MARYLAND

TECHNICAL RESEARCH REPORT

SRC TR 87-141

Design of the IMC Filter by Using the Structured Singular Value Approach

by

E. Zafiriou and M. Morari

SYSTEMS RESEARCH CENTER

UNIVERSITY OF MARYLAND

COLLEGE PARK, MARYLAND 20742

DESIGN OF THE IMC FILTER BY USING THE STRUCTURED SINGULAR VALUE APPROACH

Evanghelos Zafiriou and Manfred Morari

Department of Chemical Engineering
California Institute of Technology
Pasadena, CA 91125

Abstract

The Internal Model Control (IMC) structure has been widely recognized as very useful in clarifying the issues related to the mismatch between the model used for controller design and the actual process. The structure also gives rise to a two step controller synthesis procedure, of which the second step deals with the design of a low pass filter such that robustness with respect to model-plant mismatch is guaranteed. The Structured Singular Value (SSV) was introduced recently and it allowed the non-conservative quantification of the concept of robust performance. This paper deals with the design of the IMC filter by using the SSV and it demonstrates how this approach can be used with either an H_2- or an $H_\infty-$ optimal controller.

1. Internal Model Control

The IMC structure (Fig.1a), introduced by Garcia and Morari (1982) is mathematically equivalent to the classical feedback structure (Fig.1b). The IMC controller Q and the feedback C are related through

$$Q = C(I + \tilde{P}C)^{-1} \tag{1.1}$$

$$C = Q(I - \tilde{P}Q)^{-1} \tag{1.2}$$

where $\tilde{P}$ is the process model.

The advantage of using IMC can be seen by examining the structure for $P = \tilde{P}$ and for $P \neq \tilde{P}$.

$P = \tilde{P}$.

In this case the overall transfer function connecting the set-points r and disturbances d to the errors $e = y - r$, where y are the process outputs, is

$$e = y - r = (I - PQ)(d - r) \stackrel{\text{def}}{=} \tilde{E}(d - r) \tag{1.3}$$

Hence the IMC stucture becomes effectively open-loop (Fig.2a) and the design of Q is simplified. Note that the IMC controller is identical to the parameter of the Q-parametrization (Zames, 1981). Also the addition of a diagonal filter F by writing

$$Q = \tilde{Q}F \tag{1.4}$$

introduces parameters (the filter time constants) which can be used for adjusting on-line the speed of response for each process output.

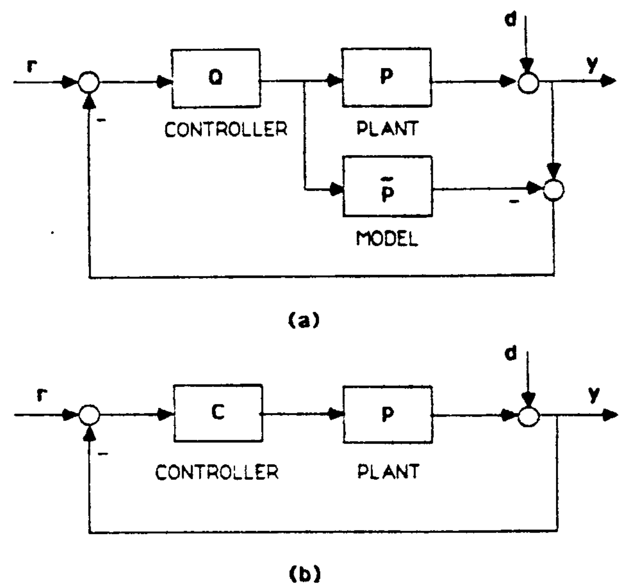


Figure 1.

$P \neq \tilde{P}$.

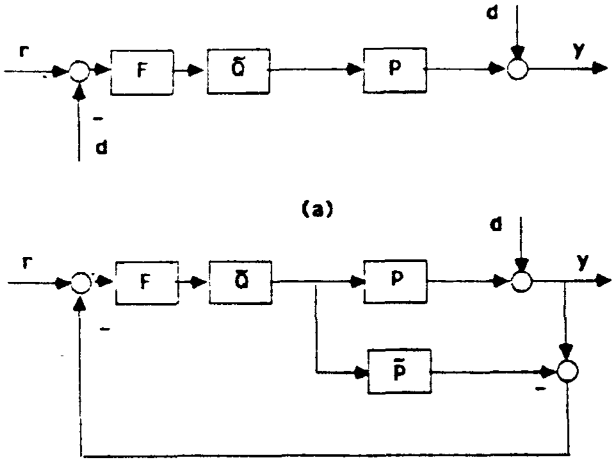
The model-plant mismatch generates a feedback signal in the IMC stucture which can cause performance deterioration or even instability. Since the relative modeling error is larger at higher frequencies, intuitively the addition of the low-pass filter F (Fig.2b) also adds robustness characteristics into the control system. In this case the closed-loop transfer function is

$$e = y - r = (I - P\tilde{Q}F)(I - (P - \tilde{P})\tilde{Q}F)^{-1}(d - r) \stackrel{\text{def}}{=} E(d - r) \tag{1.5}$$

Hence the IMC structure gives rise rather naturally to a two step design procedure:

Step 1: Design $\tilde{Q}$, assuming $P = \tilde{P}$.

Step 2: Design F so that the closed-loop characteris-



(b)

Figure 2.

tics that $\tilde{Q}$ produces in Step 1, are preserved in the presence of model-plant mismatch ($P \neq \tilde{P}$).

Finally note that the feedback controller C, given from (1.2), includes integral action if and only if Q inverts at steady-state the model $\tilde{P}$, i.e. *iff*

$$\tilde{Q}(\omega = 0) = \tilde{P}(\omega = 0)^{-1} \tag{1.6}$$

$$F(\omega = 0) = I \tag{1.7}$$

2. Structured Singular Value.

The SSV was introduced by Doyle (1982) and it allows the derivation of conditions for robust performance and stability for general structures of model uncertainty. For a constant complex matrix M the definition of the SSV $\mu_{\mathbf{\Delta}}(M)$ depends also on a certain set $\mathbf{\Delta}$. Each element Δ of $\mathbf{\Delta}$ is a block diagonal complex matrix with a specified dimension for each block, i.e.

$$\mathbf{\Delta} = \{diag(\Delta_1, \Delta_2, ..., \Delta_n) | \Delta_j \in \mathbf{C}^{m_j \times m_j}\} \tag{2.1}$$

Then

$$\frac{1}{\mu_{\mathbf{\Delta}}(M)} = \min_{\Delta \in \mathbf{\Delta}} \{\bar{\sigma}(\Delta) | det(I - M\Delta) = 0\} \tag{2.2}$$

and $\mu_{\mathbf{\Delta}}(M) = 0$ if $det(I - M\Delta) \neq 0 \quad \forall \Delta \in \mathbf{\Delta}$. Note that $\bar{\sigma}$ is the maximum singular value of the corresponding matrix.

Details on how the SSV can be used for studying the robustness of a control system can be found in Doyle (1985), where a discussion of the computational problems is also given. For three or fewer blocks in each element of $\mathbf{\Delta}$, the SSV can be computed from

$$\mu_{\mathbf{\Delta}}(M) = \inf_{D \in \mathbf{D}} \bar{\sigma}(DMD^{-1}) \tag{2.3}$$

where

$$\mathbf{D} = \{diag(d_1 I_{m_1}, d_2 I_{m_2}, ..., d_n I_{m_n}) | d_j \in \mathbf{R}_+\} \tag{2.4}$$

and I_{m_j} is the identity matrix of dimension $m_j \times m_j$. For more than three blocks, (2.3) still gives an upper bound for the SSV.

3. Filter Design

3.1. Block Structure

In order to effectively use the SSV for designing F, some rearrangement of the block structure is necessary. The IMC structure of Fig.1a can be written as that of Fig.3a, where $v = d - r$, $e = y - r$ and

$$G = \begin{pmatrix} 0 & 0 & \tilde{Q} \\ I & I & \tilde{P}\tilde{Q} \\ -I & -I & 0 \end{pmatrix} \tag{3.1.1}$$

where the blocks 0 and I have appropriate dimensions.

The structure in Fig.3a can always be transformed into that in Fig.3b, where Δ is a block diagonal matrix with the additional property that

$$\bar{\sigma}(\Delta) \leq 1 \qquad \forall \omega \tag{3.1.2}$$

The superscript u in G^u denotes the dependance of G^u not only on G but also on the specific uncertainty description available for the model $\tilde{P}$. Only some of the more common types will be covered here to demonstrate how this is done, but it is straightforward to apply the same concepts to other types of uncertainty descriptions, like parametric uncertainty.

i) Multivariable Additive Uncertainty.

The information on the model uncertainty is of the form

$$\bar{\sigma}(P - \tilde{P}) \leq l_a(\omega) \tag{3.1.3}$$

where l_a is a known function of frequency. In this case we can easily write $P - \tilde{P} = l_a \Delta$ where $\bar{\sigma}(\Delta) \leq 1$ and so obtain

$$G^u = G^a = \begin{pmatrix} l_a I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} G \tag{3.1.4}$$

ii) Multivariable Input Multiplicative Uncertainty.

$$\bar{\sigma}(\tilde{P}^{-1}(P - \tilde{P})) \leq l_i(\omega) \tag{3.1.5}$$

where l_i is known. Then

$$G^u = G^i = G \begin{pmatrix} l_i \tilde{P} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \tag{3.1.6}$$

iii) Multivariable Output Multiplicative Uncertainty.

$$\bar{\sigma}((P - \tilde{P})\tilde{P}^{-1}) \leq l_o(\omega) \tag{3.1.7}$$

$$G^u = G^o = \begin{pmatrix} l_o \tilde{P} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} G \tag{3.1.8}$$

iv) Element by Element Additive Uncertainty.

For each element p_{ij} of P we have

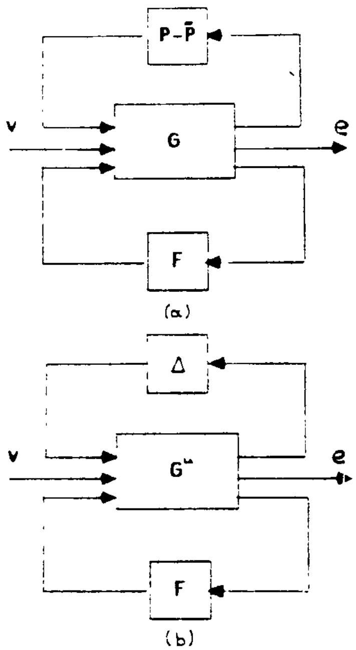


Figure 3.

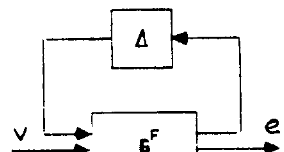


Figure 4.

$$|p_{ij} - \tilde{p}_{ij}| \le l_{ij}(\omega), \qquad i = 1,...,n; \quad j = 1,...,n \quad (3.1.9)$$

Then

$$P - \tilde{P} = J_1 \Delta L J_2 \tag{3.1.10}$$

where

$$L = diag(l_{11}, l_{12}, ..., l_{1n}, l_{21}, ..., l_{nn}) \tag{3.1.11}$$

$$J_1 = \begin{pmatrix} 1 & \dots & 1 & 0 & \dots & 0 & \dots & \dots & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & \dots & 1 & \dots & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 & \dots & \dots & 1 & \dots & 1 \end{pmatrix} \tag{3.1.12}$$

$$J_2 = \begin{pmatrix} I_n \\ I_n \\ \vdots \\ I_n \end{pmatrix} \tag{3.1.13}$$

From (3.1.10) it follows that

$$G^u = G^{ebe} = \begin{pmatrix} LJ_2 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} G \begin{pmatrix} J_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \tag{3.1.14}$$

Note that all the above relations yield a G^u already partitioned as

$$G^u = \begin{pmatrix} G^u_{11} & G^u_{12} & G^u_{13} \\ G^u_{21} & G^u_{22} & G^u_{23} \\ G^u_{31} & G^u_{32} & G^u_{33} \end{pmatrix} \tag{3.1.15}$$

Then Fig.3b can be written as Fig.4 with

$$G^F = \begin{pmatrix} G^u_{11} & G^u_{12} \\ G^u_{21} & G^u_{22} \end{pmatrix} + \begin{pmatrix} G^u_{13} \\ G^u_{23} \end{pmatrix} (I - FG^u_{33})^{-1} F \begin{pmatrix} G^u_{31} & G^u_{32} \end{pmatrix}$$
$$\stackrel{def}{=} \begin{pmatrix} G^F_{11} & G^F_{12} \\ G^F_{21} & G^F_{22} \end{pmatrix} \tag{3.1.16}$$

3.2. Robustness Conditions

3.2.1. Robust Stability

The system is stable for any of the plants in the set defined from the bounds on the model uncertainty, if and only if (Doyle,1985)

$$\mu_{\mathbf{\Delta}}(G^F_{11}) < 1 \qquad \forall \omega \tag{3.2.1}$$

3.2.2. Robust Performance

For performance, two cases will be examined; the $H_\infty-$ and the H_2-optimal. First the definitions of the L_2-norm of a vector and of the $L_\infty-$norm of a matrix will be given:

$$||v||_2 = \left(\int_{-\infty}^{+\infty} v^*(i\omega) v(i\omega) \quad d\omega \right)^{1/2} \tag{3.2.2}$$

$$||G||_\infty = \sup_{||v||_2 = 1} ||Gv||_2 = \sup_\omega \bar{\sigma}(G(i\omega)) \tag{3.2.3}$$

where the superscript $*$ indicates complex conjugate transpose.

i) $H_\infty-$optimal.

In this case, the IMC controller $\tilde{Q}$ designed in the first step, can be obtained by solving (Zames and Francis,1983,1984; Chang and Pearson, 1984; Doyle *et al*,1984):

$$\min_{\tilde{Q}} ||w\tilde{E}||_\infty \tag{3.2.4}$$

where $\tilde{E}$ was defined in (1.3) and w is a weight reflecting the frequency range of interest for the external system input v ($v = d$ for $r = 0$; $v = -r$ for $d = 0$).

In the second step of the IMC design we wish to keep $\bar{\sigma}(wE)$ bounded by a known boung $b(\omega)$ in spite of modeling error, i.e.

$$||b^{-1}wE||_\infty < 1 \qquad \forall \Delta \in \mathbf{\Delta} \tag{3.2.5}$$

Note that $E = \tilde{E}$ when $P = \tilde{P}$. The value of $\bar{\sigma}(w\tilde{E})$ for the optimal $\tilde{Q}$ obtained from (3.2.4) can serve as a guideline for the selection of the shape of $b(\omega)$. Then (Doyle,1985)

$$||b^{-1}wE||_\infty < 1 \qquad \forall \Delta \in \mathbf{\Delta} \iff \sup_\omega \mu_{\mathbf{\Delta}^o}(G^b) < 1 \tag{3.2.6}$$

where

$$G^b = \begin{pmatrix} I & 0 \\ 0 & b^{-1}wI \end{pmatrix} G^F \tag{3.2.7}$$

$$\mathbf{\Delta}^o = \{diag(\Delta, \Delta^o) | \Delta \in \mathbf{\Delta}, \Delta^o \in \mathbf{C}^{n \times n}\} \tag{3.2.8}$$

ii) H_2-optimal.

In the first step of the IMC design procedure, $\tilde{Q}$ is obtained by solving (Youla *et al*, 1976; Frank,1974; also see Morari *et al*,1986)

$$\min_{\tilde{Q}} ||W\tilde{E}v||_2 \qquad (3.2.9)$$

for a specified input v, which can be either a set-point r or a disturbance d. Note that $||W\tilde{E}v||_2$ is actually the Integral Squared Error (ISE), $||We||_2$ for this particular input v, where W is a diagonal matrix weighting each element of the error vector e differently. Also note that if one wishes the control system to behave well with both set-points and disturbances of different frequency content, then one has to implement a two-degree of freedom controller (see e.g. Morari *et al*,1986), each part of which is designed as presented here and in the corresponding references.

In the second step, the IMC filter F is designed so that the ISE ($||WEv||_2$) remains small even in the presence of model-plant mismatch. The following Theorem quantifies this objective.

Theorem 1:

For a specified v define

$$G^x \stackrel{\text{def}}{=} \begin{pmatrix} I & 0 \\ 0 & xW \end{pmatrix} G^F \begin{pmatrix} I & 0 \\ 0 & v \end{pmatrix} \qquad (3.2.10)$$

where x is a scalar function of ω and the blocks 0 have the appropriate dimensions (in general non-square). Augment G^x, which is in general a "tall" matrix, to obtain a square matrix:

$$G^x_{full} = (\,G^x \quad 0\,) \qquad (3.2.11)$$

Then

$$\mu_{\Delta^\circ}(G^x_{full}(i\omega)) = 1 \iff x(\omega) = x_0(\omega) \qquad \forall\omega \quad (3.2.12)$$

defines a function x_0 of frequency and

$$\sup_{\Delta\in\boldsymbol{\Delta}} ||WEv||_2 = ||x_0^{-1}||_2 \qquad (3.2.13)$$

Proof: For a matrix K partitioned as

$$K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \qquad (3.2.14)$$

define

$$R(K,\Delta) \stackrel{\text{def}}{=} K_{22} + K_{21}\Delta(I - K_{11}\Delta)^{-1}K_{12} \qquad (3.2.15)$$

Then the transfer function relating v to e in Fig.4 is $R(G^F,\Delta)$ and since Fig.1a and Fig.4 are equivalent, we get by using (1.5)

$$E = R(G^F,\Delta) \qquad (3.2.16)$$

The properties of the SSV and (3.2.12) imply (Doyle,1985) that

$$\sup_{\Delta\in\boldsymbol{\Delta}} \bar{\sigma}(R(G^{x_0}_{full},\Delta)) = 1 \qquad (3.2.17)$$

From (3.2.10), (3.2.11), (3.2.15), (3.2.16), it follows after some algebra that

$$R(G^{x_0}_{full},\Delta) = (\,x_0WEv \quad 0\,) \qquad (3.2.18)$$

Then from (3.2.17),(3.2.18) and the definition of the singular values, it follows, since x_0WEv is a vector:

$$\sup_{\Delta\in\boldsymbol{\Delta}} (x_0^2 v^* E^* W^* W E v) = 1 \qquad \forall\omega$$

$$\Longrightarrow \sup_{\Delta\in\boldsymbol{\Delta}} \int_{-\infty}^{+\infty} v^*E^*W^*WEv \quad d\omega \quad = \int_{-\infty}^{+\infty} x_0^{-2} \quad d\omega$$

$\iff$ (3.2.13) *QED*

Note that as it turned out $x_0^{-1} = \sup_{\Delta\in\boldsymbol{\Delta}} \bar{\sigma}(WEv)$, but the only way to compute it is through (3.2.12). Also without loss of generality x can be assumed to be positive since the value of $\mu_{\Delta^\circ}(G^x_{full})$ depends only on $|x|$.

An alternative to the above objective for designing F would be to design F with an H_∞ type of objective, even though $\tilde{Q}$ was obtained as an H_2−optimal controller in the first step of the IMC procedure. This is an interesting possibility that became available because of the two step IMC procedure and which experience showed to be of practical value. The idea behind it is that although one may expect inputs v of a particular type for which $\tilde{Q}$ is designed, one may still want to add some robustness characteristics not only with respect to model-plant mismatch but also with respect to different external inputs v entering the system. In this case one can select in (3.2.5) $w = 1$ and use as a guideline for selecting $b(\omega)$ the value of $\bar{\sigma}(\tilde{E})$ for the optimal $\tilde{Q}$ obtained from (3.2.9). From that point on, the procedure for designing F is the same as that described in the rest of this paper for the H_∞ type design.

3.3. Filter Parameter Optimization

The filter parameters can now be computed so that the robustness conditions that were discussed in §3.2 are satisfied. To do so, some structure will have to be assumed for F, which can be of any general type that the designer wishes. However in order to keep the number of variables in the optimization problem small, a rather simple structure like a diagonal F with first or second order terms would be recommended. In most cases this is not restrictive because the potentially higher orders of the model $\tilde{P}$ have been included in the controller $\tilde{Q}$ that was designed in the first step of the IMC procedure and which is in general a full matrix. The use of a full matrix F may be necessary in cases of extremely ill-conditioned systems ($\bar{\sigma}(\tilde{P})/\underline{\sigma}(\tilde{P})$ very large), but as mentioned the designer can specify such a structure for F if he so wishes. Note that F should still satisfy (1.7) for integral action. Also some additional restrictions on the filter exist in the case of an open-loop unstable plant (see Morari *et al*, 1986). Hence

$$F \stackrel{\text{def}}{=} F(s;\Lambda) \qquad (3.3.1)$$

where Λ is an array with the filter parameters. For example if an F of the form

$$F = \begin{pmatrix} 1/(\lambda_2 s^2 + \lambda_1 s + 1) & s/(\lambda_3 s + 1)^2 \\ s/(\lambda_4 s + 1)^2 & 1/(\lambda_5 s + 1) \end{pmatrix}$$

were selected, then $\Lambda = (\lambda_1 \quad \lambda_2 \quad \lambda_3 \quad \lambda_4 \quad \lambda_5)^*$

3.3.1. Problem Formulation

The problems can now be formulated as minimization problems over the elements of the array Λ. A constraint is that Λ should be such that F is a stable transfer function. However the problem can be turned into an unconstrained one by writing the denominator of each element of F as a product of polynomials of degree 2 and one of degree 1 if the order is odd, with the constant terms of the polynomials equal to 1. Then the stability requirement translates into the requirement that the coefficients (elements of Λ) are positive, which is a constraint that can be eliminated by writing λ_k^2 instead of λ_k for the corresponding filter parameters.

i) H_∞.

In this case the goal if to satisfy (3.2.6). The filter parameters can be obtained by solving

$$\min_{\Lambda} \sup_{\omega} \mu_{\mathbf{\Delta}^0}(G^b) \tag{3.3.2}$$

It may be however that the optimum values for (3.3.2), still do not manage to satisfy (3.2.6). The reason may be that an F with more parameters is required, but more often that the performance requirements set by the selection of $b(\omega)$ in (3.2.5) are too tight to satisfy in the presence of model-plant mismatch. In this case one should choose a less tight bound b and resolve (3.3.2). Note that satisfaction of the Robust Performance condition (3.2.6) implies satisfaction of the Robust Stability condition (3.2.1) as well.

ii) H_2.

The objective is to minimize (3.2.13) for a specified v (set-point or disturbance). Hence the filter parameters are obtained by solving

$$\min_{\Lambda} \|x_0^{-1}\|_2 \tag{3.3.3}$$

An additional problem here is the computation of $x_0(\omega)$ for a given Λ. This computation will be made through (3.2.12) and (2.3) will be used for computing μ. The following Theorem simplifies the problem.

Theorem 2:

Let

$$M^x = \begin{pmatrix} M_{11} & M_{12} \\ xM_{21} & xM_{22} \end{pmatrix} \tag{3.3.4}$$

where x a positive scalar.

Then $\inf_{D\in\mathbf{D}} \bar{\sigma}(DM^x D^{-1})$ is a non-decreasing function of x, where $\mathbf{D} = \{diag(D_1, D_2)\}$.

Proof: Let $0 < x_2 \le x_1$. Then we can write $x_2 = x_1\beta$, where $0 < \beta \le 1$. From (3.3.4) we have

$$DM^{x_2}D^{-1} = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \beta I \end{pmatrix} M^{x_1} D^{-1}$$

$$= \begin{pmatrix} I & 0 \\ 0 & \beta I \end{pmatrix} DM^{x_1}D^{-1} \tag{3.3.5}$$

Then the properties of the singular values yield

$$(3.3.5) \Longrightarrow \bar{\sigma}(DM^{x_2}D^{-1}) \le \bar{\sigma}\begin{pmatrix} I & 0 \\ 0 & \beta I \end{pmatrix} \quad \bar{\sigma}(DM^{x_1}D^{-1})$$

$$\Longrightarrow \bar{\sigma}(DM^{x_2}D^{-1}) \le \bar{\sigma}(DM^{x_1}D^{-1}) \qquad \forall D \in \mathbf{D}$$

$$\Longrightarrow \inf_{D\in\mathbf{D}} \bar{\sigma}(DM^{x_2}D^{-1}) \le \inf_{D\in\mathbf{D}} \bar{\sigma}(DM^{x_1}D^{-1}) \mathcal{QED}$$

Note that G^x_{full} is a special case of M in the Theorem and so Theorem 2 applies to (3.2.12).

3.3.2. Computational Issues

i) H_∞.

The computation of μ in (3.3.2) is made through (2.3); details can be found in Doyle (1982). As it was pointed out in Doyle (1985), the minimization of the Frobenious norm instead of the maximum singular value yields D's which are very close to the optimal ones for (2.3). Note that the minimization of the Frobenious norm is a very simple task. In the computation of the supremum in (3.3.2) only a finite number of frequencies is considered. Hence (3.3.2) is transformed into

$$\min_{\Lambda} \max_{\omega\in\Omega} \inf_{D\in\mathbf{D}^0} \bar{\sigma}(DG^bD^{-1}) \tag{3.3.6}$$

where Ω is a set containing a finite number of frequencies and $\mathbf{D}^0$ is the set corresponding to $\mathbf{\Delta}^0$ according to (2.1) and (2.4). Define

$$\Phi_\infty(\Lambda) \stackrel{\text{def}}{=} \max_{\omega\in\Omega} \inf_{D\in\mathbf{D}^0} \bar{\sigma}(DG^bD^{-1}) \tag{3.3.7}$$

The analytic computation of the gradient of Φ_∞ with respect to Λ is in general possible. This is not the case when the two or more largest singular values of DG^bD^{-1} are equal. However this is quite uncommon and although the computation of a generalized gradient is possible, experience has shown the use of a mean direction to be satisfactory. A similar problem appears when the $\max_{\omega\in\Omega}$ is attained at more than one frequencies, but again the use of a mean direction seems to be sufficient. We shall now proceed to obtain the expression for the gradient of $\Phi_\infty(\Lambda)$ in the general case.

Assume that for the value of Λ where the gradient of $\Phi_\infty(\Lambda)$ is computed, the $\max_{\omega\in\Omega}$ is attained at $\omega = \omega_0$ and that the $\inf_{D\in\mathbf{D}^0} \bar{\sigma}(DG^b(i\omega_0)D^{-1})$ is obtained at $D = D_0$, where only one singular value σ_1 is equal to $\bar{\sigma}$. Let the singular value decomposition (SVD) be

$$D_0G^b(i\omega_0)D_0^{-1} = (u_1 \quad U) \begin{pmatrix} \sigma_1 & 0 \\ 0 & \Sigma \end{pmatrix} \begin{pmatrix} v_1^* \\ V^* \end{pmatrix} \tag{3.3.8}$$

Then for the element of the gradient vector corresponding to the filter parameter λ_k we have under the above assumptions:

$$\frac{\partial}{\partial\lambda_k}\Phi_\infty = \frac{\partial}{\partial\lambda_k}\sigma_1(D_0G^b(i\omega_0)D_0^{-1}) \tag{3.3.9}$$

because $\nabla_{D_0}(\sigma_1) = 0$ since we are at an optimum with respect to the D's. To simplify the notation use

$$A = D_0 G^b(i\omega_0) D_0^{-1} = U_A \Sigma_A V_A^* \qquad (3.3.10)$$

By using the properties of the SVD we obtain from (3.3.8)

$$AA^* = U_A \Sigma_A^2 U_A^* \Rightarrow u_1^* \frac{\partial}{\partial \lambda_k}(AA^*) u_1 = u_1^* U_A \frac{\partial}{\partial \lambda_k}(\Sigma_A^2) U_A^* u_1$$

$$\Rightarrow u_1^* (\frac{\partial}{\partial \lambda_k}(A) A^* + A \frac{\partial}{\partial \lambda_k}(A^*)) u_1 = u_1^* U_A (2 \Sigma_A \frac{\partial}{\partial \lambda_k}(\Sigma_A)) U_A^* u_1$$

$$\Rightarrow u_1^* \frac{\partial}{\partial \lambda_k}(A) v_1 \sigma_1 + \sigma_1 v_1^* \frac{\partial}{\partial \lambda_k}(A^*) u_1 = 2\sigma_1 \frac{\partial}{\partial \lambda_k}(\sigma_1)$$

$$\Rightarrow \frac{\partial}{\partial \lambda_k}(\sigma_1) = \mathrm{Re}\left[u_1^* \frac{\partial}{\partial \lambda_k}(D_0 G^b(i\omega_0) D_0^{-1}) v_1 \right] \qquad (3.3.11)$$

Use of (3.3.9), (3.1.16), (3.2.7), (3.3.11), and of the property

$$\frac{d}{dz}(M(z)^{-1}) = -M(z)^{-1} \frac{d}{dz}(M(z)) M(z)^{-1} \qquad (3.3.12)$$

where M(z) is a matrix, yields after some algebra

$$\frac{\partial}{\partial \lambda_k} \Phi_\infty = \mathrm{Re}\left[u_1^* D_0 \begin{pmatrix} G_{13}^u \\ b^{-1} w G_{23}^u \end{pmatrix} (I - F G_{33}^u)^{-1} \frac{\partial}{\partial \lambda_k}(F(i\omega_0)) \right.$$
$$\left. (I - F G_{33}^u)^{-1} \begin{pmatrix} G_{31}^u & G_{32}^u \end{pmatrix} D_0^{-1} v_1 \right] \qquad (3.3.13)$$

where F, G_{ij}^u, b, w are computed at $\omega = \omega_0$. The derivatives of F with respect to its parameters (elements of Λ) depend on the particular form that the designer selected and they can be easily computed.

ii) H_2.

• The first issue in this case is the computation of x_0. Note that this computation has to be made at every frequency ω. In practice only a set Ω with a finite number of frequencies is used, from which $||x_0^{-1}||_2$ can be computed approximately. Theorem 2 indicates that any basic descent method should be sufficient. The fact that it is possible to obtain an analytic expression for the gradient of $\mu_{\Delta^\circ}(G_{full}^x(i\omega))$ with respect to x, simplifies the problem even further. This is possible when (2.3) is used for the computation of μ and the two largest singular values of $DG_{full}^x D^{-1}$ for the optimal D's at the value of x where the gradient is computed, are not equal to each other. If this not the case a mean direction can be used as mentioned in the H_∞ case above.

Let the $\inf_{D \in \mathcal{D}^\circ} \bar\sigma(DG_{full}^x(i\omega)D^{-1})$ be attained for $D_0 = D_0(\omega; x)$ and let σ_1 be the maximum singular value and u_1, v_1 the corresponding singular vectors. Then the same steps for obtaining (3.3.11) are valid. Hence by using (3.2.10) and (3.2.11) we get after some algebra

$$\frac{\partial}{\partial x}(\mu_{\Delta^\circ}(G_{full}^x(i\omega))) = \mathrm{Re}\left[u_1^* D_0 \begin{pmatrix} 0 & 0 & 0 \\ W G_{21}^F & W G_{22}^F v & 0 \end{pmatrix} \right.$$
$$\left. D_0^{-1} v_1 \right] \qquad (3.3.14)$$

• The second computational issue is the solution of (3.3.3). To obtain the gradient of $||x_0^{-1}||_2$ with respect to the filter parameters, we need to compute the gradient of $x_0(\omega)$ with respect to these parameters for every frequency $\omega \in \Omega$. From the definition of x_0 in (3.2.12) we see that as some filter parameter λ_k changes, $x_0(\omega)$ will also change so that $\mu_{\Delta^\circ}(G_{full}^{x}(i\omega))$ remains constantly equal to 1. Hence we can write

$$\frac{\partial \mu}{\partial x_0} \frac{\partial x_0}{\partial \lambda_k} + \frac{\partial \mu}{\partial \lambda_k} = 0 \Longrightarrow \quad \frac{\partial x_0}{\partial \lambda_k} = -\frac{\partial \mu}{\partial \lambda_k} / \frac{\partial \mu}{x_0} \qquad (3.3.15)$$

where μ is computed through (2.3). The denominator in the right hand side of (3.3.15) is given from (3.3.14). As for the numerator, it can be computed in the same way as (3.3.11) and (3.3.13) but with G_{full}^x instead of G^b:

$$\frac{\partial}{\partial \lambda_k}(\mu_{\Delta^\circ}(G_{full}^x(i\omega))) = \mathrm{Re}\left[u_1^* D_0 \begin{pmatrix} G_{13}^u \\ x W G_{23}^u \end{pmatrix} (I - F G_{33}^u)^{-1} \right.$$
$$\left. \frac{\partial}{\partial \lambda_k}(F(i\omega))(I - F G_{33}^u)^{-1} \begin{pmatrix} G_{31}^u & G_{32}^u v & 0 \end{pmatrix} D_0^{-1} v_1 \right] (3.3.16)$$

Hence $\partial x_0 / \partial \lambda_k$ can be computed from (3.3.14), (3.3.15), (3.3.16).

Acknowledgements The authors would like to thank Prof. J.C.Doyle and Dr. J.A.Mandler. Support from NSF and DOE is gratefully acknowledged.

References

[1] B.-C. Chang and J.B.Pearson,Jr., "Optimal Disturbance Rejection in Linear Multivariable Systems", *IEEE Trans. Aut. Control*, AC-29, 1984.

[2] J.C.Doyle, "Analusis of Feedback Systems with Structured Uncertainty", *IEE Proc.*,Part D,V129,1982.

[3] J.C.Doyle, *Lecture Notes*, 1984 ONR/Honeywell Workshop on Advances on Multivariable Control.

[4] J.C.Doyle, "Structured Uncertainty in Control System Design", 1985 CDC.

[5] P.M.Frank, *Entwurf von Regelkreisen mit Vorgeschrieben Verhalten*, G.Braun Verlag, Karlsruhe, 1974.

[6] C.E.Garcia and M.Morari, "Internal Model Control. A Review and Some New Results", *Ind. and Eng. Chem., Proc. Des. and Dev.*, 21, p.308, 1982.

[7] M.Morari, E.Zafiriou and C.Economou, *An Introduction to Internal Model Control*, in preparation, 1986.

[8] D.C.Youla, J.J.Bongiorno, H.A.Jabr, "Modern Viener Hopf Design of Optimal Controllers", Parts I,II: *IEEE Trans. Aut. Control*, AC-21, Feb.,June, 1976.

[9] G.Zames, "Feedback and Optimal Sensitivity: model reference transformations, Multiplicative semi-norms and approximate inverses", *IEEE Trans. Aut. Control*, AC-26, 1981.

[10] G.Zames and B.A.Francis, "Feedback, Minimax Sensitivity and Optimal Robustness", *IEEE Trans. Aut. Control*, AC-28, April, 1983.

[11] G.Zames and B.A.Francis, "H^∞-Optimal Feedback Controllers for Multivariable Systems", *IEEE Trans. Aut. Control*, AC-29, 1984.