ABSTRACT

Title of Dissertation: CATEGORIFICATION OF THE LERAY SPECTRAL SEQUENCE

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The first part of this thesis concerns the categorification of differentials in the Leray spectral sequence. Given a continuous map of topological spaces $f\colon X \to Y$ and a sheaf of abelian groups G, the Leray spectral sequence takes the form

$$E_2^{p,q} := H^p(Y, R^q f_* G) \Rightarrow H^{p+q}(X, G).$$

We provide a categorification of certain differentials in this spectral sequence, specifically those of the form

$$d_2\colon H^0(Y, R^q f_* G) \to H^2(Y, R^{q-1} f_* G),$$

by constructing a gerbe that encodes the first obstruction to a lifting problem.

The differential d_2 is first described at the level of Čech cocycles. Then, for a class $\alpha \in H^0(Y, R^q f_* G)$, we construct the sheaf of local lifts, which takes values in q-groupoids. Taking the 1-truncation of this object yields an $R^{q-1} f_* G$-banded gerbe whose cohomology class represents

the image of α under d_2. Finally, we show that this construction is compatible with the additive structure on cohomology.

Second part concerns it will the existence of non-cellular objects in the motivic stable homotopy category $\mathcal{SH}(k)$.

Categorification of the Leray Spectral Sequence

by

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Knowledge in every historical epoch belongs to the class whose labor sustains the entire society. The knowledge contained in this work, in some form, belongs to the global proletariat—you have nothing to lose and a world to win.

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Chapter 1: Introduction

This thesis has two parts. The major part concerns itself with providing a categorical description of certain differentials (d_2) of the Leray spectral sequence. The other part relates to the problem of the existence of non-cellular objects in the category of motivic spectra.

One classical motivation for studying the Leray spectral sequence arises from the description of the Brauer group of a smooth projective variety X over a field k. Recall that the Brauer group $\mathrm{Br}(X)$—the group of Morita equivalence classes of Azumaya algebras on X—is known to be isomorphic to the torsion subgroup of the étale cohomology group $H^2_{\text{ét}}(X, \mathbb{G}_m)$.

Now, consider the structure morphism:

$$f : X \to \mathrm{Spec}(k).$$

the Leray spectral sequence for this morphism with coefficients in $\mathbb{G}_m$ is

$$E_2^{p,q} = H^p(k, R^q f_* \mathbb{G}_m) \Rightarrow H^{p+q}(X, \mathbb{G}_m).$$

When X has no non-trivial global sections, i.e. $\mathcal{O}_X(X) = k^\times$, the spectral sequence induces a filtration on $H^2(X, G)$ with the following graded pieces:

$$\mathrm{Gr}^0 = H^2(k, f_*\mathbb{G}_m) = Br(k)$$

$$\mathrm{Gr}^1 = H^1(k, R^1 f_*\mathbb{G}_m) = H^1(k, Pic(\bar{X}))$$

$$\mathrm{Gr}^2 = H^0(k, R^2 f_*\mathbb{G}_m) = Br(\bar{X})^\Gamma$$

where $\Gamma := Gal(\bar{k}/k)$ is the absolute Galois group.

In particular, we obtain the following sequence of maps:

$$\mathrm{Br}(k) \to \mathrm{Br}(X) \to \mathrm{Br}(\bar{X})^\Gamma.$$

The first map is the map from the 0^{th} graded piece, corresponding to pulling back Azumaya algebras along the structure morphism $f : X \to \mathrm{Spec}(k)$. The second map is the projection to the top graded piece, given by base change along $\bar{X} \to X$. Since the pullback to $\bar{X}$ is Galois-equivariant, the image of $\mathrm{Br}(X)$ lies in $\mathrm{Br}(\bar{X})^\Gamma \subset H^2(\bar{X}, \mathbb{G}_m)^\Gamma$.

However, not every Galois-invariant class lifts to an actual Brauer class over X. The first obstruction to such a lift is measured by the differential.

$$d_2 : \mathrm{Br}(\bar{X})^\Gamma \to H^2(k, \mathrm{Pic}(\bar{X}))$$

in the Leray spectral sequence.

Our initial motivation for categorification arises from [AR21].

Let S be a smooth projective variety over a field, and let $\pi : X \to S$ be a smooth projective morphism of relative dimension 1. At the level of Chow groups, we have the following composition:

$$\mathrm{CH}^1(X) \times \mathrm{CH}^1(X) \xrightarrow{\cup} \mathrm{CH}^2(X) \xrightarrow{\pi_*} \mathrm{CH}^1(S) \tag{1.1}$$

where the first map is the intersection (cup) product and the second is the pushforward along π. Using the identification $\mathrm{CH}^1(X) = H^1(X, \mathbb{G}_m)$, we can think of elements in $\mathrm{CH}^1(X)$ as representing the equivalence classes of $\mathbb{G}_m$-torsors.

The first step towards categorification of 1.1 was taken by Deligne [Del87] who categorified the composition

$$\mathrm{CH}^1(X) \times \mathrm{CH}^1(X) \longrightarrow \mathrm{CH}^1(S)$$

by lifting it to a biadditive functor at the level of torsors:

$$\psi_{X/S} \colon \mathrm{Tors}_X(\mathbb{G}_m) \times \mathrm{Tors}_X(\mathbb{G}_m) \longrightarrow \mathrm{Tors}_S(\mathbb{G}_m)$$

where $\mathrm{Tors}_X(\mathbb{G}_m)$ denotes the category of $\mathbb{G}_m$-torsors on X.

Furthermore, let $\mathcal{K}_i$ denote the Zariski sheaf associated to the presheaf $U \mapsto K_i(U)$, where K_i is the i^{th} Quillen's algebraic K-theory group. By a theorem of Bloch and Quillen, we have $\mathrm{CH}^2(X) \cong H^2(X, \mathcal{K}_2)$, which allows us to interpret codimension-2 cycles as classes of $\mathcal{K}_2$-gerbes. This hints at the possibility of factoring the functor $\psi_{X/S}$ through the category of $\mathcal{K}_2$-gerbes.

Theorem 1.1. *[AR21] The functor $\psi_{X/S}$ factors as a bi-additive functor $\cup$ followed by an additive functor $\int_\pi$:*

$$\mathrm{Tors}_X(\mathbb{G}_m) \times \mathrm{Tors}_X(\mathbb{G}_m) \xrightarrow{\cup} \mathrm{Gerb}_X(\mathcal{K}_2) \xrightarrow{\int_\pi} \mathrm{Tors}_S(\mathbb{G}_m)$$

The functor $\int_\pi$ is the categorification of the pushforward map π_*.

The proof requires a description of the two functors involved. When it comes to the cup product, this was described in [AR16] and [AR21] explicitly in terms of Heisenberg groups. Abstractly, the map is the cup-product map on Eilenberg-Mac Lane spaces:

$$K(A,1) \otimes K(B,1) \to K(A \otimes B, 2)$$

and $K(A,1)$ and $K(A,2)$ are classifying spaces of A-torsors and A-gerbes. Also, see section 2.3.

In our case, this map is

$$\mathrm{Tors}_X(\mathbb{G}_m) \times \mathrm{Tors}_X(\mathbb{G}_m) \to \mathrm{Gerb}_X(\mathbb{G}_m \otimes \mathbb{G}_m) \to \mathrm{Gerb}_X(\mathcal{K}_2)$$

The last map is induced by $\mathbb{G}_m \times \mathbb{G}_m \to \mathcal{K}_2$, obtained using the identification $\mathbb{G}_m \simeq \mathcal{K}_1$ and the multiplication map $\mathcal{K}_1 \times \mathcal{K}_1 \to \mathcal{K}_2$.

This second functor is the one we are concerned with. This functor is the composition.

$$\mathrm{Gerb}_X(\mathcal{K}_2) \xrightarrow{\Theta} \mathrm{Tors}_S(R^1\pi_*\mathcal{K}_2) \xrightarrow{\mathrm{Norm}} \mathrm{Tors}_S(\mathcal{K}_1)$$

where the first map is the categorification of a certain differential appearing in the Leray spectral sequence.

Given a morphism of $\pi : X \to S$ of nice spaces (e.g, paracompact spaces, quasi-projective schemes, etc.), and A an abelian sheaf on X, then the Leray spectral sequence takes the form:

$$E_2^{p,q} = H^p(S, R^q\pi_*A) \Rightarrow H^{p+q}(X, A)$$

where $R^q\pi_*A$ denotes the q-th right derived functor of the pushforward π_*. In particular, filtration on the abutment $H^2(X, A)$ gives the following maps.

$$H^2(S, \pi_*A) \xrightarrow{\pi^*} H^2(X, A) \xrightarrow{\pi_*} Ker[H^0(S, R^2\pi_*A) \xrightarrow{d_2} H^2(S, R^1\pi_*A)]$$

The composition of these maps is zero, but the sequence is not exact. The obstruction to exactness lies in the next stage of the filtration. More precisely, there is a map

$$\ker(\pi_*) \longrightarrow H^1(S, R^1\pi_*A)$$

measuring the failure of exactness at the middle term.

The categorified version of this states the following([AR21, Lemma 3.1 and Definition 3.2]):

Let $\mathrm{Gerb}_X(A)'$ be the full sub(2-)category of $\mathrm{Gerb}_X(A)$ consisting of horizontal gerbes. The functor

$$\Theta_f : \mathrm{Gerb}_X(A)' \longrightarrow \mathrm{Tors}_S(R^1\pi_*A)$$

sends a gerbe $\mathcal{G}$ to $\pi_0(\pi_*(\mathcal{G}))$.

In particular, since $R^2\pi_*\mathcal{K}_2 = 0$ [AR21, Prop 4.4], every $\mathcal{K}_2$-gerbe is horizontal. This leads to our required functor

$$\mathrm{Gerb}_X(\mathcal{K}_2) \xrightarrow{\Theta} \mathrm{Tors}_S(R^1\pi_*\mathcal{K}_2)$$

It would be interesting to see how this could be generalized. In particular, if $f : Y \to S$ is a smooth proper morphism of relative dimension two, then the cup product map followed by a norm map is

$$CH^2(Y) \times CH^2(Y) \longrightarrow CH^4(Y) \xrightarrow{f_*} CH^2(S)$$

One would like to understand how to construct a natural bi-additive functor.

$$\Psi^2_{Y/S} : \mathrm{Gerb}_Y(\mathcal{K}_2) \times \mathrm{Gerb}_Y(\mathcal{K}_2) \longrightarrow \mathrm{Gerb}_S(\mathcal{K}_2)$$

which is a categorification of the pairing. This would require dealing with differentials and higher filtrations.

The central aim of this thesis is to categorify certain d_2-differentials arising in the Leray spectral sequence. Given a map of topological spaces or Grothendieck sites $f : X \to Y$ and a sheaf of abelian groups G, we focus on the differential

$$d_2 : H^0(Y, R^p f_* G) \longrightarrow H^2(Y, R^{p-1} f_* G)$$

For each class $\alpha \in H^0(Y, R^p f_* G)$, we construct a stack $\mathcal{H}_\alpha$ that satisfies the following:

Theorem 4.1: The stack $\mathcal{H}_\alpha$ is a $R^{p-1}f_*G$-banded gerbe whose class in $H^2(Y, R^{p-1}f_*G)$ is the image $d_2(\alpha)$ under the differential in the Leray spectral sequence.

Organization. In §2, we introduce the fundamental geometric objects that classify cohomology classes in various degrees: torsors in degree one, gerbes in degree two, and higher gerbes in higher degrees. §3 is devoted to recalling the construction of the Leray spectral sequence. We interpret the d_2-differential as a Yoneda product and develop a Čech cocycle formulation of the spectral sequence. In §4, we provide a concrete cocycle-level description of the image of the d_2 map and use this description to construct a gerbe representing the corresponding cohomology class. §5 is unrelated to the rest of the thesis. In this section, we demonstrate that there are motivic spectra that are non-cellular.

Chapter 2: Torsors and Gerbes

2.1 Torsors

This section develops the theory of torsors and gerbes, including their cocycle descriptions and classification via Čech cohomology. We discuss the representability of these objects by stacks and higher stacks, and show that the categories of torsors and gerbes, equipped with the symmetric monoidal structure given by the contracted product, form Picard and Picard 2-categories, respectively. Throughout, all sheaves of groups are assumed to be abelian, so that no distinction is made between left and right group actions.

Definition 2.1. *Let $\mathcal{C}$ be a Grothendieck site, and let G be a sheaf of abelian groups on $\mathcal{C}$. A* G-torsor *on $\mathcal{C}$ is a sheaf of sets $\mathcal{P}$ on $\mathcal{C}$ together with an action of G such that:*

1. ***Local Triviality:*** *There exists a covering family $\{U_i \to U\}$ in $\mathcal{C}$ such that for each i, the restriction $\mathcal{P}|_{U_i}$ is isomorphic to $G|_{U_i}$ as G-sheaves. That is, there exists an isomorphism of sheaves $\mathcal{P}|_{U_i} \cong G|_{U_i}$ compatible with the G-action.*

2. ***Principal action:*** *The action of G on $\mathcal{P}$ is free and transitive. i,e the map of sheaves*

$$\mathcal{P} \times G \longrightarrow \mathcal{P} \times \mathcal{P}, \quad (p, g) \mapsto (p, p \cdot g)$$

is an isomorphism.

Morphisms between G-torsors are G-equivariant maps of sheaves. Given such a morphism $\varphi : \mathcal{P} \to \mathcal{Q}$, we may choose a cover that trivializes both torsors. Locally, φ becomes a G-equivariant map between trivial torsors, which is an isomorphism. Since morphisms of sheaves are local, φ is an isomorphism globally. Thus, the category of G-torsors $\mathrm{Tors}_G(\mathcal{C})$ is a groupoid. In fact, we can construct a stack of torsors on $\mathcal{C}$.

Theorem 2.2. *Let $\mathcal{C}$ be a site and let G be a sheaf of groups on $\mathcal{C}$. Then the prestack*

$$U \mapsto \mathrm{Tors}_{\mathcal{C}/U}(G)$$

which assigns to each object $U \in \mathcal{C}$ the groupoid of $G|_U$-torsors on the slice category $\mathcal{C}/U$ above U, viewed as a site, is a stack *in groupoids over $\mathcal{C}$.*

This is called the *classifying stack* of G-torsors and is usually denoted by the stack quotient $BG := [*/G]$.

Remark. *The classifying stack BG is conceptually easiest to understand in the setting of an ∞-topos as the first Eilenberg–MacLane object $K(G, 1)$ internal to the topos. More precisely, for any object $X \in \mathcal{X}$, where $\mathcal{X}$ is an ∞-topos, the set of connected components $\pi_0(\mathrm{Map}(X, BG))$ classifies isomorphism classes of G-torsors on X.*

Example 2.2.1. *The sheaf of groups G is a torsor over itself. This is called the trivial torsor.*

Example 2.2.2. *A line bundle on a topological space X (or a ringed topos) is a torsor for the multiplicative group $\mathbb{G}_m$. In general, let $E \to X$ be a vector bundle of rank n over a site X. The associated frame bundle $\mathrm{Fr}(E)$ is a torsor over the general linear group $\mathrm{GL}(n)$.*

Given two G-torsors $\mathcal{P}$ and $\mathcal{Q}$ over a site $\mathcal{C}$, we can construct an internal Hom torsor $\underline{Hom}(\mathcal{P}, \mathcal{Q}) := \underline{\mathrm{Isom}}(\mathcal{P}, \mathcal{Q})$ defined by:

$$\underline{\mathrm{Isom}}(\mathcal{P}, \mathcal{Q})(V) := \mathrm{Isom}_{G|_V}(\mathcal{P}|_V, \mathcal{Q}|_V); ,$$

for each object V in $\mathcal{C}$, the G-action is the action of maps between G-sheaves. Local triviality follows because $\underline{Hom}(G, G) = G$ as G-sheaves, and we can choose a cover that trivializes both torsors. Freeness and transitivity of action can be deduced from the fact that the action on individual torsors is free and transitive.

The category of torsors is a symmetric monoidal category under the tensor product, with the trivial torsor serving as a tensor unit:

Definition 2.3. *Let G be a sheaf of abelian groups on a site $\mathcal{C}$. Given two G-torsors $\mathcal{P}$ and $\mathcal{Q}$, their tensor product $\mathcal{P} \otimes \mathcal{Q}$ is defined as the quotient*

$$\mathcal{P} \otimes \mathcal{Q} := (\mathcal{P} \times \mathcal{Q})/G,$$

where G acts diagonally by

$$(p, q) \cdot g := (p \cdot g, q \cdot g),$$

for $p \in \mathcal{P}(U)$, $q \in \mathcal{Q}(U)$, and $g \in G(U)$, for any $U \in \mathcal{C}$. This tensor product is again a G-torsor.

In fact, for a torsor $\mathcal{P}$ we have a tensor inverse $\mathcal{P}^{-1}$ which is the same sheaf $\mathcal{P}$ equipped with the G-action given by

$$p \cdot g := p \cdot g^{-1},$$

for $p \in \mathcal{P}(U)$, $g \in G(U)$, and $U \in \mathcal{C}$.

The tensor product satisfies the following properties:

$$\mathcal{P} \otimes \mathcal{P}^{-1} \cong G,$$

where G denotes the trivial G-torsor. This makes the groupoid of G-torsors a *Picard category* and BG a *Picard stack*.

The tensor product interacts with internal homs in the way we expect:

$$\mathcal{P} \otimes \underline{\mathrm{Hom}}(\mathcal{P}, \mathcal{Q}) \cong \mathcal{Q}.$$

Theorem 2.4. *Let G be a sheaf of groups on a site $\mathcal{C}$, and let $\mathcal{P}$ be a G-torsor on $\mathcal{C}$. If $\mathcal{P}$ admits a global section, i.e., a section over the final object of $\mathcal{C}$, then $\mathcal{P}$ is isomorphic to the trivial torsor.*

Proof. Let $s \in \mathcal{P}(X)$ be a global section. Define a morphism of sheaves:

$$G \to \mathcal{P}, \text{defined by } g \mapsto s \cdot g \text{ on sections.}$$

This map is an isomorphism of G-sheaves, since the action of G on $\mathcal{P}$ is free and transitive on sections. Hence, $\mathcal{P} \cong G$ is a G-torsor. □

2.1.1 Cocyclic description of torsors

The isomorphism classes of torsors are in bijection with the first Čech cohomology group. So, given a G-torsor, we can construct a 1-cocycle as follows:

Let $\mathcal{U} := \{U_i \to X\}$ be a covering of X such that the torsor $\mathcal{P}$ is trivial over each U_i, i.e., there exist local sections $s_i \in \mathcal{P}(U_i)$. On the overlaps $U_{ij} := U_i \times_X U_j$, the free and transitive

G-action gives us unique elements $g_{ij} \in G(U_{ij})$ such that

$$s_i|_{U_{ij}} = s_j|_{U_{ij}} \cdot g_{ij}.$$

The cocycle condition follows from the associativity of the group action: on triple overlaps U_{ijk}, $g_{ij}g_{jk} = g_{ik}$.

Thus, $\{g_{ij}\}$ defines a 1-cocycle in $\check{Z}^1(\mathcal{U}, G)$, and the torsor determines a cohomology class in $\mathrm{H}^1(\mathcal{U}, G)$. Now, consider a different choice of local trivializations $s_i' \in \mathcal{P}(U_i)$. Since $\mathcal{P}$ is a torsor, there exists a unique $h_i \in G(U_i)$ such that $s_i' = s_i \cdot h_i$. On the overlaps U_{ij}, we compute:

$$\begin{aligned} s_i'|_{U_{ij}} &= s_i|_{U_{ij}} \cdot h_i|_{U_{ij}} = \left(s_j|_{U_{ij}} \cdot g_{ij}\right) \cdot h_i|_{U_{ij}} \\ &= s_j|_{U_{ij}} \cdot \left(g_{ij} \cdot h_i|_{U_{ij}}\right), \\ s_i'|_{U_{ij}} &= s_j'|_{U_{ij}} \cdot g_{ij}' = \left(s_j|_{U_{ij}} \cdot h_j|_{U_{ij}}\right) \cdot g_{ij}'. \end{aligned}$$

Comparing both expressions and using the freeness of the group action, we deduce:

$$g_{ij}' = h_j^{-1}|_{U_{ij}} \cdot g_{ij} \cdot h_i|_{U_{ij}}.$$

Thus, the new cocycle $\{g_{ij}'\}$ differs from the old one $\{g_{ij}\}$ by the coboundary of $\{h_i\}$, and thus defines the same class in $\check{H}^1(\mathcal{U}, G)$.

Suppose $\mathcal{V} := \{V_a \to X\}$ is a refinement of $\mathcal{U}$, we can restrict the cocycle to its refinement and get $H^1(\mathcal{U}, G) \to H^1(\mathcal{V}, G)$. Taking the colimit over refinements provides us with a well-defined map:

$$\mathrm{Tors_X(G)} \to H^1(X, G).$$

This map, in fact, descends down to the isomorphism classes of torsors. Suppose $\mathcal{P} \cong \mathcal{P}'$ as G-torsors. Let $\mathcal{U} = \{U_i \to X\}$ be a common refinement of covers trivializing both. Then for each i, we can choose trivializations $s_i \in \mathcal{P}(U_i)$ and $s_i' \in \mathcal{P}'(U_i)$ such that the isomorphism $\varphi : \mathcal{P} \to \mathcal{P}'$ satisfies $\varphi(s_i) = s_i' = s_i \cdot h_i$ for some $h_i \in G(U_i)$. Then the cocycles g_{ij} and g_{ij}' associated to $\mathcal{P}$ and $\mathcal{P}'$ satisfy

$$g_{ij}' = h_j^{-1} g_{ij} h_i,$$

which means $\{g_{ij}'\}$ is cohomologous to $\{g_{ij}\}$, so they define the same cohomology class in $\check{H}^1(\mathcal{U}, G)$, and thus in $H^1(X, G)$ after taking the colimit over refinements. Thus, we get a map

$$\pi_0(\mathrm{Tors_X(G)}) \to H^1(X, G)$$

which is, in fact, a bijection.

Theorem 2.5. *Let G be a sheaf of groups on a site X. Then the set of isomorphism classes of G-torsors on X is in bijection with the first Čech cohomology group:*

$$\pi_0(\mathrm{Tors_X(G)}) \xrightarrow{\sim} H^1(X, G).$$

Proof. To show injectivity, suppose two G-torsors $\mathcal{P}, \mathcal{Q}$ determine the same cohomology class.

Then there exists a cover $\mathcal{U} = \{U_i\}$ trivializing both torsors and cocycles $\{g_{ij}\}$, $\{g'_{ij}\}$ such that

$$g'_{ij} = h_j^{-1} g_{ij} h_i$$

for some $h_i \in G(U_i)$. Using these h_i, we define a map $\varphi : \mathcal{P} \to \mathcal{Q}$ locally by sending $s_i \in \mathcal{P}(U_i) \cong G|_{U_i}$ to $s_i \cdot h_i \in \mathcal{Q}(U_i)$. One checks that this glues over overlaps using the cocycle condition, and defines a global G-equivariant isomorphism $\mathcal{P} \cong \mathcal{Q}$.

Proving surjectivity is straightforward. Given a Čech 1-cocycle $\{g_{ij}\} \in \check{Z}^1(\mathcal{U}, G)$, define a presheaf $\mathcal{P}$ by gluing trivial torsors $G|_{U_i}$ along overlaps U_{ij} with $\{g_{ij}\}$ serving as the transition functions. This gluing is compatible on triple intersections because $\{g_{ij}\}$ satisfies the cocycle condition $g_{ij} g_{jk} = g_{ik}$. It is clear that $\mathcal{P}$ inherits a transitive and free G-action. □

2.1.2 Functoriality of the category of torsors

Let $f : \mathcal{C} \to \mathcal{D}$ be a morphism of sites and G a sheaf of abelian groups on $\mathcal{C}$. It induces an adjunction on the topoi:

$$f^* : \mathrm{Shv}(\mathcal{D}) \rightleftarrows \mathrm{Shv}(\mathcal{C}) : f_*,$$

with $f^* \dashv f_*$. We call this a geometric morphism of topoi when f^* is left exact (preserves finite limits).

A geometric morphism of topoi induces a pullback functor.

$$f^* : \mathrm{Tors}_G(\mathcal{C}) \to \mathrm{Tors}_{f^*G}(\mathcal{D}),$$

which preserves the torsor structure. It is in fact a morphism of stacks $BG \to Bf^*G$

However, the pushforward functor f_* does not, in general, preserve torsors: the image of a G-torsor under f_* may fail to be a f_*G-torsor. This failure is measured by an obstruction class in the Leray spectral sequence.

We also have functoriality arising from morphisms of groups. Let G and H be sheaves of groups on a site $\mathcal{C}$, and let $\phi : G \to H$ be a morphism of sheaves of groups. Given a G-torsor $\mathcal{P}$, we define the *extension of structure group* to H by the contracted product, which is the sheaf associated to the quotient of $\mathcal{P} \times H$ by the equivalence relation.

$$(g.f, h) \sim (f, \phi(g)h) \,,$$

We denote this by $\mathcal{P} \times^G H$. The resulting sheaf carries a natural H-action $h.(f, h_1) = (f, hh_1)$. It is not difficult to check that this is a H-torsor, since $G \times^G H \cong H$. This results in a map $\mathrm{Tors}_G(\mathcal{C}) \to \mathrm{Tors}_H(\mathcal{C})$

Theorem 2.6. *Let*

$$0 \to G \xrightarrow{i} H \xrightarrow{p} K \to 0$$

be a short exact sequence of abelian sheaves on a site (or of abelian groups in the classical case).

Then the sequence of groupoid-valued functors:

$$\mathrm{Tors}_G(X) \longrightarrow \mathrm{Tors}_H(X) \longrightarrow \mathrm{Tors}_K(X)$$

forms a fiber sequence of groupoids: the fiber over any object $Q \in \mathrm{Tors}_K(X)$ (i.e., the groupoid of H-torsors lifting Q) is equivalent to $\mathrm{Tors}_G(X)$, provided such a lift exists.

Proof. It is enough to analyze the fiber over the trivial K-torsor. Let $P \in \mathrm{Tors}_H(X)$ be an H-

torsor such that the contracted product $P \times^H K \cong K$, i.e., P maps to the trivial K-torsor under the functor $\mathrm{Tors}_H(X) \to \mathrm{Tors}_K(X)$.

We want to classify such H-torsors P, up to isomorphism, and show they form a groupoid equivalent to $\mathrm{Tors}_G(X)$.

Choose an open cover $\{U_i\}$ of X that trivializes P. Then over $U_{ij} = U_i \cap U_j$, the torsor P is described by transition functions $h_{ij} \in H(U_{ij})$, satisfying the cocycle condition:

$$h_{ij}h_{jk} = h_{ik} \quad \text{in } H(U_{ijk}).$$

Since $P \times^H K \cong K$, the associated K-torsor is trivial. This implies that the image $p(h_{ij}) \in K(U_{ij})$ is the trivial cocycle; that is,

$$p(h_{ij}) = 1 \quad \text{for all } i, j.$$

By exactness of the sequence, this implies that each $h_{ij} \in H(U_{ij})$ lies in the image of $G(U_{ij}) \subset H(U_{ij})$, i.e., there exist $g_{ij} \in G(U_{ij})$ such that $h_{ij} = i(g_{ij})$.

Since i is injective, $\{g_{ij}\}$ satisfies the cocycle condition and defines a G-torsor T, such that $T \times^G H \cong P$.

□

Remark. $\mathrm{Tors}_H(X) \longrightarrow \mathrm{Tors}_K(X)$ *is not surjective on objects. This isn't surprising since we would expect a long exact sequence in cohomology. However, the geometric interpretation of this lack of surjectivity can be captured by a gerbe. See [AR16, section 2.2].*

2.2 Gerbes

The story of gerbes is that they are a 2-categorical generalization of torsors. Heuristically, a G-gerbe is a stack that 'looks like' the category of G-torsors locally. More precisely:

Definition 2.7. *Let G be a sheaf of abelian groups on a site X. A G-*gerbe *over X is a stack $\mathcal{G}$ of groupoids such that:*

- *$\mathcal{G}$ is locally non-empty: there exists a cover $\{U_i\}$ with $\mathcal{G}(U_i) \neq \emptyset$.*
- *$\mathcal{G}$ is locally connected: for any two objects $x, y \in \mathcal{G}(U)$, there exists a cover $\{V_j \to U\}$ such that $x|_{V_j} \cong y|_{V_j}$.*
- *(Band) For all $x \in \mathcal{G}(U)$, we have an isomorphism $\underline{\mathrm{Aut}}(x) \xrightarrow{\sim} G|_U$ between the sheaf of automorphisms and G on U in compatible way, i,e if $x, y \in ob(\mathcal{G}(U))$ and $x \to y$, then induced map $\underline{\mathrm{Aut}}(x) \to \underline{\mathrm{Aut}}(y)$ is G-equivariant map of sheaves on U.*

Maps between G-gerbes are maps of stacks that preserve the band. So, a map of gerbes

$$\varphi : \mathcal{G} \to \mathcal{H}$$

is a map of underlying stacks such that, for every object $x \in \mathcal{G}(U)$, the induced map on automorphism sheaves

$$\underline{\mathrm{Aut}}_{\mathcal{G}}(x) \xrightarrow{\varphi_*} \underline{\mathrm{Aut}}_{\mathcal{H}}(\varphi(x))$$

is such that the diagram

$$\begin{tikzcd}
\underline{\mathrm{Aut}}_{\mathcal{G}}(x) \arrow[rr, "\varphi_*"] \arrow[dr, "\sim"'] & & \underline{\mathrm{Aut}}_{\mathcal{H}}(\varphi(x)) \arrow[dl, "\sim"] \\
& G|_U &
\end{tikzcd}$$

commutes.

Example 2.7.1. *For G be a sheaf of abelian groups on a site X, the stack of G-torsors Tors_G is a G-gerbe. This is referred to as a trivial G-gerbe.*

Since BG is a Picard stack (i.e., a group object in the 2-category of stacks), G -gerbes can be interpreted as BG -torsors in a higher categorical sense.([NSS, Section 3])

We can explicitly define the twisting action of G-torsors on G-gerbes (see the discussion after 5.2.4 in [Bry]).

2.2.1 Twisting a gerbe by a torsor

Let $\mathcal{G}$ be a G-gerbe over X. Let U be an open set of X. Suppose $P_1, P_2 \in \mathcal{G}(U)$ are two objects in the gerbe. Then the sheaf

$$\underline{\mathrm{Isom}}(P_1, P_2)$$

is an G-torsor on U, with the natural left action of $\underline{\mathrm{Aut}}(P_2) \cong G$.

Conversely, given an object $P \in \mathcal{G}(U)$, and a G-torsor $\mathcal{I}$ on U, we can construct a new object $Q \in \mathcal{C}(U)$, denoted $P \wedge^G \mathcal{I}$, together with an isomorphism of torsors:

$$\mathcal{I} \cong \underline{\mathrm{Isom}}(P, Q).$$

in the following way:

Choose a cover $\mathcal{U} \xrightarrow{g} U$ such that $g^*\mathcal{I}$ admits a section $s \in \mathcal{I}(\mathcal{U})$. Over $\mathcal{U} \times_U \mathcal{U}$, we have two pullbacks $p_1^* s$ and $p_2^* s$. Their difference defines a section.

$$u := p_1^* s - p_2^* s \in G(\mathcal{U} \times_U \mathcal{U}),$$

which satisfies the cocycle condition $p_{12}^* u + p_{23}^* u = p_{13}^* u$ on the triple intersection.

This cocycle u defines an isomorphism between the pullbacks of $g^* P$ to $\mathcal{U} \times_U \mathcal{U}$, which descends down to U. Hence, there exists an object $Q \in \mathcal{G}(U)$ whose pullback to $\mathcal{U}$ is identified with g^*P, and the descent data glues via u. This object is denoted $Q := P \wedge^G \mathcal{I}$ and satisfies

$$\mathcal{I} \xrightarrow{\sim} \underline{\mathrm{Isom}}(P, Q).$$

where the section s goes to the identity map, which makes sense since on the cover $\mathcal{U}$, the pullback of Q is identical to the pullback of P.

Different choices of trivialization $s' \in \mathcal{I}(\mathcal{U})$ yield isomorphic objects Q'. The isomorphism $\beta : Q \to Q'$ is uniquely determined by the requirement that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{I} \arrow[rr, "\sim"] \arrow[dr, "\sim"'] & & \underline{\mathrm{Isom}}(P, Q') \\
& \underline{\mathrm{Isom}}(P, Q) \arrow[ur, "\beta\circ(-)"'] &
\end{tikzcd}$$

Thus, the twisted object $P \wedge^G \mathcal{I}$ is well-defined up to canonical isomorphism.

We can formalize the above discussion as follows.

Proposition 2.7.1. *(cf. [Bry, Prop 5.2.5]) Let G be a sheaf of abelian groups and $\mathcal{G}$ a G-gerbe on X.*

1. *Let P_1 and P_2 be two objects of $\mathcal{G}(U)$ on open $U \subseteq X$, then sheaf $\underline{\mathrm{Isom}}(P_1, P_2)$ is a G-torsor over U.*

2. *Given an object $P \in \mathcal{G}(U)$ and a G-torsor $\mathcal{I}$ over U, there exists an object $Q \in \mathcal{G}(U)$ such that*

$$\underline{\mathrm{Isom}}(P, Q) \cong \mathcal{I}$$

as G-torsors. The object Q is unique up to a unique isomorphism. It is denoted $P \wedge^G \mathcal{I}$ and is called the object obtained by twisting P by the G-torsor $\mathcal{I}$.

We can now show that a G-gerbe is locally equivalent to the category of G-torsors.

Theorem 2.8. *Let $\mathcal{G}$ be a G-gerbe that admits a global section on a site X. Then $\mathcal{G} \xrightarrow{\sim} \mathrm{Tors_G}$.*

Proof. Suppose $\mathcal{G} \cong \mathrm{Tors}_G$, the stack of G-torsors. Then the trivial torsor $G \in \mathrm{Tors}_G(X)$ corresponds, under the equivalence, to an object $\xi \in \mathcal{G}(X)$. Hence $\mathcal{G}$ admits a global section.

Suppose $K \in \mathcal{G}(X)$ is a global object. We construct a functor $F : \mathcal{G} \to \mathrm{Tors}_G$ as follows. For each object $P \in \mathcal{G}(U)$, define

$$F(P) := \underline{\mathrm{Isom}}(K|_U, P),$$

by (1) in 2.7.1, this is a G-torsor.

To show that it is fully faithful, let $P_1, P_2 \in \mathcal{G}(U)$. A morphism $\phi : P_1 \to P_2$ induces a map

$$\underline{\mathrm{Isom}}(K|_U, P_1) \to \underline{\mathrm{Isom}}(K|_U, P_2), \quad \psi \mapsto \phi \circ \psi$$

which is G-equivariant. This defines a morphism of G-torsors. So, we get $\mathrm{Hom}_{\mathcal{G}(U)}(P_1, P_2) \to$

$\mathrm{Hom}_{\mathrm{Tors}_G}(F(P_1), F(P_2))$. It is easy to check that this functor is fully faithful.

Essential surjectivity follows from (2) of Proposition 2.7.1. Given $\mathcal{I}$, we can construct $Q := K \wedge^G \mathcal{I}$ such that $\underline{Isom}(K, Q) \cong \mathcal{I}$.

$\square$

In particular, the theorem states that if $\mathcal{G}$ has a section over U, then it is equivalent to the category of G-torsors on U. Since $\mathcal{G}$ is locally non-empty, there exists a cover of $\mathcal{U} \xrightarrow{p} X$ such that $p^*\mathcal{G}$ is equivalent to the trivial G-gerbe.

We observed that the category $\mathrm{Tors}_X(G)$ of G-torsors on X carries a natural symmetric monoidal structure via the contracted product, with the trivial torsor serving as the unit. This monoidal structure extends to the classifying stack BG, viewed as a stack of G-torsors.

Since gerbes are obtained by gluing local data modeled on BG, and the contracted product preserves colimits, this symmetric monoidal structure on BG descends to a symmetric monoidal structure on the 2-category $\mathrm{Gerb}_X(G)$ of G-gerbes on X. The tensor product of gerbes is defined by descent from the local tensor product on BG, and the trivial gerbe BG serves as the unit object.

Furthermore, given a G-gerbe $\mathcal{G}$, one defines its inverse $\mathcal{G}^{-1}$ as the internal Hom stack:

$$\mathcal{G}^{-1} := \underline{\mathrm{Hom}}(\mathcal{G}, BG),$$

so that there exists a canonical equivalence

$$\mathcal{G} \otimes \mathcal{G}^{-1} \xrightarrow{\sim} BG,$$

given by the evaluation map. This construction endows $\mathrm{Gerb}_X(G)$ with the structure of a *Picard 2-category*, that is, a symmetric monoidal 2-category in which every object has a (weak) inverse up to equivalence.

Gerbes banded by an abelian sheaf G are classified by the 2-stack B^2G, which assigns to each open subset $U \subseteq X$ the 2-groupoid $B^2G(U)$ of G-gerbes over U. The 2-stack B^2G carries a natural structure of a Picard 2-stack, coming from the symmetric monoidal structure on the category of G-gerbes.

Remark. *While the classifying stack BG exists for any group G, including non-abelian ones, the higher stack B^2G only exists when G is abelian. This is because the homotopy sheaves of B^2G satisfy $\pi_0(B^2G) = \pi_1(B^2G) = *$ and $\pi_2(B^2G) = G$, which requires G to be a commutative group.*

2.2.2 From gerbes to 2-cocycles and back

Now consider a G-gerbe $\mathcal{G}$ over X. Choose an open cover $\{U_i\}$ such that each category $\mathcal{G}(U_i)$ is non-empty, and pick an object $x_i \in \mathcal{G}(U_i)$.

On overlaps U_{ij}, one might hope to find a morphism

$$\phi_{ij} : x_i|_{U_{ij}} \to x_j|_{U_{ij}},$$

but this is not guaranteed. However, the second axiom in the definition of a gerbe ensures local connectedness: there exists a refinement $\{U_{ij}^\alpha\}$ of U_{ij} such that on each $U_{ij}^\alpha \subseteq U_{ij}$, we have a morphism

$$\phi_{ij}^\alpha : x_i|_{U_{ij}^\alpha} \to x_j|_{U_{ij}^\alpha}.$$

On triple overlaps $U_{ijk} = U_i \cap U_j \cap U_k$, the morphisms $\phi_{ij}, \phi_{jk}, \phi_{ki}$ may not compose consistently. Instead, we get an obstruction defined by:

$$g_{ijk} := \phi_{ij}^{\alpha} \circ \phi_{jk}^{\beta} \circ (\phi_{ki}^{\gamma})^{-1} \in \mathrm{Aut}(x_i|_{U_{ijk}^{\alpha\beta\gamma}}) \cong G(U_{ijk}^{\alpha\beta\gamma}),$$

where $U_{ijk}^{\alpha\beta\gamma} = U_{ij}^{\alpha} \cap U_{jk}^{\beta} \cap U_{ki}^{\gamma}$.

These g_{ijk} define a Čech 2-cocycle with values in G, satisfying the relation:

$$g_{jkl} \cdot g_{ijl}^{-1} \cdot g_{ijk} = g_{ikl} \quad \text{on quadruple overlaps } U_{ijkl}.$$

Thus, isomorphism classes of G-gerbes give rise to elements in $\check{H}^2(X, G)$.

Let's see how we can construct a gerbe from a 2-cocycle. For a more detailed argument, see [Bre, 2.6]. Given an open cover $\mathcal{U} = \{U_i\}$ of a topological space X, and a Čech 2-cocycle $\{g_{ijk}\} \in \check{Z}^2(\mathcal{U}, G)$ with values in a sheaf of abelian groups G. That is, for every triple intersection $U_{ijk} := U_i \cap U_j \cap U_k$, we have sections $g_{ijk} \in G(U_{ijk})$ satisfying the cocycle condition on quadruple overlaps:

$$g_{jkl} - g_{ijl} + g_{ijk} = g_{ikl} \quad \text{on } U_{ijkl}.$$

We construct a stack $\mathcal{G}$ on X fibered in groupoids, defined by the following descent data:

For each open set U_i, let $\mathcal{G}|_{U_i} \simeq BG|_{U_i}$, the classifying stack of G-torsors on U_i. On double overlaps $U_{ij} := U_i \cap U_j$, choose equivalences of stacks.

$$\phi_{ij} : BG|_{U_{ij}} \xrightarrow{\sim} BG|_{U_{ji}},$$

On triple overlaps U_{ijk}, we impose the following coherence condition: the composition of equivalences

$$\phi_{ij} \circ \phi_{jk} \circ \phi_{ik}^{-1}$$

is required to be the natural isomorphism of the identity functor on $\mathbf{B}G|_{U_{ijk}}$ given by the 2-cocycle:

$$\phi_{ij} \circ \phi_{jk} = g_{ijk} \circ \phi_{ik}.$$

This defines a system of 1-morphisms between the local charts that fail to satisfy the cocycle condition up to a prescribed automorphism $g_{ijk} \in G(U_{ijk})$. The data $(\{BG|_{U_i}\}, \{\phi_{ij}\}, \{g_{ijk}\})$ satisfies the descent conditions for a stack and defines a G-gerbe on X.

2.3 Higher gerbes

Defining higher gerbes in a coherent and general way requires the language of ∞-categories(Refer to [HTT, section 7.2]). Just as 1-gerbes are classified by the second cohomology group $H^2(X, G)$ and correspond to maps into the classifying stack B^2G, higher gerbes correspond to higher cohomology classes. More precisely, for a sheaf of abelian groups G on a site X, a G-valued n-gerbe is classified (up to equivalence) by an element of $H^{n+1}(X, G)$, and is represented by a map from X to the classifying stack $B^{n+1}G$ in the ∞-topos of ∞-stacks over X. The stack $B^{n+1}G$ serves as the moduli ∞-stack of n-gerbes with a band G.

Definition 2.9. *[HTT, Definition 7.2.2.20] Let $\mathcal{X}$ be an ∞-topos. An n-gerbe on $\mathcal{X}$ is an object $X \in \mathcal{X}$ which is both n-connective and n-truncated.*

Remark. *The definition of n-gerbe used here is not the most general one. More commonly, an*

n-gerbe is defined as an object that is 1-connective and n-truncated. In this work, however, we consider a more restrictive class of n-gerbes: those that are both n-connective and n-truncated. These are sometimes referred to as EM n-gerbes *(see [nLab]).*

Theorem 2.10. *[HTT, Corollary 7.2.2.27] Let $\mathcal{X}$ be an ∞-topos, let $n \geq 2$, and let A be an abelian group object in the category of discrete objects $\mathrm{Disc}(\mathcal{X})$ of $\mathcal{X}$. There is a canonical bijection between $H^{n+1}(\mathcal{X}; A)$ and equivalence classes of n-gerbes on $\mathcal{X}$ banded by A.*

Remark. *Any n-gerbe that admits a global section—that is, a morphism from the terminal object of the ∞-topos—is equivalent to the classifying object B^nG, which is the n^{th} Eilenberg–MacLane object $K(G, n)$ internal to the ∞-topos. In this sense, an n-gerbe may be viewed as an object that is locally equivalent to the n-category of $(n-1)$-gerbes.*

Remark. *The terminology here is somewhat misleading, as n-gerbes correspond to cohomology classes in degree $n+1$. It would be more accurate to refer to them as $(n+1)$-torsors.*

For an n-truncated ∞-category $\mathcal{C}$, the mapping space $\underline{\mathrm{Map}}_{\mathcal{C}}(x, y)$ between objects $x, y \in \pi_0(\mathcal{C})$ is an $(n-1)$-groupoid. In particular, if $\mathcal{X}$ is an n-truncated ∞-topos, then all internal mapping objects in $\mathcal{X}$ are sheaves valued in $(n-1)$-groupoids.

Given an n -gerbe $\mathcal{P}$, and a section $x \in \mathcal{P}(U)$ on U trivializing the n -gerbe, we can look at the automorphism sheaf $\underline{Aut}(x)$, which is the loop space at x of B^nG, so $\underline{Aut}(x) \cong B^{n-1}G$.

Given an n-gerbe $\mathcal{P}$ banded by an abelian group object G, and a section $x \in \mathcal{P}(U)$ over an object U in the ∞-topos $\mathcal{X}$ that trivializes the gerbe on U, we can consider the automorphism sheaf $\underline{\mathrm{Aut}}(x)$, defined over the slice ∞-topos $\mathcal{X}_{/U}$. This is equivalent to the loop space of B^nG at the point x, and hence

$$\underline{\mathrm{Aut}}(x) \simeq \Omega B^n G \simeq B^{n-1} G$$

in the slice ∞-topos $\mathcal{X}_{/U}$.

2.3.1 Postnikov filtration

Let $f : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of sheaf ∞-topoi (See [HTT, 6.5.4]), and let G be a sheaf of abelian groups on $\mathcal{X}$. The pushforward $f_* B^n G$ is an object in $\mathcal{Y}$ with the property that $\underline{Map}(V, f_* B^n G) \simeq \underline{Map}(f^{-1}(V), B^n G) \simeq B^n G(f^{-1}(V))$ and we have that:

$$\pi_i^Y(f_* B^n G) \simeq R^{n-i} f_* G \quad \text{for } 0 \leq i \leq n,$$

and vanish for $i > n$:

$$\pi_i^Y(f_* B^n G) = 0 \quad \text{for } i > n.$$

This leads to the identification of the fiber of the Postnikov filtration internal to $\mathcal{Y}$.

The object $f_* B^n G$ in $\mathcal{Y}$ admits a Postnikov tower:

$$\cdots \to \tau_{\leq i}(f_* B^n G) \to \tau_{\leq i-1}(f_* B^n G) \to \cdots \to \tau_{\leq 0}(f_* B^n G),$$

where each $\tau_{\leq i}(f_* B^n G)$ is the i-truncation in the ∞-topos $\mathcal{Y}$.

Each layer of the tower fits into a fibration sequence:

$$K\big(R^{n-i} f_* G, i\big) \to \tau_{\leq i}(f_* B^n G) \to \tau_{\leq i-1}(f_* B^n G),$$

where $R^{n-i} f_* G = \pi_i(f_* B^n G)$ is the i-th homotopy sheaf of $f_* B^n G$.

In particular, the associated graded pieces of the Postnikov filtration are Eilenberg–MacLane objects:

$$\mathrm{gr}_i(f_*B^nG) \simeq K(R^{n-i}f_*G, i),$$

encoding the successive obstructions to lifting a section of $\tau_{\leq i-1}(f_*B^nG)$ to a section of $\tau_{\leq i}(f_*B^nG)$. The geometric meaning of the Postnikov tower can be made explicit in low degrees. For $n = 1$, we obtain a fibration sequence:

$$K(f_*G, 1) \longrightarrow f_*BG \longrightarrow R^1f_*G,$$

reflecting the fact that f_*BG is a 1-truncated object with homotopy sheaves

$$\pi_1(f_*BG) \simeq f_*G, \quad \pi_0(f_*BG) \simeq R^1f_*G.$$

We also observe that the zeroth homotopy sheaf of BG in $\mathcal{X}$ vanishes $\pi_0^{\mathcal{X}}(BG) = 0$. This is because a section $U \to BG$ corresponds to a G-torsor on U. Since G-torsors are locally trivial, we can cover U by objects over which the torsor becomes trivial. That is, each local section is locally homotopic to the basepoint (the trivial torsor), and because π_0 is a sheaf, these homotopies glue to a global null-homotopy. Hence, all sections lie in the connected component of the basepoint, and we conclude that $\pi_0^{\mathcal{X}}(BG) = 0$.

The above fibration sequence also gives a concrete geometric interpretation: it states that the obstruction for the pushforward $f_*\mathcal{P}$ of a G-torsor $\mathcal{P}$ on $\mathcal{X}$ to be a f_*G-torsor on $\mathcal{Y}$ lies in its class in R^1f_*G. In particular, $f_*\mathcal{P}$ admits local trivializations (as a f_*G-torsor) if and only if its image in R^1f_*G vanishes.

Chapter 3: Leray spectral sequence

In this chapter, we develop the Leray spectral sequence. In §3.1, we present its construction via the formalism of derived functors, interpreting the spectral sequence as arising from the Postnikov filtration induced by the standard t-structure on the derived category. This filtration yields the associated graded pieces of the spectral sequence. In §3.2, we identify the d_2 differential with the Yoneda product determined by certain 2-extensions of the filtration. Finally, in §3.3, we provide a Čech cohomological model for the spectral sequence. The d_2 maps in this setup will be described in §4.1

3.1 The derived functor setup

Let $f : X \to Y$ be a continuous map of topological spaces (or a map of sites). As noted before, we obtain the following adjunction: it induces an adjunction on the topoi:

$$f^* : \mathrm{Shv}(Y) \rightleftarrows \mathrm{Shv}(X) : f_*,$$

and an adjunction at the level of bounded above derived categories

$$\mathbb{L}f^* : D^+(Y) \rightleftarrows D^+(X) : \mathbb{R}f_*,$$

We also have the global sections functor:

$$\Gamma(Y,-) := \mathbb{R}\Gamma(Y,-) : D^+(Y) \to D^+(\mathbb{Z}),$$

and similarly for X:

$$\Gamma(X,-) := \mathbb{R}\Gamma(X,-) : D^+(X) \to D^+(\mathbb{Z}).$$

The following diagram commutes up to natural isomorphism:

$$\begin{tikzcd} D^+(X) \arrow[r, "\mathbb{R}f_*"] \arrow[dr, "\mathbb{R}\Gamma(X,-)"] & D^+(Y) \arrow[d, "\mathbb{R}\Gamma(Y,-)"] \\ & D^+(\mathbb{Z}) \end{tikzcd}$$

That is, for any object $\mathcal{F} \in D^+(X)$, we have a canonical equivalence:

$$\mathbb{R}\Gamma(Y, \mathbb{R}f_*\mathcal{F}) \simeq \mathbb{R}\Gamma(X, \mathcal{F}).$$

The derived category $D^+(Y)$ carries a natural t-structure, induced from the standard truncation on chain complexes of sheaves of abelian groups.

We define the nonpositive and nonnegative parts of the t -structure as follows:

$$D^+(Y)^{\leq 0} := \{\mathcal{F} \in D^+(Y) \mid \mathcal{H}^i(\mathcal{F}) = 0 \text{ for all } i > 0\},$$

$$D^+(Y)^{\geq 0} := \{\mathcal{F} \in D^+(Y) \mid \mathcal{H}^i(\mathcal{F}) = 0 \text{ for all } i < 0\}.$$

where $\mathcal{H}^i(\mathcal{F})$ is the i-th cohomology sheaf.

The truncation functors $\tau^{\leq n}$ and $\tau^{\geq n}$ define a t-structure in the sense that:

- For every $\mathcal{F} \in D^+(Y)$, there exists a distinguished triangle

$$\tau^{\leq 0}\mathcal{F} \to \mathcal{F} \to \tau^{\geq 1}\mathcal{F} \to (\tau^{\leq 0}\mathcal{F})[1],$$

functorial in $\mathcal{F}$.

- $\mathrm{Hom}_{D^+(Y)}(\mathcal{F}, \mathcal{G}) = 0$ for all $\mathcal{F} \in D^+(Y)^{\leq 0}$ and $\mathcal{G} \in D^+(Y)^{\geq 1}$.

The heart of the t-structure is the abelian category of sheaves of abelian groups:

$$D^+(Y)^{\heartsuit} = \mathrm{Shv}(Y; \mathrm{Ab}),$$

identified with the full subcategory of $D^+(Y)$, consisting of complexes concentrated in degree zero.

This t-structure allows us to speak of the cohomology sheaves $\mathcal{H}^i(\mathcal{F})$ and induces a Postnikov tower on any object $\mathcal{F} \in D^+(Y)$, whose heart is the category of sheaves of abelian groups on Y. Using this t structure, we apply the canonical Postnikov filtration to the object $\mathbb{R}f_*\mathcal{F} \in D^+(Y)$:

$$\cdots \to \tau_{\leq n}\mathbb{R}f_*\mathcal{F} \to \tau_{\leq n-1}\mathbb{R}f_*\mathcal{F} \to \cdots \to \tau_{\leq 0}\mathbb{R}f_*\mathcal{F}.$$

The cofibers of this tower are:

$$\mathrm{cofib}\,(\tau_{\leq n-1}f_*\mathcal{F} \to \tau_{\leq n}f_*\mathcal{F}) \simeq \pi_n(f_*\mathcal{F})[-n],$$

where $\pi_n(f_*\mathcal{F})$ denotes the n-th homotopy sheaf of $f_*\mathcal{F}$.

Applying the derived global sections functor to the Postnikov filtration and taking the cohomology, we get filtrations

$$F^p H^i(X, \mathcal{F}) := Image(H^i(Y, \tau_{\leq i-p}\mathbb{R}f_*\mathcal{F}) \to H^i(Y, \mathbb{R}f_*\mathcal{F}))$$

.

with the graded piece: $\mathrm{gr}^p \simeq H^i(Y, R^{i-p}f_*\mathcal{F})$. See [Ara]) and [Del, section 1] for details)

3.2 d_2 map and Yoneda product

3.2.1 The Yoneda product

Let $\mathcal{A}$ be an abelian category with enough injectives (e.g., the category of abelian sheaves on a topological space, or modules over a ring). For objects $A, B, C \in \mathcal{A}$, the *Yoneda product* is a bilinear pairing

$$\mathrm{Ext}^i(A, B) \times \mathrm{Ext}^j(B, C) \longrightarrow \mathrm{Ext}^{i+j}(A, C),$$

which is functorial in all variables and associative up to sign. Given extension classes

$$\alpha \in \mathrm{Ext}^i(A, B), \quad \beta \in \mathrm{Ext}^j(B, C),$$

their Yoneda product $\beta \circ \alpha$ (or sometimes denoted $\alpha \cup \beta$) is defined as the class obtained by *splicing* the corresponding extensions. Concretely, if α and β are represented by exact sequences

$$0 \to B \to E_1 \to \cdots \to E_{i-1} \to A \to 0, \quad 0 \to C \to F_1 \to \cdots \to F_{j-1} \to B \to 0,$$

then $\beta \circ \alpha$ is the class of the total extension

$$0 \to C \to F_1 \to \cdots \to F_{j-1} \to E_1 \to \cdots \to E_{i-1} \to A \to 0.$$

Alternatively, using derived categories, one can view the Yoneda product as the composition of morphisms:

$$\mathrm{Ext}^i(A, B) = \mathrm{Hom}_{D(\mathcal{A})}(A, B[i]), \quad \mathrm{Ext}^j(B, C) = \mathrm{Hom}_{D(\mathcal{A})}(B, C[j]),$$

so that their composition yields

$$\mathrm{Hom}_{D(\mathcal{A})}(A, C[i+j]) = \mathrm{Ext}^{i+j}(A, C).$$

3.2.2 The d_2 differential as a Yoneda product

We now describe the differential

$$d_2^{i,j} \colon H^i(Y, R^j f_* \mathcal{F}) \longrightarrow H^{i+2}(Y, R^{j-1} f_* \mathcal{F})$$

in terms of the Yoneda product, following the formalism of Ext groups.

Let $\mathcal{A}$ be an abelian category with enough injectives and $f^* : \mathcal{B} \leftrightarrows \mathcal{A} : f_*$ an adjunction, with f_* left exact. Let $\mathcal{G} \in \mathcal{A}$ and $\mathcal{H} \in \mathcal{B}$. Then the Grothendieck spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p_{\mathcal{B}}(\mathcal{H}, R^q f_* \mathcal{G}) \Rightarrow \mathrm{Ext}^{p+q}_{\mathcal{A}}(f^* \mathcal{H}, \mathcal{G})$$

has differentials $d_2^{p,q}$ expressible via a canonical element in $\mathrm{Ext}^2_{\mathcal{B}}(R^j f_*\mathcal{G}, R^{j-1} f_*\mathcal{G})$, obtained from the identity morphism of $R^j f_*\mathcal{G}$ under a secondary differential

$$\partial\colon \mathrm{Hom}(R^j f_*\mathcal{G}, R^j f_*\mathcal{G}) \longrightarrow \mathrm{Ext}^2(R^j f_*\mathcal{G}, R^{j-1} f_*\mathcal{G}).$$

Theorem 3.1 ([Sko], Prop 1.1)**.** *Let* $\alpha \in \mathrm{Ext}^i_{\mathcal{B}}(\mathcal{H}, R^j f_*\mathcal{G})$. *Then*

$$d_2^{i,j}(\alpha) = (-1)^i \alpha \cup \partial(\mathrm{id}_{R^j f_*\mathcal{G}}),$$

where the product is the Yoneda product.

3.3 Čech hypercohomology and the Leray spectral sequence

We would like to relate the abstract derived functor styled setup of the Leray spectral sequence to the more concrete Čech cocycle styled definition. The construction involves the use of a double complex built from a Čech resolution and an injective resolution of G, followed by passage to the total complex and the associated spectral sequence.

Begin with an injective resolution of G on X:

$$0 \to G \to I^0 \to I^1 \to I^2 \to \cdots ,$$

which resolves G into a complex of injective sheaves $I^\bullet$. Applying the pushforward functor f_*

termwise yields a complex of sheaves on Y:

$$f_* I^\bullet := \left\{ f_* I^0 \to f_* I^1 \to f_* I^2 \to \cdots \right\}.$$

The cohomology sheaves of this complex are the higher direct image sheaves $R^q f_* G$.

We will need to resolve this complex further and apply the global section functor. This is where we use Čech resolutions. Let $\mathcal{V} = \{V_i\}$ be an open cover of Y. For each sheaf $f_* I^q$, we take its Čech resolution with respect to the cover $\mathcal{V}$:

$$\check{C}^p(\mathcal{V}, f_* I^q) := \prod_{i_0 < \cdots < i_p} f_* I^q(V_{i_0 \cdots i_p}) = \prod_{i_0 < \cdots < i_p} \Gamma(f^{-1}(V_{i_0 \cdots i_p}), I^q).$$

This forms a double complex $C^{p,q} := \check{C}^p(f_* I^q)$,

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & C^{p,q-1} \arrow[r, "\delta"] \arrow[d, "d"] & C^{p+1,q-1} \arrow[r, "\delta"] \arrow[d, "d"] & C^{p+2,q-1} \arrow[r, "\delta"] \arrow[d, "d"] & \cdots \\
\cdots \arrow[r] & C^{p,q} \arrow[r, "\delta"] \arrow[d, "d"] & C^{p+1,q} \arrow[r, "\delta"] \arrow[d, "d"] & C^{p+2,q} \arrow[r, "\delta"] \arrow[d, "d"] & \cdots \\
\cdots \arrow[r] & C^{p,q+1} \arrow[r, "\delta"] \arrow[d, "d"] & C^{p+1,q+1} \arrow[r, "\delta"] \arrow[d, "d"] & C^{p+2,q+1} \arrow[r, "\delta"] \arrow[d, "d"] & \cdots \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

with:

- Horizontal differential $\delta : C^{p,q} \to C^{p+1,q}$, given by the Čech differential:

$$(\delta\alpha)_{i_0 \cdots i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \alpha_{i_0 \cdots \hat{i}_j \cdots i_{p+1}}.$$

- Vertical differential $d : C^{p,q} \to C^{p,q+1}$, given by the differential in the complex $I^\bullet$.

such that $\delta d + d\delta = 0$.

We define the total complex associated to the double complex $C^{p,q}$ as:

$$\mathrm{Tot}^n(C^{\bullet,\bullet}) := \bigoplus_{p+q=n} C^{p,q}.$$

The cohomology of this total complex computes the hypercohomology $\mathbb{H}^n(X, G)$.

We have two filtrations on the total complex $\mathrm{Tot}^\bullet(C^{\bullet,\bullet})$, which lead to two spectral sequences computing hypercohomology.

The row (horizontal) filtration is given by:

$$F^r_{\text{row}} \mathrm{Tot}^n := \bigoplus_{\substack{q \geq r \\ p+q=n}} C^{p,q}.$$

This filtration yields a spectral sequence whose E_1-page is the Čech cohomology group:

$$E_1^{p,q} = \check{H}^p(Y, f_* I^q),$$

This is zero for $p > 0$ since the pushforward of injective objects is injective. Hence, the spectral sequence degenerates at the E_2-page and $E_2^{p,0} = H^p(\mathcal{U}, I^\bullet)$. This E_2-term is the cohomology of the complex $\Gamma(X, I^\bullet)$ and hence computes $H^n(X, G)$.

We also have the column (vertical) filtration on the total complex as:

$$F^r_{\text{col}} \mathrm{Tot}^n := \bigoplus_{\substack{p \geq r \\ p+q=n}} C^{p,q}.$$

This filtration induces a spectral sequence whose E_1-page is $E_1^{p,q} = \check{C}^p(\mathcal{V}, R^q f_* G)$, and whose horizontal differentials d_1 come from the Čech differential δ. Taking cohomology in the vertical direction yields:

$$E_2^{p,q} = H^p(Y, R^q f_* G),$$

which is the second page of the Leray spectral sequence. This spectral sequence converges to the hypercohomology:

$$E_2^{p,q} \Rightarrow \mathbb{H}^{p+q}(X, G).$$

Chapter 4: Categorification of d_2

In this chapter, we provide an explicit cocycle-level description of the d_2 differential in the Leray spectral sequence and construct a categorification of certain cases of this map. We further demonstrate that the group structure on cohomology corresponds, under categorification, to a symmetric monoidal structure given by the Baer sum. This generalizes a theorem of Giraud when G is abelian [Gir, Chapter V Prop 3.1.6].

4.1 Cocycle level calculations

Our aim in this section is to explicitly describe the d_2 differential in terms of Čech cocycles. To do this, we will trace the structure of the double complex and the induced spectral sequence.

Given $\alpha \in H^p(Y, R^q f_* G)$, we can represent it by a cocycle $\tilde{\alpha} \in C^{p,q}$. Since α survives to the E_1-page, we have $d(\tilde{\alpha}) = 0$, where d is the *vertical* differential. Furthermore, since α survives to the E_2-page, it means that $\delta(\tilde{\alpha})$ is cohomologous to zero, i.e., $\delta(\tilde{\alpha}) = d(\beta)$ for some $\beta \in C^{p+1,q-1}$. Applying the horizontal differential again gives $\delta(\beta) \in C^{p+2,q-1}$, and this is a cocycle representing the class

$$d_2(\alpha) \in H^{p+2}(Y, R^{q-1} f_* G), \;\; d_2(\alpha) = [\delta(\beta)]$$

Thus, the d_2 differential is realized concretely as the map taking a class $\alpha \in H^p(Y, R^q f_* G)$, lifting it to a cocycle $\tilde{\alpha} \in C^{p,q}$, finding $\beta \in C^{p+1,q-1}$ such that $\delta(\tilde{\alpha}) = d(\beta)$, and then setting

$$d_2(\alpha) = [\delta(\beta)].$$

We can make this description more concrete as follows. To keep track of the differentials, we denote d_X as the differential on X and d_Y as the differential on Y. Consider first the case of the following d_2 differential.

$$d_2 : H^0\big(Y,\, R^2 f_* G\big) \;\longrightarrow\; H^2\big(Y,\, R^1 f_* G\big).$$

Let $\mathcal{V} := \{V_i\}$ be an open cover of Y, and let $\alpha \;\in\; H^0\big(\mathcal{V},\, R^2 f_* G\big)$ be a 0-cocycle with respect to this cover. That is, α is given by a collection of sections

$$\alpha_i \;\in\; H^2\big(f^{-1}(V_i),\, G\big),$$

possibly after refining the cover. In other words, we have sections of the presheaf $f_*(H^2\big(\_\,,\, G\big))$

Let $\mathcal{U} := \{\, U_i = f^{-1}(V_i)\,\}$ be the induced cover of X, and choose representatives $\tilde{\alpha}_i \;\in\; C^2\big(\mathcal{U},\, G\big)$ for each class α_i. The α_i agree on overlaps up to coboundaries. Thus, for each double overlap $U_{ij} := U_i \cap U_j$, the difference $\tilde{\alpha}_i|_{U_{ij}} \;-\; \tilde{\alpha}_j|_{U_{ij}}$ is a coboundary in $C^2(U_{ij}, G)$. That is, there exists $x_{ij} \in C^1(U_{ij}, G)$ such that

$$\tilde{\alpha}_i|_{U_{ij}} - \tilde{\alpha}_j|_{U_{ij}} = d_X(x_{ij}).$$

Now consider the triple overlaps $U_{ijk} = U_i \cap U_j \cap U_k$. Define the 2-cochain

$$\eta_{ijk} := x_{ij} + x_{jk} + x_{ki} \in C^1(U_{ijk}, G).$$

Applying the differential d_X, we find that

$$d_X(\eta_{ijk}) = d_X(x_{ij}) + d_X(x_{jk}) + d_X(x_{ki}) = 0,$$

because the coboundaries cancel due to the cocycle condition. Therefore,

$$\eta_{ijk} \in Z^1(U_{ijk}, G),$$

and so defines a class in $H^1(U_{ijk}, G)$.. If $\{y_{ij}\}$ is a different choice, we can show that $\gamma_{ijk} := y_{ij} + y_{jk} + y_{ki}$ differs from η_{ijk} by a coboundary on Y.

Using the natural map from the presheaf $V \mapsto H^1(f^{-1}(V), G)$ to its sheafification $R^1 f_* G$, we can view each class

$$\eta_{ijk} \in H^1(f^{-1}(V_{ijk}), G)$$

as defining a section of $R^1 f_* G$ over V_{ijk}, that is, $\eta_{ijk} \in R^1 f_* G(V_{ijk})$. Thus, the collection $\eta = (\eta_{ijk})$ defines a 2-cochain in the Čech complex $\check{C}^2(\mathcal{V}, R^1 f_* G)$.

It remains to verify that this cochain is a cocycle on Y. It is easy to check that on quadruple overlaps $V_{ijk\ell} := V_i \cap V_j \cap V_k \cap V_\ell$, the Čech differential

$$(d_Y \eta)_{ijk\ell} = \eta_{jkl} - \eta_{ikl} + \eta_{ijl} - \eta_{ijk}$$

vanishes. Hence, $\eta \in Z^2(\mathcal{V}, R^1 f_* G)$, and defines a class $[\eta] \in H^2(Y, R^1 f_* G)$, which is the image of α under the differential d_2.

This cocyclic description works for general d_2 maps $H^p(Y, R^q f_* G) \to H^{p+2}(Y, R^{q-1} f_* G)$, although it can be notationally cumbersome. But we will spell it out here just for completeness.

Given a class $\alpha \in H^p(Y, R^q f_* G)$, represent it by a Čech p-cocycle $\{\alpha_{i_0 \dots i_p}\}$ with respect to an open cover $\mathcal{V} = \{V_i\}$ of Y. That is, $\alpha_{i_0 \dots i_p} \in \Gamma(V_{i_0 \dots i_p}, R^q f_* G)$, where $V_{i_0 \dots i_p} = V_{i_0} \cap \dots \cap V_{i_p}$. We can refine the cover such that these are infact sections of $H^q(f^{-1}(V_{i_0 \dots i_p}), G)$.

The Čech differential (with respect to the resolution of the sheaf $R^q f_* G$ on Y) d_Y of α, denoted $(d_Y \alpha)_{i_0 \dots i_{p+1}}$, is given by the alternating sum:

$$(d_Y(\alpha))_{i_0 \dots i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0 \dots \hat{i}_k \dots i_{p+1}} \big|_{V_{i_0 \dots i_{p+1}}},$$

where $\hat{i}_k$ means that the index i_k is omitted.

Since α is a p-cocycle (its lift to $C^p(Y, R^q f_* G)$ to be more precise), we have $d_Y(\alpha) = 0$, i.e., this alternating sum vanishes on each $(p+1)$-fold intersection.

Since $\alpha_{i_0 \dots i_p}$ is a section of the sheaf $R^q f_* G$, we can lift it to a representative $\tilde{\alpha}_{i_0 \dots i_p} \in C^q(f^{-1}(V_{i_0 \dots i_p}), G)$, i.e., a q-cocycle on the preimage $f^{-1}(V_{i_0 \dots i_p})$. Since $d_Y(\alpha) = 0$, The collection $\tilde{\alpha} := \{\tilde{\alpha}_{i_0 \dots i_p}\}$ has the property that $d_Y(\tilde{\alpha})$ is a q-coboundary on X. i,e $d_Y(\tilde{\alpha}_{i_0 \dots i_p}) = d_X(\tilde{\beta}_{i_0 \dots i_{p+1}})$, where $\tilde{\beta}_{i_0 \dots i_{p+1}} \in C^{q-1}(f^{-1}(V_{i_0 \dots i_{p+1}}), G)$.

The coboundary $d_Y(\tilde{\beta})$ is then a cocycle in total degree $p+2$ and internal degree $q-1$, and represents the image of α under the differential

$$d_2 : H^p(Y, R^q f_* G) \to H^{p+2}(Y, R^{q-1} f_* G).$$

4.2 Categorification

Fix a sheaf of abelian groups G on a topological space (or site) X, and let $f : X \longrightarrow Y$ be a continuous map (or a geometric morphism of topoi). We want to start by categorifying the first few terms of the following low term exact sequence (See [Mil, Appendix II])

$$0 \to E_2^{1,0} \to H^1(X,G) \to E_2^{0,1} \xrightarrow{d_2} E_2^{2,0} \to Ker[H^2(X,G) \to E_2^{0,2}] \to E_2^{1,1}$$

The first map in this low term exact sequence is induced by pullback along f. More precisely, given a f_*G-torsor $\mathcal{P}$ on Y, we obtain a f^*f_*G-torsor $f^*\mathcal{P}$ on X. There is a natural morphism of sheaves of groups:

$$f^*f_*G \longrightarrow G$$

which is the counit of the adjunction $f^* \dashv f_*$. Using this morphism, we can push forward the structure of the torsor $f^*\mathcal{P}$ to obtain a G-torsor on X. This gives the following composition of functors:

$$Tors_Y(f_*G) \to Tors_X(f^*f_*G) \to Tors_X(G)$$

The next map $H^1(X,G) \to H^0(Y, R^1f_*G)$ is the pushforward map. Let $\alpha \in H^1(X,G)$ and P be a torsor on X so that $[P] = \alpha$. The pushforward sheaf $f_*(P)$ has an f_*G-action and the action is free and transitive but need not satisfy local triviality on Y. But this gives a global section $\alpha \in R^1f_*G(Y)$. It is clear that if this global section is zero, it means that the pushforward is locally trivial and hence a torsor. This proves exactness at $H^1(X,G)$

The next map is the first d_2-map. We start by giving a general description of certain d_2

maps, namely the ones of the form $E_2^{0,p} \to E_2^{2,p-1}$

The obstruction gerbes and the differential d_2

The portion of the Leray spectral sequence we wish to analyze is:

$$H^1(X, G) \longrightarrow H^0(Y, R^1 f_* G) \xrightarrow{d_2} H^2(Y, f_* G).$$

Given a class $\alpha \in H^0(Y, R^1 f_* G)$, we can choose an open cover $\{V_i\}$ of Y such that α is represented by local cohomology classes $\alpha_i \in H^1(f^{-1}(V_i), G)$ satisfying the compatibility condition $\alpha_i|_{V_i \cap V_j} = \alpha_j|_{V_i \cap V_j}$. Let $U_i := f^{-1}(V_i)$.

The question of exactness at $H^0(Y, R^1 f_* G)$ can then be phrased as follows: Do there exist G-torsors $\mathcal{P}_i$ over U_i, representing the classes α_i which are pairwise isomorphic on the overlaps $U_i \cap U_j$, and such that these local torsors glue to a global G-torsor $\mathcal{P}$ over X, with

$$\mathcal{P}|_{U_i} \cong \mathcal{P}_i?$$

In other words, does the collection $\{\alpha_i\}$ lift to a global class in $H^1(X, G)$?

This is a classical descent problem. If we can choose isomorphisms of G-torsors over the overlaps:

$$\varphi_{ij} : \mathcal{P}_i|_{U_{ij}} \xrightarrow{\sim} \mathcal{P}_j|_{U_{ij}},$$

such that on triple intersections $U_{ijk} := U_i \cap U_j \cap U_k$, the cocycle condition holds:

$$\varphi_{jk}|_{U_{ijk}} \circ \varphi_{ij}|_{U_{ijk}} = \varphi_{ik}|_{U_{ijk}},$$

then we can glue the $\mathcal{P}_i$ into a global G-torsor $\mathcal{P}$ on X.

Note that while the classes α_i are fixed, the choice of representatives $\mathcal{P}_i$ is not unique. Instead of fixing arbitrary representatives, we may package all such choices into a fibered category $\mathcal{G}$ over the site. We need a stack $\mathcal{G}$ such that given a cover $\{U_i\}$, $\mathcal{G}(U_i)$ is the groupoid of G-torsors over U_i with class $[\mathcal{P}_i] = \alpha_i$.

To obtain a global torsor whose restriction to each U_i is in the class α_i, we are asking whether the descent data for objects in $\mathcal{G}$ glue to a global object. In other words, we seek a global section of the stack $\mathcal{G}$—i.e., a global G-torsor $\mathcal{P}$ on X such that $[\mathcal{P}|_{U_i}] = \alpha_i$ and the local data are glued via isomorphisms satisfying descent. We will show that this is not merely a stack but a gerbe, which we shall refer to as the *obstruction gerbe*, since the descent problem is equivalent to the existence of a global section—or equivalently, a global trivialization—of this gerbe.

We now provide a geometric interpretation of the differential

$$d_2 : H^0(Y, R^p f_* G) \to H^2(Y, R^{p-1} f_* G)$$

in the Leray spectral sequence, via higher-order obstruction gerbes.

Let $\alpha \in H^0(Y, R^p f_* G)$ be a global section. This means we can choose a cover $\{V_i\}$ of Y such that the restriction $\alpha_i := \alpha|_{V_i} \in H^p(f^{-1}(V_i), G)$ gives cohomology classes $\{\alpha_i\}$ that are compatible under restriction. Our goal is to determine whether this local data arises from a global class in $H^p(X, G)$. The first obstruction to such a global lift is measured by the differential

$$d_2(\alpha) \in H^2(Y, R^{p-1} f_* G).$$

We can understand this obstruction as follows. Define a category $\mathbb{H}_\alpha$ fibered in $(p+1)$-groupoids, whose sections are specified by the following data: for every open set $V \subseteq Y$, consider the full subcategory of G-valued p-gerbes on $f^{-1}(V)$ consisting of those gerbes whose class restricts to $\alpha|_{f^{-1}(V)}$. That is,

$$\mathbb{H}_\alpha(V) := \left\{ \mathcal{E} \in \mathrm{Gerb}^p_G(f^{-1}(V)) \;\middle|\; [\mathcal{E}] = \alpha|_{f^{-1}(V)} \right\}.$$

One should think of this as encoding all possible ways to lift α to a G-valued p-gerbe on X. This data naturally takes values in $(p+1)$-groupoids. Let $\mathcal{H}_\alpha := \tau_{\leq 1}\mathbb{H}_\alpha$ denote the 1-truncation of $\mathbb{H}_\alpha$, taken internally in the ∞-category of sheaves. Then $\mathcal{H}_\alpha$ is a stack on Y.

Remark. *Since sheafification is a left exact localization, by [HTT, prop. 5.5.6.28], truncating in the sheaf category is truncation in the presheaf category followed by sheafification.*

Remark. *Caution is required in defining $\mathbb{H}_\alpha$, as α is not a global cohomology class but only a collection of local classes. Consequently, for some open subsets V, the fiber $\mathbb{H}_\alpha(V)$ may be empty if α does not restrict to a cohomology class over V.*

Theorem 4.1. *The stack $\mathcal{H}_\alpha$ is an $R^{p-1}f_*G$-gerbe whose class in $H^2(Y, R^{p-1}f_*G)$ is the image $d_2(\alpha)$ under the d_2 differential in the Leray spectral sequence.*

Proof. First, we want to show that $\mathcal{H}_\alpha$ is locally non-empty and locally connected.

Let $\alpha \in H^0(Y, R^p f_*G)$. By definition, there exists a cover $\{V_i\}$ for Y such that $\alpha|_{V_i} \in H^p(f^{-1}(V_i), G)$. So, we may choose $p-1$-gerbes $\mathcal{E}_i$ on $f^{-1}(V_i)$ whose class is $\alpha|_{V_i}$, such that there is a 1-morphism $\mathcal{E}_i|_j \xrightarrow{\sim} \mathcal{E}_j|_i$ on the intersection which is preserved by the truncation. Sheafification preserves local sections (after refining the cover, if needed). Hence, it is locally non-empty.

To show local connectedness, consider two objects $\mathcal{E}_1, \mathcal{E}_2 \in \mathcal{H}_\alpha(V)$. Now, these might not be connected on $V \subset Y$. But by the definition of $\mathcal{H}_\alpha$, we have $V' \subset V$ such that $[\mathcal{E}_1|_{V'}] = \alpha|_{V'}$ $= [\mathcal{E}_2|_{V'}]$, hence there is a 1-morphism connecting them, proving local connectivity.

Now, we check the band of the stack $\mathcal{H}_\alpha$. For a section $\mathcal{E} \in \mathrm{Gerb}^{p-1}(U)$ with $U \subset X$, we saw in §2.3 that the automorphism ∞-sheaf is given by $\underline{\mathrm{Aut}}_X(\mathcal{E}) \simeq B^{p-1}G$, and this identification is compatible with restriction. Viewing this as a sheaf on Y, i.e., by applying the pushforward, we obtain

$$\underline{\mathrm{Aut}}_Y(\mathcal{E}) \simeq f_* B^{p-1}G.$$

In particular, for any open set $V \subset Y$, we have

$$\underline{\mathrm{Aut}}_Y(\mathcal{E}(V)) = B^{p-1}G(f^{-1}(V)).$$

Taking connected components, the sheaf $\pi_0^Y(f_* B^{p-1}G)$ is isomorphic to $R^{p-1}f_*G$, and this identification is compatible with restriction, since both pushforward and taking connected components are functorial.

Now suppose we are given a general section $x \in \mathcal{H}_\alpha(V)$. Then, locally on a cover $\mathcal{V}_\bullet \to V$, we may assume that $x_i := x|_{V_i}$ comes from an actual $p-1$-gerbe $\mathcal{E}_i$ over $f^{-1}(V_i)$. We then have a diagram:

$$\begin{tikzcd}
 & R^{p-1}f_*G \arrow[dl, dotted] \arrow[d, "\sim"'] \arrow[dr, "\sim"] & \\
\underline{\mathrm{Aut}}_{\mathcal{H}_\alpha}(x) \arrow[r] & \prod_i \underline{\mathrm{Aut}}_{\mathcal{H}_\alpha}(x_i) \arrow[r, shift left] \arrow[r, shift right] & \prod_{i,j} \underline{\mathrm{Aut}}_{\mathcal{H}_\alpha}(x_{ij})
\end{tikzcd}$$

where the isomorphisms on the right are defined componentwise and are compatible with restriction maps. Since $\underline{\mathrm{Aut}}_{\mathcal{H}_\alpha}(x)$ is the limit of the descent diagram of the automorphism sheaves of the local sections x_i, and each local automorphism sheaf is identified with $R^{p-1}f_*G$, it follows that the induced dotted arrow is also an isomorphism. Therefore, the band of the stack $\mathcal{H}_\alpha$ is $R^{p-1}f_*G$.

Now, we need to show that the gerbe we have constructed is, in fact, the image $d_2(\alpha)$. We can do this by mimicking the cocycle-level construction of the d_2 map described previously. It is enough to do this at the level of cocycles on a cover. Let $\mathcal{V} := \{V_i\}$ be a cover such that we get sections $\mathcal{E}_i \in \mathcal{H}_\alpha(V_i)$. We can refine the cover such that these sections are, in fact, $p-1$-gerbes up to isomorphism on $f^{-1}(V_i)$. By definition, $[\mathcal{E}_i] = \alpha|_{f^{-1}(V_i)} =: \alpha_i$. Since $\alpha_i = \alpha_j$ on the intersections, we have a 1-isomorphism:

$$\phi_{ij} : \mathcal{E}_i|_{f^{-1}(V_{ij})} \xrightarrow{\sim} \mathcal{E}_j|_{f^{-1}(V_{ij})}.$$

in the category of $p-1$-gerbes. They need not satisfy the cocycle condition since the choice of sections and morphisms was arbitrary.

This failure is measured by the coboundary $x_{ij} \in C^{p-1}(f^{-1}(V_{ij}), G)$ such that on the triple intersection, $d_X(\eta_{ijk}) = d_X(x_{ij} + x_{jk} + x_{ki}) = 0$. Since the construction doesn't depend on the choice of x_{ij}, $\{\eta_{ijk}\}$ is the required cocycle.

□

This construction preserves the group structure on $H^0(Y, R^q f_*G)$. This is because the d_2 map preserves the group structure, and the cocycle representing the Baer sum of two p-gerbs is the sum of their cocycles. So, if $[\mathcal{H}_\alpha]$ is represented by a Čech cocycle $\{\eta_{ijk}\}$ and $[\mathcal{H}_\beta]$ is represented by a Čech cocycle $\{\eta^{'}_{ijk}\}$, then the $\{\eta_{ijk} + \eta^{'}_{ijk}\}$ represents the cocycle of the Baer sum $\mathcal{H}_\alpha \oplus \mathcal{H}_\beta$. Since d_2 is a group homomorphism, we get that $d_2(\alpha+\beta) = \{\eta_{ijk} + \eta'_{ijk}\}$. So, in particular, $\mathcal{H}_\alpha \oplus \mathcal{H}_\beta$ and $\mathcal{H}_{\alpha+\beta}$ are described by the same 2-cocycle and hence are equivalent. In the case of $d_2 : H^0(Y, R^1 f_* G) \to H^2(Y, f_* G)$, this construction recovers the maximal subgerbe construction of Giraud [Gir] when the coefficient sheaf is abelian.

Chapter 5: Non-cellularity in motivic spectra

5.1 Introduction

In motivic homotopy theory, the subcategory of cellular motivic spectra plays a fundamental role in both the computational and structural aspects of the theory. It is the part of the theory that most closely parallels the classical stable homotopy category. The cellular motivic category can be understood as a hybrid of stable homotopy theory and algebraic data structures; see [GWX] and [BKWX]. Moreover, purely topological models for this category have been constructed, facilitating explicit computations; see [Pst] and [BHS].

An object in the motivic stable homotopy category $\mathcal{SH}(k)$ is said to be *cellular* if it lies in the smallest localizing subcategory generated by the motivic spheres $S^{p,q}$. The class of cellular spectra is closed under suspensions, homotopy colimits, and retracts. It includes many classical examples, such as projective spaces, Grassmannians, and various smooth affine schemes.

Although the existence of non-cellular objects in $\mathcal{SH}(k)$ has long been known to experts, explicit constructions and proofs have largely remained unwritten for some time. In this work, we provide a construction of such an example. In particular, we show the following:

Theorem 5.1. *Let k be a field admitting an embedding into $\mathbb{C}$. Then an elliptic curve over k is not stably cellular in $\mathcal{SH}(k)$.*

Proof of a similar result originally appeared in [Tot, Corollary 7.3]. We provide an alternate proof of it here.

We will closely follow the presentation of [DI], working in the setting of model categories rather than ∞-categories. All results stated in this framework are expected to admit reformulations in the language of ∞-categories, and such translations should pose no essential difficulties.

In §5.2, we recall the construction of motivic spaces and motivic spectra, and we review the flasque model structure on them. §5.3 is devoted to the study of compact objects in the stable motivic homotopy category, and proving that smooth schemes are compact in $\mathcal{SH}(k)$. In Theorem 5.14, we prove that every compact cellular object is a retract of a finite cellular motivic spectrum. §5.4 discusses the relationship between the stable motivic homotopy category $\mathcal{SH}(k)$ and Voevodsky's triangulated category of motives DM_k.

5.2 Construction of motivic spaces and spectra

We describe the flasque model structure from [I]. The idea is to describe the objectwise flasque model structure on the category $sPre(Sm_k)$ of simplicial presheaves over the category Sm_k of smooth schemes over k and localize by inverting Nisnevich squares and $* \to \mathbb{A}^1$. This model category is Quillen equivalent to Morel-Voevodsky's category $\mathcal{MV}$ [I].

For now, we will restrict our attention to the category of smooth schemes Sm_k, but, the ideas in this section apply to more general sites.

Given $X \in Sm_k$ and a collection of monomorphisms $U_i \to X$ in Sm_k, we define $\bigcup U_i$ as the coequalizer of

$$\coprod U_i \times_X U_j \rightrightarrows \coprod U_i$$

in $sPre(Sm_k)$. This is not the same as taking the coequalizers in Sm_k. We have a canonical map $\bigcup U_i \to X$.

Given a simplicial model category $\mathcal{C}$ with $X \xrightarrow{f} Y \in Mor(\mathcal{C})$ and $K \xrightarrow{g} L \in sSet$, we can define the pushout product as the map $f\Box g$, denoted as

$$X \otimes L \coprod_{X\otimes K} Y \otimes K \longrightarrow Y \otimes L$$

.

Definition 5.2. *(Objectwise flasque model structure) Let* $X, Y \in sPre(Sm_k)$, $n \in \mathbb{Z}_+ \cup \{0\}$, *and* $0 \leq k \leq n$

1. ***I*** *be the collection of maps of the form* $f\Box g$ *where* $\bigcup U_i \xrightarrow{f} X$ *and* $\partial\Delta^n \xrightarrow{g} \Delta^n$
2. ***J*** *be the collection of maps of the form* $f\Box h$ *where* $\bigcup U_i \xrightarrow{f} X$ *and* $\Lambda^n_k \xrightarrow{h} \Delta^n$

By Theorem 3.7 in [I], $sPre(Sm_k)$ has a flasque model structure that is cofibrantly generated with generating cofibrations I, generating trivial cofibrations J, and object-wise weak equivalences.

Recall that I-inj are the maps that have the right lifting property with respect to maps in I, and I-proj are the maps that have the left lifting property with respect to maps in I. The I-cofibrations are the maps in I-inj-proj (similarly for J-cofibrations). This gives us an explicit description of fibrations, which are maps in J-inj, and similarly, trivial fibrations are maps in I-inj. (Refer to [Hov, Chapter 2])

The flasque model structure falls between the objectwise projective and injective model structures in the following sense which is [I, Lemma 3.8]:

Theorem 5.3. *A projective cofibration is a flasque cofibration, and a flasque cofibration is an injective cofibration.*

With this, one can deduce that an injective fibration is a flasque fibration, and a flasque fibration is a projective fibration.

The objectwise model structure is refined to a local model structure with Nisnevich topology by inverting Nisnevich hypercovers (refer to Definition 4.1 of [I] and Section 6 of [DHI] for a general introduction to localization of hypercovers). Proposition 3.1.4 of [MV] shows that it is sufficient to invert certain kinds of covers arising from Nisnevich squares.

A Nisnevich square of X is a cartesian square in $(Sm_k)_{Nis}$ of the form:

$$\begin{tikzcd} U \times_X V \arrow[r] \arrow[d] & U \arrow[d, "i"] \\ V \arrow[r, "p"] & X \end{tikzcd}$$

where p is étale and i is an open immersion such that $p^{-1}(X-U) \to X-U$ is an isomorphism ($X-U$ has the reduced induced subscheme structure).

Using prop 3.1.4 in [MV], a presheaf $\mathcal{F}$ on Sm_k is a Nisnevich sheaf if and only if the diagram

$$\begin{tikzcd} \mathcal{F}(X) \arrow[r] \arrow[d] & \mathcal{F}(U) \arrow[d] \\ \mathcal{F}(V) \arrow[r] & \mathcal{F}(U \times_X V) \end{tikzcd}$$

is Cartesian for every Nisnevich square. (As we are working with presheaves of simplicial sets, we need homotopy Cartesian).

This leads to the definition of a local flasque model structure.

Definition 5.4. *The local flasque model structure is the left Bousfield localization of the objectwise flasque model structure with respect to $W \to X$, where W is the homotopy pushout (considered as constant simplicial sheaves) for all Nisnevich squares as described above.*

For a left Bousfield localization, the cofibrations remain the same as in the original category, but the number of fibrations decreases. The fibrant objects are those in the original category that are also local, i.e., A is fibrant if and only if for any $W \to X$ as above, **Map**$(X, A) \to$ **Map**(W, A) is a weak equivalence of simplicial sets. Refer to [Hir] for more details.

One of the reasons it is useful to work with the flasque model structures is that we can describe flasque fibrations concretely, as opposed to fibrations in the injective model structure. However, unlike in the projective model structure, $U \coprod_X V \to X$ is a cofibration, so the homotopy pushout coincides with the ordinary pushout. Therefore, the description of local flasque fibrant objects becomes simpler. Refer to [I] Theorem 4.9 and Corollary 4.10.

Theorem 5.5. *An object $\mathcal{F}$ in $sPre(SM_k)$ is locally flasque fibrant if it satisfies the following conditions:*

1. *$\mathcal{F}$ is objectwise flasque fibrant*
2. *The map $\mathcal{F}(X) \to \mathcal{F}(U) \times_{\mathcal{F}(X)} \mathcal{F}(V)$ is a trivial fibration for all elementary Nisnevich squares*

We further invert maps of the form $X \to X \times \mathbb{A}^1$ coming from $0 \to \mathbb{A}^1$ for every X in Sm_k to obtain the motivic flasque model structure.

Definition 5.6. *The motivic flasque model structure on $sPre(Sm_k)$ is the left Bousfield localization of the local flasque model structure on $sPre(Sm_k)$ with respect to maps $X \to X \times \mathbb{A}^1$ for every*

X in Sm_k.

Again, by the general theory of left Bousfield localization [Hir], the motivic flasque fibrant objects are precisely the local flasque fibrant objects $\mathcal{F}$ such that the map $\mathcal{F}(X \times \mathbb{A}^1) \to \mathcal{F}(X)$ is a trivial fibration in $\mathbf{sSets}$.

The important aspect of using the motivic flasque model structure is that filtered colimits of motivic flasque fibrant objects are motivic flasque fibrant. (Refer [I, Prop 5.3])

We can talk about the pointed analogue of these model structures in an obvious way. See, [Hir, Thm 7.6.5]. This will again be a simplicial model category in the following sense. Given a pointed presheaf $\mathcal{F}$ and a simplicial set $\mathbf{K}$, we define $(\mathcal{F} \otimes \mathbf{K})(U) := \mathcal{F}(U) \wedge \mathbf{K}_+$.

Notice that if $X \to Y$ is a flasque cofibration, then $* \to Y/X$ is a flasque cofibration, as cofibrations preserve pushouts. In particular, for a monomorphism $X \to Y$ in Sm_S, $* \to Y/X$ is a flasque cofibration in $sPre(Sm_k)_*$. Thus, $(\mathbb{P}^1, \infty)$, $(\mathbb{A}^1 - \{0\}, 1)$, and $\mathbb{A}^1/\mathbb{A}^1 - \{0\}$ are all cofibrant.

One can describe the flasque model structure in the case of motivic spectra. We will only consider a few facts related to the model structure, but a complete version can be found in [J, Hov2]. Here, we will consider naive spectra, but these also work in the case of symmetric spectra [J].

A spectrum is a sequence of spaces E_n with structure maps $E_n \to \Omega^{2,1}E_{n+1}$. A spectrum E is fibrant if and only if the E_n's are flasque fibrant and the structure maps $E_n \to \Omega^{2,1}E_{n+1}$ are motivic weak equivalences. $E \to F$ is a stable weak equivalence between fibrant objects if $E_n \to F_n$ is a motivic weak equivalence for each n.

5.3 Compact objects and cellularity

In this section, our aim is to set up the ingredients and prove the following result:

Theorem 5.7. *[DI, Theorem 9.1] Let X be a pointed smooth scheme. For $n \in \mathbb{Z}$, $\Sigma^{(2n,n)+\infty}X$ is a compact object in the motivic stable homotopy category $\mathcal{SH}(k)$.*

Before we do that, we need a few results that come as a consequence of using the flasque fibrant model structure.

Theorem 5.8. *Let $\{E_i\}$ be a directed system of flasque-fibrant motivic spectra. Then the canonical map colim$E_i \longrightarrow$ hocolimE_i is a weak equivalence in $Spectra(\textbf{\textit{Spc}}^{\mathbb{A}^1})$*

Proof. For a diagram $\{E_i\}$, the homotopy colimit in a model category is the colimit of the diagram after replacing $\{E_i\}$ with cofibrant objects. So it is enough to prove that, given a directed system of flasque fibrant objects $\{E_i\}$ and $\{F_i\}$ such that $\{E_i\} \to \{F_i\}$ is a stable weak equivalence, the canonical map colimit $E_i \to$ to colimit F_i is a stable weak equivalence. A stable weak equivalence $E_i \to F_i$ is precisely one that induces a motivic weak equivalence $[E_i]_n \to [F_i]_n$ of spaces for all n. Since $[E_i]_n \to [F_i]_n$ is a weak equivalence between fibrant objects, by the characterization of fibrant objects in the left Bousfield localization [Hir, Prop 3.3.16], $[E_i]_n \to [F_i]_n$ is in fact an objectwise weak equivalence. Therefore, colim $[E_i]_n \to$ colim $[F_i]_n$ is a weak equivalence, which implies that colimit $E_i \to$ to colimit F_i is a stable weak equivalence.

As both the motivic flasque category and motivic spectra are simplicial model categories, they have a mapping space functor that is related to the maps in the homotopy category, i.e., for a cofibrant A and a fibrant B, $Ho(A, B)$ is isomorphic to $\pi_0(A, B)$.

$\square$

Remark. *Let M_* be the category of pointed motivic spaces. Notice that for a directed system $\{F_i\}$ of pointed motivic spaces, $\boldsymbol{Map}_{M_*}(X, colim_i F_i)$ is isomorphic to $colim_i \boldsymbol{Map}_{M_*}(X, F_i)$. The unpointed version follows from the fact that $\boldsymbol{Map}_M(X, F) = F(X)$. For the pointed version, we have a pullback diagram,*

$$\begin{tikzcd}
\mathbf{Map}_{M_*}(X,F) \arrow[r] \arrow[d] & \mathbf{Map}_M(X,F) \arrow[d] \\
* \arrow[r] & \mathbf{Map}_M(*,F)
\end{tikzcd}$$

And finite products commute with directed colimits.

On the category of simplicial presheaves, we have the internal hom $\underline{\mathbf{Hom}}(X, Y)$, which is defined as $U \mapsto \mathbf{Map}_{sSet}(X(U), Y(U))$. This gives us the definition of the loop space, $\Omega^{2,1}F$, which is defined as $\underline{\mathbf{Hom}}(\mathbb{P}^1, F)$.

Definition 5.9. *Suppose $\mathcal{C}$ is a category with small colimits, and let λ be an ordinal which we will think of as a category with maps $\beta \to \alpha$ if $\beta < \alpha$. A λ-sequence in $\mathcal{C}$ is a colimit-preserving functor $X : \lambda \to \mathcal{C}$, commonly denoted as*

$$X_0 \to X_1 \to ... \to X_\beta \to ...$$

.

Theorem 5.10. *Let E_α be a λ-sequence in* Spt *for some ordinal λ. Then, for any pointed smooth* *scheme X, the natural map*

$$\textit{colim}\ Ho(\Sigma^{(2n,n)+\infty}X, E_\alpha) \to Ho(\Sigma^{(2n,n)+\infty}X, \textit{hocolim}\ E_\alpha)$$

is an isomorphism.

Proof. We take the flasque fibrant $E_\alpha^{'}$ replacement of E_α and as the fibrant replacement is functorial, $E_\alpha^{'}$ remains a λ-sequence (hence directed). So without loss of generality, we can assume that E_α's is a λ-sequence of motivic flasque fibrant. For a pointed scheme $\Sigma^{(2n,n)+\infty}X$, is flasque cofibrant [I], So we have $Ho(\Sigma^{(2n,n)+\infty}X, \text{colim } E_\alpha) \longrightarrow Ho(\Sigma^{(2n,n)+\infty}X, \text{hocolim } E_\alpha)$, which is an isomorphism by Theorem 2.2. Therefore, it is sufficient to prove that

$$\text{colim } \pi_0(\mathbf{Map}(\Sigma^{(2n,n)+\infty}X, E_\alpha)) \longrightarrow \pi_0(\mathbf{Map}(\Sigma^{(2n,n)+\infty}X, \text{colim } E_\alpha))$$

is an isomorphism.

Because $\Sigma^{(2n,n)+\infty}$ is a suspension spectrum of a scheme, giving a map at the level of spectra is the same as giving a map on spaces as follows. If $n < 0$, then the map at the level of spectra comes from *colimit* $\mathbf{Map}_{M_*}(X, [E_\alpha]_{-n}) \to \mathbf{Map}_{M_*}(X, (\textit{colimit}\ [E_\alpha])_{-n})$, which is an isomorphism by the above remark. $\square$

Similarly, if $n > 0$, by the suspension-loop space adjunction, the map on spectra comes from

$$\text{colim } \mathbf{Map}_{M_*}(X, \Omega^{(2n,n)}[E_\alpha]_0) \to \mathbf{Map}_{M_*}(X, \Omega^{(2n,n)}\ (\text{colim } [E_\alpha]_0))$$

which is an isomorphism by the remark above.

Now we can prove theorem 5.7:

Proof. Consider a collection $\{E_i\}$ of motivic spectra. We can assume that they are cofibrant. Notice that $E_i \vee E_j$ is cofibrant because $E_i \to E_i \vee E_j$ is a pushout of the cofibration $0 \to E_j$ and hence a cofibration. This gives us a unique map $0 \to E_i \to E_i \vee E_j$ which is a composition of cofibrations and hence a cofibration.

To prove that X is compact, we need to prove that

$$\bigoplus Ho(\Sigma^{(2n,n)+\infty} X, E_i) \to Ho(\Sigma^{(2n,n)+\infty} X, \bigvee E_i)$$

is an isomorphism. The map is clearly injective. To prove the surjection, notice that $E_0 \vee E_1 ... \vee E_{n-1} \to E_0 \vee E_1 ... \vee E_{n-1} \vee E_n$ is a cofibration between cofibrant obejcts. So we get a directed sequence $E_0 \to E_0 \vee E_1 \to E_0 \vee E_1 \vee E_2 ...$ of cofibrations between cofibrant objects. So, by definition, the homotopy colimit $\bigvee E_i$ is weakly equivalent to the colimit. So by the previous theorem, the map to $\bigvee E_i$ factors through some $\bigvee_{0 \geq j \geq n} E_j$. But as we are in the stable homotopy category which is a triangulated category, finite coproducts and products coincide. So this gives a map in $Ho(\Sigma^{(2n,n)+\infty} X, \prod_{i=0}^{j} E_i)$ which is isomophic to d $\bigoplus_{i=0}^{j} Ho(\Sigma^{(2n,n)+\infty} X, E_i)$. This proves the surjection.

$\square$

Definition 5.11. *The category of cellular spectra* ***Cell*** *is the smallest sub-category generated in* $Spectra(\textbf{\textit{Spc}}^{\mathbb{A}^1})$ *such that:*

- *Spheres* $S^{p,q} \in$ ***Cell***

- *It is closed under weak equivalence*

- *It is closed under taking homotopy colimits*

The category $Ho(\mathbf{Cell})$ is the smallest triangulated subcategory in $Ho(Spectra(\mathcal{MV}_*))$ which contains the spheres.

The good thing about working in the category **Cell** is that weak equivalences are detected at the level of homotopy groups.

Theorem 5.12. *([DI, Prop 7.1] Let $E \to F$ be a map between cellular spectra which induces isomorphism $\pi_{p,q}(E) \xrightarrow{\sim} \pi_{p,q}(F)$ between the homotopy groups for all p, q. Then $E \to F$ is a weak equivalence.*

Proof. In the stable setting, a map f is a weak equivalence iff its cofiber is contractible. So it is enough to prove that for a cellular spectrum E, $\pi_{p,q}(E) = 0$ implies that E is contractible. We can consider E to be a cofibrant-fibrant object. Consider the collection of objects X such that $\mathbf{Map}(X, E)$ is contractible. This collection is closed under weak equivalences, homotopy colimits and contains the spheres $S^{p,q}$. Since E is cellular, it belongs to this collection. So $\mathbf{Map}(E, E)$ is contractible which implies that the identity map is nullhomotopic which can only happen if E is contractible. □

Theorem 5.13. *Every object M in **Cell** is the homotopy colimit of a diagram of the form $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \ldots$ where X_j and cones of f_j are (infinite) coproducts of $S^{p,q}$*

Proof. Consider all maps $S^{p,q} \xrightarrow{g_i^0} M$ that are not nullhomotopic. Let $X_0 = \bigvee_{g_i^0} S^{p,q}$. Consider all $S^{p,q} \xrightarrow{g_1^i} X_0$ such that $S^{p,q} \xrightarrow{g_1^i} X_0 \to M$ it is nullhomotopic. Let X_1 be the cofiber of the canonical map $\bigvee_{g_i^1} S^{p,q} \to X_0$. By the property of cofiber, this gives us a canonical map $X_1 \to M$ such that the following diagram commutes.

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & M \\ X_1 \arrow[ur, dashed] & \end{tikzcd}$$

We continue by taking all maps $S^{p,q} \xrightarrow{g_2^i} X_1$, such that when composed with $X_1 \to M$, they are nullhomotopic. Let X_2 be the cofiber of the canonical map $\bigvee_{g_i^2} S^{p,q} \to X_1$. We continue this construction and get our X_ns. Let X be the *hocolim*(X_i). By definition of hocolim, we have a map $X \to M$. We prove that this induces an isomorphism on the homotopy groups $\pi_{p,q}$ and the result follows from Theorem 5.12.

For surjectivity, notice that any non-zero element in $\pi_{p,q}(M)$ factors through X_0 by the definition of X_0 and hence comes from an element $\pi_{p,q}(X)$. □

Theorem 5.14. *If M is a compact object in* ***Cell****, then M is a retract of a finite cellular spectrum in $Ho(Spectra(\mathcal{MV}))$.*

Proof. Let M be the sequential homotopy colimit of the diagram $\{X_n\}$ as in Theorem 5.13 and hence are isomorphic in the homotopy category to $hocolim\ \{X_n\}$. As M is compact, the identity map $M \to hocolim\ \{X_n\} \to M$ factors through some X_i, and we obtain a retract $M \to X_i \to M$. So we are reduced to proving that M is a retract of a finite subobject of X_i.

For $i = 0$, X_0 is an (infinite) coproduct of spheres $\bigvee_I S^{p,q}$ and as M is compact, the map $Hom(M, \bigvee_I S_i^{p,q}) \to \bigoplus_I Hom(M, S_i^{p,q})$ is bijective. Hence, the identity map is a retract of a coproduct of finitely many spheres.

For $i > 0$ consider the diagram,

$$\begin{tikzcd} M' \arrow[r] \arrow[d] & M \arrow[d, "i"] \\ 0 \arrow[r] & X_i \end{tikzcd}$$

where $M^{'}$ is obtained by lemma 5.15 below for the map $0 \to X_i$. Notice that the map $0 \to X_i$ satisfies the hypothesis of Lemma 5.15 as X_i's are extensions of (infinite) coproducts of spheres by construction. As i is split mono, $M^{'} \to M$ is a zero map and the cofiber of this map is M. By lemma 5.15 we conclude that M is built out of finitely many extensions of spheres and so is a finite cell complex.

□

Lemma 5.15. *Let τ be a triangulated category with coproducts with $\mathcal{K}$ being the set of compact generators. Let $X' \xrightarrow{a} X$ be a map with cone which is an extension of (infinite) coproducts of objects $\{\Sigma^n K\}$ with $K \in \mathcal{K}$. Given M, a compact object with a map to X, we can complete the diagram*

$$\begin{tikzcd} M' \arrow[r] \arrow[d] & M \arrow[d, "i"] \\ X' \arrow[r, "a"] & X \end{tikzcd}$$

such that the cone of $M^{'} \to M$ is a finite extension of $\{\Sigma^n K\}$

Proof. We do this by induction on the length of extension l of the cone $X^{''}$ of $X' \to X$. For $l = 0$, $X^{''}$ is the coproduct of spheres. Because M is compact, the map $M \to X \to X^{''}$ factors through some $M^{''}$ which is a finite coproduct.

Now we can complete the diagram below:

$$\begin{tikzcd}
M' \arrow[r] \arrow[d, dashed] & M \arrow[r] \arrow[d, "i"] & M'' \arrow[r] \arrow[d] & \Sigma M' \arrow[d] \\
X' \arrow[r, "a"] & X \arrow[r] & X'' \arrow[r] & \Sigma X'
\end{tikzcd}$$

with $M^{'}$ being the fiber of $M' \to M$.

For $l > 0$, let $X^{''}$ be generated by $X_0^{''} \to X^{''} \to X_1^{''}$ with a length of $X_0^{''}$ and $X_1^{''}$ being less than l. Considering the octahedron over $X \to X^{''} \to X_1^{''}$ gives a triangle

$$X^{'} \to X_1 \to X_0^{''} \to \Sigma X^{'}$$

such that we obtain a diagram.

$$\begin{tikzcd}
X' \arrow[r, "a_0"] \arrow[d, "id"] & X_1 \arrow[r] \arrow[d, dashed, "a_1"] & X_0'' \arrow[r] \arrow[d] & \Sigma X' \arrow[d] \\
X' \arrow[r, "a"] & X \arrow[r] & X'' \arrow[r] & \Sigma X'
\end{tikzcd}$$

We see that the map a is the composition of a_0 and a_1 with cones $X_0^{''}$ and $X_1^{''}$, respectively. Now, by induction, we obtain the diagram.

$\square$

$$\begin{tikzcd}
M' \arrow[r, "b_0"] \arrow[d] & M_1 \arrow[r, "b_1"] \arrow[d] & M \arrow[d] \\
X' \arrow[r, "a_0"] & X_1 \arrow[r, "a_1"] & X
\end{tikzcd}$$

and by taking the octahedron over $b_1 \circ b_0$, we see that the above diagram gives the required $M^{'}$

5.4 $H\mathbb{Z}$-spectra and mixed Tate objects

Throughout this section, we assume that the group field k is of characteristic zero. We use the idea that the category DM_k is the homotopy category $\mathrm{Ho}(M\mathbb{Z} - \mathrm{Mod})$ of modules over motivic Eilenberg-MacLane spectra $M\mathbb{Z}$([OR, Section 2.4])

Let **MSS** be the category of motivic symmetric spectra. This is a category of motivic spectra with a symmetric monoidal smash product at the level of spectra. We refer to [J] for the construction of the motivic version of symmetric spectra. This category has a model structure that is given as follows:

Definition 5.16. *A map between motivic spectra $E \to F$ is said to be a level-wise weak equivalence(resp. level-wise fibration) if it is a weak equivalence (resp. fibration) at every level $E_n \to F_n$.*

Let $(\_\_)^c$ be the cofibrant replacement in the level model category.

Definition 5.17. *A spectrum G is said to be stably fibrant if the map $G_n \to \Omega_{\mathbb{P}^1} G_{n+1}$ is a weak equivalence of spaces, and we define the stable weak equivalences as maps $E \to F$ that give*

$$\boldsymbol{Map}(F^c, G) \to \boldsymbol{Map}(E^c, G)$$

a weak equivalence for every stably fibrant G. A map is a stable fibration if it has a left lifting property with respect to stable weak equivalences and cofibrations.

By theorem 15 of [OR], the stable fibrations and stable weak equivalences make **MSS** into a symmetric monoidal model category, such that smashing with a cofibrant object preserves stable weak equivalences.

Now we describe the category of modules over motivic Eilenberg-MacLane spectra and look at the model category generated by the induced model structure coming from **MSS**.

The objects in the category M$\mathbb{Z}$-mod are spectra E such that there is an action of M$\mathbb{Z}$, i.e., M$\mathbb{Z} \wedge E \longrightarrow E$. The morphisms are morphisms of spectra that are compatible with the M$\mathbb{Z}$ action. By forgetting the module structure, we obtain a lax symmetric monoidal functor $\mathcal{U}$: M$\mathbb{Z}$-mod $\longrightarrow$ **MSS** with a strict symmetric monoidal left adjoint M$\mathbb{Z} \wedge$ _ : **MSS** $\longrightarrow$ M$\mathbb{Z}$-mod.

A map $E \to F$ in M$\mathbb{Z}$-mod is a weak equivalence(resp. fibration) if it is a weak equivalence(resp. fibration) on the underlying spectra. This gives a proper monoidal model structure on H$\mathbb{Z}$-mod [OR, Prop 2.36]

Notice that this makes $\mathcal{U}$ into a right Quillen functor, giving the Quillen pair

$$\mathrm{M}\mathbb{Z} \wedge \_ : \mathrm{M}\mathbb{Z}\text{-mod} \rightleftarrows \mathbf{MSS} : \mathcal{U}$$

.

So on the homotopy categories, we get the maps

$$\mathbf{L}\mathrm{M}\mathbb{Z} \wedge \_ : DM_k \rightleftarrows SH(k) : \mathbf{R}\mathcal{U}$$

The total left derived functor commutes with homotopy colimits as seen in Proposition 3.15 of [Har]. So in particular, as $S^{p,q}$ goes to $\mathbb{Z}(q)[p]$ in DM_k, we see that the finite cellular objects realize to finite extensions of objects of the form $\mathbb{Z}(q)[p]$ in DM_k; i,e., they are mixed Tate motives.

By theorem 5.14 we see that every compact object of the category of stably cellular objects is a retract of finite cellular objects. A stably cellular scheme is a compact object and hence, a retract of finite cellular objects. This means that it becomes a retract of a mixed Tate motive in DM_k, which again is mixed Tate. Therefore, any scheme that is not of mixed Tate type is not stably cellular. In particular, if the base field k admits an embedding into $\mathbb{C}$, then elliptic curves in $\mathcal{SH}(k)$ are not stably cellular and hence not cellular.

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