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Non-Smooth Simultaneous Stabilization of Nonlinear Systems: Interpolation of Feedback Laws

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Non-Smooth Simultaneous Stabilization of Nonlinear Systems: Interpolation of Feedback Laws

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Abstract

In this paper, we introduce a method that enables us to construct a continuous simultaneous stabilizer for pairs of systems in the plane that cannot be simultaneously stabilized by smooth feedback. We extend this method to higher dimensional systems and prove that any pair of asymptotically stabilizable *nonlinear* systems can be simultaneously stabilized (not asymptotically) by means of continuous feedback. The resulting simultaneous stabilizer depends on a partition of unity and we show how to circumvent the computation of this partition of unity by constructing an explicit simultaneous stabilizer.

Keywords: Continuous feedback, simultaneous stabilization, partition of unity, nonlinear systems.

1 Introduction

The simultaneous stabilization problem was first introduced in [13] and consists in finding a controller that stabilizes each one of the systems of a finite collection of systems. In [13, 14], tractable necessary and sufficient conditions for the simultaneous stabilizability of two linear systems by linear feedback are given. Necessary and sufficient conditions for the existence of a linear feedback law that simultaneously stabilizes three linear systems are proposed in [1, 3, 14], but none of them is tractable.

To overcome the limitation of linear time invariant controller, the use of alternative feedback need to be investigated. For linear time invariant systems, Khargonekar et al. [10] proved that a finite family of stabilizable linear systems can be simultaneously stabilized by linear periodically time-varying feedback, while Kabamba et al. [8] showed that such a family can be simultaneously stabilized by a controller based on generalized sample and hold functions. On the other hand, by using a sampler and zeroth order hold functions, Zhang et al. [15] proposed a scheme for the simultaneous stabilization of single input single output linear systems. Finally, in [12] an implicit (and therefore not really applicable) necessary and sufficient condition for the simultaneous quadratic stabilization of single input linear systems by means of continuous

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feedback is given.

To the best of our knowledge, the simultaneous stabilization of a collection of nonlinear systems has not been addressed in the literature.

Before introducing our results, we need some definitions and notation. A feedback law $u : \mathbb{R}^n \to \mathbb{R}^m$ is said to be continuous if it is continuous on a neighborhood of the origin with $u(0) = 0$ and almost C^k, if it is C^k on a neighborhood of the origin except at the origin. For a given system, we let $x(\cdot, x_0)$ denote its trajectory that starts from x_0 at time $t = 0$.

Definition 1.1 *Consider the system* $(S) : \dot{x} = f(x)$ *where the state* x *is in* $\mathbb{R}^n$ *and the mapping* $f : \mathbb{R}^n \to \mathbb{R}^n$ *is continuous with* $f(0) = 0$.

i) *The system* (S) *is stable if for each* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that for each* $t \geq 0$ *and each solution* $x(., x_0)$ *of* (S), *we have* $\|x(t, x_0)\| < \varepsilon$ *whenever* $\|x_0\| < \delta$.

ii) *The system* (S) *is locally asymptotically stable if it is stable and if there exists* $\delta_0 > 0$ *such that* $x(t, x_0) \to 0$ *as* $t \to \infty$ *whenever* $\|x_0\| < \delta_0$.

A feedback law u **stabilizes** a system if the closed-loop system obtained once u is fed back into the system, is **stable** per Definition 1.1 **(i)**.

We emphasize that we use the word "stable" in the basic sense, i.e, stable according to Definition 1.1 **(i)**, while in the control theory literature it is commonly used to denote "locally asymptotically stable" per Definition 1.1 **(ii)**.

Finally a feedback law u simultaneously stabilizes (resp. asymptotically stabilizes) a collection of controlled systems if u stabilizes (resp. asymptotically stabilizes) each system of the collection.

Inspired by the elegant results that have been recently obtained in the context of stabilization of nonlinear systems by continuous feedback [2, 9], we investigate in this paper some simultaneous stabilization issues by using merely continuous feedback. More precisely, we first consider the pair of systems

$$S_- : \begin{cases} \dot{x}_1 &= a_- x_1 + b_- x_2 \\ \dot{x}_2 &= u \end{cases} \qquad \text{and} \qquad S_+ : \begin{cases} \dot{x}_1 &= a_+ x_1 + b_+ x_2 \\ \dot{x}_2 &= u \end{cases},$$

where a_-, a_+, b_+ are positive, b_- is negative and u is a scalar control. Upon noticing that these two systems are not simultaneously asymptotically stabilizable by smooth feedback, we introduce a method that enables us to construct a continuous stabilizer by using two particular feedback laws that asymptotically stabilizes S_- and S_+ respectively. Secondly, we consider a pair $\{S_i : i = 1, 2\}$ of locally asymptotically stabilizable [at the origin] systems

$$S_i : \quad \dot{x} = f_i(x, u), \quad i = 1, 2,$$

where the state x lies in $\mathbb{R}^n$, the input u, is in $\mathbb{R}^m$ and for each $i = 1, 2$ the mapping $f_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous on a neighborhood of the origin with $f_i(0, 0) = 0$. By extending the aforementioned method, we exhibit a continuous feedback law that simultaneously stabilizes S_1 and S_2. The resulting simultaneous stabilizer depends on a partition of unity and we show how to circumvent the computation of this partition of unity by constructing an explicit simultaneous stabilizer.

The paper is organized as follows. The simultaneous asymptotic stabilization of S_- and S_+ is solved in section 2, while the simultaneous stabilization of S_1 and S_2 is addressed in section 3 and 4. After some concluding remarks in section 5, we present in section 6 the technical lemmas used in the proofs of the main theorems.

2 Simultaneous stabilization in the Plane

In this section, we consider two systems in the plane

$$S_- : \quad \left\{ \begin{array}{rcl} \dot{x}_1 & = & a_- x_1 + b_- x_2 \\ \dot{x}_2 & = & u \end{array} \right. \qquad \text{and} \qquad S_+ : \quad \left\{ \begin{array}{rcl} \dot{x}_1 & = & a_+ x_1 + b_+ x_2 \\ \dot{x}_2 & = & u \end{array} \right. ,$$

where a_-, a_+, b_+ are positive reals, b_- is a negative real, and u is a scalar control. By using elementary linear algebra, it is easily checked that there exists no smooth feedback law that simultaneously locally asymptotically stabilizes S_- and S_+. However, as we shall see below there exists a merely continuous feedback law that simultaneously globally asymptotically stabilizes S_- and S_+.

This result is proved in the following theorem. The general line of the proof is to construct two feedback laws $u^-_{k_0}$ and $u^+_{k_0}$ that globally asymptotically stabilize S_- and S_+ respectively. We introduce two bases at the origin $\{W^-_\beta\}_{\beta>0}$ and $\{W^+_\beta\}_{\beta>0}$ such that for each $\beta > 0$, the neighborhoods W^-_β and W^+_β are invariant with respect to the systems S_- (with $u = u^-_{k_0}$) and S_+ (with $u = u^+_{k_0}$) respectively. We then construct a new base at the origin $\{W_j\}_{j\in\mathbb{Z}}$ such that the odd (resp. even) sets W_{2n+1} (resp. W_{2n}) belong to the family $\{W^-_\beta\}_{\beta>0}$ (resp. $\{W^+_\beta\}_{\beta>0}$). Finally, we define a continuous feedback law u_{k_0} which is equal to $u^-_{k_0}$ (resp. $u^+_{k_0}$) on the boundary of the odd sets W_{2n+1} (resp. even sets W_{2n}). It follows that the closure of each neighborhood of the base at the origin $\{W_{2n+1}\}_{n\in\mathbb{Z}}$ (resp. $\{W_{2n}\}_{n\in\mathbb{Z}}$) is invariant with respect to the closed-loop system obtained once u_{k_0} is fed back into S_- (resp. S_+). This implies that u_{k_0} simultaneously stabilizes S_- and S_+. Asymptotic stability is then obtained by proving that the only positive limit set of the system S_- (resp. S_+) with $u = u_{k_0}$, in the sets $\overline{W}_{2n+1}$ (resp. $\overline{W}_{2n}$), is the origin.

Theorem 2.1 *Let a_-, a_+ and b_+ be positive, and let b_- be negative. Then, there exists a continuous and almost smooth feedback law that simultaneously globally asymptotically stabilizes the systems S_- and S_+.*

Proof: Throughout the proof, we use the following notation: The set $\mathbb{Z}$ denotes the set of integers. For each x in $\mathbb{R}^2$, we denote by x_1 and x_2 its coordinates, and we define the mappings, $f_-, f_+ : \mathbb{R}^2 \to \mathbb{R}$ by setting $f_-(x) = a_- x_1 + b_- x_2$ and $f_+(x) = a_+ x_1 + b_+ x_2$ respectively. For a subset Y of $\mathbb{R}^2$, we denote by $\widehat{Y}$ its symmetric with respect to the origin, i.e., $\widehat{Y} \triangleq \{-y : \ y \in Y\}$. Finally, for each real α, we let Σ_α denote the half-line

$$\Sigma_\alpha \ \triangleq \ \{x \in \mathbb{R}^2 : \ x_1 = \alpha x_2, \ x_1 > 0\}.$$

Construction of u^-_k and u^+_k:

Let θ, μ and δ be fixed positive reals with $\theta > \max\left(\frac{b_+}{a_+}, \frac{-b_-}{a_-}\right)$, $\delta > 2\theta$ and $\mu < \min\left(\frac{b_+}{a_+}, \frac{-b_-}{a_-}\right)$. Consider Fig. 1, and for each $\beta > 0$, let W^-_β and W^+_β be the open subsets of $\mathbb{R}^2$ bounded by the closed curves in bold. The sets W^-_β and W^+_β are symmetric with respect to the origin. The segments $[\widehat{A}_5, A_1]$ and $[A_2, A_3]$ are respectively horizontal and vertical, while the segments $[A_5, A_4]$ and $[A_4, A_3]$ have respective slopes $\dfrac{dx_1}{dx_2} = -\delta$ and $\dfrac{dx_1}{dx_2} = \mu$. Furthermore, the segments $[\widehat{B}_5, B_1]$ and $[B_3, B_4]$ are respectively horizontal and vertical, while the segments $[B_1, B_2]$ and $[B_2, B_3]$ have respective slopes $\dfrac{dx_1}{dx_2} = \delta$ and $\dfrac{dx_1}{dx_2} = -\mu$. From the assumptions made on θ, δ and μ, it is easily checked that W^-_β and W^+_β are well-defined for each $\beta > 0$. We now define the following open subsets of $\mathbb{R}^2 \setminus \{0\}$:

R_1 : region between the half-lines $\widehat{\Sigma}_{-\frac{b_+}{a_+}}$ and $\Sigma_{-\frac{b_-}{a_-}}$,

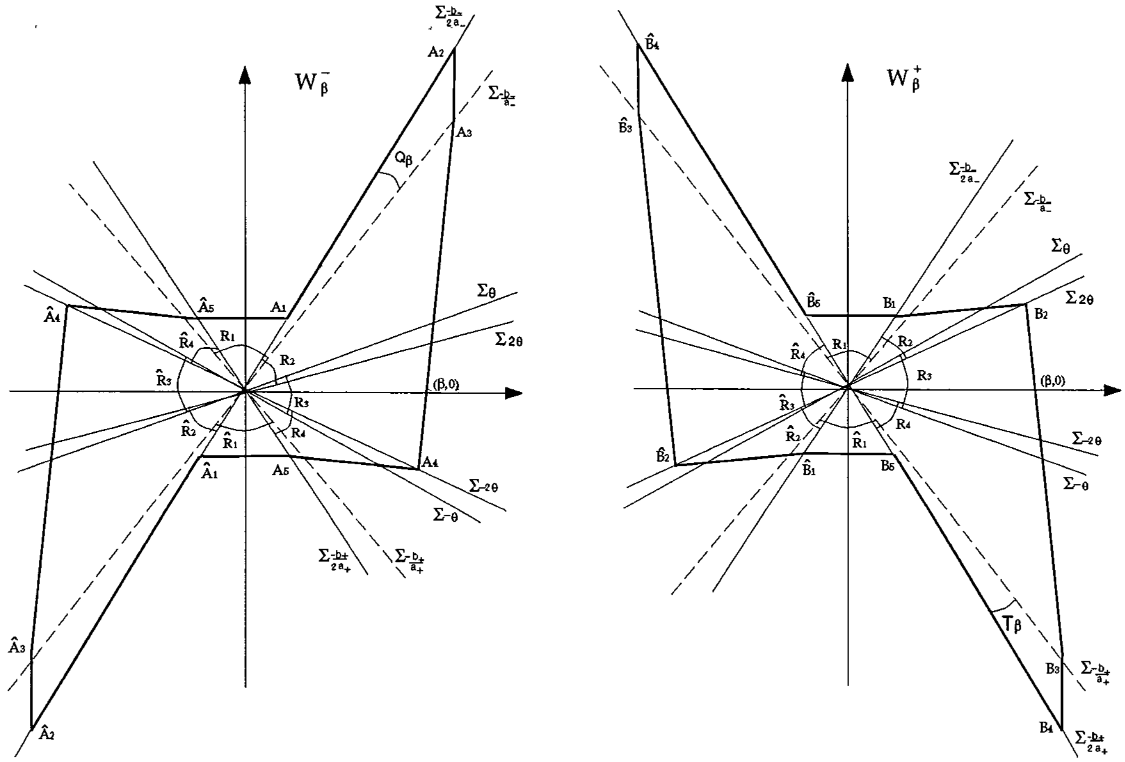

Figure 1: Neighborhoods W_β^- and W_β^+

R_2 : region between the half-lines $\Sigma_{-\frac{b_-}{2a_-}}$ and $\Sigma_{2\theta}$,

R_3 : region between the half-lines Σ_θ and $\Sigma_{-\theta}$,

R_4 : region between the half-lines $\Sigma_{-2\theta}$, $\Sigma_{-\frac{b_+}{2a_+}}$,

Q_β : region delimited by $\Sigma_{-\frac{b_-}{a_-}}$, $\Sigma_{-\frac{b_-}{2a_-}}$ and the segment $[A_2, A_3]$, (1)

T_β : region delimited by $\Sigma_{-\frac{b_+}{a_+}}$, $\Sigma_{-\frac{b_+}{2a_+}}$ and the segment $[B_3, B_4]$. (2)

Because $\{R_1, .., R_4, \widehat{R}_1, .., \widehat{R}_4\}$ [where $\widehat{R}_i$ is the symmetric set of R_i with respect to the origin for each $i = 1, 2, 3, 4$] is an open cover of $\mathbb{R}^2 \setminus \{0\}$, there exists a C^∞ partition of unity $\{p_1, .., p_4, \widehat{p}_1, .., \widehat{p}_4\}$ subordinate to it such that the support of p_i (resp. $\widehat{p}_i$) is included in R_i (resp. $\widehat{R}_i$) for each $i = 1, 2, 3, 4$ [4].

For each $k > 0$, we now define the mappings $u_k^-, u_k^+ : \mathbb{R}^2 \to \mathbb{R}$, by setting

$$u_k^-(x) = \begin{cases} 0 & \text{if } x = 0 \\ -kx_2\,(p_1(x) + \widehat{p}_1(x)) + \frac{1}{\mu}(2a_-x_1 + b_-x_2)\,(p_2(x) + p_3(x) + \widehat{p}_2(x) + \widehat{p}_3(x)) & \\ -\frac{1}{\delta}(\frac{a_-}{2}x_1 + b_-x_2)\,(p_4(x) + \widehat{p}_4(x)) & \text{otherwise}, \end{cases}$$

and

$$u_k^+(x) = \begin{cases} 0 & \text{if } x = 0 \\ -kx_2\,(p_1(x) + \widehat{p}_1(x)) + \frac{1}{\delta}(\frac{a_+}{2}x_1 + b_+x_2)\,(p_2(x) + \widehat{p}_2(x)) & \\ -\frac{1}{\mu}(2a_+x_1 + b_+x_2)\,(p_3(x) + p_4(x) + \widehat{p}_3(x) + \widehat{p}_4(x)) & \text{otherwise}. \end{cases}$$

Because the mapping p_i (resp. $\widehat{p}_i$) is smooth on $\mathbb{R}^2\setminus\{0\}$ for each $i=1,2,3,4$, it is plain that u_k^- and u_k^+ are smooth on $\mathbb{R}^2\setminus\{0\}$. Furthermore, the mappings of a partition of unity summing up to 1, it is readily seen from the definition of u_k^- and u_k^+ that

$$|u_k^-(x)| \;\leq\; \max\left(|kx_2|,\, \frac{1}{\mu}|2a_-x_1+b_-x_2|,\, \frac{1}{\delta}|\frac{a_-}{2}x_1+b_-x_2|\right), \quad x\in\mathbb{R}^2,$$

and

$$|u_k^+(x)| \;\leq\; \max\left(|kx_2|,\, \frac{1}{\mu}|2a_+x_1+b_+x_2|,\, \frac{1}{\delta}|\frac{a_+}{2}x_1+b_+x_2|\right), \quad x\in\mathbb{R}^2,$$

and continuity of u_k^- and u_k^+ at the origin follows for each $k>0$.

Construction of u_k:

We first note that the families $\{W_\beta^-\}_{\beta>0}$ and $\{W_\beta^+\}_{\beta>0}$ are bases at the origin such that $W_\beta^-\subset W_{\beta'}^-$ and $W_\beta^+\subset W_{\beta'}^+$ for all $\beta<\beta'$. This, together with the fact that for each bounded subset U of $\mathbb{R}^2$ there exists $\beta>0$ such that $U\subset W_\beta^-$ and $U\subset W_\beta^+$, yield the existence of a sequence of positive reals $\{\beta_j\}_{j\in\mathbb{Z}}$ satisfying

$$\beta_j \to 0 \;\; as \;\; j\to+\infty \qquad \text{and} \qquad \beta_j\to+\infty \;\; as \;\; j\to-\infty, \tag{3}$$

with

$$\overline{W}^-_{\beta_{2n+1}} \subset W^+_{\beta_{2n}} \qquad \text{and} \qquad \overline{W}^+_{\beta_{2n}} \subset W^-_{\beta_{2n-1}}, \quad n\in\mathbb{Z}. \tag{4}$$

With the notation

$$W_{2n} \triangleq W^+_{\beta_{2n}} \qquad \text{and} \qquad W_{2n+1}\triangleq W^-_{\beta_{2n+1}}, \quad n\in\mathbb{Z},$$

the inclusions (4) translate to

$$\overline{W}_{j+1}\subset W_j, \quad j\in\mathbb{Z}. \tag{5}$$

It is easily checked from (3) and (5) that $\{W_{j-1}\backslash\overline{W}_{j+1}\}_{j\in\mathbb{Z}}$ is an open cover of $\mathbb{R}^2\setminus\{0\}$. Let $\{q_j\}_{j\in\mathbb{Z}}$ be a partition of unity subordinate to $\{W_{j-1}\backslash\overline{W}_{j+1}\}_{j\in\mathbb{Z}}$ such that the support of q_j is included in $W_{j-1}\backslash\overline{W}_{j+1}$, for each j in $\mathbb{Z}$ [4].

Finally, for each $k>0$, we define the feedback law $u_k:\mathbb{R}^2\to\mathbb{R}$ by setting

$$u_k(x)=\begin{cases} 0 & \text{if } x=0\\ u_k^+(x)\displaystyle\sum_{n\in\mathbb{Z}} q_{2n}(x) \;+\; u_k^-(x)\sum_{n\in\mathbb{Z}} q_{2n+1}(x) & \text{otherwise.}\end{cases}$$

Fix $k>0$ and let x be in $\mathbb{R}^2\setminus\{0\}$. It is easily checked that there exists a neighborhood U_x of x such that U_x intersects with at most three sets of the collection $\{W_{j-1}\backslash\overline{W}_{j+1}\}_{j\in\mathbb{Z}}$. As the support of the mapping q_j is included in $W_{j-1}\backslash\overline{W}_{j+1}$ for each j in $\mathbb{Z}$, it follows that on U_x, the infinite sums in the expressions of $u_k(x)$ reduce to the sum of at most three terms. This last comment combined with the smoothness on $\mathbb{R}^2\setminus\{0\}$ of the mappings u_k^-, u_k^+ and q_j, $j\in\mathbb{Z}$ implies that u_k is smooth on $\mathbb{R}^2\setminus\{0\}$. Furthermore, the mappings q_j summing up to 1, we get

$$|u_k(x)| \;\leq \max\left(|u_k^+(x)|, |u_k^-(x)|\right), \quad x\in\mathbb{R}^2,$$

and for each $k>0$, continuity of u_k at the origin follows from that of u_k^- and u_k^+.

Invariance of the sets W_j :

We now show that there exists $k_0>0$ such that for each n in $\mathbb{Z}$, the sets $\overline{W}_{2n+1}$ and $\overline{Q}_{\beta_{2n+1}}$ (resp. $\overline{W}_{2n}$ and $\overline{T}_{\beta_{2n}}$) are invariant with respect to the vector-field $[f_-,u_{k_0}]^t$ (resp. $[f_+,u_{k_0}]^t$).

Recall that $Q_{\beta_{2n+1}}$ and $T_{\beta_{2n}}$ denote the sectors of W_{2n+1} and W_{2n} [or equivalently $W^-_{\beta_{2n+1}}$ and $W^+_{\beta_{2n}}$] defined by (1) and (2) respectively.

We note that for each m in $\mathbb{Z}$, the boundary ∂W_m is included in $W_{m-1}\backslash\overline{W}_{m+1}$ and does not intersect with any other set $W_{j-1}\backslash\overline{W}_{j+1}$. Because the support of the mapping q_j is included in $W_{j-1}\backslash\overline{W}_{j+1}$ for each j in $\mathbb{Z}$, and the mappings q_j sum up to 1, we obtain

$$u_k(x) = u_k^+(x), \quad x \in \partial W_{2n} \qquad \text{and} \qquad u_k(x) = u_k^-(x), \quad x \in \partial W_{2n+1}, \quad n \in \mathbb{Z}. \tag{6}$$

Next, because $u_k^-(x)$ and $u_k^+(x)$ are both equal to $-kx_2$ for x in the set $\Sigma_{-\frac{b_-}{2a_-}} \cup \Sigma_{-\frac{b_+}{2a_+}}$, we get

$$u_k(x) = -kx_2, \quad x \in \Sigma_{-\frac{b_-}{2a_-}} \cup \Sigma_{-\frac{b_+}{2a_+}}. \tag{7}$$

By definition of u_k we also have

$$u_k(x) > 0, \quad x \in \Sigma_{-\frac{b_-}{a_-}} \cup \widehat{\Sigma}_{-\frac{b_+}{a_+}} \qquad \text{and} \qquad u_k(x) < 0, \quad x \in \widehat{\Sigma}_{-\frac{b_-}{a_-}} \cup \Sigma_{-\frac{b_+}{a_+}}. \tag{8}$$

Let k_- and k_+ be obtained through Lemmas 6.1 and 6.2 [with μ and δ as defined here] and set $k_0 \triangleq \max(k_-, k_+)$. We now fix n in $\mathbb{Z}$ and show that the set $\overline{W}_{2n+1}$ is invariant with respect to the vector-field $[f_-, u_{k_0}]^t$. This will be proved if for each x in the boundary ∂W_{2n+1}, the vector $[f_-(x), u_{k_0}(x)]^t$ points inside the set W_{2n+1}.

Because the intersection of more than two sets of the family $\{R_1, \dots, R_4, \widehat{R}_1, \dots, \widehat{R}_4\}$ is empty, for each x in ∂W_{2n+1}, the vector $[f_-(x), u_{k_0}^-(x)]^t$ either reduces to one of the vectors listed in the different assertions of Lemma 6.1, and therefore points inside W_{2n+1}, or is a convex combination of two of them. In the latter case, $[f_-(x), u_{k_0}^-(x)]^t$ points inside W_{2n+1} either because we have a convex combination, or because we have $f_-(x) < 0$ (resp. $f_-(x) > 0$) on the segments $[A_2, A_3]$ (resp. $[\widehat{A}_2, \widehat{A}_3]$) of ∂W_{2n+1}. By (6), we have $u_{k_0} = u_{k_0}^-$ on ∂W_{2n+1} and it follows that the vector $[f_-(x), u_{k_0}(x)]^t$ points inside W_{2n+1} for each x in ∂W_{2n+1}.

Therefore, the set $\overline{W}_{2n+1}$ is invariant with respect to the vector-field $[f_-, u_{k_0}]^t$, for each n in $\mathbb{Z}$.

Similarly, (7), (8) and Assertion **(i)** of Lemma 6.1 yield the invariance of the set $\overline{Q}_{\beta_{2n+1}}$ with respect to the vector-field $[f_-, u_{k_0}]^t$, for each n in $\mathbb{Z}$.

On the other hand, (6), (7), (8) and Lemma 6.2 imply that the sets $\overline{W}_{2n}$ and $\overline{T}_{\beta_{2n}}$ are invariant with respect to the vector-field $[f_+, u_{k_0}]^t$, for each n in $\mathbb{Z}$.

Asymptotic stability:

We now show that the feedback law u_{k_0} globally asymptotically stabilizes the system S_-. Let $\widetilde{S}_-$ denote the system obtained once u_{k_0} is fed back into S_-. Fix n in $\mathbb{Z}$ and let x_0 be in $\overline{W}_{2n+1}$. In view of (8), we have $u_{k_0}(x) \neq 0$ for all x in $\mathbb{R}^2 \setminus \{0\}$ with $f_-(x) = 0$, so that the origin is the unique equilibrium point of $\widetilde{S}_-$ in $\overline{W}_{2n+1}$. Thus, by the invariance with respect to $\widetilde{S}_-$ of the compact set $\overline{W}_{2n+1}$, and the Poincaré-Bendixson Theorem [5, p. 151], the positive limit set $\mathcal{P}(x_0)$ of x_0 in $\overline{W}_{2n+1}$ is either equal to $\{0\}$ or to a nontrivial periodic orbit $\mathcal{O}$. If we assume that $\mathcal{P}(x_0) = \mathcal{O}$, then by Theorem 3.1 in [5, p. 150], $\mathcal{O}$ encircles the origin. This contradicts the invariance of $\overline{Q}_{\beta_{2n+1}}$ with respect to $\widetilde{S}_-$, and we conclude that $\mathcal{P}(x_0) = \{0\}$. Therefore, each trajectory of $\widetilde{S}_-$ starting in $\overline{W}_{2n+1}$ remains in $\overline{W}_{2n+1}$ and converges to the origin [5, Corollary 1.1 p. 146].

Because this last result holds for each n in $\mathbb{Z}$, and the family $\{W_{2n+1}\}_{n\in\mathbb{Z}}$ is a base at the origin that covers $\mathbb{R}^2$, the feedback law u_k globally asymptotically stabilizes the system S_-.

Similarly, from the invariance of the sets $\overline{W}_{2n}$ and $\overline{T}_{\beta_{2n}}$ with respect to the vector-field $[f_+, u_{k_0}]^t$ for each n in $\mathbb{Z}$, and the Poincaré-Bendixson Theorem [5], it follows that u_{k_0} globally asymptotically stabilizes the system S_+. Hence the theorem. ■

3 Simultaneous stabilization of nonlinear systems

Throughout this section we consider a more general pair $\{S_i,\ i = 1,2\}$ of systems

$$S_i: \quad \dot{x} \;=\; f_i(x,u), \quad i = 1,2,$$

where the state x lies in $\mathbb{R}^n$, the input u, is in $\mathbb{R}^m$, and for each $i = 1,2$, the mapping $f_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous on a neighborhood of the origin with $f_i(0,0) = 0$.

In the following theorem, we show that if there exist continuous feedback laws u_1 and u_2 that locally asymptotically stabilize the systems S_1 and S_2 respectively, then there exists a continuous feedback law v that simultaneously stabilizes S_1 and S_2. To prove this result we extend the interpolation method introduced in the proof of Theorem 2.1 as follows: For each $i = 1,2$, we let V_i denote a Lyapunov function for the system $\dot{x} = f_i(x, u_i(x))$. We define a sequence of neighborhoods of the origin $\{W_j\}_{j=1}^{\infty}$ such that the boundaries of the odd sets W_{2n-1} (resp. even sets W_{2n}) are level sets of V_1 (resp. V_2). Then, we design a continuous feedback law v which is equal to u_1 (resp. to u_2) on the boundaries of the sets W_{2n-1} (resp. W_{2n}). It follows that the sets $\overline{W}_{2n-1}$ (resp. $\overline{W}_{2n}$) are invariant with respect to the system obtained once v is fed back into S_1 (resp. S_2). We conclude that v stabilizes S_1 (resp. S_2) upon noting that the family $\{W_{2n-1}\}_{n=1}^{\infty}$ (resp. $\{W_{2n}\}_{n=1}^{\infty}$) is a base at the origin.

Theorem 3.1 *Assume that for each $i = 1,2$, there exists a continuous and almost C^k feedback law u_i that locally asymptotically stabilizes the system S_i at the origin. Then, there exists a continuous and almost C^k feedback law that simultaneously stabilizes the systems S_1 and S_2.*

Proof: Throughout the proof, we adopt the usual convention that the infimum of a real-valued mapping over the empty set is $+\infty$.

Note that for each $i = 1,2$, the mapping $x \mapsto f_i(x, u_i(x))$ is continuous. By the Converse Lyapunov Theorem [11], the local asymptotic stability (resp. global asymptotic stability) of the system $\dot{x} = f_i(x, u_i(x))$ for each $i = 1,2$ yields the existence of a Lyapunov function (resp. radially unbounded Lyapunov function) $V_i : D_i \to [0,\infty)$, where D_i is a bounded neighborhood of the origin (resp. $D_i = \mathbb{R}^n$). If $D_1 = D_2 = \mathbb{R}^n$, we set $D \triangleq \mathbb{R}^n$, and otherwise we let D be a bounded neighborhood of the origin such that $\overline{D} \subset D_1 \cap D_2$. In both cases, we have

$$\nabla V_i(x)\, f_i(x, u_i(x)) \;<\; 0, \quad x \in \overline{D}\backslash\{0\}, \quad i = 1,2. \tag{9}$$

For each $i = 1,2$ and each $\beta > 0$, we define W_i^{β} by setting

$$W_i^{\beta} \;\triangleq\; \{x \in D : V_i(x) \;<\; \beta\},$$

and we construct a sequence of neighborhoods of the origin as follows: By applying Lemma 6.6 (with D, V_1 and V_2), we obtain a sequence of positive reals $\{\beta_j\}_{j=1}^{\infty}$ satisfying

$$\beta_{2n-1} \;<\; \inf_{x\in\partial D} V_1(x), \quad n = 1,2,\ldots, \tag{10}$$

$$\beta_{2n} \;<\; \inf_{x\in\partial D} V_2(x), \quad n = 1,2,\ldots, \tag{11}$$

$$\beta_j \;\to\; 0 \;\text{ as }\; j \to \infty, \tag{12}$$

with

$$\overline{W}_2^{\beta_{2n}} \subset W_1^{\beta_{2n-1}} \qquad \text{and} \qquad \overline{W}_1^{\beta_{2n+1}} \subset W_2^{\beta_{2n}}, \quad n = 1, 2, \ldots. \tag{13}$$

Upon setting

$$W_{2n-1} \triangleq W_1^{\beta_{2n-1}} \qquad \text{and} \qquad W_{2n} \triangleq W_2^{\beta_{2n}}, \quad n = 1, 2, \ldots.$$

the inclusions (13) translate to

$$\overline{W}_{j+1} \subset W_j, \quad j = 1, 2, \ldots. \tag{14}$$

From (10) and Lemma 6.6 **(i)** we get $\overline{W}_1 \subset D$ and we therefore have a sequence of nested neighborhoods

$$D \supset W_1 \supset W_2 \supset W_3 \supset W_4 \supset \ldots$$

such that each neighborhood contains the closure of the neighborhood that follows. We now define the set Δ_j for each $j = 1, 2, \ldots$, by setting

$$\Delta_1 \triangleq D \backslash \overline{W}_2 \qquad \text{and} \qquad \Delta_j \triangleq W_{j-1} \backslash \overline{W}_{j+1}, \quad j = 2, 3, \ldots.$$

By combining the inclusion (14) with the fact that $\{W_j\}_{j=1}^{\infty}$ is a base at the origin [which follows from (12) and Lemma 6.3], it is not hard to check that $\{\Delta_j\}_{j=1}^{\infty}$ is an open cover of $D \backslash \{0\}$. Let $\{q_j\}_{j=1}^{\infty}$ be a partition of unity subordinate to $\{\Delta_j\}_{j=1}^{\infty}$ such that the support of q_j is included in Δ_j, for each $j = 1, 2, \ldots$ [4].

Finally, we let the feedback law $v : D \to \mathbb{R}^m$ be given by

$$v(x) = \begin{cases} 0, & x = 0 \\ u_1(x) \sum_{n=1}^{\infty} q_{2n-1}(x) + u_2(x) \sum_{n=1}^{\infty} q_{2n}(x), & x \in D \backslash \{0\}. \end{cases}$$

By a similar argument to that used in the proof of Theorem 2.1 to prove continuity of the feedback law u_k, it is easily checked that v is almost C^k and continuous on D.

Stability :

From the definitions of the sets W_j and Δ_j, it is not hard to see that for each $j = 1, 2, \ldots$, the boundary ∂W_j is included in Δ_j and does not intersect with any other set Δ_m. Thus, because the support of the mapping q_j is included in Δ_j for each $j = 1, 2 \ldots$, it follows from the definition of v that for each $n = 1, 2, \ldots$, we have

$$v(x) = u_1(x), \quad x \in \partial W_{2n-1} \qquad \text{and} \qquad v(x) = u_2(x), \quad x \in \partial W_{2n}.$$

This, together with (9) and the fact that $\overline{W}_j$ is included in D for each $j = 1, 2, \ldots$, yield

$$\nabla V_1(x) f_1(x, v(x)) < 0, \quad x \in \partial W_{2n-1}, \quad n = 1, 2, \ldots \tag{15}$$

and

$$\nabla V_2(x) f_2(x, v(x)) < 0, \quad x \in \partial W_{2n}, \quad n = 1, 2, \ldots. \tag{16}$$

For each $n = 1, 2, \ldots$, by combining (15) with Lemma 6.5 applied with D, V_1, f_1 and β_{2n-1} we obtain that the set $\overline{W}_{2n-1}$ is invariant with respect to the system $\dot{x} = f_1(x, v(x))$.

Finally, because the family $\{W_{2n-1}\}_{n=1}^{\infty}$ is a base at the origin, by using the invariance of the sets $\overline{W}_{2n-1}$, $n = 1, 2, \ldots$, it is easily checked that v stabilizes the system S_1.

Similarly, from (16) together with Lemma 6.5 and the fact that $\{W_{2n}\}_{n=1}^{\infty}$ is a base at the origin, we deduce that v stabilizes S_2. Hence the theorem. ■

4 An explicit simultaneous stabilizer

The simultaneous stabilizing feedback law v defined in the proof of Theorem 3.1 is based on the partition of unity $\{q_j\}_{j=1}^{\infty}$ which might be difficult to explicitly express. This prevents us from obtaining an explicit expression for the stabilizing feedback law v. In this section, we show that we can circumvent this problem, and give an explicit stabilizing feedback law.

Throughout we assume that for each $i = 1, 2$, there exists a continuous and almost $C^{k'}$ feedback law u_i that locally asymptotically stabilizes the system S_i as defined in section 3. As mentioned in Theorem 3.1, this yields the existence of a bounded neighborhood of the origin (resp. $D = \mathbb{R}^n$) and a $C^{k''}$ Lyapunov function (resp. a $C^{k''}$ radially unbounded Lyapunov function) $V_i : D \to [0, \infty)$ for each $i = 1, 2$, such that

$$\nabla V_i(x)\, f_i(x, u_i(x)) \;<\; 0, \quad x \in \overline{D}\backslash\{0\}. \tag{17}$$

We let k denote the integer $k \triangleq \min(k', k'')$ and for each $i = 1, 2$ and each $\beta > 0$ we set $W_i^{\beta} \triangleq D \cap V_i^{-1}([0, \beta))$. By applying Lemma 6.7 (with D, V_1 and V_2), we obtain three sequences of positive reals $\{\alpha_j\}_{j=1}^{\infty}$, $\{\beta_j\}_{j=1}^{\infty}$, and $\{\gamma_j\}_{j=1}^{\infty}$ such that

$$\alpha_j, \beta_j, \gamma_j \to 0 \;\; as \;\; j \to \infty \tag{18}$$

with

$$\alpha_j \;<\; \beta_j \;<\; \gamma_j, \quad j = 1, 2, \ldots . \tag{19}$$

We also have

$$\gamma_{2n-1} \;<\; \inf_{x \in \partial D} V_1(x) \qquad \text{and} \qquad \gamma_{2n} \;<\; \inf_{x \in \partial D} V_2(x), \quad n = 1, 2, \ldots, \tag{20}$$

with

$$\overline{W_2^{\gamma_{2n}}} \subset W_1^{\alpha_{2n-1}} \qquad \text{and} \qquad \overline{W_1^{\gamma_{2n+1}}} \subset W_2^{\alpha_{2n}}, \quad n = 1, 2, \ldots . \tag{21}$$

For each $n = 1, 2, \ldots$, we now define the mappings $\overline{q}_{2n}, \overline{q}_{2n-1} : D \to [0, 1]$ by setting

$$\overline{q}_{2n-1}(x) = \begin{cases} e^{\frac{(V_1(x)-\beta_{2n-1})^2}{(V_1(x)-\beta_{2n-1})^2-(\beta_{2n-1}-\alpha_{2n-1})^2}} & \text{if } V_1(x) \in (\alpha_{2n-1}, \beta_{2n-1}) \\ e^{\frac{(V_1(x)-\beta_{2n-1})^2}{(V_1(x)-\beta_{2n-1})^2-(\gamma_{2n-1}-\beta_{2n-1})^2}} & \text{if } V_1(x) \in [\beta_{2n-1}, \gamma_{2n-1}) \\ 0\,, & \text{otherwise} \end{cases},$$

and

$$\overline{q}_{2n}(x) = \begin{cases} e^{\frac{(V_2(x)-\beta_{2n})^2}{(V_2(x)-\beta_{2n})^2-(\beta_{2n}-\alpha_{2n})^2}} & \text{if } V_2(x) \in (\alpha_{2n}, \beta_{2n}) \\ e^{\frac{(V_2(x)-\beta_{2n})^2}{(V_2(x)-\beta_{2n})^2-(\gamma_{2n}-\beta_{2n})^2}} & \text{if } V_2(x) \in [\beta_{2n}, \gamma_{2n}) \\ 0\,, & \text{otherwise} \end{cases}.$$

Finally, we let $\overline{v} : D \to \mathbb{R}^m$ be given by

$$\overline{v}(x) = u_1(x) \sum_{n=1}^{\infty} \overline{q}_{2n-1}(x) \;+\; u_2(x) \sum_{n=1}^{\infty} \overline{q}_{2n}(x), \quad x \in D \tag{22}$$

Theorem 4.1 *The feedback law $\overline{v}$ is continuous and almost C^k on D. Moreover, $\overline{v}$ simultaneously stabilizes S_1 and S_2.*

Proof: For a given sequence of positive real $\{\delta_j\}_{j=1}^{\infty}$ we set

$$W^{\delta_{2n-1}} \triangleq W_1^{\delta_{2n-1}} \qquad \text{and} \qquad W^{\delta_{2n}} \triangleq W_2^{\delta_{2n}}, \quad n = 1, 2, \ldots .$$

With this notation (19), (20), (21) and Lemma 6.4 **(i)** imply that we have a sequence of neighborhoods

$$D \supset W^{\gamma_1} \supset W^{\beta_1} \supset W^{\alpha_1} \supset W^{\gamma_2} \supset W^{\beta_2} \supset W^{\alpha_2} \supset W_1^{\gamma_3} \supset \ldots , \tag{23}$$

such that each neighborhood contains the closure of the neighborhood that follows. For each $j = 1, 2, \ldots$, we let Π_j denote the set

$$\Pi_j \triangleq \{x \in D : \overline{q}_j(x) \neq 0\}.$$

The inequalities (20) together with Lemma 6.4 **(i)** then yield

$$\Pi_j = W^{\gamma_j} \backslash \overline{W}^{\alpha_j}, \quad j = 1, 2, \ldots . \tag{24}$$

and it follows that

$$\Pi_j \cap \Pi_m = \emptyset, \quad j \neq m. \tag{25}$$

Let x be in $D\backslash\{0\}$ and let r be in $(0, \|x\|)$. Because $\{W^{\gamma_j}\}_{j=1}^{\infty}$ is a base at the origin composed of nested neighborhoods [from (18), (23) and Lemma 6.3], there exists an integer n_r such that

$$W^{\gamma_j} \subset \overline{B_r(0)}, \quad j = 2n_r + 1,\, 2n_r + 2, \ldots ,$$

where $B_r(0)$ denote the set $B_r(0) \triangleq \{x \in \mathbb{R}^n : \|x\| < r\}$. It follows from the definition of $\overline{v}$ together with the fact that Π_j is included in W^{γ_j} for each $j = 1, 2, \ldots$ [by (24)], that

$$\overline{v}(y) = u_1(y) \sum_{n=1}^{n_r} \overline{q}_{2n-1}(y) + u_2(y) \sum_{n=1}^{n_r} \overline{q}_{2n}(y), \quad y \in D\backslash\overline{B_r(0)}. \tag{26}$$

Because u_1, u_2 and the mappings $\overline{q}_j, j = 1, 2, \ldots$ are C^k on $D\backslash\{0\}$ [follows from Lemma 6.8], we easily obtain from (26) that $\overline{v}$ is C^k on $D\backslash\{0\}$. Furthermore, (25) implies that

$$\|\overline{v}(x)\| \leq \max(\|u_1(x)\|, \|u_2(x)\|), \quad x \in D,$$

and continuity of $\overline{v}$ at the origin follows from that of u_1 and u_2.

Stability:

From (25) and the definition of the mappings $\overline{q}_j$, we deduce that for each $j = 1, 2, \ldots$

$$\overline{q}_j(x) = 1 \quad \text{with} \quad \overline{q}_m(x) = 0, \quad x \in \partial W^{\beta_j}, \quad m \neq j,$$

so that the definition of $\overline{v}$ yields for each $n = 1, 2, \ldots$,

$$\overline{v}(x) = u_1(x), \quad x \in \partial W^{\beta_{2n-1}} \qquad \text{and} \qquad \overline{v}(x) = u_2(x), \quad x \in \partial W^{\beta_{2n}}.$$

Thus, it follows from (17) that

$$\nabla V_1(x)\, f_1(x, \overline{v}(x)) < 0, \quad x \in \partial W^{\beta_{2n-1}}, \quad n = 1, 2, \ldots , \tag{27}$$

and

$$\nabla V_2(x)\, f_2(x, \overline{v}(x)) < 0, \quad x \in \partial W^{\beta_{2n}}, \quad n = 1, 2, \ldots . \tag{28}$$

Finally, by combining (27) with Lemma 6.5 applied with D, V_1, f_1 and β_{2n-1} for each $n = 1, 2, \ldots$, we obtain that the set $\overline{W}^{\beta_{2n-1}}$ is invariant with respect to the system $\dot{x} = f_1(x, \overline{v}(x))$, and stability of this system follows from the fact that $\{W^{\beta_{2n-1}}\}_{n=1}^{\infty}$ is a base at the origin.

Similarly, because $\{W^{\beta_{2n}}\}_{n=1}^{\infty}$ is a base at the origin, it is easily checked from (28) and Lemma 6.5 that v stabilizes S_2. Hence the theorem. ■

This theorem reduces the problem of designing a simultaneous stabilizer for S_1 and S_2 to that of finding the sequences of reals $\{\alpha_j\}_{j=1}^{\infty}$, $\{\beta_j\}_{j=1}^{\infty}$ and $\{\gamma_j\}_{j=1}^{\infty}$. To obtain these sequences in case the system S_i and the feedback law u_i are linear for each $i = 1, 2$, we proceed as described below.

The linear case:

Assume that for each $i = 1, 2$, the system S_i and the feedback law u_i are linear and let $V_i : \mathbb{R}^n \to [0, \infty)$ be a Lyapunov function for the system $\dot{x} = f_i(x, u_i(x))$ given by $V_i(x) = x^t P_i x$, where P_i is a positive definite matrix. By applying Lemma 6.9 with P_1 and P_2, we obtain explicit sequences $\{\alpha_j\}_{j=1}^{\infty}$, $\{\beta_j\}_{j=1}^{\infty}$ and $\{\gamma_j\}_{j=1}^{\infty}$ such that the assertions of Lemma 6.7 are satisfied. Therefore, we can use these sequences to define $\overline{v}$ as given by (22) and Theorem 4.1 implies that $\overline{v}$ simultaneously stabilizes S_1 and S_2.

5 Concluding remark

In this paper, we have introduced a feedback law interpolation method that has enabled us to prove that any pair of locally asymptotically stabilizable systems is simultaneously stabilizable by continuous feedback. The resulting simultaneous stabilizer depending on a partition of unity, we then show how to circumvent the computation of this partition of unity and construct an explicit simultaneous stabilizer. These results in fact extend to a countable number of systems [6].

Following the results presented here as well as those of [7], we believe that the few techniques developed recently in the context of stabilization of nonlinear systems are good sources of inspiration for robust and simultaneous stabilization issues. We hope that this work will yield some new insight into these problems.

6 Appendix

We present here several technical lemmas that were used in the proof of Theorems 2.1 and 3.1. The next two lemmas were needed in Theorem 2.1.

Lemma 6.1 *Let a_-, a_+ and b_+ be positive, and let b_- be negative. Let μ and δ be some positive reals with $\mu < -\frac{b_-}{a_-}$ and $\frac{2b_+}{a_+} < \delta$. Then, there exists $k_- > 0$ such that the following holds.*

i) *For each $k \geq k_-$, the vector $[f_-(x), -kx_2]^t$ points into the region below $\Sigma_{-\frac{b_-}{2a_-}}$ for each x in $\Sigma_{-\frac{b_-}{2a_-}}$, and into the region above $\widehat{\Sigma}_{-\frac{b_-}{2a_-}}$ for each x in $\widehat{\Sigma}_{-\frac{b_-}{2a_-}}$.*

ii) *For each $\beta > 0$, let D_β denote the set $D_\beta \triangleq \{x \in \mathbb{R}^2 : x_1 = \mu x_2 + \beta,\, x_1 > 0\}$. Then, for each $\beta > 0$, the vector $[f_-(x), \frac{1}{\mu}(2a_-x_1 + b_-x_2)]^t$ points towards the left of D_β for each x in D_β below $\Sigma_{\frac{-b_-}{a_-}}$, and towards the right of $\widehat{D}_\beta$ for each x in $\widehat{D}_\beta$ above $\widehat{\Sigma}_{\frac{-b_-}{a_-}}$.*

iii) *For each $\tau > 0$, let L_τ denote the set $L_\tau \triangleq \{x \in \mathbb{R}^2 : x_1 = -\delta x_2 - \tau,\, x_1 > 0\}$. Then, for each $\tau > 0$, the vector $[f_-(x), -\frac{1}{\delta}(\frac{a_-}{2}x_1 + b_- x_2)]^t$ points into the region above L_τ for each x in L_τ, and into the region below $\widehat{L}_\tau$ for each x in $\widehat{L}_\tau$.*

Proof: We only prove the first part of the assertions of the lemma as the arguments carry over to the second part of the assertion.

(i) Let x be in $\Sigma_{\frac{-b_-}{2a_-}}$. We have $f_-(x) = \frac{b_-}{2}x_2$, so that $-\frac{f_-(x)}{kx_2} = -\frac{b_-}{2k}$. Because $-\frac{f_-(x)}{kx_2}$ is less than $-\frac{b_-}{2a_-}$ for k large enough, the claim follows.

(ii) Let $\beta > 0$ and let x be in D_β below $\Sigma_{\frac{-b_-}{a_-}}$. As we have $f_-(x) > 0$ and $a_- x_1 > 0$, we immediately obtain $\frac{f_-(x)}{\frac{2a_- x_1 + b_- x_2}{\mu}} < \mu$ for all $\beta > 0$. Hence the claim.

(iii) Let $\tau > 0$ and let x be in D_τ. Because we have $\frac{a_-}{2\delta}x_1 > 0$, we easily get $-\frac{-\frac{1}{\delta}(\frac{a_-}{2}x_1 + b_- x_2)}{f_-(x)} < \frac{1}{\delta}$ for all $\tau > 0$. Hence the result. ■

The proof of the following lemma is similar to that of Lemma 6.1 and is therefore omitted.

Lemma 6.2 *Let a_-, a_+ and b_+ be positive, and let b_- be negative. Let μ and δ be some positive reals with $\mu < \frac{b_+}{a_+}$ and $-\frac{2b_-}{a_-} < \delta$. Then, there exists $k_+ > 0$ such that the following holds.*

i) *For each $k \geq k_+$, the vector $[f_+(x), -kx_2]^t$ points into the region above $\Sigma_{-\frac{b_+}{2a_+}}$ for each x in $\Sigma_{-\frac{b_+}{2a_+}}$, and into the region below $\widehat{\Sigma}_{-\frac{b_+}{2a_+}}$ for each x in $\widehat{\Sigma}_{-\frac{b_+}{2a_+}}$.*

ii) *For each $\beta > 0$, let D_β denote the set $D_\beta \triangleq \{x \in \mathbb{R}^2 : x_1 = -\mu x_2 + \beta,\, x_1 > 0\}$. Then, for each $\beta > 0$, the vector $[f_+(x), -\frac{1}{\mu}(2a_+ x_1 + b_+ x_2)]^t$ points towards the left of D_β for each x in D_β above $\Sigma_{\frac{-b_+}{a_+}}$, and towards the right of $\widehat{D}_\beta$ for each x in $\widehat{D}_\beta$ below $\widehat{\Sigma}_{\frac{-b_+}{a_+}}$.*

iii) *For each $\tau > 0$, let L_τ denote the set $L_\tau \triangleq \{x \in \mathbb{R}^2 : x_1 = \delta x_2 - \tau,\, x_1 > 0\}$. Then, for each $\tau > 0$, the vector $[f_+(x), \frac{1}{\delta}(\frac{a_+}{2}x_1 + b_+ x_2)]^t$ points into the region below L_τ for each x in L_τ, and into the region above $\widehat{L}_\tau$ for each x in $\widehat{L}_\tau$.*

The next three lemmas are used in the proofs of Theorem 3.1 and Theorem 4.1. The proof of the next two lemmas being elementary, we omit them; details are available in [6].

Lemma 6.3 *Let D be a bounded neighborhood of the origin in $\mathbb{R}^n$ (resp. $D = \mathbb{R}^n$) and let $V : D \to [0, \infty)$ be a Lyapunov function (resp. a radially unbounded Lyapunov function). For each $\beta > 0$, let W^β denote the set $W^\beta \triangleq \{x \in D : V(x) < \beta\}$. Then, the family $\{W^\beta\}_{\beta > 0}$ is a base at the origin such that $W^\alpha \subset W^\beta$ whenever $\alpha \leq \beta$.*

Lemma 6.4 *Let D be a bounded neighborhood of the origin in $\mathbb{R}^n$ (resp. $D = \mathbb{R}^n$) and let $V : \overline{D} \to [0, \infty)$ be a Lyapunov function (resp. a radially unbounded Lyapunov function). Let β be in the interval $(0, \inf_{x \in \partial D} V(x))$ and define the set U by setting $U \triangleq D \cap V^{-1}([0, \beta))$. Then, the following holds:*

i) $\overline{U} = D \cap V^{-1}([0, \beta])$.

ii) *Let x_0 be in $\overline{U}$ and let the mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuous. If the trajectory $x(\cdot, x_0)$ of the system $\dot{x} = f(x)$ does not remain in $\overline{U}$ forever, then there exists $\widehat{t} \geq 0$ and $\widehat{h} > 0$ such that*

$$x(\widehat{t}, x_0) \in \partial U \qquad \textit{with} \qquad V(x(\widehat{t}, x_0)) = \beta,$$

$$
\begin{aligned}
x(\widehat{t}+h, x_0) &\notin V^{-1}([0,\beta]), \quad h \in (0,\widehat{h}),\\
x(\widehat{t}+h, x_0) &\in D, \quad h \in (0,\widehat{h}).
\end{aligned}
$$

Lemma 6.5 *Let D be a bounded neighborhood of the origin (resp. $D = \mathbb{R}^n$) and let $V : \overline{D} \to [0,\infty)$ be a Lyapunov function. Let $f : \overline{D} \to \mathbb{R}^n$ be a continuous mapping and denote by (S) the system $\dot{x} = f(x)$. Let β be in $(0, \inf_{x\in\partial D} V(x))$ and let W^β denote the set $W^\beta \triangleq D \cap V^{-1}([0,\beta))$. Assume that*

$$
\nabla V(x)\, f(x) \ < \ 0, \quad x \in \partial W^\beta, \tag{29}
$$

then, the set $\overline{W}^\beta$ is invariant with respect to the system (S).

Proof: We prove the lemma by a contradiction argument. Assume that $\overline{W}^\beta$ is not invariant with respect to (S). Then, there exists x_0 in $\overline{W}^\beta$ such that the trajectory $x(\cdot, x_0)$ of (S) does not remain forever in $\overline{W}^\beta$. By combining Lemma 6.4 (with D, V and W^β) with the fact that $\beta < \inf_{x\in\partial D} V(x)$, we obtain $\widehat{t} \geq 0$ and $\widehat{h} > 0$ such that

$$
x(\widehat{t}, x_0) \in \partial W^\beta \qquad \text{and} \qquad V(x(\widehat{t}, x_0)) = \beta. \tag{30}
$$

For each h in $(0, \widehat{h})$, we also have

$$
x(\widehat{t}+h, x_0) \ \notin \ V^{-1}([0,\beta]) \qquad \text{and} \qquad x(\widehat{t}+h, x_0) \ \in \ D. \tag{31}
$$

From (29) and (30) we get

$$
\frac{\partial V(x(t,x_0))}{\partial t}\Big|_{t=\widehat{t}} \ = \ \nabla V(x(\widehat{t}, x_0))\, f(\,x(\widehat{t}, x_0)\,) \ < \ 0.
$$

Thus, in view of (30), there exists $\widetilde{h}$ in $(0, \widehat{h}]$ such that

$$
V(x(\widehat{t}+h, x_0)) \ < \ V(x(\widehat{t}, x_0)) \ = \ \beta, \quad h \in (0, \widetilde{h}),
$$

a contradiction with the strict inequality

$$
V(x(\widehat{t}+h, x_0)) \ > \ \beta, \quad h \in (0, \widehat{h}),
$$

which follows from (31). Hence the result. ∎

Lemma 6.6 *Let D be a bounded neighborhood of the origin (resp. $D = \mathbb{R}^n$). For each $i = 1, 2$, let $V_i : \overline{D} \to [0,\infty)$ be a Lyapunov function (resp. a radially unbounded Lyapunov function) and let W_i^β denote the set $W_i^\beta \triangleq \{x \in D : V_i(x) < \beta\}$ for each $\beta > 0$. Then, there exists a sequence of positive reals $\{\beta_j\}_{j=1}^\infty$ converging to 0 such that for each $n = 1, 2, \ldots,$ we have:*

$$
\beta_{2n-1} < \inf_{x\in\partial D} V_1(x) \qquad \textit{and} \qquad \beta_{2n} < \inf_{x\in\partial D} V_2(x),
$$

with

$$
\overline{W}_2^{\beta_{2n}} \subset W_1^{\beta_{2n-1}} \qquad \textit{and} \qquad \overline{W}_1^{\beta_{2n+1}} \subset W_2^{\beta_{2n}}.
$$

Proof: For each $n = 1, 2 \ldots,$ we define β_{2n-1} and β_{2n} by induction on n.

For $n = 1$, we first pick β_1 in $(0, \inf_{x\in\partial D} V_1(x)\,)$. Then, because the family $\{W_2^\beta\}_{\beta>0}$ is a base at the origin with $W_2^\alpha \subset W_2^\beta$ for all $\alpha < \beta$ (Lemma 6.3), there exists β_2 in $(0, \inf_{x\in\partial D} V_2(x)\,)$ such that

$$
\overline{W}_2^{\beta_2} \ \subset \ W_1^{\beta_1}.
$$

For each $n = 2, 3, \ldots$, we define β_{2n-1} and β_{2n} from $\beta_{2(n-1)-1}$ and $\beta_{2(n-1)}$ as follows: By lemma 6.3 applied to $\{W_1^\beta\}_{\beta>0}$, there exists β_{2n-1} in the interval $(0, \frac{\beta_{2(n-1)-1}}{2}]$ such that

$$\overline{W}_1^{\beta_{2n-1}} \subset W_2^{\beta_{2(n-1)}}.$$

Similarly, using Lemma 6.3, we select β_{2n} in $(0, \frac{\beta_{2(n-1)}}{2}]$ such that

$$\overline{W}_2^{\beta_{2n}} \subset W_1^{\beta_{2n-1}}.$$

It is now plain from this construction that the assertions of the lemma hold. ∎

Lemma 6.7 *Let D be a bounded neighborhood of the origin (resp. $D = \mathbb{R}^n$). For each $i = 1, 2$, let $V_i : \overline{D} \to [0, \infty)$ be a-Lyapunov function (resp. a radially unbounded Lyapunov function) and let W_i^β denote the set $W_i^\beta \triangleq \{x \in D : V_i(x) < \beta\}$ for each $\beta > 0$. Then, there exist sequences of positive reals $\{\alpha_j\}_{j=1}^\infty$, $\{\beta_j\}_{j=1}^\infty$, and $\{\gamma_j\}_{j=1}^\infty$ converging to the origin with $\gamma_{2n-1} < \inf_{x\partial D} V_1(x)$ and $\gamma_{2n} < \inf_{x\partial D} V_2(x)$ for each $n = 1, 2, \ldots$, satisfying*

$$\alpha_j < \beta_j < \gamma_j, \quad j = 1, 2, \ldots$$

with

$$\overline{W}_2^{\gamma_{2n}} \subset W_1^{\alpha_{2n-1}} \qquad \text{and} \qquad \overline{W}_1^{\gamma_{2n+1}} \subset W_2^{\alpha_{2n}}, \quad n = 1, 2, \ldots.$$

Proof: For each $n = 1, 2 \ldots$, we define γ_{2n-1}, β_{2n-1}, α_{2n-1}, γ_{2n}, β_{2n} and α_{2n} by induction on n.

For $n = 1$, we pick γ_1 in $(0, \inf_{x \in \partial D} V_1(x))$, and we choose β_1 and α_1 such that $0 < \alpha_1 < \beta_1 < \gamma_1$. Then, because the family $\{W_2^\gamma\}_{\gamma>0}$ is a neighborhood base at the origin with $W_2^{\gamma'} \subset W_2^{\gamma''}$ for all $\gamma' < \gamma''$ (Lemma 6.3), there exists γ_2 in $(0, \inf_{x \in \partial D} V_2(x))$ such that

$$\overline{W}_2^{\gamma_2} \subset W_1^{\alpha_1}.$$

We choose β_2 and α_2 such that $0 < \alpha_2 < \beta_2 < \gamma_2$.

For each $n = 2, 3, \ldots$, we define γ_{2n-1}, $\beta_{2n-1}, \alpha_{2n-1}$, γ_{2n}, β_{2n} and α_{2n} from $\gamma_{2(n-1)-1}$, $\gamma_{2(n-1)}$ and $\alpha_{2(n-1)}$ as follows: By lemma 6.3 applied to $\{W_1^\gamma\}_{\gamma>0}$, there exists γ_{2n-1} in the interval $(0, \frac{\gamma_{2(n-1)-1}}{2}]$ such that

$$\overline{W}_1^{\gamma_{2n-1}} \subset W_2^{\alpha_{2(n-1)}}.$$

We then pick α_{2n-1} and β_{2n-1} such that $0 < \alpha_{2n-1} < \beta_{2n-1} < \gamma_{2n-1}$.

Similarly, by using Lemma 6.3, we select γ_{2n} in $(0, \frac{\gamma_{2(n-1)}}{2}]$ such that

$$\overline{W}_2^{\gamma_{2n}} \subset W_1^{\alpha_{2n-1}},$$

and we choose α_{2n} and β_{2n} such that $0 < \alpha_{2n} < \beta_{2n} < \gamma_{2n}$. It is now clear from this construction that the assertions of the lemma hold. ∎

The remaining two lemmas are used in section 4. The proof of the next lemma being elementary, we omit it and refer the reader to [6] for further details.

Lemma 6.8 *Let X be a subset of $\mathbb{R}^n$ and let $V : X \to \mathbb{R}$ be C^k on X. Let α, β and γ be some positive reals such that $\alpha < \beta < \gamma$. Then, the mapping $h : X \to [0, 1]$ given by*

$$h(x) = \begin{cases} e^{\frac{(V(x)-\beta)^2}{(V(x)-\beta)^2-(\beta-\alpha)^2}} & \text{if } V(x) \in (\alpha, \beta) \\ e^{\frac{(V(x)-\beta)^2}{(V(x)-\beta)^2-(\gamma-\beta)^2}} & \text{if } V(x) \in [\beta, \gamma) \\ 0\,, & \text{otherwise} \end{cases},$$

is C^k on X.

Lemma 6.9 *For each $i = 1,2$, let P_i be a positive definite matrix, let $\frac{1}{M_i}$ and $\frac{1}{m_i}$ denote respectively the smallest and the largest eigenvalue of P_i, let θ_i be in $(0,1)$, and let $V_i : \mathbb{R}^n \to (0,\infty)$ be given by $V_i(x) = x^t P_i x$. Furthermore, let π_1 and π_2 be in $(0, \frac{M_2}{m_1})$ and $(0, \frac{M_1}{m_2})$ respectively, and let $\overline{\gamma}_1$ be an arbitrary positive real. Assume that*

$$\pi_1 \pi_2 \theta_1^2 \theta_2^2 < 1, \tag{32}$$

and let the sequences of positive reals $\{\gamma_j\}_{j=1}^{\infty}$, $\{\beta_j\}_{j=1}^{\infty}$, and $\{\alpha_j\}_{j=1}^{\infty}$ be defined by

$$\gamma_1 \triangleq \overline{\gamma}_1 \quad \text{with} \quad \beta_{2n-1} \triangleq \theta_1 \gamma_{2n-1}, \quad \alpha_{2n-1} \triangleq \theta_1 \beta_{2n-1}, \quad \gamma_{2n} \triangleq \pi_2 \alpha_{2n-1}, \quad n = 1,2,\ldots$$

and

$$\beta_{2n} \triangleq \theta_2 \gamma_{2n}, \quad \alpha_{2n} \triangleq \theta_2 \beta_{2n}, \quad \gamma_{2n+1} \triangleq \pi_1 \alpha_{2n}, \quad n = 1,2,\ldots.$$

Then, we have

$$\alpha_j, \beta_j, \gamma_j \to 0 \quad \text{as} \quad j \to \infty \tag{33}$$

and for each $n = 1,2,\ldots$, we have

$$V_2^{-1}([0,\gamma_{2n}]) \subset V_1^{-1}([0,\alpha_{2n-1}]) \quad \text{and} \quad V_1^{-1}([0,\gamma_{2n+1}]) \subset V_2^{-1}([0,\alpha_{2n}]). \tag{34}$$

Proof: In what follows we fix n in $\mathbb{Z}$. Let $\delta > 0$. It is well known that for each $i = 1,2$, the set $V_i^{-1}([0,\delta])$ is the volume bounded by the ellipsoid centered at the origin with smallest axis $\sqrt{m_i \delta}$ and largest axis $\sqrt{M_i \delta}$. Thus, (34) will hold if

$$\gamma_{2n} < \frac{M_1}{m_2} \alpha_{2n} \quad \text{and} \quad \gamma_{2n+1} < \frac{M_2}{m_1} \alpha_{2n}$$

respectively. Because π_1 and π_2 are in $(0, \frac{M_2}{m_1})$ and $(0, \frac{M_1}{m_2})$ respectively, the definitions of γ_{2n} and γ_{2n+1} clearly yield (34). On the other hand, it is easily checked from the assumptions that we have

$$\beta_{j+2} = (\pi_1 \pi_2 \theta_1^2 \theta_2^2) \beta_j, \quad j = 1,2,\ldots$$

so that (33) follows from (32) and the definitions of γ_j, β_j and α_j. ■

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