=1}^{d} \lambda_j \otimes (\ln |t_j|). \end{aligned}$$

This map can be viewed as the quotient map of $T(\mathbb{C})$ by its maximal compact subgroup. More generally, suppose $X \to S$ is an analytic $T(\mathbb{C})$-torus over a complex space S. Let $K \subset T(\mathbb{C})$ be its maximal compact subgroup. Then $X/K \to S$ is a $T/K \xrightarrow{\sim} Y_*(T)_{\mathbb{R}}$-torsor. Since we can patch up local sections via a partition of unity, it can be smoothly trivialized. Any trivialization yields a map

$$\mathrm{ord}_{\mathrm{X}} : X \to X/K \xrightarrow{\sim} Y_*(T)_{\mathbb{R}} \times S \to Y_*(T)_{\mathbb{R}}.$$

Any two trivializations differ by a smooth map $S \to Y_*(T)_{\mathbb{R}}$ which can be lifted to $T(\mathbb{C})$ (e.g. via $\mathbb{R} \to \mathbb{C}^\times, t \mapsto e^t$). Thus ord_X is defined up to translation by a smooth map $X \to S \to T(\mathbb{C})$.

4.6 Toroidal compactifications

We are in a position to construct the toroidal compactifications of mixed Shimura varieties. The process bears some resemblance to the construction of the Baily-Borel compactification of pure Shimura varieties but is more involved.

Before we dive into the technical details, let's outline the rough idea behind this project. Let $(P, \mathcal{X})$ be a mixed Shimura datum and $K_f \subseteq P(\mathbb{A}_f)$ an open compact subgroup. We will see that the mixed Shimura variety $M^{K_f}(P, \mathcal{X})(\mathbb{C})$ can be viewed as the quotient $\mathcal{U}/\sim$ of a covering $\mathcal{U}$ obtained out of all rational boundary components $(P_1, \mathcal{X}_1)$ of $(P, \mathcal{X})$. This covering embeds as an open dense subset into a larger space: $\mathcal{U} \rightarrowtail \bar{\mathcal{U}}$. The equivalence relation $\sim$ extends to $\bar{\mathcal{U}}$ and the quotient $\bar{\mathcal{U}}/\sim$ is our desired toroidal compactification of $M^{K_f}(P, \mathcal{X})(\mathbb{C})$ with respect to a certain cone decomposition. The following diagram summarizes the situation.

$$\begin{tikzcd}
\mathcal{U} \arrow[r, tail] \arrow[d, two heads] & \bar{\mathcal{U}} \arrow[d, two heads] \\
\mathcal{U}/\sim \arrow[r, tail] & \bar{\mathcal{U}}/\sim
\end{tikzcd}$$

We begin by defining a K_f-admissible partial cone decomposition Σ on $\mathcal{C}(P, \mathcal{X}) \times P(\mathbb{A}_f)$ and the relevant decompositions associated with Σ. Then we construct the compactification in the case that Σ is concentrated on the unipotent fiber. Next we obtain the covering $\mathcal{U}$ of the mixed Shimura variety associated with all its rational boundary components and define the equivalence relation $\sim$ on this covering. This covering is then embedded into the larger covering $\overline{U}$ with the equivalence relation extended. At the end, we briefly mention the smoothness and completeness properties.

This subsection contains materials from Chapters 6 and 9 in [Pin90].

Definition 4.6.1. Let $(P, \mathcal{X})$ be a mixed Shimura datum and $K_f \subseteq P(\mathbb{A}_f)$ an open compact subgroup. Let Σ be a collection of subsets of $\mathcal{C}(P, \mathcal{X}) \times P(\mathbb{A}_f)$. For a rational boundary component $(P_1, \mathcal{X}_1)$ and $\mathcal{X}^0 \subseteq \mathcal{X}^+_{(P_1, \mathcal{X}_1)}$, define $\Sigma(\mathcal{X}^0, P_1, p_f) \coloneqq \{\sigma \in \Sigma \,|\, \sigma \subset -\mathcal{C}^*(\mathcal{X}^0, P_1) \times \{p_f\}\}$. Then Σ is called a *K_f-admissible partial cone decomposition for* $(P, \mathcal{X})$ if it satisfies the following conditions.

(1) $\Sigma = \bigcup_{(\mathcal{X}^0, P_1, p_f)} \Sigma(\mathcal{X}^0, P_1, p_f)$. In other words, any σ is contained in some $\Sigma(\mathcal{X}^0, P_1, p_f)$.

(2) Every $\Sigma(\mathcal{X}^0, P_1, p_f) \subseteq U_1(\mathbb{R})(-1) \times \{p_f\}$ is a partial rational polyhedral decomposition of $-C^*(\mathcal{X}^0, P_1) \times \{p_f\}$.

(3) Σ is invariant under right multiplication of K_f on the second factor.

(4) Σ is invariant under left multiplication of $P(\mathbb{Q})$ on both factors.

(5) For a fixed $(P_1, \mathcal{X}_1)$ and $\mathcal{X}^0 \subseteq \mathcal{X}^+_{(P_1, \mathcal{X}_1)}$, the subcollection $\bigcup_{p_f \in P(\mathbb{A}_f)} \Sigma(\mathcal{X}^0, P_1, p_f)$ is invariant under left multiplication of $P_1(\mathbb{A}_f)$ on the second factor.

Furthermore,

(6) Σ is *finite* if $P(\mathbb{Q}) \backslash \Sigma / K_f$ is finite.

(7) Σ is *complete* if it is finite and every $\Sigma(\mathcal{X}^0, P_1, p_f) \subseteq U_1(\mathbb{R})(-1) \times \{p_f\}$ is a complete rational polyhedral decomposition of $-C^*(\mathcal{X}^0, P_1) \times \{p_f\}$

(8) Σ is *smooth with respect to* K_f if for every rational boundary component $(P_1, \mathcal{X}_1)$ and $\mathcal{X}^0 \subseteq \mathcal{X}^+_{(P_1, \mathcal{X}_1)}$, $\Sigma(\mathcal{X}^0, P_1, p_f)$ is smooth with respect to the arithmetic lattice $\Gamma_{U_1}(-1) \subset U_1(\mathbb{Q})(-1)$, where Γ_{U_1} is the image under the projection $Z(P) \times U_1 \to U_1$ of the group $\big(\mathrm{Id}_{Z(P)(\mathbb{Q})}(\mathcal{X}) \times U_1(\mathbb{Q})\big) \cap p_f K_f p_f^{-1}$.

One can construct new admissible cone decompositions out of a given one. Let $(P, \mathcal{X})$ be a mixed Shimura datum, $K_f \subset P(\mathbb{A}_f)$ an open compact subgroup and Σ a K_f-admissible partial cone decomposition for $(P, \mathcal{X})$.

(1) For all $p_f \in P(\mathbb{A}_f)$, define

$$[\cdot p_f]^*\Sigma \coloneqq \{(u, p'_f \cdot p_f^{-1}) | (u, p'_f) \in \sigma \, \text{for some} \, \sigma \in \Sigma\}$$

In other words, $[\cdot p_f]^*\Sigma(\mathcal{X}^0, P_1, p'_f) = \Sigma(\mathcal{X}^0, P_1, p'_f p_f) \cdot p_f^{-1}$. This is a $p_f K_f p_f^{-1}$-admissible partial cone decomposition for $(P, \mathcal{X})$. It inherits the finiteness or completeness property of Σ if any.

(2) If $\varphi : (P, \mathcal{X}) \to (P.\mathcal{X})$ is an automorphism, then it induces an automorphism of $\mathcal{C}(P, \mathcal{X}) \times P(\mathbb{A}_f)$. Define $\varphi^*\Sigma \coloneqq \{\varphi^{-1}(\sigma) | \sigma \in \Sigma\}$. This is a $\varphi^{-1}(K_f)$-admissible partial cone decomposition and it inherits the finiteness or completeness property of Σ if any.

(3) For a rational boundary component $(P_1, \mathcal{X}_1)$ of $(P, \mathcal{X})$, we can obtain a $P_1(\mathbb{A}_f) \cap K_f$-admissible cone decomposition $\Sigma|_{(P_1, \mathcal{X}_1)}$ as follows: For all rational boundary components $(P_2, \mathcal{X}_2)$ and $\mathcal{X}_1^0 \subseteq (\mathcal{X}_1)^+_{(P_2, \mathcal{X}_2)}$, there corresponds a unique $\mathcal{X}^0 \subseteq \mathcal{X}^+_{(P_2, \mathcal{X}_2)} \subseteq \mathcal{X}^+_{(P_1, \mathcal{X}_1)}$, then

$$\Sigma|_{(P_1, \mathcal{X}_1)}(\mathcal{X}_1^0, P_2, p_{1,f}) \coloneqq \Sigma(\mathcal{X}^0, P_2, p_{1,f})$$

Note that $\Sigma|_{(P_1, \mathcal{X}_1)}$ inherits neither finiteness nor completeness property in general.

(4) The *restriction of* Σ *to the unipotent fiber* is defined to be

$$\Sigma^0 := \Sigma \cap \left(\cup_{(\mathcal{X}^0,(P_1,\mathcal{X}_1))}(-C(\mathcal{X}^0,P_1)) \times P(\mathbb{A}_f)\right),$$

where $(P_1,\mathcal{X}_1)$ runs over all improper rational boundary components and $\mathcal{X}^0$ runs over all connected components in $\mathcal{X}_1 \subseteq \mathcal{X}$. Thus

$$\bigcup_{(\mathcal{X}^0,(P_1,\mathcal{X}_1))} (-C(\mathcal{X}^0,P_1)) \times P(\mathbb{A}_f) = \bigcup_{(\mathcal{X}^0,(P_1,\mathcal{X}_1))} U(\mathbb{R})(-1) \times P(\mathbb{A}_f),$$

whence the name. In case $\Sigma = \Sigma^0$, Σ is said to be concentrated on the unipotent fiber.

Let $(P,\mathcal{X})$ be a mixed Shimura datum with neat open compact subgroup $K_f \subset P(\mathbb{A}_f)$ and Σ a K_f-admissible partial cone decomposition for $(P,\mathcal{X})$ concentrated on the unipotent fiber. Let $(P',\mathcal{X}') = (P,\mathcal{X})/U$ and $K'_f \subset P'(\mathbb{A}_f)$ be the image of K_f under the projection $\pi_U : (P,\mathcal{X}) \to (P',\mathcal{X}')$. Suppose Σ is a K_f-admissible partial cone decomposition for $(P,\mathcal{X})$ concentrated on the unipotent fiber. Fix $[x,p_f] \in M^{K_f}(P,\mathcal{X})(\mathbb{C})$. Then by the results from Subsection 4.3, the fiber over $\pi_U([x,p_f])$ is isomorphic to $\Gamma_U\backslash U(\mathbb{C})$, where Γ_U is the image of

$$\left(\mathrm{Id}_{Z(P)(\mathbb{Q})}(\mathcal{X}) \times U(\mathbb{Q})\right) \cap p_f K_f p_f^{-1}$$

under $Z(P) \times U \to U$, which is an arithmetic lattice. $\Gamma_U\backslash U(\mathbb{C})$ is an algebraic torus.

Regarding $C^\times$ as $\mathbb{C}/\mathbb{Z}(1)$ via the exponential map, its cocharacter group is

$$\mathrm{Hom}(\frac{\mathbb{C}}{\mathbb{Z}(1)}, \Gamma_U\backslash U(\mathbb{C})) = \mathrm{Hom}(\mathbb{Z}(1), \Gamma_U) = \Gamma_U(-1) \subset \Gamma_U \otimes \mathbb{R}(-1) = U(\mathbb{R})(-1)$$

Let $\mathcal{X}^0$ be the connected component containing x and $(P_1, \mathcal{X}_1)$ the unique improper boundary component containing $\mathcal{X}^0$. Since $\Sigma(\mathcal{X}^0, P_1, p_f) \subseteq U(\mathbb{R})(-1)$ is a rational partial polyhedral decomposition of $U(\mathbb{R})(-1)$, it defines a torus embedding of each fiber $\Gamma_U\backslash U(\mathbb{C})$ up to isomorphism. It follows that we have a relative torus embedding over $M^{K'_f}(P', \mathcal{X}')(\mathbb{C})$. Denote this torus embedding by $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$. We thus have a diagram.

$$\begin{tikzcd} M^{K_f}(P,\mathcal{X})(\mathbb{C}) \arrow[rr, tail] \arrow[dr, two heads, "\pi_U"'] & & M^{K_f}(P,\mathcal{X},\Sigma)(\mathbb{C}) \arrow[dl, two heads] \\ & M^{K'_f}(P',\mathcal{X}')(\mathbb{C}) & \end{tikzcd}$$

Neatness of K_f implies smoothness of $M^{K_f}(P, \mathcal{X})(\mathbb{C})$. In this case, $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ is smooth if and only if Σ is smooth with respect to K_f. If K_f is not neat, we define $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ to be the quotient of $M^{\tilde{K}_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ under the right action of $K_f/\tilde{K}_f$, where $\tilde{K}_f \trianglelefteq K_f$ is any neat open compact normal subgroup. This makes sense since we can extend the action of $K_f/\tilde{K}_f$ on $M^{\tilde{K}_f}(P, \mathcal{X})(\mathbb{C})$ to $M^{\tilde{K}_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ by the K_f-admissibility of Σ. The isomorphism class of the resulting $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C}) \to M^{K'_f}(P', \mathcal{X}')(\mathbb{C})$ does not depend on the choice of $\tilde{K}_f$.

Now we construct the covering $\mathcal{U}$ mentioned at the beginning of this section. Again let $(P, \mathcal{X})$ be a mixed Shimura datum with open compact subgroup $K_f \subset P(\mathbb{A}_f)$ and Σ a K_f-admissible partial cone decomposition. Suppose $(P_1, \mathcal{X}_1)$ is a rational boundary component of $(P, \mathcal{X})$ and $p_f \in P(\mathbb{A}_f)$. We know that there is an open embedding $\mathcal{X}^+_{(P_1,\mathcal{X}_1)} \rightarrowtail \mathcal{X}_1$, bijective on connected components and $P_1(\mathbb{R}) \cdot Q(\mathbb{R})^0 \cdot U(\mathbb{C})$-equivariant with Q the $\mathbb{Q}$-admissible parabolic subgroup for $(P_1, \mathcal{X}_1)$. Define $K^1_f \coloneqq P_1(\mathbb{A}_f) \cap p_f K_f p_f^{-1}$ and let $\mathcal{U}(P_1, \mathcal{X}_1, p_f) \coloneqq P_1(\mathbb{Q}) \backslash \mathcal{X}^+_{(P_1,\mathcal{X}_1)} \times P_1(\mathbb{A}_f)/K^1_f$. Since

$$P_1(\mathbb{Q}) \backslash \mathcal{X}^+_{(P_1,\mathcal{X}_1)} \times P_1(\mathbb{A}_f)/K^1_f \rightarrowtail P_1(\mathbb{Q}) \backslash \mathcal{X}_1 \times P_1(\mathbb{A}_f)/K^1_f,$$

$\mathcal{U}(P_1, \mathcal{X}_1, p_f) \rightarrowtail M^{K^1_f}(P_1, \mathcal{X}_1)(\mathbb{C})$ as an open subspace. Further, there is a map to $M^{K_f}(P, \mathcal{X})(\mathbb{C})$ as follows.

$$\begin{aligned} \beta(P_1, \mathcal{X}_1, p_f) : \mathcal{U}(P_1, \mathcal{X}_1, p_f) &\to M^{K_f}(P, \mathcal{X})(\mathbb{C}) \\ [x, p_{1,f}] \quad &\longmapsto \quad [x, p_{1,f} \cdot p_f] \end{aligned}$$

We then take $\mathcal{U}$ to be the disjoint union of $\mathcal{U}(P_1, \mathcal{X}_1, p_f)$ for all $(P_1, \mathcal{X}_1)$ and p_f.

$$\mathcal{U} \coloneqq \bigsqcup_{(P_1,\mathcal{X},p_f)} \mathcal{U}(P_1, \mathcal{X}_1, p_f).$$

The various $\beta(P_1, \mathcal{X}_1, p_f)$ combine to give a surjective map $\beta : \mathcal{U} \twoheadrightarrow M^{K_f}(P, \mathcal{X})(\mathbb{C})$. As explained in [Pin90, 6.10], β is a quotient map and $M^{K_f}(P, \mathcal{X})(\mathbb{C})$ inherits its structure of a complex space from that of $\mathcal{U}$. Let $\sim$ be the equivalence relation on $\mathcal{U}$ defined by this map. Since $M^{K_f}(P, \mathcal{X})(\mathbb{C})$ is Hausdorff, $\sim$ is a closed subset of $\mathcal{U} \times \mathcal{U}$.

Let $(P'_1, \mathcal{X}'_1, p'_f)$ be a pair similar to $(P_1, \mathcal{X}_1, p_f)$. One can find several basic morphisms $\mathcal{U}(P_1, \mathcal{X}_1, p_f) \to \mathcal{U}(P'_1, \mathcal{X}'_1, p'_f)$.

(1) If $(P'_1, \mathcal{X}'_1) = (P_1, \mathcal{X}_1)$ and $p'_f = p_f \cdot k_f$ for $k_f \in K_f$, then $K^1_f = {K^1_f}'$ and the

identity map on $M^{K^1_f}(P_1, \mathcal{X}_1)(\mathbb{C})$ induces

$$[\mathrm{id}] : \mathcal{U}(P_1, \mathcal{X}_1, p_f) \to \mathcal{U}(P_1, \mathcal{X}_1, p_f \cdot k_f).$$

(2) If $(P_1', \mathcal{X}_1') = (P_1, \mathcal{X}_1)$ and $p_f' = p_{1,f}^{-1} \cdot p_f$ for some $p_{1,f} \in P_1(\mathbb{A}_f)$, then right multiplication by $p_{1,f}$ induces

$$[\cdot p_{1,f}] : \mathcal{U}(P_1, \mathcal{X}_1, p_f) \to \mathcal{U}(P_1, \mathcal{X}_1, p_{1,f}^{-1} \cdot p_f).$$

(3) If $(P_1', \mathcal{X}_1') = \mathrm{int}(p)(P_1, \mathcal{X}_1)$ and $p_f' = p \cdot p_f$ for some $p \in P(\mathbb{A}_f)$, then $\mathrm{int}(p)$ also induces

$$[\mathrm{int}(p)] : \mathcal{U}(P_1, \mathcal{X}_1, p_f) \to \mathcal{U}(p \cdot P_1, p \cdot \mathcal{X}_1, p \cdot p_f).$$

(Here $p \cdot P_1$ means pP_1p^{-1}.)

(4) If $(P_1, \mathcal{X}_1)$ is a rational boundary component of $(P_1', \mathcal{X}_1')$ and $p_f' = p_f$, then $\mathcal{X}^+_{(P_1,\mathcal{X}_1)} \subseteq \mathcal{X}^+_{(P_1',\mathcal{X}_1')}$ and this inclusion map induces

$$\begin{aligned} \beta(P_1', \mathcal{X}_1', P_1, \mathcal{X}_1, p_f) : \mathcal{U}(P_1, \mathcal{X}_1, p_f) &\to \mathcal{U}(P_1', \mathcal{X}_1', p_f), \\ [x, p_{1,f}] &\mapsto [x, p_{1,f}]. \end{aligned}$$

Remark 4.6.2. Note that only the fourth type can possibly change the rational boundary component involved.

These maps are subject to certain commutativity laws (e.g. $[\mathrm{id}] \circ [\cdot p_{1,f}] = [\cdot p_{1,f}] \circ [\mathrm{id}]$). We will omit them and leave the verification to the reader. If $f : \mathcal{U}(P_1, \mathcal{X}_1, p_f) \to \mathcal{U}(P_1', \mathcal{X}_1', p_f')$ is one of the above four types of maps, then it is clear that $\beta(P_1', \mathcal{X}_1', p_f') \circ f = \beta(P_1, \mathcal{X}_1, p_f)$. Thus the equivalence relation on $\mathcal{U} \times \mathcal{U}$ generated by the graphs of these maps is contained in $\sim$. The lemma below states that the two equivalence relations indeed coincide.

Lemma 4.6.3. *Let $u \in \mathcal{U}(P_1, \mathcal{X}_1, p_f)$ and $u' \in \mathcal{U}(P_1', \mathcal{X}', P_f')$. Then $u \sim u'$ if and only if there exists a rational boundary component $(P_2, \mathcal{X}_2)$ of $(P, \mathcal{X})$, $p \in P(\mathbb{Q})$, $p_{2,f} \in P_2(\mathbb{A}_f)$ and $k_f \in K_f$ such that*

(1) $(P_1, \mathcal{X}_1)$ is a rational boundary component of $(P_2, \mathcal{X}_2)$;

(2) $(P_1', \mathcal{X}_1')$ is a rational boundary component of $\mathrm{int}(p)(P_2, \mathcal{X}_2)$;

(3) $p_f' = p \cdot p_{2,f} \cdot p_f \cdot k_f$;

(4) u and u' map to the same image under the following diagram.

$$\begin{tikzcd}
\mathcal{U}(P_1, \mathcal{X}_1, p_f) \ni u \arrow[d, "\beta(P_2,\mathcal{X}_2,P_1,\mathcal{X}_1,p_f)"'] & u' \in \mathcal{U}(P_1', \mathcal{X}_1', p_f') \arrow[d, "\beta(p\cdot P_2, p\cdot\mathcal{X}_2, P_1', \mathcal{X}_1', p_f')"] \\
\mathcal{U}(P_2, \mathcal{X}_2, p_f) & \mathcal{U}(p \cdot P_2, p \cdot \mathcal{X}_2, p_f') \\
\mathcal{U}(P_2, \mathcal{X}_2, p_{2,f} \cdot p_f) \arrow[u, "{[\cdot p_{2,f}]}", "\wr"'] \arrow[r, "{[\mathrm{id}]}", "\sim"'] & \mathcal{U}(P_2, \mathcal{X}_2, p_{2,f} \cdot p_f \cdot k_f) \arrow[u, "{[\mathrm{int}(p)]}"', "\wr"]
\end{tikzcd}$$

Given $\mathcal{U}$ and $\sim$, we would like to obtain a larger covering $\bar{\mathcal{U}}$ such that $\sim$ can be extended to a reasonably good equivalence relation on $\bar{\mathcal{U}}$. For this consider a rational boundary component $(P_1, \mathcal{X}_1)$ and $p_f \in P(\mathbb{A}_f)$. Define

$$\Sigma_{(P_1,\mathcal{X}_1,p_f)} \coloneqq \left(([\cdot p_f]^*\Sigma)\,|_{(P_1,\mathcal{X}_1)}\right)^0.$$

Thus

$$\begin{aligned}\Sigma_{(P_1,\mathcal{X}_1,p_f)}(\mathcal{X}_1^0, P_2, p_{1,f}) &= \left(\Sigma(\mathcal{X}^0, P_2, p_{1,f}\cdot p_f)p_f^{-1}\right)\bigcap(-C^*(\mathcal{X}_1^0, P_1))\times\{p_{1,f}\}\\ &= \left(\Sigma(\mathcal{X}^0, P_2, p_{1,f}\cdot p_f)p_f^{-1}\right)\bigcap U_1(\mathbb{R})(-1)\times\{p_{1,f}\},\end{aligned}$$

for all rational boundary components $(P_2, \mathcal{X}_2)$ of $(P_1, \mathcal{X}_1)$ and $\mathcal{X}^0 \rightarrowtail \mathcal{X}_1^0 \subseteq (\mathcal{X}_1)^+_{(P_2,\mathcal{X}_2)}$. $\Sigma_{(P_1,\mathcal{X}_1,p_f)}$ is a K_f^1-admissible cone decomposition for $(P_1, \mathcal{X}_1)$ concentrated on the unipotent fiber. We have

$$M^{K_f^1}(P_1, \mathcal{X}_1)(\mathbb{C}) \rightarrowtail M^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_{(P_1,\mathcal{X}_1,p_f)})(\mathbb{C})$$

as an open dense subset. Let $\overline{\mathcal{U}}(P_1, \mathcal{X}_1, p_f)$ be the interior of the closure of $\mathcal{U}(P_1, \mathcal{X}_1, p_f)$ $\subseteq M^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_{(P_1,\mathcal{X}_1,p_f)})(\mathbb{C})$, which contains $\mathcal{U}(P_1, \mathcal{X}_1, p_f)$ as an open dense subset (in the analytic topology).

Let $\sigma \in \Sigma_{(P_1,\mathcal{X}_1,p_f)}(\mathcal{X}_1^0, P_1, p_{1,f}) = \Sigma(\mathcal{X}^0, P_1, p_{1,f}\cdot p_f)p_f^{-1}$. We denote by

$$M^{K_f^1}(P_1, \mathcal{X}_1)(\mathbb{C})_\sigma \subset M^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_{(P_1,\mathcal{X}_1,p_f)})$$

the σ-stratum. (Strictly speaking, distinct connected components of $M^{K_f^1}(P_1, \mathcal{X}_1)(\mathbb{C})$ correspond to distinct $p_{1,f}$. However, since $\Sigma_{(P_1,\mathcal{X}_1,p_f)}$ is concentrated on the unipotent fiber and $(P_1, \mathcal{X}_1)$ is irreducible, the cone decomposition $\Sigma_{(P_1,\mathcal{X}_1,p_f)} \subset \mathcal{C}(P_1, \mathcal{X}_1)\times$ $P_1(\mathbb{A}_f)$ doesn't depend on the second factor.) In fact, as claimed in [Pin90, 6.13], the part $\bar{\mathcal{U}}(P_1, \mathcal{X}_1, p_f) \cap M^{K_f^1}(P_1, \mathcal{X}_1)(\mathbb{C})_\sigma$ has an explicit description. We will discuss this again in Subsection 5.2.

We take $\bar{\mathcal{U}} = \bigsqcup_{(P_1,\mathcal{X}_1,p_f)} \mathcal{U}(P_1, \mathcal{X}_1, p_f)$ to be the disjoint union of all $\bar{\mathcal{U}}(P_1, \mathcal{X}_1, p_f)$. It contains $\mathcal{U}$ as an open dense subset. The next step is to extend the equivalence relation $\sim$ to $\bar{\mathcal{U}}$. For this, we need to first extend the above four types of maps $\mathcal{U}(P_1, \mathcal{X}_1, p_f) \to \mathcal{U}(P_1', \mathcal{X}_1', p_f')$.

Each of the types (1), (2), and (3) maps is induced by an isomorphism $M^{K_f^1}(P_1, \mathcal{X}_1, p_f)$ $\to M^{K_f^{1\prime}}(P_1', \mathcal{X}_1', p_f')$ that maps $\Sigma_{(P_1,\mathcal{X}_1,p_f)}$ into $\Sigma_{(P_1',\mathcal{X}_1',p_f')}$. Thus this isomorphism extends to an isomorphism

$$M^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_{(P_1,\mathcal{X}_1,p_f)}) \to M^{K_f^{1\prime}}(P_1', \mathcal{X}_1', \Sigma_{(P_1',\mathcal{X}_1',p_f')})$$

and hence induces a canonical map $\bar{\mathcal{U}}(P_1, \mathcal{X}_1, p_f) \to \bar{\mathcal{U}}(P_1', \mathcal{X}_1', p_f')$.

To extend the type (4) map, consider $\Sigma_{(P_1',\mathcal{X}_1',p_f)}\big|_{(P_1,\mathcal{X}_1)}$. By definition,

$$
\begin{aligned}
&\Sigma_{(P_1',\mathcal{X}_1',p_f)}\big|_{(P_1,\mathcal{X}_1)}(\mathcal{X}_1^0, P_1, p_{1,f})\\
&= \left(\Sigma(\mathcal{X}^0, P_1, p_{1,f}\cdot p_f)p_f^{-1}\right) \bigcap U_1'(\mathbb{R})(-1) \times \{p_{1,f}\}\\
&= \left(\Sigma \cap \left((-C^*(\mathcal{X}^0, P_1)) \times \{p_{1,f}\cdot p_f\}\right)\right) p_f^{-1} \bigcap \left(U_1'(\mathbb{R})(-1) \times \{p_{1,f}\cdot p_f\}\right) p_f^{-1}\\
&= \left(\Sigma \bigcap (-C^*(\mathcal{X}^0, P_1)) \times \{p_{1,f}\cdot p_f\} \bigcap U_1'(\mathbb{R})(-1) \times \{p_{1,f}\cdot p_f\}\right) p_f^{-1}\\
&= \left(\Sigma \bigcap (-C^*(\mathcal{X}^0, P_1')) \times \{p_{1,f}\cdot p_f\}\right) p_f^{-1} \text{ (Proposition 4.4.13)}\\
&= \Sigma(\mathcal{X}^0, P_1', p_{1,f}\cdot p_f)p_f^{-1}\\
&= \Sigma_{(P_1',\mathcal{X}_1',p_f)}(\mathcal{X}_1'^0, P_1', p_{1,f}).
\end{aligned}
$$

It follows that $\beta(P_1', \mathcal{X}_1', P_1, \mathcal{X}_1, p_f) : \mathcal{U}(P_1, \mathcal{X}_1, p_f) \to \mathcal{U}(P_1', \mathcal{X}_1', p_f)$ extends to a map

$$\overline{\beta}(P_1', \mathcal{X}_1', P_1, \mathcal{X}_1, p_f) : \overline{\mathcal{U}}(P_1, \mathcal{X}_1, p_f) \cap M^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_{(P_1', \mathcal{X}_1', p_f)}\big|_{(P_1, \mathcal{X}_1)}) \to \overline{\mathcal{U}}(P_1', \mathcal{X}_1', p_f)$$

The image of this map is a union of connected components of $\overline{\mathcal{U}}(P_1', \mathcal{X}_1', p_f)$ (cf. [Pin90, 6.15]). We will denote the intersection $\overline{\mathcal{U}}(P_1, \mathcal{X}_1, p_f) \cap M^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_{(P_1', \mathcal{X}_1', p_f)}\big|_{(P_1, \mathcal{X}_1)})$ by $\overline{\mathcal{U}}(P_1, \mathcal{X}_1, p_f)_{\Sigma(P_1', \mathcal{X}_1', p_f)|_{(P_1, \mathcal{X}_1)}}$.

Now with these extended maps, we may define an extended equivalence relation on $\overline{\mathcal{U}} \times \overline{\mathcal{U}}$ as follows.

Lemma 4.6.4 ([Pin90, 6.17])**.** *Let $u \in \overline{\mathcal{U}}(P_1, \mathcal{X}_1, p_f)$ and $u' \in \overline{\mathcal{U}}(P_1', \mathcal{X}', P_f')$. Define $u \bar{\sim} u'$ if and only if there exists a rational boundary component $(P_2, \mathcal{X}_2)$ of $(P, \mathcal{X})$, $p \in P(\mathbb{Q})$, $p_{2,f} \in P_2(\mathbb{A}_f)$ and $k_f \in K_f$ such that*

(1) $(P_1, \mathcal{X}_1)$ is a rational boundary component of $(P_2, \mathcal{X}_2)$;

(2) $(P_1', \mathcal{X}_1')$ is a rational boundary component of $\mathrm{int}(p)(P_2, \mathcal{X}_2)$;

(3) $p_f' = p \cdot p_{2,f} \cdot p_f \cdot k_f$;

(4) u and u' map to the same image under the following diagram.

$$\begin{tikzcd}
\overline{\mathcal{U}}(P_1, \mathcal{X}_1, p_f)_{\Sigma(P_2, \mathcal{X}_2, p_f)|_{(P_1, \mathcal{X}_1)}} \ni u \arrow[d, "\overline{\beta}(P_2, \mathcal{X}_2, P_1, \mathcal{X}_1, p_f)"'] & u' \in \overline{\mathcal{U}}(P_1', \mathcal{X}_1', p_f')_{\Sigma(p \cdot P_2, p \cdot \mathcal{X}_2, p_f')|_{(P_1', \mathcal{X}_1')}} \arrow[d, "\overline{\beta}(p \cdot P_2, p \cdot \mathcal{X}_2, P_1', \mathcal{X}_1', p_f')"] \\
\overline{\mathcal{U}}(P_2, \mathcal{X}_2, p_f) & \overline{\mathcal{U}}(p \cdot P_2, p \cdot \mathcal{X}_2, p_f') \\
\overline{\mathcal{U}}(P_2, \mathcal{X}_2, p_{2,f} \cdot p_f) \arrow[u, "{[\cdot p_{2,f}]}", "\wr"'] \arrow[r, "{[\mathrm{id}]}", "\sim"'] & \overline{\mathcal{U}}(P_2, \mathcal{X}_2, p_{2,f} \cdot p_f \cdot k_f) \arrow[u, "{[\mathrm{int}(p)]}"', "\wr"]
\end{tikzcd}$$

This is indeed an equivalence relation on $\overline{\mathcal{U}} \times \overline{\mathcal{U}}$, containing $\sim \subset \mathcal{U} \times \mathcal{U}$ and *contained in the closure of* $\sim$ *in* $\overline{\mathcal{U}} \times \overline{\mathcal{U}}$.

Theorem 4.6.5 ([Pin90, 6.22-6.24])**.** *Let* $(P, \mathcal{X})$ *be a mixed Shimura datum,* $K_f \subset P(\mathbb{A}_f)$ *an open compact subgroup and* Σ *a* K_f*-admissible partial polyhedral decomposition for* $(P, \mathcal{X})$*. Let* $(P_1, \mathcal{X}_1)$ *be any rational boundary component of* $(P, \mathcal{X})$ *and* $Q \subseteq P$ *the* $\mathbb{Q}$*-admissible parabolic subgroup for* $(P_1, \mathcal{X}_1)$*.*

(1) The equivalence relation $\eqsim \subset \overline{\mathcal{U}} \times \overline{\mathcal{U}}$ *is the closure of* $\sim \subset \mathcal{U} \times \mathcal{U}$*. Thus the quotient space* $\overline{\mathcal{U}}/\eqsim$ *is Hausdorff.*

(2) $\overline{\mathcal{U}}/\eqsim$ *is locally isomorphic to an open subset* $\Delta_1 \backslash \mathcal{W}_1 \subset \Delta_1 \backslash \overline{\mathcal{U}}(P_1, \mathcal{X}_1, p_f)$*, where* Δ_1 *is an arithmetic subgroup of* $(Q/P_1)(\mathbb{Q})$ *that acts properly discontinuously and analytically. Therefore* $\overline{\mathcal{U}}/\eqsim$ *is a normal complex space which contains* $M^{K_f}(P, \mathcal{X})(\mathbb{C}) = \mathcal{U}/\sim$ *as an open dense subset.*

(3) If Σ *is concentrated on the unipotent fiber, it is canonically isomorphic to the* $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ *defined before.*

Define $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C}) \coloneqq \overline{\mathcal{U}}/\eqsim$*. This is the* ***toroidal compactification of*** $\boldsymbol{M^{K_f}(P, \mathcal{X})(\mathbb{C})}$ ***with respect to*** $\boldsymbol{\Sigma}$*.*

(4) $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ *is compact if* Σ *is a complete partial polyhedral decomposition of* K_f*.*

(5) If K_f *is neat and* Σ *is smooth with respect to* K_f*, then* $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ *is smooth.*

Remark 4.6.6. Let $\delta_Q \coloneqq \mathrm{Stab}_{Q(\mathbb{Q})}(\mathcal{X}_1) \cap P_1(\mathbb{A}_f) \cdot p_f \cdot K_f \cdot p_f^{-1}$. Since $P_1(\mathbb{A}_f)/K_f^1 \xrightarrow{\sim}$

$P_1(\mathbb{A}_f) \cdot p_f \cdot K_f/K_f$, $\Delta_Q \coloneqq \delta_{\mathbb{Q}}/P_1(\mathbb{Q})$ acts equivariantly on

$$\mathcal{U}(P_1, \mathcal{X}_1, p_f) \to M^{K_f^1}(P_1, \mathcal{X}_1)(\mathbb{C}) \to M^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_{(P_1,\mathcal{X}_1,p_f)})(\mathbb{C})$$

via left multiplication on both factors, it is not hard to see that Δ_1 acts through a composition of morphisms of type (1), (2) and (3) and hence this action extends to one on $\bar{\mathcal{U}}(P_1, \mathcal{X}_1, p_f)$. Therefore, the quotient map $\bar{\mathcal{U}}(P_1, \mathcal{X}_1, p_f) \to M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ factors through $\Delta_Q \backslash \bar{\mathcal{U}}(P_1, \mathcal{X}_1, p_f)$.

As usual, the toroidal compactification has certain functorial properties. Let $K_f', K_f \subset P(\mathbb{A}_f)$ be open compact subgroups with $K_f' \subseteq p_f K_f p_f^{-1}$ and Σ (resp. Σ') be a K_f-admissible (resp. K_f'-admissible) cone decomposition. Suppose that for all $\tau \in \Sigma'$, there exists $\sigma \in [\cdot p_f]^*\Sigma$ such that $\tau \subseteq \sigma$. Then we have a holomorphic map

$$[\cdot p_f] : M^{K_f'}(P, \mathcal{X}, \Sigma')(\mathbb{C}) \to \left(M^{p_f K_f p_f^{-1}}(P, \mathcal{X}, [\cdot p_f]^*\Sigma)(\mathbb{C}) \xrightarrow{\sim}\right) M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C}).$$

If $K_f' = p_f K_f p_f^{-1}$ and $\Sigma' = [\cdot p_f]^*\Sigma$, this is an isomorphism. If $K_f' = p_f K_f p_f^{-1}$ and $\Sigma' \subseteq [\cdot p_f]^*\Sigma$, this is an open embedding with the complement of the image a closed analytic subset. If $\Sigma' = \Sigma$ and $K_f' \trianglelefteq K_f$ is a normal subgroup, then the finite group K_f/K_f' acts from the right on $M^{K_f'}(P, \mathcal{X}, \Sigma)(\mathbb{C})$. The canonical projection $M^{K_f'}(P, \mathcal{X}, \Sigma)(\mathbb{C}) \to M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ is the quotient map under this action.

Consider a morphism of mixed Shimura data $\varphi : (P, \mathcal{X}) \to (P', \mathcal{X}')$ and open compact subgroups $K_f \subset P(\mathbb{A}_f), K_f' \subset P'(\mathbb{A}_f)$ with cone decompositions Σ, Σ' respectively, such that $\varphi(K_f) \subseteq K_f'$. Suppose that for all $\sigma \in \Sigma$, there exists a $\sigma' \in \Sigma'$

such that $\varphi(\sigma) \subseteq \sigma'$. Then we have an induced holomorphic map

$$[\varphi] : M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C}) \to M^{K'_f}(P', \mathcal{X}', \Sigma')(\mathbb{C}).$$

In case φ is an automorphism of $(P, \mathcal{X})$, $K'_f = \varphi(K_f)$ and $\Sigma = \varphi^*(\Sigma')$, $[\varphi]$ is an isomorphism.

The toroidal compactification $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ is related to the Baily-Borel compactification in the following way. According to [Pin90, 6.2], if $(P, \mathcal{X})$ is a pure Shimura datum with open compact $K_f \subset P(\mathbb{A}_f)$, then the Baily-Borel compactification of $M^{K_f}(P, \mathcal{X})(\mathbb{C})$ can be defined as

$$M^{K_f}(P, \mathcal{X})^*(\mathbb{C}) \coloneqq P(\mathbb{Q}) \backslash \mathcal{X}^* \times P(\mathbb{A}_f)/K_f$$

where $\mathcal{X}^* = \bigsqcup_{(P_1, \mathcal{X}_1)} \mathcal{X}_1/W_1$. $\mathcal{X}^*$ as a set is the disjoint union of $\mathcal{X}_1/W_1$ as $(P_1, \mathcal{X}_1)$ runs over all rational boundary components of $(P, \mathcal{X})$. It carries the Satake topology generated by

$$\bigsqcup_{(P_2, \mathcal{X}_2) \geq (P_1, \mathcal{X}_1)} \psi_2(\mathrm{im}^{-1}(\Lambda \cdot D) \cap \psi_1^{-1}(\mathcal{Y}))$$

where $\psi_i : \mathcal{X}_i \to \mathcal{X}_i/W_i$ is the projection, $\Lambda \in \mathbb{R}_{>0}$ and $D \subseteq C(\mathcal{X}^0, P_1)$ is a convex core. The sum runs through all rational boundary components $((P, \mathcal{X}) \dashrightarrow) \, (P_2, \mathcal{X}_2) \dashrightarrow (P_1, \mathcal{X}_1)$ We will not discuss the topology in detail but the notion of a core will be introduced in Subsection 5.1.

Proposition 4.6.7. *Let $(P, \mathcal{X})$, K_f and Σ be as above, with $\pi : (P, \mathcal{X}) \to (P, \mathcal{X})/W$ the canonical projection. Then the canonical map $[\pi] : \mathcal{U} \to M^{K_f}(P, \mathcal{X})(\mathbb{C}) \to$*

$M^{\pi(K_f)}(P/W, \mathcal{X}/W)(\mathbb{C})$ *extends uniquely to a holomorphic map*

$$[\pi]^* : \overline{\mathcal{U}} \to M^{\pi(K_f)}(P/W, \mathcal{X}/W)^*(\mathbb{C}).$$

This map factors through $\overline{\mathcal{U}}/\!\approx$*. In other words, we have a diagram as below.*

$$\begin{tikzcd}
\mathcal{U} \arrow[r, tail] \arrow[d, two heads] & \overline{\mathcal{U}} \arrow[d, two heads] \\
M^{K_f}(P, \mathcal{X})(\mathbb{C}) \arrow[r, tail] \arrow[d, two heads, "{[\pi]}"'] & M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C}) \arrow[d, "{[\pi]^*}"] \\
M^{\pi(K_f)}(P/W, \mathcal{X}/W)(\mathbb{C}) \arrow[r, tail] & M^{\pi(K_f)}(P/W, \mathcal{X}/W)^*(\mathbb{C})
\end{tikzcd}$$

We have now constructed the toroidal compactification, though we assumed there exists a K_f-admissible partial polyhedral decomposition Σ. Actually, the existence of a suitable Σ is a non-trivial question and in Chapters 8 and 9 of [Pin90] it takes considerable efforts to obtain one. We conclude this subsection by listing the main theorems therein.

Theorem 4.6.8 ([Pin90, 9.21])**.** *Let* $(P, \mathcal{X})$ *be a mixed Shimura datum and* $K_f \subset P(\mathbb{A}_f)$ *a neat open compact subgroup. There exists a* K_f*-admissible complete cone decomposition* Σ *for* $(P, \mathcal{X})$ *such that* $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ *is smooth, projective, and the complement* $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C}) \backslash M^{K_f}(P, \mathcal{X})(\mathbb{C})$ *is a union of smooth divisors with only normal crossings.*

If Σ' *is any* K_f*-admissible complete cone decomposition for* $(P, \mathcal{X})$*, then* Σ *can be chosen to be a refinement of* Σ'*.*

Theorem 4.6.9 ([Pin90, 9.24, 9.25])**.** *Let* $(P, \mathcal{X})$ *be a mixed Shimura datum and*

$K_f \subset P(\mathbb{A}_f)$ an open compact subgroup.

(1) The mixed Shimura variety $M^{K_f}(P,\mathcal{X})(\mathbb{C})$ admits a canonical structure of a normal quasi-projective algebraic variety over $\mathbb{C}$, such that all the functorial maps discussed in Subsection 4.3 become algebraic morphisms.

(2) If Σ is a complete K_f-admissible cone decomposition such that $M^{K_f}(P,\mathcal{X},\Sigma)(\mathbb{C})$ is projective, the open embedding $M^{K_f}(P,\mathcal{X})(\mathbb{C}) \rightarrowtail M^{K_f}(P,\mathcal{X},\Sigma)(\mathbb{C})$ is algebraic.

We denote by $M^{K_f}_{\mathbb{C}}(P,\mathcal{X})$ the normal complex algebraic variety above underlying $M^{K_f}(P,\mathcal{X})(\mathbb{C})$. Similarly, $M^{K_f}_{\mathbb{C}}(P,\mathcal{X},\Sigma)$ denotes the complex algebraic variety underlying $M^{K_f}(P,\mathcal{X},\Sigma)(\mathbb{C})$ if there exists one. When $(P,\mathcal{X})$ is a pure Shimura datum, $M^{K_f}_{\mathbb{C}}(P,\mathcal{X})^*$ denotes the normal projective variety underlying the Baily-Borel compactification $M^{K_f}(P,\mathcal{X})^*(\mathbb{C})$.

(3) $M^{K_f}_{\mathbb{C}}(P,\mathcal{X})$, $M^{K_f}_{\mathbb{C}}(P,\mathcal{X},\Sigma)$ (if exists) and $M^{K_f}_{\mathbb{C}}(P,\mathcal{X})^*$ are unique up to canonical isomorphism.

Theorem 4.6.10 ([Pin90, 9.21])**.** *Every mixed Shimura datum $(P,\mathcal{X})$ admits a canonical model (in the sense of [Pin90, 11.5]) over its reflex field $E(P,\mathcal{X})$.*

This finishes our quick tour into the theory of mixed Shimura varieties and their toroidal compactifications. We start the proof of our main theorem in the next section, after establishing several technical lemmas.

5 Proof of the main theorem

A crucial step in the proof of our main theorem is to demonstrate that some neighborhood of a certain stratum in a torus embedding maps into the toroidal compactification. For this purpose we need a few lemmas.

Since a toroidal compactification is constructed with respect to some K_f-admissible cone decomposition, we first study cones and cores in euclidean spaces. Our essential goal is to establish Lemma 5.2.2, which characterizes $\overline{\mathcal{U}}(P_1, \mathcal{X}_1, p_f)$ (and certain subspaces thereof) on each σ-stratum.

5.1 Cones and cores in euclidean spaces

The result below may be well-known but we produce a proof here for the convenience of the reader.

Lemma 5.1.1. *Let V be a finite-dimensional real vector space and $C \subseteq V$ be an open convex subset. Suppose $\pi : V \to W$ is a surjective linear map of $\mathbb{R}$-vector spaces. Denote by $\overline{C}$ the closure of C in V.*

(1) $\pi(C)$ is the interior of $\overline{\pi(C)}$ in W.

(2) In case C is an open convex cone, $\pi(\overline{C}) = \overline{\pi(C)}$.

Proof.

(1) $\pi(C)$ is an open convex subset of W as the projection is open and the image under a linear map of a convex subset is also convex. Thus we are reduced to showing that the interior of the closure of an open convex subset $D \subset \mathbb{R}^m$ is D

itself. Let y be a point in the interior of $\overline{D}$. Take $x \in D$ to be an arbitrary point. We may assume $y \neq x$. There exists an open ball neighborhood of y contained in $\overline{D}$. Thus by convexity, for $\epsilon > 0$ small enough, $z := x + (1+\epsilon)(y-x) \in \overline{D}$. Then y lies on the the segment connecting x and z. Any small open ball neighborhood $B_\delta \ni z$ contains a point w of D. We fix an open ball neighborhood B_x of x contained in D. As δ goes to zero, $w \to z$. Since the convex hull of $B_x \cup \{z\}$ contains y, so does the convex hull of $B_x \cup \{w\}$ if w is close enough to z. But this latter convex hull is contained in D by convexity.

(2) Since in this case $\pi(C)$ is an open convex cone, it is equal to the interior of $\overline{\pi(C)}$ and thus dense in $\overline{\pi(C)}$. We just have to show $\pi(\overline{C})$ is closed.

Consider the unit balls $B^{n+m} \subset \mathbb{R}^{n+m}$ and $B^m \subset \mathbb{R}^m$ respectively. Any convex cone is determined by its intersection with the unit ball. $\overline{C} \cap B^{n+m}$ is a closed and hence compact subset. Thus $\pi(\overline{C} \cap B^{n+m})$ is compact and hence closed. $\pi(\overline{C})$ is the cone spanned by $\pi(\overline{C} \cap B^{n+m})$ and is thus closed.

□

Before the next lemma, we introduce a new notion.

Definition 5.1.2. Let V be a finite-dimensional $\mathbb{Q}$ vector space. Fix a lattice $V(\mathbb{Z}) \subset V$. Let $C \subset V_{\mathbb{R}}$ be an open convex cone. Define $D_0 \subset C$ to be the convex closure of $C \cap V(\mathbb{Z})$. A subset $D \subset C$ is called a *core* if there exists $\lambda_0, \lambda_0' \in \mathbb{R}_{>0}$ such that $\lambda_0 \cdot D_0 \subseteq D \subseteq \lambda_0' \cdot D_0$.

It is not hard to see that $D_0 \supseteq \lambda_0 \cdot D_0$ for all $\lambda_0 \geq 1$. Thus it can be shown that the above definition doesn't depend on the choice of the lattice. Further, If $D \subset C$

is a core, then so are the following: the interior of D, the closure $\overline{D}$ of D, the convex closure of D, $\mathbb{R}_{\geq 1} \cdot D$ and $D + C$. Also, $C = \mathbb{R}_{>0} \cdot D$ (cf. [Pin90, 6.1]).

Remark 5.1.3. Observe that if D is convex, then $D = \mathbb{R}_{\geq 1} \cdot D$. If D is open, then $D + C = D + \overline{C}$. For $D + C$ and any $v \in C$, $D + C + v \subset D + C$. In this case, we say the core $D + C$ is C-invariant. Hence when D is open and C-invariant, we have $D + C = D + \overline{C} = D$. In fact, any open convex core D is C-invariant. (For all $v_D \in D$ and all $v \in C$, there exists a small enough $1 > \varepsilon_1 > 0$ and a large enough $\varepsilon_2 > 1$ such that $v_D + \varepsilon_1 \cdot v \in D$ and $v_D + \varepsilon_2 \cdot v \in D$. By convexity, $v_D + v \in D$. So $D + C \subseteq D \subseteq D + \overline{C} = D + C$.)

For the product space $\mathbb{R}^n \times \mathbb{R}^m$, we take the standard lattice $\mathbb{Z}^n \times \mathbb{Z}^m$ formed by integral points. For a subset $S \subseteq \mathbb{R}^l$, $\mathrm{Int}(S)$ denotes its interior. Define $\pi_n : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $\pi_m : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ be the projections onto the left and the right factor, respectively. For a subset $S \subset \mathbb{R}^n \times \mathbb{R}^m$, a subset $\sigma^0 \subset \mathbb{R}^n$ and a point $z_0 \in \mathbb{R}^m$, denote the level set $S \cap \mathbb{R}^n \times \{z_0\}$ by S_{z_0} while $S_{\sigma, z_0} \coloneqq S \cap \sigma^0 \times \{z_0\}$ and $S(\sigma^0) \coloneqq S \cap \sigma^0 \times \mathbb{R}^m$.

Lemma 5.1.4. *Let $C \subset \mathbb{R}^n \times \mathbb{R}^m$ be an open convex cone with $D \subset C$ an open convex core. Write $\overline{C}$ for the closure of C in $\mathbb{R}^n \times \mathbb{R}^m$ and σ^0 for the interior of the intersection $\overline{C} \cap (\mathbb{R}^n \times \{0\})$ in $\mathbb{R}^n \times \{0\}$. Suppose for the open convex cone $\sigma^0 \subset \mathbb{R}^n \times \{0\}$, either of the following two conditions are satisfied.*

(A) $C \cap \mathbb{R}^n \times \{0\} = \varnothing$.

(B) $\sigma^0 \subset C$.

Then

(1) *If* $C_{z_0} \neq \varnothing$, *then the open convex subset* $\pi_n(C_{\sigma^0, z_0}) \subset \sigma^0 \times \{0\}$ *generates the open cone* σ^0. *That is, if* $(v, z_0) \in C$, *then for any* $v_\sigma \in \sigma^0$, *there exists a* $\lambda > 0$ *such that* $(\lambda \cdot v_\sigma, z_0) \in C$.

In particular, $\pi_m(C) = \pi_m(C(\tau^0))$ *for any open convex cone* $\tau^0 \subseteq \sigma^0$.

In Case (B), $\pi_m(C) = \mathbb{R}^m$.

(2) *For all* $(v, z_0) \in C$ *and* $(v', 0) \in \overline{C} \cap \mathbb{R}^n \times \{0\}$, $(v, z_0) + \lambda(v', 0) \in C$ *for all* $\lambda \geq 1$.

D *generates* C *as an open convex cone, i.e. for all* $(v, z_0) \in C$ *there exists* $\lambda \geq 1$, *such that* $\lambda(v, z_0) \in D$.

Let $f : C \to \mathbb{R} \cup \{\pm\infty\}$ *be the function* $f(v, z) \coloneqq \inf\{t \mid (v, t \cdot z) \in D\}$. *(*$f(v, z) = \infty$ *if* $v + \mathbb{R} \cdot z$ *doesn't meet* D.*) We have*

(3) *Let* $(v_\sigma, z_0) \in C(\sigma^0)$. *For all bounded subset* $K \subset \mathbb{R}^n \times \{0\}$, *there exists a* $\mu > 0$ *such that, for all* $v' \in \mu \cdot v_\sigma + K$ *and all* $t \geq 1$, $f(t \cdot v', z_0) < \mu$.

(4) *If* $(v_0, z_0) \in D$, *for all* $v_\sigma \in \sigma^0$, *there exists a* $\lambda > 0$, *such that* $(\lambda v_\sigma, z_0) \in D$. *Equivalently,* $D_{z_0} \neq \varnothing$ *implies that* $\pi_n(D(\sigma^0))$ *generates the open cone* σ^0.

In particular, $\pi_m(D) = \pi_m(D(\tau^0))$ *for any open convex cone* $\tau^0 \subseteq \sigma^0$.

In Case (B), $\pi_m(D) = \mathbb{R}^m$.

(5) *If* $(v_\sigma, z_0) \in D(\sigma^0)$, *then the function* $f_{z_0}(\lambda) = f(\lambda v_\sigma, z_0)$ *with* $\lambda \geq 1$ *is a decreasing convex continuous function. In Case (A), it is positive and continuous for* $\lambda \geq 1$. *In Case (B), either* $f_{z_0}(\lambda) = -\infty$ *for all* $\lambda \geq 1$ *or* $f_{z_0}(\lambda) \neq -\infty$ *(and thus real-valued) for all* $\lambda \geq 1$.

In particular, $(\lambda v_\sigma, z_0) \in D$ *for all* $\lambda \geq 1$.

Proof. We first prove everything in Case (A).

(1) Denote the subset $\sigma^0 \times \mathbb{R}^m$ by $C(\sigma^0)$. Given $(v, z_0) \in C$, let $v_\sigma \in \sigma^0$ be any point. Consider the line segment $\zeta(t) = (1-t)(v_\sigma, 0) + t(v, z_0) = ((1-t)v_\sigma + tv, tz_0)(0 \leq t \leq 1)$ connecting the two points. As C is an open convex cone and $(v_\sigma, 0)$ lies on the boundary, $\zeta(t) \in C$ for all $0 < t \leq 1$. If $t > 0$ is sufficiently small, as σ^0 is open in $\mathbb{R}^n \times \{0\}$, $\zeta(t)$ will lie in $C(\sigma^0)$. Then $t^{-1}\zeta(t) \in C_{\sigma^0, z_0}$. Indeed, we have shown that, any v_σ can be approached by the open (convex) cone $\mathrm{Cone}(\pi_n(C_{\sigma^0,z_0})) = \mathbb{R}_{>0} \cdot \pi_n(\sigma^0 \times \{z_0\} \cap C)$. So σ^0 is contained in the closure of $\mathrm{Cone}(\pi_n(C_{\sigma^0,z_0}))$ and must be actually contained in $\mathrm{Cone}(\pi_n(C_{\sigma^0,z_0}))$ as any open convex cone is the interior of its closure. But $\mathrm{Cone}(\pi_n(C_{\sigma^0,z_0})) \subseteq \sigma^0$. Thus we have the equality as desired.

(2) As $\overline{C}$ is a closed convex cone, $\lambda(v', 0) \in \overline{C}$ and so $\frac{1}{2}(v, z_0) + \frac{1}{2}\lambda(v', 0) \in C$. The second assertion follows from the fact that $\lambda(v, z) \in \lambda_0 \cdot (\mathbb{Z}^n \times \mathbb{Z}^m) \subseteq D$ for some $\lambda \geq 1$ and all $(v, z) \in C$.

(3) Replace K with any closed ball centered at $(0, 0)$ containing K. Recall that for some fixed $\lambda_0, \lambda_0' \geq 0$, we have $\lambda_0 \cdot D_0 \subseteq D \subseteq \lambda_0' \cdot D_0$ where D_0 is the convex closure of $C \cap \mathbb{Z}^n \times \mathbb{Z}^m$. A subset of the form $v + \lambda_0(I^n \times I^m)$ (for I^n, I^m the unit cubes in $\mathbb{R}^n$, $\mathbb{R}^m$ respectively) is called a lattice cube if $v \in \lambda_0 \cdot (\mathbb{Z}^n \times \mathbb{Z}^m)$. Since C and σ^0 are open convex cones, we may choose $\mu > 0$ large enough so that (1) any lattice cube meeting $(\mu \cdot v_\sigma, \mu \cdot z_0) + K$ is contained in C and (2) any lattice cube on $\mathbb{R}^n \times \{0\}$ meeting $\mu \cdot v_\sigma + K$ is contained in σ^0. Now take any $v' \in \mu \cdot v_\sigma + K$ and any $t \geq 1$. As $(t \cdot v', \mu \cdot z_0) = (v', \mu \cdot z_0) + ((t-1)v', 0)$, any lattice cube meeting $(t \cdot v', \mu \cdot z_0)$ will also be contained in C (and thus in D) by (2) just

proved. In particular, $f(t \cdot v', \mu \cdot z_0) < 1$. But $f(t \cdot v', \mu \cdot z_0) = \mu^{-1} f(t \cdot v', z_0)$, and so $f(t \cdot v', \mu \cdot z_0) < \mu$ as needed.

(4) First note that since $D \subseteq \lambda_0' \cdot D_0 = \overline{\lambda_0' \cdot D_0}$ is convex and C doesn't meet $\mathbb{R}^n \times \{0\}$, $f(v, z)$ is a positive function on C and convex on the level set C_z.

Let $(v_0, z_0) \in D(\implies f(v_0, z_0) < 1 - \varepsilon$ for some $\varepsilon > 0$ small enough. Let $v_\sigma \in \sigma^0$. By (1), we may assume (after extending v_σ) that $(v_\sigma, z_0) \in C$. By (2) and the fact that σ^0 is an open convex cone, we may also assume that $v_\sigma' := v_\sigma - v_0 \in \sigma^0$.

Let K be a closed ball in $\mathbb{R}^n \times \{0\}$ centered at $(0, 0)$ containing $(v_0, 0)$. By (3), we may take $\lambda \geq 1$ large enough so that for all $v \in \lambda \cdot v_\sigma' + K$ and $t \geq 1$, $f(t \cdot v, z_0) \leq M$ for some fixed $M > 0$. Now that $t\lambda v_\sigma' + v_0$ is on the segment connecting $t\lambda v_\sigma'$ and $t(\lambda v_\sigma' + v_0)$, $f(t\lambda v_\sigma' + v_0, z_0) \leq M$ for all $t \geq 1$ as well.

Let $u_{t\lambda} = (1 - t\lambda)v_0 + t\lambda v_\sigma$. Then $u_{t\lambda} = t\lambda v_\sigma' + v_0$ and $f(u_{t\lambda}, z_0) \leq M$. However, $v_\sigma = \frac{1}{t\lambda} u_{t\lambda} + (1 - \frac{1}{t\lambda})v_0$. By convexity, we have, for all $t \geq 1$

$$
\begin{aligned}
f(v_\sigma, z_0) &= f(\frac{1}{t\lambda} u_{t\lambda} + (1 - \frac{1}{t\lambda})v_0, z_0) \\
&\leq \frac{1}{t\lambda} f(u_{t\lambda}, z_0) + (1 - \frac{1}{t\lambda}) f(v_0, z_0) \\
&\leq \frac{1}{t\lambda} M + (1 - \frac{1}{t\lambda})(1 - \varepsilon).
\end{aligned}
$$

Let $t \to \infty$, we obtain $f(v_\sigma, z_0) \leq 1 - \epsilon$. Thus $(v_\sigma, \delta z_0) \in D$ for some $1 - \epsilon < \delta < 1$. Then $(\delta^{-1} v_\sigma, z_0) \in D$.

(5) We've seen above that $f_{z_0}(\lambda)$ is convex positive. Suppose there exists $\lambda_1 < \lambda_2$

such that $f_{z_0}(\lambda_1) < f_{z_0}(\lambda_2)$, then for every $\lambda > \lambda_2$, $(\lambda, f_{z_0}(\lambda))$ must be above the line passing through $(\lambda_1, f_{z_0}(\lambda_1))$ and $(\lambda_2, f_{z_0}(\lambda_2))$, which has positive slope. This contradicts (3). Therefore f_{z_0} is decreasing. A decreasing convex function must be continuous.

We now turn to the proof in Case (B).

(1) Take $(v, z_0) \in C$. For any $v_\sigma \in \sigma^0 \subset C$, we can take $\lambda > 0$ such that $\lambda(v_\sigma, 0) + (0, z_0) \in C$. Then $(\lambda v_\sigma, z_0) \in C$.

(2) Note that (2) doesn't depend on σ^0.

(3) The proof of (3) remains the same.

(4) First by (1) take $(\lambda v_\sigma, z_0) \in C$. Choose $\mu > 0$ large enough so that $\mu(v_\sigma, 0) \in D$ and then further $\mu(v_\sigma, 0) + (\lambda v_\sigma, z_0) \in D$ (as D is open convex). Hence $((\mu + \lambda)v_\sigma, z_0) \in D$.

(5) Finally, notice that on the plane $\mathbb{R} \cdot v_\sigma + \mathbb{R} \cdot z_0$, the points of C form an open convex cone containing the positive x-axis while that of D form an open convex subset of this cone. The point $(1, 1)$ is in D. That $f_{z_0}(\lambda)$ is decreasing still follows from (3). If λ is large enough, $f_{z_0}(\lambda) < \infty$. This implies $f_{z_0}(\lambda) < \infty$ for all $\lambda \geq 1$. If $f_{z_0}(\lambda_1) = -\infty$ for some λ_1, then for all $\lambda \geq \lambda_1$, $f_{z_0}(\lambda) = -\infty$ by that f_{z_0} is decreasing. By convexity and openness of D we deduce $f_{z_0}(\lambda) = -\infty$ for all $\lambda \geq 1$. Thus the inequality $f_{z_0}(1-t)\cdot\lambda_1+t\cdot\lambda_2) \leq (1-t)f_{z_0}(\lambda_1)+tf_{z_0}(\lambda_2)$ holds for $0 \leq t \leq 1$, $\lambda_1, \lambda_2 \geq 1$ in all cases. We conclude that f_{z_0} is convex and continuous.

This finishes the proof of the lemma. $\square$

5.2 $\overline{\mathcal{U}}(P_1, \mathcal{X}_1, p_f)$ on each stratum

Let $(P, \mathcal{X})$ be a mixed Shimura datum with $K_f \subset P(\mathbb{A}_f)$ a neat open compact subgroup and Σ a K_f-admissible partial polyhedral cone decomposition for $(P, \mathcal{X})$(cf. Definition 4.6.1). Take any rational boundary component $(P_1, \mathcal{X}_1)$ of $(P, \mathcal{X})$ (associated with a $\mathbb{Q}$-admissible parabolic subgroup $Q \subseteq P$). Let $\mathcal{X}^+_{(P_1,\mathcal{X}_1)} \rightarrowtail \mathcal{X}_1$ be the $P_1(\mathbb{R}) \cdot Q(\mathbb{R})^0 \cdot U(\mathbb{C})$-equivariant open embedding as in Proposition 4.4.8 and 4.4.10. Define $K^1_f = P_1(\mathbb{A}_f) \cap p_f K_f p_f^{-1}$. This is a neat open compact subgroup of $P_1(\mathbb{A}_f)$.

Given any connected component $\mathcal{X}^0_1 \subset \mathcal{X}_1$, let $\mathrm{im} : \mathcal{X}^0_1 \to U_1(\mathbb{R})(-1)$ be the restriction of the map "imaginary part" as defined in Subsection 4.4. For $p_{1,f} \in P_1(\mathbb{A}_f)$, $\Gamma_1 := \mathrm{Stab}_{P_1(\mathbb{Q})}(\mathcal{X}^0_1) \cap p_{1,f} K^1_f p^{-1}_{1,f}$ is a neat arithmetic subgroup of $P_1(\mathbb{Q})$ acting on $\mathcal{X}^0_1$. Since $(P_1, \mathcal{X}_1)$ is irreducible, P_1 acts on U_1 by scalars and thus Γ_1 acts trivially on U_1. Therefore im factors through a map $\overline{\mathrm{im}} : \Gamma_1 \backslash \mathcal{X}^0_1 \to U_1(\mathbb{R})(-1)$. (Note that $\Gamma_1 \backslash \mathcal{X}^0_1$ is a connected component of $M^{K^1_f}(P_1, \mathcal{X}_1)(\mathbb{C})$, being the image of $\mathcal{X}^0_1 \times \{p_{1,f}\}$.) Then $\Gamma_1 \backslash \mathcal{X}^0 = \overline{\mathrm{im}}^{-1}(C(\mathcal{X}^0, P_1))$.

Remark 5.2.1. If K_f is not neat, Γ_1 acts on $U_1(\mathbb{R})(-1)$ through $\{\pm 1\}$. When $(P_1, \mathcal{X}_1)$ is a proper boundary component, $C(\mathcal{X}^0, P_1)$ doesn't contain the origin and is stabilized by Γ_1. Thus Γ_1 still acts trivially on $U_1(\mathbb{R})(-1)$ and we still have a map $\overline{\mathrm{im}} : \Gamma_1 \backslash \mathcal{X}^0_1 \to U_1(\mathbb{R})(-1)$. When $(P_1, \mathcal{X}_1)$ is an improper boundary component, $C(\mathcal{X}^0, P_1) = U_1(\mathbb{R})(-1)$ and any core $D \subset C(\mathcal{X}^0, P_1)$ is just $U_1(\mathbb{R})(-1)$. In summary, the subsets $\mathrm{im}^{-1}(C(\mathcal{X}^0, P_1)) = \mathcal{X}^0$ and $\mathrm{im}^{-1}(D)$ are always Γ_1-invariant.

Since $\mathcal{X}^+_{(P_1,\mathcal{X}_1)} \rightarrowtail \mathcal{X}_1$ is bijective on connected components, the open embedding $\mathcal{U}(P_1, \mathcal{X}_1, p_f) = P_1(\mathbb{Q}) \backslash \mathcal{X}^+_{(P_1,\mathcal{X}_1)} \times P_1(\mathbb{A}_f)/K^1_f \rightarrowtail M^{K^1_f}(P_1, \mathcal{X}_1)(\mathbb{C}) = P_1(\mathbb{Q}) \backslash \mathcal{X}_1 \times$

$P_1(\mathbb{A}_f)/K_f^1$ (as defined in Subsection 4.6) is also bijective on connected components. By equivariance, it follows that $\Gamma_1\backslash\mathcal{X}^0 \rightarrowtail \Gamma_1\backslash\mathcal{X}_1^0$ is the open embedding of the corresponding connected components, where $\mathcal{X}^+_{(P_1,\mathcal{X}_1)} \supset \mathcal{X}^0 \rightarrowtail \mathcal{X}_1^0$.

We also defined, in Subsection 4.6, the cone decomposition

$$\Sigma_{(P_1,\mathcal{X}_1,p_f)} = \Big(([\cdot p_f]^*\Sigma)|_{(P_1,\mathcal{X}_1)}\Big)^0.$$

It is K_f^1-admissible and concentrated on the unipotent fiber. As $(P_1,\mathcal{X}_1)$ is irreducible, $\Sigma_{(P_1,\mathcal{X}_1,p_f)}$ only depends on the connected component $\mathcal{X}^0 \subset \mathcal{X}$. Moreover, the projection $M^{K_f^1}(P_1,\mathcal{X}_1)(\mathbb{C}) \to M^{\pi_{U_1}(K_f^1)}((P_1,\mathcal{X}_1)/U_1)(\mathbb{C})$ is bijective on connected components as it has connected fibers (cf. Subsection 4.3). We see that the relative torus embedding

$$M^{K_f^1}(P_1,\mathcal{X}_1)(\mathbb{C}) \rightarrowtail M^{K_f^1}(P_1,\mathcal{X}_1,\Sigma_{(P_1,\mathcal{X}_1,p_f)})(\mathbb{C})$$

can be viewed as constructed componentwise. From now on, we will denote $\Sigma_{(P_1,\mathcal{X}_1,p_f)}$ by Σ_1^0 and write $\Gamma_1\backslash\mathcal{X}_1^0 \rightarrowtail \big(\Gamma_1\backslash\mathcal{X}_1^0\big)_{\Sigma_1^0}$ for the torus embedding on the connected component $\Gamma_1\backslash\mathcal{X}_1^0$. For $\sigma \in \Sigma_1^0(\mathcal{X}_1^0, P_1, p_{1,f})$ (which we just abbreviate as $\sigma \in \Sigma_1^0$), $\big(\Gamma_1\backslash\mathcal{X}_1^0\big)_\sigma$ is the embedding relative to σ and $\big(\Gamma_1\backslash\mathcal{X}_1^0\big)_\sigma^\sim \subset \big(\Gamma_1\backslash\mathcal{X}_1^0\big)_\sigma$ is the σ-stratum on $\big(\Gamma_1\backslash\mathcal{X}_1^0\big)_\sigma \subseteq \big(\Gamma_1\backslash\mathcal{X}_1^0\big)_{\Sigma_1^0}$. Use $\pi_\sigma : \big(\Gamma_1\backslash\mathcal{X}_1^0\big)_\sigma \to \big(\Gamma_1\backslash\mathcal{X}_1^0\big)_\sigma^\sim$ for the projection onto the σ-stratum. It restricts to a surjective projection $\Gamma_1\backslash\mathcal{X}_1^0 \to \big(\Gamma_1\backslash\mathcal{X}_1^0\big)_\sigma^\sim$ and to an isomorphism on $\big(\Gamma_1\backslash\mathcal{X}_1^0\big)_\sigma^\sim \subset \big(\Gamma_1\backslash\mathcal{X}_1^0\big)_\sigma$. Recall that $\overline{\mathcal{U}}(P_1,\mathcal{X}_1,p_f)$ is defined to be the interior of the closure of $\mathcal{U}(P_1,\mathcal{X}_1,p_f)$ in $M^{K_f^1}(P_1,\mathcal{X}_1,\Sigma_{(P_1,\mathcal{X}_1,p_f)})(\mathbb{C})$. From above we deduce that $\overline{\mathcal{U}}(P_1,\mathcal{X}_1,p_f)$ is also constructed componentwise. Its part in

$\left(\Gamma_1\backslash\mathcal{X}_1^0\right)_{\Sigma_1^0}$ is just the interior of the closure of $\Gamma_1\backslash\mathcal{X}^0$ in $\left(\Gamma_1\backslash\mathcal{X}_1^0\right)_\sigma$.

Let $[x_1, p_{1,f}] \in M^{K_f^1}(P_1, \mathcal{X}_1)(\mathbb{C})$. We know from Subsection 4.3 that, the fiber over $\pi_{U_1}([x_1, p_{1,f}])$ under $M^{K_f^1}(P_1, \mathcal{X}_1)(\mathbb{C}) \to M^{\pi_{U_1}(K_f^1)}((P_1, \mathcal{X}_1)/U_1)(\mathbb{C})$ is isomorphic to $T(\mathbb{C}) = \Omega_{U_1}\backslash U_1(\mathbb{C})$ where Ω_{U_1} is the image under the map $Z(P_1) \times U_1 \to U_1$ of the group $\left(\mathrm{Id}_{Z(P_1)(\mathbb{Q})}(\mathcal{X}_1) \times U_1(\mathbb{Q})\right) \cap p_{1,f} K_f^1 p_{1,f}^{-1}$. $T(\mathbb{C})$ is an algebraic torus. For every fiber we may choose a suitable x_1 such that the composition $\Omega_{U_1}\backslash U_1(\mathbb{C}) \rightarrowtail \Gamma_1\backslash\mathcal{X}_1^0 \overset{\overline{\mathrm{im}}}{\to} U_1(\mathbb{R})(-1)$ is just the map

$$\begin{aligned}\Omega_{U_1}\backslash U_1(\mathbb{C}) = (\Omega_{U_1}\backslash U_1(\mathbb{R})) \oplus U_1(\mathbb{R})(-1) &\to U_1(\mathbb{R})(-1)\\ (b, a) &\mapsto a.\end{aligned}$$

The above map is just the map ord_T defined in Subsection 4.5. The induced map $\overline{\mathrm{im}}$ is also just one ord-map as defined there. From now on, we write $\overline{\mathrm{im}}$ as ord.

The next result will be crucial to the proof of our main theorem. (Again we will use $\mathrm{Int}(S)$ to denote the interior of a subset S.)

Lemma 5.2.2. *Given a mixed Shimura datum $(P, \mathcal{X})$ with $K_f \subset P(\mathbb{A}_f)$ an open compact subgroup and Σ a K_f-admissible partial polyhedral decomposition, let $\mathcal{U}(P_1, \mathcal{X}_1, p_f) \rightarrowtail M^{K_f^1}(P_1, \mathcal{X}_1)(\mathbb{C}) \rightarrowtail M^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_1^0)(\mathbb{C})$ be the open embeddings associated with any rational boundary component $(P_1, \mathcal{X}_1)$ and Σ as above. Fix a connected component $\mathcal{X}^0$ and $p_{1,f} \in P_1(\mathbb{A}_f)$, let $\Gamma_1\backslash\mathcal{X}^0 \rightarrowtail \Gamma_1\backslash\mathcal{X}_1^0 \rightarrowtail \left(\Gamma_1\backslash\mathcal{X}_1^0\right)_{\Sigma_1^0}$ be the corresponding open embeddings. Suppose $D \subset C(\mathcal{X}^0, P_1)$ is any open convex core and $\sigma \in \Sigma_1^0$. (The interior of) the closure of* $\mathrm{ord}^{-1}(D) = \Gamma_1\backslash\mathrm{im}^{-1}(D)$ *and* $\mathrm{ord}^{-1}(C(\mathcal{X}^0, P_1)) = \Gamma_1\backslash\mathrm{im}^{-1}(C(\mathcal{X}^0, P_1)) = \Gamma_1\backslash\mathcal{X}^0$ *in $\left(\Gamma_1\backslash\mathcal{X}_1^0\right)_{\Sigma_1^0}$ meets the σ-stratum $\left(\Gamma_1\backslash\mathcal{X}_1^0\right)_\sigma^\sim$ in the* *following way. (We denote the closure in* $\left(\Gamma_1\backslash\mathcal{X}_1^0\right)_{\Sigma_1^0}$ *by* $\mathrm{Cl}(S)$ *and the closure in an* $\mathbb{R}$*-vector space by* $\overline{S}$*.)*

(1) $\mathrm{Int}\left(\mathrm{Cl}\left(\mathrm{ord}^{-1}\left(C(\mathcal{X}^0,P_1)\right)\right)\right)\cap\left(\Gamma_1\backslash\mathcal{X}_1^0\right)_\sigma^\sim=\pi_\sigma\left(\mathrm{ord}^{-1}\left(C(\mathcal{X}^0,P_1)\right)\right)$.

That is, $\mathrm{Int}\left(\mathrm{Cl}\left(\Gamma_1\backslash\mathcal{X}^0\right)\right)\cap\left(\Gamma_1\backslash\mathcal{X}_1^0\right)_\sigma^\sim=\pi_\sigma\left(\Gamma_1\backslash\mathcal{X}^0\right)$.

(2) $\mathrm{Int}\left(\mathrm{Cl}\left(\mathrm{ord}^{-1}(D)\right)\right)\cap(\Gamma_1\backslash\mathcal{X}_1^0)_\sigma^\sim=\pi_\sigma\left(\mathrm{ord}^{-1}(D)\right)$.

(3) $\mathrm{Cl}\left(\Gamma_1\backslash\mathcal{X}^0\right)\cap\left(\Gamma_1\backslash\mathcal{X}_1^0\right)_\sigma^\sim=\pi_\sigma\left(\mathrm{ord}^{-1}\left(\overline{C(\mathcal{X}^0,P_1)}\right)\right)$.

In particular, if $-\sigma^0\subset C(\mathcal{X}^0,P_1)$*,* $\left(\Gamma_1\backslash\mathcal{X}_1^0\right)_\sigma^\sim\subset\mathrm{Int}\left(\mathrm{Cl}\left(\Gamma_1\backslash\mathcal{X}^0\right)\right)\subset\bar{\mathcal{U}}(P_1,\mathcal{X}_1,p_f)$*. (In this case even* $\left(\Gamma_1\backslash\mathcal{X}_1^0\right)_\sigma^\sim\subset\mathrm{Int}\left(\mathrm{Cl}\left(\mathrm{ord}^{-1}(D)\right)\right)$*.*

Remark 5.2.3. The analogue of (3) does not hold for D. That is, it is in general not true that $\mathrm{Cl}\left(\mathrm{ord}^{-1}(D)\right)\cap\left(\Gamma_1\backslash\mathcal{X}_1^0\right)_\sigma^\sim=\pi_\sigma\left(\mathrm{ord}^{-1}(\overline{D})\right)$.

Remark 5.2.4. The special case for $C(\mathcal{X}^0,P_1)$ of Lemma 5.2.2 is also claimed in [Pin90, 6.13]. Note that we don't assume K_f is neat.

Before the proof of Lemma 5.2.2, let's reduce it to a simpler case and introduce two more notions.

Lemma 5.2.5. *Given* $f:X\to S$ *an analytic* $T(\mathbb{C})$*-torsor (for a* $\mathbb{C}$*-torus* T*) and* V *an open subset of* X*, let* $\overline{V}$ *be the closure of* V *in* X_σ *where* $\sigma\subset Y_*(T)_\mathbb{R}$ *is a convex rational polyhedral cone and* X_σ *the relative torus embedding. Suppose* $\{U_i\}$ *is an open covering of* S *(e.g. on which the torsor is trivial). In order to show that* $\pi_\sigma(\overline{V}\cap X_\sigma^\sim)=\pi_\sigma(\overline{V}\cap X)$ *and* $\pi_\sigma(\mathrm{Int}\left(\overline{V}\right)\cap X_\sigma^\sim)=\pi_\sigma(V)$*, it suffices to prove the two properties for every* $f^{-1}(U_i)\to U_i$*.*

Proof. Let $X_i:=f^{-1}(U_i)$; so $(X_i)_\sigma$ and $(X_i)_\sigma^\sim$ are well-defined. Suppose we are already done over each U_i, then

$$\begin{aligned}\pi_\sigma\left(\overline{V}\cap X_\sigma^\sim\right) &= \bigcup_i \pi_\sigma(\overline{V}\cap (X_i)_\sigma^\sim)\\ &= \bigcup_i \pi_\sigma(\overline{V}\cap (X_i)_\sigma\cap (X_i)_\sigma^\sim)\\ &\quad (\overline{V}\cap (X_i)_\sigma \text{ is the closure of } V\cap X_i \text{ in } (X_i)_\sigma\\ &= \bigcup_i \pi_\sigma(\overline{V}\cap X_i) = \pi_\sigma(\overline{V}\cap X).\end{aligned}$$

As for any subset S and open subset U in a total space X, $\mathrm{Int}^X(S)\cap U = \mathrm{Int}^U(S\cap U)$, we obtain

$$\begin{aligned}\pi_\sigma\left(\mathrm{Int}^{X_\sigma}\left(\overline{V}\right)\cap X_\sigma^\sim\right) &= \bigcup_i \pi_\sigma(\mathrm{Int}^{X_\sigma}\left(\overline{V}\right)\cap (X_i)_\sigma\cap (X_i)_\sigma^\sim)\\ &= \bigcup_i \pi_\sigma(\mathrm{Int}^{(X_i)_\sigma}\left(\overline{V}\cap (X_i)_\sigma\right)\cap (X_i)_\sigma^\sim)\\ &= \bigcup_i \pi_\sigma(V\cap X_i) = \pi_\sigma(V).\end{aligned}$$

□

Next, suppose we have a diargram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & X' \arrow[d, "g'"] \\ Y \arrow[r, "f'"] & Y' \end{tikzcd}$$

in the category of topological spaces. We say f is fiberwise surjective if for all $y\in Y$, $g^{-1}(y)\to (g')^{-1}(f'(y))$ is surjective. In this situation, it is easy to verify that for any $U'\subseteq X'$, $g(f^{-1}(U')) = (f')^{-1}(g'(U'))$.

Given a product space $\mathbb{R}^n\times\mathbb{R}^m$, a subset $S\subseteq\mathbb{R}^n\times\mathbb{R}^m$ and an open convex cone $\sigma^0\subset\mathbb{R}^n\times\{0\}$, we say that a point $(*, z_0)\in\{*\}\times\mathbb{R}^m$ can be σ^0-*approached within* S

if there is a sequence $\{(v_n, z_n)\}_{n=1}^{\infty} \subset (\sigma \times \mathbb{R}^m) \cap S$ such that $z_n \to z_0$ and $|v_n| \to \infty$ as $n \to \infty$. Basically, the sequence converges to z_0 in the second coordinate and goes to infinity in the first coordinate along σ^0 within S.

Proof of Lemma 5.2.2. First assume that K_f is neat. Then we have the map $\overline{\mathrm{im}} = \mathrm{ord} : \Gamma_1 \backslash \mathcal{X}_1^0 \to U_1(\mathbb{R})(-1)$ while $\Gamma_1 \backslash \mathrm{im}^{-1}\left(C(\mathcal{X}^0, P_1)\right) = \mathrm{ord}^{-1}(C(\mathcal{X}^0, P_1))$ and $\Gamma_1 \backslash \mathrm{im}^{-1}(D) = \mathrm{ord}^{-1}(D)$. Let $T(\mathbb{C}) = \Omega_{U_1} \backslash U_1(\mathbb{C})$ be as defined before Lemma 5.2.2 and $K = \Omega_{U_1} \backslash U_1(\mathbb{R})$ be its maximal compact subgroup. Consider the analytic $T(\mathbb{C})$-torsor $M^{K_f^1}(P_1, \mathcal{X}_1)(\mathbb{C}) \to M^{\pi_{U_1}(K_f^1)}(P_1/U_1, \mathcal{X}_1/U_1)(\mathbb{C})$, which we denote by $X \to Y$. By definition, the map ord is the composition $X \to X/K \xrightarrow{\sim} Y \times (T(\mathbb{C})/K) \to Y_*(T)_{\mathbb{R}} = U_1(\mathbb{R})(-1)$. Since X/K as a T/K-torsor can be globally trivialized, we see that it is possible to choose a local section $\mathcal{N} \to \Gamma_1 \backslash \mathcal{X}_1^0 \subset M^{K_f^1}(P_1, \mathcal{X}_1)(\mathbb{C})$ over open subset $\mathcal{N} \subset M^{\pi_{U_1}(K_f^1)}(P_1/U_1, \mathcal{X}^1/U_1)(\mathbb{C})$ such that $\mathrm{ord}_T^{-1}(C(\mathcal{X}^0, P_1)) \times \mathcal{N} \to \Gamma_1 \backslash \mathcal{X}_1^0$ maps (isomorphically) onto $(\mathrm{ord}|_{\mathcal{X}_1^0})^{-1}(C(\mathcal{X}^0, P_1))$ and similarly for the core D. Hence we just need to prove the parallel statements for $\mathrm{ord}_T^{-1}(C(\mathcal{X}^0, P_1)) \times \mathcal{N} \subset T(\mathbb{C}) \times \mathcal{N}$ with respect to the torus embedding $T(\mathbb{C})_\sigma \times \mathcal{N}$ for some convex rational polyhedral cone $\sigma \subset Y_*(T)_{\mathbb{R}}$. (After all, $(\Gamma_1 \backslash \mathcal{X}_1^0)_{\Sigma_1^0}$ is also constructed locally.) This reduces us to the case of a single $\mathrm{ord}_T^{-1}(C(\mathcal{X}^0, P_1)) \subset T(\mathbb{C}) \subset T(\mathbb{C})_\sigma$.

Let $T' \subset T$ be the subtorus such that $\mathbb{R} \cdot \sigma = Y_*(T')_{\mathbb{R}}$. Note that $T_\sigma^\sim \cong \operatorname{Spec} k[X^*(T)_{\sigma=0}]$. Thus we have a split short exact sequence $1 \to T' \to T \to T_\sigma^\sim \to 1$. After choosing a splitting, $T' \to T \to T_\sigma \xrightarrow{\pi_\sigma} T_\sigma^\sim$ may be identified with $T' \to T' \times T_\sigma^\sim \to T'_\sigma \times T_\sigma^\sim \to T_\sigma^\sim$.

Let $K' \subseteq T'$ (resp. $K_\sigma^\sim \subset T_\sigma^\sim$) be the maximal compact subgroup. We have a diagram as below, where write T for $T(\mathbb{C})$ to save space.

$$
\begin{tikzcd}
T \arrow[r, "\sim"] \arrow[d, tail] & T' \times T_\sigma^{\sim} \arrow[r, two heads, "q", "\text{f.s.o.}"'] \arrow[d, tail] & T'/K' \times T_\sigma^{\sim}/K_\sigma^{\sim} \arrow[r, "\sim"] \arrow[d, tail] & Y_*(T')_{\mathbb{R}} \times Y_*(T_\sigma^{\sim})_{\mathbb{R}} \arrow[dd, two heads] \\
T_\sigma \arrow[r, "\sim"] \arrow[d, two heads, "\pi_\sigma"'] & T'_\sigma \times T_\sigma^{\sim} \arrow[r, two heads, "q_\sigma", "\text{f.s.o.}"'] \arrow[d, two heads] & T'_\sigma/K' \times T_\sigma^{\sim}/K_\sigma^{\sim} \arrow[d, two heads, "j_\sigma"] & \\
T_\sigma^{\sim} \arrow[r, "="] & T_\sigma^{\sim} \arrow[r, two heads] & T_\sigma^{\sim}/K_\sigma^{\sim} \arrow[r, "\sim"] & Y_*(T_\sigma^{\sim})_{\mathbb{R}}
\end{tikzcd}
$$

Here f.s.o means “fiberwise surjective and open”. Under the splitting, $T_\sigma^{\sim}$ sits in T_σ as follows, where $\{*\}$ is the image of $(T')_\sigma^{\sim}$.

$$
\begin{tikzcd}
T_\sigma^{\sim} \arrow[r, "\sim"] \arrow[d, tail] \arrow[dd, bend right, "\wr"'] & (T')_\sigma^{\sim} \times T_\sigma^{\sim} \arrow[r, two heads] \arrow[d, tail] & \{*\} \times T_\sigma^{\sim}/K_\sigma^{\sim} \arrow[d, tail] \arrow[dd, bend left, "\wr"] \\
T_\sigma \arrow[r, "\sim"] \arrow[d, two heads, "\pi_\sigma"] & T'_\sigma \times T_\sigma^{\sim} \arrow[r, two heads, "q_\sigma", "\text{f.s.o.}"'] \arrow[d, two heads] & T'_\sigma/K' \times T_\sigma^{\sim}/K_\sigma^{\sim} \arrow[d, two heads, "j_\sigma"] \\
T_\sigma^{\sim} \arrow[r, "="] & T_\sigma^{\sim} \arrow[r, two heads] & T_\sigma^{\sim}/K_\sigma^{\sim}
\end{tikzcd}
$$

For an open map $q : X \to Y$, $q^{-1}(\mathrm{Cl}^Y(S)) = \mathrm{Cl}^X(q^{-1}(S))$ and $q^{-1}(\mathrm{Int}^Y(S)) = \mathrm{Int}^X(q^{-1}(S))$. Let $A \coloneqq \{*\} \times T_\sigma^{\sim}/K_\sigma^{\sim}$. A little diagram chasing shows that we just have to prove both statements

$$\mathrm{Int}(\mathrm{Cl}(S)) \cap A = j_\sigma(S)$$

$$\mathrm{Cl}(S) \cap A = j_\sigma(\overline{S})$$

for $S = C(\mathcal{X}^0, P_1)$, and only the first statement for $S = D$ (an open convex core), as a subset of $T'/K' \times T_\sigma^{\sim}/K_\sigma^{\sim} \cong Y_*(T')_{\mathbb{R}} \times Y_*(T_\sigma^{\sim})_{\mathbb{R}}$. We will write $\mathrm{Cl}(S)$ for the closure in $T'_\sigma/K' \times T_\sigma^{\sim}/K_\sigma^{\sim}$ and $\overline{S}$ for the closure in the Euclidean space $T'_\sigma/K' \times T_\sigma^{\sim}/K_\sigma^{\sim} \cong Y_*(T')_{\mathbb{R}} \times Y_*(T_\sigma^{\sim})_{\mathbb{R}}$. Note $\overline{S} = \mathrm{Cl}(S) \cap (T'_\sigma/K' \times T_\sigma^{\sim}/K_\sigma^{\sim})$.

How can points $\{(\overline{t_n}, z_n)\}_{n=1}^{\infty} \subset T'/K' \times T_\sigma^{\sim}/K_\sigma^{\sim}$ approach a point $(*, z_0) \in \{*\} \times T_\sigma^{\sim}/K_\sigma^{\sim} = A$? Certainly, the second coordinate must converge: $z_n \to z_0$. For the first coordinate, choose $\{t_n\}_{n=1}^{\infty} \subset T'$ lifting $\overline{t_n}$. Note that under the open map $T'_\sigma \to T'_\sigma/K'$, the fiber over ∞ is the singleton $(T')_\sigma^{\sim}$, as every σ-stratum is a T'-orbit and stable under K'. Since K' is compact and T'_σ is Hausdorff, it follows that any such lifting $\{t_n\}_{n=1}^{\infty}$ must converge to $(T')_\sigma^{\sim}$. Suppose $\chi_1, \ldots, \chi_r$ is a set of generators for $X^*(T')_{\sigma \geq 0}$ as a monoid. As $T'_\sigma = \operatorname{Spec} k[X^*(T')_{\sigma \geq 0}]$, $(\chi_1, \ldots, \chi_r)$ defines a closed embedding $T'_\sigma \rightarrowtail \mathbb{A}^r_{\mathbb{C}}$ sending $(T')_\sigma^{\sim}$ to the origin. Thus $t_n \to (T')_\sigma^{\sim}$ if and only if $|\chi_i(t_n)| \to 0$ for all $1 \leq i \leq r$.

Recall that the map ord is defined to be $T(\mathbb{C}) \cong Y_*(T) \otimes \mathbb{C}^\times \to Y_*(T) \otimes \mathbb{R}, \lambda \otimes z \mapsto \lambda \otimes (\ln |z|)$. Thus in terms of the splitting, it is $T' \times T_\sigma^{\sim} \to T'/K' \times T_\sigma^{\sim}/K_\sigma^{\sim} \xrightarrow{\sim} Y_*(T')_{\mathbb{R}} \times Y_*(T_\sigma^{\sim})_{\mathbb{R}}$. Let $\lambda_1, \ldots, \lambda_l$ be a basis of $Y_*(T')$. Given $t \in T'(\mathbb{C})$, suppose its image in $Y_*(T) \otimes \mathbb{C}^\times$ is $\sum_{j=1}^{l} \lambda_j \otimes t_j$, then $\operatorname{ord}(t) = \sum_{j=1}^{l} \lambda_j \otimes (\ln |t_j|)$.

$$\begin{aligned}\chi_i(\operatorname{ord}(t)) &= \sum_{j=1}^{l} \langle \chi_i, \lambda_j \rangle \ln |t_j| \\ &= \ln \Big| \prod_{j=1}^{l} t_j^{\langle \chi_i, \lambda_j \rangle} \Big| = \ln |\chi_i(t)|\end{aligned}$$

Thus $\overline{t_n} \to *$ if and only if any lifting $t_n \to (T')_\sigma^{\sim}$ if and only if $|\chi_i(t_n)| \to 0$ if and only if $\chi_i(\operatorname{ord}(t_n)) \to -\infty \, (1 \leq i \leq r)$. Denote $\operatorname{ord}(t_n)$ by $v_n (\in Y_*(T')_{\mathbb{R}})$. This means

$\{v_n\}$ goes to ∞ along $-\sigma^0$. In summary,

$$\begin{aligned}
&\text{points in } \{*\} \times T_\sigma^\sim/K_\sigma^\sim \\
&\text{that can be approached by } S \subset T'/K' \times T_\sigma^\sim/K_\sigma^\sim \\
\Longleftrightarrow\ &\text{points in } \{*\} \times Y_*(T_\sigma^\sim)_\mathbb{R} \\
&\text{that can be } (-\sigma^0)\text{-approached by } S \subset Y_*(T')_\mathbb{R} \times Y_*(T_\sigma^\sim)_\mathbb{R}.
\end{aligned}$$

Our S in $Y_*(T')_\mathbb{R} \times Y_*(T_\sigma^\sim)_\mathbb{R}$ is $C(\mathcal{X}^0, P_1)$ or D. But $\sigma \in \Sigma_1^0(\mathcal{X}_1^0, P_1, p_{1,f}) = \Sigma(\mathcal{X}^0, P_1, p_{1,f} \cdot p_f)p_f^{-1}$, which is a complete cone decomposition of $-C(\mathcal{X}^0, P_1)$. And so $-\sigma \subset C^*(\mathcal{X}^0, P_1) \subset \overline{C(\mathcal{X}^0, P_1)}$. By Proposition 4.4.11, we see that there is a unique rational boundary component $(P_2, \mathcal{X}_2) \geq (P_1, \mathcal{X}_1)$ such that $-\sigma^0 \subset C(\mathcal{X}^0, P_2)(\subset U_2(\mathbb{R})(-1))$ and so $-\sigma \subset U_2(\mathbb{R})(-1)$. If $(P_2, \mathcal{X}_2) \neq (P_1, \mathcal{X}_1)$. Note that $C^*(\mathcal{X}^0, P_1) \cap U_2(\mathbb{R})(-1) = C^*(\mathcal{X}^0, P_2)$. Hence

$$\begin{aligned}
C(\mathcal{X}^0, P_1) \cap (-\sigma) &= C(\mathcal{X}^0, P_1) \cap C^*(\mathcal{X}^0, P_1) \cap (-\sigma) \\
&\subset C(\mathcal{X}^0, P_1) \cap C^*(\mathcal{X}^0, P_1) \cap U_2(\mathbb{R})(-1) \\
&= C(\mathcal{X}^0, P_1) \cap C^*(\mathcal{X}^0, P_2) = \varnothing
\end{aligned}$$

and we are in Case (A) of Lemma 5.1.4 ($-\sigma^0$ contained in the interior of $\overline{C(\mathcal{X}^0, P_1)} \cap Y_*(T')_\mathbb{R}$). If $(P_2, \mathcal{X}_2) = (P_1, \mathcal{X}_1)$, $-\sigma^0 \subset C(\mathcal{X}^0, P_1)$ and we are in Case (B) of Lemma 5.1.4.

Let $C := C(\mathcal{X}^0, P_1)$. With the above analysis, we claim that the set $E := \mathrm{Cl}(C) \cap A$ of points in $A = \{*\} \times Y_*(T_\sigma^\sim)_\mathbb{R}$ that can be $(-\sigma^0)$-approached within $C(\mathcal{X}^0, P_1) \subset$

$Y_*(T')_{\mathbb{R}} \times Y_*(T_\sigma^\sim)_{\mathbb{R}}$ is exactly $\overline{j_\sigma(C)}$. Namely, $E \subset \overline{j_\sigma(C)}$ by definition of $(-\sigma^0)$-approaching and $E \supset \overline{j_\sigma(C)}$ by (2) of Lemma 5.1.4. By Lemma 5.1.1, $E = \overline{j_\sigma(C)} = j_\sigma(\overline{C})$ and we have done (3).

In particular, $j_\sigma(C) \subset \mathrm{Cl}(C)$. Thus we have shown, for all $\sigma \in \Sigma_1^0$, that $\pi_\sigma(\mathrm{ord}^{-1}(C)) \subset \mathrm{Cl}(\mathrm{ord}^{-1}(C))$.

Points in $F \coloneqq \mathrm{Int}(\mathrm{Cl}(C)) \cap A$ should lie in the interior

$$\mathrm{Int}^A(\mathrm{Cl}(C) \cap A) = \mathrm{Int}^A(j_\sigma(\overline{C})) = \mathrm{Int}^A(\overline{j_\sigma(C)}) = j_\sigma(C),$$

by Lemma 5.1.1 and so $F \subset j_\sigma(C)$.

Let $\tau \leq \sigma$ be any face of σ. Then $-\tau \subset \overline{C}$ and so $-\tau + C = -\tau^0 + C = C$. By [Pin90, 5.9], $\bigcup_{\tau \leq \sigma} \pi_\tau(\mathrm{ord}^{-1}(C))$ is an open subset, contained in $\mathrm{Cl}(\mathrm{ord}^{-1}(C))$. This implies $\pi_\sigma(\mathrm{ord}^{-1}(C)) \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{ord}^{-1}(C)))$ or equivalently, $j_\sigma(C) \subset \mathrm{Int}(\mathrm{Cl}(C))$. Thus $j_\sigma(C) \subset \mathrm{Int}(\mathrm{Cl}(C)) \cap A = F$ and $F = j_\sigma(C)$ as desired. We have done (1).

In the case of an open convex core D, a similar argument using (4) and (5) of Lemma 5.1.4 gives $\mathrm{Cl}(D) \cap A = \overline{j_\sigma(D)}$ and in particular $j_\sigma(D) \subset \mathrm{Cl}(D)$. By [Pin90, 5.9] again $j_\sigma(D) \subset \mathrm{Int}(\mathrm{Cl}(D))$. Thus

$$\begin{aligned} j_\sigma(D) &\subset \mathrm{Int}(\mathrm{Cl}(D)) \cap A \\ &\subset \mathrm{Int}^A\big(\mathrm{Cl}(D) \cap A\big) \\ &= \mathrm{Int}^A(\overline{j_\sigma(D)}) = j_\sigma(D) \end{aligned}$$

by Lemma 5.1.1. We have done (2).

If $-\sigma^0 \subset C(\mathcal{X}^0, P_1)$, we have $j_\sigma(C) = j_\sigma(D) = T_\sigma^\sim / K_\sigma^\sim$, hence $\pi_\sigma(\mathrm{ord}^{-1}(C)) = \pi_\sigma(\mathrm{ord}^{-1}(D)) = T_\sigma^\sim \subset \mathrm{Int}(\mathrm{Cl}(D)) \subset \mathrm{Int}(\mathrm{Cl}(C))$. This proves the last statement for a neat K_f.

When K_f is not assumed neat, take a neat open compact normal subgroup $K_{f,n} \trianglelefteq K_f$. By the K_f-admissibility of Σ, the action of the finite group $K_f^1/K_{f,n}^1$ on $M^{K_{f,n}^1}(P_1, \mathcal{X}_1)(\mathbb{C})$ (over $M^{\pi_{U_1}(K_{f,n}^1)}(P_1/U_1, \mathcal{X}_1/U_1)(\mathbb{C})$) extends to the relative torus embedding $M^{K_{f,n}^1}(P_1, \mathcal{X}_1, \Sigma_1^0)(\mathbb{C})$. This action preserves the σ-stratum and is compatible with π_σ. Therefore all the assertions follow from taking quotients. This finishes the proof of Lemma 5.2.2. $\square$

We will also need the following result from [BF24].

Lemma 5.2.6 ([BF24, Lemma 49])**.** *Let U be a real vector space with $C \subset U$ an open convex cone and $\Gamma \subset U$ a lattice. Let $p : U_{\mathbb{C}} \to \Gamma \backslash U_{\mathbb{C}} =: T(\mathbb{C})$ be the quotient map. ($\Gamma \backslash U_{\mathbb{C}} = (\Gamma \backslash U) \oplus (i \cdot U)$.) Suppose $\sigma \subset C$ is a smooth rational polyhedral convex cone of top dimension. Let $T \to T_\sigma$ be the torus embedding associated with σ. Given $c \in U$, define $D_c \coloneqq U + i \cdot (c + C)$. Then for sufficiently small polydisc D centered at $T_\sigma^\sim$, $D^\circ \coloneqq D \cap p(D_c)$ is a product of punctured discs. In particular, this applies to $D^\circ = D \cap p(U + iC)$.*

5.3 Proof of Theorem 1.1

We now turn to the proof of Theorem 1.1.

Let $(P, \mathcal{X})$ be a mixed Shimura datum with Σ a K_f-admissible complete polyhedral cone decomposition such that $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ is a projective variety (with the complement $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C}) \backslash M^{K_f}(P, \mathcal{X})(\mathbb{C})$ a union of smooth divisors with only normal crossings) as in Theorem 4.6.8. Consider the rational boundary component $(P_1, \mathcal{X}_1)$ related to the parabolic subgroup $Q \subset P$ with the $P_1(\mathbb{R})\cdot\mathrm{Stab}_{Q(\mathbb{R})}(\mathcal{X}_1)\cdot U(\mathbb{C})$-equivariant open embedding $\mathcal{X}^+_{(P_1,\mathcal{X}_1)} \to \mathcal{X}_1$. Suppose $\mathcal{X}_1^0 \subset \mathcal{X}_1$ is the connected component corresponding to $\mathcal{X}^0$. $C(\mathcal{X}^0, P_1) \subset U_1(\mathbb{R})(-1)$ is the open convex cone as in Proposition 4.4.10. An element $P_1(\mathbb{Q})$ stabilizes $\mathcal{X}^0$ if and only if it stabilizes $\mathcal{X}_1^0$. We defined the following groups in the introduction.

$$\begin{aligned}
\Gamma &= \mathrm{Stab}_{P(\mathbb{Q})}(\mathcal{X}^0) \cap p_f K_f p_f^{-1} \\
\Gamma_{ZU_1} &= \big(\mathrm{Id}_{Z(P)(\mathbb{Q})}(\mathcal{X}) \times U_1(\mathbb{Q})\big) \cap p_f K_f p_f^{-1} \\
\Gamma_Z &= \big(\mathrm{Id}_{Z(P)(\mathbb{Q})}(\mathcal{X})\big) \cap p_f K_f p_f^{-1} \\
\Gamma_{U_1} &= \frac{\Gamma_{ZU_1}}{\Gamma_Z} \subset \frac{\Gamma}{\Gamma_Z} \\
&\quad \big(\Gamma_{U_1} = p_{U_1}(\Gamma_{ZU_1}) \text{ for } p_{U_1} : Z(P) \times U_1 \to U_1\big)
\end{aligned}$$

We need a few more analogous groups.

$$\begin{aligned}
\Gamma_1 &\coloneqq \mathrm{Stab}_{P_1(\mathbb{Q})}(\mathcal{X}^0) \cap p_f K_f p_f^{-1} \\
&= \Gamma \cap P_1(\mathbb{Q}) \subset \Gamma \\
\Omega_{Z_1} &\coloneqq \mathrm{Id}_{Z(P_1)(\mathbb{Q})}(\mathcal{X}_1) \cap p_f K_f p_f^{-1} \\
&= \Gamma_Z \cap P_1(\mathbb{Q}) \subset \Gamma_Z \\
\Omega_{Z_1U_1} &\coloneqq (\mathrm{Id}_{Z(P_1)(\mathbb{Q})}(\mathcal{X}_1) \times U_1(\mathbb{Q})) \cap p_f K_f p_f^{-1} \\
&= \Gamma_{ZU_1} \cap P_1(\mathbb{Q}) \subset \Gamma_{ZU_1}
\end{aligned}$$

$$\begin{aligned}\Omega_{U_1} &\coloneqq \Omega_{Z_1U_1}/\Omega_{Z_1} \\ &= p_{U_1}(\Omega_{Z_1U_1}) \text{ for } p_{U_1} : Z(P_1)\times U_1 \to U_1\end{aligned}$$

Note that $\Omega_{U_1} = \Omega_{Z_1U_1}/\Omega_{Z_1} \subset \Gamma_{ZU_1}/\Gamma_Z = \Gamma_{U_1}$, as $\Omega_{Z_1U_1}\cap\Gamma_Z = \Omega_{Z_1}$ (by Proposition 4.4.8, $\mathrm{Id}_{Z(P_1)(\mathbb{Q})}(\mathcal{X}_1) = P_1(\mathbb{Q})\cap \mathrm{Id}_{Z(P)(\mathbb{Q})}(\mathbb{Q})$). Also, $\Omega_{Z_1}\subset\Omega_{Z_1U_1}\subset\Gamma_1$.

Let $K_f^1 \coloneqq P_1(\mathbb{A}_f)\cap p_fK_fp_f^{-1}$. Recall that

$$\mathcal{U}(P_1,\mathcal{X}_1,p_f) \coloneqq P_1(\mathbb{Q})\backslash\mathcal{X}^+_{(P_1,\mathcal{X}_1)}\times P_1(\mathbb{A}_f)/K_f^1$$
$$\Sigma_1^0 \coloneqq \left(([\cdot p_f]^*\Sigma)\,|_{(P_1,\mathcal{X}_1)}\right)^0$$

and $\overline{\mathcal{U}}(P_1,\mathcal{X}_1,p_f) \coloneqq \mathrm{Int}(\mathrm{Cl}(\mathcal{U}(P_1,\mathcal{X}_1,p_f)))$ is the interior of the closure of $\mathcal{U}(P_1,\mathcal{X}_1,p_f)$ in $M^{K_f^1}(P_1,\mathcal{X}_1,\Sigma_1^0)(\mathbb{C})$.

Proof of Theorem 1.1. (1) By definition, Γ_Z acts trivially on $\mathcal{X}^0$. Thus $\Gamma\backslash\mathcal{X}^0$ is actually $(\Gamma/\Gamma_Z)\backslash\mathcal{X}^0$. Further, let $\gamma\in\Gamma$ fix a point $x_0\in\mathcal{X}^0$. Then by Lemma 4.3.5, γ lies in $Z(P)(\mathbb{Q})$ and hence must fix every point $x\in\mathcal{X}$, i.e. $\gamma\in\mathrm{Id}_{Z(P)(\mathbb{Q})}(\mathcal{X})\cap\Gamma = \Gamma_Z$. We have shown that Γ/Γ_Z acts freely on $\mathcal{X}^0$. Since by Theorem 4.6.9, $\Gamma\backslash\mathcal{X}^0$ is a (connected) quasi-projective normal variety over $\mathbb{C}$, and $\Gamma'/\Gamma'_Z \trianglelefteq \Gamma/\Gamma_Z$, we see that $\frac{\Gamma'}{\Gamma'_Z}\backslash\mathcal{X}^0 \to \frac{\Gamma}{\Gamma_Z}\backslash\mathcal{X}^0$ is a Galois cover with Galois group $\frac{\Gamma/\Gamma_Z}{\Gamma'/\Gamma'_Z}$.

(2) Let $\delta_Q \coloneqq \mathrm{Stab}_{Q(\mathbb{Q})}(\mathcal{X}_1)\cap\left(P_1(\mathbb{A}_f)\cdot p_fK_fp_f^{-1}\right)$ and $\Delta_Q\coloneqq\delta_Q/P_1(\mathbb{Q})$. By Remark 4.6.6, the map $\overline{\mathcal{U}}(P_1,\mathcal{X}_1,p_f)\to M^{K_f}(P,\mathcal{X},\Sigma)(\mathbb{C})$ factors through $\Delta_Q\backslash\overline{\mathcal{U}}(P_1,\mathcal{X}_1,p_f)$ (resp. $\mathcal{U}(P_1,\mathcal{X}_1,p_f)\to M^{K_f}(P,\mathcal{X})(\mathbb{C})$ factors through $\Delta_Q\backslash\mathcal{U}(P_1,\mathcal{X}_1,p_f)$). The

above objects fit into a diagram.

$$
\begin{tikzcd}
\mathcal{U}(P_1, \mathcal{X}_1, p_f) \arrow[r, two heads] \arrow[d, tail] \arrow[dr, tail] & \Delta_Q \backslash \mathcal{U}(P_1, \mathcal{X}_1, p_f) \arrow[rr, "{[\cdot p_f]}"] \arrow[dr, tail] & & M^{K_f}(P, \mathcal{X})(\mathbb{C}) \arrow[d, tail] \\
M^{K_f^1}(P_1, \mathcal{X}_1)(\mathbb{C}) \arrow[d, tail] & \overline{\mathcal{U}}(P_1, \mathcal{X}_1, p_f) \arrow[r, two heads] \arrow[dl, tail] & \Delta_Q \backslash \overline{\mathcal{U}}(P_1, \mathcal{X}_1, p_f) \arrow[r] & M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C}) \\
M^{K_f^1}(P_1, \mathcal{X}_1, \Sigma_1^0)(\mathbb{C}) & & &
\end{tikzcd}
$$

In terms of connected components, the above diagram translates into a diagram as below.

$$
\begin{tikzcd}
\Gamma_1 \backslash \mathcal{X}^0 \arrow[r, two heads] \arrow[d, tail] \arrow[dr, tail] & \Gamma_Q \backslash \mathcal{X}^0 = \frac{\Gamma_Q}{\Gamma_1} \backslash (\Gamma_1 \backslash \mathcal{X}^0) \arrow[rr, two heads] \arrow[dr, tail] & & \Gamma \backslash \mathcal{X}^0 \arrow[d, tail] \\
\Gamma_1 \backslash \mathcal{X}_1^0 \arrow[d, tail] & (\Gamma_1 \backslash \mathcal{X}^0)_{\Sigma_1^0} \arrow[r, two heads] \arrow[dl, tail] & \frac{\Gamma_Q}{\Gamma_1} \Big\backslash (\Gamma_1 \backslash \mathcal{X}^0)_{\Sigma_1^0} \arrow[r] & (\Gamma \backslash \mathcal{X}^0)_{\Sigma} \\
(\Gamma_1 \backslash \mathcal{X}_1^0)_{\Sigma_1^0} & & &
\end{tikzcd}
$$

where $\Gamma_Q = \mathrm{Stab}_{Q(\mathbb{Q})}(\mathcal{X}^0) \cap p_f K_f p_f^{-1} = Q(\mathbb{Q}) \cap \Gamma$ and $(\Gamma_1 \backslash \mathcal{X}^0)_{\Sigma_1^0} \subset \overline{\mathcal{U}}(P_1, \mathcal{X}_1, p_f)$ is the interior of the closure of $\Gamma_1 \backslash \mathcal{X}^0 \subset (\Gamma_1 \backslash \mathcal{X}_1^0)_{\Sigma_1^0}$.

Now choose a point $x_0 \in \mathcal{X}^0$. The $U_1(\mathbb{C})$-orbit map $U_1(\mathbb{C}) \to \mathcal{X}_1^0$, $u_1 \mapsto u_1 \cdot x_0$ descends to a map $\Omega_{U_1} \backslash U_1(\mathbb{C}) \to \Gamma_1 \backslash \mathcal{X}_1^0$. According to Subsections 4.3 and 4.6, this is the inclusion of a closed fiber into the analytic $\Omega_{U_1} \backslash U_1(\mathbb{C})$-torsor $M^{K_f^1}(P_1, \mathcal{X}_1)(\mathbb{C}) \to M^{\pi_{U_1}(K_f^1)}(P_1/U_1, \mathcal{X}_1/U_1)(\mathbb{C})$, relative to which the torus embedding $\Gamma_1 \backslash \mathcal{X}_1^0 \rightarrowtail (\Gamma_1 \backslash \mathcal{X}_1^0)_{\Sigma_1^0}$ is constructed. (Note $T \coloneqq \Omega_{U_1} \backslash U_1(\mathbb{C})$ is an algebraic torus.)

We claim that under this orbit map, $U_1(\mathbb{R}) + C(\mathcal{X}^0, P_1)$ is mapped into $\mathcal{X}^0$.

Indeed, let $u_1 \cdot c \in U_1(\mathbb{R}) + C(\mathcal{X}^0, P_1)$ and $\mathrm{im} : \mathcal{X}_1^0 \to U_1(\mathbb{R})(-1)$ be the map of imaginary part. Then $\mathrm{im}(u_1 \cdot c \cdot x_0) = (u_1 \cdot c) \cdot \mathrm{im}(x_0) = c + \mathrm{im}(x_0) \in C(\mathcal{X}^0, P_1) + C(\mathcal{X}^0, P_1) \subset C(\mathcal{X}^0, P_1)$ as $C(\mathcal{X}^0, P_1)$ is an open convex cone. Denote $C(\mathcal{X}^0, P_1)$ by C. It follows that we have an induced map $\Omega_{U_1} \backslash (U_1(\mathbb{R}) + C) \rightarrowtail \Gamma_1 \backslash \mathcal{X}^0$, compatible with $\Omega_{U_1} \backslash U_1(\mathbb{C}) \rightarrowtail \Gamma_1 \backslash \mathcal{X}_1^0$.

By assumption, $\Sigma_1^0(\mathcal{X}_1^0, P_1, 1) = \Sigma(\mathcal{X}^0, P_1, p_f) p_f^{-1}$ is a complete rational polyhedral cone decomposition of $C^*(\mathcal{X}^0, P_1) \subset U_1(\mathbb{R})(-1)$. In particular, there exists a $\sigma \in \Sigma_1^0$ of top dimension such that $-\sigma^0 \subset C(\mathcal{X}^0, P_1)$. Then $T_\sigma \rightarrowtail (\Gamma_1 \backslash \mathcal{X}_1^0)_\sigma \subset (\Gamma_1 \backslash \mathcal{X}_1^0)_{\Sigma_1^0}$.

By the last assertion of Lemma 5.2.2, $(\Gamma_1 \backslash \mathcal{X}^0)_{\Sigma_1^0}$ is an open neighborhood of the entire σ-stratum $(\Gamma_1 \backslash \mathcal{X}_1^0)_\sigma^\sim$. In particular, it is a neighborhood of the point $T_\sigma^\sim$. Therefore, a neighborhood $D \subset T_\sigma$ of this point will map into $(\Gamma_1 \backslash \mathcal{X}^0)_{\Sigma_1^0}$ and then into $(\Gamma \backslash \mathcal{X}^0)_\Sigma$, as shown above.

We take $\sigma_s \subset \sigma$ to be a rational convex polyhedral cone smooth with respect to the larger lattice $\Gamma_{U_1} \supset \Omega_{U_1}$. This gives a morphism $T_{\sigma_s} \to T_\sigma$ mapping $T_{\sigma_s}^\sim$ to $T_\sigma^\sim$. Thus a neighborhood of $T_{\sigma_s}^\sim$ will also map into $(\Gamma \backslash \mathcal{X}^0)_\Sigma$.

However, $\mathrm{Id}_{Z(P)(\mathbb{Q})}(\mathcal{X}) \times U_1(\mathbb{Q}) \subset \mathrm{Stab}_{Q(\mathbb{Q})}(\mathcal{X}^0)$, so $\Gamma_{ZU_1} \subset \Gamma_Q$ and $\Gamma_{ZU_1}/\Omega_{Z_1 U_1} \subset \Gamma_Q/\Gamma_1$. It follows that the maps $T_{\sigma_s} \to (\Gamma_1 \backslash \mathcal{X}_1^0)_{\Sigma_1^0}$ and $\Omega_{U_1} \backslash (U_1(\mathbb{R}) + C) \to \Gamma_1 \backslash \mathcal{X}^0$ descend to the corresponding quotients, inducing the following diagram.

$$\begin{tikzcd}
D^{\circ} \arrow[r, tail] \arrow[dd, tail] & \Gamma_{U_1}\backslash(U_1(\mathbb{R})+C) \arrow[r] \arrow[d, tail] & \frac{\Gamma_Q}{\Gamma_1}\backslash(\Gamma_1\backslash\mathcal{X}^0) \arrow[rr] \arrow[d, tail] \arrow[dr, tail] & & \Gamma\backslash\mathcal{X}^0 \arrow[d, tail] \\
& \Gamma_{U_1}\backslash U_1(\mathbb{C}) \arrow[r] \arrow[d, tail] & \frac{\Gamma_Q}{\Gamma_1}\backslash(\Gamma_1\backslash\mathcal{X}^0_1) \arrow[d, tail] & \frac{\Gamma_Q}{\Gamma_1}\backslash(\Gamma_1\backslash\mathcal{X}^0)_{\Sigma^0_1} \arrow[r] \arrow[dl, tail] & (\Gamma\backslash\mathcal{X}^0)_{\Sigma} \\
D \arrow[r, tail] \arrow[urrr, dashed] & (\Gamma_{U_1}\backslash U_1(\mathbb{C}))_{\sigma_s} \arrow[r] & \frac{\Gamma_Q}{\Gamma_1}\backslash(\Gamma_1\backslash\mathcal{X}^0_1)_{\Sigma^0_1} & &
\end{tikzcd}$$

As $\Gamma_{U_1}/\Omega_{U_1}$ is finite, there exists a $\Gamma_{U_1}/\Omega_{U_1}$-invariant neighborhood of $T^{\sim}_{\sigma}$ that maps into $(\Gamma_1\backslash\mathcal{X}^0)_{\Sigma^0_1}$. This shows a neighborhood of $(\Gamma_{U_1}\backslash U_1(\mathbb{C}))^{\sim}_{\sigma_s}$ maps into $\frac{\Gamma_Q}{\Gamma_1}\backslash(\Gamma_1\backslash\mathcal{X}^0_1)_{\Sigma^0_1}$, viewing $\Gamma_{U_1}\backslash U_1(\mathbb{C})$ and $(\Gamma_{U_1}\backslash U_1(\mathbb{C}))_{\sigma_s}$ as quotients of $T = \Omega_{U_1}\backslash U_1(\mathbb{C})$ and T_{σ_s} respectively by $\Gamma_{U_1}/\Omega_{U_1} \subset T$.

Now that σ_s is smooth of top dimension with respect to Γ_{U_1}, by Lemma 5.2.6, a polydisc neighborhood D of $(\Gamma_{U_1}\backslash U_1(\mathbb{C}))^{\sim}_{\sigma_s}$ intersects $\Gamma_{U_1}\backslash(U_1(\mathbb{R})+C)$ in a product of punctured discs D°.

Let's compute the image of $\pi_1(D^{\circ}) \to \pi_1(\Gamma\backslash\mathcal{X}^0) \to \mathrm{Aut}(\Gamma'\backslash\mathcal{X}^0 \to \Gamma\backslash\mathcal{X}^0)$. By Proposition 4.4.10, C is $(W \cap U_1)(\mathbb{R})(-1)$-translation invariant and $C/(W \cap U_1)(\mathbb{R})(-1)$ is a self-adjoint homogeneous open cone. Thus

$$\frac{U_1(\mathbb{R})}{(W \cap U_1)(\mathbb{R})} + \frac{C}{(W \cap U_1)(\mathbb{R})(-1)}$$

is a tube domain and in particular, topologically simply connected. Then $U_1(\mathbb{R}) + C$ is a $(W \cap U_1)(\mathbb{C})$-vector bundle over a simply connected base and so simply connected as well. The map $U(\mathbb{R}) + C \to \mathcal{X}^0$ is Γ_{ZU_1}-equivariant, where Γ_{ZU_1} acts via $\Gamma_{ZU_1} \rightarrowtail \Gamma_{U_1}$. By Proposition 4.3.2, Γ/Γ_Z acts properly discontinuously (and freely) on $\mathcal{X}^0$. Thus the image on π_1 is just

$$\begin{aligned} &= \mathrm{Im}\left(\frac{\Gamma_{ZU_1}}{\Gamma_Z} \rightarrowtail \frac{\Gamma}{\Gamma_Z} \twoheadrightarrow \frac{\Gamma/\Gamma_Z}{\Gamma'/\Gamma'_Z} = \frac{\Gamma}{\Gamma_Z \cdot \Gamma'}\right) \\ &= \frac{\Gamma_{ZU_1}}{\Gamma_Z \cdot \Gamma'_{ZU_1}} = \frac{\Gamma_{ZU_1}/\Gamma_Z}{\Gamma'_{ZU_1}/\Gamma'_Z} = \frac{\Gamma_{U_1}}{\Gamma'_{U_1}}. \end{aligned}$$

Apply Theorem 3.7 and we have done the proof of the second part.

(3) The assertion will be a direct consequence of the second part combined with Theorem 3.3 once we show there exists a Γ/Γ'-equivariant smooth compactification $Y \to (\Gamma\backslash\mathcal{X}^0)_\Sigma$ extending $\Gamma'\backslash\mathcal{X}^0 \to \Gamma\backslash\mathcal{X}^0$.

Since Σ is a complete K_f-admissible cone decomposition, it is automatically a complete $K'_f(\trianglelefteq K_f)$-admissible cone decomposition. Thus (by Theorem 4.6.5) $M^{K'_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ is a compact normal complex space with a finite morphism to $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$, the quotient of $M^{K'_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ by the finite group K_f/K'_f. Now the Generalized Riemann Existence Theorem (cf. [Har77, Appendix B, Theorem 3.2]) implies that $M^{K'_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$ is a (proper) normal scheme admitting a finite morphism to a projective variety, hence projective itself. Now equivariant resolution of singularities produces a K_f/K'_f-equivariant smooth compactification $p : Y' \to M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$.

By Proposition 4.6.7, $(\Gamma\backslash\mathcal{X}^0)_\Sigma$ is a smooth projective variety, being a connected component of $M^{K_f}(P, \mathcal{X}, \Sigma)(\mathbb{C})$. Taking the preimage $Y := p^{-1}((\Gamma\backslash\mathcal{X}^0)_\Sigma) \subset Y'$, we get a Γ/Γ'-equivariant compactification $Y \to (\Gamma\backslash\mathcal{X}^0)_\Sigma$.

(4) We just need to find $K'_f \trianglelefteq K_f$ such that $\Gamma'_{U_1} \subset p \cdot \Gamma_{U_1}$. Define the following

groups.

$$
\begin{aligned}
K_f^{ZU_1} &\coloneqq (Z(P)(\mathbb{A}_f) \times U_1(\mathbb{A}_f)) \cap p_f K_f p_f^{-1} \\
K_f^{U_1} &\coloneqq U_1(\mathbb{A}_f) \cap p_f K_f p_f^{-1} \\
K_f^{Z} &\coloneqq Z(P)(\mathbb{A}_f) \cap p_f K_f p_f^{-1}
\end{aligned}
$$

Let $\gamma_{U_1} \coloneqq U_1(\mathbb{Q}) \cap K_f^{U_1}$. Then $\gamma_{U_1} \subset \Gamma_{U_1}$. We may choose a smaller $\widetilde{K}_f^{U_1} \subset K_f^{U_1}$ such that $\widetilde{\gamma}_{U_1} \coloneqq U_1(\mathbb{Q}) \cap \widetilde{K}_f^{U_1} (\subset \gamma_{U_1})$ is of the form $(n_1\mathbb{Z}) \times \cdots \times (n_r\mathbb{Z})$ (regarding $U_1(\mathbb{Q})$ as $\mathbb{Q}^r$ with $r = \dim U_1$). Then $p \cdot \widetilde{\gamma}_{U_1} = U_1(\mathbb{Q}) \cap \widehat{K}_f^{U_1}$ for some $\widehat{K}_f^{U_1} \subset \widetilde{K}_f^{U_1}$. But $K_f^Z \times \widehat{K}_f^{U_1}$ is an open compact subgroup of $Z(P)(\mathbb{A}_f) \times U_1(\mathbb{A}_f)$. Since $Z(P) \times U_1$ is a (closed) subgroup of P, we may find neat open compact $K'_f \trianglelefteq K_f$ such that ${K_f^{ZU_1}}' \coloneqq p_f K'_f p_f^{-1} \cap (Z(P)(\mathbb{A}_f) \times U_1(\mathbb{A}_f)) \subset K_f^Z \times \widehat{K}_f^{U_1}$. Then $\Gamma'_{ZU_1} \subset \left(\mathrm{Id}_{Z(P)(\mathbb{Q})}(\mathcal{X}) \times U_1(\mathbb{Q})\right) \cap \left(K_f^Z \times \widehat{K}_f^{U_1}\right) = \Gamma_Z \times (p \cdot \widetilde{\gamma}_{U_1})$. Thus $\Gamma'_{U_1} \subset p \cdot \widetilde{\gamma}_{U_1} \subset p \cdot \gamma_{U_1} \subset p \cdot \Gamma_{U_1}$. The inclusive relations among the groups are as follows.

$$
\begin{tikzcd}
\Gamma_{U_1} & \Gamma_{ZU_1} \arrow[l, two heads] \arrow[r, tail] & K_f^{ZU_1} \arrow[r, tail] & p_f K_f p_f^{-1} \\
\gamma_{U_1} \arrow[u, tail] & \Gamma_Z \times \gamma_{U_1} \arrow[l, two heads] \arrow[r, tail] \arrow[u, tail] & K_f^Z \times K_f^{U_1} \arrow[u, tail] & \\
\widetilde{\gamma}_{U_1} \arrow[u, tail] & \Gamma_Z \times \widetilde{\gamma}_{U_1} \arrow[l, two heads] \arrow[r, tail] \arrow[u, tail] & K_f^Z \times \widetilde{K}_f^{U_1} \arrow[u, tail] & \\
p \cdot \widetilde{\gamma}_{U_1} \arrow[u, tail] & \Gamma_Z \times (p \cdot \widetilde{\gamma}_{U_1}) \arrow[l, two heads] \arrow[r, tail] \arrow[u, tail] & K_f^Z \times \widehat{K}_f^{U_1} \arrow[u, tail] & \\
\Gamma'_{U_1} \arrow[u, tail] & \Gamma'_{ZU_1} \arrow[l, two heads] \arrow[r, tail] \arrow[u, tail] & {K_f^{ZU_1}}' \arrow[r, tail] \arrow[u, tail] & p_f K'_f p_f^{-1} \arrow[uuuu, tail]
\end{tikzcd}
$$

The surjections $\Gamma_{U_1} \twoheadrightarrow \Gamma_{U_1}/\Gamma'_{U_1} \twoheadrightarrow \Gamma/(p \cdot \Gamma_{U_1})$ implies that

$$\frac{\Gamma_{U_1}}{\Gamma'_{U_1}} \otimes \frac{\mathbb{Z}}{p\mathbb{Z}} \xrightarrow{\sim} \frac{\Gamma_{U_1}}{p \cdot \Gamma_{U_1}} \otimes \frac{\mathbb{Z}}{p\mathbb{Z}} \Longrightarrow \operatorname{rank} \frac{\Gamma_{U_1}}{\Gamma'_{U_1}} \otimes \frac{\mathbb{Z}}{p\mathbb{Z}} = \operatorname{rank} \frac{\Gamma_{U_1}}{p \cdot \Gamma_{U_1}} \otimes \frac{\mathbb{Z}}{p\mathbb{Z}} = \dim U_1.$$

This finishes the proof of our main theorem. $\square$

6 Examples of lower bounds

Note that, given the notations in Theorem 1.1, if we take $(P_1, \mathcal{X}_1)$ to be the (unique) improper boundary component with $\mathcal{X}^0 \subset \mathcal{X}^+_{(P_1,\mathcal{X}_1)}$, then $U_1 = U$ and $\mathrm{ed}_{\mathbb{C}}(\Gamma'\backslash\mathcal{X}^0 \to \Gamma\backslash\mathcal{X}^0; p) \geq \mathrm{rank}_p(\Gamma_U/\Gamma'_U)$ where Γ_U is the image of

$$\left(\mathrm{Id}_{Z(P)(\mathbb{Q})}(\mathcal{X}) \times U(\mathbb{Q})\right) \cap p_f K_f p_f^{-1}$$

under $Z(P) \times U \to U$. This gives a lower bound without computing any proper boundary components.

Let's look at a few examples.

6.1 The torus case of dimension 1

Recall that in Example 4.2.11, we defined the mixed Shimura datum $(P_0, \mathcal{X}_0)$ as the unipotent extension of $(\mathbb{G}_m, \mathcal{H}_0)$, where $U_0 = \mathbb{G}_{a,\mathbb{Q}}$ and $P_0 = U_0 \rtimes \mathbb{G}_{m,\mathbb{Q}}$ with $\mathbb{G}_{m,\mathbb{Q}}$ acting via multiplication. The image $h(\mathcal{H}_0) = \{N\}$ with $N : \mathbb{S} \to \mathbb{G}_{m,\mathbb{R}}, z \mapsto z\bar{z}$. Let $\iota : \mathbb{G}_{m,\mathbb{Q}} \rightarrowtail P_0$ be the inclusion, then $\mathcal{X}_0$ consists of two connected components $\mathcal{X}_0^{\pm}$, each of which projects isomorphically onto $U_0(\mathbb{C}) \cdot (\iota \circ N) \xleftarrow{\sim} U_0(\mathbb{C})$. Take a neat open compact subgroup $K_f^* \subset \mathbb{G}_m(\hat{\mathbb{Z}})$ and take $K_f^U(d) := d \cdot U_0(\hat{\mathbb{Z}})$. Then $K_f^P(d) := K_f^U(d) \rtimes K_f^* \subset P_0(\mathbb{A}_f)$ is a neat open compact subgroup.

Since $Z(P_0) = 1$, (for $p_f = (0, g_f) \in \mathbb{G}_m(\hat{\mathbb{Z}})$) one sees that $\Gamma = \Gamma_U = U_0(\mathbb{Q}) \cap K_f^P(d) = d \cdot U_0(\mathbb{Z}) =: \Gamma_U(d)$. The $d\mathbb{Z}/nd\mathbb{Z}$-torsor $\Gamma_U(nd)\backslash U_0(\mathbb{C}) \to \Gamma_U(d)\backslash U_0(\mathbb{C})$ is just the $\mathbb{Z}/n\mathbb{Z}$-torsor $\mathbb{G}_{m,\mathbb{C}} \xrightarrow{n} \mathbb{G}_{m,\mathbb{C}}$. By our main theorem, we find that $(1 \geq$

) $\mathrm{ed}_{\mathbb{C}}(\mathbb{G}_{m,\mathbb{C}} \xrightarrow{n} \mathbb{G}_{m,\mathbb{C}}; p) \geq 1$ (hence equal to 1, i.e. p-incompressible) if and only if $p \,|\, n$.

This assertion can also be proved directly. If $p \nmid n$, then base change by $\mathbb{G}_{m,\mathbb{C}} \xrightarrow{n} \mathbb{G}_{m,\mathbb{C}}$ itself makes the torsor trivial and thus $\mathrm{ed}_{\mathbb{C}}(\mathbb{G}_{m,\mathbb{C}} \xrightarrow{n} \mathbb{G}_{m,\mathbb{C}}; p) = 0$. Otherwise, it cannot become trivial after a prime to p extension, by the following elementary result (applied to the case that $p \,|\, a$, $p \nmid b$ and thus $d < a$).

Proposition 6.1. *Let k be a field and $a, b \in \mathbb{Z}_{\geq 0}$. Write $\mathbb{G}_m$ for $\mathbb{G}_{m,k}$. Denote by $\mathbb{G}_m \times_{(a,b)} \mathbb{G}_m$ the fiber product of the diagram $\mathbb{G}_m \xrightarrow{a} \mathbb{G}_m \xleftarrow{b} \mathbb{G}_m$.*

(1) If $a = b = 0$, $\mathbb{G}_m \times_{(a,b)} \mathbb{G}_m = \mathbb{G}_m \times \mathbb{G}_m$.

(2) If a, b are not both 0, let $d = \gcd(a, b)$. Then $\mathbb{G}_m \times_{(a,b)} \mathbb{G}_m$ has at most d connected components.

Proof. Let $a = a'd$ and $b = b'd$. It suffices to assume k is algebraically closed. It is not hard to verify that the following two diagrams are pullbacks.

$$\gcd(a,b) = 1 \qquad\qquad \begin{tikzcd} \mathbb{G}_m \arrow[r, "a"] \arrow[d, "b"] & \mathbb{G}_m \arrow[d, "b"] \\ \mathbb{G}_m \arrow[r, "a"] & \mathbb{G}_m \end{tikzcd}$$

$$a|b(= b'a) \qquad\qquad \begin{tikzcd} \mu_d \times \mathbb{G}_m \arrow[r, "\mathrm{pr}_2"] \arrow[d, "u \cdot y^{b'}"] & \mathbb{G} \arrow[d, "b"] \\ \mathbb{G}_m \arrow[r, "a"] & \mathbb{G}_m \end{tikzcd}$$

It follows that $\mathbb{G}_m \times_{(a,b)} \mathbb{G}_m$ is the pullback $\mathbb{G}_m \times_{(a', uy^{b'})} (\mu_d \times \mathbb{G}_m)$ of $\mathbb{G}_m \xrightarrow{a'} \mathbb{G}_m \leftarrow \mu_d \times \mathbb{G}_m$ where the leftward arrow is $u \cdot y^{b'} \hookleftarrow (u, y)$. For each connected component $\{u_i\} \times \mathbb{G}_m$ with $u_i = \zeta_{b'd}^{b'i}$ (and $\zeta_{b'd} = \zeta_b$ any primitive b-th root of unity), the composition $\mathbb{G}_m \to \{u_i\} \times \mathbb{G}_m \to \mathbb{G}_m$ with the first arrow defined by $y \mapsto (u_i, \zeta_{b'd}^{-i} \cdot y)$ is just $y \mapsto y^{b'}$. By the first diagram, the preimage in $\mathbb{G}_m \times_{(a', uy^{b'})} (\mu_d \times \mathbb{G}_m)$ above

$\{u_i\} \times \mathbb{G}_m$ is isomorphic to $\mathbb{G}_m$. Thus there are d connected components. The case $a = b = 0$ is straightforward. $\square$

6.2 The torus case of dimension n

Generalize the first example to the unipotent extension $(P, \mathcal{X}) \to (\mathbb{G}_{m,\mathbb{Q}}, \mathcal{H}_0)$ with $U = \mathbb{Q}^r$ and $P = U \rtimes \mathbb{G}_{m,\mathbb{Q}}$. We take $K_f^U(d_1, \ldots, d_r) \subset U(\hat{\mathbb{Z}}) = \hat{\mathbb{Z}}^r$ to be the open compact subgroup $(d_1\hat{\mathbb{Z}}) \times \cdots \times (d_r\hat{\mathbb{Z}})$. Let $K_f^* \subset \mathbb{G}_m(\hat{\mathbb{Z}})$ be defined as above and $K_f^P = K_f^U(d_1, \ldots, d_r) \rtimes K_f^*$. For $p_f = (0, g_f) \in \mathbb{G}_m(\hat{\mathbb{Z}})$, $p_f K_f^P p_f^{-1} = K_f^P$. Thus $\Gamma = \Gamma_U = U(\mathbb{Q}) \cap K_f^P = (d_1\mathbb{Z}) \times \cdots (d_r\mathbb{Z}) =: \Gamma_U(d_1, \ldots, d_r)$.

Again $\mathcal{X}$ has two connected components, each of which is isomorphic to $U(\mathbb{C})$. The $(\mathbb{Z}/n_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/n_r\mathbb{Z})$-torsor

$$\Gamma_U(n_1d_1, \ldots, n_rd_r)\backslash U(\mathbb{C}) \to \Gamma_U(d_1, \ldots, d_r)\backslash U(\mathbb{C})$$

is just $\mathbb{G}_{m,\mathbb{C}} \times \cdots \times \mathbb{G}_{m,\mathbb{C}} \xrightarrow{(n_1,\ldots,n_r)} \mathbb{G}_{m,\mathbb{C}} \times \cdots \times \mathbb{G}_{m,\mathbb{C}}$. Denote $(n_1, \ldots, n_r)$ by (n). Theorem 1.1 implies that $\mathrm{ed}_{\mathbb{C}}(\mathbb{G}_{m,\mathbb{C}}^r \overset{(n)}{\to} \mathbb{G}_{m,\mathbb{C}}^r; p) \geq \mathrm{rank}_p((\mathbb{Z}/n_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/n_r\mathbb{Z}))$

Note that by Theorem 2.4 and Remark 2.6, we can deduce the upper bound $\mathrm{ed}_{\mathbb{C}}(\mathbb{G}_{m,\mathbb{C}}^r \overset{(n)}{\to} \mathbb{G}_{m,\mathbb{C}}^r; p) \leq \mathrm{rank}_p((\mathbb{Z}/n_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/n_r\mathbb{Z}))$. Hence equality holds, i.e.

$$\mathrm{ed}_{\mathbb{C}}(\mathbb{G}_{m,\mathbb{C}}^r \overset{(n)}{\to} \mathbb{G}_{m,\mathbb{C}}^r; p) = \mathrm{rank}_p(\frac{\mathbb{Z}}{n_1\mathbb{Z}} \times \cdots \times \frac{\mathbb{Z}}{n_r\mathbb{Z}}).$$

Thus this torsor is p-incompressible if and only if $\mathrm{rank}_p(\frac{\mathbb{Z}}{n_1\mathbb{Z}} \times \cdots \times \frac{\mathbb{Z}}{n_r\mathbb{Z}}) = r$, or equivalently $p \,|\, n_i$ for all $1 \leq i \leq r$.

Remark 6.2. Certainly, we may as well apply the fixed point method directly to the torsor $\mathbb{G}_{m,\mathbb{C}}^r \stackrel{(n)}{\to} \mathbb{G}_{m,\mathbb{C}}^r$ since as a toric variety $\mathbb{G}_{m,\mathbb{C}}^r$ possesses the compactification $(\mathbb{P}^1_{\mathbb{C}})^r$. The proof is similar to that of [BF24, Theorem 34].

6.3 Siegel modular varieties

Let $(CSp_{2n}, \mathcal{H}_{2n})$ be the pure Shimura datum defined as in Example 4.2.5, for $V = \mathbb{Q}^{2n}$ and $\psi : V \times V \to \mathbb{Q}$ the alternating form. (The standard basis $e_{\pm i}(1 \leq i \leq n)$ is a symplectic basis for ψ. Thus CSp_{2n} is naturally defined over $\mathbb{Z}$.) In order to get a better lower bound, we need to know more about the rational boundary components of $(CSp_{2n}, \mathcal{H}_{2n})$.

Suppose $V_0 \subset V$ is a totally isotropic subspace and let $Q \coloneqq \mathrm{Stab}_{CSp_{2n}}(V_0)$. Then Q is a $\mathbb{Q}$-admissible parabolic subgroup of CSp_{2n} and this gives a one-to-one correspondence between the class of totally isotropic subspaces and $\mathbb{Q}$-admissible parabolic subgroups (cf. [Pin90, 4.25]). (We may assume without loss of generality that $V_0 = \langle e_1, \ldots, e_r \rangle$ for $0 \leq r \leq n$.)

Fix such a V_0 and Q. Then the rational boundary component $(P_1, \mathcal{X}_1)$ of $(P, \mathcal{X})$ associated with Q can be described as follows. Let $x \in \mathcal{X}^+_{(P_1, \mathcal{X}_1)}$ and $x_1 \in \mathcal{X}_1$ be its image. Then $h_{x_1} : \mathbb{S}_{\mathbb{C}} \to P_{1,\mathbb{C}} \subset Q_{\mathbb{C}} \subset CSp_{2n,\mathbb{C}}$ induces a Hodge structure on V with weight filtration as follows.

$$W_{-2}^{h_{x_1}} V = V_0,$$

$$W_{-1}^{h_{x_1}} V = V_0^{\perp},$$

$$W_0^{h_{x_1}} V = V.$$

(Note that this is consistent with the fact that Q should be the subgroup of CSp_{2n} preserving the above weight filtration.) Then we have

$$P_1 = \{g \in Q \,|\, g|_{V/V_0^\perp} = \mathrm{id}\}$$

$$W_1 = \left\{ g \in Q \;\middle|\; \begin{array}{l} g|_{V/V_0^\perp} = \mathrm{id} \\ g|_{V_0^\perp/V_0} = \mathrm{id} \\ g|_{V_0} = \mathrm{id} \end{array} \right\}$$

$$U_1 = \left\{ g \in Q \;\middle|\; \begin{array}{l} g|_{V/V_0} = \mathrm{id} \\ g|_{V_0^\perp} = \mathrm{id} \end{array} \right\}$$

Let $G \coloneqq CSp_{2n}$. Define $K_f(d) \coloneqq \{g \in G(\hat{\mathbb{Z}}) \,|\, g \equiv 1 \bmod d\}(d \geq 3)$. By [Pin90, 0.1], $K_f(d) \trianglelefteq G(\hat{\mathbb{Z}})$ is neat. It is immediate that $\mathrm{Id}_{Z(G)(\mathbb{Q})}(\mathcal{H}_{2n}) = Z(G)(\mathbb{Q}) = \mathbb{G}_m(\mathbb{Q}) \subset GL_{2n}(\mathbb{Q})$. For $g_f \in G(\hat{\mathbb{Z}})$, we have $(\mathbb{G}_m(\mathbb{Q}) \times U_1(\mathbb{Q})) \cap g_f K_f(d) g_f^{-1} = d \cdot U_1(\mathbb{Z}) \eqqcolon \Gamma_{U_1}(d)$. Take $\mathcal{H}^0 \coloneqq \mathcal{H}_{2n}^+$ to be the connected component that induces positive definite forms. Then $\mathrm{Stab}_{G(\mathbb{Q})}(\mathcal{H}^0) \cap K_f(d) = Sp_{2n}(\mathbb{Q}) \cap K_f(d) = Sp_{2n}(\mathbb{Z}) \cap K_f(d) \eqqcolon \Gamma_{Sp}(d)$. Theorem 1.1 implies that

$$\mathrm{ed}_{\mathbb{C}}(\Gamma_{Sp}(md)\backslash\mathcal{H}^0 \to \Gamma_{Sp}(d)\backslash\mathcal{H}^0; p) \geq \mathrm{rank}_p \frac{d \cdot U_1(\mathbb{Z})}{md \cdot U_1(\mathbb{Z})} = \mathrm{rank}_p \frac{U_1(\mathbb{Z})}{m \cdot U_1(\mathbb{Z})}.$$

An inspection of the Lie algebra of U_1 reveals that $\dim U_1 = \frac{r(r+1)}{2}$. We find $\mathrm{ed}_{\mathbb{C}}(\Gamma_{Sp}(md)\backslash\mathcal{H}^0 \to \Gamma_{Sp}(d)\backslash\mathcal{H}^0; p) \geq \frac{r(r+1)}{2}$ if $p \,|\, m$. Now take $V_0 = \langle e_1, \ldots, e_n \rangle$ to be a maximal totally isotropic subspace. This gives

$$\mathrm{ed}_{\mathbb{C}}(\Gamma_{Sp}(md)\backslash\mathcal{H}^0 \to \Gamma_{Sp}(d)\backslash\mathcal{H}^0; p) \geq \frac{n(n+1)}{2},\ p \,|\, m.$$

But it is well-known that $\dim_{\mathbb{C}} \mathcal{H}_{2n} = \frac{n(n+1)}{2}$. Therefore, for $p \,|\, m$ we obtain the p-incompressibility of $\Gamma_{Sp}(md)\backslash\mathcal{H}^0 \to \Gamma_{Sp}(d)\backslash\mathcal{H}^0$. This example illustrates that larger U_1 produces better lower bound of the essential dimension.

Finally, Let $M(d) \coloneqq M^{K_f(d)}(G, \mathcal{H}_{2n})(\mathbb{C})$, $d \geq 3$. Then $M(d)$ is the moduli space of principally polarized abelian varieties of dimension n with level-d structure. We have shown the p-incompressibility of $M(pd) \to M(d)$ (or that of $\Gamma_{Sp}(md)\backslash\mathcal{H}_{2n}^+ \to \Gamma_{Sp}(d)\backslash\mathcal{H}_{2n}^+$). Thereby we have retrieved [BF24, 41] and [FKW21, 3.2.9].

Remark 6.3.1. Note that when $\gcd(m, d) = 1$, $\Gamma_{Sp}(d)/\Gamma_{Sp}(md)$ is canonically isomorphic to $Sp_{2n}(\frac{\mathbb{Z}}{m\cdot\mathbb{Z}})$. Take $m = p$ to be an odd prime. The p-incompressibility of $\Gamma_{Sp}(pd)\backslash\mathcal{H}^0 \to \Gamma_{Sp}(d)\backslash\mathcal{H}^0$ implies that $\mathrm{ed}_{\mathbb{C}}(Sp_{2n}(\mathbb{F}_p); p) \geq \frac{n(n+1)}{2}$. However, as computed in [BF24, 79], $\mathrm{ed}_{\mathbb{C}}(Sp_{2n}(\mathbb{F}_p); p) = p^{n-1}$. We see that there is an exponential difference. The reason for this is that we essentially used a subgroup $\Gamma_{U_1}(p)/\Gamma_{U_1}(pd) \subset \Gamma_{Sp}(d)/\Gamma_{Sp}(pd) = Sp_{2n}(\mathbb{F}_p)$ to deduce the lower bound. This subgroup may be relatively small compared to $Sp_{2n}(\mathbb{F}_p)$.

6.4 Universal families over Siegel modular varieties

Given (V, ψ), $G = CSp_{2n}$ as in the previous example, let $U \coloneqq \mathbb{Q}$ and $W \coloneqq U \times V$ with multiplication defined by $(u, v) \cdot (u', v') \coloneqq (u + u' + \frac{1}{2}\psi(v, v'), v + v')$. G acts on W via $g \cdot (u, v) \coloneqq (g(u), g(v)) = (\mu(g) \cdot u, g(v))$, where $\mu : G \to \mathbb{G}_{m,\mathbb{Z}}$ is the multiplier morphism. We have a short exact sequence $1 \to U \to W \to V \to 1$ leading to unipotent extensions $P_{2n} \coloneqq W \rtimes G$ and $V \rtimes G$. This gives unipotent extensions of mixed Shimura data

$$(P_{2n}, \mathcal{X}_{2n}) \to (V \rtimes G, \mathcal{Y}_{2n}) \to (G, \mathcal{H}_{2n})$$

as in Example 4.2.12.

For $d \geq 4$ even, let $K_f(d) \trianglelefteq G(\hat{\mathbb{Z}})$ be as in Subsection 6.3. Define $K_f^W(d) \coloneqq K_f^U(d) \times K_f^V(d) = (d \cdot U(\hat{\mathbb{Z}})) \times (d \cdot V(\hat{\mathbb{Z}}))$ and $K_f^P(d) \coloneqq K_f^W \rtimes K_f(d)$. (By assumption K_f^W is really a subgroup normalized by $K_f(d)$.) Consider the following mixed Shimura varieties.

$$M(d) \coloneqq M_{\mathbb{C}}^{K_f(d)}(G, \mathcal{H}_{2n})$$

$$M_V(d) \coloneqq M_{\mathbb{C}}^{K_f^V \rtimes K_f(d)}(V \rtimes G, \mathcal{H}'_{2n})$$

$$M_W(d) \coloneqq M_{\mathbb{C}}^{K_f^P(d)}(P_{2n}, \mathcal{X}_{2n})$$

Remark 6.4.1. According to [Pin90, 10.9], $M_W(d) \to M_V(d) \to M(d)$ is the universal family of morphisms $X \to A \to S$, where $A \to S$ is an abelian scheme over S of relative dimension n with a symplectic level d-structure and $X \to A$ is a normalized totally symmetric $\mathbb{G}_m$-torsor over A.

We have described the rational boundary components of $(G, \mathcal{H}_{2n})$ in Subsection 6.3. Indeed, this enables us to obtain the rational boundary components of $(P_{2n}, \mathcal{X}_{2n})$ conveniently. Let $V_0 \subset V$ and Q be as in Subsection 6.3. Then $W \rtimes Q \subset P$ is a $\mathbb{Q}$-admissible parabolic subgroup. Recall that by the definition in Subsection 4.4, the rational boundary component $(P_{2n,1}, \mathcal{X}_{2n,1})$ associated with $W \rtimes Q$ should concern the smallest $\mathbb{Q}$-normal subroup $P_{2n,1} \trianglelefteq W \rtimes Q$ containing the image of $\iota \circ h_{x_1}$ Here $\iota : G \rightarrowtail P_{2n} = W \rtimes G$ is the inclusion. $(P_1, \mathcal{X}_1)$ is the rational boundary component of $(G, \mathcal{H}_{2n})$ associated with $Q \subset G$ and x_1 the image of x under $\mathcal{X}^+_{(P_1,\mathcal{X}_1)} \rightarrowtail \mathcal{X}_1$.

Proposition 6.4.2. *Let the rational boundary component associated with the admissible $\mathbb{Q}$-parabolic subgroup $W \rtimes Q \subset W \rtimes G$ be $(P_{2n,1}, \mathcal{X}_{2n,1})$. Then*

(1) $P_{2n,1} = (U \times V_0^{\perp}) \rtimes P_1$.

(2) $U_{2n,1} = (U \times V_0) \times U_1$.

Proof.

(1) $U_{2n,1} = (U \times V_0) \times U_1$ just by considering the weight, as described in Subsection 6.3.

(2) In order to compute $P_{2n,1}$, first observe that by the description of the weight filtration in Subsection 6.3, the unipotent radical of $P_{2n,1}$ is $W_{2n,1} = (U \times V_0^{\perp}) \rtimes W_1$. ($W_1$ is the unipotent radical of P_1.)

Next, we claim that $(U \times V_0^{\perp}) \rtimes P_1$ is a $\mathbb{Q}$-normal subgroup of $W \rtimes Q$. In fact it is clearly normalized by Q. For $(u, v) \in W(\mathbb{Q})$. Since P_1 acts trivially on $V/V_0^{\perp}$, we have

$$\begin{aligned}(u, v, 1)(0, 0, p_1)(-u, -v, 1) &= \big(u - p_1(u) - \frac{1}{2}\psi(v, -p_1(v)), v - p_1(v), p_1\big) \\ &\in \big((U \times V_0^{\perp}) \rtimes P_1\big)(\mathbb{Q}).\end{aligned}$$

Thus $(U \times V_0^{\perp}) \rtimes W_1 \subseteq P_{2n,1} \subseteq (U \times V_0^{\perp}) \rtimes P_1$. But $P_{2n,1}$ must project onto P_1 as its image under the projection contains $h_{x_1}(\mathbb{S}_{\mathbb{C}})$ (and P_1 is the smallest such $\mathbb{Q}$-normal subgroup in Q). Hence $P_{2n,1} = (U \times V_0^{\perp}) \rtimes P_1$. $\square$

Let $K_f^{P_{2n}}(d) := K_f^W(d) \rtimes K_f(d)$ and $\mathcal{X}_{2n}^+$ the connected component that maps onto $\mathcal{H}_{2n}^+$. We have

$$\mathrm{Stab}_{P_{2n}(\mathbb{Q})}(\mathcal{X}_{2n}^+) \cap K_f^{P_{2n}}(d) = (d{\cdot}U(\mathbb{Z}) \times d{\cdot}V(\mathbb{Z})) \rtimes \Gamma_{Sp}(d) =: \Gamma_{P_{2n}}(d);$$

$$\big(\mathrm{Id}_{Z(P_{2n})(\mathbb{Q})}(\mathcal{X}_{2n}) \times U_{2n,1}(\mathbb{Q})\big) \cap K_f^{P_{2n}}(d) = (d{\cdot}U(\mathbb{Z}) \times d{\cdot}V_0(\mathbb{Z})) \times (d{\cdot}U_1(\mathbb{Z})),$$

$$\text{as } \mathrm{Id}_{Z(P_{2n})(\mathbb{Q})}(\mathcal{X}_{2n}) = 1.$$

If $p \mid m$, Theorem 1.1 concludes that

$$\mathrm{ed}_{\mathbb{C}}(\Gamma_{P_{2n}}(md)\backslash\mathcal{X}_{2n}^+ \to \Gamma_{P_{2n}}(d)\backslash\mathcal{X}_{2n}^+; p) \geq 1 + r + \frac{r(r+1)}{2}.$$

A maximal totally isotropic subspace V_0 of dimension $r = n$ gives the lower bound $1 + n + \frac{n(n+1)}{2}$. But since $\dim_{\mathbb{C}} \mathcal{X}_{2n} = 1 + n + \frac{n(n+1)}{2}$, we obtain

Proposition 6.4.3. *Let $d \geq 4$ be even. The $\Gamma_{P_{2n}}(d)/\Gamma_{P_{2n}}(md)$-torsor $\Gamma_{P_{2n}}(md)\backslash\mathcal{X}_{2n}^+ \to \Gamma_{P_{2n}}(d)\backslash\mathcal{X}_{2n}^+$ is p-incompressible for $p \mid m$.*

6.5 Kuga varieties

A *Kuga datum* $(P, \mathcal{X})$ is a mixed Shimura datum with $U = 1$. That is, there is no subspace of $\mathrm{Lie}\, P$ of weight -2. A *Kuga variety* is just a mixed Shimura variety $M^{K_f}(P, \mathcal{X})(\mathbb{C})$ from a Kuga datum. For more properties of Kuga data, the reader can refer to [Che13].

Here we consider the Kuga datum $(V \rtimes G, \mathcal{Y}_{2n})$ as defined in Subsection 6.4.

Similar to Proposition 6.4.2, we have the following

Proposition 6.5.1. *Let rational boundary component associated with the admissible $\mathbb{Q}$-parabolic subgroup $V \rtimes Q \subset V \rtimes G$ be $(P'_{2n,1}, \mathcal{Y}_{2n,1})$. Then*

(1) $P'_{2n,1} = V_0^{\perp} \rtimes P_1$.

(2) $U'_{2n,1} = V_0 \times U_1$.

Let $\mathcal{Y}_{2n}^{+}$ be the connected component above $\mathcal{H}_{2n}^{+}$. Let $K'_f(d) \coloneqq K_f^V(d) \rtimes K_f(d)$.

$$\mathrm{Stab}_{(V \rtimes G)(\mathbb{Q})}(\mathcal{Y}_{2n}^{+}) \cap K'_f(d) = (d{\cdot}V(\mathbb{Z})) \rtimes \Gamma_{Sp}(d) =: \Gamma'(d);$$

$$\big(\mathrm{Id}_{Z(V \rtimes G)(\mathbb{Q})}(\mathcal{Y}_{2n}) \times U'_{2n,1}(\mathbb{Q})\big) \cap K'_f(d) = (d{\cdot}V_0(\mathbb{Z})) \times (d{\cdot}U_1(\mathbb{Z})),$$

$$\text{as } \mathrm{Id}_{Z(V \rtimes G)(\mathbb{Q})}(\mathcal{Y}_{2n}) = 1.$$

Again, a maximal totally isotropic subspace V_0 of dimension $r = n$ gives the lower bound $n + \frac{n(n+1)}{2}$. Since $\dim_{\mathbb{C}} \mathcal{Y}_{2n} = n + \frac{n(n+1)}{2}$, we get

Proposition 6.5.2. *Let $d \geq 3$. The $\Gamma'(d)/\Gamma'(md)$-torsor $\Gamma'(md)\backslash\mathcal{Y}_{2n}^{+} \to \Gamma'(d)\backslash\mathcal{Y}_{2n}^{+}$ is p-incompressible for $p \,|\, m$.*

Remark 6.5.3. Since V is already a group, we don't need to assume that d is even as in Subsection 6.4.

Let's analyze the structure of the underlying Kuga varieties. First, using the splitting $G \rightarrowtail V \rtimes G$, we have an inclusion $\mathcal{H}_{2n} \rightarrowtail \mathcal{Y}_{2n}$. Since $V(\mathbb{R}) \rtimes G(\mathbb{R})$ acts transitively on $\mathcal{Y}_{2n}$ with $V(\mathbb{R})$ acting freely, we can deduce that the map

$$V(\mathbb{R}) \times \mathcal{H}_{2n} \to \mathcal{Y}_{2n}$$

$$(v, x) \mapsto v \cdot x$$

is a $V(\mathbb{R}) \rtimes G(\mathbb{R})$-equivariant analytic isomorphism. Here $V(\mathbb{R}) \rtimes G(\mathbb{R})$ acts on $V(\mathbb{R}) \times \mathcal{H}_{2n}$ via the formula $(v, g) \cdot (w, x) := (v + g(w), g \cdot x)$.

Consider the action of the groups $\Gamma_V(l) \rtimes \Gamma_{Sp}(l)$ $(l = pd, d)$, we have the following diagram

$$\begin{tikzcd}
\Gamma_V(pd)\backslash V(\mathbb{R}) \times \{x\} \arrow[r, two heads] \arrow[d, hook, "\text{fiber}"'] & \Gamma_V(d)\backslash V(\mathbb{R}) \times \{x\} \arrow[r, two heads, "\sim"] \arrow[d, hook, "\text{fiber}"'] & \Gamma_V(d)\backslash V(\mathbb{R}) \times \{x\} \arrow[d, hook, "\text{fiber}"'] \\
\Gamma_V(pd) \rtimes \Gamma_{Sp}(pd)\backslash V(\mathbb{R}) \times \mathcal{H}_{2n}^{+} \arrow[r, two heads] \arrow[dr, two heads] & \Gamma_V(d) \rtimes \Gamma_{Sp}(pd)\backslash V(\mathbb{R}) \times \mathcal{H}_{2n}^{+} \arrow[r, two heads] \arrow[d, two heads] & \Gamma_V(d) \rtimes \Gamma_{Sp}(d)\backslash V(\mathbb{R}) \times \mathcal{H}_{2n}^{+} \arrow[d, two heads] \\
& \Gamma_{Sp}(pd)\backslash \mathcal{H}_{2n}^{+} \arrow[r, two heads] & \Gamma_{Sp}(d)\backslash \mathcal{H}_{2n}^{+}
\end{tikzcd}$$

It is not hard to see that the bottom right square is a pullback. Thus the congruence cover $M_V(pd) \to M_V(d)$ is just the pullback of $M(pd) \to M(d)$ along $M_V(d) \to M(d)$ composed with the map of multiplication by p on each fiber of $M_V(d)$. The incompressibility result Proposition 6.5.2 can be viewed as an analogue of Brosnan's conjecture that the map of multiplication by p on an abelian variety is p-incompressible. The interested readers can refer to, say, [FS22, 6.1], [FKW24, 2.3.5], and [KZ25].

If we use the simpler notations $\mathcal{Y}_{n,d} := M_V(d)$ and $\mathcal{A}_{n,d} := M(d)$. The above diagram becomes the one that appeared in the introduction.

$$\begin{tikzcd}
\mathcal{Y}_{n,pd} \arrow[r, "p"] \arrow[dr] & \mathcal{Y}_{n,pd} \arrow[r] \arrow[d] & \mathcal{Y}_{n,d} \arrow[d] \\
& \mathcal{A}_{n,pd} \arrow[r] & \mathcal{A}_{n,d}
\end{tikzcd}$$

This finishes our discussion on the examples of lower bounds.

7 Bibliography

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