ABSTRACT

Title of Dissertation: AGE-STRUCTURED TUMOR GROWTH MODELS

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Doctor of Philosophy, 2026

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Mathematical models of tumor growth are essential for complementing experimental findings and furthering the understanding of cancer development and spread, as well as creating treatment plans. This dissertation focuses on an age-structured partial differential equation (PDE) model. The underlying process for tumor growth is similar to classical models, where growth is driven by pressure-limited cell division, and Darcy's law, describing cell movement away from regions of high internal pressure. By introducing an age-structure, the model accounts for the life-cycle of tumor cells. This allows for more accurate medical predictions and optimal therapy design.

There are three main results in this dissertation. The first is the existence of weak solutions. The second is a proof of the convergence of solutions to an incompressible limit satisfying a Hele-Shaw free boundary problem, describing the geometric growth of the tumor. The third is a proof of the existence of traveling wave solutions of the model, providing insight into the tumor spreading speed.

AGE-STRUCTURED TUMOR GROWTH MODELS

by

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Dissertation submitted to the Faculty of the Graduate School of the
University of Maryland, College Park in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
2026

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Acknowledgments

I would like to thank my advisor, Dr. Antoine Mellet, as well as my committee members, Dr. Doron Levy, Dr. Christopher Henderson, Dr. Huy Nguyen, and Dr. William Fagan. In addition, I owe a huge amount of gratitude to my incredible friends, family, and loved ones who have supported me during the writing of this dissertation.

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Chapter 1: Introduction

1.1 Introduction to Mechanical Tumor Growth Models

There is a rich history of research on mathematical models of tumor growth, e.g., [1, 74]. Mathematical models serve to complement experimental findings, improve our understanding of cancer development and spread, and assist with the design of effective therapies. Tumor models can represent various levels of complexity and can take into account various mechanisms, e.g., competition for space with other tumor cells and healthy cells [9, 11], cellular movement towards higher concentrations of nutrients [34, 61, 63], and phenotypic traits that affect cell growth [18, 48]. In this dissertation, I will focus on a partial differential equation (PDE) model that describes tumor growth due to two factors: pressure-limited cell proliferation, and Darcy's law [34], which describes movement away from regions of high internal pressure.

A PDE model of this type was first proposed by Byrne and Drasdo in 2008 [11]. In this work, the PDE is derived as a continuum of an individual-based model, i.e., a model in which the unit is a single cell. The individual-based model describes cell interactions and proliferation rates based on experimental studies [13, 20, 66]. The number of cells is taken to infinity to derive the *mean field limit*. This yields a continuum PDE model that shares the same growth properties as the individual-based model. The resulting continuum model describes the cell density, $\rho(t, x)$, at time $t > 0$ and position

$x \in \mathbb{R}^d$, $d \geq 1$. The dynamics are described by the following equation:

$$\partial_t \rho - \text{div}_x(\rho \nabla p) = \rho F(p), \quad p = \frac{m}{m-1}\rho^{m-1}, \quad (t,x) \in Q = (0,\infty) \times \mathbb{R}^d, \tag{1.1}$$

where p can be thought of as the pressure inside the tissue. The growth rate, $F(p)$, is a decreasing function of p satisfying $F(p_M) = 0$ for some $p_M > 0$ known as the *homeostatic pressure*, i.e., the pressure at which proliferation ceases. The equation (1.1) is a nonlinear degenerate diffusion equation called the *porous medium equation*, an equation with an established in-depth theory, including the existence and uniqueness of weak solutions [69]. The parameter m is taken to be $m > 1$ and describes the "stiffness" of the cells. For large values of m, pressure disappears when $\rho < 1$ and goes to infinity when $\rho > 1$. In the limit $m \to \infty$, a maximum density of $\rho \leq 1$ is enforced.

An important aspect of (1.1) is that it exhibits *pressure limited growth*, as dictated by the growth term, $F(p)$. Biologically, pressure is caused by cells competing for space, i.e., high cell density in an area will lead to high pressure. The properties of the growth term, $F(p)$, ensure that under higher pressure, cells duplicate at a lower rate, and under high enough pressure (precisely when $p = p_M$), cell duplication stops completely. In addition, cells move away from high pressure regions, as governed by Darcy's law, resulting in tumor growth (this process is described by the term $\text{div}_x(\rho \nabla p)$ in (1.1)). It is important to note that the movement away from high pressure regions is not driven by cells moving themselves, but by cells growing and dividing, resulting in congestion and an outward "pushing" motion.

The model (1.1) has been studied extensively, i.e., [18, 19, 51, 61]. Of particular interest is the link between the cell density equation, (1.1), and a Hele-Shaw free boundary problem, with is obtained by taking $m \to \infty$. The free boundary problem describes the geometric growth of the solid tumor,

providing a more "macroscopic" view of tumor growth. This link was first investigated by Perthame, Quirós, and Vázquez in 2014 [61]. Free boundary problems make up a second commonly studied class of tumor growth models, where instead of tracking cell density, the geometric expansion of the entire tumor is tracked over time through the movement of its boundary [17, 29]. The relationship between the two classes of models will be discussed in Chapter 2.

1.2 Cancer Overview

Cancer is characterized by uncontrolled proliferation of cells, resulting from defects in the regulatory systems that control the cell cycle, proliferation, and apoptosis (programmed cell death). These defects lead to unregulated growth and malignant tumor formation. In the landmark work of Hanahan and Weinberg [37], the authors identify six physiological traits that cause these defects: self-sufficiency in growth signals, insensitivity to antigrowth signals, evasion of apoptosis, unlimited reproductive potential, sustained angiogenesis (the formation of blood vessels), and the ability to metastasize.

Tumor development typically begins with a single cell that develops a mutation affecting its regulatory system. When the cell duplicates, the mutation is propagated to its offspring. The expansion of mutated cells can be exacerbated by external factors such as hormonal imbalance, chronic inflammation, viruses, environmental toxins, etc. [60, 64].

Once several mutated cells are collected, the tumor enters its stage of *avascular growth*, where its volume increases exponentially. Experimental studies have shown that most proliferating cells are located near the outer boundary of the tumor, while a necrotic core is formed in the center of the tumor. This structure is due to access to oxygen, nutrients, and space [28, 42, 54]. These aspects of tumor growth motivate the mathematical models introduced in Section 1.3, and discussed throughout

this dissertation. These models typically describe tumor in the avascular stage.

The final stage of tumor growth is the *vascular growth* stage, where angiogenesis occurs. Angiogenesis is the formation of blood vessels, providing cells with access to oxygen and nutrients. Further, angiogenesis allows cancer cells to spread to different locations in the body, leading to metastasis [27, 58].

1.3 Age-Structured Models

Significant effort has been made to develop models that take into account the life-cycle of tumor cells, e.g., [21, 36, 47, 50]. The typical cell cycle is made of two main phases: *interphase*, where the cell grows and synthesizes its genetic material, and *mitosis*, where the cell divides into two daughter cells. It is essential to develop models that consider the cell cycle for several reasons. The first reason is for accurate prediction of tumor growth: the breakdown of the typical regulated cell cycle leads to uncontrolled proliferation, which is one of the key features of cancer [37, 57]. The second reason is to aid in the design of effective cancer therapies. Many therapies target cells at a specific stage of their cell cycle (e.g., anti-mitotic drugs), so having a model describing the age distribution of cells is critical for therapy development [49].

1.3.1 Space-Homogeneous Models

Tumor cells are known to have age-dependent proliferation and death rates. With that as motivation, many age-structured tumor models have been introduced and studied in the space-homogeneous case, i.e., where cell populations lack spatial dependence. The following *growth-fragmentation equa-*

tion is a simple model describing the density $n(t,\theta)$ of cells of age $\theta > 0$ at time $t > 0$ [53, 62]:

$$\begin{cases} \partial_t n + \partial_\theta n = -\beta(\theta)n - \mu(\theta)n, & t, \theta > 0 \\ n(t,0) = 2\int_0^\infty \beta(\theta)n(t,\theta)d\theta, & t > 0, \end{cases} \tag{1.2}$$

where $\beta(\theta)$ is the proliferation rate and $\mu(\theta)$ is the death rate.

This model has been extended to consider cells that can switch between a proliferating and quiescent (non-proliferating) state. For example, in [21], the following model is introduced, describing the dynamics of $n(t,\theta)$ and $q(t,\theta)$, the densities of proliferating and quiescent cells, respectively, at time $t > 0$ and age $\theta > 0$.

$$\begin{cases} \partial_t n + \partial_\theta n = -(\beta(\theta) + \sigma(\theta))n + \tau(\theta)q, & t > 0, 0 < \theta < \theta^+ \\ \partial_t q + \partial_\theta q = \sigma(\theta)n - \tau(\theta)q, & t > 0, 0 < \theta < \theta^+, \\ n(t,0) = 2f\int_0^{\theta^+} \beta(\theta)n(t,\theta)d\theta, & t > 0, \\ q(t,0) = 2(1-f)\int_0^{\theta^+} \beta(\theta)n(t,\theta)d\theta, & t > 0, \end{cases}$$

where θ^+ is the maximum age, $\beta(\theta)$ is the proliferation rate, $\sigma(\theta)$ is the rate at which cells change from proliferating to quiescent, $\tau(\theta)$ is the rate at which cells change from quiescent to proliferating, and $f \in [0,1]$ is the probability that a newborn cell will be in a proliferating state. The authors show that the long-time age distribution and the long-time distribution of proliferating and quiescent cells are independent of the initial distributions. Further, the authors show that for large times, the population follows asynchronous exponential growth. This system has been studied thoroughly, both analytically and numerically, e.g., [31, 36, 47].

The growth-fragmentation model, (1.2), has also been extended to incorporate the response of immune cells. In [50], tumor cell density, $n(t,\theta)$, and effector cells (immune cells can that clear tumor cells), $E(t)$, are described by the following age-structured model:

$$\begin{cases} \partial_t n + \partial_\theta n = -\beta(\theta)n - \mu(\theta)n - \sigma(\theta)E(t)n, & t,\theta > 0, \\ \frac{dE}{dt} = s - d_E E(t) + \tau E(t)\int_0^\infty n(t,\theta)d\theta - \int_0^\infty \kappa(\theta)E(t)n(t,\theta)d\theta, & t > 0, \\ n(t,0) = 2\int_0^\infty \beta(\theta)n(t,\theta)d\theta, & t > 0, \end{cases}$$

where $\beta(\theta)$ is the proliferation rate, $\mu(\theta)$ is the death rate, and $\sigma(\theta)$ is the rate of clearance of tumor cells of age θ by effector cells. The parameter s represents the rate at which effector cells enter the tumor, d_E is the death rate of effector cells, τ represents the rate of promotion of effector cells when coming in contact with tumor cells, and $\kappa(\theta)$ is the inactivation rate of effector cells as a result of interaction with tumor cells. The authors derive a threshold, depending on the parameters, for which a tumor-free steady state exists.

A commonality among these age-structured models is that the effect of pressure, as caused by cell congestion, is not taken into consideration. In [47], this concept is incorporated in part. The authors introduce a model, considering again populations of proliferating cells, $n(t,\theta)$, and quiescent

cells, $q(t,\theta)$, where the proliferation rate is a decreasing function of the total cell density.

$$\begin{cases} \partial_t n + \partial_\theta n = -\mu(\theta)n - \beta(t,N(t))n + \sigma(\theta)q, & t>0, 0<\theta<\theta^+, \\ \partial_t q + \partial_\theta q = -\mu(\theta)q - \sigma(\theta)q, & t>0, 0<\theta<\theta^+, \\ n(t,0) = 2f\int_0^{\theta^+}\beta(\theta,N(t))n(t,\theta)d\theta, & t>0, \\ q(t,0) = 2(1-f)\int_0^{\theta^+}\beta(\theta,N(t))n(t,\theta)d\theta, & t>0, \end{cases}$$

where $N(t) = \int_0^{\theta^+} n(t,\theta) + q(t,\theta)d\theta$ is the total cell density at time t. Here, the proliferation rate satisfies $\partial_N \beta < 0$. This model includes pressure-limited growth, however, it is still space-homogeneous, and does not consider the role of cell movement away from regions of high pressure.

1.3.2 Novel Age-Structured Tumor Model

Motivated by the previous work mentioned above, one of the main contributions of this dissertation is the introduction of a novel age-structured tumor model, which includes a space variable. It is well-known that tumors are not spatially homogeneous. In fact, an important property of many tumors is that most cell proliferation occurs near the tumor boundary, while very little occurs in the center [11, 44, 59, 68]. This highlights the importance of developing a tumor model that is both age-dependent and space-dependent.

The model discussed here is introduced in a joint work by Doron Levy, Hyunah Lim, Antoine Mellet, and myself [43]. The model combines aspects of the cell density model, (1.1), and the growth fragmentation model, (1.2). The goal of this new model is to address the limitations of these two models by incorporating both age-structure and spatial structure.

We associate an age $\theta \geq 0$ with each cell, and define the cell distribution function, $n(t, \theta, x)$, which can be interpreted as the probability of finding a cell of age θ at time $t \geq 0$ and position $x \in \mathbb{R}^d$. We define the volume of a cell, $V(\theta)$, to be a non-decreasing function of age, and define the *volume density* as

$$\rho(t, x) = \int_0^\infty V(\theta) n(t, \theta, x)\, d\theta. \tag{1.3}$$

The pressure $p(t, x)$ is given by $p = \frac{m}{m-1}\rho^{m-1}$, so the pressure depends not only on the number of cells in an area but also on their volume. The dynamics of n under the effects of pressure and cell duplication are described by the following boundary value problem:

$$\begin{cases} \partial_t n + r(p)\partial_\theta n - \text{div}_x(n\nabla_x p) = -r(p)\nu(\theta, p)n - \mu(\theta)n & t > 0,\ \theta > 0,\ x \in \mathbb{R}^d, \\ n(t, 0, x) = 2\int_0^\infty \nu(\theta, p) n(t, \theta, x) d\theta & t \geq 0,\ x \in \mathbb{R}^d, \\ n(0, \theta, x) = n_0(\theta, x) & \theta \geq 0,\ x \in \mathbb{R}^d. \end{cases} \tag{1.4}$$

In this equation, the term $\partial_\theta n$ represents cell aging, and $r(p)$ represents the pressure-dependent aging rate. When r is non-constant, age is not linear in time and represents not simply the amount of time since a cell's birth, but rather how far the cell has progressed in its cell-cycle. Because aging is assumed to be pressure-limited (cells will only grow if they have enough space), the function $r(p)$ satisfies

$$r(p) \geq 0, \quad r'(p) < 0 \text{ for all } p \in [0, p_M], \quad r(p) = 0 \text{ for all } p \geq p_M, \tag{1.5}$$

for the homeostatic pressure, $p_M > 0$. The term $\mu(\theta) > 0$ is the the age-dependent death rate. The term $\nu(\theta, p)$ is the pressure- and age-dependent division rate, and can be thought of as the probability that a cell of age θ in the presence of local pressure p enters its mitotic phase. Cell division is pressure-

limited, so ν satisfies

$$\nu(\theta,p)\geq 0,\quad \partial_p\nu(\theta,p)<0 \text{ for all } p\in[0,p_M],\quad \nu(\theta,p)=0 \text{ for all } p\geq p_M. \tag{1.6}$$

When a cell enters its mitotic phase, the cell is lost (the term $-\nu n$ on the right-hand side of the first line), and is replaced by two cells of age $\theta = 0$ (the second line). The function $n_0(\theta, x)$ represents the initial condition.

The age-structured model, (1.4), is related to the classical tumor growth model, (1.1). In the case where V and ν are constant in θ, r is constant, and $\mu = 0$ (essentially removing all of the age-dependent parameters), multiplying (1.4) by V and integrating in θ exactly recovers equation (1.1), with $F(p) = r\nu(p)$. This shows that the underlying dynamics are the same as in the classical model.

However, when these parameters are non-constant, the behaviour is quite different from that of (1.1). For example, a biologically-relevant assumption to make is that mitosis is volume preserving, i.e., the combined volume of the two daughter cells is equal to the volume of the parent cell. This can be imposed by assuming that

$$\nu(\theta,p)(2V(0)-V(\theta))=0, \tag{1.7}$$

for all $\theta > 0$. In this case, multiplying (1.4) by $V(\theta)$ and integrating in θ yields the following equation on ρ :

$$\partial_t\rho-\text{div}_x(\rho\nabla p)=r(p)\int_0^\infty (V'(\theta)-\mu(\theta)V(\theta))n(t,\theta,x)d\theta.$$

In this case, tumor growth is driven primarily by the growth of individual cells prior to mitosis, rather than by cell division, as is the case in (1.1).

1.4 Outline of the Dissertation

In Chapter 2, I will discuss previous results relevant to the work presented in this dissertation. In Chapter 3, I will further discuss the age-structured model, (1.4), including the existence of weak solutions, extensions of the model, and numerical results. In Chapter 4, I will prove that solutions of (1.4) converge to an incompressible limit. In Chapter 5, I will prove that traveling wave solutions to (1.4) exist under certain assumptions.

1.5 Future Directions

There are several natural extensions of my work, primarily related to traveling waves. Traveling waves are of interest when studying reaction-diffusion equations because of their ability to predict long-time behavior and speed of propagation of the support of solutions. For this reason, there are several interesting future directions from my work on traveling wave solutions to (1.4).

Long-time behaviour and the role of death rate. One extension is exploring the role of the death rate, μ, on the long-time behavior of (1.4). An important question is whether there is a threshold value for μ that characterizes the long-time behaviour of the tumor as either expanding or regressing. That is, is it true that for μ small enough, the tumor will continue to grow for all time (expanding), and for μ large enough, the tumor will eventually disappear (regressing)? This is conjectured to be true in our paper [43]. Furthermore, it is established numerically in one dimension that in certain cases, for sufficiently small μ, $n(t, \theta, x)$ approaches a space-homogeneous profile $\phi_\infty(\theta)$ as $t \to \infty$. From (1.4),

if such a profile exists, it must solve:

$$\begin{cases} r(p_0)\phi'(\theta) = -r(p_0)\nu(\theta, p_0)\phi(\theta) - \mu(\theta)\phi(\theta), \\ \phi(0) = 2\int_0^\infty \nu(\theta, p_0)\phi(\theta), \\ p_0 = \frac{m}{m-1}(\int_0^\infty V(\theta)\phi(\theta)d\theta)^{m-1}, \end{cases}$$

for every $x \in \mathbb{R}$.

This question is related to the existence of traveling wave solutions. It is conjectured that for small enough μ, the long-time behaviour of the tumor is described by a traveling wave solution $(\bar{n}, \bar{p}, c)$ of (1.4) with the following boundary conditions:

$$\lim_{x\to-\infty} \bar{p}(x) = p_0, \quad \lim_{x\to-\infty} \bar{n}(\theta, x) = \phi_\infty(\theta), \quad \lim_{x\to\infty} \bar{p}(x) = \lim_{x\to\infty} \bar{n}(\theta, x) = 0.$$

In Chapter 5, we prove that traveling waves exist when there is no cell death, i.e., the death rate $\mu(\theta) = 0$. This has proved to be much more difficult when $\mu > 0$. When $\mu > 0$, $\bar{p}$ will no longer satisfy the simple boundary condition $\lim_{x\to-\infty} \bar{p}(x) = p_M$, i.e., the inside of the tumor will not have maximum pressure, because cells will die and be removed, causing pressure to decrease. The first step in this direction is to show the existence of traveling wave solutions of (1.4) when $\mu > 0$ is sufficiently small.

The role of external factors. Considering the role of the death rate is a natural first step before taking external factors into account, e.g., immune system response and/or therapeutic treatment. This next step could be accomplished by coupling the tumor cell equations with equations describing these responses, and investigating the long-time behaviour. Incorporating these factors is of importance

when developing tumor models, and is an extension of the previous discussion on death rate, because immune cells and therapeutic treatments impact the death rate of tumor cells. Similar work has been done in this vein, but without spatial dependence, e.g., [45, 47, 50].

Nutrient availability. Another factor that is commonly considered when modeling cellular behaviour is the availability of nutrients, e.g., in [33, 38, 52, 63]. Previous work has been done along these lines: In [63], traveling wave solutions are established for a system coupling an equation for tumor cells with an equation for nutrient concentration. In [38], traveling wave solutions are established for a standard reaction-diffusion equation coupled with an equation for nutrient concentration. A future direction is to explore the long-time behaviour and existence of traveling wave solutions to the age-structured model, (1.4) coupled with an equation describing nutrient concentration.

Chapter 2: Background

2.1 Incompressible Limit

There are two commonly studied classes of tumor growth models. The first class consists of cell density models such as (1.1). These models track the cell density at a specific point in time and space. The second class describes the geometric growth of a solid tumor via a free boundary problem, rather than describing cell density [17,30]. In the seminal work of Perthame, Quirós, and Vázquez [61], a link between the two classes is established. The authors show that the asymptotic limit of (1.1) as $m \to \infty$ yields a free boundary problem of Hele-Shaw type. This limit is referred to as an *incompressible limit*, referencing the increasing "stiffness" of cells as m increases. In particular, in the limit, the cells are completely incompressible, so the condition $\rho \leq 1$ is enforced.

The free boundary problem partitions the region into the tumor region, where the density is at its maximum, $\rho = 1$ and the pressure is positive, and the healthy region, where $\rho < 1$ and the pressure is zero. This model focuses on the role of congestion: when $\rho < 1$, growth is exponential, but when $\rho = 1$, cells are congested and pressure takes over as the primary cause of growth. The free boundary problem tracks the expansion of the tumor region by modeling the movement of its boundary.

The equation (1.1) can be rewritten as a porous medium equation:

$$\partial_t \rho - \Delta \rho^m = \rho F(p).$$

It is well known that for classical porous medium equations without a growth term, i.e., $\partial_t \rho - \Delta \rho^m = 0$, the limit of the solution as $m \to \infty$ solves a standard Hele-Shaw free boundary problem [2, 22, 69]. The support of a solution to the porous medium equation spreads with a *finite speed of propagation*, meaning that if the initial condition is compactly supported, the solution remains compactly supported for any finite time [69]. Consequently, it is natural that in the limit $m \to \infty$, a free boundary problem arises, where the boundary separates the tumor region from the healthy region.

2.1.1 Classical Convergence Result

In [61], Perthame, Quirós, and Vázquez extend the convergence result for classical porous medium equations to tumor models, i.e., equations of the form (1.1). In this paper, the authors prove the following theorem, with $Q_T = [0, T] \times \mathbb{R}^d$.

Theorem 2.1.1 ([61], Theorem 2.1)**.** *Let ρ_m be a solution of (1.1), and make the following assumptions on the initial data, $\rho_{m,0}(x)$:*

$$\rho_{m,0}(x) \geq 0, \quad \frac{m}{m-1}(\rho_{m,0}(x))^{m-1} \leq p_M, \quad ||\rho_{m,0} - \rho_0||_{L^1(\mathbb{R}^d)} \xrightarrow[m\to\infty]{} 0,$$
$$||\rho_{m,0}||_{L^1(\mathbb{R}^d)} \leq C_1, \quad ||\partial_{x_i}\rho_{m,0}||_{L^1(\mathbb{R}^d)} \leq C_2,$$

for all $i \in \{1, ..., d\}$, for constants C_1 and C_2 independent of m, and for some $\rho_0 \in L^1_+(\mathbb{R}^d)$. Then, up to a subsequence, for all $T > 0$, the solutions ρ_m and p_m converge strongly in $L^1(Q_T)$ to limits $\rho_\infty \in C([0, T]; L^1(\mathbb{R}^d)) \cap BV(Q_T)$ and $p_\infty \in BV(Q_T)$. The limits satisfy $0 \leq \rho_\infty \leq 1$, $0 \leq p_\infty \leq p_M$,

$$\partial_t \rho_\infty - \Delta p_\infty = \rho_\infty F(p_\infty) \text{ in } \mathcal{D}'(Q_T), \tag{2.1}$$

and $p_\infty \in P_\infty(\rho_\infty)$, where P_∞ is the Hele-Shaw graph, given by

$$P_\infty(\rho) = \begin{cases} 0, & 0 \le \rho < 1 \\ [0, \infty), & \rho = 1. \end{cases} \tag{2.2}$$

Further, if the initial data is compactly supported uniformly in m, the limiting functions ρ_∞, p_∞ are also compactly supported.

Simple calculations can yield the following equation on p_m :

$$\partial_t p_m = (m-1)p_m \Delta p_m + |\nabla p_m|^2 + (m-1)p_m F(p_m). \tag{2.3}$$

Dividing by $m-1$ and taking the limit $m \to \infty$ establishes the following *complementarity formula*.

Theorem 2.1.2 ([61],Theorem 2.3)**.** *Under the assumptions of Theorem 2.1.1, the pressure function p_∞ satisfies*

$$p_\infty(\Delta p_\infty + F(p_\infty)) = 0, \quad \text{in } \mathcal{D}'(Q_T),$$

in the sense that

$$\int_{\mathbb{R}^d} \varphi\big(-|\nabla p_\infty|^2 + p_\infty F(p_\infty)\big) - p\nabla\varphi \cdot \nabla p \, dx = 0,$$

for any test function $\varphi \in C^1(Q_T)$.

These results give rise to a free boundary problem, where $\Omega(t) = \{x \in \mathbb{R}^d : p(t,x) > 0\}$ can be thought of as the tumor region. The pressure equation (2.3) shows, formally, that on the boundary, $\partial\Omega(t)$, where $p_m = 0$, it holds that

$$\partial_t p_\infty = |\nabla p_\infty|^2.$$

Thus, the authors show that the normal velocity at the boundary of $\Omega(t)$ is given by

$$V_n = |\nabla p_\infty|,$$

yielding a Hele-Shaw problem.

2.1.1.1 Heuristics for Theorem 2.1.1

In order to prove Theorem 2.1.1, the authors of [61] prove strong L^1 convergence, up to a subsequence. In order to do so, they establish bounds on ρ_m and p_m. The necessary bounds are as follows:

$L^\infty(Q_T)$ bounds on p_m, ρ_m. Using the comparison principle, the bounds

$$0 \leq \rho_m \leq (\frac{m-1}{m} p_M)^{\frac{1}{m-1}} \xrightarrow[m\to\infty]{} 1, \quad 0 \leq p_m \leq p_M,$$

hold for all $m > 1$.

$L^1(Q_T)$ bounds on p_m, ρ_m. Integrating (1.1) in x,

$$\frac{d}{dt}\int_{\mathbb{R}^d} \rho_m \, dx = \int_{\mathbb{R}^d} \rho_m F(p_m) \, dx \leq \int_{\mathbb{R}^d} \rho_m F(0),$$

and by Gronwall's inequality,

$$\int_{\mathbb{R}^d} \rho_m(t,x) \, dx \leq e^{F(0)t} \int_{\mathbb{R}^d} \rho_{m,0}(x) \, dx,$$

which gives

$$||\rho_m||_{L^1(Q_T)} \leq C(T),$$

for a constant $C(T)$ independent of m. The bounds on p_m is achieved by writing

$$p_m = \frac{m}{m-1}\rho_m(\frac{m-1}{m}p_m)^{\frac{m-2}{m-1}},$$

and using the L^1 bound on ρ_m and the L^∞ bound on p_m.

$L^1(Q_T)$ bounds on $\partial_t p_m, \partial_t \rho_m$. These bounds come from the L^1 bounds of ρ_m, p_m, and lower bounds on $\partial_t \rho_m$, $\partial_t p_m$.

$L^1(Q_T)$ bounds on $\nabla p_m, \nabla \rho_m$. These bounds are established by taking the x_i derivative of (1.1), for all $i = 1, ..., d$, and proceeding in the same way as the bounds on p_m, ρ_m.

With these bounds, by the compact embedding of $BV(K)$ in $L^1(K)$, for any $K \subset \mathbb{R}^d$, there is a subsequence of (ρ_m, p_m) that converges strongly in $L^1_{loc}(Q_T)$ to some (ρ_∞, p_∞). To get strong convergence in $L^1(Q_T)$, the authors show that the mass is uniformly small in $t \in [0, \frac{1}{R}]$ and $|x| > R$, for R large enough.

Finally, to see that the limits (ρ_∞, p_∞) satisfy the Hele-Shaw graph, (2.2), notice that

$$\rho_m p_m = p_m^{(1+m)/m},$$

and passing to the limit yields

$$(1 - \rho_\infty)p_\infty = 0.$$

The authors also prove that the solution of the free boundary equation is unique. The following theorem is proved.

Theorem 2.1.3 ([61], Theorem 2.4)**.** *There exist a unique pair* (ρ, p) *in* $L^\infty((0,\infty); L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$ *solving* (2.1)*, with* $p \in P_\infty(\rho)$*, such that for all* $T > 0$*,* $\rho(t, \cdot)$ *is uniformly compactly supported for*

$t \in [0, T]$, *and* $\nabla p \in L^2(Q_T)$.

In particular, this theorem shows that if the initial data $\rho_{m,0}$ are compactly supported, uniformly in m, the limit (ρ_∞, p_∞) is unique. The proof of Theorem 2.1.3 uses Hilbert's duality method [3,16,61].

The authors extend the convergence results to the following model with nutrients.

$$\begin{cases} \partial_t \rho - \text{div}(\rho \nabla p) = \rho \Phi(p, c), \\ \partial_t c - \Delta c = -\rho \Psi(p, c), \end{cases} \tag{2.4}$$

where and $c(t, x)$ represents the density of nutrients at time t and position x. The following theorem is proved:

Theorem 2.1.4 ([61], Theorem 4.1)**.** *Under appropriate assumptions on initial conditions,* Ψ, *and* Φ, *up to a subsequence, the solutions* (ρ_m, p_m, c_m) *of* (2.4) *approach a limit* $(\rho_\infty, p_\infty, c_\infty)$ *as* $m \to \infty$, *which is the unique distributional solution of*

$$\begin{cases} \partial_t \rho_\infty = \Delta p_\infty + \rho_\infty \Phi(p_\infty, c_\infty), \\ \partial_t c_\infty = \Delta c_\infty - \rho_\infty \Psi(p_\infty, c_\infty), \end{cases} \tag{2.5}$$

with $p_\infty \in P_\infty(\rho_\infty)$.

The proof of existence and uniqueness of this system presented in [61] is similar to the case without nutrients.

2.1.2 Extensions

Since the classical convergence result was established in 2014, many extensions of Theorem 2.1.1 have been proved. I will list several examples.

In [35], a porous medium equation with a right hand side that can be either a source or a sink is considered. Precisely, the authors consider the boundary value problem:

$$\begin{cases} \partial_t \rho - \Delta \rho^m = \lambda(t,x)\rho, & t > 0, x \in \Omega, \\ \rho(t,x) = f(t,x)^{\frac{1}{m-1}}, & t > 0, x \in \partial K, \\ \rho(0,x) = \rho_0(x), & x \in \Omega, \end{cases} \tag{2.6}$$

where $\lambda(t,x)$ is bounded and can take both positive and negative values, and $\Omega = \mathbb{R}^d \backslash K$, where $K \subset \mathbb{R}^d$ is bounded, and $f > 0$. The authors prove a convergence result for this system.

Theorem 2.1.5 ([35], Theorem 2.4)**.** *Under appropriate conditions on the initial data, up to a subsequence, for all $T > 0$, the solutions ρ_m and p_m of (2.6) converge strongly in $L^1([0,T] \times \Omega)$ to limits $\rho_\infty, p_\infty \in BV([0,T] \times \Omega)$, which are bounded and satisfy the Hele-Shaw graph, (2.2).*

In [40], the incompressible limit of (1.1) with a more general growth rate F is studied. In this work, F can be bistable or monostable, meaning it need not be monotone or nonnegative. It is shown that for a more general F, the solutions still converge to a limit, but the convergence is weaker and the limit cannot be entirely characterized. The following theorem is proved.

Theorem 2.1.6 ([40], Proposition 1.1-1.2)**.** *Suppose that $F(p)$ is Lipschitz and satisfies $F(p) \leq 0$ for $p \geq p_M$. Under appropriate conditions on the initial data, up to a subsequence, for all $T > 0$, the solutions ρ_m and p_m of (1.1) converge weakly-* in $L^\infty(Q_T)$ to ρ_∞ and p_∞, which satisfy (2.2) and*

solve

$$\partial \rho_\infty - \Delta p_\infty = \rho_\infty g, \text{ in } \mathcal{D}'(Q_T),$$

for some function $g \in L^\infty(Q_T)$.

Because of the lack of strong convergence of p_m, it is not possible to pass to the limit in the nonlinearity $F(p_m)$.

Convergence to an incompressible limit has been extended to cross-diffusion equations, i.e., systems that consider multiple cell populations. In [10], two interacting populations, n^1 and n^2, are considered, as described by the following one-dimensional system:

$$\begin{cases} \partial_t n^1 = \partial_x(n^2 \partial_x p) + n^1 F_1(p) + n^2 G_1(p), \\ \partial_t n^2 = \partial_x(n^2 \partial_x p) + n^1 F_2(p) + n^2 G_2(p), \end{cases} \tag{2.7}$$

for $t \geq 0, x \in \mathbb{R}$, where the pressure p is given by $p = (n^1 + n^2)^m$. The authors prove that under appropriate assumptions on F_i, G_i, $i = 1, 2$, and the initial data, solutions n^1_m, n^2_m, and p_m of (2.7) converge, up to a subsequence, to limits satisfying a Hele-Shaw free boundary problem. A similar problem is considered in [46]. In these cross-diffusion systems, uniqueness of the limit is a much more difficult and largely open problem.

2.1.3 Convergence for a Structured Model

Building off of the cross-diffusion systems with a finite number of populations, in [18], David introduces a tumor growth model structured by *phenotypic traits*. This is a cross-diffusion model describing an infinite number of interacting cell populations, where each population has a phenotypic

trait $y \in [0,1]$. It is assumed that the phenotype of a given cell does not change. The density of cells with phenotype y at time $t \geq 0$ and $x \in \mathbb{R}^d$ is denoted by $n(t,y,x)$, the total cell density is

$$\rho(t,x) = \int_0^1 n(t,y,x)\, dy,$$

and the pressure is given by

$$p(t,x) = (\rho(t,x))^m.$$

The dynamics are described by the following model:

$$\begin{cases} \partial_t n - \operatorname{div}_x(n \nabla_x p) = nR(y,p), & t > 0, y \in [0,1], x \in \mathbb{R}^d, \\ n(0,y,x) = n_0(y,x), & y \in [0,1], x \in \mathbb{R}^d, \end{cases} \tag{2.8}$$

with initial data $n_0 \in L^\infty([0,1] \times \mathbb{R}^d) \cap L^1([0,1] \times \mathbb{R}^d)$, and R satisfying

$$\partial_p R \leq 0, \quad R(\cdot, p_M) \leq 0, \tag{2.9}$$

with $p_M > 0$, describing pressure-limited growth. David proves that solutions (n_m, ρ_m, p_m) to (2.8) exist. Further, by assuming that the initial data, $n_{m,0}$, is compactly supported uniformly in m, the solutions are also compactly supported uniformly in m, because of finite speed of propagation. Hence, the domain of the problem can be reduced to a bounded domain $K \subset \mathbb{R}^d$. The following theorems are proved, denoting $K_T = [0,T] \times K$.

Theorem 2.1.7 ([18], Theorem 2.2)**.** *Let (n_m, ρ_m, p_m) be a solution of (2.8). Under appropriate assumptions on the initial data, assuming R satisfies (2.9), up to a subsequence, for all $T > 0$,*

$n_m \rightharpoonup n_\infty$ *weakly-* in* $L^\infty((0,1)\times K_T)$, $\rho_m \rightharpoonup \rho_\infty$ *weakly-* in* $L^\infty(K_T)$, *and* $p_m \to p_\infty$ *strongly in* $L^2(K_T)$. *The limits satisfy*

$$p_\infty(1-\rho_\infty) = 0 \text{ almost everywhere in } K_T,$$

and

$$\partial_t \rho_\infty = \Delta p_\infty + \int_0^1 n_\infty F(y, p_\infty)\, dy \text{ in } \mathcal{D}'(K_T).$$

The following complementarity formula is also established.

Theorem 2.1.8 ([18], Theorem 2.3)**.** *The limit solution satisfies*

$$p_\infty(\Delta p_\infty + \int_0^1 n_\infty R(y, p_\infty)\, dy) = 0 \text{ in } \mathcal{D}'(K_T).$$

The theorems are proved by establishing apriori bounds, independent of m, on n_m, ρ_m, and p_m. In order to pass to the limit in (2.8), strong convergence of ∇p_m is required to pass to the limit in the divergence term, and strong convergence of p_m is needed to pass to the limit in the nonlinearity R. Obtaining strong convergence of ∇p_m is the biggest challenge in the proof, and the main difference between this proof and the proof of the incompressible limit for the classical tumor growth model, (1.1). The proof relies vitally on a compensated compactness theorem [23, 55, 67].

2.2 Traveling Waves

Traveling wave (TW) solutions are particular solutions of PDEs involving two variables, $t \geq 0$ and $x \in \mathbb{R}^d$, of the form

$$u(t, x) = \bar{u}(x \cdot e - ct) = \bar{u}(z),$$

where $e \in \mathbb{S}^{d-1}$ is the direction of propagation, $c \in \mathbb{R}$ is the *wave speed*, and $\bar{u}$ is the *profile*, which is fixed in space. Traveling waves are solutions that are one-dimensional in space, making convenient to work with, while still capturing crucial information about a solution's shape and spreading speed.

TW solutions to reaction-diffusion equations, i.e., equations of the form

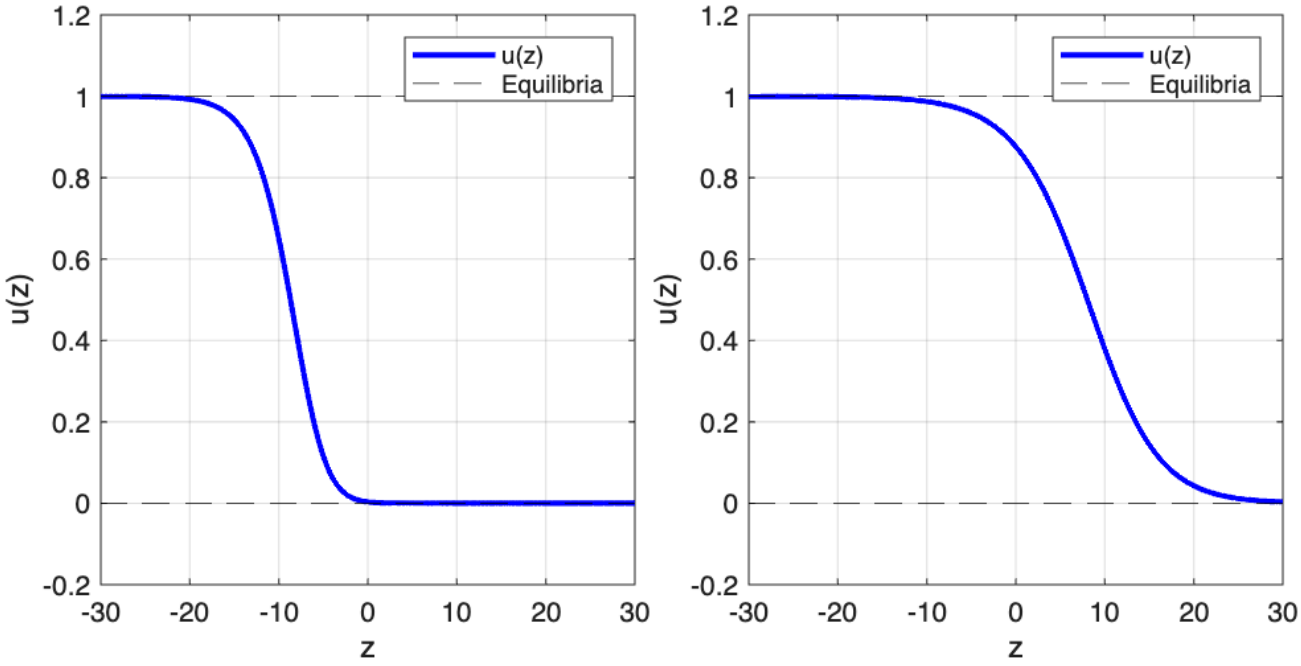
$$u_t - u_{xx} = f(u), \tag{2.10}$$

have been of interest for nearly 100 years, and are still commonly studied today, due to their ability to predict long-time behaviour of solutions and provide insight into the speed of propagation of solutions. Typical formulations of this problem have $f \in C^1$ satisfying $f(0) = f(1) = 0$. This area of study is motivated by population dynamics, an archetype being the standard Fisher-KPP nonlinearity, $f(u) = u(1-u)$ [26, 41]. In this formulation, traveling wave solutions connect the states $\bar{u} = 1$ and $\bar{u} = 0$, and solve the following Fisher-KPP equation:

$$\begin{cases} -c\bar{u}' - \bar{u}'' = \bar{u}(1-\bar{u}), \quad z \in \mathbb{R}, \\ \lim_{z\to-\infty} \bar{u}(z) = 1, \quad \lim_{z\to\infty} \bar{u}(z) = 0. \end{cases} \tag{2.11}$$

2.2.1 Traveling Wave Solutions to Reaction-Diffusion Equations

It was shown in the classical 1937 papers by Fisher and Kolmogorov, Petrovsky, and Piskunov that solutions to (2.11) exist for all wave speeds $c \geq 2$ [26, 41]. The comparison between traveling waves with minimal speed at higher speeds can be seen in Figure 2.1. The existence of traveling wave



(a) Traveling wave with $c = 2$.

(b) Traveling wave with $c = 4$.

Figure 2.1: Traveling wave solutions to the Fisher-KPP equation with minimal speed (left) and higher speed (right).

solutions to (2.10) has been shown for more general families of f. For example, for

$$f \in C^1, f(0) = f(1) = 0, f'(0) > 0, f'(1) < 0, f'(s) \leq f'(0), \text{ and } f(s) > 0 \text{ for all } s \in (0, 1), \tag{2.12}$$

the following result was shown in the original paper by Kolmogorov, Petrovsky and Piskunov.

Theorem 2.2.1 ([41]). *Under the assumptions* (2.12)*, traveling wave solutions to* (2.10) *exist for all wave speeds* $c \geq c^* = 2\sqrt{f'(0)}$.

The book by Gilding and Kersner, [32], provides an exhaustive list of conditions on f for the existence of traveling waves, as well as the ranges of possible wave speeds. The minimal speed, c^* is of primary interest: solutions of reaction-diffusion equations with compactly supported initial data approach the TW with minimal speed as $t \to \infty$, making this TW the "physically relevant" one.

2.2.1.1 Heuristics for Theorem 2.2.1

There are various methods that have been used to prove the existence of TW solutions to reaction-diffusion equations. The first method is to transform the PDE into a system of ODEs, and solve using phase plane analysis, connecting the two states $\bar{u} = 0$ to $\bar{u} = 1$, e.g., [41, 56]

The second commonly used method, e.g., [4, 5], is similar to the method used in Chapter 5 of this dissertation. It involves truncated the domain, and solving the following PDE:

$$\begin{cases} -c_a \bar{u}_a' - \bar{u}_a'' = f(\bar{u}), & z \in [-a, a], \\ \bar{u}_a(-a) = 1, \ \bar{u}_a(a) = 0, \\ \bar{u}_a(0) = \theta. \end{cases}$$

Then, sub- and supersolutions are found, bounds independent of a are established on $\bar{u}_a$ and c_a, and the limit $a \to \infty$ is taken.

2.2.1.2 Convergence of Solutions to Traveling Waves

Traveling wave solutions are of significant interest due to their ability to approximate long-time behaviour of solutions to reaction-diffusion equations. In fact, under some assumptions on the initial data and the reaction term, f, solutions of (2.10) approach traveling wave solutions as $t \to \infty$. In the

classical 1937 paper by Kolmogorov, Petrovsky, and Piskunov, [41], the following convergence result is proved.

Theorem 2.2.2 ([41]). *Assume that f satisfies (2.12), and the initial data, u_0, is chosen to be a Heaviside function, i.e. $u_0(x) = 0$ for $x < 0$, $u_0(x) = 1$ for $x \geq 0$. Then there exists a traveling wave profile $\bar{u}$, a function ψ such that*

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |u(t,x) - \bar{u}(x - c^*t - \psi(t))| = 0,$$

where $\bar{u}$ is the TW with minimal speed c^*.

This result has been extended by many authors to generalize the requirements on f and u_0, e.g., [8, 25, 39]. The following result was proved by Bramson in 1983.

Theorem 2.2.3 ([8]). *Assume that f satisfies (2.12), and u_0 satisfies*

$$0 \leq u_0 \leq 1,\ u_0 \not\equiv 0,\ u_0(x) = 0 \text{ for } x \geq A,$$

for some $A \in \mathbb{R}$. Then,

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |u(t,x) - \bar{u}(x - c^*t - \psi(t))|,$$

for some function ψ, where $\bar{u}$ is the TW with minimal speed c^*.

2.2.2 Traveling Wave Solutions to Porous Medium Equations

Traveling wave solutions to porous medium equations, i.e., equations of the form

$$\partial_t \rho - (\rho^m)'' = \rho F(p), \tag{2.13}$$

have also been shown to exist for all wave speeds $c \geq c^*$, for some minimal speed $c^* > 0$, under appropriate assumptions on F [32]. TW solutions, defined by $\rho(t,x) = \bar{\rho}(x - ct) = \bar{\rho}(z)$, solve the following equation:

$$\begin{cases} -c\bar{\rho}' - (\bar{\rho}^m)'' = \bar{\rho}F(\bar{p}), \\ \bar{p} = \frac{m}{m-1}\bar{\rho}^{m-1}, \\ \lim_{z\to-\infty} \bar{p}(z) = p_M, \ \lim_{z\to\infty} \bar{p}(z) = 0. \end{cases} \tag{2.14}$$

An important property of these traveling waves, in contrast with the traveling waves for standard reaction-diffusion equations, is that the wave profile for the minimal speed, c^*, is qualitatively different from the profile for speeds $c > c^*$. For the wave with minimal speed c^*, there exists some $z_0 \in \mathbb{R}$ such that for all $z \geq z_0$, the wave satisfies $\bar{\rho}(z) = 0$, meaning solutions are supported on the half-line. This is called a *sharp* traveling wave. For speeds $c > c^*$, the wave solutions are positive for all $z \in \mathbb{R}$, and are called *smooth* traveling waves. This matter is notable because it means that traveling waves with minimal speed capture the finite speed of propagation of support, a defining feature of porous medium equations [69]. This suggests that the traveling waves with minimal speeds are the most physically relevant solutions. A visual comparison of the two traveling waves can be seen in Figure 2.2.

The precise result from Gilding and Kersner is as follows.

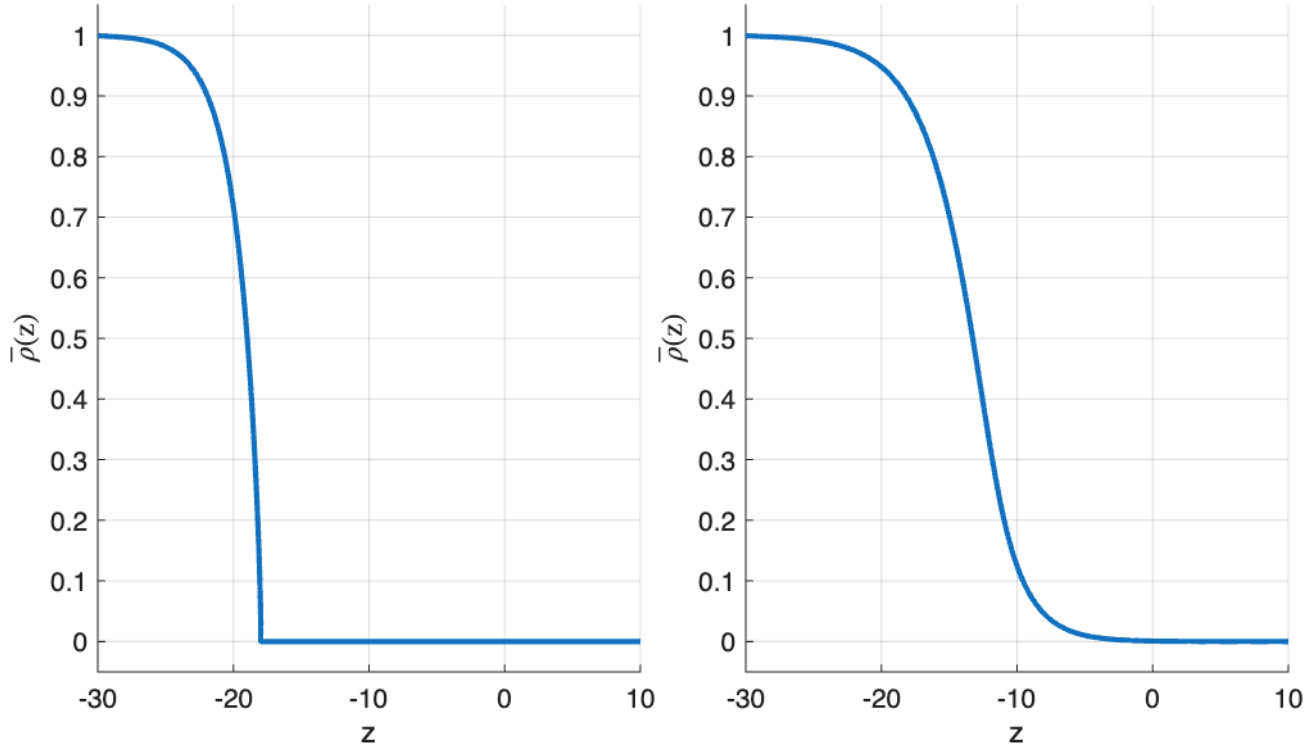
Theorem 2.2.4 ([32], Theorem 33, Theorem 38)**.** *Assume that F is positive and decreasing on $[0, p_M]$*

and $F(p) = 0$ *for all* $p \geq p_M$. *Then there exists some* $c^* > 0$ *such that for all* $c \geq c^*$, *a traveling wave solution to* (2.14) *exists. Further, any solution with wave speed* c^* *satisfies*

$$\bar{\rho}(z) = 0 \text{ for all } z \geq z_0, \text{ for some } z_0 \in \mathbb{R},$$

and any solution with wave speed $c > c^*$ *satisfies*

$$\bar{\rho}(z) > 0 \text{ for all } z \in \mathbb{R}.$$



(a) Sharp traveling wave with minimal speed c^*.

(b) Smooth traveling wave with speed $c > c^*$.

Figure 2.2: Sharp (left) and smooth (right) traveling wave solutions to the porous medium equation.

Similarly to standard reaction diffusion equations, solutions to porous medium equations converge to traveling waves as $t \to \infty$, under some assumptions on the initial data and F. The following

result holds for convergence to sharp traveling waves.

Theorem 2.2.5 ([7], Theorem 4, [72], Theorem 1.1)**.** *Let $\rho(t,x)$ be a solution to (2.13) where F satisfies the conditions in Theorem 2.2.4, and the initial data ρ_0 satisfies*

$$\rho_0 \in L^\infty(\mathbb{R}),\ \rho_0 \geq 0,\ \ \rho_0 \text{ is piecewise continuous,}$$

$$\operatorname{Supp} u_0 \subset (-\infty, x_0] \text{ for some } x_0 \in \mathbb{R},\ \lim_{x\to-\infty} \rho_0(x) > 0.$$

Then, there exists $x_0 \in \mathbb{R}$ such that

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |\rho(t,x) - \bar{\rho}(x - c^* t - x_0)| = 0,$$

where $\bar{\rho}$ is the sharp traveling wave with minimal speed c^*, given in Theorem 2.2.5, with $z_0 = 0$.

This theorem demonstrates that solutions to the porous medium equation with initial data that is supported on the half line will converge to a sharp traveling wave. In other words, the support propagates with finite speed.

Chapter 3: Age-Structured Mechanical Model

The contents of this chapter are from a joint work by Doron Levy, Hyunah Lim, Antoine Mellet, and myself [43]. The mathematics in this chapter are a joint effort by all authors of this paper.

3.1 Definitions and Notations

In this chapter, I will discuss the age-structured model, (1.4). First, a simplified version of the model is presented. In this version, there is no death rate and aging is linear, so the age of a cell, θ, simply represents the amount of time that has passed since the cell's birth. The simplified equation is given by:

$$\begin{cases} \partial_t n + \partial_\theta n - \operatorname{div}_x(n\nabla_x p) = -\nu(\theta,p)\, n & t>0,\ \theta>0,\ x\in\mathbb{R}^d, \\ n(t,0,x) = 2\displaystyle\int_0^\infty \nu(\theta,p)\, n(t,\theta,x)\, d\theta & t>0,\ x\in\mathbb{R}^d, \\ n(0,\theta,x) = n_0(\theta,x) & \theta>0,\ x\in\mathbb{R}^d, \end{cases} \tag{3.1}$$

where $p(t,x) = \frac{m}{m-1}\rho(t,x)^{m-1}$, with volume density $\rho(t,x) = \int_0^\infty V(\theta) n(t,\theta,x)\, d\theta$. The equation (1.4) is an extension of (3.1) to include an age-dependent death rate and a pressure-dependent aging rate. Realistically, cells do not age linearly, as they might stay in interphase for an extended period of time. In (1.4), where the aging rate depends on pressure, the age of a cell, θ, can be thought of as the "physiological age", i.e., how far it has progressed in its cell cycle.

This model can be extended to include interacting populations of proliferating cells, n, and quiescent cells, q, as has been done in the space-homogeneous case, e.g., [36, 47]. This leads to the following system of equations:

$$\begin{cases} \partial_t n + r_1(p)\partial_\theta n - \operatorname{div}_x(n\nabla_x p) = -\mu(\theta)n - r_1(p)\nu(\theta,p)n + \sigma_1 q - \sigma_2 n, \\ \partial_t q + r_2(p)\partial_\theta q - \operatorname{div}_x(q\nabla_x p) = -\mu(\theta)q - \sigma_1 q + \sigma_2 n, \\ n(t,0,x) = 2(1-\lambda)\displaystyle\int_0^\infty \nu(\theta,p)n(t,\theta,x)\,d\theta, \\ q(t,0,x) = 2\lambda\displaystyle\int_0^\infty \nu(\theta,p)n(t,\theta,x)\,d\theta. \end{cases} \tag{3.2}$$

In the above, r_1 and r_2 are the pressure-dependent aging rates of proliferating and quiescent cells, respectively, λ is the fraction of newborn cells that are quiescent, σ_1, σ_2 are the rates at which cells switch between proliferating and quiescent states, and

$$p(t,x) = \frac{m}{m-1}\rho(t,x)^{m-1}, \qquad \rho(t,x) = \int_0^\infty V(\theta)(n(t,\theta,x) + q(t,\theta,x))\,d\theta.$$

In this chapter, the existence of weak solutions to the simplified equation, (3.1) is proved. Adding the death rate μ and aging rate r do not result in any major difficulties in the proof, and are only omitted for simplicity.

3.1.1 Equation for the Density

To get an equation on $\rho(t,x) = \int_0^\infty V(\theta)n(t,\theta,x)\,d\theta$, (3.1) is multiplied by by $V(\theta)$ and integrated in θ to get:

$$\partial_t\rho - \operatorname{div}(\rho\nabla p) = \int_0^\infty V'(\theta)n(x,\theta,t)\,d\theta + \int_0^\infty \nu(\theta,p)(2V(0)-V(\theta))n(x,\theta,t)\,d\theta. \tag{3.3}$$

The first term on the right hand side describes the expansion of the tumor due to the volume change of individual cells prior to mitosis, while the second term accounts for the growth of the tumor due to cell division. As mentioned in the introduction, mitosis events are often assumed to be volume preserving (i.e. the total volume of the two daughter cells is equal to the volume of the dividing cell), which can be enforced by assuming that $\nu(\theta,\rho)(2V(0)-V(\theta)) = 0$ for all $\theta > 0$. A simple form for the coefficients ν and V describing this situation is as follows:

$$V(\theta) = \begin{cases} V_0 + \alpha\theta & \text{if } \theta \in [0, V_0/\alpha], \\ 2V_0 & \text{if } \theta \geq V_0/\alpha, \end{cases} \tag{3.4}$$

$$\nu(\theta,p) = 0 \quad \text{for } \theta < V_0/\alpha, \tag{3.5}$$

in which case (3.3) reduces to

$$\partial_t\rho - \operatorname{div}(\rho\nabla p) = \alpha\int_0^{V_0/\alpha} n(x,\theta,t)\,d\theta.$$

3.1.2 Outline of the Chapter

In Section 3.2, the existence of weak solutions to (3.1) is proved. In Section 3.3, some images of numerical solutions to the equation are shown. These numerics were not done by me, but by the other authors of [43], and are included to demonstrate the behaviour of the solutions and the importance of age-structured models.

3.2 Existence of Weak Solutions

In this section, we consider solutions to the equation (3.1). We assume that the coefficients satisfy the following general assumptions: Assume ν satisfies (1.6). Assume further that there exist positive constants V_0, C such that

$$V_0 \leq V(\theta) \leq C, \quad 0 \leq V'(\theta) \leq C, \quad \forall \theta \geq 0, \qquad \text{and} \qquad 0 \leq \nu(\theta, p) \leq C, \quad \forall \theta,\ p \geq 0, \tag{3.6}$$

and

$$|V''(\theta)|, |\partial_\theta \nu(\theta, p)| \leq C. \tag{3.7}$$

Assumption (3.6) implies in particular that the volume density $\rho(t, x)$ is bounded above and below by the usual number density $\int_0^\infty n(t, \theta, x)\, d\theta$. Our main result is the following:

Theorem 3.2.1. *For all $m > 2$ and for any initial condition $n_0(\theta, x)$ such that*

$$n_0 \in L^1 \cap L \log L((0,\infty)\times\mathbb{R}^d), \quad (|x|^2+\theta)n_0 \in L^1((0,\infty)\times\mathbb{R}^d), \quad \rho_0 := \int_0^\infty V(\theta) n_0 d\theta \in L^\infty(\mathbb{R}^d), \tag{3.8}$$

there exists a weak solution $n(t,\theta,x)$ *of* (3.1)*. The function* $n(t,\theta,x)$ *is such that*

$$n \in L^\infty(0,T;L^1 \cap L\log L((0,\infty)\times\mathbb{R}^d)), \quad (|x|^2+\theta)n \in L^\infty(0,T;L^1((0,\infty)\times\mathbb{R}^d)),$$

while the density $\rho(t,x)$ *and pressure* $p(t,x) = \frac{m}{m-1}\rho(t,x)^{m-1}$ *are such that*

$$\rho \in L^\infty(0,T;L^1 \cap L^\infty(\mathbb{R}^d)), \qquad \nabla p \in L^2(0,T;L^2(\mathbb{R}^d))$$

and the equation (3.1) *is satisfied in the following distributional sense:*

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^d}\int_0^\infty n(t,\theta,x)\big[-\partial_t\psi(t,\theta,x) - \partial_\theta\psi(t,\theta,x) \\
&\quad + \nabla_x p(t,x)\cdot\nabla_x\psi(t,\theta,x) + \nu(\theta,p(t,x))\psi(t,\theta,x)\big]\,d\theta\,dx\,dt \\
&\quad = \int_{\mathbb{R}^d}\int_0^\infty n_0(\theta,x)\psi(0,\theta,x)\,d\theta\,dx + \int_0^T\int_{\mathbb{R}^d}\psi(t,0,x)2\int_0^\infty \nu(\theta,p(t,x))n(t,\theta,x)\,d\theta\,dx\,dt
\end{aligned} \tag{3.9}$$

for all $\psi \in \mathcal{D}([0,T)\times[0,\infty)\times\mathbb{R}^d)$*. Furthermore, the density* $\rho(t,x)$ *solves* (3.3)*.*

We will prove Theorem 3.2.1 in two steps: first we will show that the following regularized system

$$\begin{cases}
\partial_t n + \partial_\theta n - \operatorname{div}_x(n\nabla_x p) - \varepsilon\Delta_x n = -\nu(\theta,p)\,n & t>0,\theta>0,x\in\mathbb{R}^d, \\
n(t,0,x) = 2\displaystyle\int_0^\infty \nu(\theta,p)\,n(t,\theta,x)\,d\theta & t>0,x\in\mathbb{R}^d, \\
n(0,\theta,x) = n_0(\theta,x) & \theta>0,x\in\mathbb{R}^d,
\end{cases} \tag{3.10}$$

has a solution for all $\varepsilon > 0$ where we recall that

$$p = \frac{m}{m-1}\rho^{m-1}, \qquad \rho(t,x) = \int_0^\infty V(\theta) n(t,\theta,x)\, d\theta.$$

Then we will pass to the limit $\varepsilon \to 0$ to construct a solution of (3.1) and prove our main result.

3.2.1 Existence of a Solution for the Regularized System (3.10)

The existence of a solution for (3.10) will be proved by using a discrete time scheme which splits the transport in θ and the diffusion in x. We initialize the scheme with $n_0(\theta, x)$ and for all $k \geq 0$, we proceed as follows:

Step 1: Given $n_k(\theta, x)$, define $n_{k+\frac{1}{2}}(\theta, x)$ solution of

$$\begin{cases} n_{k+\frac{1}{2}} - n_k + \tau \partial_\theta n_{k+\frac{1}{2}} = -\tau \nu(\theta, p_k)\, n_{k+\frac{1}{2}} & \theta > 0, x \in \mathbb{R}^d, \\ n_{k+\frac{1}{2}}(0,x) = 2\displaystyle\int_0^\infty \nu(\theta, p_k)\, n_k(\theta, x)\, d\theta & x \in \mathbb{R}^d, \end{cases} \tag{3.11}$$

where $p_k(x) = \frac{m}{m-1}\rho_k(x)^{m-1}$ and $\rho_k(x) = \int_0^\infty V(\theta) n_k(\theta, x)\, d\theta$. Note that the space variable x is a parameter in this equation.

Step 2: Then define $n_{k+1}(\theta, x)$ solution of

$$n_{k+1} - n_{k+\frac{1}{2}} - \tau \mathrm{div}_x(n_{k+1} \nabla_x p_{k+1}) - \varepsilon\tau \Delta n_{k+1} = 0 \qquad \theta > 0, x \in \mathbb{R}^d. \tag{3.12}$$

Note that the age variable θ is a parameter in this equation.

This iterative scheme defines $n_k(\theta, x)$ for all $k \in \mathbb{N}$. We then define $n_\tau(t, \theta, x)$ piecewise con-

stant function in t by

$$n_\tau(t,\theta,x) = n_k(\theta,x), \qquad t \in [k\tau,(k+1)\tau) \tag{3.13}$$

and

$$\rho_\tau(t,x) = \int_0^\infty V(\theta) n_\tau(t,\theta,x)\, d\theta.$$

We also define the piecewise linear function $\widetilde{n}_\tau(t,\theta,x)$ which satisfies $\widetilde{n}_\tau(k\tau,\theta,x) = n_k(\theta,x)$:

$$\widetilde{n}_\tau(t,\theta,x) = \left(\frac{(k+1)\tau - t}{\tau}\right) n_k(\theta,x) + \left(\frac{t-k\tau}{\tau}\right) n_{k+1}(\theta,x), \qquad t \in [k\tau,(k+1)\tau) \tag{3.14}$$

and the corresponding volume density

$$\widetilde{\rho}_\tau(t,x) = \int_0^\infty \widetilde{n}_\tau(t,\theta,x)\, d\theta.$$

The well-posedness of the scheme (i.e. the existence and uniqueness of a solution to (3.11) and (3.12)) will be proved below. We then note that by combining (3.11) and (3.12), we find

$$\begin{cases} \frac{n_{k+1}-n_k}{\tau} + \partial_\theta n_{k+\frac{1}{2}} - \operatorname{div}_x(n_{k+1}\nabla_x p_{k+1}) - \varepsilon\Delta n_{k+1} = -\nu(\theta,p_k)\, n_{k+\frac{1}{2}} & \theta > 0, x \in \mathbb{R}^d \\ n_{k+\frac{1}{2}}(0,x) = 2\displaystyle\int_0^\infty \nu(\theta,p_k)\, n_k(\theta,x)\, d\theta & x \in \mathbb{R}^d. \end{cases} \tag{3.15}$$

Our goal is then to derive appropriate bounds on n_k, ρ_k and on the interpolations n_τ, ρ_τ and $\tilde{\rho}_\tau$ in order to pass to the limit $\tau \to 0$ in (3.15)

3.2.2 Iterative Scheme: Existence and Estimates from n_k, ρ_k

The proof of our main result will require several a priori estimates. Throughout these proofs, we use the letter C to denote a constant that might change from line to line, but does not depend on k and τ.

Proposition 3.2.2 (Well-posedness of Step 1)**.** *Given $n_k(\theta, x)$ non-negative function in $L^1(\mathbb{R}^d \times (0,\infty))$, there exists $n_{k+\frac{1}{2}}(\theta, x)$ solution of (3.11). Furthermore, the volume density satisfies*

$$\rho_{k+\frac{1}{2}}(x) \leq \frac{1+C\tau}{1-C\tau}\rho_k(x) \qquad \forall x \in \mathbb{R}^d, \tag{3.16}$$

$$|\rho_{k+\frac{1}{2}}(x) - \rho_k(x)| \leq C\tau\rho_k(x) \qquad \forall x \in \mathbb{R}^d, \tag{3.17}$$

for some constant C independent of k and τ and for all $\tau > 0$ small enough. Finally, we also have

$$\Big|\int_0^\infty \varphi(\theta)[n_{k+\frac{1}{2}}(\theta, x) - n_k(\theta, x)]\, d\theta\Big| \leq C\|\varphi'(\theta)\|_\infty \rho_k(x)\tau \qquad \forall x \in \mathbb{R}^d, \tag{3.18}$$

for any $\varphi \in C^2$.

Proof. We rewrite (3.11) as

$$\begin{cases} (1+\tau\nu(\theta, p_k))n_{k+\frac{1}{2}} + \tau\partial_\theta n_{k+\frac{1}{2}} = n_k & \theta > 0, \\ n_{k+\frac{1}{2}}(0, x) = 2\displaystyle\int_0^\infty \nu(\theta, p_k)\, n_k(\theta, x)\, d\theta. \end{cases}$$

This equation can be solved explicitly: with $A(\theta) = \int_0^\theta 1 + \tau\nu(\theta', p_k)\, d\theta' \geq \theta$, the unique solution of

(3.11) is given by:

$$n_{k+\frac{1}{2}}(\theta,x) = 2e^{-\frac{1}{\tau}A(\theta)}\int_0^\infty \nu(\theta',p_k)\,n_k(\theta',x)\,d\theta' + \frac{1}{\tau}\int_0^\theta e^{-\frac{1}{\tau}[A(\theta)-A(\theta')]}n_k(\theta',x)\,d\theta'.$$

Multiplying the equation (3.11) by $V(\theta)$ and integrating with respect to θ, we get the following relation (using (3.6)):

$$\begin{aligned}\rho_{k+\frac{1}{2}}(x)-\rho_k(x) = \tau\Bigg[&\int_0^\infty \nu(\theta,p_k)2V(0)n_k(\theta,x)\,d\theta - \int_0^\infty \nu(\theta,p_k)V(\theta)n_{k+\frac{1}{2}}(\theta,x)\,d\theta \\ &+ \int_0^\infty V'(\theta)n_{k+\frac{1}{2}}(\theta,x)\,d\theta\Bigg] \qquad (3.19)\\ &\le C\tau[\rho_k+\rho_{k+\frac{1}{2}}], \qquad (3.20)\end{aligned}$$

which yields first (3.16) and then (3.17). Multiplying the equation (3.11) by $\varphi(\theta)$ and integrating with respect to θ also gives (3.18). □

Proposition 3.2.3 (Well posedness of Step 2)**.** *Given $n_{k+\frac{1}{2}}$ non-negative function in $L^1((0,\infty)\times\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d;L^1((0,\infty))$, there exists a non-negative function $n_{k+1}\in L^1((0,\infty)\times\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d;L^1((0,\infty))$ solution of (3.12). Furthermore the density satisfies:*

$$\int_{\mathbb{R}^d}\rho_{k+1}^q\,dx + \frac{4q(q-1)}{(m+q-1)^2}\tau\int_{\mathbb{R}^d}\left|\nabla(\rho_{k+1}^{\frac{m+q-1}{2}})\right|^2 dx + \varepsilon\tau\frac{4(q-1)}{q^2}\int_{\mathbb{R}^d}\left|\nabla\rho_{k+1}^{\frac{q}{2}}\right|^2 dx \le \int_{\mathbb{R}^d}\rho_{k+\frac{1}{2}}^q\,dx \qquad (3.21)$$

for all $q\in[1,\infty)$ and

$$\|\rho_{k+1}\|_{L^\infty(\mathbb{R}^d)} \le \|\rho_{k+\frac{1}{2}}\|_{L^\infty(\mathbb{R}^d)}. \qquad (3.22)$$

Proof. In order to prove that a solution exists, we first notice that by multiplying (3.12) by $V(\theta)$ and

integrating with respect to θ, we get

$$\rho_{k+1} - \tau \mathrm{div}_x(\rho_{k+1}\nabla_x p_{k+1}) - \tau\varepsilon\Delta\rho_{k+1} = \rho_{k+\frac{1}{2}} \qquad x \in \mathbb{R}^d, \tag{3.23}$$

which is a classical nonlinear elliptic equation. Given $\rho_{k+\frac{1}{2}} \in L^1 \cap L^\infty(\mathbb{R}^d)$, this equation has a unique solution with $\rho_{k+1} \in L^1 \cap L^\infty(\mathbb{R}^d)$. We then define $n_{k+1}(\theta, x)$ the solution of (3.12) with p_{k+1} given by the solution of (3.23). This solution exists: indeed, θ is a parameter in this equation, so we only need to solve this equation for a fixed $\theta \geq 0$, and when $\varepsilon > 0$, the solution ρ_{k+1} of (3.23) is smooth, so the advection term $\nabla_x p_{k+1}$ is smooth and the existence of a solution is straightforward.

Since we first found $\rho_{k+1}(x)$ and then $n_{k+1}(\theta, x)$, we need to make sure that

$$\int_0^\infty V(\theta) n_{k+1}(\theta, \cdot)\, d\theta = \rho_{k+1},$$

to show that $n_k(x)$ solves (3.12).

To do so, we note that the function $w(x) = \rho_{k+1}(x) - \int_0^\infty V(\theta) n_{k+1}(\theta, x)\, d\theta$ solves

$$w - \tau \mathrm{div}\,(w\nabla p_{k+1}) - \tau\varepsilon\Delta w = 0.$$

In order to prove that $w = 0$, we take $\beta_\delta(s) = \sqrt{s^2 + \delta^2} - \delta$ and multiply this equation by $\beta'_\delta(w)$ to

find:

$$\begin{aligned}\int_{\mathbb{R}^d} w\beta_\delta'(w)\,dx &= -\tau\varepsilon\int_{\mathbb{R}^d}\beta_\delta''|\nabla w|^2\,dx - \tau\int_{\mathbb{R}^d} w\beta_\delta''(w)\nabla p_{k+1}\cdot\nabla w\,dx\\ &\le -\tau\int_{\mathbb{R}^d}\nabla p_{k+1}\cdot\nabla(w\beta_\delta'(w)-\beta_\delta(w))\,dx\\ &\le \tau\int_{\mathbb{R}^d}\Delta p_{k+1}(w\beta_\delta'(w)-\beta_\delta(w))\,dx.\end{aligned}$$

Since $|s\beta_\delta'(s)-\beta_\delta(s)|\le\delta$ and $s\beta_\delta'(s)\to|s|$ with $s\beta_\delta'(s)\le|s|$, we deduce $\int_{\mathbb{R}^d}|w|\,dx\le 0$, that is, $w=0$ and so

$$\int_0^\infty V(\theta)n_{k+1}(\theta,x)\,d\theta=\rho_{k+1}(x)\qquad\text{a.e. } x\in\mathbb{R}^d,$$

thus proving that $n_k(x)$ is the solution of (3.12).

Next, we derive (3.21). We multiply (3.23) by ρ_{k+1}^{q-1} (for $q>1$) and integrate in x to get:

$$\int_{\mathbb{R}^d}\rho_{k+1}^q\,dx-\tau\int_{\mathbb{R}^d}\rho_{k+1}^{q-1}\mathrm{div}\,(\rho_{k+1}\nabla p_{k+1})\,dx-\tau\varepsilon\int_{\mathbb{R}^d}\rho_{k+1}^{q-1}\Delta\rho_{k+1}\,dx=\int_{\mathbb{R}^d}\rho_{k+\frac12}\rho_{k+1}^{q-1}\,dx.$$

Rearranging, using the fact that $p_{k+1}=\frac{m}{m-1}\rho_{k+1}^{m-1}$, and using the convexity of the function $s\mapsto s^q$ for any $q>1$ and $s\ge0$, we deduce:

$$\begin{aligned}\tau q(q-1)\int_{\mathbb{R}^d}\rho_{k+1}^{m+q-3}|\nabla\rho_{k+1}|^2\,dx+\tau\varepsilon(q-1)\int_{\mathbb{R}^d}\rho_{k+1}^{q-2}|\nabla\rho_{k+1}|^2\,dx &= \int_{\mathbb{R}^d}\rho_{k+1}^{q-1}(\rho_{k+\frac12}-\rho_{k+1})\,dx\\ &\le\int_{\mathbb{R}^d}\frac1q\rho_{k+\frac12}^q-\frac1q\rho_{k+1}^q\,dx.\end{aligned}$$

which implies (3.21). To get (3.22), we can apply the maximum principle to (3.23) or pass to the limit $q\to\infty$ in the inequality $\|\rho_{k+1}\|_{L^q(\mathbb{R}^d)}\le\|\rho_k\|_{L^q(\mathbb{R}^d)}$. □

3.2.3 Estimates for n_τ and ρ_τ

We recall that once n_k and ρ_k have been iteratively constructed, we can define the piecewise constant functions $n_\tau(t,\theta,x)$ and $\rho_\tau(t,x)$ and the piecewise linear interpolations $\widetilde{n}_\tau(t,\theta,x)$, $\widetilde{\rho}_\tau(t,x)$ (see (3.13) and (3.14)). In what follows, we fix $T = K\tau > 0$. We start with the following straightforward consequence of Propositions 3.2.2 and 3.2.3:

Proposition 3.2.4. *There exists C such that if $\tau \leq \frac{1}{C}$, then $\rho_\tau \geq 0$ is bounded in $L^\infty(0,T;L^q(\mathbb{R}^d))$ for all $q \in [1,\infty]$ uniformly with respect to τ.*

Furthermore, for all $q \in (1,\infty)$, there exists C_q such that if $\tau \leq \frac{1}{C_q}$, then

$$\|\nabla \rho_\tau^{\frac{m+q-1}{2}}\|^2_{L^2(0,T;L^2(\mathbb{R}^d))} + \varepsilon \|\nabla \rho_\tau^{\frac{q}{2}}\|^2_{L^2(0,T;L^2(\mathbb{R}^d))} \leq C_q. \tag{3.24}$$

Proof. Combining (3.21) with (3.16) yields

$$\begin{aligned} \int_{\mathbb{R}^d} \rho_{k+1}^q \, dx + \frac{4q(q-1)}{(m+q-1)^2}\tau \int_{\mathbb{R}^d} |\nabla(\rho_{k+1}^{\frac{m+q-1}{2}})|^2 \, dx \\ + \varepsilon\tau \frac{4(q-1)}{q^2} \int_{\mathbb{R}^d} |\nabla \rho_{k+1}^{\frac{q}{2}}|^2 \, dx \leq \left(\frac{1+C\tau}{1-C\tau}\right)^q \int_{\mathbb{R}^d} \rho_k^q \, dx. \end{aligned} \tag{3.25}$$

Similarly, combining (3.22) with (3.16) yields:

$$\|\rho_{k+1}\|_{L^\infty(\mathbb{R}^d)} \leq \frac{1+C\tau}{1-C\tau} \|\rho_k\|_{L^\infty(\mathbb{R}^d)}. \tag{3.26}$$

Using the fact that $\frac{1+x}{1-x} \leq 1 + 4x$ when $x \in [0, 1/2]$, and iterating (3.26), we get

$$\|\rho_K\|_{L^\infty(\mathbb{R}^d)} \leq (1+4C\tau)^K \|\rho_0\|_{L^\infty(\mathbb{R}^d)} \leq e^{4CK\tau} \|\rho_0\|_{L^\infty(\mathbb{R}^d)},$$

as long as $\tau < \frac{1}{2C}$. Similarly, (3.25) gives

$$\|\rho_K\|_{L^q(\mathbb{R}^d)} \leq (1+4C\tau)^K \|\rho_0\|_{L^q(\mathbb{R}^d)} \leq e^{4CK\tau}\|\rho_0\|_{L^q(\mathbb{R}^d)}.$$

These inequalities imply the first statement.

Next, we can rewrite (3.25) as

$$\begin{aligned}
&\int_{\mathbb{R}^d} \rho_{k+1}^q \, dx + \frac{4q(q-1)}{(m+q-1)^2}\tau \int_{\mathbb{R}^d} |\nabla(\rho_{k+1}^{\frac{m+q-1}{2}})|^2 \, dx + \varepsilon\tau \frac{4(q-1)}{q^2} \int_{\mathbb{R}^d} |\nabla \rho_{k+1}^{\frac{q}{2}}|^2 \, dx \\
&\qquad \leq \int_{\mathbb{R}^d} \rho_k^q \, dx + \left[\left(\frac{1+C\tau}{1-C\tau}\right)^q - 1\right] \int_{\mathbb{R}^d} \rho_k^q \, dx \\
&\qquad \leq \int_{\mathbb{R}^d} \rho_k^q \, dx + C_q\tau \int_{\mathbb{R}^d} \rho_k^q \, dx,
\end{aligned}$$

as long as $C_q\tau \leq 1$ for some constant C_q depending on q. Summing up for $k = 0, \dots K$, we deduce:

$$\begin{aligned}
&\frac{4q(q-1)}{(m+q-1)^2} \sum_{k=0}^{K} \tau \int_{\mathbb{R}^d} |\nabla(\rho_{k+1}^{\frac{m+q-1}{2}})|^2 \, dx \\
&\qquad + \varepsilon \frac{4(q-1)}{q^2} \sum_{k=0}^{K} \tau \int_{\mathbb{R}^d} |\nabla \rho_{k+1}^{\frac{q}{2}}|^2 \, dx \leq \left(1 + C_q K\tau e^{4qCK\tau}\right) \|\rho_0\|_{L^q(\mathbb{R}^d)}^q,
\end{aligned}$$

which implies (3.24). □

Next, we prove the following result, which will be used to control the values of n_τ at infinity (in x and θ):

Proposition 3.2.5. *There exists a constant C such that if $\tau \leq \frac{1}{C}$, then*

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^d} \int_0^\infty (|x|^2 + \theta) n_\tau(t,\theta,x) \, d\theta \, dx \leq C, \tag{3.27}$$

with C dependent on T but not on τ.

Proof. Multiplying (3.11) by $(|x|^2+\theta) \geq 0$ and integrating in x and θ gives

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^d} (|x|^2+\theta) n_{k+\frac{1}{2}} \, dx \, d\theta \\
&\quad \leq -\tau \int_0^\infty \int_{\mathbb{R}^d} \partial_\theta n_{k+\frac{1}{2}} (|x|^2+\theta) \, dx \, d\theta + \int_0^\infty \int_{\mathbb{R}^d} (|x|^2+\theta) n_k \, dx \, d\theta \\
&\quad \leq \tau \int_0^\infty \int_{\mathbb{R}^d} n_{k+\frac{1}{2}} \, dx \, d\theta + \tau \int_{\mathbb{R}^d} n_{k+\frac{1}{2}}(0,x)|x|^2 \, dx + \int_0^\infty \int_{\mathbb{R}^d} (|x|^2+\theta) n_k \, dx \, d\theta \\
&\quad \leq \tau \int_{\mathbb{R}^d} \rho_{k+\frac{1}{2}} \, dx + \tau \int_{\mathbb{R}^d} 2 \int_0^\infty \nu(\theta, p_k) n_k(\theta,x)|x|^2 \, d\theta \, dx + \int_0^\infty \int_{\mathbb{R}^d} (|x|^2+\theta) n_k \, dx \, d\theta \\
&\quad \leq \tau \int_{\mathbb{R}^d} \rho_{k+\frac{1}{2}} \, dx + (1+C\tau) \int_0^\infty \int_{\mathbb{R}^d} (|x|^2+\theta) n_k \, dx \, d\theta. \qquad (3.28)
\end{aligned}$$

Similarly, (3.12) yields:

$$\int_0^\infty \int_{\mathbb{R}^d} (|x|^2+\theta) n_{k+1} \, dx \, d\theta = \int_0^\infty \int_{\mathbb{R}^d} (|x|^2+\theta) n_{k+\frac{1}{2}} \, dx \, d\theta + 2d\tau \int_{\mathbb{R}^d} \rho_{k+1}^m + \varepsilon \rho_{k+1} \, dx.$$

Combining these inequalities give:

$$\int_0^\infty \int_{\mathbb{R}^d} (|x|^2+\theta) n_{k+1} \, dx \, d\theta \leq (1+C\tau) \int_0^\infty \int_{\mathbb{R}^d} (|x|^2+\theta) n_k \, dx \, d\theta + C\tau \int_{\mathbb{R}^d} \rho_{k+\frac{1}{2}} + \rho_{k+1}^m + \varepsilon \rho_{k+1} \, dx,$$

and the bounds from Proposition 3.2.4 (together with (3.16) to control $\rho_{k+\frac{1}{2}}$) implies

$$\int_0^\infty \int_{\mathbb{R}^d} (|x|^2+\theta) n_{k+1} \, dx \, d\theta \leq (1+C\tau) \int_0^\infty \int_{\mathbb{R}^d} (|x|^2+\theta) n_k \, dx \, d\theta + C(T)\tau,$$

for all $k \leq K$ (with $K\tau = T$), which implies

$$\int_0^\infty \int_{\mathbb{R}^d} (|x|^2 + \theta) n_k \, dx \, d\theta \leq (1 + C\tau)^k \int_0^\infty \int_{\mathbb{R}^d} (|x|^2 + \theta) n_0 \, dx + C(T)(1 + C\tau)^{k+1} \leq C(1 + C\tau)^k, \tag{3.29}$$

and the result follows.

□

Finally, we note that while ρ_τ is bounded in $L^\infty(0, T; L^p(\mathbb{R}^d))$ for $p \in [1, \infty]$, we only have n_τ bounded in $L^\infty(0, T; L^1((0, \infty) \times \mathbb{R}^d)$. In order to show that we do not get a measure (in θ) in the limit $\tau \to 0$, we will make use of the following result:

Proposition 3.2.6. *The piecewise constant function $n_\tau \geq 0$ is bounded in $L^\infty(0, T; L^1((0, \infty) \times \mathbb{R}^d))$ and satisfies:*

$$\sup_{t \in (0,T)} \int_{\mathbb{R}^d} \int_0^\infty n_\tau(t, \theta, x) \log_+ n_\tau(t, \theta, x) \, d\theta \, dx \leq C, \tag{3.30}$$

for some constant depending only on T and the initial condition.

This proposition follows from the following Lemma which we prove below:

Lemma 3.2.7. *There exists a constant C such that*

$$\begin{aligned} &\int_{\mathbb{R}^d} \int_0^\infty n_{k+1}(\theta, x) \log(n_{k+1}(\theta, x)) V(\theta) \, d\theta \, dx \\ &\qquad + \frac{4\tau}{m} \int_{\mathbb{R}^d} |\nabla (\rho_{k+1})^{m/2}(x)|^2 \, dx \\ &\qquad\qquad \leq (1 + C\tau) \int_{\mathbb{R}^d} \int_0^\infty n_k(\theta, x) \log(n_k(\theta, x)) V(\theta) \, d\theta \, dx + C\tau(1 + C\tau)^k. \end{aligned} \tag{3.31}$$

Note that inequality (3.31) also yields a bound on $\nabla \rho_\tau^{m/2}$. We did not include this estimate in Proposition 3.2.6 since we will not be using it.

Proof of Proposition 3.2.6. Iterating (3.31), we get:

$$\int_{\mathbb{R}^d}\int_0^\infty n_k(\theta,x)\log(n_k(\theta,x))V(\theta)\,d\theta\,dx \le (1+C\tau)^k\int_{\mathbb{R}^d}\int_0^\infty n_0(\theta,x)\log(n_0(\theta,x))V(\theta)\,d\theta\,dx + Ck\tau(1+C\tau)^k.$$

Recalling that $k\tau = T$, we deduce

$$\sup_{k=1,\dots,K}\int_{\mathbb{R}^d}\int_0^\infty n_k(\theta,x)\log(n_k(\theta,x))V(\theta)\,d\theta\,dx \le e^{CT}\int_{\mathbb{R}^d}\int_0^\infty n_0(\theta,x)\log(n_0(\theta,x))V(\theta)\,d\theta\,dx + CTe^{CT}.$$

Next, we recall the classical inequality $|s\log s|\chi_{0\le s\le 1} \le s\omega + Ce^{-\omega/2}$ (for all ω) with $\omega = \theta + |x|^2$. We deduce:

$$\int_{\mathbb{R}^d}\int_0^\infty n_\tau\log_+(n_\tau)V(\theta)\,d\theta\,dx \le \int_{\mathbb{R}^d}\int_0^\infty n_\tau\log(n_\tau)V(\theta)\,d\theta\,dx + \int_{\mathbb{R}^d}\int_0^\infty \left((\theta+|x|^2)n_\tau + Ce^{-(\theta+|x|^2)/2}\right)V(\theta)\,d\theta\,dx,$$

and so (3.27) implies

$$\int_{\mathbb{R}^d}\int_0^\infty n_\tau\log_+(n_\tau)V(\theta)\,d\theta\,dx \le \int_{\mathbb{R}^d}\int_0^\infty n_\tau\log(n_\tau)V(\theta)\,d\theta\,dx + C(T) \tag{3.32}$$

for all $t\in[0,T]$ and the result follows.

□

Proof of Lemma 3.2.7. Multiplying (3.12) by $V(\theta)(\log(n_{k+1})+1)$ and integrating with respect to θ

and x yields:

$$\int_{\mathbb{R}^d}\int_0^\infty (n_{k+1}-n_{k+\frac{1}{2}})(\log(n_{k+1})+1)V(\theta)\,d\theta\,dx+\tau\int_{\mathbb{R}^d}\nabla p_{k+1}\cdot\nabla\rho_{k+1}\,dx=0,$$

and so the convexity of $s\mapsto s\log s$ and the definition of $p(\rho)$ gives

$$\int_{\mathbb{R}^d}\int_0^\infty n_{k+1}\log(n_{k+1})V(\theta)\,d\theta\,dx+\frac{4\tau}{m}\int_{\mathbb{R}^d}|\nabla(\rho_{k+1})^{m/2}|^2\,dx\le\int_{\mathbb{R}^d}\int_0^\infty n_{k+\frac{1}{2}}\log(n_{k+\frac{1}{2}})\,V(\theta)d\theta\,dx. \tag{3.33}$$

Similarly, multiplying (3.11) by $V(\theta)(\log(n_{k+\frac{1}{2}})+1)$, we get

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_0^\infty (n_{k+\frac{1}{2}}-n_k)(\log(n_{k+\frac{1}{2}})+1)V(\theta)\,d\theta\,dx\\
&=-\tau\int_{\mathbb{R}^d}\int_0^\infty \partial_\theta(n_{k+\frac{1}{2}}\log(n_{k+\frac{1}{2}}))V(\theta)\,d\theta\,dx-\tau\int_{\mathbb{R}^d}\int_0^\infty \nu(\theta,p_k)\,n_{k+\frac{1}{2}}(\log(n_{k+\frac{1}{2}})+1)V(\theta)\,d\theta\,dx\\
&\le\tau\int_{\mathbb{R}^d} n_{k+\frac{1}{2}}(0,x)\log(n_{k+\frac{1}{2}}(0,x))V(0)\,dx+\tau\int_{\mathbb{R}^d}\int_0^\infty (n_{k+\frac{1}{2}}\log(n_{k+\frac{1}{2}}))V'(\theta)\,d\theta\,dx\\
&\quad-\tau\int_{\mathbb{R}^d}\int_0^\infty \nu(\theta,p_k)\,n_{k+\frac{1}{2}}\log(n_{k+\frac{1}{2}})V(\theta)\,d\theta\,dx.
\end{aligned}$$

Using the convexity of $s\mapsto s\log s$ again, we deduce

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_0^\infty n_{k+\frac{1}{2}}\log(n_{k+\frac{1}{2}})V(\theta)\,d\theta\,dx-\int_{\mathbb{R}^d}\int_0^\infty n_k\log(n_k)V(\theta)\,d\theta\,dx\\
&\qquad\le\tau\int_{\mathbb{R}^d} n_{k+\frac{1}{2}}(0,x)\log_+(n_{k+\frac{1}{2}}(0,x))V(0)\,dx\\
&\qquad\quad+\tau\int_{\mathbb{R}^d}\int_0^\infty (n_{k+\frac{1}{2}}\log(n_{k+\frac{1}{2}}))[V'(\theta)-\nu(\theta,p_k)V(\theta)]\,d\theta\,dx.
\end{aligned}$$

The first term can be bounded using the monotonicity of $s\mapsto s\log_+ s$ and the fact that $n_{k+\frac{1}{2}}(0,x)\le C\rho_k(x)$ which is bounded in $L^1\cap L^\infty(\mathbb{R}^d)$ (by a constant depending only on T). For the second term,

we use the fact that $|V'(\theta) - \nu(\theta, p_k)V(\theta)| \leq CV(\theta)$ and get:

$$\int_{\mathbb{R}^d}\int_0^\infty n_{k+\frac{1}{2}} \log(n_{k+\frac{1}{2}})V(\theta)\, d\theta\, dx - \int_{\mathbb{R}^d}\int_0^\infty n_k \log(n_k)V(\theta)\, d\theta\, dx$$
$$\leq C\tau + C\tau \int_{\mathbb{R}^d}\int_0^\infty \left| n_{k+\frac{1}{2}} \log(n_{k+\frac{1}{2}}))\right| V(\theta)\, d\theta\, dx,$$

and we can proceed as in (3.32) (we combine (3.28) and (3.29) to get the required bound on the moments of $n_{k+\frac{1}{2}}$) to get:

$$\int_{\mathbb{R}^d}\int_0^\infty n_{k+\frac{1}{2}} \log(n_{k+\frac{1}{2}})V(\theta)\, d\theta\, dx - \int_{\mathbb{R}^d}\int_0^\infty n_k \log(n_k)V(\theta)\, d\theta\, dx$$
$$\leq C\tau + C\tau \int_{\mathbb{R}^d}\int_0^\infty (n_{k+\frac{1}{2}} \log(n_{k+\frac{1}{2}}))V(\theta)\, d\theta\, dx + C\tau(1+C\tau)^k, \tag{3.34}$$

which implies (assuming τ is small enough so that $C\tau < 1/2$ and $\frac{1}{1-C\tau} \leq (1+C\tau)$):

$$\int_{\mathbb{R}^d}\int_0^\infty n_{k+\frac{1}{2}} \log(n_{k+\frac{1}{2}})V(\theta)\, d\theta\, dx \leq (1+C\tau)\int_{\mathbb{R}^d}\int_0^\infty n_k \log(n_k)V(\theta)\, d\theta\, dx + C\tau(1+C\tau)^k. \tag{3.35}$$

We combine this inequality with (3.33) to get (3.31). □

3.2.4 Weak Convergence of n_τ and Strong Convergence of ρ_τ as $\tau \to 0$

Since n_τ is bounded in $L^\infty(0,T;L^1((0,\infty)\times\mathbb{R}^d)$, Propositions 3.2.5 implies that it converges up to a subsequence for the narrow topology to $n \in L^\infty(0,T;\mathcal{M}((0,\infty)\times\mathbb{R}^d)$. From now on, we

thus fix a subsequence (still denoted n_τ for simplicity) such that

$$\int_0^T \int_{\mathbb{R}^d} \int_0^\infty n_\tau(t,\theta,x)\psi(t,\theta,x)\,d\theta\,dx\,dt \to \int_0^T \int_{\mathbb{R}^d} \int_0^\infty n(t,\theta,x)\psi(t,\theta,x)\,d\theta\,dx\,dt \tag{3.36}$$

for all $\psi \in C_b^0([0,T]\times(0,\infty)\times\mathbb{R}^d)$. Furthermore 3.2.6 and Dunford Pettis theorem imply that $n \in L^\infty(0,T;L^1((0,\infty)\times\mathbb{R}^d)$ and that the limit (3.36) holds for function $\psi \in L^\infty([0,T]\times(0,\infty)\times\mathbb{R}^d)$. If we take $\psi(t,\theta,x) = V(\theta)\phi(t,x)$ in (3.36), we deduce (along the same subsequence)

$$\rho_\tau(t,x) \rightharpoonup \rho(t,x) = \int_0^\infty n(t,\theta,x)V(\theta)\,d\theta.$$

In order to pass to the limit $\tau \to 0$ and derive (3.10), we need to establish the strong convergence of ρ_τ and $\nabla\rho_\tau^{m-1}$. A key ingredient for that is the following equation for ρ_k, which we obtain by combining (3.19) and (3.23):

$$\rho_{k+1} - \rho_k - \tau \mathrm{div}_x(\rho_{k+1}\nabla_x p_{k+1}) - \tau\varepsilon\Delta\rho_{k+1} = \tau F_k \tag{3.37}$$

where

$$F_k(x) = \int_0^\infty \nu(\theta,p_k)2V(0)n_k(\theta,x)\,d\theta - \int_0^\infty \nu(\theta,p_k)V(\theta)n_{k+\frac{1}{2}}(\theta,x)\,d\theta + \int_0^\infty V'(\theta)n_{k+\frac{1}{2}}(\theta,x)\,d\theta.$$

We have

$$|F_k(x)| \le C\rho_k(x) + C\rho_{k+\frac{1}{2}}(x),$$

and so F_τ is bounded in $L^\infty(0,T;L^1\cap L^\infty(\mathbb{R}^d))$. We will deduce the following proposition:

Proposition 3.2.8. *Up to another subsequence, $\rho_\tau(t,x)$ converges strongly to $\rho(t,x)$ in $L^2(0,T;L^2(\mathbb{R}^d))$ and almost everywhere in $[0,T]\times\mathbb{R}^d$.*

Proof. First, we will prove the convergence of the piecewise linear function $\tilde{\rho}_\tau(t,x)$: The bound (3.24) with $q=2$ gives

$$\varepsilon\|\nabla\rho_\tau\|_{L^2(0,T;L^2(\mathbb{R}^d))}\leq C.$$

(This bound is the main reason we added the regularization $\varepsilon\Delta n$). Since $\tilde{\rho}_\tau$ is the linear interpolation with the same value as ρ_τ, it is easy to check that we also have

$$\varepsilon\|\nabla\tilde{\rho}_\tau\|_{L^2(0,T;L^2(\mathbb{R}^d)}\leq C.$$

Furthermore, we have

$$\partial_t\tilde{\rho}_\tau=\frac{\rho_{k+1}-\rho_k}{\tau}=\operatorname{div}_x(\rho_{k+1}\nabla_x p_{k+1})+\varepsilon\Delta\rho_{k+1}+\tau F_k\qquad\text{for } t\in(k\tau,(k+1)\tau)$$

and so

$$\|\partial_t\tilde{\rho}_\tau\|_{L^2(0,T;H^{-1}(\mathbb{R}^d))}\leq C.$$

Together with the bound on $\int_{\mathbb{R}}^d|x|^2\tilde{\rho}_\tau\,dx$ (see Proposition 3.2.5), we have all we need to apply Aubin-Lions and show that $\{\tilde{\rho}_\tau\}_{\tau>0}$ is pre-compact in $L^2(0,T;L^2(\mathbb{R}^d))$: There exists $\rho\in L^2(0,T;H^1(\mathbb{R}^d))$ such that $\tilde{\rho}_\tau$ converges (up to a subsequence) to ρ strongly in $L^2(0,T;L^2(\mathbb{R}^d))$.

Finally, note that for $t \in [k\tau, (k+1)\tau)$, we have

$$\begin{aligned}\rho_\tau(t,x) - \tilde{\rho}_\tau(t,x) &= \frac{t-k\tau}{\tau}(\rho_{k+1} - \rho_k) \\ &= (t-k\tau)[\mathrm{div}_x(\rho_{k+1}\nabla_x p_{k+1}) + \varepsilon\Delta\rho_{k+1} + F_k,\end{aligned}$$

and so

$$\|\rho_\tau - \tilde{\rho}_\tau\|_{L^2(0,T;H^{-1}(\mathbb{R}^d))} \leq C\tau.$$

It follows that ρ_τ converges to ρ in $L^2(0,T;H^{-1}(\mathbb{R}^d))$ and since ρ_τ is bounded in $L^2(0,T;H^1(\mathbb{R}^d))$, it also converges strongly in $L^2(0,T;L^2(\mathbb{R}^d))$.

□

Finally, since ρ_τ is bounded in $L^\infty(0,T,L^\infty(\mathbb{R}^d))$, we deduce the convergence of the pressure $p_\tau = \frac{m}{m-1}\rho_\tau^{m-1}$:

Corollary 3.2.8.1. *Along the same subsequence as above, ρ_τ converges to ρ strongly in $L^p((0,T)\times\mathbb{R}^d)$ and so $p_\tau(t,x) = \frac{m}{m-1}\rho_\tau(t,x)^{m-1}$ converges to $p(t,x) = \frac{m}{m-1}\rho(t,x)^{m-1}$ strongly in L^p and almost everywhere in $(0,T)\times\mathbb{R}^d$.*

3.2.5 Strong Convergence of ∇p_τ as $\tau \to 0$

The strong convergence of p_τ allows us to pass to the limit in most terms in the weak formulation of (3.10) except for the term $n_\tau\nabla p_\tau$. Since n_τ only converges weakly, we have to show that ∇p_τ converges strongly. This is proved by using the ρ_τ equation (3.37) and its limit:

First we define the piecewise constant function

$$F_\tau(t,x) = F_k(x) \qquad t \in [k\tau, (k+1)\tau).$$

We note that (3.18) implies

$$F_k(x) = \int_0^\infty \nu(\theta, p_k) 2V(0) n_k(\theta, x)\, d\theta - \int_0^\infty \nu(\theta, p_k) V(\theta) n_k(\theta, x)\, d\theta + \int_0^\infty V'(\theta) n_k(\theta, x)\, d\theta + \mathcal{O}(C\rho_k(x)\tau)$$

with C depending on $\|V\|_\infty$, $\|V'\|_\infty$, $\|\partial_\theta \nu\|_\infty$ and $\|V''\|_\infty$. We can then write:

$$F_\tau(t,x) = \int_0^\infty \nu(\theta, p_\tau) 2V(0) n_\tau(t,\theta,x)\, d\theta - \int_0^\infty \nu(\theta, p_\tau) V(\theta) n_\tau(t,\theta,x)\, d\theta + \int_0^\infty V'(\theta) n_\tau(t,\theta,x)\, d\theta + \mathcal{O}(\tau).$$

The weak convergence of n_τ and the strong convergence of p_τ imply that F_τ converges weakly in $L^2(0,T;L^2(\mathbb{R}^d))$ to

$$F(x) = \int_0^\infty \nu(\theta, p) 2V(0) n(\theta, x)\, d\theta - \int_0^\infty \nu(\theta, p) V(\theta) n(\theta, x)\, d\theta + \int_0^\infty V'(\theta) n(\theta, x)\, d\theta.$$

Furthermore, we can rewrite equation (3.37) as

$$\partial_t \tilde{\rho}_\tau - \Delta \rho_\tau^m - \varepsilon \Delta \rho_\tau = F_\tau$$

and passing to the limit $\tau \to 0$ (since $\rho_\tau^m \to \rho^m$) in the sense of distribution, we see that the limit

$\rho(x,t)$ solves

$$\partial_t \rho - \Delta \rho^m - \varepsilon \Delta \rho = F \tag{3.38}$$

in $\mathcal{D}'((0,T)\times\mathbb{R}^d)$. This will be the key to proving the following proposition:

Proposition 3.2.9. *Up to another subsequence, $\nabla \rho_\tau^{m-1}$ converges strongly to $\nabla \rho^{m-1}$.*

Proof. Multiplying (3.37) by ρ_k^{m-2}, and using the convexity of $s \mapsto s^m - 1$, we get

$$\begin{aligned}
&\int \frac{\rho_{k+1}^{m-1}}{m-1} - \frac{\rho_k^{m-1}}{m-1}\,dx + \tau \int \rho_{k+1} \nabla_x p_{k+1} \nabla \rho_{k+1}^{m-2}\,dx \\
&\qquad \le \int (\rho_{k+1} - \rho_k)\,\rho_{k+1}^{m-2}\,dx + \tau \int \rho_{k+1} \nabla_x p_{k+1} \nabla \rho_{k+1}^{m-2}\,dx \\
&\qquad \le \tau \int F_k \rho_{k+1}^{m-2}\,dx,
\end{aligned}$$

which yields

$$\int \frac{\rho_{K+1}^{m-1}}{m-1} - \frac{\rho_0^{m-1}}{m-1}\,dx + \frac{m-2}{m} \sum_{k=0}^{K} \tau \int |\nabla_x p_{k+1}|^2\,dx \le \sum_{k=0}^{K} \tau \int F_k \rho_{k+1}^{m-2}\,dx,$$

that is

$$\int \frac{\rho_\tau(T)^{m-1}}{m-1} - \frac{\rho_0^{m-1}}{m-1}\,dx + \frac{m-2}{m} \int_0^T \int |\nabla_x p_\tau|^2\,dx \le \int_0^T \int F_\tau \rho_\tau^{m-2}\,dx + \mathcal{O}(\tau),$$

so that

$$\liminf_{\tau\to 0} \frac{m-2}{m} \int_0^T \int |\nabla_x p_\tau|^2\,dx \le -\int \frac{\rho(T)^{m-1}}{m-1}\,dx + \int \frac{\rho_0^{m-1}}{m-1}\,dx + \int_0^T \int F \rho^{m-2}\,dx.$$

We now use the limiting equation for ρ, (3.38). Multiplying that equation by ρ^{m-2} and integrating in

x and t gives

$$\int \frac{\rho(T)^{m-1}}{m-1} - \frac{\rho_0^{m-1}}{m-1}\,dx + \frac{m-2}{m}\int_0^T \int |\nabla_x p|^2\,dx = \int_0^T \int F\rho^{m-2}\,dx.$$

Together, these two relations imply

$$\liminf_{\tau\to 0} \frac{m-2}{m}\int_0^T \int |\nabla_x p_\tau|^2\,dx \le \frac{m-2}{m}\int_0^T \int |\nabla_x p|^2\,dx,$$

which in turn means that $\nabla_x p_\tau$ converges strongly to $\nabla_x p$ in $L^2(0,T;L^2(\mathbb{R}^d))$. □

3.2.6 The Limit $\tau \to 0$: Solutions of (3.10)

Given a smooth test function $\varphi(t,\theta,x)$, compactly supported in $[0,\infty)\times[0,\infty)\times\mathbb{R}^d$, we multiply (3.15) by $\varphi_k = \varphi(k\tau,\theta,x)$, integrate with respect to x and θ and sum over k:

$$\begin{aligned}&\sum_{k=0}^{\infty}\tau\int_{\mathbb{R}^d}\int_0^\infty \frac{1}{\tau}(n_{k+1}\varphi_k - n_k\varphi_k) - n_{k+\frac12}\partial_\theta\varphi_k + n_{k+1}\nabla_x p_{k+1}\cdot\nabla_x\varphi_k - \varepsilon n_{k+1}\Delta_x\varphi_k\,d\theta\,dx\\ &= -\sum_{k=0}^{\infty}\tau\int_{\mathbb{R}^d} 2\int_0^\infty \nu(\theta,p_k)\,n_k(\theta,x)\,d\theta\varphi_k(0)\,dx - \sum_{k=0}^{\infty}\tau\int_{\mathbb{R}^d}\int_0^\infty \nu(\theta,p_k)\,n_{k+\frac12}\varphi_k\,d\theta\,dx,\end{aligned}$$

and writing $\sum_{k=0}^{\infty} n_{k+1}\varphi_k - n_k\varphi_k = \sum_{k=1}^{\infty} n_k(\varphi_{k-1}-\varphi_k) - n_0\varphi_0$, we deduce:

$$\begin{aligned}&\int_0^\infty\int_{\mathbb{R}^d}\int_0^\infty -n_\tau\partial_t\varphi - \bar n_\tau\partial_\theta\varphi + n_\tau\nabla_x p_\tau\cdot\nabla_x\varphi\,d\theta - \varepsilon n_\tau\Delta_x\varphi\,dx\,dt\\ &\quad= -\int_0^\infty\int_{\mathbb{R}^d} 2\int_0^\infty \nu(\theta,p_\tau)\,n_\tau(t,\theta,x)\,d\theta\varphi(t,0,x)\,dx\,dt\\ &\qquad -\int_0^\infty\int_{\mathbb{R}^d}\int_0^\infty \nu(\theta,p_\tau)\,\bar n_\tau\varphi\,d\theta\,dx\,dt + \mathcal{O}(\tau),\end{aligned}$$

where $\bar{n}_\tau$ is the piecewise constant function equal to $n_{k+\frac{1}{2}}$ on the interval $[k\tau, (k+1)\tau)$.

We note (see (3.28) and (3.35)) that $n_{k+\frac{1}{2}}$ satisfies the same bounds on n_k so that $\bar{n}_\tau(t,\theta,x)$ converges weakly in $L^1(0,T;L^1((0,\infty)\times\mathbb{R}^d)$ as well (up to another subsequence) and (3.18) implies that this limit must be $n(t,\theta,x)$ (and therefore that the original subsequence converges).

We can thus pass to the limit in all the terms. For the diffusion term, we have to write

$$\int_0^\infty \int_{\mathbb{R}^d} \int_0^\infty n_\tau \nabla_x p_\tau \cdot \nabla_x \varphi \, d\theta \, dx \, dt = \int_0^\infty \int_{\mathbb{R}^d} \int_0^\infty n_\tau \nabla_x \varphi \, d\theta \cdot \nabla_x p_\tau \, dx \, dt,$$

and note that $\int_0^\infty n_\tau \nabla_x \varphi \, d\theta$ converges weakly in $L^2(0,T;L^2(\mathbb{R}^d))$ to $\int_0^\infty n \nabla_x \varphi \, d\theta$ in order to pass to the limit in that term (using the fact that $\nabla_x p_\tau$ converges strongly).

We deduce that $n(t,\theta,x)$ is a weak solution of (3.10) in the sense that

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}^d} \int_0^\infty -n\partial_t\varphi - n\partial_\theta\varphi + n\nabla_x p \cdot \nabla_x \varphi \, d\theta - \varepsilon n \Delta_x \varphi \, dx \, dt \\ &= -\int_0^\infty \int_{\mathbb{R}^d} 2 \int_0^\infty \nu(\theta,p)\, n(t,\theta,x)\, d\theta \varphi(t,0,x) \, dx \, dt - \int_0^\infty \int_{\mathbb{R}^d} \int_0^\infty \nu(\theta,p)\, \bar{n} \varphi \, d\theta \, dx \, dt. \end{aligned} \tag{3.39}$$

3.2.7 The Limit $\varepsilon \to 0$: Proof of Theorem 3.2.1

From now on, we denote by $n_\varepsilon(t,\theta,x)$ the solution of (3.10) constructed in the previous section and by $\rho_\varepsilon(t,x)$ the corresponding volume density. We saw that n_ε, ρ_ε satisfies (3.39) and we now need to pass to the limit $\varepsilon \to 0$.

Much of the arguments are similar to the limit $\tau \to 0$. Indeed, Proposition 3.2.4 implies that ρ_ε is bounded in $L^\infty(0,T;L^q(\mathbb{R}^d))$ for all $q \in [1,\infty]$ and by passing to the limit $\tau \to 0$ in (3.24), (3.27)

and (3.30), we easily get the following bounds (uniform in ε):

$$\|\nabla \rho_\varepsilon^{\frac{m+q-1}{2}}\|^2_{L^2(0,T;L^2(\mathbb{R}^d))} \leq C_q \qquad \forall q > 1, \tag{3.40}$$

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^d} \int_0^\infty (|x|^2 + \theta) n_\varepsilon(t,\theta,x)\, d\theta\, dx \leq C, \tag{3.41}$$

$$\int_{\mathbb{R}^d} \int_0^\infty |n_\varepsilon \log n_\varepsilon|\, dx\, d\theta \leq C. \tag{3.42}$$

In particular, we can proceed as Section 3.2.4 to prove the existence of a subsequence $\varepsilon \to 0$ along which

$$\int_0^T \int_{\mathbb{R}^d} \int_0^\infty n_\varepsilon(t,\theta,x)\psi(t,\theta,x)\, d\theta\, dx\, dt \to \int_0^T \int_{\mathbb{R}^d} \int_0^\infty n(t,\theta,x)\psi(t,\theta,x)\, d\theta\, dx\, dt$$

for all $\psi \in L^\infty([0,T]\times(0,\infty)\times\mathbb{R}^d)$ and

$$\rho_\varepsilon(t,x) \rightharpoonup \rho(t,x) = \int_0^\infty n(t,\theta,x)V(\theta)\, d\theta.$$

The main difference is in the way we prove the strong convergence of ρ_ε, since in the proof of Proposition 3.2.8, we used the bound on $\nabla\rho_\tau$ which was not uniform in ε. Instead, we will use (3.40) to prove the strong convergence of ρ_ε^s for s large enough (the argument below does not appear to work with the time approximation, which is the reason we had to introduce the regularization $\varepsilon > 0$ in the system):

Proposition 3.2.10. *The sequence ρ_ε^{m+2} is precompact in $L^2(0,T;L^2(\mathbb{R}^d))$.*

We immediately note that this implies the existence of a subsequence along which ρ_ε^{m+2} con-

verges strongly in L^2 and almost everywhere. This implies the convergence almost everywhere of ρ_ε and the uniform bounds in $L^\infty(0,T;L^q(\mathbb{R}^d))$ implies that ρ_ε converges strongly in $L^q(0,T;L^q(\mathbb{R}^d))$ for all $q \in [1,\infty)$.

We can then prove the strong convergence of $\nabla p_\varepsilon = \frac{m}{m-1}\nabla \rho_\varepsilon^{m-1}$ as in Proposition 3.2.9 and pass to the limit in (3.39) to get (3.9) and complete the proof of Theorem 3.2.1.

Proof of Proposition 3.2.10. Taking $q = m+5$ in (3.40) implies that ρ_ε^{m+2} is bounded in $L^2(0,T;H^1(\mathbb{R}^d))$. Furthermore, we recall that ρ_ε solves (see (3.38))

$$\partial_t \rho_\varepsilon - \Delta \rho_\varepsilon^m - \varepsilon \Delta \rho_\varepsilon = F_\varepsilon,$$

with F_ε bounded in $L^2(0,T;L^2(\mathbb{R}^d))$ and so

$$\frac{1}{m+2}\partial_t \rho_\varepsilon^{m+2} = \rho_\varepsilon^{m+1}\Delta \rho_\varepsilon^m + \varepsilon \rho_\varepsilon^{m+1}\Delta \rho_\varepsilon + F_\varepsilon \rho_\varepsilon^{m-1}.$$

To show that the right-hand-side is bounded, we rewrite

$$\rho_\varepsilon^{m+1}\Delta \rho_\varepsilon^m = \operatorname{div}\left(\rho_\varepsilon^{m+1}\nabla \rho_\varepsilon^m\right) - \nabla \rho_\varepsilon^{m+1}\cdot \nabla \rho_\varepsilon^m$$

and

$$\rho_\varepsilon^{m+1}\Delta \rho_\varepsilon = \operatorname{div}\left(\rho_\varepsilon^{m+1}\nabla \rho_\varepsilon\right) - \nabla \rho_\varepsilon^{m+1}\cdot \nabla \rho_\varepsilon = \frac{1}{m+2}\Delta \rho_\varepsilon^{m+2} - \frac{2(m+1)}{(m+2)^2}\left|\nabla \rho_\varepsilon^{\frac{m+2}{2}}\right|^2.$$

Since (3.40) implies that $\nabla \rho_\varepsilon^s$ is bounded in $L^2(0,T;L^2(\mathbb{R}^d))$ for all $s > \frac{m}{2}$, we see that these two terms are bounded in $L^2(0,T;H^{-1}(\mathbb{R}^d)) + L^1(0,T;L^1(\mathbb{R}^d))$.

We have thus shown that ρ_ε^{m+2} is bounded in $L^2(0,T;H^1(\mathbb{R}^d))$ and that $\partial_t \rho_\varepsilon^{m+2}$ is bounded in $L^2(0,T;H^{-1}(\mathbb{R}^d)) + L^1(0,T;L^1(\mathbb{R}^d))$. Together with (3.41), this is enough to apply Aubin-Lions lemma to prove that ρ_ε^{m+2} is precompact in $L^2(0,T;L^2(\mathbb{R}^d))$. □

3.3 Numerical Results

In this section, several numerical results for the equation (3.1) are presented. These are not my work, but are the work of my collaborators in [43], and are included to give the reader a visual of the age-distribution throughout tumor growth.

In Figure 3.1, the total density $\rho(t,x) = \int_0^\infty V(\theta) n(t,\theta,x) d\theta$ is pictured at time steps $t = 0, 10, 20, 40,$ and 50. This figure demonstrates that solutions with asymmetric initial data will converge to a disk shape as t increases, as is expected for porous medium equations [69]. In addition, it demonstrates that the cell density is highest near the center of the tumor, and decreases to the edge.

In Figure 3.2, the density of cells of age $\theta = 0$ ("new borns") throughout the tumor is shown at $t = 15, 30, 40,$ and 50. It can be seen that most of the cell division occurs in an annulus near, but not at, the tumor boundary. This is consistent with experimental observations, e.g., [65, 71]. This demonstrates that very little cell division occurs in the center of the tumor. Further, the presence of "surfing cells" can be seen at the tumor boundary. These are cells that are not proliferating but are pushed outward by the proliferation of internal cells, another process that has been experimentally observed [24].

In Figure 3.3, the average age of the cells,

$$\Theta(t,x) := \frac{\int_0^\infty \theta n(t,\theta,x)\, d\theta}{\int_0^\infty n(t,\theta,x)\, d\theta}, \tag{3.43}$$

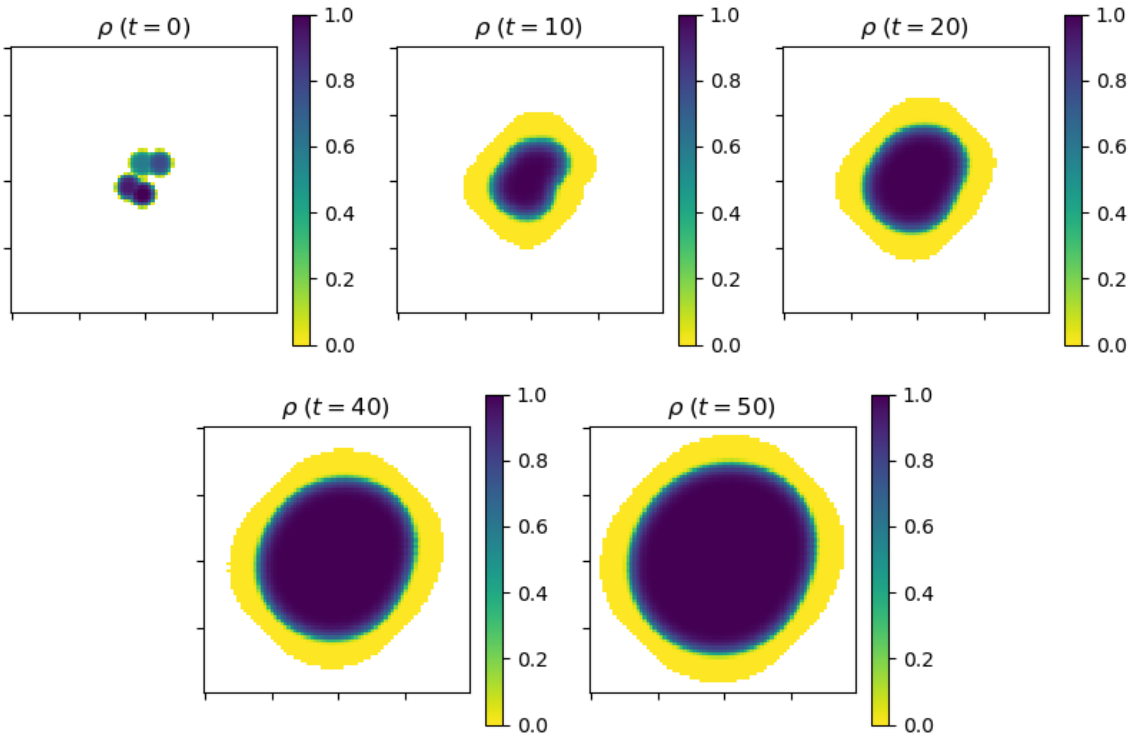


Figure 3.1: The density $x \mapsto \rho(t, x)$ for a 2-dimensional tumor at various times. The initial data (top left plot) is not radially symmetric, but the shape of the tumor converges to a disk as t increases.

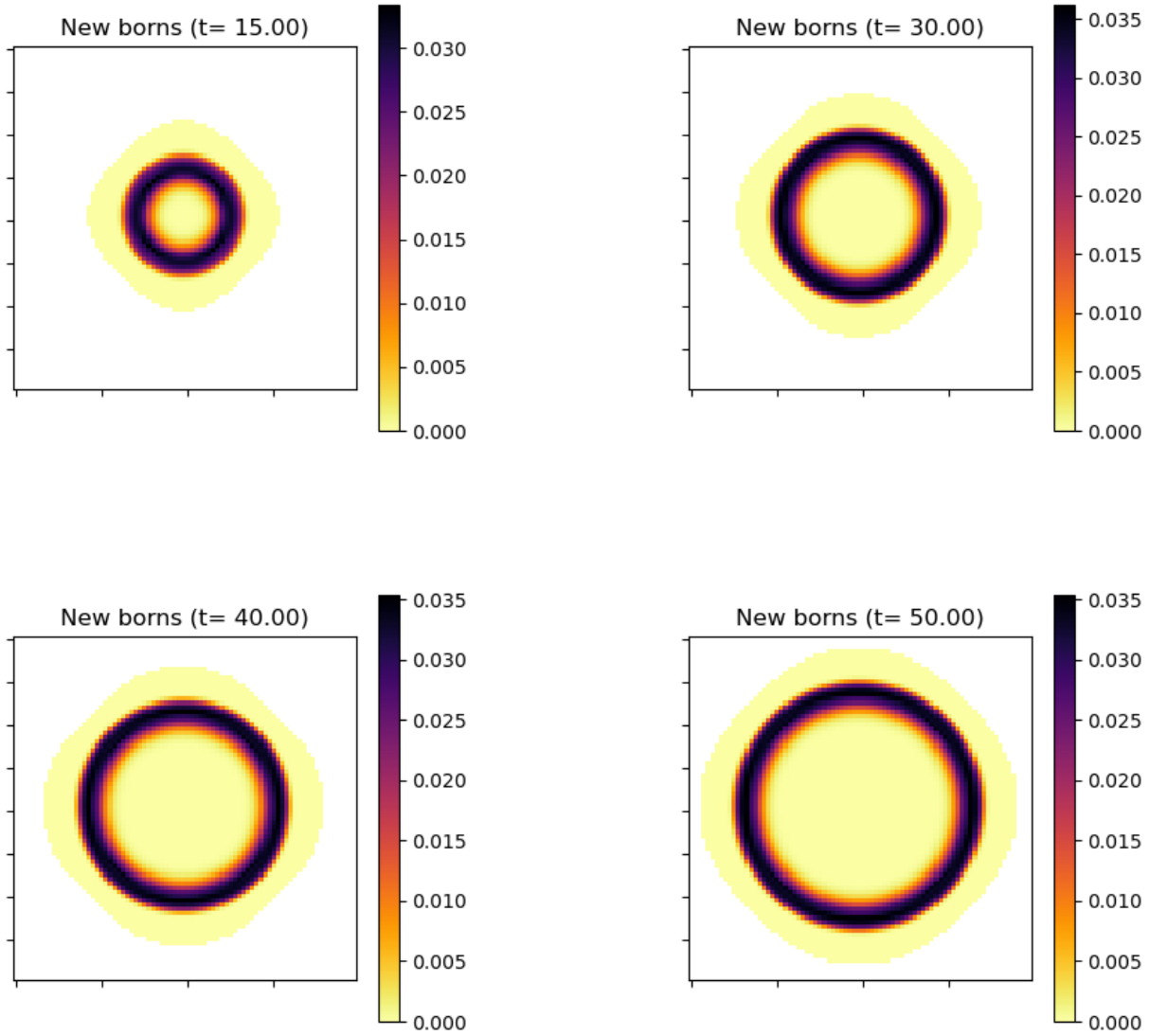


Figure 3.2: Number of mitosis events inside the tumor at various stages. The highest concentration of mitosis events can be found in an annulus which has constant width and stays at some positive distance from the edge of the tumor.

is shown at times $t = 0, 20, 40,$ and 50. One can observe that the average age in the center of the tumor is highest, and the lowest average age is in an annulus near the boundary, consistent with Figure 3.2.

In Figure 3.4, the density of cells of maximum age is shown throughout the tumor at times $t = 20, 30, 40,$ and 50. It can be seen that the cells of maximum age accumulate in the center of the tumor. As time goes on, this "necrotic core" becomes larger. This formation of a necrotic core is consistent with experimental findings, e.g., [12, 73].

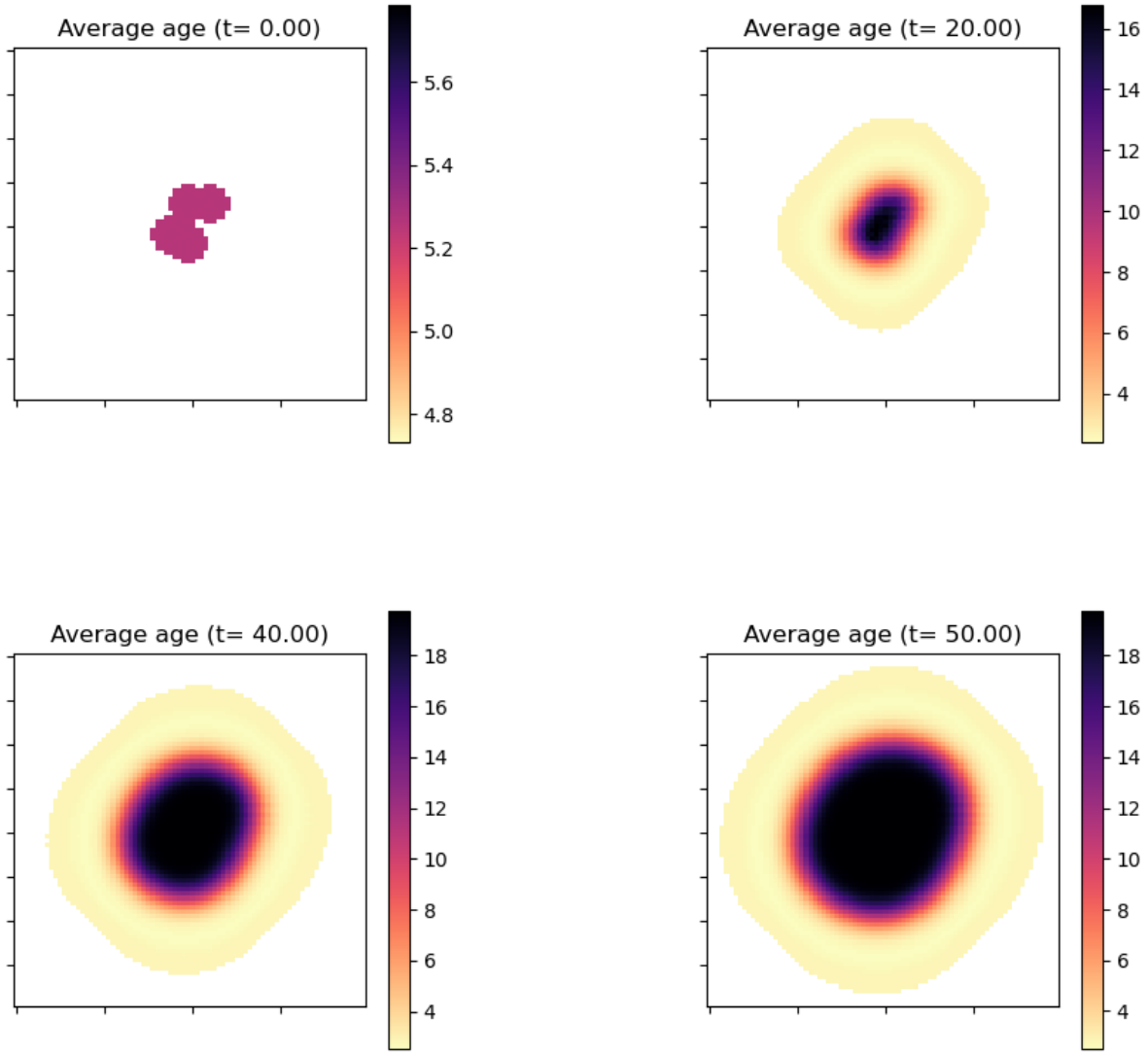


Figure 3.3: Average age $\Theta(t, x)$ of the cells across the tumor at various stages. The lowest average age (light yellow) is found in a ring that has constant width and stays at some positive distance from the edge of the tumor.

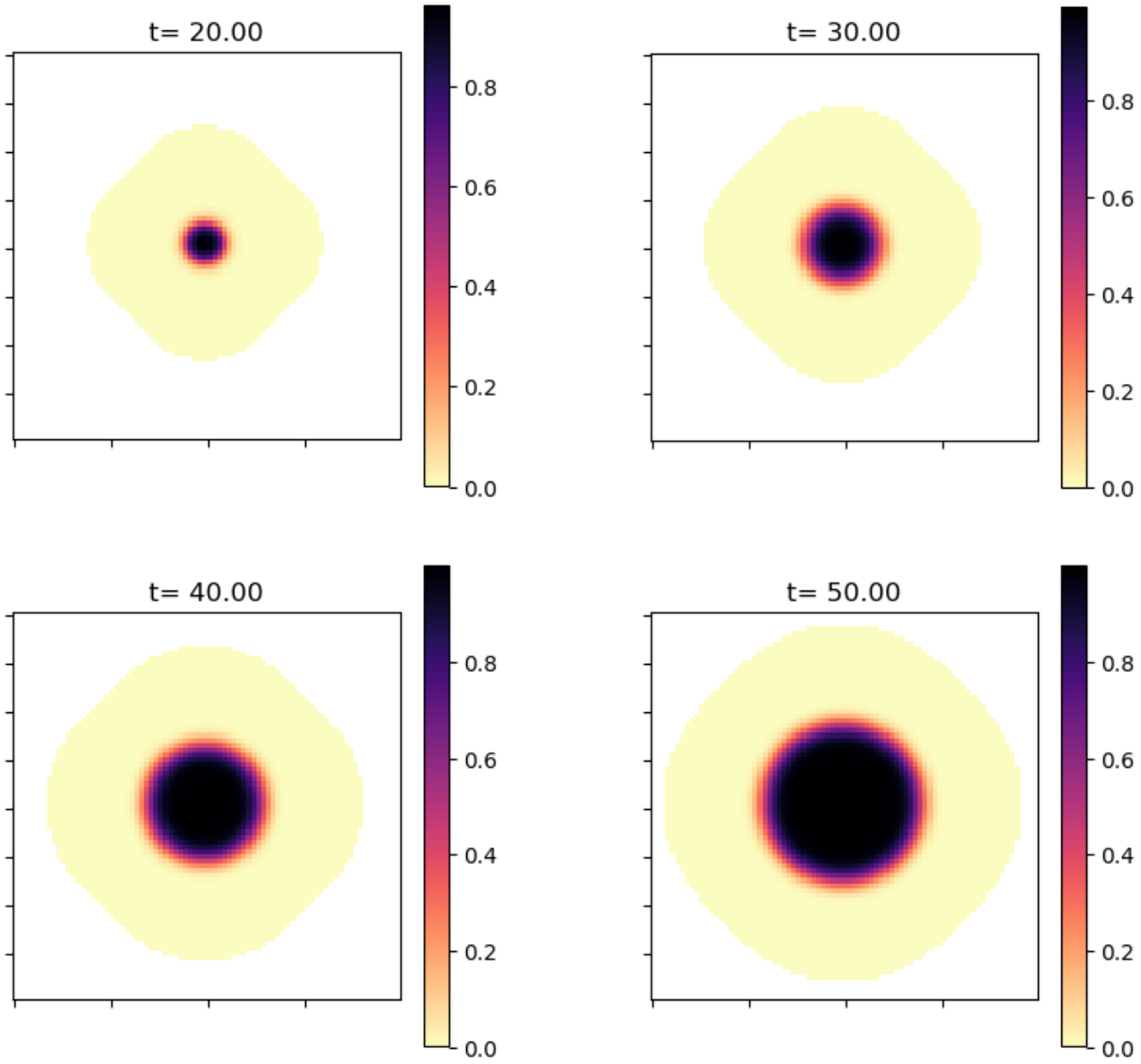


Figure 3.4: The spatial distribution of the cells of the maximum numerical age $\theta_{max} = 20$ shows an accumulation of old cells in the core of the tumor (necrotic core).

Chapter 4: Incompressible Limit

In this chapter, the convergence of solutions of (1.4) to a limit as $m \to \infty$ is proved. It is shown that the limit solves a Hele-Shaw free boundary problem and satisfies a complementarity formula. The content in this chapter can be found in [70].

Recall the definition of pressure, $p(t,x) = \frac{m}{m-1}\rho(t,x)^{m-1}$, where $\rho(t,x) = \int_0^\infty V(\theta)n(t,\theta,x)\,d\theta$. Heuristically, it can be observed that as $m \to \infty$, p blows up when $\rho > 1$, and approaches zero when $\rho < 1$. Therefore, the limit $m \to \infty$ enforces the condition that $\rho \leq 1$. This means that volume density cannot be greater than 1, meaning that cells cannot be compressed, hence the name, "incompressible limit".

The convergence of (1.4) to an incompressible limit was motivated by the convergence of the structured model (2.8) to an incompressible limit, proved by David in [18]. In this chapter, we follow the outline of David's proof. The main point where our proof diverges from David's is in establishing the strong convergence of ∇p_m. In the age-structured case, the addition of the transport term in θ means that θ cannot be treated as a parameter, as y could. Another difference is that $\theta \in [0,\infty)$, an unbounded domain, rather than $y \in [0,1]$. These differences cause a challenge in proving strong convergence. The key to overcoming this obstacle is to show that n_m is compactly supported in x and θ, uniformly in m. Additionally, this work extends David's by incorporating an age-dependent volume, meaning ρ depends not only on the number of cells ($\rho(t,x) = \int_0^1 n(t,y,x)\,dy$ in (2.8)), but

instead on the volume they take up ($\rho(x) = \int_0^\infty V(\theta)n(t,\theta,x)\,d\theta$ in (1.4)).

4.1 Definitions and Notations

In this chapter, we consider (1.4) with the same assumptions that were made in Chapter 1, precisely, assume (1.5) and (1.6). We begin by presenting the equation that describes the evolution of the density, ρ. Multiplying (1.4) by $V(\theta)$ and integrating in θ yields the following equation for ρ:

$$\begin{cases} \partial_t \rho - \text{div}(\rho \nabla p) = \int_0^\infty n\Big(r(p)[V'(\theta) + (2V(0) - V(\theta))\nu(\theta,p)] - \mu(\theta)V(\theta)\Big)\,d\theta, \\ \rho(0,x) = \rho_0(x) = \int_0^\infty V(\theta)n_0(\theta,x)\,d\theta. \end{cases}$$

For simplicity of notation, we define

$$F(\theta,p) := r(p)[V'(\theta) + (2V(0) - V(\theta))\nu(\theta,p)] - \mu(\theta)V(\theta), \tag{4.1}$$

so the equation for ρ becomes

$$\begin{cases} \partial_t \rho - \text{div}(\rho \nabla p) = \int_0^\infty n(t,\theta,x)F(\theta,p)\,d\theta & t \in [0,T],\ x \in \mathbb{R}^d \\ \rho(0,x) = \rho_0(x) = \int_0^\infty V(\theta)n_0(\theta,x)\,d\theta & x \in \mathbb{R}^d. \end{cases} \tag{4.2}$$

From assumptions (1.5) and (1.6) on r and ν, the function $F(\cdot,p) \leq 0$ for $p \geq p_M$. We highlight two particular cases of $F(\theta,p)$ that have different biological meanings and result in very different dynamics.

Case 1: Consider cells that have constant volume, that is, $V(\theta) = V_0$ for all $\theta \geq 0$. Then,

$$F(\theta, p) = V_0 r(p)\nu(\theta, p) - \mu(\theta)V_0,$$

and it is apparent that tumor growth is driven primarily by cell division. If we assume further that $\nu(\theta, p) = \nu(p)$, and $\mu(\theta) = \mu_0$, the equation on ρ exactly recovers the original porous media tumor growth model, (1.1), with $F(p) = r(p)\nu(p)V_0 - \mu_0 V_0$.

Case 2: Consider cells with age-dependent volume, and assume that mitosis is volume preserving, that is, the total size of two daughter cells is equal to the size of the parent cell. This can be done by imposing that

$$\nu(\theta, p)(2V(0) - V(\theta)) = 0,$$

for all $\theta \geq 0$. In this case,

$$F(\theta, p) = r(p)V'(\theta) - \mu(\theta)V(\theta).$$

It can be seen that tumor growth is now driven by the increase in volume of individual cells prior to cell division, as opposed to being driven by cell division, as it was in Case 1.

4.1.1 Outline of the Chapter

In Section 4.2, we will discuss assumptions and state the main results of the chapter. In Section 4.3, we prove the convergence to an incompressible limit and the complementarity formula for the limit.

4.2 Assumptions and Main Results

Given $m > 2$, we consider n_m, a weak solution of the following boundary value problem (a precise definition of a weak solution is given by (3.9)).

$$\begin{cases} \partial_t n_m + r(p_m)\partial_\theta n_m - \text{div}(n_m \nabla p_m) = -r(p_m)\nu(\theta, p_m)n_m - \mu(\theta)n_m & t \in (0,T],\ \theta \in (0,\infty),\ x \in \mathbb{R}^d, \\ n_m(t,0,x) = 2\int_0^\infty \nu(\theta, p_m)n_m(t,\theta,x)\,d\theta & t \in (0,T],\ x \in \mathbb{R}^d, \\ p_m(t,x) = \frac{m}{m-1}\rho_m(t,x)^{m-1}, & t \in [0,T], x \in \mathbb{R}^d, \\ n_m(0,\theta,x) = n_{0,m}(\theta,x) & \theta \in (0,\infty),\ x \in \mathbb{R}^d. \end{cases} \tag{4.3}$$

We make the following assumptions on the various parameters of the problem:

$$0 \le \mu(\theta) \le \mu_{max}, \quad |\mu'(\theta)| \le C, \tag{4.4}$$

$$0 \le r(p) \le r_{max}, \quad r'(p) \le 0, \quad r(p_M) = 0, \tag{4.5}$$

$$\nu \in C^1([0,\infty) \times [0,p_M]),\ 0 \le \nu(\theta,p) \le C,\ \partial_p \nu(\theta,p) \le 0,\ |\partial_\theta \nu(\theta,p)| \le C,\ \nu(\theta,p_M) = 0, \tag{4.6}$$

and

$$V \in C^2([0,\infty)), \quad 0 \le V'(\theta) \le C, \quad |V''| \le C, \quad \text{and} \quad V_0 := V(0) \le V(\theta) \le C, \tag{4.7}$$

for constants $\mu_{max}, r_{max}, p_M, V_0 > 0$, and various constants all called $C > 0$. Multiplying (4.3) by $V(\theta)$ and integrating in θ yields that $\rho_m(t,x) = \int_0^\infty V(\theta) n_m(t,\theta,x)\, d\theta$ solves

$$\begin{cases} \partial_t \rho_m - \text{div}(\rho_m \nabla p_m) = \int_0^\infty n_m F(\theta, p_m)\, d\theta & t \in [0,T],\ x \in \mathbb{R}^d \\ \rho_m(0,x) = \rho_{0,m}(x) = \int_0^\infty V(\theta) n_{0,m}(\theta,x)\, d\theta & x \in \mathbb{R}^d, \end{cases} \tag{4.8}$$

where F is defined by (4.1). We make the following assumptions on the initial data. Assume that

$$n_{0,m} \in L^1 \cap L\log L((0,\infty)\times\mathbb{R}^d), \quad (|x|^2+\theta) n_{0,m} \in L^1((0,\infty)\times\mathbb{R}^d), \quad 0 \le \rho_{0,m} \le (\frac{m-1}{m} p_M)^{\frac{1}{m-1}}, \tag{4.9}$$

where the bounds are independent of m. Assume there exists $n_0 \in L^1([0,\infty)\times\mathbb{R}^d)$ such that

$$||n_{0,m} - n_0||_{L^1([0,\infty)\times\mathbb{R}^d)} \xrightarrow[m\to\infty]{} 0. \tag{4.10}$$

In this chapter, we show that, up to a subsequence, for all $T > 0$, as $m \to \infty$, n_m, ρ_m, and p_m converge to limits n_∞, ρ_∞, and p_∞, solving the following equation:

$$\begin{cases} \partial_t n_\infty + r(p_\infty)\partial_\theta n_\infty - \text{div}(n_\infty \nabla p_\infty) = -r(p_\infty)\nu(\theta, p_\infty) n_\infty - \mu(\theta) n_\infty, \\ n_\infty(t,0,x) = 2\int_0^\infty \nu(\theta, p_\infty) n_\infty(t,\theta,x)\, d\theta, \\ p_\infty \in P_\infty(\rho_\infty), \\ n_\infty(0,\theta,x) = n_0(\theta,x), \end{cases} \tag{4.11}$$

in $\mathcal{D}'([0,T]\times(0,\infty)\times\mathbb{R}^d)$, where

$$\rho_\infty(t,x)=\int_0^\infty V(\theta)n_\infty(t,\theta,x)d\theta, \tag{4.12}$$

and $P_\infty(\rho)$ denotes the Hele-Shaw graph,

$$P_\infty(\rho)=\begin{cases}0, & 0\le\rho<1,\\ [0,\infty), & \rho=1.\end{cases} \tag{4.13}$$

Further, ρ_∞ solves

$$\begin{cases}\partial_t\rho_\infty-\Delta p_\infty=\int_0^\infty n_\infty F(\theta,p_\infty))\,d\theta & \text{in } \mathcal{D}'([0,T]\times\mathbb{R}^d),\\ \rho_\infty(0,x)=\rho_0(x)=\int_0^\infty V(\theta)n_0(\theta,x)\,d\theta.\end{cases} \tag{4.14}$$

We now state the two main convergence results.

Theorem 4.2.1 (Weak Free-Boundary Limit)**.** *Let (n_m,p_m,ρ_m) be a weak solution of (4.3), under the assumptions (4.4), (4.5), (4.6), and (4.7). Assume that the initial data satisfies (4.9)-(4.10). Further, assume that $n_{0,m}$ is compactly supported in x and θ, uniformly in m. Then, up to a subsequence, as $m\to\infty$, for all $T>0$, p_m converges strongly in $L^2([0,T]\times\mathbb{R}^d)$, ρ_m converges weakly in $L^\infty(0,T;L^q(\mathbb{R}^d))$ and n_m converges weakly-* in $L^\infty(0,T;L^q(\mathbb{R}^d;L^1([0,\infty))))$, for all $q\in[1,\infty]$, to limits $p_\infty,\rho_\infty,n_\infty$ satisfying (4.11).*

Remark 4.2.2. *In Theorem 4.2.1, n_m converges weakly-* in $L^\infty(0,T;L^q(\mathbb{R}^d);L^1([0,\infty)))$. This*

means that for any $\varphi \in L^1(0,T;L^{q'}(\mathbb{R}^d;L^\infty([0,\infty)))$*, up to a subsequence,*

$$\int_0^T \int_{\mathbb{R}^d} \int_0^\infty n_m(t,\theta,x)\varphi(t,\theta,x)d\theta dx dt \to \int_0^T \int_{\mathbb{R}^d} \int_0^\infty n_\infty(t,\theta,x)\varphi(t,\theta,x)d\theta dx dt,$$

for some $n_\infty \in L^\infty(0,T;L^q(\mathbb{R}^d;L^1([0,\infty)))$.

Theorem 4.2.3 (Complementarity Formula)**.** *The limits* p_∞*,* n_∞ *satisfy the relation*

$$p_\infty\Big(\Delta p_\infty + \int_0^\infty n_\infty\big(F(\theta,p_\infty) - \mu(\theta)V(\theta)\big)\,d\theta\Big) = 0 \text{ in } \mathcal{D}'([0,T]\times\mathbb{R}^d). \tag{4.15}$$

These results will be proved in Section 4.3 using several lemmas that establish the convergence of p_m, ρ_m, and n_m.

4.3 Incompressible Limit Proof

In order to prove Theorems 4.2.1 and 4.2.3, we will prove several lemmas which establish a priori bounds on ρ_m, p_m and n_m.

We begin by remarking that p_m solves

$$\partial_t p_m = (m-1)p_m\Delta p_m + |\nabla p_m|^2 + m(\frac{m-1}{m}p_m)^{\frac{m-2}{m-1}}\int_0^\infty n_m F(\theta,p_m)\,d\theta, \tag{4.16}$$

and by recalling the definition of F, (4.1), and the assumptions on V and ν, it holds that $F(\theta,p) \leq Cr(p)$, for a constant C independent of m. Then,

$$\partial_t p_m \leq (m-1)p_m\Delta p_m + |\nabla p_m|^2 + C(m-1)p_m r(p_m). \tag{4.17}$$

Lemma 4.3.1. *Assume that* $\rho_{0,m} \geq 0$ *and* $\frac{m}{m-1}\rho_{0,m}^{m-1} \leq p_M$. *Then* ρ_m *and* p_m *are bounded in* $L^\infty([0,T] \times \mathbb{R}^d)$ *uniformly in* m, *that is, for every* $m \geq 3$,

$$0 \leq p_m \leq p_M \text{ and } 0 \leq \rho_m \leq (\frac{m-1}{m} p_M)^{1/(m-1)} := \rho_{m,M} \leq \max\{p_M, 1\}.$$

Proof. From the assumptions made on V, r, μ, and ν, we have that $|F(\theta, p_m)| \leq C$ for a constant C only depending on the bounds of V, r, μ, and ν. From (4.16) and (4.17), the assumptions on $\rho_{0,m}$, and the fact that $r(p_M) = 0$, 0 is a subsolution and p_M is a supersolution of (4.16). By the comparison principle, the desired bound on p_m holds, i.e., $0 \leq p_m \leq p_M$. The desired bound on ρ_m then follows from the definition of p_m. □

Lemma 4.3.2. *Assuming* $\rho_{0,m} \in L^1(\mathbb{R}^d)$, *bounded uniformly in* m, *then* ρ_m, p_m *are bounded in* $L^\infty(0,T; L^1(\mathbb{R}^d))$ *uniformly in* m, *and* n_m *is bounded in* $L^\infty(0,T; L^1([0,\infty) \times \mathbb{R}^d))$ *uniformly in* m.

Proof. Integrating (4.8) with respect to x,

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^d} \rho_m(t,x)\,dx &= \int_{\mathbb{R}^d}\int_0^\infty n_m F(\theta, p_m)\,d\theta dx \\ &\leq C\int_{\mathbb{R}^d} \rho_m(t,x)\,dx, \end{aligned}$$

for a C depending only on the bounds of V, V', ν, and r, from the definition of F. By Gronwall's inequality,

$$\int_{\mathbb{R}^d} \rho_m(t,x)\,dx \leq e^{Ct}\int_{\mathbb{R}^d} \rho_{0,m}(x)\,dx,$$

and so

$$\sup_{t\in[0,T]}\int_{\mathbb{R}^d}\rho_m(t,x)\,dxdt\le C(T).$$

For the bound on p_m,

$$p_m=\rho_m\frac{m}{m-1}(\frac{m-1}{m}p_m)^{\frac{m-2}{m-1}},$$

so

$$\int_{\mathbb{R}^d}p_m\,dx\le\frac{m}{m-1}(\frac{m-1}{m})^{\frac{m-2}{m-1}}p_M^{\frac{m-2}{m-1}}\int_{\mathbb{R}^d}\rho_m(t,x)\,dx\le\tilde{C}e^{Ct}\int_{\mathbb{R}^d}\rho_{0,m}\,dx,$$

for $\tilde{C}$ depending only on p_M, and so,

$$\sup_{t\in[0,T]}\int_{\mathbb{R}^d}p_m(t,x)\,dx\le C(T).$$

For the bound on n_m,

$$\begin{aligned}\sup_{t\in[0,T]}\int_0^\infty\int_{\mathbb{R}^d}n_m(t,\theta,x)\,dxd\theta&\le\sup_{t\in[0,T]}\frac{1}{V_0}\int_0^\infty\int_{\mathbb{R}^d}V(\theta)n_m(t,\theta,x)dxd\theta\\&=\sup_{t\in[0,T]}\frac{1}{V_0}\int_{\mathbb{R}^d}\rho_m(t,x)\,dx\\&\le C(T).\end{aligned}$$

□

Lemma 4.3.3. *For any $q\ge1$, ρ_m is bounded in $L^\infty(0,T;L^q(\mathbb{R}^d))$.*

Proof. By multiplying (4.8) by ρ^{q-1} and integrating in x yields,

$$\frac{d}{dt}\int_{\mathbb{R}^d} \rho_m^q \, dx + mq(q-1)\int_{\mathbb{R}^d} \rho_m^{m+q-3}|\nabla \rho_m|^2 \, dx \leq Cq \int_{\mathbb{R}^d} \rho_m^q \, dx. \tag{4.18}$$

By Gronwall's inequality and the non-negativity of ρ_m,

$$\int_{\mathbb{R}^d} \rho_m^q \, dx \leq e^{Cqt} \int_{\mathbb{R}^d} \rho_{0,m}^q \, dx.$$

□

Remark 4.3.4. *It is clear from the proofs of the previous two lemmas that ρ_m, p_m, and ρ_m^q, $q \geq 1$, are also bounded in $L^1([0,T] \times \mathbb{R}^d)$.*

We now state two lemmas that demonstrate the compact support of solutions in both space x and age θ. Although we are working on unbounded domains $\mathbb{R}^d$ in space and $[0, \infty)$ in age, after showing that solutions have compact support, we can restrict our argument to bounded domains. The compact support in space is due to the assumptions that $\rho_{0,m}$ is compactly supported uniformly in m, and the finite speed of propagation of porous medium equations [69]. The compact support in age is due to the assumption that $n_{0,m}$ is compactly supported in θ uniformly in m.

Lemma 4.3.5 ([61], Lemma 2.6)**.** *Assume that $\rho_{0,m}$ is compactly supported uniformly in m, that is, the supports of $\rho_{0,m}$ for all m are contained in a ball B_R. Then for all $T > 0$, there exists a constant C_T, depending only on T and R, such that for any $|x| \geq C_T$, $\rho_m(t,x) = 0$ for all $m > 2$ and for all $t \in [0,T]$.*

Proof. From (4.8), we have that

$$\partial_t \rho_m - \operatorname{div}(\rho_m \nabla p_m) \leq C\rho_m,$$

where C is a constant depending only on the bounds on ν, r, V and V'. It holds that ρ_m is a subsolution of the equation considered in [61], so by Lemma 2.6 in that work, the solutions ρ_m are compactly supported in space, uniformly in m. □

Lemma 4.3.6. *Assume that $n_{0,m}$ is compactly supported in θ uniformly in m, that is, there exists $\theta_{in} > 0$ such that for all $m > 2$, $n_{0,m}(\theta, x) = 0$ for all $\theta \geq \theta_{in}$. Then, for all $T > 0$, $n_m(t, \theta, x) = 0$ for any $\theta \geq \theta_{in} + r_{max}T$, for all $m > 2$ and for all $t \in [0, T]$, $x \in \mathbb{R}^d$.*

Proof. We have that for all m,

$$\int_{\mathbb{R}^d} \int_{\theta_{in}}^{\infty} n_{0,m}(\theta, x) d\theta dx = 0. \tag{4.19}$$

Consider

$$\tilde{\rho}_m(t, x) := \int_{\theta_{in} + r_{max}t}^{\infty} n_m(t, \theta, x)\, d\theta,$$

where r_{max} is the maximum value of $r(p)$ (see (4.5)). Then $\int_{\mathbb{R}^d} \tilde{\rho}_m(0, x)\, dx = 0$ from (4.19). Integrat-

ing in x and taking the derivative in t,

$$\begin{aligned}
\partial_t \int_{\mathbb{R}^d} \tilde{\rho}_m dx &= \partial_t \int_{\mathbb{R}^d} \int_{\theta_{in}+r_{max}t}^{\infty} n_m(t,\theta,x) d\theta dx \\
&= \int_{\mathbb{R}^d} \int_{\theta_{in}+r_{max}t}^{\infty} \partial_t n_m(t,\theta,x) d\theta dx - \int_{\mathbb{R}^d} n_m(t,\theta_{in}+r_{max}t,x) r_{max} dx \\
&\leq \int_{\mathbb{R}^d} \int_{\theta_{in}+r_{max}t}^{\infty} -r(p_m)\partial_\theta n_m + \operatorname{div}(n_m \nabla p_m) d\theta - n_m(t,\theta_{in}+r_{max}t,x) r_{max} dx \\
&= \int_{\mathbb{R}^d} (r(p_m) - r_{max}) n_m(t,\theta_{in}+r_{max}t,x) + \operatorname{div}(\tilde{\rho}_m \nabla p_m) dx \\
&\leq 0,
\end{aligned}$$

meaning that $\int_{\mathbb{R}^d} \tilde{\rho}_m(t,x)dx = 0$ for all $t \geq 0$. It follows directly that

$$\int_{\mathbb{R}^d} \int_{\theta_{in}+r_{max}T}^{\infty} n_m(t,x,\theta)\, d\theta\, dx = 0,$$

for all $t \geq 0$ and for all m. □

Next, we will prove some bounds of ρ_m and p_m that will provide weak convergence.

Lemma 4.3.7. *Assume $p_{0,m} \in L^1(\mathbb{R}^d)$. For $m \geq 3$, p_m is bounded in $L^2(0,T;H^1(\mathbb{R}^d))$ and $\partial_t \rho_m$ is bounded in $L^2(0,T;H^{-1}(\mathbb{R}^d))$, uniformly in m. Further, by defining $v_m = \rho_m^m$, ∇v_m is bounded in $L^2([0,T]\times\mathbb{R}^d)$ uniformly in m.*

Proof. Taking $q = m-1$ in (4.18), yields

$$\frac{d}{dt}\int_{\mathbb{R}^d} p_m\, dx + (m-2)\int_{\mathbb{R}^d} |\nabla p_m|^2\, dx \leq C(m-1)\int_{\mathbb{R}^d} p_m\, dx.$$

Then, integrating from 0 to T,

$$\int_{\mathbb{R}^d} p_m(T,x)\,dx - \int_{\mathbb{R}^d} p_m(0,x)\,dx + (m-2)\int_0^T \int_{\mathbb{R}^d} |\nabla p_m|^2\,dxdt \leq C(m-1)\int_0^T \int_{\mathbb{R}^d} p_m\,dxdt,$$

and by the non-negativity of p_m,

$$\int_0^T \int_{\mathbb{R}^d} |\nabla p_m|^2\,dxdt \leq C\frac{m-1}{m-2}\int_0^T \int_{\mathbb{R}^d} p_m\,dxdt + \frac{1}{m-2}\int_{\mathbb{R}^d} p_m(0,x)\,dx,$$

so from the L^1 bounds on p_m and $p_{0,m}$, it holds that $p_m \in L^2(0,T;H^1(\mathbb{R}^d))$. It follows that $\partial_t \rho_m \in L^2(0,T;H^{-1}(\mathbb{R}^d))$. Finally,

$$\nabla v_m = \rho_m \nabla p_m \in L^2([0,T]\times\mathbb{R}^d).$$

□

Remark 4.3.8. *Recall that*

$$\int_0^\infty |n_m(t,\theta,x)|d\theta \leq \frac{1}{V_0}\rho_m(t,x),$$

so from the bounds on ρ_m, it holds that $\int_0^\infty n_m(\cdot,\theta,\cdot)d\theta$ is bounded uniformly in m in $L^\infty(0,T;L^q(\mathbb{R}^d))$ for all $q \in [1,\infty]$. It follows that n_m converges weakly- in $L^\infty(0,T;L^q(\mathbb{R}^d;\mathcal{M}([0,\infty))))$, for all $q \in [1,\infty]$, where $\mathcal{M}$ is the space of radon measures, in the following sense:*

For any $\varphi \in L^1(0,T;L^{q'}(\mathbb{R}^d;C^0([0,\infty))))$,

$$\int_0^T \int_{\mathbb{R}^d} \int_0^\infty n_m(t,\theta,x)\varphi(t,\theta,x)d\theta dxdt \to \int_0^T \int_{\mathbb{R}^d} \int_0^\infty n_\infty(t,\theta,x)\varphi(t,\theta,x)d\theta dxdt,$$

for $n_\infty \in L^\infty(0,T;L^q(\mathbb{R}^d;\mathcal{M}([0,\infty))))$.

We will show in Lemma 4.3.14 that a stronger result holds, and in fact, n_m converges weakly-* in $L^\infty(0,T;L^q(\mathbb{R}^d;L^1([0,\infty))))$, meaning that n_∞ is a function which is L^1 in θ, not merely a measure.

From the above lemmas, it follows that there exist limit functions $\rho_\infty, p_\infty, v_\infty, n_\infty$ such that, up to a subsequence:

$$\begin{cases} \rho_m \rightharpoonup \rho_\infty \text{ weakly-* in } L^\infty(0,T;L^q(\mathbb{R}^d)) \text{ for all } q \in [1,\infty], \\ p_m \rightharpoonup p_\infty \text{ weakly-* in } L^\infty(0,T;L^q(\mathbb{R}^d)) \text{ for all } q \in [1,\infty], \\ n_m \rightharpoonup n_\infty \text{ weakly-* in } L^\infty(0,T;L^q(\mathbb{R}^d;\mathcal{M}([0,\infty)))), \text{ for all } q \in [1,\infty], \\ \nabla p_m \rightharpoonup \nabla p_\infty \text{ weakly in } L^2([0,T]\times\mathbb{R}^d), \\ \nabla v_m \rightharpoonup \nabla v_\infty \text{ weakly in } L^2([0,T]\times\mathbb{R}^d), \\ v_m \rightharpoonup v_\infty \text{ weakly in } L^\infty(0,T;L^2(\mathbb{R}^d)). \end{cases}$$

Lemma 4.3.9. *The limit functions ρ_∞, n_∞ satisfy*

$$\rho_\infty(t,x) = \int_0^\infty V(\theta) n_\infty(t,\theta,x) d\theta$$

almost everywhere.

Proof. This follows from the fact that $\rho_m(t,x) = \int_0^\infty V(\theta) n_m(t,\theta,x) d\theta$, the weak-* convergences of ρ_m and n_m, and Lemma 4.3.6. □

Remark that (4.8), the equation on ρ_m, can be re-written as

$$\partial_t \rho_m - \Delta v_m = \int_0^\infty n_m F(\theta, p_m) d\theta. \tag{4.20}$$

Using the convergences listed above, we are able to show that the left-hand side of equation (4.8) can be passed to the limit. However, to pass to the limit on the right-hand side, strong convergence in p_m is required (because of the non-linearity F). As such, we are only able to show so far that the right hand side converges to some L^∞ function, $\mathcal{H}_\infty$. This is outlined in the following lemma.

Lemma 4.3.10 ([18], Corollary 4.5)**.** *The limit functions satisfy* $0 \leq \rho_\infty \leq 1$, $0 \leq p_\infty \leq p_M$, $v_\infty = p_\infty$, *and* $p_\infty(1-\rho_\infty) = 0$ *almost everywhere in* $[0,T] \times \mathbb{R}^d$. *The limit functions,* p_∞, ρ_∞ *satisfy* $p_\infty \in L^2(0,T;H^1(\mathbb{R}^d))$, $\partial_t \rho_\infty \in L^2(0,T;H^{-1}(\mathbb{R}^d))$, *and solve*

$$\partial_t \rho_\infty - \Delta p_\infty = \mathcal{H}_\infty \tag{4.21}$$

in $\mathcal{D}'([0,T] \times \mathbb{R}^d)$, *for some* $\mathcal{H}_\infty(t,x) \in L^\infty([0,T] \times \mathbb{R}^d)$, *where* $\mathcal{H}_\infty$ *is the weak-* limit in* L^∞ *of* $\int_0^\infty n_m(F(\theta,p_m) - \mu(\theta)V(\theta))d\theta$.

Proof. The bounds on ρ_∞ and p_∞ follow from the L^∞ bounds on ρ_m and p_m. To show that $p_\infty = p_\infty \rho_\infty$ almost everywhere, consider $v_m = \rho_m^m = \frac{m-1}{m}\rho_m p_m$. From Lemma 4.3.3, we have that $\rho_m, p_m \in L^\infty(0,T;L^2(\mathbb{R}^d))$ with weak L^2 limits ρ_∞, p_∞. Further, from Lemma 4.3.7, $\partial_t \rho_m \in L^2(0,T;H^{-1}(\mathbb{R}^d))$, and $p_m \in L^2(0,T;H^1(\mathbb{R}^d))$. By the compensated compactness theorem ([18], Appendix A), up to a subsequence,

$$\lim_{m\to\infty} \int_0^T \int_{\mathbb{R}^d} v_m \varphi \, dxdt = \lim_{m\to\infty} \int_0^T \int_{\mathbb{R}^d} \frac{m-1}{m} \rho_m p_m \varphi \, dxdt = \int_0^T \int_{\mathbb{R}^d} \rho_\infty p_\infty \varphi \, dxdt,$$

for all $\varphi \in C(0,T;C^1(\mathbb{R}^d))$. But since $v_m \rightharpoonup v_\infty$ in $L^\infty(0,T;L^2(\mathbb{R}^d))$, by uniqueness of the weak limit, $v_\infty = \rho_\infty p_\infty$ almost everywhere.

Further, since $x \mapsto x^{\frac{m}{m-1}}$ is convex, by weak lower-semicontinuity, we can show that

$$v_\infty = \liminf_{m\to\infty} (\frac{m-1}{m} p_m)^{\frac{m}{m-1}} \geq p_\infty.$$

To see this, define $\Psi_m(x) = \frac{m-1}{m} x^{\frac{m}{m-1}}$, and define $\psi_\delta(x) = \max\{x - \delta, 0\}$. Notice that ψ_δ is convex and $\psi_\delta(x) \leq \Psi_m(x)$ for large m. Then, by lower-semicontinuity of ψ_δ,

$$\psi_\delta(p_\infty) \leq \liminf_{m\to\infty} \psi_\delta(p_m) \leq \liminf_{m\to\infty} \Psi_m(p_m) = \liminf_{m\to\infty} v_m = v_\infty.$$

Taking $\delta \to 0$, we have $p_\infty \leq v_\infty$. At the same time, because $\rho_\infty \leq 1$, it holds that

$$v_\infty = \rho_\infty p_\infty \leq p_\infty,$$

meaning that $p_\infty = v_\infty$. From here, we conclude that $p_\infty = \rho_\infty p_\infty$ a.e. In particular, this proves that p_∞ satisfies (4.13).

To show (4.21), using the weak-* $L^\infty([0,T] \times \mathbb{R}^d)$ convergence of ρ_m and the weak $L^2([0,T] \times \mathbb{R}^d)$ convergence of ∇v_m, the left hand side of (4.20) can be passed to the weak-* limit. On the right hand side, recall that

$$\int_0^\infty n_m F(\theta, p_m)\, d\theta \leq C\rho_m,$$

so this term is bounded in $L^\infty([0,T] \times \mathbb{R}^d)$, and converges weakly-* to some limit $\mathcal{H}_\infty$ in $L^\infty([0,T] \times \mathbb{R}^d)$. □

What remains is to identify the function $\mathcal{H}_\infty$, and pass to the limit in the equation on n_m, (4.3). In order to do so, strong convergence is required in both p_m (to pass to the limit in the reaction term

to identify $\mathcal{H}_\infty$) and ∇p_m (to pass to the limit in $\operatorname{div}(n_m \nabla p_m)$). It is easier to first show that ∇v_m converges strongly to ∇v_∞, and because $v_\infty = p_\infty$, the strong convergence of p_m and ∇p_m will follow easily.

The proof of the following lemma follows the structure of Lemma 4.6 in [18]. Some parts of the proof are identical, but are included for completion. The main difference in our case appears due to the transport in θ, which means that it cannot be treated as a parameter. Because of this, the compensated compactness theorem cannot be directly applied, as it could in [18].

Lemma 4.3.11. *Up to a subsequence, as $m \to \infty$, $\nabla v_m \to \nabla v_\infty$ strongly in $L^2((0,T) \times \mathbb{R}^d)$.*

Proof. To show the strong convergence of ∇v_m,

$$\begin{aligned}\limsup_{m\to\infty} \int_0^T \int_{\mathbb{R}^d} |\nabla(v_m - v_\infty)|^2 \, dxdt &= \limsup_{m\to\infty} \int_0^T \int_{\mathbb{R}^d} \nabla v_m \cdot \nabla(v_m - v_\infty) \\ &\quad - \nabla v_\infty \cdot \nabla(v_m - v_\infty) \, dxdt \\ &\leq \limsup_{m\to\infty} \int_0^T \int_{\mathbb{R}^d} \nabla v_m \cdot \nabla(v_m - v_\infty) \, dxdt,\end{aligned}$$

because $\nabla(v_m - v_\infty) \rightharpoonup 0$ weakly in $L^2([0,T] \times \mathbb{R}^d)$. Testing (4.20) against $v_m - v_\infty$,

$$\int_{\mathbb{R}^d} \partial_t \rho_m (v_m - v_\infty) \, dx + \int_{\mathbb{R}^d} \nabla v_m \cdot \nabla(v_m - v_\infty) \, dx = \int_{\mathbb{R}^d} (v_m - v_\infty) \int_0^\infty n_m F(\theta, p_m) \, d\theta dx.$$

Rearranging,

$$\begin{aligned}\limsup_{m\to\infty} \int_0^T \int_{\mathbb{R}^d} \nabla v_m \cdot \nabla(v_m - v_\infty) \, dxdt &= \limsup_{m\to\infty} \int_0^T \int_{\mathbb{R}^d} (v_m - v_\infty) \int_0^\infty F(\theta, p_m) n_m \, d\theta dx \\ &\quad + \limsup_{m\to\infty} (- \int_0^T \int_{\mathbb{R}^d} \partial_t \rho_m (v_m - v_\infty) \, dxdt).\end{aligned}$$

Notice that

$$\partial_t \rho_m v_m = \frac{1}{m+1} \partial_t (\rho_m^{m+1}),$$

so

$$\begin{aligned}
\limsup_{m\to\infty}(-\int_0^T \int_{\mathbb{R}^d} \partial_t \rho_m v_m \, dxdt) &= \limsup_{m\to\infty}(-\frac{d}{dt}\int_0^T \int_{\mathbb{R}^d} \frac{1}{m+1}\rho_m^{m+1} \, dxdt) \\
&= \limsup_{m\to\infty} \frac{1}{m+1} \int_{\mathbb{R}^d} \rho_m^{m+1}(0,x) - \rho_m^{m+1}(T,x) \, dx \\
&= 0,
\end{aligned}$$

since $\rho_m^{m+1}(T)$ and $\rho_m^{m+1}(0)$ are bounded in $L^1(\mathbb{R}^d)$ uniformly in m. Then,

$$\begin{aligned}
\limsup_{m\to\infty} \int_0^T \int_{\mathbb{R}^d} \nabla v_m \cdot \nabla (v_m - v_\infty) \, dxdt &= \limsup_{m\to\infty} \int_0^T \int_{\mathbb{R}^d} (v_m - v_\infty) \int_0^\infty F(\theta, p_m) n_m \, d\theta dx \\
&\quad + \limsup_{m\to\infty} \int_0^T \int_{\mathbb{R}^d} \partial_t \rho_m v_\infty \, dxdt \\
&= \limsup_{m\to\infty} \int_0^T \int_{\mathbb{R}^d} (v_m - v_\infty) \int_0^\infty (F(\theta, p_m) - F(\theta, p_\infty)) n_m \, d\theta dxdt \qquad (4.22) \\
&\quad + \limsup_{m\to\infty} \int_0^T \int_{\mathbb{R}^d} (v_m - v_\infty) \int_0^\infty F(\theta, p_\infty) n_m \, d\theta dxdt \qquad (4.23) \\
&\quad + \int_0^T \int_{\mathbb{R}^d} \partial_t \rho_\infty v_\infty \, dxdt, \qquad (4.24)
\end{aligned}$$

using the fact that $\partial_t \rho_m$ is bounded in $L^2(0,T; H^{-1}(\mathbb{R}^d))$ to get (4.24).

To conclude the lemma, it remains to show that all three terms (4.22), (4.23), (4.24) are zero. The proofs of (4.22) and (4.24) follow closely the proof in [18]. The proof of (4.23) is the main difference between this proof and the one in [18], and requires a generalized version of the compensated

compactness theorem.

Proof that (4.22) **equals zero.** We have:

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^d} \int_0^\infty (F(\theta, p_m) - F(\theta, p_\infty)) n_m (v_m - v_\infty)\, d\theta dx dt \\
&= \int_0^T \int_{\mathbb{R}^d} \int_0^\infty (F(\theta, p_m) - F(\theta, p_\infty)) n_m (p_m (\frac{m-1}{m} \rho_m - 1) + p_m - p_\infty)\, d\theta dx dt \\
&\leq \int_0^T \int_{\mathbb{R}^d} \int_0^\infty (F(\theta, p_m) - F(\theta, p_\infty)) n_m (p_m (\frac{m-1}{m} \rho_m - 1))\, d\theta dx dt. \qquad (4.25)
\end{aligned}$$

In the above calculation, the first line comes from the definition of v_m, and the fact that $p_\infty = v_\infty$. The second line comes from the fact that $F(\theta, p)$ is nonincreasing in p so $(F(\theta, p_m) - F(\theta, p_\infty))(p_m - p_\infty) \leq 0$. Then, for a fixed $\varepsilon > 0$,

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^d} \int_0^\infty (F(\theta, p_m) - F(\theta, p_\infty)) n_m (p_m (\frac{m-1}{m} \rho_m - 1))\, d\theta dx dt \\
&\qquad = \iint_{\rho_m \leq 1-\varepsilon} \int_0^\infty (F(\theta, p_m) - F(\theta, p_\infty)) n_m\, d\theta (\rho_m^{m-1} (\rho_m - \frac{m}{m-1}))\, dx dt \\
&\qquad + \iint_{\rho_m > 1-\varepsilon} \int_0^\infty (F(\theta, p_m) - F(\theta, p_\infty)) n_m\, d\theta (p_m (\frac{m-1}{m} \rho_m - 1))\, dx dt.
\end{aligned}$$

For the first term,

$$\begin{aligned}
&\iint_{\rho_m \leq 1-\varepsilon} \int_0^\infty (F(\theta, p_m) - F(\theta, p_\infty)) n_m\, d\theta (\rho_m^{m-1} (\rho_m - \frac{m}{m-1}))\, dx dt \\
&\qquad \leq C \rho_M (1-\varepsilon)^{m-1} \iint_{\rho_m \leq 1-\varepsilon} \rho_m + \frac{m}{m-1}\, dx dt \\
&\qquad \leq C(T)(1-\varepsilon)^{(m-1)},
\end{aligned}$$

using the bound on F in the first inequality, and the $L^1((0,T)\times\mathbb{R}^d)$ bound on ρ_m, and the compact support of ρ_m in the second inequality. Also,

$$\begin{aligned}\iint_{\rho_m>1-\varepsilon}\int_0^\infty (F(\theta,p_m)-F(\theta,p_\infty))n_m\,d\theta(p_m(\frac{m-1}{m}\rho_m-1))\,dxdt\\ \leq C\rho_M\iint_{\rho_m>1-\varepsilon} p_m|\frac{m-1}{m}\rho_m-1|\,dxdt\\ \leq C(T)\sup_{\rho_m>1-\varepsilon}|\frac{m-1}{m}\rho_m-1|,\end{aligned}$$

using the $L^1([0,T]\times\mathbb{R}^d)$ bound on p_m. To bound $\sup_{\rho_m>1-\varepsilon}|\frac{m-1}{m}\rho_m-1|$, first notice that $\rho_m\leq e^{\frac{1}{m-1}\ln(p_M)}\leq 1+\frac{1}{m-1}\ln(p_M)+o(\frac{1}{m-1})$ and so $\frac{m-1}{m}\rho_m-1\leq o(\frac{1}{m-1})$.

Also, when $\rho_m-1>-\varepsilon$, $\frac{m-1}{m}\rho_m-1>-\varepsilon-\frac{1}{m}\rho_M$. Combining these bounds, for $\rho_m>1-\varepsilon$, it holds that

$$|\frac{m-1}{m}\rho_m-1|\leq\max(\frac{1}{m}\rho_M+\varepsilon,o(\frac{1}{m-1})).$$

Taking $\varepsilon=\frac{1}{\sqrt{m-1}}$,

$$\begin{aligned}\limsup_{m\to\infty}\int_0^T\int_{\mathbb{R}^d}\int_0^\infty (F(\theta,p_m)-F(\theta,p_\infty))n_m(p_m(\frac{m-1}{m}\rho_m-1))\,d\theta dxdt\\ \leq\limsup_{m\to\infty}C(T)(1-\frac{1}{\sqrt{m-1}})^{m-1}+C(T)\max(\frac{1}{m}\rho_M+\frac{1}{\sqrt{m-1}},o(\frac{1}{m-1}))=0.\end{aligned}$$

Proof that (4.23) equals zero. This will be done using an extended version of the proof of the Compensated Compactness Theorem found in Appendix A of [18], and is the main difference between this

Lemma and Lemma 4.6 in [18].

For simplicity of notation, define $w_m(t,\theta,x) = (v_m - v_\infty)F(\theta, p_\infty)$. Recall that $w_m \in L^\infty([0,T] \times \mathbb{R}^d; W^{1,\infty}([0,\infty))) \cap L^\infty(0,\infty; L^2(0,T; H^1(\mathbb{R}^d)))$ and $w_m \rightharpoonup 0$ weakly in that space. In what follows, we will use mollification to obtain a strong convergence, and show that the product $w_m n_m$ converges. We have

$$\int_0^T \int_{\mathbb{R}^d} \int_0^\infty (v_m - v_\infty)F(\theta, p_\infty) n_m d\theta dx dt = \int_0^T \int_{\mathbb{R}^d} \int_0^\infty w_m n_m d\theta dx dt.$$

Take $\psi_\varepsilon(x) = \frac{1}{\varepsilon^d}\psi(\frac{x}{\varepsilon})$, and $\zeta_\sigma(t) = \frac{1}{\sigma}\zeta(\frac{t}{\sigma})$, where ψ and ζ are standard mollifiers.

Then,

$$\begin{aligned}\int_0^T \int_{\mathbb{R}^d} \int_0^\infty w_m n_m d\theta dx dt = &\underbrace{\int_0^T \int_{\mathbb{R}^d} \int_0^\infty n_m(w_m - w_m *_x \psi_\varepsilon) d\theta dx dt}_{\mathcal{A}_m} + \\ &\underbrace{\int_0^T \int_{\mathbb{R}^d} \int_0^\infty (n_m - n_m *_t \zeta_\sigma)(w_m *_x \psi_\varepsilon) d\theta dx dt}_{\mathcal{B}_m} \\ &+ \underbrace{\int_0^T \int_{\mathbb{R}^d} \int_0^\infty (n_m *_t \zeta_\sigma)(w_m *_x \psi_\varepsilon) d\theta dx dt}_{\mathcal{C}_m}.\end{aligned}$$

We will show that

$$\limsup_{m\to\infty} \left| \int_0^T \int_{\mathbb{R}^d} \int_0^\infty w_m n_m d\theta dx dt \right| \leq C(\varepsilon, \sigma),$$

where $\lim_{\varepsilon,\sigma\to 0} C(\varepsilon,\sigma) = 0$, proving that (4.23) is zero.

First, consider the term $\mathcal{C}_m$:

$$\begin{aligned}|\mathcal{C}_m| &= |\int_0^T \int_{\mathbb{R}^d} \int_0^\infty (n_m *_t \zeta_\sigma)(w_m *_x \psi_\varepsilon) d\theta dx dt| \\ &= |\int_0^T \int_{\mathbb{R}^d} \int_0^\infty n_m w_m *_x \psi_\varepsilon *_t \zeta_\sigma d\theta dx dt| \\ &\leq ||w_m *_x \psi_\varepsilon *_t \zeta_\sigma||_{L^\infty([0,T]\times[0,\infty)\times\mathbb{R}^d)} ||n_m||_{L^\infty([0,T]\times\mathbb{R}^d; L^1([0,\infty)))}.\end{aligned}$$

Thanks to the mollification, for any $\varepsilon, \sigma > 0$, $w_m *_x \psi_\varepsilon *_t \zeta_\sigma$ has bounded derivatives in x and t, so,

$$w_m *_x \psi_\varepsilon *_t \zeta_\sigma \in W^{1,\infty}([0,T] \times [0,\infty) \times \mathbb{R}^d).$$

By Arzela-Ascoli, and using the fact that w_m is compactly supported in all three variables (from Lemmas 4.3.5, 4.3.6), $w_m *_x \psi_\varepsilon *_t \zeta_\sigma$ converges uniformly up to a subsequence. Using the fact that $w_m \rightharpoonup 0$ weakly, it must be that $w_m *_x \psi_\varepsilon *_t \zeta_\sigma \to 0$ uniformly.

Using the fact that n_m is bounded in $L^\infty([0,T] \times \mathbb{R}^d; L^1([0,\infty))$ uniformly in m, it follows that

$$\limsup_{m\to\infty} |\mathcal{C}_m| = 0.$$

Next, consider the term $\mathcal{A}_m$:

$$
\begin{aligned}
|\mathcal{A}_m| &= |\int_0^T \int_{\mathbb{R}^d} \int_0^\infty n_m(w_m - w_m *_x \psi_\varepsilon) d\theta dx dt| \\
&= |\int_0^T \int_{\mathbb{R}^d} \int_0^\infty \Big(\int_{\mathbb{R}^d} (w_m(t,\theta,x) - w_m(t,\theta,x-\varepsilon y))\psi(y) dy \Big) n_m(t,\theta,x) d\theta dx dt| \\
&= |\int_0^T \int_{\mathbb{R}^d} \int_0^\infty \int_{\mathbb{R}^d} \int_{x-\varepsilon y}^x \nabla_x w_m(t,\theta,z) dz \psi(y) n_m(t,\theta,x) dy d\theta dx dt| \\
&\leq \int_0^T \int_{\mathbb{R}^d} \int_0^\infty \int_{\mathbb{R}^d} ||\nabla_x w_m(t,\theta,\cdot)||_{L^2(\mathbb{R}^d)} (\varepsilon|y|)^{\frac{1}{2}} n_m(t,\theta,x) dy d\theta dx dt \\
&\leq \int_{\mathbb{R}^d} ||\nabla_x w_m||_{L^\infty(0,\infty;L^2([0,T]\times\mathbb{R}^d))} (\varepsilon|y|)^{\frac{1}{2}} \psi(y) ||n_m||_{L^\infty(0,T;L^1([0,\infty)\times\mathbb{R}^d))} dy \\
&\leq C \int_{\mathbb{R}^d} \sqrt{(\varepsilon|y|)} \psi(y) dy \\
&\leq C\sqrt{\varepsilon},
\end{aligned}
$$

using the fact that $w_m \in L^\infty(0,\infty; L^2(0,T; H^1(\mathbb{R}^d)))$, and $n_m \in L^\infty(0,T; L^1([0,\infty)\times\mathbb{R}^d)$, both uniformly in m. So,

$$\limsup_{m\to\infty} |\mathcal{A}_m| \leq C\sqrt{\varepsilon},$$

for a constant C not depending on m.

Lastly, consider the term $\mathcal{B}_m$:

$$
\begin{aligned}
|\mathcal{B}_m| &= |\int_0^T \int_{\mathbb{R}^d} \int_0^\infty (n_m - n_m *_t \zeta_\sigma)(w_m *_x \psi_\varepsilon) d\theta dx dt| \\
&= |\int_0^T \int_{\mathbb{R}^d} \int_0^\infty \Big(\int_0^\infty (n_m(t) - n_m(t-\sigma\tau))\zeta(\tau) d\tau \Big)(w_m *_x \psi_\varepsilon) d\theta dx dt| \\
&= |\int_0^T \int_{\mathbb{R}^d} \int_0^\infty \Big(\int_0^\infty \int_{t-\sigma\tau}^t \partial_t n_m(s) ds \zeta(\tau) d\tau \Big)(w_m *_x \psi_\varepsilon) d\theta dx dt| \\
&= |\int_0^T \int_{\mathbb{R}^d} \int_0^\infty \int_0^\infty \int_{t-\sigma\tau}^t (\underbrace{-r(p_m)\partial_\theta n_m(s)}_{i} + \underbrace{\operatorname{div}(n_m \nabla p_m(s))}_{ii} \\
&\quad \underbrace{-r(p_m)\nu(\theta, p_m) n_m(s) - \mu(\theta) n_m(s)}_{iii}) ds \zeta(\tau) d\tau (w_m *_x \psi_\varepsilon) d\theta dx dt|.
\end{aligned}
$$

For i, taking the θ integral and integrating by parts,

$$
\begin{aligned}
&|\int_0^T \int_{\mathbb{R}^d} \int_0^\infty \int_0^\infty \int_{t-\sigma\tau}^t -r(p_m(s,x))\partial_\theta n_m(s,\theta,x)\zeta(\tau)(w_m *_x \psi_\varepsilon)(t,\theta,x) ds d\tau d\theta dx dt| \\
&= |\int_0^T \int_{\mathbb{R}^d} \int_0^\infty \int_{t-\sigma\tau}^t [(w_m *_x \psi_\varepsilon)(t,0,x) 2r(p_m(s,x)) \int_0^\infty \nu(\theta, p_m) n_m(s,\theta,x) d\theta \\
&+ r(p_m(s,x)) \int_0^\infty n_m(s,\theta,x) \partial_\theta (w_m *_x \psi_\varepsilon)(t,\theta,x) d\theta] \zeta(\tau) ds d\tau dx dt| \\
&\leq \Big(||C(w_m *_x \psi_\varepsilon) n_m||_{L^\infty([0,T]\times\mathbb{R}^d; L^1([0,\infty)))} \\
&+ ||C n_m \partial_\theta (w_m *_x \psi_\varepsilon)||_{L^\infty([0,T]\times\mathbb{R}^d; L^1([0,\infty)))} \Big) \int_0^T \int_0^\infty \sigma\tau\zeta(\tau) d\tau dt \\
&\leq C(T) \int_0^\infty \sigma\tau\zeta(\tau) d\tau \\
&\leq C\sigma,
\end{aligned}
$$

using that $\partial_\theta w_m, w_m \in L^\infty([0,T]\times[0,\infty)\times\mathbb{R}^d)$, and $n_m \in L^\infty([0,T]\times\mathbb{R}^d; L^1([0,\infty)))$, bounded

uniformly in m, and the compact support of solutions. For ii, taking the integral in x,

$$
\begin{aligned}
&|\int_0^T \int_{\mathbb{R}^d} \int_0^\infty \int_0^\infty \int_{t-\sigma\tau}^t \operatorname{div}(n_m \nabla p_m(s)) ds \zeta(\tau) d\tau (w_m *_x \psi_\varepsilon) d\theta dx dt| \\
&= |-\int_0^T \int_{\mathbb{R}^d} \int_0^\infty \int_0^\infty \int_{t-\sigma\tau}^t n_m(s) \nabla p_m(s) \nabla (w_m *_x \psi_\varepsilon)(t) \zeta(\tau) ds d\tau d\theta dx dt| \\
&\leq \int_0^\infty \int_{t-\sigma\tau}^t ||\rho_m(s)||_{L^\infty(\mathbb{R}^d)} ||\nabla p_m(s)||_{L^2(\mathbb{R}^d)} ||\nabla (w_m *_x \psi_\varepsilon)(t)||_{L^\infty(0,\infty;L^2(\mathbb{R}^d))} \zeta(\tau) d\tau ds dt \\
&\leq C(T) \int_0^\infty ||\rho_m||_{L^\infty([0,T]\times\mathbb{R}^d} ||\nabla p_m||_{L^2([0,T]\times\mathbb{R}^d)} ||\nabla (w_m *_x \psi_\varepsilon)||_{L^\infty(0,\infty;L^2([0,T]\times\mathbb{R}^d))} \zeta(\tau) \sqrt{\sigma\tau} d\tau \\
&\leq C \int_0^\infty \sqrt{\sigma\tau} \zeta(\tau) d\tau \\
&\leq C\sqrt{\sigma},
\end{aligned}
$$

using that $\nabla p_m \in L^2([0,T]\times\mathbb{R}^d)$, $\nabla(w_m *_x \psi_\varepsilon) \in L^\infty(0,\infty;L^2([0,T]\times\mathbb{R}^d))$ and $n_m \in L^\infty([0,T]\times\mathbb{R}^d; L^1([0,\infty)))$, bounded uniformly in m. For iii,

$$
\begin{aligned}
&|\int_0^T \int_{\mathbb{R}^d} \int_0^\infty \int_0^\infty \int_{t-\sigma\tau}^t (-r(p_m)\nu(\theta,p_m) - \mu(\theta)) n_m(s) \zeta(\tau) (w_m *_x \psi_\varepsilon) ds d\tau d\theta dx dt| \\
&\leq C \int_0^\infty \tau\sigma\zeta(\tau) d\tau \\
&\leq C\sigma,
\end{aligned}
$$

using that $(w_m *_x \psi_\varepsilon) \in L^\infty([0,T]\times[0,\infty)\times\mathbb{R}^d)$ and $(r(p_m)\nu(\theta,p_m) - \mu(\theta))n_m \in L^\infty([0,T]\times\mathbb{R}^d; L^1([0,\infty)))$, bounded uniformly in m, and the compact support of solutions. So,

$$\limsup_{m\to\infty} |\mathcal{B}_m| \leq C\sigma,$$

for a constant C independent of m.

Combining $\mathcal{A}_m, \mathcal{B}_m$, and $\mathcal{C}_m$, we have

$$\limsup_{m\to\infty} \left| \int_0^T \int_{\mathbb{R}^d} \int_0^\infty w_m n_m d\theta dx dt \right| \leq \limsup_{m\to\infty} |\mathcal{A}_m| + |\mathcal{B}_m| + |\mathcal{C}_m| \leq C\sqrt{\varepsilon} + C\sigma,$$

for constants C not depending on m. Taking $\varepsilon, \sigma \to 0$, we have that (4.23) is equal to zero.

Proof that (4.24) **equals zero.** Formally, this should hold because $v_\infty > 0$ only when $\rho_\infty = 1$, i.e., $\partial_t \rho_\infty = 0$. However, because $\partial_t \rho_\infty$ is only in $L^2(0,T; H^{-1}(\mathbb{R}^d))$, this needs to be justified. This proof follows closely from [18], where further details can be found. First, notice that

$$\begin{aligned}
&\frac{1}{\varepsilon} \int_0^T \int_{\mathbb{R}^d} (\rho_\infty(t+\varepsilon, x) - \rho_\infty(t,x)) v_\infty \, dx dt \\
&\qquad = \frac{1}{\varepsilon} \int_0^T \int_{\mathbb{R}^d} (\rho_\infty(t+\varepsilon, x) - 1) v_\infty + (1 - \rho_\infty(t,x)) v_\infty \, dx dt \\
&\qquad \leq 0,
\end{aligned}$$

because $(1-\rho_\infty)v_\infty = 0$ and $\rho_\infty \leq 1$. Similarly, and

$$\begin{aligned}
&\frac{1}{\varepsilon} \int_\varepsilon^T \int_{\mathbb{R}^d} (\rho_\infty(t, x) - \rho_\infty(t-\varepsilon,x)) v_\infty \, dx dt \\
&\qquad = \frac{1}{\varepsilon} \int_\varepsilon^T \int_{\mathbb{R}^d} (\rho_\infty(t, x) - 1) v_\infty + (1 - \rho_\infty(t-\varepsilon,x)) v_\infty \, dx dt \\
&\qquad \geq 0.
\end{aligned}$$

Recall from Lemma 4.3.10 that $v_\infty \in L^2(0,T;H^1(\mathbb{R}^d))$ and $\partial_t \rho_\infty \in L^2(0,T;H^{-1}(\mathbb{R}^d))$. Then,

$$\begin{aligned}
0 &= \lim_{\varepsilon \to 0} \int_0^T \int_{\mathbb{R}^d} (\rho_\infty(t+\varepsilon,x) - \rho_\infty(t,x)) v_\infty dx dt \\
&= \lim_{\varepsilon \to 0} \int_0^T \langle \frac{1}{\varepsilon}(\rho_\infty(t+\varepsilon,x) - \rho_\infty(t,x), v_\infty \rangle_{H^{-1}(\mathbb{R}^d),H^1(\mathbb{R}^d)} dt \\
&= \int_0^T \langle \partial_t \rho_\infty(t,x), v_\infty \rangle_{H^{-1}(\mathbb{R}^d),H^1(\mathbb{R}^d)} dt \\
&= \int_0^T \int_{\mathbb{R}^d} \partial_t \rho_\infty v_\infty dx dt.
\end{aligned}$$

Because (4.22), (4.23) and (4.24) are all 0, the proof of the lemma is complete.

□

Lemma 4.3.12 ([18], Corollary 4.7)**.** *Up to a subsequence, $p_m \to p_\infty$ strongly in $L^2([0,T] \times \mathbb{R}^d)$.*

Proof. From Lemma 4.3.11, and Lemma 4.3.5, and Poincare's inequality, $v_m \to v_\infty$ strongly in $L^2([0,T] \times \mathbb{R}^d)$. Then since $p_m = \frac{m-1}{m} v_m^{\frac{m-1}{m}}$, $p_m \to v_\infty = p_\infty$ as $m \to \infty$, strongly in $L^2([0,T] \times \mathbb{R}^d)$. □

Lemma 4.3.13. *Up to a subsequence, $\nabla p_m \to \nabla p_\infty$ strongly in $L^2([0,T] \times \mathbb{R}^d)$.*

Proof. Assume $m > 4$. First,

$$|\nabla p_m|^2 = \nabla v_m \cdot \nabla (\frac{m}{m-2} \rho_m^{m-2}).$$

Because $\nabla v_m \to \nabla v_\infty$ strongly in $L^2([0,T] \times \mathbb{R}^d)$, it suffices to show that $\nabla(\frac{m}{m-2}\rho_m^{m-2})$ is bounded in $L^2([0,T] \times \mathbb{R}^d)$. Once this is shown, it will hold that $||\nabla p_m||_{L^2} \to ||\nabla p_\infty||_{L^2}$. To see this, notice that $\frac{m}{m-2}\rho_m^{m-2} = \frac{m}{m-2} v_m^{\frac{m-2}{m}} \to p_\infty$, because v_m converges strongly to $v_\infty = p_\infty$. Since it already is known that $\nabla p_m \rightharpoonup \nabla p_\infty$ weakly in $L^2([0,T] \times \mathbb{R}^d)$, the weak convergence combined with the convergence

in norm implies that $\nabla p_m \to \nabla p_\infty$ strongly in $L^2([0,T]\times\mathbb{R}^d)$.

To show the desired L^2 bound,

$$|\frac{m}{m-2}\nabla\rho_m^{m-2}|^2 = m^2\rho_m^{2m-6}|\nabla\rho_m|^2.$$

Then, using (4.18) with $q = m-3$,

$$\frac{d}{dt}\int_{\mathbb{R}^d}\rho_m^{m-3}\,dx + m(m-3)(m-4)\int_{\mathbb{R}^d}\rho_m^{2m-6}|\nabla\rho_m|^2\,dx \le (m-3)\int_{\mathbb{R}^d}\rho_m^{m-3}\,dx.$$

Multiplying by $\frac{m}{m-2}$,

$$\frac{d}{dt}\int_{\mathbb{R}^d}\frac{m}{m-2}\rho_m^{m-3}dx + \frac{m-3}{m-2}(m-4)\int_{\mathbb{R}^d}|\nabla(\frac{m}{m-2}\rho_m^{m-2})|^2\,dx \le (m-3)\int_{\mathbb{R}^d}\frac{m}{m-2}\rho_m^{m-3}\,dx.$$

Integrating in time,

$$\int_0^T\int_{\mathbb{R}^d}|\nabla(\frac{m}{m-2}\rho_m^{m-2})|^2dxdt \le \frac{m}{m-4}\int_0^T\int_{\mathbb{R}^d}\rho_m^{m-3}dxdt + \frac{m}{(m-3)(m-4)}\int_{\mathbb{R}^d}\rho_m^{m-3}(0,x)\,dx.$$

Then, since $\rho_m^{m-3} = (\frac{m-1}{m}p_m)^{\frac{m-3}{m-1}} \le \max\{p_M, 1\}$, and the compact support of solutions is uniform in m, the right hand side is bounded independent of m. □

Finally, we present a lemma that demonstrates that $n_m \log n_m$ is bounded in $L^1([0,\infty))$ in θ, and by Dunford-Pettis theorem, this will imply that n_∞ is not a measure, and the sequence n_m converges in $L^1([0,\infty))$ in θ. This lemma is similar to Lemma 4.5 in [43].

Lemma 4.3.14. *The function n_m satisfies the following:*

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^d} \int_0^\infty n_m(t,\theta,x) \log_+ n_m(t,\theta,x)\, d\theta dx \leq C(T), \tag{4.26}$$

for a constant $C(T)$ independent of m, only depending on T and the uniform in m bound on the initial condition.

Proof. Begin by multiplying (4.3) by $(1+\log n_m)V(\theta)$ and integrating in θ and x to get the following equation:

$$\begin{aligned}\int_{\mathbb{R}^d} \int_0^\infty \partial_t (n_m \log n_m) V(\theta) + \int_{\mathbb{R}^d} \int_0^\infty r(p_m) \partial_\theta (n_m \log n_m) V(\theta) d\theta dx \\ = \int_{\mathbb{R}^d} \int_0^\infty -r(p_m)\nu(\theta,p_m) n_m (\log n_m + 1) V(\theta) - \mu(\theta) n_m (\log n_m + 1) V(\theta) d\theta dx \\ - \int_{\mathbb{R}^d} \nabla \rho_m \cdot \nabla p_m dx.\end{aligned}$$

Then, integrating $\partial_\theta (n_m \log n_m) V(\theta)$ in θ, recalling that $n_m \geq 0$, and noticing that $\nabla \rho_m \cdot \nabla p_m = \frac{m^2}{4} \rho_m^{m-1} |\nabla \rho_m|^2 \geq 0$, rearranging, we have,

$$\begin{aligned}\int_{\mathbb{R}^d} \int_0^\infty \partial_t (n_m \log n_m) V(\theta) d\theta dx &\leq \int_{\mathbb{R}^d} r(p_m) V(0) n_m(t,0,x) \log n_m(t,0,x) dx \\ &+ \int_{\mathbb{R}^d} \int_0^\infty n_m \log n_m (V'(\theta) - r(p_m)\nu(\theta,p_m)V(\theta) - \mu(\theta)V(\theta)) d\theta dx \\ &\leq \int_{\mathbb{R}^d} r(p_m) V(0) n_m(t,0,x) \log_+ n_m(t,0,x) dx \\ &+ \int_{\mathbb{R}^d} \int_0^\infty n_m \log n_m (V'(\theta) - r(p_m)\nu(\theta,p_m)V(\theta) - \mu(\theta)V(\theta)) d\theta dx.\end{aligned}$$

Then, because $n_m(t,0,x) \leq C\rho_m$, for a constant C depending only on ν and V, from the monotonicity

of $s \mapsto s \log_+ s$,

$$\begin{aligned}\int_{\mathbb{R}^d} \int_0^\infty \partial_t (n_m \log n_m) V(\theta) d\theta dx &\leq C \int_{\mathbb{R}^d} \rho_m \log_+ \rho_m dx + C \int_{\mathbb{R}^d} \int_0^\infty |n_m \log n_m| V(\theta) \\ &\leq C(T) + C \int_{\mathbb{R}^d} \int_0^\infty |n_m \log n_m| V(\theta) d\theta dx,\end{aligned}$$

using the $L^1 \cap L^\infty(\mathbb{R}^d)$ bound on ρ_m, for varying constants C depending only on ν, V, μ, r, and the uniform in m L^1 and L^∞ bounds of ρ_m. Then, notice that $|n_m \log n_m|$ can be written as

$$|n_m \log n_m| = n_m \log n_m + 2|n_m \log n_m|_{\chi_{0 \leq n_m \leq 1}}.$$

Using the property $|s \log s|_{\chi_{0 \leq s \leq 1}} \leq sC_1 + C_2$ for arbitrary constants $C_1, C_2 > 0$, we have

$$\begin{aligned}\int_{\mathbb{R}^d} \int_0^\infty \partial_t (n_m \log n_m) V(\theta) d\theta dx &\leq C(T) + C \int_{\mathbb{R}^d} \int_0^\infty n_m \log n_m V(\theta) d\theta dx \\ &\quad + C \int_{\mathbb{R}^d} \int_0^\infty n_m C_1 + C_2 \, d\theta dx \\ &\leq C(T) + C \int_{\mathbb{R}^d} \int_0^\infty n_m \log n_m V(\theta) d\theta dx \\ &\quad + C||n_m||_{L^\infty(0,T;L^1([0,\infty)\times\mathbb{R}^d))} + C|\text{Supp}\, n_m| \\ &\leq C(T) + C \int_{\mathbb{R}^d} \int_0^\infty n_m \log n_m V(\theta) d\theta dx,\end{aligned}$$

using the compact support of n_m in θ and x and the $L^\infty(0,T;L^1([0,\infty) \times \mathbb{R}^d)$ bound of n_m. By Gronwall's inequality,

$$\int_{\mathbb{R}^d} \int_0^\infty n_m \log n_m d\theta dx \leq C(T) + C(T) \int_{\mathbb{R}^d} \int_0^\infty n_{0,m} \log n_{0,m} d\theta dx \leq C(T),$$

using the uniform in m bound of $n_{0,m}$. Finally, using again the property $|s\log s|_{\chi_{0\le s\le 1}} \le s\omega + Ce^{-\omega/2}$, we have

$$\begin{aligned} n_m \log_+ n_m &= n_m \log n_m + |n_m \log n_m|_{\chi_{0\le n_m\le 1}} \\ &\le n_m \log n_m + (n_m\omega + Ce^{-\omega/2}), \end{aligned}$$

and using again the compact support of n_m, and the $L^\infty(0,T;L^1([0,\infty)\times\mathbb{R}^d))$ bounds of n_m,

$$\int_{\mathbb{R}^d}\int_0^\infty n_m \log_+ n_m d\theta dx \le C(T),$$

for a constant $C(T)$ depending only on the uniform bound of the initial data. □

Proof of Theorem 4.2.1. The equation (4.3) can be written in the weak format

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^d}\int_0^\infty n_m[-\partial_t\varphi - r(p_m)\partial_\theta\varphi + \nabla p_m\cdot\nabla\varphi + r(p_m)\nu(\theta,p_m)\varphi + \mu(\theta)\varphi]\,d\theta dx dt \\ &\quad = \int_{\mathbb{R}^d}\int_0^\infty n_{0,m}(x,\theta)\varphi(0,\theta,x)\,d\theta dx + \int_0^T\int_{\mathbb{R}^d}\int_0^\infty 2\nu(\theta,p_m)n_m\varphi(t,0,x)\,d\theta dx dt, \quad (4.27) \end{aligned}$$

for any $\varphi\in C^1([0,T]\times[0,\infty)\times\mathbb{R}^d)$. Since n_m converges weakly in $L^\infty(0,T;L^1((0,\infty)\times\mathbb{R}^d)$, p_m and ∇p_m converge strongly in $L^2([0,T]\times\mathbb{R}^d)$, and solutions are compactly supported, all of the above terms can be passed to the limit, which solves

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^d}\int_0^\infty n_\infty[-\partial_t\varphi - r(p_\infty)\partial_\theta\varphi + \nabla p_\infty\cdot\nabla\varphi + r(p_\infty)\nu(\theta,p_\infty)\varphi + \mu(\theta)\varphi]\,d\theta dx dt \\ &\quad = \int_{\mathbb{R}^d}\int_0^\infty n_0(\theta,x)\varphi(0,\theta,x)\,d\theta dx + \int_0^T\int_{\mathbb{R}^d}\int_0^\infty 2\nu(\theta,p_\infty)n_\infty\varphi(t,0,x)\,d\theta dx dt. \quad (4.28) \end{aligned}$$

Further, the equation on ρ_m, p_m can be written in the weak format

$$-\int_0^T \int_{\mathbb{R}^d} \rho_m \partial_t \varphi \, dxdt + \int_0^T \int_{\mathbb{R}^d} \nabla\varphi \cdot \nabla \rho_m^m \, dxdt - \int_0^T \int_{\mathbb{R}^d} \int_0^\infty n_m(F(\theta, p_m) - \mu(\theta)V(\theta)) \, d\theta\varphi \, dxdt = \int_{\mathbb{R}^d} \rho_{0,m}(x)\varphi(0,x) \, dx$$

for $\varphi \in C^1([0,T] \times \mathbb{R}^d)$. Passing to the limit,

$$-\int_0^T \int_{\mathbb{R}^d} \rho_\infty \partial_t \varphi \, dxdt + \int_0^T \int_{\mathbb{R}^d} \nabla\varphi \cdot \nabla p_\infty \, dxdt - \int_0^T \int_{\mathbb{R}^d} \int_0^\infty n_\infty(F(\theta, p_\infty) - \mu(\theta)V(\theta)) \, d\theta\varphi \, dxdt = \int_{\mathbb{R}^d} \rho_0(x)\varphi(0,x) \, dx.$$

□

Finally, we prove the complementarity formula, (4.15).

Proof of Theorem 4.2.3. This proof relies on the strong convergence of p_m and ∇p_m. Dividing (4.16) by $m-1$, for a test function $\xi \in C^1([0,T] \times R^d)$, the following identity holds.

$$\int_0^T \int_{\mathbb{R}^d} \xi p_m \Delta p_m + \xi \frac{m}{m-1} \left(\frac{m-1}{m} p_m\right)^{\frac{m-2}{m-1}} \int_0^\infty n_m(F(\theta, p_m) - \mu(\theta)V(\theta)) d\theta \, dx \, dt = \frac{1}{m-1} \int_0^T \int_{\mathbb{R}^d} -p_m \partial_t \xi - \xi |\nabla p_m|^2 \, dx \, dt.$$

Taking the $\limsup$ of both sides, from the boundedness of $p_m \in L^1([0,T] \times \mathbb{R}^d)$ and $\nabla p_m \in L^2([0,T] \times \mathbb{R}^d)$,

$$\limsup_{m \to \infty} \int_0^T \int_{\mathbb{R}^d} \xi p_m \Delta p_m + \xi \frac{m}{m-1} \left(\frac{m-1}{m} p_m\right)^{\frac{m-2}{m-1}} \int_0^\infty n_m(F(\theta, p_m) - \mu(\theta)V(\theta)) d\theta \, dx \, dt = 0.$$

Integrating by parts and using the strong convergence of p_m and ∇p_m, and the weak convergence of n_m,

$$\begin{aligned}
0 &= \limsup_{m\to\infty} \int_0^T \int_{\mathbb{R}^d} -\nabla\xi \cdot \nabla p_m \, p_m - \xi |\nabla p_m|^2 \\
&\quad + \xi \left(\frac{m}{m-1}\right)^{\frac{1}{m-1}} p_m^{\frac{m-2}{m-1}} \int_0^\infty n_m (F(\theta, p_m) - \mu(\theta) V(\theta)) d\theta \, dx \, dt \\
&= \int_0^T \int_{\mathbb{R}^d} -\nabla\xi \cdot \nabla p_\infty p_\infty - \xi |\nabla p_\infty|^2 + \xi p_\infty \int_0^\infty n_\infty (F(\theta, p_\infty) - \mu(\theta) V(\theta)) d\theta \, dx \, dt.
\end{aligned}$$

Integrating by parts again yields

$$0 = \int_0^T \int_{\mathbb{R}^d} \xi \Delta p_\infty p_\infty + \xi p_\infty \int_0^\infty n_\infty (F(\theta, p_\infty) - \mu(\theta) V(\theta)) d\theta,$$

which completes the proof. □

Chapter 5: Traveling Waves

In this chapter, we establish the existence of traveling wave solutions to the incompressible limit equations (4.11)-(4.13) that were derived in Chapter 4. We assume that there is no cell death, i.e., $\mu(\theta) = 0$ for all θ. Existence is shown for two particular cases, given by specific assumptions on ν, V, and r, which will be made precise later on.

5.1 Definitions and Notations

In Chapter 4, it is shown that the incompressible limit functions, (n, p, ρ), solve the following equation:

$$\begin{cases} \partial_t n + r(p)\partial_\theta n - \operatorname{div}_x(n\nabla_x p) = -r(p)\nu(\theta, p)n, & t \in (0,T], \theta \in (0,\infty), x \in \mathbb{R}^d, \\ n(t,0,x) = 2\int_0^\infty \nu(\theta,p)n(t,\theta,x)\,d\theta, & t \in (0,T], x \in \mathbb{R}^d, \\ p \in P_\infty(\rho), \end{cases} \tag{5.1}$$

with initial data $n(0,\theta,x) = n_0(\theta,x)$, where $\rho(t,x) = \int_0^\infty V(\theta)n(t,\theta,x)d\theta$ and $P_\infty(\rho)$ is the Hele-Shaw graph,

$$P_\infty(\rho) = \begin{cases} 0, & \rho < 1 \\ [0,\infty), & \rho = 1. \end{cases} \tag{5.2}$$

The constraint $p \in P_\infty(\rho)$ is equivalent to the assumptions

$$p \geq 0, \rho \leq 1, p(1-\rho) = 0.$$

Multiplying the top line in (5.1) by $V(\theta)$ and integrating in θ yields the following equation on ρ :

$$\partial_t \rho - \operatorname{div}_x(\rho \nabla_x p) = r(p) \int_0^\infty n[V'(\theta) + (2V(0) - V(\theta))\nu(\theta,p)]d\theta. \tag{5.3}$$

The Hele-Shaw condition, $p \in P_\infty(\rho)$, naturally partitions the domain of this problem into two regions: the "tumor region", given by $\Omega(t) = \{x \in \mathbb{R}^d : p(t,x) > 0\}$, and the "healthy region", $\mathbb{R}^d \backslash \Omega$ (see for example [14, 61, 63]). Then, (5.1) can be simplified by noting that $\rho = 1$ on $\Omega(t)$, and $p = 0$ on $\mathbb{R}^d \backslash \Omega(t)$, and the problem can be solved separately in the two regions.

On $\Omega(t)$, $\rho = 1$ and, from (5.3), p solves

$$\begin{cases} -\Delta_x p = \int_0^\infty n r(p)[V'(\theta) + (2V(0) - V(\theta))\nu(\theta,p)]d\theta, & \text{in } \Omega(t), \\ p = 0, & \text{on } \partial\Omega(t), \end{cases}$$

and on $\mathbb{R}^d \backslash \Omega(t)$, $p = 0$ and ρ solves

$$\partial_t \rho = r(0) \int_0^\infty n[V'(\theta) + (2V(0) - V(\theta))\nu(\theta, 0)]d\theta, \quad \text{in } \mathbb{R}^d \backslash \Omega(t).$$

On both sets $\Omega(t)$ and $\mathbb{R}^d \backslash \Omega(t)$, the evolution of p and ρ requires the knowledge of $n(t, \theta, x)$ for all θ, so this is still a non-trivial problem.

We will consider two cases of (5.1) that give rise to very different dynamics. In the first case, V and r are constant. The assumption on r means that aging is linear in time and cells always age, regardless of local pressure forces. Additionally, the assumption on V means that cells have a constant volume, so mitosis is the main process that creates an increase in volume of the tumor: when a mitosis event occurs, one cell splits into two, whose total volume is twice that of the parent cell. In this case, (5.3) becomes:

$$\partial_t \rho - \operatorname{div}_x(\rho \nabla_x p) = rV \int_0^\infty n(t, \theta, x)\nu(\theta, p)d\theta,$$

yielding an equation reminiscent of (1.1). In this equation, one can observe that tumor growth is driven by cell duplication. If ν is independent of θ, this equation exactly recovers the original tumor growth model, (1.1).

A second case we consider is when V and r are non-constant, and mitosis is assumed to be volume preserving, which is a biologically natural assumption. This can be imposed by taking $(2V(0) - V(\theta))\nu(\theta, p) = 0$, meaning that when a cell divides, the total volume of two daughter cells equals the volume of the parent cell. In this case, (5.3) becomes

$$\partial_t \rho - \operatorname{div}_x(\rho \nabla_x p) = r(p) \int_0^\infty n(t, \theta, x)V'(\theta)d\theta.$$

In contrast with the previous case, tumor growth is now driven not by cell proliferation, but by the increase in volume of individual cells, as seen by $V'(\theta)$.

In the first case, for simplicity, we take $r = 1$, $V = 1$, and $\nu(\theta, p) = (1-p)\beta(\theta)$, where $\beta(\theta)$ is a bounded function representing the age-dependent rate of cell division. The system (5.1) becomes:

$$\begin{cases} \partial_t n + \partial_\theta n - \operatorname{div}_x(n\nabla_x p) = -(1-p)\beta(\theta)n, & t > 0, \theta > 0, x \in \mathbb{R}^d, \\ n(t,0,x) = 2(1-p)\int_0^\infty \beta(\theta)n(t,\theta,x)d\theta, & t > 0, x \in \mathbb{R}^d, \\ p \in P_\infty(\rho), \end{cases}$$

where $\rho(t,x) = \int_0^\infty n(t,\theta,x)d\theta$.

In the second case, we take $r(p) = 1-p$, $\nu(\theta,p) = \beta(\theta)$. Then (5.1) becomes:

$$\begin{cases} \partial_t n + (1-p)\partial_\theta n - \operatorname{div}_x(n\nabla_x p) = -(1-p)\beta(\theta)n, & t > 0, \theta > 0, x \in \mathbb{R}^d, \\ n(t,0,x) = 2\int_0^\infty \beta(\theta)n(t,\theta,x)d\theta, & t > 0, x \in \mathbb{R}^d, \\ p \in P_\infty(\rho), \end{cases}$$

where $\rho(t,x) = \int_0^\infty V(\theta)n(t,\theta,x)d\theta$.

In the case of constant volume, we prove that a family of traveling waves with different wave speeds exists. We also prove that one of the traveling waves semi-compactly supported, and the rest are strictly positive, similar to the known result for traveling wave solutions of porous medium equations [32]. In the case of non-constant volume, we prove the existence of a traveling wave that is semi-compactly supported. In both cases, because n, p, and wave speed c all depend on each other, the proofs are quite a bit different than those for classical reaction-diffusion equations, and require the

Leray-Schauder fixed point theorem.

5.1.1 Limitations

One limitation is that we are only considering the free boundary model, not the porous medium model, where p is given by $p = \frac{m}{m-1}\rho^{m-1}$. The existence of traveling waves is conjectured to hold in this case, but is not addressed in this work.

A second limitation of this work is that for the case of non-constant volume, we only show the existence of a traveling wave that is supported on the half-line. We conjecture that traveling waves exist for a family of speeds, as in the constant volume case, however, the traveling wave that we construct is expected to be the minimal speed traveling wave, describing the behaviour of compactly-supported solutions.

A third limitation is the assumption that $\nu(\theta, p) = (1-p)\beta(\theta)$. The product form of the division rate separates the actions of p and θ and allows for the use of a fixed point theorem. This assumption is necessary for the proof, however, we argue that this is not a major limitation. For the purpose of the application of tumor growth, most reaction terms would be of this form, so this assumption is not restricting many interesting cases from an application point of view. On the other hand, we remark that the assumption can easily be extended to $\nu(\theta, p) = \alpha(p)\beta(\theta)$, where $\alpha'(p) < 0$ and $\alpha(p_M) = 0$. The assumption of $\alpha(p) = 1 - p$ is purely technical, for ease of the argument.

5.1.2 Outline of the Chapter

In Section 5.2, we prove the existence of traveling wave solutions when $V(\theta)$ and $r(p)$ are constant. In Section 5.3, we extend the result to $r(p)$ and $V(\theta)$ non-constant.

5.2 Existence of Traveling Waves: The Case of Constant Volume

In this section, we take $r = 1$, meaning that the age of a cell simply represents the amount of time since its birth. We take volume to be $V = 1$, and take the division rate to be $\nu(\theta, p) = (1-p)\beta(\theta)$. The term $1-p$ is reminiscent of the original Fisher-KPP equation, which is standard in the literature [26,41]. With these assumptions, we consider the following version of (5.1).

$$\begin{cases} \partial_t n + \partial_\theta n - \operatorname{div}_x(n\nabla_x p) = -(1-p)\beta(\theta)n, & t \in (0,T], \theta \in (0,\infty), x \in \mathbb{R}^d, \\ n(t,0,x) = 2(1-p)\int_0^\infty \beta(\theta)n(t,\theta,x)d\theta, & t \in (0,T], x \in \mathbb{R}^d, \\ p \in P_\infty(\rho), \end{cases} \tag{5.4}$$

where $\rho(t,x) = \int_0^\infty n(t,\theta,x)d\theta$. The term $\beta(\theta)$ is the age-dependent rate of cell division, with the following assumptions: there exist constants $\beta_1, \beta_2 > 0$ and $C > 0$ such that

$$|\beta'(\theta)| \le C, \quad \beta_1 \le \beta(\theta) \le \beta_2,$$

for all $\theta \ge 0$. We consider traveling wave solutions of (5.4) by taking $n(t,\theta,x) = \bar{n}(\theta, x \cdot e - ct)$, where $e \in \mathbb{S}^{d-1}$ is the unit vector in the direction of propagation and $c > 0$ is the wave speed. We will drop the overline for convenience of notation. The traveling wave equation, to be solved for n, p and

c, is given by:

$$\begin{cases} \partial_\theta n - \partial_x((c + \partial_x p)n) = -(1-p)\beta(\theta)n, & \theta > 0,\ x \in \mathbb{R}, \\ n(0,x) = 2(1-p)\int_0^\infty \beta(\theta)n(\theta,x)\,d\theta, & x \in \mathbb{R}, \\ p \in P_\infty(\rho), \end{cases} \tag{5.5}$$

where $\rho(x) = \int_0^\infty n(\theta, x)d\theta$. Integrating (5.5) in θ yields the following equation for ρ.

$$-c\rho' - (\rho p')' = (1-p)\int_0^\infty \beta(\theta)n(\theta,x)\,d\theta \qquad x \in \mathbb{R}. \tag{5.6}$$

The condition $p \in P_\infty(\rho)$ implies that $\rho(x) = 1$ in the region $\{x : p(x) > 0\}$, so (5.6) reduces to

$$-p'' = (1-p)\int_0^\infty \beta(\theta)n(\theta,x)d\theta, \quad \text{in } \{x : p(x) > 0\}.$$

In order to find traveling wave solutions to (5.5), we impose the following boundary values, meaning that the traveling wave connects the two states $\rho = 1$ (tumor region) and $\rho = 0$ (tumor free region).

$$\lim_{x\to-\infty} \rho(x) = 1, \quad \lim_{x\to\infty} \rho(x) = 0.$$

Because $P_\infty(\rho)$ partitions $\mathbb{R}$ into two regions, one where $\rho = 1$ and one where $\rho < 1$, using the boundary conditions and invariance by spatial translation, we can assume that $\rho = 1$ for $x < 0$ and $\rho < 1$ (and so $p = 0$) for $x \geq 0$. This suggests that a family of traveling waves exist, parametrized by the value $\rho(0^+)$. We now present the main existence result for traveling wave solutions to this problem.

Theorem 5.2.1. *For each $\ell \in [0,1)$, there exists a traveling wave solution (n,p,c) of (5.5) with*

$\rho(0^+) = \ell$, *satisfying*

$$\lim_{x\to-\infty} \rho(x) = 1, \ \lim_{x\to\infty} \rho(x) = 0,$$

and

$$\rho = 1 \textit{ for } x < 0, \quad p = 0 \textit{ for } x \geq 0.$$

The wave speed c *is given by* $c = \frac{-p'(0^-)}{1-\ell}$,

Further, the traveling wave with $\ell = 0$ *is semi-compactly supported, that is,* $n(\theta, x) = 0$ *for all* $x \in [0, \infty)$.

Solutions to (5.5) can be found separately in the regions $x < 0$ (the tumor region) and $x \geq 0$ (the healthy region). To see what is happening at the interface $x = 0$, we set $\ell = \rho(0^+)$. From (5.6) we have $-(c\rho + \rho p')' = (1-p)\int_0^\infty \beta n d\theta$. Because the right-hand side is bounded, $c\rho + \rho p'$ is continuous, i.e.,

$$c\rho(0^-) + \rho(0^-)p'(0^-) = c\rho(0^+) + \rho(0^+)p'(0^+).$$

Because $\rho(0^-) = 1$ and $p'(0^+) = 0$, the above simplifies to

$$c = \frac{-p'(0^-)}{1 - \rho(0^+)} = \frac{-p'(0-)}{1-\ell}, \tag{5.7}$$

giving us the wave speed for all $\ell \in [0, 1)$. In the region $x > 0$, by taking $p = 0$, (5.6) becomes

$$-c\rho' = \int_0^\infty \beta(\theta) n(\theta, x) d\theta, \quad x > 0. \tag{5.8}$$

The traveling wave corresponding to $\ell = 0$ captures the finite speed of propagation of support of porous medium equations, making it the "physically relevant" solution, and the one of primary

interest here. To see this, notice from (5.7) that this traveling wave has speed $c = -p'(0^-)$. From (5.8), $\rho'(x) \leq 0$, so because $\rho(0^+) = 0$, it holds that

$$\rho(x) = 0 \text{ and } n(\theta, x) = 0 \text{ for all } x > 0.$$

When $\ell \in (0, 1)$, traveling waves can be found with speeds given by (5.7). In these cases, (5.8) suggests that the solution has a "tail", meaning the solution is always positive. Informed by the literature, e.g., [32], we conjecture that the traveling wave given by $\ell = 0$ is the one with minimal speed. However, this is difficult to show, and is related to uniqueness of the problem, a largely difficult and open problem on cross-diffusion models such as this one. We can show the following:

Proposition 5.2.2. *For every $\ell \in [0, 1)$, let $c(\ell)$ be the corresponding speed. Then for all $\ell \in [0, 1)$, $c(\ell) \geq c_1$ for some $c_1 > 0$ and $\lim_{\ell \to 1} c(\ell) = \infty$.*

This proposition suggests that there is a minimal speed traveling wave with speed c^*, and there exists a traveling wave for all $c \geq c^*$. This is in line with previous results about traveling waves solutions to reaction-diffusion equations [32].

We will split the proof of Theorem 5.2.1 into two lemmas, one corresponding to $\ell = 0$ and one to $\ell > 0$.

Lemma 5.2.3. *There exists a traveling wave solution (n, p, c) of (5.5) where $\rho(x) = 0$ for $x > 0$, with*

$$\lim_{x \to -\infty} \rho(x) = 1, \ \lim_{x \to \infty} \rho(x) = 0.$$

The wave speed is given by $c = -p'(0^-)$.

Lemma 5.2.4. *For each $\ell \in (0,1)$, there exists a traveling wave solution (n,p,c) of (5.5) where $\rho(0^+) = \ell$, with*

$$\lim_{x\to-\infty} \rho(x) = 1, \ \lim_{x\to\infty} \rho(x) = 0.$$

The wave speed is given by $c = \frac{-p'(0^-)}{1-\ell}$.

5.2.1 Proof of Lemma 5.2.3

We will begin with the proof of Lemma 5.2.3. It has already been established that $\rho(x) = 0$ for all $x > 0$, and because $n(\theta, x) \geq 0$ for all $x \in \mathbb{R}, \theta \geq 0$, it holds that $n(\theta, x) = 0$ for all $x > 0$.

When $x < 0$, $\rho = 1$ and so the solutions (n,p) of (5.5) will solve

$$\begin{cases} -p'' = (1-p)\int_0^\infty \beta(\theta) n(\theta,x)\, d\theta, & x < 0, \\ \partial_\theta n - \partial_x((c+\partial_x p)n) = -(1-p)\beta(\theta)n, & \theta > 0, x < 0, \\ n(0,x) = 2(1-p)\int_0^\infty \beta(\theta) n(\theta,x) d\theta, & x < 0, \\ c = -p'(0^-). \end{cases} \tag{5.9}$$

Remark 5.2.5. *The system (5.9) is introduced because its formulation allows for a fixed-point method to be used to find solutions. We will show that the system (5.5), along with the requirement that $\rho(x) = 0, n(\theta,x) = 0$ for $x \in [0,\infty)$ is equivalent to the system (5.9). It is clear that solutions of the system (5.5) will be solutions of the system (5.9): the condition $p \in P_\infty(\rho)$ can be replaced by the top equation for p in (5.9). To show that the two are equivalent, i.e., that solutions of (5.9) are solutions of (5.5), we must show that solutions of (5.9) satisfy the requirement $p \in P_\infty(\rho)$. To do this, we will show that the solution $n(\theta,x)$ of (5.9) satisfies $\int_0^\infty n(\theta,x)dx = 1$ for $x \in (-\infty, 0)$. This is nontrivial*

and will be the final step in the proof of Lemma 5.2.3

The formulation of the problem given in (5.9) suggests that a fixed-point method can be used to find the solution. We will use Leray-Schauder fixed point theorem: the steps are briefly outlined below, and will be detailed in several lemmas.

Given $\varphi \geq 0$ such that $\int_{-\infty}^{0} \varphi(x)dx < \infty$, we consider the weighted L^2 space given by

$$X = L^2_\varphi((-\infty, 0)) = \{u : (-\infty, 0) \to \mathbb{R}, \int_{-\infty}^{0} u(x)^2\varphi(x)dx < \infty\}.$$

Given $u \in X$, we define the truncated function $\bar{u} = \max(\min(u, \beta_2), \beta_1)$, and solve the equation

$$\begin{cases} -p'' = (1-p)\bar{u}, & x < 0, \\ p(0) = 0,\ p(-\infty) = 1. \end{cases} \tag{5.10}$$

Then using p, the unique solution of (5.10), we define $c = -p'(0^-)$ and solve

$$\begin{cases} \partial_\theta n - \partial_x((c+p')n) = -(1-p)\beta(\theta)n & \theta > 0,\ x < 0 \\ n(0,x) = 2(1-p)\bar{u} & x < 0. \end{cases} \tag{5.11}$$

Finally, we define the map $T : L^2_\varphi((-\infty, 0)) \to L^2_\varphi((-\infty, 0))$ by

$$T[u] := \int_0^\infty \beta(\theta)n(\theta, x)\, d\theta, \tag{5.12}$$

where n is the solution of (5.11). Then, we will show using Leray-Schauder fixed point theorem that T has a fixed point. We will then prove that this fixed point is in fact a solution of (5.5). To do this (see

Remark 5.2.5), we recover the Hele-Shaw condition, $p \in P_\infty(\rho)$ by showing that $\int_0^\infty n(\theta, x)d\theta = 1$ in $\{x : p(x) > 0\} = (-\infty, 0)$. This will conclude the proof of Lemma 5.2.3.

We now prove several lemmas that will be used to prove Lemma 5.2.3.

Lemma 5.2.6. *Given $u \in L^2_\varphi((-\infty, 0))$, there exists a unique solution to (5.10) on $(-\infty, 0)$.*

Proof. Consider the equation

$$\begin{cases} -\bar{p}'' = (1 - \bar{p})\beta_2, & x < 0, \\ \bar{p}(-\infty) = 1, \bar{p}(0) = 0. \end{cases}$$

This equation can be solved explicitly to get

$$\bar{p}(x) = 1 - e^{\sqrt{\beta_2}x}.$$

Because $\bar{u} \leq \beta_2$,

$$-\bar{p}'' \geq (1 - \bar{p})\bar{u},$$

and because it satisfies the same boundary conditions, $\bar{p}$ is a supersolution of (5.10).

Similarly, the function

$$\underline{p}(x) = 1 - e^{\sqrt{\beta_1}x}$$

satisfies

$$-\underline{p}'' = (1 - p)\beta_2,$$

and therefore is a subsolution of (5.10).

Because $\underline{p} \leq \bar{p}$ on $(-\infty, 0)$, by Perron's method [15], there exists some p, defined as the supre-

mum of all the viscosity subsolutions of (5.10), that is a viscosity solution of (5.10) and satisfies

$$1 - e^{\sqrt{\beta_1}x} \leq p(x) \leq 1 - e^{\sqrt{\beta_2}x} \quad \text{for all } x \in (-\infty, 0). \tag{5.13}$$

Furthermore, because p'' is bounded, $p \in W^{2,\infty}$ and so p is a strong solution to (5.10).

To show that p is the unique solution, we use a sliding method [6]. Suppose that $\tilde{p}$ and $\hat{p}$ are solutions to (5.10). Let

$$t_0 = \inf\{t : \tilde{p} \leq \hat{p} + t\}.$$

Then $\tilde{p}(x) \leq \hat{p}(x) + t_0$ for all $x \leq 0$. If $t_0 > 0$, there exists some $x_0 \in (-\infty, 0)$ such that $\tilde{p}(x_0) = \hat{p}(x_0) + t_0$. This implies that $-\hat{p}''(x_0) \leq -\tilde{p}''(x_0)$ and so $(1 - \hat{p}(x_0))\bar{u}(x_0) \leq (1 - \tilde{p}(x_0))\bar{u}(x_0)$.

Rearranging,

$$\hat{p}(x_0) + t_0 = \tilde{p}(x_0) \leq \hat{p}(x_0),$$

yielding $t_0 = 0$. From the definition of t_0, this means $\tilde{p} \leq \hat{p}$. Switching the roles of $\hat{p}$ and $\tilde{p}$ concludes that $\tilde{p} = \hat{p}$. □

Remark 5.2.7. *By defining $c = -p'(0)$, the inequality (5.13) implies that $\sqrt{\beta_1} \leq c \leq \sqrt{\beta_2}$. Further, because $p'' \leq 0$, it holds that $p'(x) + c \geq 0$ for all $x \leq 0$.*

Lemma 5.2.8. *Given p, the unique solution to (5.10), there exists a unique solution to (5.11).*

Proof. The equation (5.11) can be rewritten as

$$\begin{cases} \partial_\theta n - (c + p')\partial_x n = -(1 - p)(\beta(\theta) + \bar{u})n \\ n(0, x) = 2(1 - p)\bar{u}, \end{cases}$$

and solved as a transport equation. Let

$$\begin{cases} X'(\theta) = -(p'(X) + c) \\ X(0) = x_0. \end{cases}$$

The above ODE can be solved because p'' is bounded, so $p' + c$ is Lipschitz. From (5.11), we have

$$\begin{cases} \frac{d}{d\theta} n(\theta, X(\theta)) = -(1 - p(X(\theta))(\beta(\theta) + \bar{u}(X(\theta)))n(\theta, X(\theta)) \\ n(0, x_0) = 2(1 - p(x_0))\bar{u}(x_0). \end{cases}$$

Recall that $1 - p(X(\theta)) \geq e^{\sqrt{\beta_2} X(\theta)}$ and $\beta(\theta) + \bar{u}(X(\theta)) \geq 2\beta_1$, yielding

$$\frac{d}{d\theta} n(\theta, X(\theta)) \leq -2\beta_1 e^{\sqrt{\beta_2} X(\theta)} n(\theta, X(\theta)). \tag{5.14}$$

Given some fixed $x < 0$ and $\theta > 0$, there exists $x_0 \in (x, 0)$ such that $X(\theta) = x$ when $X(0) = x_0$. Because $X'(\theta) \leq 0$, it follows that $X(\tilde{\theta}) \geq X(\theta)$ for all $\tilde{\theta} \leq \theta$. Integrating (5.14) and using $X(\theta) = x$,

$$\begin{aligned} n(\theta, X(\theta)) &\leq 2(1 - p(x_0))\bar{u}(x_0) e^{-2\beta_1 \int_0^\theta e^{\sqrt{\beta_2} X(\tilde{\theta})} d\tilde{\theta}} \\ &\leq 2(1 - p(x_0))\bar{u}(x_0) e^{-2\beta_1 e^{\sqrt{\beta_2} x} \theta} \\ &\leq 2\beta_2 e^{-2\beta_1 e^{\sqrt{\beta_2} x} \theta}. \end{aligned}$$

Because $X(\theta) = x$, we get

$$n(\theta, x) \leq 2\beta_2 e^{-2\beta_1 e^{\sqrt{\beta_2} x} \theta}. \tag{5.15}$$

Because x and θ were chosen arbitrarily, (5.15) holds for all $x < 0$ and all $\theta > 0$.

Integrating (5.15) in θ,

$$\bar{\rho}(x) := \int_0^\infty n(\theta, x) d\theta \leq \int_0^\infty 2\beta_2 e^{-2\beta_1 e^{\sqrt{\beta_2}x}\theta} d\theta = \frac{\beta_2}{\beta_1 e^{\sqrt{\beta_2}x}}, \tag{5.16}$$

which is finite for any fixed $x < 0$. The uniqueness of n follows from the method of characteristics and the uniqueness of p. □

Lemma 5.2.9. *Given $u \in L^2_\varphi((-\infty, 0))$, $\bar{\rho}(x) = \int_0^\infty n(\theta, x) d\theta \leq C$ for some constant $C > 0$ depending only on β_1 and β_2.*

Proof. We want to show that $\bar{\rho} \leq C(\beta_1, \beta_2)$. Unfortunately, (5.16) blows up when $x \to -\infty$, so the bound will be shown in two steps.

First, consider $x \geq -1$. From (5.16),

$$\bar{\rho}(x) \leq \frac{\beta_2}{\beta_1} e^{-\sqrt{\beta_2}x} \leq \frac{\beta_2}{\beta_1} e^{\sqrt{\beta_2}} \text{ for all } x \in [-1, 0].$$

Second, consider $x \leq -1$. Integrating (5.11) in θ gives

$$-2(1-p)\bar{u} - ((c+p')\bar{\rho})' = -(1-p)\int_0^\infty \beta(\theta) n d\theta, \tag{5.17}$$

and rearranging,

$$-((c+p')\bar{\rho})' \leq 2(1-p)\bar{u} \leq 2\beta_2(1-p).$$

For any $x < 0$, integrating from x to 0 and using the fact that $c + p'(0) = 0$ yields

$$(c + p'(x))\bar{\rho}(x) \leq 2\beta_2 \int_x^0 (1 - p(y))dy,$$

and from (5.13),

$$(c + p'(x))\bar{\rho}(x) \leq 2\beta_2 \int_x^0 e^{\sqrt{\beta_1}y}\, dy = \frac{2\beta_2}{\sqrt{\beta_1}}(1 - e^{\sqrt{\beta_1}x}) \leq \frac{2\beta_2}{\sqrt{\beta_1}}. \tag{5.18}$$

At the same time, because $-p'' = (1 - p)\bar{u}$, integrating from x to 0 gives

$$-p'(0) + p'(x) = \int_x^0 (1 - p(y))\bar{u}(y)dy$$

so

$$p'(x) = -c + \int_x^0 (1 - p(y))\bar{u}(y)dy \geq -c + \beta_1 \int_x^0 e^{y\sqrt{\beta_2}}dy = -c + \frac{\beta_1}{\sqrt{\beta_2}}(1 - e^{x\sqrt{\beta_2}}), \tag{5.19}$$

so for $x < 0$, $p' + c > 0$. In particular, for $x \leq -1$, $p'(x) + c \geq \frac{\beta_1}{\sqrt{\beta_2}}(1 - e^{-\sqrt{\beta_2}}) := \delta$.

Because $c + p'(x) \geq \delta$ for $x \leq -1$, dividing by $c + p'(x)$ in (5.18) gives

$$\bar{\rho}(x) \leq \frac{2\beta_2}{\delta\sqrt{\beta_1}} \leq C(\beta_1, \beta_2) \text{ for all } x \in (-\infty, 1].$$

□

Remark 5.2.10. *Notice that*

$$T[u] = \int_0^\infty \beta(\theta) n(\theta, x) d\theta \leq \beta_2 \bar{\rho}(x) \leq C(\beta_1, \beta_2),$$

and $T[u] \geq 0$, *so* $T[u]$ *is bounded in* $L^\infty((-\infty, 0))$, *independently of* u. *Then, using the fact that* $\int_{-\infty}^0 \varphi(x) dx < \infty$, *it follows that* $T[u]$ *is bounded in* $L^2_\varphi((-\infty, 0))$, *meaning that* T *maps functions from* $L^2_\varphi((-\infty, 0))$ *to a bounded subset of* $L^2_\varphi(-\infty, 0))$.

In order to apply the Leray-Schauder fixed point theorem, we need to show that T is a compact and continuous operator such that the set $\{u \in X : u = \lambda T[u], 0 \leq \lambda \leq 1\}$ is bounded. This is shown in the following lemmas. We start with showing compactness.

Lemma 5.2.11. *The map* $T : L^2_\varphi((-\infty, 0)) \to L^2_\varphi((-\infty, 0))$ *is compact.*

Proof. Let $\{u_k\}_{k=1}^\infty \in L^2_\varphi((-\infty, 0))$ be a bounded sequence. Let $v_k = T[u_k]$. From the definition of T,

$$v_k = \int_0^\infty \beta(\theta) n_k d\theta \leq \beta_2 \bar{\rho}_k \leq C(\beta_1, \beta_2). \tag{5.20}$$

From Remark 5.2.10, we deduce

$$\int_{-\infty}^0 v_k^2 \varphi dx \leq C(\beta_1, \beta_2)^2 \int_{-\infty}^0 \varphi dx \leq C(\beta_1, \beta_2, \varphi),$$

so v_k is bounded in L^2_φ. Next, it needs to be shown that $v_k'(x) = \int_0^\infty \beta(\theta) \partial_x n_k(\theta, x) d\theta$ is bounded in L^2_φ.

Multiplying (5.11) by $\beta(\theta)$ and integrating in θ, we get

$$-2\beta(0)(1-p_k)\bar{u}_k-\int_0^\infty \beta'(\theta)n_k d\theta-(c_k+p_k')\int_0^\infty \beta(\theta)\partial_x n_k d\theta = -(1-p_k)\int_0^\infty \beta(\theta)n_k(\beta(\theta)+\bar{u}_k)d\theta.$$

Then since $p_k, \bar{\rho}_k, \bar{u}_k, \beta$, and β' are all bounded independent of k in L^∞, it follows from the above that

$$|(c_k + p_k'(x))v_k'(x)| \le C_1 \quad \text{for all } x < 0,$$

for some constant C_1 depending only on β_1 and β_2. Then,

$$|v_k'(x)| \le \frac{C_1}{p_k'(x) + c_k} \quad \text{for all } x < 0.$$

For any $\varepsilon > 0$, from (5.19), $p_k'(x) + c_k \ge \frac{\beta_1}{\sqrt{\beta_2}}(1 - e^{-\varepsilon\sqrt{\beta_2}})$ for $x \le -\varepsilon$, and so

$$|v_k'(x)| \le C_2 \quad \text{for all } x \le -\varepsilon,$$

for a constant C_2 independent of k. Therefore, there exists a $v \in L^\infty((-\infty, 0))$ such that v_k converges locally uniformly to v as $k \to \infty$. Then for any $\varepsilon > 0$, we can write

$$\begin{aligned}
\limsup_{k\to\infty} ||v_k - v||_{L^2_\varphi}^2 &= \limsup_{k\to\infty} \int_{-\infty}^0 |v_k - v|^2 \varphi dx \\
&\le \lim_{k\to\infty} \int_{-1/\varepsilon}^{-\varepsilon} |v_k - v|^2 \varphi dx + \limsup_{k\to\infty} \int_{-\infty}^{-1/\varepsilon} |v_k - v|^2 \varphi dx + \limsup_{k\to\infty} \int_{-\varepsilon}^0 |v_k - v|^2 \varphi dx \\
&\le 0 + 2||v||_{L^\infty}^2 \int_{-\infty}^{-1/\varepsilon} \varphi dx + 2||v||_{L^\infty}^2 \int_{-\varepsilon}^0 \varphi dx.
\end{aligned}$$

Because this holds for all $\varepsilon > 0$, taking $\varepsilon \to 0$, we find

$$\limsup_{k\to\infty} ||v_k - v||^2_{L^2_\varphi} = 0.$$

We have shown that along a subsequence, v_k converges to v strongly in L^2_φ, and therefore that T is compact. □

Next, we prove that T is continuous.

Lemma 5.2.12. *The map T is continuous.*

Proof. Let $\{u_k\}_{k=1}^\infty$ be a sequence in $L^2_\varphi((-\infty, 0))$ such that $u_k \to u$ for some $u \in L^2_\varphi((-\infty, 0))$. From Lemma 5.2.11, $v_k = T[u_k]$ is pre-compact in $L^2_\varphi((-\infty, 0))$, meaning there exists a subsequence that converges to some $v \in L^2_\varphi((-\infty, 0))$.

To establish continuity of T, it remains to show that $v = T[u]$. The uniqueness of $T[u]$, obtained in Lemma 5.2.8 will then imply that the whole sequence $\{v_k\}$ converges to v.

Because $u_k \to u$ in L^2_φ, it also converges locally uniformly, up to a subsequence. It follows that up to a subsequence, $\bar{u}_k \to \bar{u}$ locally uniformly. Let p_k and n_k be the solutions to (5.10) and (5.11) with $\bar{u}_k$.

First, it needs to be shown that $p_k \to p$. It is clear that $0 \le p_k \le 1$. From Remark 5.2.7, c_k is bounded independent of k and $0 \ge p_k'(x) \ge -c_k$, so

$$|p_k'(x)| \le c_k \le \sqrt{\beta_2} \text{ for all } x < 0.$$

Therefore p_k and p_k' are bounded independent of k, and by Arzela-Ascoli, up to a subsequence, $p_k \to p$ locally uniformly. Also, since $|p_k''| \le \beta_2$, p_k'' converges weakly-* in L^∞, and p_k' converges locally

uniformly, up to a subsequence. Then since p_k solves

$$\begin{cases} -p_k'' = (1-p_k)\bar{u}_k, & x<0, \\ p_k(-\infty)=1, p_k(0)=0, \end{cases}$$

and taking the weak limit as $k\to\infty$, p solves

$$\begin{cases} -p'' = (1-p)\bar{u}, & x<0, \\ p(-\infty)=1, p(0)=0, \end{cases}$$

in the sense of distributions. The sequence c_k is bounded, so up to another subsequence, we can assume that $c_k \to c$ for some c. Further,

$$c = \lim_{k\to\infty} c_k = -\lim_{k\to\infty} p_k'(0) = -p'(0).$$

For a bound on the sequence n_k, from (5.15),

$$n(\theta,x) \le 2\beta_2 \quad \text{for all } \theta>0, x<0,$$

so n is bounded uniformly in k in $L^\infty_x L^\infty_\theta$. This implies $n_k \rightharpoonup n$ weakly-* in $L^\infty_x L^\infty_\theta$. Recall that n_k solves the equation

$$\begin{cases} \partial_\theta n_k - \partial_x((c_k+p_k')n_k) = -(1-p_k)\beta(\theta)n_k, & \theta>0, x<0, \\ n_k(0,x) = 2(1-p_k)\bar{u}_k, & x<0. \end{cases}$$

The above can be passed the limit as $k \to \infty$, because $n_k \to n$ weakly-*, $c_k \to c$, and $p_k' \to p'$, $\bar{u}_k \to \bar{u}$, and $p_k \to p$ all locally uniformly, up to a subsequence. The limit solves

$$\begin{cases} \partial_\theta n - \partial_x((c+p')n) = -(1-p)\beta(\theta)n, & \theta > 0, x < 0, \\ n(0,x) = 2(1-p)\bar{u} & x < 0, \end{cases}$$

in the sense of distributions. From Lemma 5.2.8, n is unique, so each of n_k, c_k and p_k converges along the whole sequence. Finally,

$$v = \lim_{k\to\infty} v_k = \lim_{k\to\infty} T[u_k] = \lim_{k\to\infty} \int_0^\infty \beta(\theta) n_k d\theta = \int_0^\infty \beta(\theta) n d\theta = T[u],$$

concluding that T is continuous. □

Next, we prove the final requirement for the Leray-Schauder fixed point theorem:

Lemma 5.2.13. *The set* $\{u \in L^2_\varphi : u = \lambda T[u], 0 \le \lambda \le 1\}$ *is bounded.*

Proof. This follows directly from Lemma 5.2.9. □

Proof of Lemma 5.2.3. From Lemmas 5.2.11, 5.2.12, 5.2.13, and the Leray-Schauder fixed point theorem, $T[u]$ has a fixed point, that is, there is some $u \in L^2_\varphi((-\infty, 0))$ such that $u = T[u]$. To conclude that (5.5) has a solution, it remains to show $p \in P_\infty(\rho)$, which will be done by showing that $\bar{\rho} = 1$, and it will follow that $u = \bar{u}$. To this end, assume that u is a fixed point of T, i.e.,

$$u(x) = \int_0^\infty \beta(\theta) n(\theta, x) d\theta.$$

Then from (5.17),

$$-((c+p')\bar{\rho})' = (1-p)(\bar{u}-u) - p''$$

and rearranging,

$$-((c+p')(\bar{\rho}-1))' = (1-p)(\bar{u}-u). \tag{5.21}$$

Suppose that $\bar{\rho}(x) \geq 1$ for some $x < 0$. Then $u(x) \geq \beta_1 \bar{\rho}(x) \geq \beta_1$, so by the definition of $\bar{u}$, $\bar{u}(x) \leq u(x)$. Using (5.21), we find

$$-((c+p')(\bar{\rho}-1))' \leq 0, \quad \text{if } \bar{\rho}(x) - 1 \geq 0. \tag{5.22}$$

Similarly, if $\bar{\rho}(x) \leq 1$, then $u(x) \leq \beta_2 \bar{\rho}(x) \leq \beta_2$, so $\bar{u}(x) \geq u(x)$, and (5.21) implies:

$$-((c+p')(\bar{\rho}-1))' \geq 0, \quad \text{if } \bar{\rho}(x) - 1 \leq 0. \tag{5.23}$$

Let $w(x) = (c+p'(x))(\bar{\rho}(x)-1)$. Combining (5.22)-(5.23) and the fact that $c + p' \geq 0$ implies that $w(x)w'(x) \geq 0$. We deduce:

$$0 \leq \int_x^0 w(x)w'(x)dx = \int_x^0 (\frac{w(x)^2}{2})'dx = \frac{w(0)^2}{2} - \frac{w(x)^2}{2},$$

and because $w(0) = 0$, it follows that $w(x)^2 \leq 0$ and $w(x) = 0$ for all $x \leq 0$, meaning $\bar{\rho}(x) = 1$ for all $x \leq 0$. Thus,

$$u(x) = \int_0^\infty \beta(\theta)n(\theta, x)d\theta \leq \beta_2 \quad \text{for all } x \leq 0,$$

and

$$u(x) = \int_0^\infty \beta(\theta) n(\theta, x) d\theta \geq \beta_1, \qquad \text{for all } x \leq 0,$$

so $u = \bar{u}$. Finally, recalling that $n(\theta, x) = 0$ for $x \in [0, \infty)$, this implies that (5.5) has a solution on $\mathbb{R}$.

□

5.2.2 Proof of Lemma 5.2.4

In this lemma, it is no longer the case that solutions of (5.5) satisfy $n(\theta, x) = 0$, $\rho(x) = 0$ for $x \in [0, \infty)$. Instead, we have that $\rho(0^+) = \ell$, and the solution has a positive tail for $x \in [0, \infty)$.

Proof of Lemma 5.2.4. In this lemma, $\ell \in (0, 1)$ and $\rho(0^+) = \ell$. We recall (see (5.7)) that this implies the relation $c = \frac{-p(0^-)}{1-\ell}$. We will again consider $x > 0$ and $x < 0$ separately. Because the right hand side of the equation (5.5) is bounded, it must be that $(c + p')n$ is continuous across $x = 0$, i.e.,

$$(c + p'(0^-))n(\theta, 0^-) = (c + p'(0^+))n(\theta, 0^+).$$

Using that $p'(0^+) = 0$ and $-p'(0^-) = c(1 - \ell)$, we find

$$\ell n(\theta, 0^-) = n(\theta, 0^+) \tag{5.24}$$

for all $\theta > 0$.

First consider $x \in (0, \infty)$. Because $\rho(x) < 1$ and $p(x) = 0$ for $x > 0$, the solution n of (5.5)

will solve:

$$\begin{cases} \partial_\theta n - c\partial_x n = -\beta(\theta) n, & \theta > 0, x > 0, \\ n(0,x) = 2\int_0^\infty \beta(\theta) n(\theta, x) d\theta, & x > 0. \end{cases}$$

This transport equation can be solved explicitly to get

$$n(\theta, x) = n(0, x + c\theta) e^{-\int_0^\theta \beta(\theta') d\theta'},$$

and the boundary condition at $\theta = 0$ is then equivalent to

$$n(0,x) = 2\int_0^\infty \beta(\theta) n(0, x + c\theta) e^{-\int_0^\theta \beta(\theta') d\theta'} d\theta.$$

Setting $H(\theta) = \beta(\theta) e^{-\int_0^\theta \beta(\theta') d\theta'}$, and $u(x) = e^{\alpha x} n(0,x)$, for some $\alpha > 0$ to be defined, this condition can be rewritten as:

$$u(x) = 2\int_0^\infty H(\theta) u(x + c\theta) e^{-\alpha c\theta} d\theta. \tag{5.25}$$

Now note that $\int_0^\infty H(\theta) d\theta = 1$, because $\int_0^\infty \beta(\theta) d\theta = \infty$ (from the lower-bound on β). Given $\alpha > 0$, we consider the integral

$$A(\alpha) = \int_0^\infty 2H(\theta) e^{-\alpha c\theta} d\theta.$$

Then, $A(0) = 2$ and $\lim_{\alpha\to\infty} A(\alpha) = 0$, so from the intermediate value theorem, there is some $\alpha^* \in (0, \infty)$ such that $A(\alpha^*) = 1$. By taking $\alpha = \alpha^*$ in (5.25), it holds that $u(x) = 1$ is a solution to

(5.25), which gives $n(0,x) = e^{-\alpha^* x}$. The explicit solution for n can now be given:

$$n(\theta, x) = e^{-\alpha^*(x+c\theta) - \int_0^\theta \beta(\theta')d\theta'}, \quad \theta > 0, x > 0. \tag{5.26}$$

From (5.24),

$$n(\theta, 0^-) = \frac{1}{\ell} e^{-\alpha^* c\theta - \int_0^\theta \beta(\theta')d\theta'}, \quad \theta > 0. \tag{5.27}$$

The equation (5.26) provides a explicit form for the solution $n(\theta, x)$ of (5.5) for $x \in [0, \infty)$.

Now, consider $x < 0$. We will consider the following system of equations:

$$\begin{cases} -p'' = (1-p)\int_0^\infty \beta(\theta) n(\theta, x) d\theta, & x < 0, \\ \partial_\theta n - \partial_x((c + \partial_x p)n) = -(1-p)\beta(\theta)n, & \theta > 0, x < 0, \\ n(0,x) = 2(1-p)\int_0^\infty \beta(\theta) n(\theta, x) d\theta, & x < 0, \\ c = -\frac{-p'(0^-)}{1-\ell}. \end{cases} \tag{5.28}$$

Similarly to Lemma 5.2.3, this system can be solved using a fixed-point method.

Remark 5.2.14. *The system* (5.28) *is equivalent to the system* (5.5)*, along with the known form of* $n(\theta, x)$ *on* $x \in [0, \infty)$ *given by* (5.26)*. It is clear that solutions of* (5.5) *are solutions of* (5.28)*, and to show the inverse, we will prove that the solution of* (5.28) *satisfies* $p \in P_\infty(\rho)$*. This will be done by showing that* $\int_0^\infty n(\theta, x) d\theta = 1$*, where* $n(\theta, x)$ *is the solution to* (5.28)*.*

The exact same proof as in Lemma 5.2.3 will go through, using the same fixed point operator, except for the following changes, to account for the fact that now $c + p'(0) > 0$.

i) In Remark 5.2.7, the bound on c is now $\frac{\sqrt{\beta_1}}{1-\ell} \leq c \leq \frac{\sqrt{\beta_2}}{1-\ell}$.

ii) In Lemma 5.2.8, the equation for n now has boundary conditions at $\theta = 0$ and at $x = 0$. The condition at $x = 0$ is given by (5.27). This curve is non-characteristic, and so the equation for n can be solved. (Note that there will be a discontinuity at $(0, 0)$.)

iii) In Lemma 5.2.9, $c + p'(0) = c\ell \in [\frac{\sqrt{\beta_1}\ell}{1-\ell}, \frac{\sqrt{\beta_2}\ell}{1-\ell}]$, and $\bar{\rho}(0) = 1$, which is enough for the bounds on $\bar{\rho}$.

iv) In the proof of Lemma 5.2.3, the same process goes through, except using (5.24), meaning that $\bar{\rho}(0^-) = 1$, instead of $c + p'(0) = 0$ to get that $w(0) = 0$.

□

5.2.3 Proof of Theorem 5.2.1 and Proposition 5.2.2

Proof of Theorem 5.2.1. This follows from combining Lemmas 5.2.3 and 5.2.4. □

Proof of Proposition 5.2.2. From the proof of Lemmas 5.2.3 and 5.2.4, we have that $c(\ell) = \frac{-p'(0)}{1-\ell} \geq \frac{\sqrt{\beta_1}}{1-\ell} \geq \sqrt{\beta_1}$. Further, for any ℓ, $-p'(0) \geq \sqrt{\beta_1}$, so $\lim_{\ell\to 1} c(\ell) = \lim_{\ell\to 1} \frac{-p'(0)}{1-\ell} \geq \lim_{\ell\to 1} \frac{\sqrt{\beta_1}}{1-\ell} = \infty$. □

5.2.4 Example with ν Independent of θ

To demonstrate the way that traveling waves encode information about the age distribution of cells, even when ν is independent of age, we consider a simple example where $\nu(\theta, p) = 1 - p$. In this case, the traveling wave solutions can be explicitly calculated.

The equation (5.5) becomes:

$$\begin{cases} \partial_\theta n - \partial_x((c + \partial_x p)n) = -(1-p)n, & \theta > 0,\ x \in \mathbb{R}, \\ n(0,x) = 2(1-p)\int_0^\infty n(\theta, x)\, d\theta, & x \in \mathbb{R}, \\ p \in P_\infty(\rho), \end{cases}$$

and so when $x < 0$, equation (5.9) becomes:

$$\begin{cases} \partial_\theta n - \partial_x((c + \partial_x p)n) = -(1-p)n & \theta > 0, x < 0, \\ n(0,x) = 2(1-p)\int_0^\infty n(\theta, x) d\theta & x < 0, \\ -p'' = (1-p) & x < 0, \\ p(-\infty) = 1, p(0) = 0, \end{cases}$$

In this simplified setting, p can be solved independently from n, so a fixed point method is not needed. We once again have that for all $\ell \in [0, 1)$, there exists a traveling wave with wave speed given by $c = \frac{-p'(0^-)}{1-\ell}$, where $\ell = \rho(0^+)$.

Because the equation for p is independent of c, for all c, we get the explicit solution for p:

$$p(x) = \begin{cases} 1 - e^x, & x < 0, \\ 0, & x \geq 0, \end{cases}$$

and so $-p'(0^-) = 1$. From there, we have that the TW with minimal speed has speed $c^* = 1$, and the

solution has semi-compact support. The solution is given explicitly by:

$$n(\theta, x) = \begin{cases} \frac{2e^{\theta-2x}}{(e^{-x}-1+e^{\theta})^3}, & x < 0, \\ 0, & x \geq 0, \end{cases}$$

$$\rho(x) = \begin{cases} 1, & x < 0, \\ 0, & x \geq 0. \end{cases}$$

For the traveling wave with speed $c = 1$, figure 5.1 shows the age distribution of cells at varying distances from the wavefront, $x = 0$. Near the front, at $x = -0.8$, most cells are newborn, while further away from the front, at $x = -5$ and $x = -15$, older cells can be found. This shows that most cell duplication occurs near the front, demonstrating the role that pressure plays in limiting cell duplication.

For the higher speeds, we have

$$c = \frac{1}{1-\ell},$$

for all $\ell \in (0, 1)$, by taking $\ell = \rho(0^+)$, giving a family of traveling waves with wave speeds $c \geq 1$. For the higher speeds, we have that p is the same as when $c = 1$, $\rho(x)$ is given by:

$$\rho(x) = \begin{cases} 1, & x < 0, \\ \ell e^{-\frac{1}{c}x}, & x \geq 0, \end{cases}$$

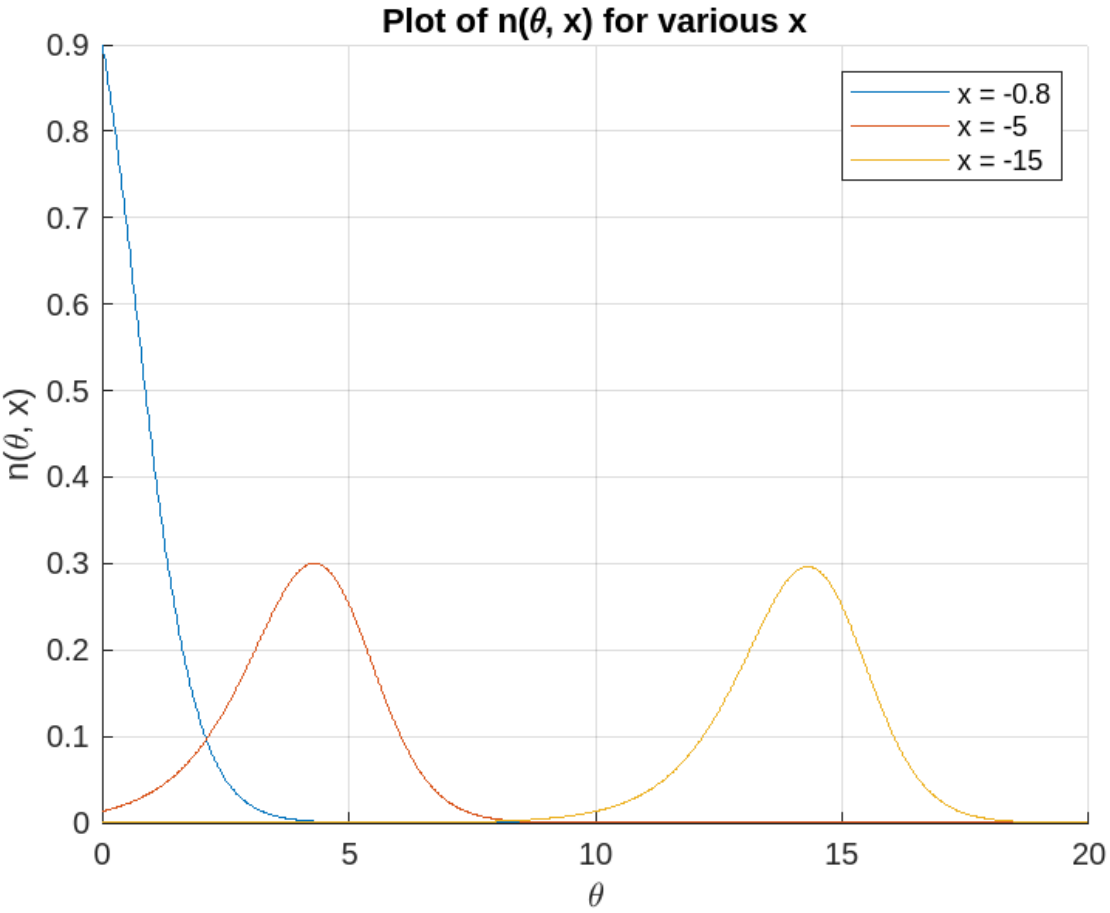


Figure 5.1: Age distribution of tumor cells in TW with minimal speed $c = 1$.

and n is given by:

$$n(\theta, x) = \begin{cases} \frac{2e^{\theta-2x}}{(e^{-x}-1+e^{\theta})^3}, & x < 0, \\ 2\ell e^{-\frac{x}{c}-2\theta}, & x \geq 0. \end{cases}$$

The traveling waves $\rho(x)$ for wave speeds $c = 1$ and $c = 2$ are given in Figure 5.2.

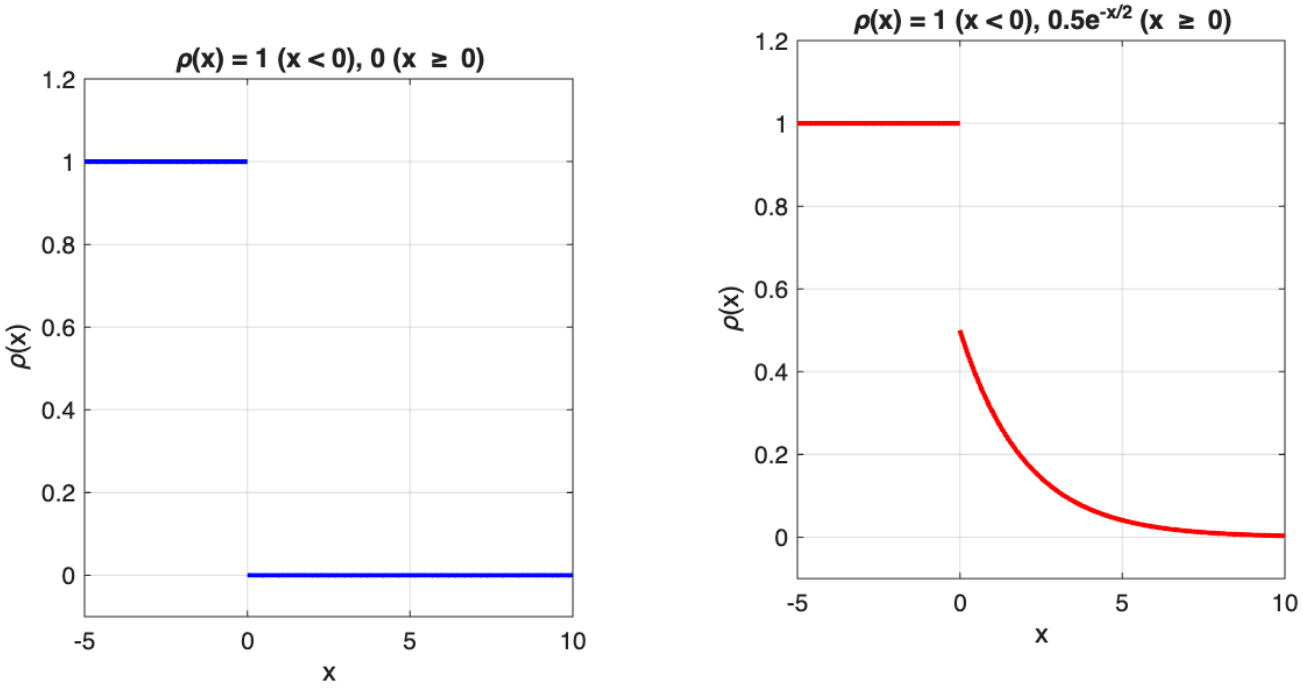


Figure 5.2: Traveling waves with minimal speed $c = 1$ (left) and speed $c = 2$ (right).

5.3 Existence of Traveling Waves: The Case of Non-constant Volume

In Section 5.2, it was assumed that the volume was constant with age, that is, $V(\theta) = 1$. In this section, we consider volume that depends on θ. Additionally, we assume that the aging rate, $r(p)$ is non-constant, so age is no longer linear in time, but instead is pressure-dependent. Cells in the presence of high pressure will stop their aging process, so θ now represents the "physiological age" of a cell, i.e., how far along in the cell-cycle it has progressed, as opposed to simply representing the amount of time since birth, which was the case in the previous section. Consider the following formulation,

which is (5.1) with $\nu(\theta, p) = \beta(\theta)$ and $r(p) = 1 - p$.

$$\begin{cases} \partial_t n + (1-p)\partial_\theta n - \operatorname{div}_x(n\nabla_x p) = -(1-p)\beta(\theta)n, & t \in (0,T], \theta \in (0,\infty), x \in \mathbb{R}^d \\ n(t,0,x) = 2\int_0^\infty \beta(\theta) n \, d\theta, & t \in (0,T], x \in \mathbb{R}^d, \\ p \in P_\infty(\rho). \end{cases} \tag{5.29}$$

Here, $P_\infty(\rho)$ is defined in (5.2), and the volume density ρ is defined as

$$\rho(x) = \int_0^\infty V(\theta) n(t,\theta,x) d\theta,$$

where V satisfies:

$$V(0) = V_0 \le V(\theta) \le 2V_0 := V_M, \quad 0 \le V'(\theta) \le V_M' \text{ for } \theta \ge 0, \tag{5.30}$$

and β satisfies:

$$0 \le \beta(\theta) \le \beta_2 \text{ for } \theta \ge 0. \tag{5.31}$$

Then, traveling wave solutions $n(\theta, x - ct)$, $p(x - ct)$ of (5.29) solve

$$\begin{cases} (1-p)\partial_\theta n - \partial_x((c + \partial_x p)n) = -(1-p)\beta(\theta)n, & \theta \ge 0, x \in \mathbb{R} \\ n(0,x) = 2\int_0^\infty \beta(\theta) n(\theta, x) \, d\theta, & x \in \mathbb{R}, \\ p \in P_\infty(\rho), \end{cases} \tag{5.32}$$

with boundary conditions

$$\lim_{x\to-\infty}\rho(x)=1,\lim_{x\to\infty}\rho(x)=0. \tag{5.33}$$

Multiplying (5.32) by $V(\theta)$ and integrating in θ, we obtain the following equation for the volume density $\rho(x)$.

$$-c\rho'-(\rho p')'=(1-p)\int_0^\infty V'(\theta)n(\theta,x)\,d\theta+(1-p)\int_0^\infty(2V(0)-V(\theta))\beta(\theta)n\,d\theta. \tag{5.34}$$

It is standard to assume in this case that mitosis is volume preserving, meaning that

$$(2V(0)-V(\theta))\beta(\theta)=0,\qquad \text{for all } \theta\geq 0. \tag{5.35}$$

In this case (5.34) becomes

$$-c\rho'-(\rho p')'=(1-p)\int_0^\infty V'(\theta)n(\theta,x)d\theta.$$

Remark 5.3.1. *A simple form for V,β satisfying this assumption is*

$$V(\theta)=\begin{cases}V_0+\gamma\theta, & \theta\leq\theta_0,\\ 2V_0, & \theta\geq\theta_0,\end{cases} \tag{5.36}$$

$$\beta(\theta)=\begin{cases}0, & \theta\leq\theta_0,\\ 1, & \theta\geq\theta_0,\end{cases} \tag{5.37}$$

for $\theta_0>0$ and $\gamma=\frac{V_0}{\theta_0}$.

We will now state the existence result for traveling wave solutions of (5.32). We will prove the existence of a traveling wave with semi-compact support. Similarly to Section 5.2, the traveling wave with semi-compact support corresponds to the traveling wave with wavespeed $c = -p'(0^-)$.

Theorem 5.3.2. *Assume β and V satisfy (5.30), (5.31), and (5.35). There exists a traveling wave solution (n, p, c) of (5.32) satisfying the boundary conditions (5.33), such that $n(\theta, x) = 0$ for all $x \in [0, \infty), \theta \in [0, \infty)$. Further, the wave speed is given by $c = -p'(0^-)$.*

5.3.1 Existence of traveling waves in the non-degenerate case

Proving the existence of traveling waves under condition (5.35) is difficult because of the degeneracy of the equation. The proof uses a fixed-point theorem, and compactness of the fixed-point operator (to be defined) relies on $V'(\theta) > 0$ for all $\theta \in [0, \infty)$. However, we do not want to make this assumption, because it is natural that volume will be constant for some ages. Instead, we will first consider the non-degenerate case, where we assume that there exists a small $\alpha > 0$ such that

$$(2V(0) - V(\theta))\beta(\theta) = \alpha \text{ for all } \theta \geq 0. \tag{5.38}$$

Then (5.34) becomes

$$-c\rho' - (\rho p')' = (1 - p) \int_0^\infty (V'(\theta) + \alpha) n(\theta, x) d\theta. \tag{5.39}$$

Because $p \in P_\infty(\rho)$, $\rho = 1$ when $p > 0$, and so (5.39) becomes:

$$-p'' = (1 - p) \int_0^\infty (V'(\theta) + \alpha) n(\theta, x) d\theta, \quad \text{in } \{x : p(x) > 0\}. \tag{5.40}$$

We will first prove the existence of traveling waves in the non-degenerate case with the assumption (5.38). (In Subsection 5.3.2, we will extend to the degenerate case and prove Theorem 5.3.2. This is done by taking the limit as $\alpha \to 0$.) In the existence proof, we will use a fixed-point method, which requires the following additional assumptions on β, V :

$$|V''(\theta)| \leq C, \quad |\beta'(\theta)| \leq C. \tag{5.41}$$

Because $p \in P_\infty(\rho)$, we can partition $\mathbb{R}$ into two regions, similar to Section 5.2. We assume that $\rho = 1$ for $x < 0$ and $p = 0$ for $x \geq 0$, and solve separately in the two regions. First, consider $\{x : p(x) > 0\} = (-\infty, 0)$. In this region, under the assumption (5.38), by substituting the condition $p \in P_\infty(\rho)$ in (5.32) with the equation (5.40) for p, it is clear that solutions of (5.32) are solutions of:

$$\begin{cases} -p'' = (1-p)\int_0^\infty (V'(\theta) + \alpha) n(\theta, x) d\theta, & x < 0, \\ (1-p)\partial_\theta n - \partial_x((c + \partial_x p)n) = -(1-p)\beta(\theta) n, & \theta > 0, x < 0, \\ n(0, x) = 2\int_0^\infty \beta(\theta) n(\theta, x) d\theta, & x < 0, \\ c = -p'(0^-). \end{cases} \tag{5.42}$$

We will use Leray-Schauder fixed point theorem to find a solution to (5.42) with $c = -p'(0^-)$. Then, by proving that $\int_0^\infty V(\theta) n(\theta, x) d\theta = 1$ on $(-\infty, 0)$, we recover the condition $p \in P_\infty(\rho)$. We will eventually show that this solution corresponds with $n(\theta, x) = 0$ for $x \geq 0$, and from there, we can conclude that this is a solution to (5.32).

Proposition 5.3.3. *There exists a solution* (n, p) *of* (5.42)*, for* $x \in (-\infty, 0), \theta \in [0, \infty)$*, with* $p(-\infty) = 1, p(0) = 0$*, and* $c = -p'(0^-)$.

This proposition will be proved via a fixed point argument. Given a fixed function $\varphi(x)$ such that $\varphi \geq 0$ and $\int_{-\infty}^{0} \varphi(x)dx < \infty$, consider the cone

$$X = \{u : \int_{-\infty}^{0} u(x)^2 \varphi(x)\, dx < \infty, u \geq 0\},$$

and let $u, w \in X$. Take $\bar{\bar{u}} = \min(\max(u, \frac{\alpha}{V_M}), A)$, for $A > 0$ to be defined later on, $\bar{u} = \max(u, \frac{\alpha}{V_M})$ and $\bar{w} = \min(w, \frac{\beta_2}{V(0)})$. Then we will show that there exist functions p, n that are the solutions to:

$$\begin{cases} -p'' = (1-p)\bar{\bar{u}}, & x \leq 0, \\ -\partial_x((c+p')n) + (1-p)\partial_\theta n = -(1-p)\beta(\theta)n, & x \leq 0, \theta \geq 0, \\ n(0,x) = 2\bar{w} + \frac{1}{V(0)}(\bar{u} - u), & x \leq 0, \end{cases} \tag{5.43}$$

where $c = -p'(0^-)$. We then define the map $T : X \times X \to X \times X$ by

$$T[u, w] = \left(\int_0^\infty (V'(\theta) + \alpha) n(\theta, \cdot) d\theta, \int_0^\infty \beta(\theta) n(\theta, \cdot) d\theta \right).$$

However, we cannot directly apply Leray-Schauder fixed point theorem to the map T, because it is a map on the cone X which is not a Banach space. Instead, we need to define the map

$$\tilde{T}[u, w] = T[u_+, w_+],$$

where u_+, w_+ are the positive parts of u and w, i.e., $u_+(x) = \max\{u(x), 0\}$. $\tilde{T}$ is a map on the Banach space, $L^2_\varphi((-\infty, 0)) \times L^2_\varphi((-\infty, 0))$, where $L^2_\varphi((-\infty, 0)) = \{u : \int_{-\infty}^{0} u(x)^2 \varphi(x) dx < \infty\}$.

We will show using the Leray-Schauder fixed point theorem that $\tilde{T}$ has a fixed point. This

implies that there exist $(u, w) = T[u_+, w_+]$, so $u, w \geq 0$, meaning that $u = u_+, w = w_+$, and so (u, w) is also a fixed point of T. We will then show that these provide a solution to (5.42). Finally, we will show that $\bar{\rho}(x) = 1$ for all $x < 1$, which implies that $p \in P_\infty(\rho)$.

Lemma 5.3.4. *Given $(u, w) \in X$, there exist $p(x)$, $n(\theta, x)$, and c solving (5.43) such that $p(-\infty) = 1, p(0) = 0$.*

Proof. We have that

$$-p'' \geq (1 - p)\frac{\alpha}{V_M}$$

and

$$-p'' \leq (1 - p)A,$$

so solving, we have that $\bar{p}(x) = 1 - e^{\sqrt{A}x}$ is a supersolution and $\underline{p}(x) = 1 - e^{\sqrt{\frac{\alpha}{V_M}}x}$ is a subsolution, and by Perron's Method, there exists a solution p such that

$$1 - e^{\sqrt{\frac{\alpha}{V_M}}x} \leq p(x) \leq 1 - e^{\sqrt{A}x} \tag{5.44}$$

and by taking $c = p'(0^-)$,

$$\sqrt{\frac{\alpha}{V_M}} \leq c \leq \sqrt{A}.$$

Uniqueness for p follows from the proof in Lemma 5.2.6. The equation for n can be solved uniquely by method of characteristics, because $1 - p$ and $c + p'$ are Lipschitz continuous. □

Lemma 5.3.5. *The solution (n, p) of (5.43) satisfies $\int_0^\infty n(\theta, x)d\theta \leq C$ for all $x \in [-1, 0]$, for some C depending only on $\alpha, \beta_2, V_M, V_0$.*

We will consider (5.43) on the interval $x \in [-1, 0]$. The equation for n can be written as:

$$\begin{cases} (1-p)\partial_\theta n - (c+p')\partial_x n = -(1-p)(\beta(\theta) + \bar{\bar{u}})n, & \theta \geq 0, x \in [-1, 0] \\ n(0, x) = 2\bar{w} + \frac{1}{V_0}(\bar{u} - u), & x \in [-1, 0]. \end{cases}$$

Because $p < 1 - e^{-\sqrt{A}}$ on $[-1, 0]$, the equation can be divided by $1 - p$ to get

$$\partial_\theta n - \frac{c+p'}{1-p}\partial_x n = -(\beta(\theta) + \bar{\bar{u}})n \leq -\frac{\alpha}{V_M} n. \tag{5.45}$$

Solving along characteristics,

$$n(\theta, x) \leq n(0, x)e^{-\frac{\alpha}{V_M}\theta} \leq (2\frac{\beta_2}{V_0} + \frac{\alpha}{V_M V_0})e^{-\frac{\alpha}{V_M}\theta}, \quad \text{for all } x \in [-1, 0], \theta \in [0, \infty),$$

so integrating in θ gives

$$\int_0^\infty n(\theta, x)d\theta \leq (2\frac{\beta_2}{V_0} + \frac{\alpha}{V_M V_0})\frac{V_M}{\alpha} \quad \text{for all } x \in [-1, 0],$$

and in particular $\int_0^\infty n(\theta, 0)d\theta$ is finite.

Lemma 5.3.6. *The solution for* (5.43) *satisfies* $0 \leq \bar{\rho}(x) := \int_0^\infty V(\theta)n(\theta, x)d\theta \leq C$, *for all* $x \in (-\infty, 0]$, *for some constant* C, *depending only on* $\alpha, \beta_2, V_0, V_M$.

Proof. Take $h(x) = \int_0^\infty n(\theta, x)d\theta$. It is clear that $h(x) \geq 0$ because $n(\theta, x) \geq 0$. Integrating the

equation for n in (5.43) with respect to θ, we find

$$-\partial_x((c+p')h) = (1-p)(2\bar{w} + \frac{1}{V_0}(\bar{u}-u)) - (1-p)\int_0^\infty \beta(\theta)n(\theta,x)d\theta.$$

Integrating in x from x to 0 and using the fact that h is finite at 0 and $c+p'(0)=0$, gives

$$(c+p'(x))h(x) \leq (2\frac{\beta_2}{V_0} + \frac{\alpha}{V_0 V_M})\int_x^0 (1-p(y))dy \quad \text{for all } x \in (-\infty, 0).$$

On the other hand, integrating the equation for p in (5.43) with respect to x yields:

$$c+p'(x) = \int_x^0 (1-p)\bar{\bar{u}}dy \geq \frac{\alpha}{V_M}\int_x^0 (1-p)dy, \quad \text{for all } x \in (-\infty, 0).$$

Putting these two inequalities together, we obtain:

$$h(x) \leq (2\frac{\beta_2 V_M}{\alpha V_0} + \frac{1}{V_0}) \quad \text{for all } x \in (-\infty, 0).$$

It follows from this bound and Lemma 5.3.5 that $\int_0^\infty V(\theta)n(\theta,x)d\theta \leq V_M h(x) \leq C$ for all $x \in (-\infty, 0]$. □

Remark 5.3.7. *Notice that because V' and β are bounded, $T[u_+, w_+]$ is bounded in $L^\infty((-\infty,0)) \times L^\infty((-\infty,0))$. Because $\int_{-\infty}^0 \varphi(x)dx < \infty$, $T[u_+,w_+]$ will also be bounded in $L^2_\varphi((-\infty,0))\times L^2_\varphi((-\infty,0))$, independently of u and w. At the same time, $T[u_+,w_+] \geq 0$, meaning T is a map from $X \times X$ to itself, and so $\tilde{T}$ is a nonnegative map from $L^2_\varphi \times L^2_\varphi$ to itself.*

We now wish to apply the Leray-Schauder fixed point theorem to the map $\tilde{T}$. To do so, we need to show that $\tilde{T}$ is a compact and continuous operator such that the set $\{(u,w) \in L^2_\varphi \times L^2_\varphi : (u,w) =$

$\lambda\tilde{T}[u,w], 0 \leq \lambda \leq 1\}$ is bounded. We start by proving compactness.

Lemma 5.3.8. *The map $\tilde{T}$ is compact.*

Proof. Let $\{(u_k, w_k)\}$ be a bounded set in $L^2_\varphi \times L^2_\varphi$. Then $\{(u_{k_+}, v_{k_+})\}$ is bounded as well. Let $(v_k, z_k) = \tilde{T}[u_k, w_k] = T[u_{k_+}, w_{k_+}]$. From the previous remark, v_k and z_k are bounded in X independent of k. Multiplying the equation for n by $V'(\theta) + \alpha$ gives

$$\begin{aligned} -(c_k + p_k')v_k' = -(1-p_k)\bar{\bar{u}}_k v_k - (1-p_k)\int_0^\infty \beta(\theta)(V'(\theta)+\alpha)n_k d\theta \\ + \int_0^\infty V''(\theta)n_k d\theta + (1-p_k)(V'(0)+\alpha)(2\bar{w}_k + \frac{1}{V_0}(\bar{u_k} - u_k)). \end{aligned}$$

Then, because $\bar{w}_k, \bar{\bar{u}}_k, \bar{u_k} - u_k, p_k, v_k, V, V'$, and V'' are bounded independent of k, we have that

$$|(c_k + p_k'(x))v_k'(x)| \leq C, \quad \text{for all } x \in (-\infty, 0),$$

for a constant independent of k. Then, recall that

$$c_k + p_k'(x) \geq \frac{\alpha}{V_M}\int_x^0 (1-p)dy \geq \frac{\alpha}{V_M}\int_x^0 e^{\sqrt{A}y}dy = \frac{\alpha}{V_M}(1 - e^{\sqrt{A}x}),$$

so for a small $\varepsilon > 0$, $c_k + p_k'(x) \geq \frac{\alpha}{V_M}(1 - e^{-\varepsilon\sqrt{A}})$ for $x \leq -\varepsilon$. Also, $c_k + p_k'(x) \leq c_k \leq \sqrt{A}$ for all $x < 0$, so

$$|v_k'(x)| \leq C \quad \text{for all } x \in (-\infty, -\varepsilon),$$

for a different constant, also independent from k. A similar argument when multiplying the n equation by $\beta(\theta)$ gives that $|z_k'(x)| \leq C$ for all $x \in (-\infty, -\varepsilon)$, for another constant C independent of k.

These bounds, along with Arzela-Ascoli theorem imply that there exists subsequences of v_k

and z_k, and functions $v, z \in L^\infty((-\infty, 0))$, such that v_k converges to v and z_k converges to z locally uniformly. Then, it can be concluded that along a subsequence, v_k converges to v and z_k converges to z strongly in L^2_φ using the same argument as in Lemma 5.2.11. Therefore, $\{T[u_{k_+}, w_{k_+}]\} = \{\tilde{T}[u_k, w_k]\}$ is relatively compact in $L^2_\varphi \times L^2_\varphi$, so $\tilde{T}$ is compact. □

Next, we prove that $\tilde{T}$ is continuous.

Lemma 5.3.9. *The map $\tilde{T}$ is continuous.*

Proof. If $u_k \to u$ and $w_k \to w$, then $u_{k_+} \to u_+$ and $w_{k_+} \to w_+$. The proof that T is continuous is essentially identical to the proof of Lemma 5.2.12, so it is omitted. Then, it follows directly that $\tilde{T}$ is continuous. □

The final requirement for applying the fixed point theorem is the following:

Lemma 5.3.10. *The set $\{(u, w) \in L^2_\varphi \times L^2_\varphi : (u, w) = \lambda \tilde{T}[u, w], 0 \leq \lambda \leq 1\}$ is bounded.*

Proof. Consider $(u, w) = \lambda \tilde{T}[u, w] = \lambda T[u_+, w_+]$. Then, $u, w \geq 0$, so the result follows from this lower bound and the upper bound in Lemma 5.3.6. □

Now that we have shown the requirements for Leray-Schauder fixed point theorem, we will apply the theorem and conclude the proof of Proposition 5.3.3.

Proof of Proposition 5.3.3. This proof consists of showing that the system (5.42) has a solution. From the previous three lemmas and Leray-Schauder fixed point theorem, there exists a fixed point of $\tilde{T}$, i.e., there is $(u, w) = \tilde{T}[u, w] = T[u_+, w_+]$, so $u, w \geq 0$ meaning that (u, w) is a fixed point of T as well. This implies that there exists $u, v \in X$ such that

$$u(x) = \int_0^\infty (V'(\theta) + \alpha) n(\theta, x) d\theta,$$

$$w(x) = \int_0^\infty \beta(\theta) n(\theta, x) d\theta.$$

To conclude, it remains to show that $u = \bar{\bar{u}}$, $w = \bar{w}$, and $\bar{\rho}(x) := \int_0^\infty V(\theta) n(\theta, x) d\theta = 1$ for all $x < 0$. First, we have

$$u(x) \leq (V_M' + \alpha) h(x) \leq 2(V_M' + \alpha)(\frac{\beta_2 V_M}{\alpha V_0} + \frac{1}{V_0}) \text{ for all } x < 0,$$

from the bound on h from Lemma 5.3.6. By taking $A \geq 2(V_M' + \alpha)(\frac{\beta_2 V_M}{\alpha V_0} + \frac{1}{V_0})$, we have $\bar{u} = \bar{\bar{u}}$. Then, using (5.43),

$$\begin{aligned}
-\partial_x((c + p')(\bar{\rho} - 1)) &= -(1-p)\bar{\bar{u}} + (1-p)\int_0^\infty V'(\theta) n d\theta + (1-p)V_0(2\bar{w} + \frac{1}{V_0}(\bar{u} - u)) \\
&\quad - (1-p)\int_0^\infty V(\theta)\beta(\theta) n d\theta \\
&= -(1-p)\bar{u} + (1-p)\int_0^\infty V'(\theta) n d\theta + (1-p)V_0 2(\bar{w} - w) + (1-p)V_0 2w \\
&\quad + (1-p)(\bar{u} - u) - (1-p)\int_0^\infty V(\theta)\beta(\theta) n d\theta \\
&= 2V_0(1-p)(\bar{w} - w),
\end{aligned}$$

recalling that $(2V_0 - V(\theta))\beta(\theta) = \alpha$ From the definition of $\bar{w}$, $\bar{w} \leq w$, so by integrating the above from x to 0, and using $c + p'(0) = 0$,

$$(c + p'(x))(\bar{\rho}(x) - 1) \leq 0,$$

and because $c + p' \geq 0$, it holds that

$$\bar{\rho}(x) \leq 1 \text{ for all } x < 0.$$

Then

$$w(x) = \int_0^\infty \beta(\theta) n(\theta, x) d\theta \leq \frac{\beta_2}{V_0} \bar{\rho}(x) < \frac{\beta_2}{V_0} \leq \bar{w}(x),$$

so $w = \bar{w}$ and $(c + p')(\bar{\rho} - 1)$ is constant, i.e.,

$$(c + p'(x))(\bar{\rho}(x) - 1) = (c + p'(0))(\bar{\rho}(0) - 1) = 0.$$

Because $c = -p'(0)$ and $c > -p'(x)$ for all $x < 0$, it holds that

$$\bar{\rho}(x) = 1 \text{ for all } x < 0. \tag{5.46}$$

Also,

$$u(x) = \int_0^\infty (V'(\theta) + \alpha) n(\theta, x) d\theta \geq \frac{\alpha}{V_M} \bar{\rho}(x) = \frac{\alpha}{V_M},$$

so $u \geq \bar{u}$, and

$$u = \bar{\bar{u}}.$$

□

5.3.2 Extension to the degenerate case

So far, we have shown that there exists a solution of the system (5.42) on the half line $(-\infty, 0)$. In order to extend this result to prove our main theorem, Theorem 5.3.2, we need to show that a solution of (5.32) with assumption (5.35) exists, and the solution is zero on $[0, \infty)$. To do so, we will first take $\alpha \to 0$ in the solution we found for (5.42). This consists of showing that n_α, p_α, and c_α are bounded independently of α, and that c_α has a strictly positive lower bound, to ensure that the limit does not converge to $c = 0$. Then, we use (5.46) to recover $p \in P_\infty(\rho)$ and finally show that $n(\theta, x) = 0$ for all $x \geq 0$, concluding the proof of the theorem.

Proof of Theorem 5.3.2. To prove the existence of a solution of (5.32) with assumption (5.35), we need to show that the limit $\alpha \to 0$ can be taken. Let $(n_\alpha, p_\alpha, c_\alpha)$ be the solution that we found of (5.42) on $(-\infty, 0)$ with assumption (5.38), where $V = V_\alpha$ and $\beta = \beta_\alpha$, for some functions V_α and β_α. This means $(2V_\alpha(0) - V_\alpha(\theta))\beta_\alpha(\theta) = \alpha$ for all $\theta \geq 0$, and the following assumptions hold:

$$0 \leq \beta_\alpha(\theta) \leq \beta_2, V_{0,\alpha} = V_\alpha(0) \leq V_\alpha(\theta) \leq V_{M,\alpha}.$$

We will make the following additional assumptions: Suppose $\alpha \leq \frac{1}{4}$ and that there exist constants $C_1, C_2, C_3, C_4, C_5, C_6, C_7 > 0$ independent of α, such that for all α,

$$V_{0,\alpha} \geq C_1, \quad C_2 \leq V_{M,\alpha} \leq C_3, \quad 0 \leq V'_\alpha(\theta) \leq C_4,$$

and

$$\beta_\alpha(\theta) \geq C_5 \text{ for } \theta \geq \theta_2, \quad V'_\alpha(\theta) \geq C_6 \text{ for } \theta \leq \theta_1, \quad 0 \leq \theta_2 - \theta_1 \leq C_7.$$

This "discreteness" of β and V will be necessary to prove a uniform lower bound on c_α. Finally, we suppose that $\lim_{\alpha\to 0}\beta_\alpha(\theta)=\beta(\theta)$ and $\lim_{\alpha\to 0}V_\alpha(\theta)=V(\theta)$ exist and satisfy $(2V(0)-V(\theta))\beta(\theta)=0$.

Then, for all α, using (5.45) to see that $n_\alpha(\theta,x)\le n_\alpha(0,x)$ for all θ, we have that

$$0\le n_\alpha(\theta,x)\le 2\frac{\beta_2}{V_\alpha(0)}\le\frac{2\beta_2}{C_1}, \tag{5.47}$$

and from (5.44), we have that

$$0\le p_\alpha(x)\le 1.$$

Also, because $\rho_\alpha\le 1$,

$$\begin{aligned}-p_\alpha'' &= (1-p_\alpha)\int_0^\infty (V_\alpha'(\theta)+\alpha)n\,d\theta\\ &\le (1-p_\alpha)\frac{C_4+\frac{1}{4}}{V_\alpha(0)}\\ &\le \frac{C_4+\frac{1}{4}}{V_\alpha(0)}\\ &\le \frac{C_4+\frac{1}{4}}{C_1},\end{aligned}$$

and

$$\begin{aligned}-p_\alpha'' &\ge (1-p_\alpha)\frac{\alpha}{V_{M,\alpha}}\\ &\ge 0.\end{aligned}$$

The bounds

$$(1-p_\alpha)\frac{\alpha}{V_{M,\alpha}} \leq -p_\alpha'' \leq (1-p_\alpha)\frac{C_4+\frac{1}{4}}{C_1}$$

also imply, by the sub- and super-solution method of Lemma 5.3.4, that

$$0 \leq \sqrt{\frac{\alpha}{V_{M,\alpha}}} \leq c_\alpha \leq \sqrt{\frac{C_4+\frac{1}{4}}{C_1}}$$

and because p_α' is decreasing,

$$0 \geq p_\alpha'(x) \geq p_\alpha'(0) \geq -\sqrt{\frac{C_4+\frac{1}{4}}{C_1}}.$$

These bounds are enough to pass $n_\alpha, p_\alpha, c_\alpha$ to the limit as $\alpha \to 0$ by the Arzela-Ascoli theorem. It remains to verify that c_α is bounded below independent of α, so that we ensure as $\alpha \to 0$, c_α doesn't converge to 0.

To do so, assume that $c_\alpha < \varepsilon \leq \frac{1}{2}$, for $\varepsilon > 0$ independent of α, to be defined later on. Using this and the fact that p_α' is decreasing, we have that $p_\alpha'(x) \geq -\frac{1}{2}$ for all $x \in [-1, 0]$, and because $p_\alpha(0) = 0$, $p_\alpha(x) \leq \frac{1}{2}$ for all $x \in [-1, 0]$. Using the non-positivity of p_α', integrating the equation for p_α in x on

$[-1, 0]$, we have

$$\begin{aligned}
\varepsilon &\geq c_\alpha + p'_\alpha(-1) \\
&= \int_{-1}^{0} -p''_\alpha dx \\
&= \int_{-1}^{0} (1 - p_\alpha) \int_0^\infty (V'_\alpha(\theta) + \alpha) n_\alpha d\theta dx \\
&\geq \frac{1}{2} \int_{-1}^{0} \int_0^{\theta_1} V'_\alpha(\theta) n_\alpha d\theta dx \\
&\geq \frac{C_6}{2} \int_{-1}^{0} \int_0^{\theta_1} n_\alpha d\theta dx,
\end{aligned}$$

and so

$$\int_{-1}^{0} \int_0^{\theta_1} n_\alpha d\theta dx \leq \frac{2\varepsilon}{C_6}. \tag{5.48}$$

Integrating the equation for n_α in x on $[-1, 0]$, we have

$$\begin{aligned}
\frac{d}{d\theta} \int_{-1}^{0} (1 - p_\alpha) n_\alpha dx &= -(c_\alpha + p'_\alpha(-1)) n_\alpha(\theta, -1) - \int_{-1}^{0} \beta_\alpha(\theta)(1 - p_\alpha) n_\alpha dx \\
&\geq -2\frac{\beta_2}{C_1}\varepsilon - \int_{-1}^{0} \beta_\alpha(\theta)(1 - p_\alpha) n_\alpha dx,
\end{aligned}$$

using the fact that $c_\alpha + p'_\alpha(-1) \leq c_\alpha \leq \varepsilon$ and the bound (5.47) on n_α. Integrating from 0 to $\theta \leq \theta_2$,

$$\begin{aligned}
\int_{-1}^{0} (1 - p_\alpha) n_\alpha(\theta, x) dx - \int_{-1}^{0} (1 - p_\alpha) 2 \int_0^\infty \beta_\alpha(\tilde{\theta}) n_\alpha d\tilde{\theta} dx & \\
\geq -2\frac{\beta_2}{C_1}\varepsilon\theta - \int_{-1}^{0} \int_0^{\theta} \beta_\alpha(\tilde{\theta})(1 - p_\alpha) n_\alpha d\tilde{\theta} dx &
\end{aligned}$$

Then using that $p_\alpha(x) \leq \frac{1}{2}$ for $x \in [-1, 0]$ and $1 - p_\alpha(x) \leq 1$ for all $x \in (-\infty, 0]$,

$$\begin{aligned}\int_{-1}^{0} n_\alpha(\theta, x)dx &\geq -2\frac{\beta_2}{C_1}\varepsilon\theta + \int_{-1}^{0}\int_{\theta}^{\infty} \beta_\alpha(\tilde{\theta})n_\alpha(\tilde{\theta}, x)d\tilde{\theta}dx \\ &\geq -2\frac{\beta_2}{C_1}\varepsilon\theta_2 + \int_{-1}^{0}\int_{\theta_2}^{\infty} \beta_\alpha(\tilde{\theta})n_\alpha(\tilde{\theta}, x)d\tilde{\theta}dx \\ &\geq -2\frac{\beta_2}{C_1}\varepsilon\theta_2 + C_5\int_{-1}^{0}\int_{\theta_2}^{\infty} n_\alpha(\tilde{\theta}, x)d\tilde{\theta}dx.\end{aligned}$$

Then integrating from 0 to θ_2,

$$\int_0^{\theta_2}\int_{-1}^{0} n_\alpha(\theta, x)dxd\theta \geq -2\frac{\beta_2}{C_1}\varepsilon\theta_2^2 + C_5\theta_2\int_{\theta_2}^{\infty}\int_{-1}^{0} n_\alpha(\theta, x)dxd\theta. \tag{5.49}$$

Also, using the fact that $\rho_\alpha = 1$,

$$\int_0^{\theta_2}\int_{-1}^{0} n_\alpha(\theta, x)dxd\theta + \int_{\theta_2}^{\infty}\int_{-1}^{0} n_\alpha(\theta, x)d\theta \geq \int_{-1}^{0}\frac{1}{V_{M,\alpha}}\int_0^{\infty} V_\alpha(\theta)n_\alpha(\theta, x)d\theta dx = \frac{1}{V_{M,\alpha}} \geq \frac{1}{C_3}.$$

Combining this with (5.49),

$$\begin{aligned}\frac{\theta_2}{C_3} &\leq \theta_2\Big(\int_0^{\theta_2}\int_{-1}^{0} n_\alpha(\theta, x)dxd\theta + \int_{\theta_2}^{\infty}\int_{-1}^{0} n_\alpha(\theta, x)d\theta\Big) \\ &\leq (\frac{1}{C_5} + \theta_2)\int_0^{\theta_2}\int_{-1}^{0} n_\alpha(\theta, x)dxd\theta + 2\frac{\beta_2}{C_1}\varepsilon\theta_2^2,\end{aligned}$$

meaning that

$$\int_0^{\theta_2}\int_{-1}^{0} n_\alpha(\theta, x)dxd\theta \geq \frac{\frac{\theta_2}{C_3} - 2\frac{\beta_2}{C_1}\varepsilon\theta_2^2}{\frac{1}{C_5} + \theta_2} = \frac{C_5\theta_2 C_1 - 2C_5C_3\beta_2\theta_2^2\varepsilon}{C_1C_3(1 + C_5\theta_2)}.$$

But from (5.48) and (5.47),

$$\int_0^{\theta_2}\int_{-1}^{0} n_\alpha(\theta,x)dxd\theta = \int_0^{\theta_1}\int_{-1}^{0} n_\alpha(\theta,x)dxd\theta + \int_{\theta_1}^{\theta_2}\int_{-1}^{0} n_\alpha(\theta,x)dxd\theta \leq \frac{2\varepsilon}{C_6} + \frac{2\beta_2}{C_1}(\theta_2-\theta_1),$$

so that

$$\frac{C_5\theta_2 C_1 - 2C_5C_3\beta_2\theta_2^2\varepsilon}{C_1C_3(1+C_5\theta_2)} \leq \frac{2\varepsilon}{C_6} + \frac{2\beta_2}{C_1}(\theta_2-\theta_1) \leq \frac{2\varepsilon}{C_6} + \frac{2\beta_2 C_7}{C_1}$$

and rearranging,

$$\varepsilon \geq \frac{\frac{C_5\theta_2}{C_3(1+C_5\theta_2)} - \frac{2\beta_2C_7}{C_1}}{\frac{2}{C_6} + \frac{2C_5\beta_2\theta_2^2}{C_1(1+C_5\theta_2)}},$$

which is a contradiction by taking C_7 small enough and

$$\varepsilon = \frac{\frac{C_5\theta_2}{C_3(1+C_5\theta_2)} - \frac{2\beta_2C_7}{C_1}}{\frac{4}{C_6} + \frac{4C_5\beta_2\theta_2^2}{C_1(1+C_5\theta_2)}}$$

Then we have that for all α,

$$c_\alpha \geq \varepsilon.$$

Finally, we can take $\alpha \to 0$ to get a limit (n,p,c), with $c = -p'(0^-)$. Using (5.46) to recover $p \in P_\infty(\rho)$, we conclude that this limit is a solution to (5.32) in the region $x \in (-\infty, 0)$. To take care of the region $x \in [0,\infty)$, we use the same logic as in Theorem 5.2.1. We have that $(c\rho + \rho p')$ is continuous, meaning that $c = \frac{-p'(0^-)}{1-\rho(0^+)}$, so for the solution with $c = -p'(0)$, it holds that $\rho(0^+) = 0$. Also, on $(0,\infty)$, $p = 0$ so (5.34) becomes

$$\rho'(x) = -\int_0^\infty V'(\theta)n(\theta,x)d\theta \leq 0,$$

meaning that

$$\rho(x) = 0, \text{ for } x > 0, \text{ and } n(\theta, x) = 0 \text{ for } x > 0, \theta \geq 0.$$

□

Remark 5.3.11. *An example of a pair V_α, β_α that satisfy the requirements for this theorem when V, β are given by* (5.36)-(5.37) *is:*

$$V_\alpha(\theta) = \begin{cases} V(0) + (\gamma - \alpha)\theta, & \theta \leq \theta_0 - \omega(\alpha), \\ \textit{smooth}, & \theta_0 - \omega(\alpha) \leq \theta \leq \theta_0, \\ 2V(0) - \alpha, & \theta \geq \theta_0, \end{cases}$$

$$\beta_\alpha(\theta) = \begin{cases} \frac{\alpha}{V(0) - (\gamma - \alpha)\theta}, & \theta \leq \theta_0 - \omega(\alpha), \\ \textit{smooth}, & \theta_0 - \omega(\alpha) \leq \theta \leq \theta_0, \\ 1, & \theta \geq \theta_0. \end{cases}$$

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