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A Computational Method for H∞ Controller Design in the Frequency Domain

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A Computational Method for H^∞ Controller Design in the Frequency Domain*

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Abstract

A new approach to frequency domain design of robust controllers for distributed parameter systems is presented. The central idea is to use techniques that were developed for the solution of the Corona Problem, for the solution of both the Bezout equation and an auxiliary equation that arises form the Nehari interpolation problem. An algebraic reformulation of these equations allows the solution to be computed from the solution of an inhomogeneous Cauchy Riemann equation with a Carleson measure as the inhomogeneous term. The theory is applied to a single input single output system with delay to yield the transfer function of a stabilizing controller with guaranteed H^∞ stability margin. Finally the framework is extended to handle multi-input multi-output systems.

Key words: Robust Control, Linear Systems, H^∞, Cauchy Riemann Equations, Interpolation.

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1 Introduction

This paper presents a new approach to H^∞ controller design for a general class of linear systems that are described by irrational transfer functions. The approach is based on a computational method for solving linear Diophantine equations in algebras over the ring of H^∞ functions. The term Diophantine here is used in analogy with Diophantine equations over the ring of integers to refer to algebraic equations over more general rings, in this case rings constructed from function spaces. The term linear is used to indicate that the equations that are being considered do not involve products of the unknown variable, in other words they have the same form as linear algebraic equations over fields. Two such equations occur in the design of H^∞ controllers for linear systems: the first is the Bezout identity that leads to the parameterization of all stabilizing controllers, and the second is an equation that arises from the Nehari problem. Bounded solutions to the second equation provide controllers with guaranteed robustness in the sense of H^∞ control theory.

The extension of H^∞ design techniques to distributed parameter systems has been an area of active research since the 1980's. In the report [Cur92] Curtain compares five approaches that were developed during this period. The approach that has enjoyed the greatest practical success is based on the work of Glover, Curtain and Partington [GCP88], in this paper the authors show that if a distributed system has a Hankel operator of nuclear type, then the system may be approximated by a finite dimensional system with a guaranteed bound on the L^∞ norm of the error. This result means that a finite dimensional controller designed for the approximate plant will also control the infinite dimensional plant with guaranteed (though reduced) H^∞ stability margins. For linear plants that are associated with a nuclear Hankel operator, a class that includes many cases of practical interest, this method provides a practical solution to the problem of controller design.

In spite of this result, research has continued on ways to attack the problem of controller design directly, without the initial finite dimensional approximation to the plant. As in the case of the theory that was developed for rational plants, the work on infinite dimensional systems can be divided between approaches that consider the plant as a linear state-space system with an infinite dimensional state space, and approaches that consider the plant as a transfer function that acts multiplicatively in the frequency domain on the transform of the signal space. An example of the first approach is found in the work of van Kuelen [vK93] in which the author presents a pair of infinite dimensional Riccati equations with solutions that are the basis for a state space representation of a controller. Examples of the second approach are found in the work of Foias, Tannenbaum, Özbay and Smith [FT88a], [FT88b], [OST93], [Ö93], and in the work of Dym, Geogiou and Smith [GS92], [DGS93]. The two sets of authors design H^∞ controllers using methods based on the theory of skew Toeplitz operators that is presented by Bercovici, Foias and Tannenbaum in [BFT88]. The authors Flamm, Mitter and Yang independently follow a similar approach in [FM87] and [FY94].

The method of controller design presented in this paper takes the frequency domain approach. It extends the method presented by Francis in [Fra87] for systems with rational transfer functions to systems with irrational transfer functions. The method starts with a co-prime factorization of the plant, solves the associated Bezout Equation, and from this computes a Youla parameterization of all stabilizing controllers. An associated Nehari problem is solved approximately and the solution provides a controller which, although not optimal, has guaranteed robustness in an H^∞ sense. The result of the controller computation is a numerical approximation to the controller's transfer functions rather than a closed form expression composed of elementary functions.

The main contribution of this paper is the presentation of a new method for solving both the Bezout equation that leads to the Youla parameterization and the equation that is derived from the Nehari Problem. This method is based on a constructive proof of the corona theorem presented by Garnett in [Gar81]. The constructive part involves an explicit solution to the inhomogeneous Cauchy Riemann equation $\partial b/\partial \bar{z} = \mu$, where the inhomogeneous part μ is a Carleson measure, and the solution b is a distribution with support in the right half plane and an L^∞ boundary value. The terminology is explained in detail in Section 3. The method of solution of this first order partial differential equation, which is due to Jones [Jon80] [Jon83], combines a careful decomposition of the measure μ with a Green's function method.

It seems most logical to compare the results achieved here with those achieved by the methods developed in [OST93], and to this end the example presented by Enns, Özbay and Tannenbaum in [EOT92] is reworked in this paper with the techniques developed here. The steps in the computation of a compensator transfer function are presented in detail, and an open-loop Nyquist plot for the compensated system is presented along with magnitude plots of appropriate sensitivity transfer functions for the closed loop system. Enns et al. are able to compute an optimal solution for this problem, so a comparison with the results in [EOT92] provides a good indication of the distance between the solution computed here and an optimal solution.

The paper is divided into six sections of which this introduction is the first. In the second section the example from [EOT92] is presented and an outline of the method to be employed for its solution is given, in particular this section shows how linear Diophantine equations occur in the H^∞ design problem. The third section provides the theory that is used to solve the linear Diophantine equations. This is necessarily the most technical section of the paper and includes a number of ideas that are not commonly met in the control literature. The fourth section shows how the algorithms that are described in section three are applied to the example introduced in section two. Included in this section is a description of the structure of the software that was developed to implement the algorithms and the results of the controller design. Section five contains the theory needed to extend the methods presented in the previous sections to general multiple-input multiple-output systems, and the final section contains provides some conclusions and a discussion of future directions

2 An Example

A concrete example is introduced in this section in order to establish a design method and provide motivation for the techniques that are introduced in the following section. The example uses a simplified model of the pitch-axis fast dynamics of an unstable aircraft taken from a paper by Enns, Özbay and Tannenbaum [EOT92]. The plant, which is given by the transfer function[1]

$$F(z) = \frac{e^{-\tau z}}{\sigma z - 1}, \tag{1}$$

has an unstable pole at $z = 1/\sigma$, and a delay of τ seconds, as such it is amongst the simplest unstable infinite dimensional systems. The design objective is to produce a linear feedback controller with the configuration illustrated in Figure 1 that both robustly stabilizes the plant and maintains low low-frequency sensitivity.

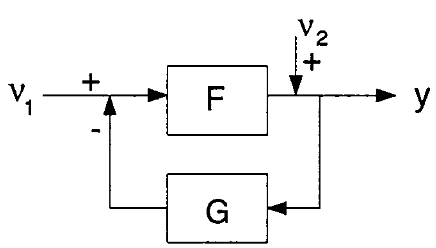

Figure 1: Feedback Controller

The design proceeds by the standard procedure [Fra87] of reformulating the problem as the minimization of the L^∞ norm of an affine expression over an H^∞ parameter. Let

$$\begin{aligned} F_1(z) &= e^{-\tau z}/(\sigma z + 1) \\ F_2(z) &= (\sigma z - 1)/(\sigma z + 1) \end{aligned} \tag{2}$$

[1] The symbol z is used throughout the paper to denote an independent complex variable in, for example, transfer functions. While this notation is at odds with the usual choice of s for the independent variable in the Laplace transform, it is standard in complex analysis, and it's adoption here offers the benefit of maintaining consistency between the notation in Section 3 and the notation in the literature, as well as an improvement in the self-consistency of the notation.

then $F(z) = F_1(z)/F_2(z)$ and $\inf_{\mathrm{Re}\, z>0}(|F_1(z)| + |F_2(z)|) > 0$ so F_1 and F_2 form a co-prime factorization of F. Let X_1 and X_2 be functions in H^∞ that solve the Bezout equation[2]

$$F_1(z)X_1(z) + F_2(z)X_2(z) = 1 \tag{3}$$

then all stabilizing controllers may be expressed in terms of an H^∞ parameter Q by the bilinear function

$$G(z) = \frac{X_1(z) + F_2(z)Q(z)}{X_2(z) - F_1(z)Q(z)}. \tag{4}$$

This parameterization was first given by Youla et al. [YBJ76] for the case of rational transfer functions and was extended to irrational transfer functions by Baras in [Bar80]. Smith gives a proof of the existence of strongly co-prime factorizations for stabilizable plants in [Smi89].

A key step in the parameterization is the solution of the Bezout equation. If the plant has rational co-prime factors, then the equation may be solved by algorithms that exploit the Euclidean domain structure of the ring of polynomials such as that given by Kailath in Chapter 2 of [Kai80], or by algorithms that rely on state space techniques such as that used by Francis in [Fra87]. Neither technique will handle irrational transfer functions however, and people working with irrational transfer functions have been restricted to systems that are sufficiently simple that the solution to the Bezout equation may be found by inspection. This is the case in the example presented in Equation (1); the authors of [EOT92] give the solution

$$\begin{aligned} X_1(z) &= e^{T/\sigma} \\ X_2(z) &= \frac{(\sigma z + 1) - 2e^{T/\sigma}e^{-Tz}}{(\sigma z - 1)}. \end{aligned}$$

The lack of a good method for computing solutions to the Bezout equation has hindered the application of frequency domain methods to controller design for plants with irrational transfer functions.

Returning to the example, and in line with the H^∞ design methodology, the design goals of robust stabilization and low low-frequency sensitivity are reformulated as the requirement that the controller should be chosen so that the weighted norms of a pair of transfer functions should be minimized. The sensitivity is given by the closed loop transfer function $(1 + FG)^{-1}$ and the

[2]The symbol 1 is used to represent the constant function $1(z) = 1$.

robust stabilization requirement is interpreted as meaning that the controller should stabilize all plants with transfer functions within an L^∞ neighborhood of the nominal plant. An argument based on the Nyquist theorem (Theorem 1 of [CD82]) assures a stable neighborhood provided the transfer function $G(1+FG)^{-1}$ is bounded above. Both objectives are satisfied with the choice of a feedback controller G that minimizes a combination of the norms of both transfer functions Such a controller attempts to optimize both the sensitivity and robustness of the closed loop system. The relative importance of the sensitivity and robustness objectives are controlled in a frequency dependent fashion by multiplying the two transfer functions by H^∞ weighting functions, $W_1(z)$ and $W_2(z)$. A good description of the H^∞ design methodology may be found in [Fra87] [DFT92], and details of its particular application to the pitch control problem are given in [EOT92].

In short the design specification is translated into the requirement that the controller should be chosen to stabilize the plant and minimize the value of the norm

$$\left\| \begin{pmatrix} W_2 G(1+FG)^{-1} \\ W_1(1+FG)^{-1} \end{pmatrix} \right\|_\infty, \tag{5}$$

where W_1 and W_2 are the weighting functions referred to earlier in the previous paragraph. If the Youla parameterization (4) is substituted for G in (5) then the objective is transformed into the norm of the affine expression

$$\left\| \begin{pmatrix} F_2(z)(X_1(z)+F_2(z)Q(z))W_2(z) \\ F_2(z)(X_2(Z)-F_1(z)Q(z))W_1(z) \end{pmatrix} \right\|_\infty. \tag{6}$$

The design problem is to find a function Q in H^∞ that produces a value for (6) within a predefined upper bound. This problem, which is related to the classical Nehari problem has been at the center of frequency domain approaches to the H^∞ control system design since the early work of Francis et al. [FHZ84], [FZ84]. The authors of [EOT92] and [FY94] use operator theoretic techniques to tackle the Nehari Problem, the method that this paper presents starts from a different point of view. If the affine expression in (6) is labeled P, then

$$P(z) = \begin{pmatrix} F_2(z)X_1(z)W_2(z) \\ F_2(z)X_2(z)W_1(z) \end{pmatrix} + Q(z)\begin{pmatrix} F_2(z)^2 W_2(z) \\ -F_2(z)F_1(z)W_1(z) \end{pmatrix},$$

which may be rewritten as,

$$\begin{pmatrix} F_2(z)X_1(z)W_2(z) \\ F_2(z)X_2(z)W_1(z) \end{pmatrix} = 1P(z) + Q(z)\begin{pmatrix} -F_2(z)^2 W_2(z) \\ F_2(z)F_1(z)W_1(z) \end{pmatrix}, \tag{7}$$

and the design problem may be viewed as the problem of solving a linear Diophantine equation with the added requirement that a norm on the solution should be minimized. More specifically, one must find Q, an H^∞ function, and P, a matrix of H^∞ functions which together solve (7) with the added constraint that the L^∞ norm for the largest singular value of P should lie within a specified bound. An optimal solution is one for which the norm on P is a minimum with respect to the values that this norm takes over all solutions P to Equation (7). The algorithm presented here will not, in general, achieve that minimum, and we demonstrate in Section 4 that the discrepancy can be large. Equation (7) resembles the Bezout equation (3) which is also a linear Diophantine equation, and the same basic technique is used to compute solutions for both of them.

3 Solving Linear Diophantine Equations

Let f_1, f_2 and h be three functions in H^∞. This section addresses the problem of finding solutions h_1 and h_2 in H^∞ for the equation

$$f_1 g_1 + f_2 g_2 = h. \tag{8}$$

Equation (8) subsumes the Bezout equation (3), and if the objective function (6) were replaced by a scalar objective function, it would subsume the linear equation (7) that arises in the Nehari problem as well. If f_1 and f_2 are outer functions, which is to say that they possess multiplicative inverses in H^∞, then the solution is easy. A family of solutions with parameter η an H^∞ function is formed by setting $g_1 = \eta h f_1^{-1}$ and $g_2 = (1-\eta)h f_2^{-1}$. When f_1 and f_2 have zeros in the right half plane, the inverses no longer exist and this method breaks down, but if the requirement that the solutions be in H^∞ is temporarily relaxed then bounded (but not analytic) solutions may be found as follows.

Let ϕ be a bounded function on the half plane with the property that the zero set of f_1 is bounded away from the support of ϕ, and the the zero set of f_2 is bounded away from the support of $1-\phi$. For such a function to exist some restriction needs to be placed on the functions f_1 and f_2, for instance, it is necessary that they should have no common zeros. Bounded solutions to Equation 8 can be constructed in a piecewise fashion by taking $\tilde{g}_1 = 0$ outside the support of ϕ, and $\tilde{g}_1 = \phi h f_1^{-1}$ on the support of ϕ and for the second function $\tilde{g}_2 = 0$ outside the support of $1-\phi$ and $\tilde{g}_2 = (1-\phi)h f_2^{-1}$ on the support of $1-\phi$.

Observe that if e is a bounded function on the right half plane, then the two functions $-ef_1$ and ef_2 satisfy the relation

$$(-ef_1)f_2 + (ef_2)f_1 = 0.$$

With this in mind the step from the bounded solutions $\tilde{g}_1$ and $\tilde{g}_2$ to H^∞ solutions can be made if a bounded function e can be found such that the functions

$$\begin{aligned} g_1 &= \tilde{g}_1 + ef_2 \\ g_2 &= \tilde{g}_2 - ef_1 \end{aligned} \tag{9}$$

are are both in H^∞. In this section it is shown that if the function ϕ is chosen appropriately, a suitable function e can be calculated as a solution to a first order partial differential equation. The H^∞ norms of the solutions, g_1 and g_2, depend on the choice of ϕ and e. In the case of the Nehari Problem described in section 2, a bound on the norm $\|P\|_\infty$ is obtained by separately bounding quantities that correspond to $\widetilde{g}_1$ and e in equation (9). As a consequence, success in calculating solutions with small $\|P\|_\infty$ norms depends critically on controlling the norm of the function e. Section 4, which presents a solution to the problem posed in Section 2, contains further discussion of this point.

3.1 Algebraic Reformulation

The appropriate setting in which to make the introductory paragraphs of this section precise is the setting of homological algebra. This setting was first presented by Hörmander [Hör67] in conjunction with the corona problem, and has been used by Berenstein, Taylor, Struppa and Yger [BS86], [BT80], [Str83], [BY89] for the investigation of linear Diophantine equations in algebras over rings of analytic functions of bounded growth. In the simple cases of equations that arise from single-input single-output systems the algebraic formalism reduces precisely to the equations of the opening paragraph. A good introduction to the analysis used in this chapter is [BG91], and the survey [BS93] describes recent applications of the methods that are used in this paper to other problems in the analysis of linear operators.

Let R denote a ring of functions (or distributions) on the half plane $\mathcal{H}$. For any positive integer m let $\Lambda(R)$ denote the graded module over R that consists of functions on $\mathbb{C}$ that take values in the exterior algebra of antisymmetric forms on an m-dimensional vector space. Let $\Lambda^k(R)$ denote the homogeneous elements in $\Lambda(R)$ of order k. $\Lambda(R)$ is a finite free module over R, and a basis element will be denoted by $e_{i_1,\ldots,i_k}$ with the indices ordered $1 \leq i_1 \leq \ldots \leq i_k \leq m$. If $j \in \{i_1,\ldots,i_k\}$ then the symbol $e_{i_1,\ldots,\widehat{j},\ldots,i_k}$ denotes the basis element of Λ^{k-1} that is formed by deleting the index j from $e_{i_1,\ldots,i_k}$; if $j \notin \{i_1,\ldots,i_k\}$ then $e_{i_1,\ldots,\widehat{j},\ldots,i_k} = 0$. Suppose that $f = \{f_1,\ldots f_m\}$ is a finite subset of R, then the operator P_f defined on $\Lambda(R)$ acts on a basis element as follows

$$P_f(e_{i_1,\ldots,i_k}) = \sum_{j=1}^{k} f_{i_j} e_{i_1,\ldots,\widehat{i_j},\ldots,i_k}$$

The operator P_f forms an exact sequence over the homogeneous submodules Λ^k called the Koszul complex. In the case of $m = 2$ the Koszul complex is represented by the diagram

$$0 \longrightarrow \Lambda^2(R) \xrightarrow{P_f} \Lambda^1(R) \xrightarrow{P_f} \Lambda^0(R) \longrightarrow 0, \tag{10}$$

and when $R = \mathrm{H}^\infty$, this corresponds exactly to the concrete algebraic setting described in the opening paragraphs of this section. Finding a solution to Equation (8) is equivalent to inverting the operator $P_f : \Lambda^1(\mathrm{H}^\infty) \to \Lambda^0(\mathrm{H}^\infty)$. The approach to inverting the operator P_f that was outlined in the introduction was first to find an inverse in a larger space than $\Lambda^1(\mathrm{H}^\infty)$, and then add an element from the image of $P_f : \Lambda^2(\mathrm{H}^\infty) \to \Lambda^1(\mathrm{H}^\infty)$ that will return the solution to H^∞. The appropriate ring in which to invert P_f is the ring of distributions which will be introduced with the following definitions from Hörmander [Hör67].

Denote the open right half plane by $\mathcal{H}$, its closure by $\bar{\mathcal{H}}$, and its boundary, the imaginary axis, by $\partial\mathcal{H}$. Let μ be a measure with support on $\mathcal{H}$, then a distribution u on $\mathcal{H}$ satisfies the equation[3]

$$\frac{\partial u}{\partial \bar{z}} = \mu \tag{11}$$

if for any continuously differentiable test function ψ with support compactly contained in $\mathcal{H}$,

$$\begin{aligned} \left\langle \frac{\partial u}{\partial \bar{z}}, \psi \right\rangle &= -\int_{\mathcal{H}} u(z) \frac{\partial \psi(z)}{\partial \bar{z}} \, dx \, dy \\ &= \int_{\mathcal{H}} \psi(z) \, d\mu. \end{aligned} \tag{12}$$

The measure $dx\,dy$ in the first integral is the Lebesgue measure on $\mathbb{C}$. A distribution u that satisfies (12) is said to have boundary value ϕ, an L^∞ function on the imaginary axis, if there exists U, an extension of u to $\bar{\mathcal{H}}$, that satisfies:

$$\frac{\partial U}{\partial \bar{z}} = \mu - \phi dz/2i. \tag{13}$$

Each side of this formula is to be interpreted as a distribution acting on test functions supported in the closed half plane $\bar{\mathcal{H}}$ and the measure $\phi dz/2i$ is a measure on $\mathbb{C}$ with support on the imaginary axis. The motivation for this definition comes from Stokes' Theorem.

[3]When $\mathbb{C}$ is considered as a homeomorphic to $\mathbb{R}^2$ in the usual way $(x, y) \to x + iy$ then the operator $\partial/\partial\bar{z}$ is expressed in local real coordinates as $\partial/\partial\bar{z} = 1/2(\partial/\partial x + i\partial/\partial y)$

A measure μ in $\mathcal{H}$ is called a Carleson measure [Gar81] with Carleson constant C if

$$\mu(S) < C\ l(S) \tag{14}$$

for every square $S \subset \mathcal{H}$ with a side of length $l(S)$ lying on an interval on the imaginary axis. The space of Carleson measures is denoted by the symbol $\mathcal{C}$.

Let $\mathcal{B}$ denote the ring of distributions over $\mathcal{H}$ with boundary value in L^∞, and with each element $b \in \mathcal{B}$ satisfying

$$\frac{\partial b}{\partial \bar{z}} = \mu \tag{15}$$

for some Carleson measure μ in $\mathcal{H}$. This is the ring in which the operator P_f will be inverted. The differential operator $\partial/\partial\bar{z} : \mathcal{B} \to \mathcal{C}$ and the canonical injection $i : \mathrm{H}^\infty \to B$ form an exact sequence

$$0 \longrightarrow \mathrm{H}^\infty \xrightarrow{\ i\ } \mathcal{B} \xrightarrow{\partial/\partial\bar{z}} \mathcal{C} \longrightarrow 0 \tag{16}$$

which when combined with the Koszul complex (10) gives a double complex which for the case $m = 2$ is represented by the commutative diagram of Figure 2. For notational convenience the modules $\Lambda^k(\mathcal{B})$ and $\Lambda^k(\mathcal{C})$ are denoted by Λ_1^k and Λ_2^k respectively. Readers familiar with complex manifold theory should recognize Equation (16) as a $\bar{\partial}$ co-homology sequence. Since the results of this paper are restricted to analytic functions defined on the complex half-plane, it is not difficult to avoid introducing the language of the co-homology of differential forms — the appropriate setting for analogous results about analytic functions of several complex variables.

The next theorem, which comes from [Hör67], explains how the complex is used to provide solutions to the Diophantine equations. The construction of the solutions is given in the proof which is repeated here for the sake of completeness.

Theorem 1 [Hörmander]
Suppose that the following conditions are satisfied:

(i) *Let s take the values 0 and 1, and r take the values 1 and 2. If $h \in \Lambda_r^s$ and $P_f h = 0$ then the equation $P_f g = h$ has a solution $g \in \Lambda_r^{s+1}$ with $\partial g/\partial\bar{z} \in \Lambda_{r+1}^{s+1}$ when $\partial h/\partial\bar{z} = 0$.*

(ii) *$\partial g/\partial\bar{z} = \mu$ has a solution $g \in \Lambda_1^2$ for every $\mu \in \Lambda_2^2$.*

$$
\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\Lambda^2(\mathrm{H}^\infty) \arrow[r, "P_f"] \arrow[d] & \Lambda^1(\mathrm{H}^\infty) \arrow[r, "P_f"] \arrow[d] & \Lambda^0(\mathrm{H}^\infty) \arrow[r, "P_f"] \arrow[d] & 0 \\
\Lambda^2_1 \arrow[r, "P_f"] \arrow[d, "\partial/\partial\bar{z}"] & \Lambda^1_1 \arrow[r, "P_f"] \arrow[d, "\partial/\partial\bar{z}"] & \Lambda^0_1 \arrow[r, "P_f"] \arrow[d, "\partial/\partial\bar{z}"] & 0 \\
\Lambda^2_2 \arrow[r, "P_f"] \arrow[d] & \Lambda^1_2 \arrow[r, "P_f"] \arrow[d] & \Lambda^0_2 \arrow[r, "P_f"] \arrow[d] & 0 \\
0 & 0 & 0 &
\end{tikzcd}
$$

Figure 2: double complex for m = 2

Then for every $h \in \Lambda^0_1$ with $\partial h/\partial\bar{z} = 0$ one can find $g \in \Lambda^1_1$ so that $\partial g/\partial\bar{z} = 0$ and $P_f g = h$.

Proof:
The result follows when premises (i) and (ii) are used to traverse the diagram in Figure 2 as follows.

Suppose that $h \in \Lambda^0_1$ is a holomorphic function with boundary value in L^∞; that is, $\partial h/\partial\bar{z} = 0$ on $\mathcal{H}$, and there exists a function $H(y) \in \mathrm{L}^\infty(\mathbb{R})$ such that for almost all $y \in \mathbb{R}$, $H(y) = \lim_{z\to iy} h(z)$ when the limit is non-tangential to the boundary. Then by the first premise there exists $g^1 \in \Lambda^1_1$ such that $P_f g^1 = h$ and $\partial g^1/\partial\bar{z} \in \Lambda^1_2$. Commutativity implies that $P_f \partial g^1/\partial\bar{z} = \partial/\partial\bar{z}\, P_f g^1 = 0$, so again by the first premise there exists $g^2 \in \Lambda^2_2$ such that $P_f g^2 = \partial g^1/\partial\bar{z}$. By the second premise the equation $\partial g^3/\partial\bar{z} = g^2$ has a solution $g^3 \in \Lambda^2_1$. Let $g = g^1 - P_f g^3$, then

$$
\begin{aligned}
\frac{\partial g}{\partial\bar{z}} &= \frac{\partial}{\partial\bar{z}}(g^1 - P_f\, g^3) \\
&= \frac{\partial g^1}{\partial\bar{z}} - P_f\,\frac{\partial g^3}{\partial\bar{z}} \\
&= 0
\end{aligned}
$$

and $P_f g = P_f(g^1 - P_f g^3) = P_f(g^1) - P_f P_f g^3 = P_f g^1 = h$ as required.

□

Before Theorem 1 can be used to construct solutions to Diophantine equations, explicit inversion formulas for the operators P_f and $\partial/\partial\bar{z}$ satisfying premises (i) and (ii) need to be presented. It was this need for inversion formulas which governed the definitions of the spaces $\mathcal{B}$ and $\mathcal{C}$. Formulae for the inversion of the operator P_f that are based on the work in [Hör67] with only small modification are presented next. Only the case of $m = 2$, the case needed for single-input single-output systems, is considered here; a more general situation that will be used for multi-input multi-output systems is dealt with in Section 5. A constructive scheme for inverting the Cauchy Riemann operator $\partial/\partial\bar{z}$ that comes from the more recent work of Jones [Jon80] [Jon83] is presented in the next section.

The inversion of the operator $P_f : \Lambda_1^1 \to \Lambda_1^0$ is dealt with first. It turns out that the requirement that is hardest to satisfy is the requirement that the anti-holomorphic derivative of the inverse should be a bounded Carleson measure. To overcome this problem (which in fact presents a major obstacle in the proof of the corona theorem) the construction is based on an application of the following Lemma from [Hör67]

Lemma 2 [Carleson - Hörmander] *Let $f_j \in \mathcal{H}$, $j = 1, \ldots, n$, and assume that for some $c > 0$*

$$|f_1(z)| + \ldots + |f_n(z)| \geq c. \tag{17}$$

Then for sufficiently small $\epsilon > 0$ one can find a partition of unity ϕ_j subordinate to the covering of $\mathcal{H}$ by open sets $\mathcal{H}_j = \{z : |f_j(z)| > \epsilon\}$ such that $\partial\phi_j/\partial\bar{z}$, defined in the sense of distribution theory, is a Carleson measure for all j.

This Lemma is a restatement of a result of Carleson's original paper [Car62] in which he directly constructs the measure. A more recent account of the construction is given in Garnett's book [Gar81]. The difficult part of the lemma is the construction of a partition of the plane into two sets each of which contains the regions of the plane where one of the two functions f_1 or f_2 becomes very small. In general, the boundary between the two sets will be a complicated curve, however in practice, the functions f_1 and f_2 may possess some regularity that allows a boundary curve to be easily chosen. For example, when solving the Bezout equation that arises from the example presented in Section 2, f_1 and f_2 are the functions $F_1(z) = e^{-\tau z}/(1+z)$ and $F_2(z) = (1-z)/(1+z)$ so the only restriction on the partition is that it separate the point $z = 1$ where $F_2(z) = 0$ from the regions of the plane where $|z|$ is large and $F_1(z)$ tends to zero, and a simple geometry suffices.

The partition of unity from Lemma 2 is used to construct a left inverse for P_f on Λ_1^0 as follows: for $h \in \Lambda_1^0$, let

$$g_i = h\frac{\phi_i}{f_i}. \tag{18}$$

then $P_f g = f_1 g_1 + f_2 g_2 = h$. The right inverse g also satisfies the premise of Theorem 1, for if $\partial h/\partial \bar{z} = 0$, then $\partial g/\partial \bar{z} = h f_i^{-1} \partial \phi_j / \partial \bar{z}$ which by Lemma 2 is a Carleson measure.

A second inversion formula is needed to invert the operator $P_f : \Lambda_2^2 \to \Lambda_2^1$. In fact, for the purposes of Theorem 1 it suffices to invert P_f on the subspace consisting of measures $\partial g^1/\partial \bar{z}$ where g^1 is a solution of $P_f g^1 = h$ for some holomorphic function h. In this case the problem may be written down explicitly as a redundant set of equations for the coefficient of g^2

$$\begin{aligned} g_{12}^2 f_2 &= h/f_1 \, \partial \phi_1 / \partial \bar{z} \\ -g_{12}^2 f_1 &= h/f_2 \, \partial \phi_2 / \partial \bar{z} \end{aligned}$$

and since f_2 and f_1 are both holomorphic functions with magnitude bounded away from zero on the support of $\partial \phi_1 / \partial \bar{z}$ a solution is given by

$$g_{12}^2 = \frac{h}{f_1 f_2} \, \partial \phi_1 / \partial \bar{z}.$$

3.2 Constructing Bounded Solutions to the Inhomogeneous Cauchy Riemann Equation

In the preceding section the construction of solutions to the Bezout Equation has been reduced to two steps: the construction of the partition of unity ϕ_j, and the construction of bounded solutions to the Cauchy Riemann equation. This section describes a technique devised by P. Jones [Jon80] for solving the Cauchy Riemann equation; the presentation is based on the account given in Garnett [Gar81].

The problem that needs to be solved is: given μ, a Carleson measure on the right half plane, find a distribution b with bounded boundary values that satisfies

$$\partial b / \partial \bar{z} = \mu.$$

The solution, which is based on a Green's function argument, has three stages: the measure μ is approximated by a sequence of measures μ_j which converge weakly to μ, each μ_j being supported on a finite set of points; the support of each measure μ_j is partitioned in such a way that the pseudo-hyperbolic distance[4] between any two points in the same partition is bounded from below, and the measure μ_j is subdivided into a corresponding sum $\sum \mu_j^k$ each μ_j^k having support on a distinct set in the partition; finally the Cauchy Riemann equation is boundedly solved for each μ_j^k and these solutions are summed to form the approximate solution b_j. The whole procedure is performed in such a way that the sequence of solutions b_j is a uniformly bounded sequence of functions in H^∞.

Before the solution is discussed in detail the fundamental solution to the Cauchy Riemann operator $\partial/\partial\bar{z}$ is introduced, and a result about interpolating Blaschke products is recounted. Let $D \subset \mathbb{C}$ be an open domain with C^1 boundary that contains the origin $z = 0$. The fundamental solution to the operator $\partial/\partial\bar{z}$ on D is a distribution b that satisfies the identity

$$-\int_D b(z)\frac{\partial\phi(z)}{\partial\bar{z}}dxdy = \phi(0)$$

for any C^∞ function ϕ with support compactly contained in D [Hör90]. In this formula the integral on the left hand side of the identity should be interpreted as the action of the distribution on a test function. The fundamental solution is computed as follows. Suppose that ϕ is an arbitrary C^∞ function with support compactly contained in D. Let $U \subset D$ have C^1 boundary and contain the support of ϕ in its interior. Consider the function $\phi(\zeta)/\zeta$, Stokes' theorem gives

$$\begin{aligned}\int_{\partial U}\frac{\phi(\zeta)}{\zeta}\,d\zeta - \int_{|\zeta|=\epsilon}\frac{\phi(\zeta)}{\zeta}\,d\zeta &= \int_{|\zeta|>\epsilon}\frac{\partial}{\partial\bar{\zeta}}\left(\frac{\phi(\zeta)}{\zeta}\right)\,d\bar{\zeta}\wedge d\zeta \\ &= -2i\int_{|\zeta|>\epsilon}\frac{1}{\zeta}\frac{\partial\phi}{\partial\bar{\zeta}}\,dxdy.\end{aligned}$$

Because $\phi(z) = 0$ on the boundary of U, the first boundary integral is zero, and as $\epsilon \to 0$ the second integral approaches the limit $i2\pi\phi(0)$. Consequently a fundamental solution for $\partial/\partial\bar{z}$ is given by the distribution $b(z) = 1/(\pi z)$.

If $z = x + iy$ is a complex number, then the real conjugate of z is defined to be the number $\tilde{z} = -x + iy$. The need for this usage results from considering Laplace transforms of system operators; the Laplace transform of a bounded causal system gives a transfer function which is analytic in the right half plane, so in places where a complex conjugate $\bar{z}$ occurs in the

[4]The pseudo-hyperbolic distance between two points in the half-plane is defined as $\rho(z_1, z_2) = |z_1 - z_2|/|z_1 - \tilde{z}_2|$

analysis of functions analytic in the upper half plane, it will be natural to substitute the real conjugate $\tilde{z}$. For instance, given a set $\{\zeta_j = \xi_j + i\eta_j, \xi_j > 0\}$ that satisfies the condition

$$\sum \frac{\xi_n}{1+|\zeta_n|^2} < \infty,$$

a Blaschke product with zeros ζ_j is defined by the expression

$$B(z) = \left(\frac{z-1}{z+1}\right)^m \prod_{\zeta_j \neq 1} \frac{|\zeta_j - 1|}{\zeta_j - 1} \frac{z-\zeta_j}{z-\tilde{\zeta}_j}.$$

The factors $|\zeta_j - 1|/(\zeta_j - 1)$ ensure that the product converges when the sequence $\{|\zeta_j|\}$ is unbounded, and for finite zero sets they may be omitted. A finite number of zeros at $\zeta = 1$ may be introduced into the Blaschke product separately in the factor in front of the product sign.

Let $B_0(z)$ be a Blaschke product with a zero set $\{\zeta_j = \xi_j + i\eta_j\}$ that satisfies the condition

$$\prod_{j, j\neq k} \left|\frac{\zeta_k - \zeta_j}{\zeta_k - \tilde{\zeta}_j}\right| \geq \delta > 0, \tag{19}$$

then the inverse $1/B_0(z)$ is an analytic function except on the zero set $\{\zeta_j\}$ and is given by the expression

$$1/B_0(z) = 1 + \sum_j \frac{1/B_0'(\zeta_j)}{z - \zeta_j}$$

If $1/B_0(z)$ is considered as a distribution on $\mathcal{H}$, then it follows from the discussion of the fundamental solution to the $\bar{\partial}$ operator that

$$\begin{aligned} \frac{\partial}{\partial \bar{z}}(1/B_0) &= \sum_j \frac{\pi}{B_0'(\zeta_j)} \delta_{\zeta_j} \\ &= \sum_j \beta_j \xi_j \delta_{\zeta_j}, \end{aligned} \tag{20}$$

where $1 \leq |\beta_j| \leq 1/\delta$.

The following theorem is quoted from Jones [Jon83] Theorem 6.

Theorem 3 [Jones]
Suppose $\{z_k\}$ is a sequence of points in the half plane that satisfies

$$\inf_j \prod_{k, k\neq j} \left|\frac{z_k - z_j}{z_k - \tilde{z}_j}\right| \geq \delta > 0, \qquad k = 1, 2, \ldots.$$

Let $B(z)$ be the the Blaschke product with zeros at the points $\{z_k\}$ and $B_j(z)$ be the Blaschke product with zeros in the set formed by removing the point z_j from the set $\{z_k\}$. Let $E_j(z)$ be the function

$$E_j(z) = c_j B_j(z) \left(\frac{y_j}{z - \bar{z}_j} \right)^2 \exp \left\{ \frac{-i}{\log 2/\delta} \sum_{y_k \le y_j} \frac{y_k}{z - \bar{z}_k} \right\} \tag{21}$$

where

$$c_j = -4(B_j(z_j))^{-1} \exp \left\{ \frac{i}{\log 2/\delta} \sum_{y_k \le y_j} \frac{y_k}{z_j - \bar{z}_k} \right\}$$

Then $E_j(z_k) = \delta_{j,k}$ and

$$\sum_j |E_j(z)| \le (C_0/\delta)\ \log(2/\delta) \tag{22}$$

for all $z \in \mathcal{H}$.

This theorem is an instance of Carleson's interpolation theorem that explicitly gives the form of the interpolating function. The bound on the norm of the interpolating function, $(C_0/\delta)\ \log(2/\delta)$ is optimal in δ up to the multiplicative factor C_0.

The application of Jones' interpolation formula requires the following lemma which is extracted from the proof of Carleson's interpolation theorem in Chapter 7 of [Gar81].

Lemma 4 (Garnett [Gar81])
Let $\{z_j\}$ be a sequence in the right half plane, with points z_j well separated in the hyperbolic metric, i.e.

$$\rho(z_k, z_j) = \left| \frac{z_k - z_j}{z_k - \bar{z}_j} \right| \ge a > 0, \qquad j \ne k,$$

and suppose that there exists a constant A such that for every square $Q = \{y_0 \le y \le y_0 + l(Q), 0 < x \le l(Q)\}$,

$$\sum_{z_j \in Q} x_j \le A\, l(Q)$$

then

$$\inf_k \prod_{j, j \ne k} \left| \frac{z_k - z_j}{z_k - \bar{z}_j} \right| \ge \delta \ge \exp\left(-40A \left(1 + 2 \log \frac{1}{a} \right) \right).$$

The bound that is given for δ in the lemma depends on the points $\{z_k\}$ having a minimum spacing a in the hyperbolic metric and on the measure $\sum x_j \delta_{z_j}$ being a Carleson measure with Carleson constant A. Unfortunately the generality of the theorem means that the bound derived will be conservative for many specific examples. This is particularly true of examples such as the one presented in this paper in which the measures have easily recognizable additional structure. Additional information about the distribution of the points $\{z_k\}$ could well be used to derive a less conservative estimate.

The next two lemmas contain the constructive solution to the Cauchy Riemann equation that is presented in Chapter 8 of [Gar81]. The proofs closely follow the work cited, but are given here because they contain the algorithms that are used to compute actual solutions. Jones' interpolation theorem and the discussion preceding it on fundamental solutions provide the basis for calculating solutions to the Cauchy Riemann equation in the following simple case.

Lemma 5 (Garnett [Gar81])
Let z_j be a finite set of points satisfying (19) and let $\mu = \sum \alpha_j x_j \delta_{z_j}$ with $|\alpha_j| \leq 1$. Then the function

$$b(z) = E(z)/B_1(z) \tag{23}$$

satisfies $\partial b/\partial \bar{z} = \mu$ where $B_1(z)$ is a Blaschke product with zeros z_j, and $E(z)$ is a function that is analytic on the right half plane and has a bound that depends on the choice of μ only through the δ of Equation (19).

Proof:
Equation (20) states that

$$\frac{\partial}{\partial \bar{z}} \frac{1}{B_1(z)} = \sum_j \beta_j x_j \delta_{z_j},$$

and that the coefficients β_j lie within the uniform bounds $1 \leq |\beta_j| \leq 1/\delta$. An application of Jones' interpolation theorem produces a function

$$E(z) = \sum \alpha_j/\beta_j E_j(z) \tag{24}$$

that is analytic in the right half plane, interpolates the values α_j/β_j at the points z_j, and is bounded on the imaginary axis by

$$|E(z)| \leq (C_0/\delta)\ \log(2/\delta)$$

in which C_0 is an absolute constant.

The result follows by taking $b(z) = E(z)/B_1(z)$

$\square$

The case of a general Carleson measure μ is tackled by constructing a sequence of approximating measures $\{\mu_n\}$ that converges (weakly) to μ; each measure in the sequence is supported on a finite set of points and has the form $\mu_n = \sum \alpha_j x_j \delta_{z_j}$. A sequence $\{b_n\}$ of uniformly bounded solutions to the equations $\partial b_n/\partial \bar{z} = \mu_n$ is calculated, and the restrictions to the imaginary axis $\{b_n(iy)\}$ form a uniformly bounded sequence of L^∞ functions. The mapping

$$f(y) \longrightarrow \int_{-\infty}^{\infty} f(y) b_n(iy)\, dy, \qquad f \in L^1$$

associates the set $\{b_n(iy)\}$ with a set of uniformly bounded functionals on L^1, and weak compactness of the unit ball ensures that $\{b_n(iy)\}$ contains a subsequence that converges in the weak* sense to an H^∞ function $b(iy)$. Choose one such subsequence, and relabel it $\{b_n(iy)\}$, then, if ψ is any $C_0^\infty(\mathbb{C})$ test function with support that intersects the half plane, the restriction to the imaginary axis $\psi(iy)$ is an L^1 function, and

$$\begin{aligned}
\int_{-\infty}^{\infty} \psi(iy) b(iy) dy &= \lim_{n\to\infty} \int_{-\infty}^{\infty} \psi(iy) b_n(iy)\, dy \\
&= \lim_{n\to\infty} \frac{1}{i} \int_{\partial\mathcal{H}} \psi b_n(z)\, dz \\
&= \lim_{n\to\infty} \frac{1}{i} \int_{\mathcal{H}} d(\psi b_n) \wedge dz \\
&= \lim_{n\to\infty} \frac{1}{i} \int_{\mathcal{H}} b_n \frac{\partial\psi}{\partial\bar{z}} d\bar{z} \wedge dz + \int_{\mathcal{H}} \psi \frac{\partial b_n}{\partial \bar{z}} d\bar{z} \wedge dz \\
&= \lim_{n\to\infty} 2 \int_{\mathcal{H}} b_n \frac{\partial\psi}{\partial\bar{z}}\, dx \wedge dy + \int_{\mathcal{H}} \psi \frac{\partial b_n}{\partial\bar{z}}\, dx \wedge dy
\end{aligned}$$

Rearranging the left-most and right-most sides, and substituting $\mu_n = \partial b_n/\partial\bar{z}$ gives the equation

$$\lim_{n\to\infty} \int_{\mathcal{H}} b_n \frac{\partial\psi}{\partial\bar{z}}\, dx \wedge dy = \left\langle \mu, \frac{\partial\psi}{\partial\bar{z}} \right\rangle - \int_{-\infty}^{\infty} \psi(iy) b(iy) dy$$

which provides a consistent definition for b, as a distributional solution to $\partial b/\partial\bar{z} = \mu$ on $\mathcal{H}$ with boundary value $b(t)$

Lemma 5 is not quite enough to provide the sequence of solutions $\{b_n\}$; the difficulty is that the bound in Lemma 5 depends on the parameter δ which, through Lemma 4, is related to the spacing (in the pseudo-hyperbolic metric) of the points in the supporting set $\{z_j\}$, and if a general Carleson measure is going to be approximated by a sequence of measures with finite point support, then the spacing of the points in the support of the approximating measures will decrease to zero as the approximations converge. What is needed is a method for decomposing the approximating measures in such a way that the spacing between points of support for each element of the decomposition remains large, yet the sum of the Carleson constants of the elements in the decomposition remains constant. The next lemma uses this approach to produce a method for solving the equations $\partial b_n/\partial\bar{z} = \mu_n$ with a uniform bound on the sequence of solutions b_n.

Lemma 6 (Jones-Garnett [Gar81])
Let $\mu = \sum_{j=1}^{M} \alpha_j x_j \delta_{z_j}$ be a measure supported on the finite set $\{z_j = x_j + iy_j\}$, with masses $\alpha_j x_j$ at the points z_j, and with Carleson constant $N(\mu) \leq C$. Then there exist an integer N, functions $b_p(z)$, and a function

$$b(z) = \frac{1}{N}\sum_{p=1}^{2N} b_p(z)$$

such that each $b_p(z)$ is a function of the type produced in Lemma 5, $\partial b/\partial\bar{z} = \widehat{\mu}$ for a measure $\widehat{\mu}$ that is arbitrarily close to μ, and $|b(it)| < KC$ for $t \in \mathbb{R}$ and K a constant independent of μ.

Proof:
First it is shown that μ may be approximated arbitrarily closely by a new measure $\widehat{\mu}$ of the form $C/N \sum x_j \delta_{z_j}$. The support of $\widehat{\mu}$ is the same as the support of μ, but each point mass $x_j\delta_{z_j}$ may be repeated a finite, and possibly large number of times in the new sum. If N is chosen to be a sufficiently large positive integer, the coefficients α_j in the finite sum μ may be uniformly approximated to arbitrary accuracy by $\alpha_j \approx n_j/NC$ in which n_j are positive integers and C is the Carleson constant of μ. If each term in the sum $\sum \alpha_j x_j \delta_{z_j}$ is expanded as

$$\alpha_j x_j \delta_{z_j} \approx \frac{C}{N}(x_j\delta z_j + \ldots\,(n_j \textit{ times})\ldots + x_j\delta z_j)$$

then a renumbering of the terms in the summation gives the approximation

$$\mu \approx \widehat{\mu} = \frac{C}{N}\sum_j x_j \delta_{z_j}$$

From here on no distinction will be made between the measure μ and the approximation $\widehat{\mu}$.

In the second part of the proof a systematic method of decomposing the measure μ is established. The point masses $x_j\delta_{z_j}$ are distributed amongst a finite number of sets in such a way that the distance between any two points in the same set is large in the hyperbolic metric.

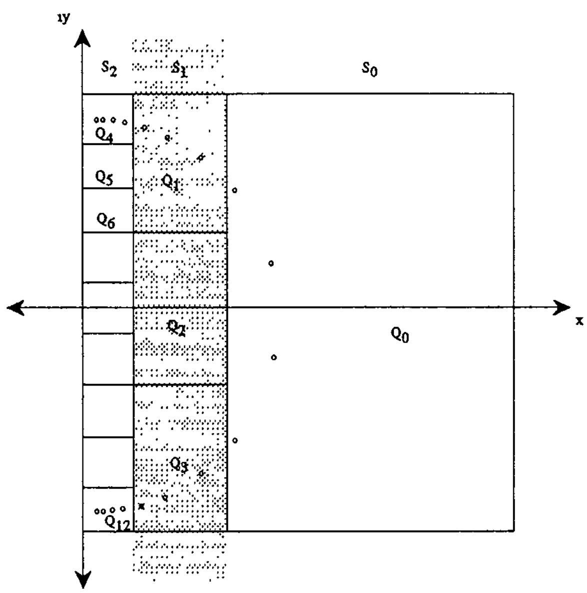

Figure 3: Dyadic Subdivision of the Half Plane

Choose a square Q_0 with $\operatorname{supp}\mu \subset Q_0 \subset \mathcal{H}$ and a side of length $l(Q_0)$ lying on the imaginary axis. This square may be subdivided to form a dyadic sequence of squares of uniform hyperbolic size as follows (Figure 3 illustrates the construction). Let Q_1, Q_2 be the two adjacent squares that comprise the left half of the square Q_0. Each have sides of length $l(Q)/2$, and each have one side on the imaginary axis; continue this subdivision process inductively on each square Q_i until the squares $Q_{2^n}, Q_{2^{n+1}-1}$ are outside the support of μ for some n (the process is guaranteed to stop because the support of μ, which is finite, is compactly contained in $\mathcal{H}$). Since the Carleson constant of μ is fixed to the constant C, a simple count shows that the right hand section of any dyadic square Q can contain at most $2N$ points z_j. This allows the points $\{z_j\}$ to be partitioned into $2N$ sets $\{S_n\}$ in such a way that the spacing between any two points in the same set is uniformly bounded from below by $a = 1/3$.

The sets S_n are explicitly defined as follows. For every n, let $S_n = \{z_j : l(Q_0)2^{-n-1} \leq x_j \leq l(Q_0)2^{-n}\}$ and order the elements of each S_n so that $S_n = \{x_{k,n} + iy_{k,n}\}$ with

$$y_{k-1,n} \leq y_{k,n} \leq y_{k+1,n}.$$

Then the set $\{z_j\}$ may be split into $2N$ sequences $Y_1, \ldots, Y_{2N}$ such that the points in each S_n are evenly distributed between the Y_r, i.e. if $z_j = x_{k,n} + iy_{k,n} \in S_n$ then $z_j \in Y_r$ if $r = k \bmod 2N$. Now suppose that $P \subset \mathcal{H}$ is a fixed square of arbitrary size with one side lying on the imaginary axis, let $M_n(P)$ be the number of points in $S_n \cap P$, then each set $Y_r \cap S_n \cap P$ must contain fewer than $1 + M_n(P)/(2N)$ points z_j, and

$$\begin{aligned}
\sum_{Y_r \cap P} x_j &\leq \sum_{n: S_n \cap P \neq \phi} \left(1 + \frac{M_n(P)}{2N}\right) 2^{-n}\, l(Q_0) \\
&\leq 2l(P) \sum_{n=0}^{\infty} 2^{-n} + \frac{1}{mN} \sum_{z_j \in P} m x_j \\
&\leq 4l(P) + \mu(P) \\
&\leq 5l(P)
\end{aligned} \tag{25}$$

Consider the sets $\{X_p\}$ defined by

$$X_r = Y_r \cap \bigcup_{n \text{ even}} S_n \tag{26}$$

and

$$X_{2r+1} = Y_r \cap \bigcup_{n \text{ odd}} S_n. \tag{27}$$

then the measures $\mu_p = \sum_{z_j \in X_p} x_i \delta_{z_j}$ satisfy $\mu = C/N \sum_p \mu_p$ and up to the factor C/N provide a decomposition of μ into measures with well spaced support and Carleson constant uniformly bounded by 5.

A bound on the separation between points of support is arrived at by the following argument. If $z_i \in S_n$ and $z_j \in S_{n-2}$ then $\rho(z_i, z_j) > 1/3$ by the definition of the sets S_n. On the other hand, if z_i and z_j are in the same set S_n then it follows, from the fact that the top half of any Q_j contains at most $2N$ points, and the way in which the set Y_r that corresponds to

X_p was constructed, that z_i and z_j must be separated by at least 1 square of length $l(Q_0)2^{-n}$. Consequently, the distance between z_i and z_j must be bounded below by

$$\begin{aligned} \rho(z_i, z_j) &\geq \frac{2^{-n}}{\sqrt{2^{-2n} + 8^{-2n}}} \\ &\geq 1/3. \end{aligned} \tag{28}$$

Lemma 5 can now be applied to the measures μ_p to produce functions b_p that satisfy

$$\frac{\partial b_p}{\partial \bar{z}} = \mu_p.$$

The constant δ in Lemma 5 which determines the bounds on the norms $\|b_p\|$ is estimated by using Lemma 4 and the inequalities (25) and (28). This gives an estimate on the norms $\|b_p\|$ of

$$\|b_p\| \leq K,$$

in which K is an absolute constant that is independent of the measure μ_p. Let

$$b(z) = \frac{C}{N} \sum_{p=1}^{2N} b_p(z).$$

Then $\partial b / \partial \bar{z} = C/N \sum z_j \delta_{z_j} = \mu$ and

$$\|b\| \leq 2CK, \tag{29}$$

which completes the proof.

□

Lemma 6 provides the last step in the constructive proof of the following theorem.

Theorem 7 (Garnett [Gar81])
Let μ be a Carleson measure with Carleson constant $N(\mu) \leq 1$. Then there is a distribution $b(z)$ with L^∞ boundary value, supported on $\bar{\mathcal{H}}$ such that

$$\frac{\partial b}{\partial \bar{z}} = \mu,$$

and the boundary value satisfies $\|b\|_\infty < C$ for C a positive constant independent of the choice of μ. Further, there exists a sequence of measures

$\{\mu_k\}$ that satisfy the criteria of lemma 6, and which converge weakly to μ. The corresponding sequence of solutions $\{b_k(z)\}$ converge in a distributional sense on $\mathcal{H}$ to $b(z)$, and have boundary values that converge in the weak-star topology to $b(iy)$ on the imaginary axis.

The reason for presenting Lemma 6 in such detail is that the construction in the proof provides a key part of the algorithm that is used to compute solutions to the Diophantine equations arising from the control problems. In this application a bound on the norm of the solution to the equation $\partial b/\partial \bar{z} = \mu$ has physical significance, and consequently a tight a priori estimate of this bound would be valuable. Unfortunately, the generality of the methods presented means that the estimates on the norms that can be obtained from Lemmas 4, 5 and 6 are too conservative to be of practical use.

4 Solution to Enns' example

The theory presented in the previous section provides a practical way to design linear compensators for a general class of linear time invariant systems. The remainder of this paper provides examples that illustrate the use of the theory. This section continues the example that was started in Section 2 by showing how a compensator is calculated for the plant that was given in the example.

The plant from the example in Section 2 is described by the transfer function in Equation (1),

$$F(z) = \frac{e^{-\tau z}}{\sigma z - 1}.$$

Values are ascribed to the parameters for the numerical calculations: σ takes the value $\sigma = 1$, making the open loop system unstable, and τ takes the values $\tau = 0.06$ or $\tau = 0.37$, the first is a small delay which has little influence on the behavior of the open loop system, and the second is a large value for the delay that makes the problem of robust stabilization significantly more challenging. The transfer function of a stabilizing controller is given in terms of an H^∞ parameter Q by the bilinear function in Equation (4). Evaluating this expression requires solutions to the Bezout equation (3) and the Diophantine equation (7), the Bezout equation is dealt with first.

4.1 Computing solutions to the Bezout equation

Using the notation of Section 3.1 the Bezout equation may be re-written as $P_f g = h$ in which h is the constant function $h(z) = 1$, f is the pair of co-prime factors $f_1(z) = N(z)$ and $f_2 = M(z)$, and g, the solution, is a function in $\Lambda^1(R)$ with components $g_1 = X_1$ and $g_2 = X_2$ which are H^∞ functions. From theorem 1 the solution may be written as $g = g^1 - P_f g^3$ in which g^1 satisfies the equations $P_f g^1 = h$ and $\partial g^1 / \partial \bar{z} = P_f g^2$ for g^2 a Carleson measure on $\mathbb{C}$, and g^3 satisfies $\partial g^3 / \partial \bar{z} = g^2$ for the same measure g^2. Substituting the notation of the problem gives the following set of equations:

$$\begin{aligned} X_1(z) &= \widetilde{X}_1(z) - b(z) F_2(z) \\ X_2(z) &= \widetilde{X}_2(z) + b(z) F_1(z) \end{aligned} \qquad (30)$$

in which $\widetilde{X}_1$ and $\widetilde{X}_2$ satisfy the equation

$$\widetilde{X}_1(z) F_1(z) + \widetilde{X}_2(z) F_2(z) = 1, \qquad (31)$$

and $b(z) = g^2_{12} = -g^2_{21}$ is the coefficient function of the 2-form g^2 and a distributional solution of

$$\frac{\partial b}{\partial \bar{z}} = \mu. \tag{32}$$

The inhomogeneous term μ in equation 32 is a Carleson measure that is supported on the half plane and that satisfies

$$\begin{aligned} \frac{\partial \tilde{X}_1}{\partial \bar{z}} &= \mu F_2 \\ \frac{\partial \tilde{X}_2}{\partial \bar{z}} &= -\mu F_1. \end{aligned} \tag{33}$$

Lemma 2 guarantees the existence of choices for $\tilde{X}_1$, $\tilde{X}_2$ and μ that satisfy equations (31) and (33). In fact, since the function $F_1(z) = e^{-hz}/(z+1)$ is bounded away from zero on any set compactly contained in the right half plane, and the function $M(z) = (1-z)/(1+z)$ has a single zero at the point $z = 1$ and is bounded away from zero on any set that excludes a neighborhood of that point, a partition of unity that satisfies the condition in lemma 2 is the following:

$$\phi_1(z) = \begin{cases} 1, & |z-1| < r \\ 0, & |z-1| \geq r \end{cases} \qquad \phi_2(z) = \begin{cases} 1, & |z-1| \geq r \\ 0, & |z-1| < r \end{cases} \tag{34}$$

With this choice for a partition the solution to (31) determined by equation (18) is $\tilde{X}_1(z) = 1/F_1(z)\ \phi_1(z)$ and $\tilde{X}_2(z) = 1/F_2(z)\ \phi_2(z)$. Taking anti-holomorphic derivatives, and substituting into the first of the equations (33) yields the measure

$$\mu = 1/(F_1(z)F_2(z))\, \partial\phi_1/\partial\bar{z}. \tag{35}$$

The meaning of the expression $\partial\phi_1/\partial\bar{z}$ may be elucidated by mollification. Let Ω denote the support of ϕ_1, and let ψ_k be a sequence of positive C^∞ functions supported on connected neighborhoods of the origin, and with the property that diameter(supp ψ_k) $\to \{0\}$ as $k \to \infty$. For each k the C^∞ function $\tilde{\phi}_k = \phi_1 * \psi_k$ is a mollification of ϕ_1; for sufficiently large k it is supported on a region slightly larger than Ω, and takes the constant value 1 on a region slightly smaller than Ω. Let D_k denote the support of the C^∞ function $\eta_k = \partial\tilde{\phi}_k/\partial\bar{z}$. It follows that D_k is a tubular neighborhood of the boundary of the support of ϕ_k, and that $\tilde{\phi}_k$ takes the value 1 on the interior part of ∂D (the boundary of D), and the value 0 on the exterior part of ∂D. The function η_k may be interpreted as a complex valued measure on $\mathbb{C}$ in the following sense. Let χ be a compactly supported C^∞ function, then an application of Stokes theorem gives

$$\begin{aligned}
\int \chi \, d\eta_k &= \int_{D_k} \chi \frac{\partial \widetilde{\phi}_k}{\partial \bar{z}} \, dx dy \\
&= \frac{i}{2} \int_{D_k} \chi \frac{\partial \widetilde{\phi}_k}{\partial \bar{z}} \, dz \wedge d\bar{z} \\
&= \frac{i}{2} \int_{\partial D_k} \widetilde{\phi}_k \chi \, dz - \frac{i}{2} \int_{D_k} \widetilde{\phi}_k \frac{\partial \chi}{\partial \bar{z}} \, dz \wedge d\bar{z} \qquad (36)
\end{aligned}$$

As $k \to 0$ the sequence $\widetilde{\phi}_k$ converges to ϕ_1 in the topology of the space of distributions, the area of the region D_k converges to 0, and the boundary ∂D_k converges to the set $\partial\Omega$. Since the function $\partial\chi/\partial\bar{z}$ is uniformly bounded, the second integral in (36) converges to 0, and the first integral, which only has a contribution from the interior part of ∂D_k, converges to $-i/2 \int_{\partial\Omega} \chi \, dz$. The negative sign on the contour integral is a consequence of the orientation of the boundary ∂D_k. The expression $\partial\phi_1/\partial\bar{z}$ is interpreted as a measure supported on the set $\partial\Omega$, which acts on a C^∞ function χ by

$$\int \chi \, d\left(\frac{\partial \phi}{\partial \bar{z}}\right) = -\frac{i}{2} \int_{\partial\Omega} \chi \, dz.$$

The shape of the region Ω is arbitrary provided that the partition of unity that it determines satisfies the condition in lemma 2. For the actual computation of the solutions to the Bezout equation, Ω was chosen to be a circle for the pragmatic reasons that it is a simple curve to describe and that it seems to give reasonable results. The radius for the circle was chosen with a view to keeping the Carleson constant of the measure μ small. It was found that $r = 0.7$ is a suitable value when the delay in the plant takes either of the values $\tau = 0.06$ or $\tau = 0.37$.

The only remaining step in determining solutions to Equation (3) is the calculation of a solution to the Cauchy Riemann equation (32). The algorithm presented in the proof of Lemma 6 provides a way to calculate numerical approximations to a solution of the equation. The approximations converge in the weak-star sense; this is a natural topology in the context of system transfer functions. In accordance with the proof of Lemma 6, the measure μ is approximated by a finitely supported measure constructed as a sum of point masses uniformly distributed on the support of μ, and the algorithm presented in Lemma 6 is used to calculate an approximate solution for $\partial b/\partial\bar{z} = \mu$.

4.2 Computing solutions to the Nehari equation

The Diophantine equation (7) is solved using the same method as was used for the Bezout equation with only small modifications. The similarity in form between the Nehari problem and the Bezout equation becomes apparent when Equation (7) is rewritten as

$$A = 1P + BQ \tag{37}$$

in which

$$A = \begin{pmatrix} F_2 X_1 W_2 \\ F_2 X_2 W_1 \end{pmatrix} \qquad B = \begin{pmatrix} -F_2^2 W_2 \\ F_2 F_1 W_1 \end{pmatrix}.$$

For purposes of comparison the choice of weighting functions W_1 and W_2 is the same as that used in [EOT92]:

$$\begin{aligned} W_1 &= 2\frac{1+z}{1+10z} \\ W_2 &= 0.2\,. \end{aligned}$$

Equation (37) may be rewritten again in the notation of Section 3.1 as $P_f g = h$. Now $h = A$ is a vector with two H^∞ components, f has components $f_1 = 1$ and $f_2 = B$, a vector, and the solution g has two components, $g_1 = P$, a vector with two H^∞ components, and $g_2 = Q$ an H^∞ function. The same formalism that was used for the Bezout equation yields the set of equations:

$$\begin{aligned} P &= \tilde{P} - b(z)B \\ Q &= \tilde{Q} + b(z) \end{aligned} \tag{38}$$

where $\tilde{P}$ and $\tilde{Q}$ satisfy

$$\tilde{P} + \tilde{Q}B = A$$

and $b(z)$ is a distributional solution of

$$\frac{\partial b}{\partial \bar{z}} = \mu$$

where μ is a Carleson measure that satisfies either of the equivalent equations

$$\begin{aligned} \frac{\partial \tilde{P}}{\partial \bar{z}} &= \mu B \\ \frac{\partial \tilde{Q}}{\partial \bar{z}} &= -\mu. \end{aligned}$$

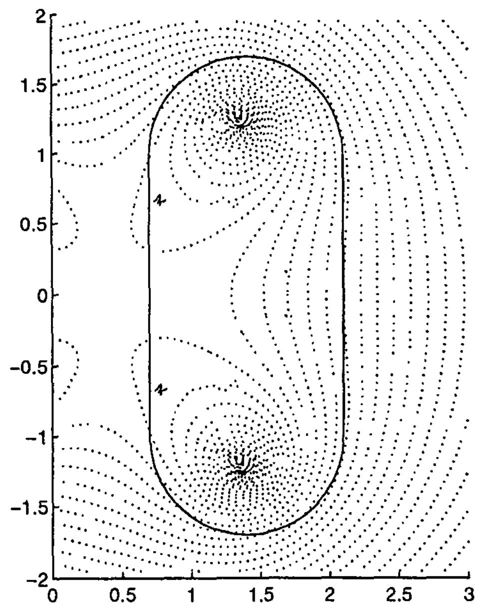

Figure 4: Contour for measure in Nehari Problem

At this point a problem becomes apparent. Consider P to be a 2×1 matrix with H^∞ entries, then the optimal solution to the control problem is the solution that minimizes the L^∞ norm of the largest singular value of $P(iy)$. The problem is that the algorithms based on the methods presented in section 3 can only guarantee an upper bound for the norm on P, they do not give information about how close the computed solution is to an optimal solution. The upper bound on the solution is obtained by the triangle inequality from bounds on the size of the two terms in the right hand size of equation (38). The bound on the size of the first term $\widetilde{P}$ is controlled directly by the choice of the distributional solutions $\widetilde{P}$ and $\widetilde{Q}$, and the bound on the second term, bB is controlled by the the Carleson constant for the measure μ and the a priori bound that theorem 7 imposes on b the solution to the Cauchy Riemann equations. In the example calculation described in the following paragraphs ad hoc methods are used in the selection of the distributions $\widetilde{P}$ and $\widetilde{Q}$. The results indicate that better methods of selection need to be combined with improved solutions for the Cauchy Riemann equations if the goal of reliable computation of near optimal controllers is to be achieved.

The distributions $\widetilde{P}$, $\tilde{Q}$ are chosen to minimize the L^∞ norm of $\widetilde{P}$ and the Carleson constant for μ. The equation $(A - QB) \cdot B = 0$ has a solution $Q = A \cdot B / B \cdot B$ which is a bounded analytic function away from the zeros of the denominator $B \cdot B$, and unbounded in a neighborhood of these zeros. Let Ω be a region in the complex plane that contains a neighborhood of the zeros of B and is bounded by an absolutely continuous closed curve, and let $\phi_1(z) = 1$ when $z \in \Omega$ and $\phi_1(z) = 0$ when $z \notin \Omega$, and let $\tilde{Q}$ and $\widetilde{P}$ be the bounded distributions:

$$
\begin{aligned}
\tilde{Q} &= \frac{A \cdot B}{B \cdot B} \phi_1 \\
\widetilde{P} &= A - \tilde{Q} B && (39) \\
&= A - \frac{A \cdot B}{B \cdot B} B \phi_1. && (40)
\end{aligned}
$$

Outside the region Ω the measure $\widetilde{P}$ is the pointwise orthogonal projection of A onto $B^\perp$. Substituting in the values for A and B gives

$$
\begin{aligned}
\tilde{Q} &= \frac{W_2^2 F_2 X_1 - W_1^2 F_1 X_2}{W_2^2 F_2^2 + W_1^2 F_1^2} \phi_1 \\
\widetilde{P_1} &= F_2 X_1 W_2 (1 - \phi_1) + \frac{F_2 F_1 W_1^2 W_2}{W_2^2 F_2^2 + W_1 W_2^2} \phi_1 \\
\widetilde{P_2} &= F_2 X_2 W_1 (1 - \phi_1) + \frac{F_2^2 W_1 W_2^2}{W_2^2 F_2^2 + W_1 W_2^2} \phi_1,
\end{aligned}
$$

and for the measure μ acting on a C^∞ function χ,

$$
\begin{aligned}
\mu(\chi) &= -\frac{i}{2} \int_{\partial\Omega} \left(\frac{A \cdot B}{B \cdot B} \right) \chi \, dz \\
&= -\frac{i}{2} \int_{\partial\Omega} \frac{W_2^2 F_2 X_1 - W_1^2 F_2 X_2}{W_2^2 F_2^2 + W_1^2 F_1^2} \chi \, dz. && (41)
\end{aligned}
$$

The criteria for choosing the region Ω are that it should include the zeros of the function $W_2^2 M^2 + W_1^2 N^2$ and that the L^∞ norm of $\widetilde{P}(iy)$ and Carleson constant of the measure μ should be minimized. For the results presented, the region Ω was chosen by plotting the weighting function $(W_2^2 MY - W_1^2 NX)/(W_2^2 M^2 + W_1^2 N^2)$ and choosing by inspection a contour that includes the singularities of the function, yet keeps the Carleson constant of μ small. The chosen contour is illustrated in Figure 4; the contour is superimposed on a logarithmically scaled contour plot of the magnitude of the weighting function in Equation (41). Once the contour is chosen, the Carleson measure in (41) can be computed and the inhomogeneous Cauchy Riemann $\partial b/\partial \bar{z} = \mu$ may be solved. The solution b is substituted into (38) to give the Youla parameter Q and the transfer function matrix P that constitute a sub-optimal solution to the Nehari Problem.

4.3 The Algorithm and its Implementation

This section contains a summary of the algorithm which indicates the exact order of the steps taken in obtaining solutions, and where the various formulae in Sections 2 – 4 are used.

The data for the problem are the transfer function F given in equation (1), and the weighting functions that appear in the norm (5). The solution algorithm can be divided into two parts, one part to calculate the solutions X_1 and X_2 to the Bezout Identity (3), and the other to calculate solutions to the Diophantine equation (7) associated with the Nehari Problem.

Part1: The Bezout Identity

1 Factor the transfer function — Equation (2).

2 Choose a partition of $\mathcal{H}$ that separates the zeros of the factors F_1 and F_2 — Equation (34).

3 Define the Carleson measure μ for the Bezout equation (35).

4 Compute a discrete approximation to the Carleson measure as a finite sequence of complex valued point masses at points on the support of μ — Theorem 7.

5 Group the points according to the algorithm presented in the proof of Lemma 6.

6 Use Equation (21) of Theorem 3 and equations (23) and (24) of Lemma (5) to compute solutions to the inhomogeneous Cauchy Riemann equation (32).

7 Use Equation (30) to calculate the solutions to the Bezout equation at points $z \in \bar{\mathcal{H}}$.

In practice, the solution is calculated over a given set of points. Steps 1 – 5, which are independent of the point z at which the solution is to be calculated, are performed first, and then the calculations in steps 6 and 7 are performed at each point z in the given set.

Part2: The Diophantine Equation for the Nehari Problem

8 Choose a partition of $\mathcal{H}$ so that the Carleson constant for the measure in Equation (41) is minimized. The Carleson constant for a measure is defined in Equation (14).

9 Define the Carleson measure μ for the Nehari problem by Equation (41).

10 Follow steps 4 – 7 above to compute the solutions to the inhomogeneous Cauchy Riemann equation associated with the measure constructed in step 9. This step requires the evaluation of X_1 and X_2 at each of the points in the discretization of the Carleson measure μ from step 9.

11 Use Equation (38) to evaluate the solutions P and Q on points $z \in \bar{\mathcal{H}}$.

The whole algorithm for computing the solutions is encoded in computer software. The heart of the computation is performed by two C programs. The first program computes the approximation to the Carleson measure, and the second computes the uniformly bounded approximations to the solution of the Cauchy Riemann equation. Matlab routines that call the C programs calculate the transfer functions for the controller on a set of points on the imaginary axis, and produce the graphical output.

The first C program produces a discrete approximation to the measure μ and orders the points of support according to the decomposition described in Lemma 6. The process occurs in a number of discrete steps. The first step is to calculate a discrete approximation to the measure. The measure μ is supported on a curve in the right half complex plane. In the example presented here, in both the case of the solution to the Bezout equation and the case of the solution to the Diophantine Equation associated with the Nehari Problem, this support is compactly contained in the open right half plane, and the weighting function is fairly regular. Consequently it suffices to choose the supporting points of the discretization to be equally spaced on the supporting curve. A case of non-compact support would require that the measure be restricted first to a compact subset of the plane, and the added complexity of non uniform spacings in the discretization may be justified in some instances by an increase in the efficiency of the approximation. The discretization of the measure is stored as a list of ordered pairs of complex numbers, the first element in each pair is the point of support $z_k = x_k + iy_k$, and the second is the value of the weight assigned to each point $-i(\Delta x_k + i\Delta y_k)/(2F_1(z_k)F_2(z_k))$. $\Delta x_k + i\Delta y_k$ is the tangent to the supporting curve at z_k with length equal to the arc-length spacing of points in the discretization.

The second part of the measure computation, the ordering of the points, is independent of the particular measure being approximated. The algorithm proceeds via a sequence of procedures that implement the steps in the proof of Lemma 6. The first procedure takes the list of weights from the discretization and and produces a list of weights of uniform size by splitting large weights. Given a small constant $\delta > 0$ there exists an integer R sufficiently large that each weight may be written as $w_k = m_k\widehat{w}_k/R$ where m_k is an integer and $1 \leq |\widehat{w}_k| \leq 1 + \delta$. So the weight (z_k, w_k) is split to form m_k weights $(z_k, \widehat{w}_k/R)$. A second procedure groups the weights into sets according to the value of the real part of their support, the weights in each band are sorted by the imaginary part of their support, and these sorted bands are passed to a procedure that selects weights from the sorted bands in a fashion that results in groups of weights with supports that are well spaced in the hyperbolic metric. One last procedure ensures that the resulting measures are symmetric about the real axis; this step helps to minimize numerical errors that manifest themselves later as small asymmetries in the final transfer functions.

The output from the first program is a file that contains sets of ordered pairs of complex numbers. The sets correspond to the partition given in Equations (26) and (27). The second C program takes as input the partitioned measure data from the first Program and a list of points at which to compute the value of the solution. The program constructs the Blaschke products and the Jones' interpolating functions of Lemma 6 for each part of the partitioned measure, and evaluates the partial solutions $b_k(z)$ at each of the data points. Summing the partial solutions at each data point produces the desired evaluations $b(z)$.

In summary, given a description of the Carleson measure, the pair of C programs will compute values of approximating solutions to the inhomogeneous Cauchy Riemann equations on any set of data points in the complex plane. The computed solutions converge in distribution, on any compact set to a bounded solution of the equations. The restriction of the approximations to the imaginary axis are uniformly bounded L^{∞} functions that converge in the weak-star topology as the discretization of the Carleson measure converges in distribution to the actual Carleson measure.

4.4 Results

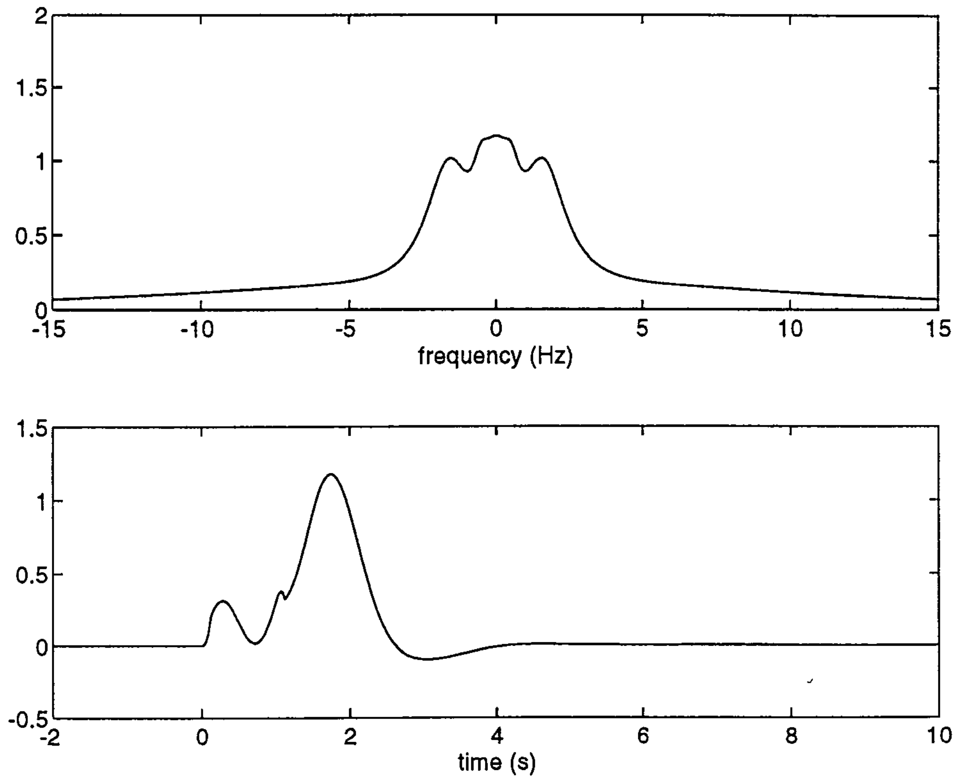

Figure 5: Transfer function and pulse response for $Q(z)$ with $\tau = 0.37$.

Figures 5 to 9 show the results of the computational solution to the problem described in section 2.

Figure 5 contains two graphs that describe the solution for the H^{∞} parameter Q in terms of a transfer function and the time domain response to a square input pulse of unit magnitude and 1 second duration. The value for the delay chosen was $\tau = 0.37$. The causal character of the pulse response verifies that the computed Youla parameter is an H^{∞} function. Figure 6 contains Nyquist plots of the open loop transfer functions of the combined

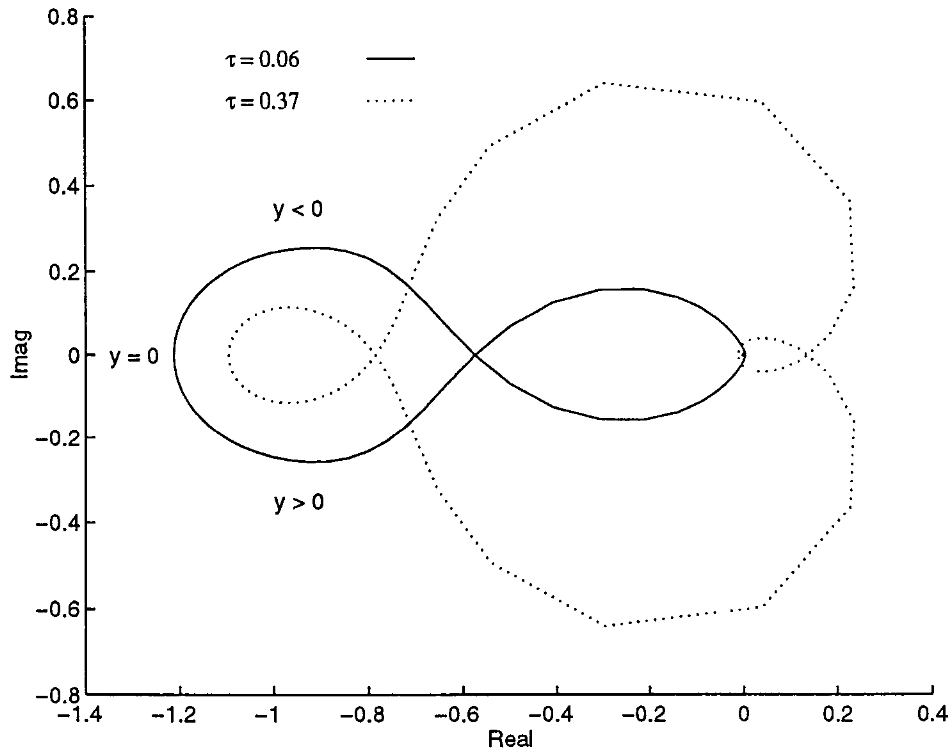

Figure 6: Nyquist plot of loop gain for controlled system.

system including Plant and feedback compensator for two values of the delay $\tau = 0.06$ and $\tau = 0.37$. The frequency variable that parameterizes the curves is y. A comparison with the corresponding Nyquist plot from [EOT92], Figure 7, illustrates well the differences in the controllers that are produced by the two different approaches. The controller of Enns et al. does have better stability and better low frequency sensitivity, this is to be expected, since the system that is considered in the example is sufficiently simple that the skew Toeplitz theory from [OST93] that Enns et al. use is able solve the Nehari problem with an optimal solution. In order to do this they require that the system be factored as a product of an H^∞ function and a rational function with inverse in H^∞; further, each factor needs to be decomposed by an inner outer factorization. Solutions under conditions more relaxed then those in [OST93] appear in the works of Flamm and Yang [FY94] and Özbay [Ö93]. Although a large number of interesting systems satisfy the conditions imposed in these works, the conditions are restrictive. While computation of inner-outer factorizations is a non-trivial problem, recent work by Flamm and Crow [FC94] which addresses the problem of computing numerical approximations to inner outer factorizations should extend the applicability of the results in [OST93] to more complicated examples. The method presented in this paper avoids both of these restrictions by avoiding the operator theoretic approach of [OST93] entirely.

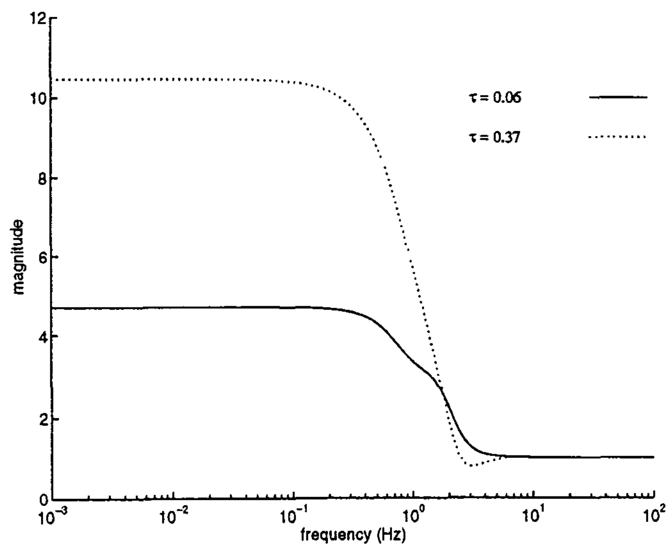

Figure 7: $|S(iy)| = |(1 + P(iy)C(iy))^{-1}|$.

Figures 7 and 8 plot the transfer functions that determine the closed loop sensitivity and the robustness. Comparison with Figures 9 and 10 of [EOT92], provides confirmation of the comments made in the previous paragraph. The graph of the largest singular value of $P(iy)$, Figure 9, provides an indication of how far the computed solution is from an optimal solution. An optimal solution would have a flat response with value less than 1.

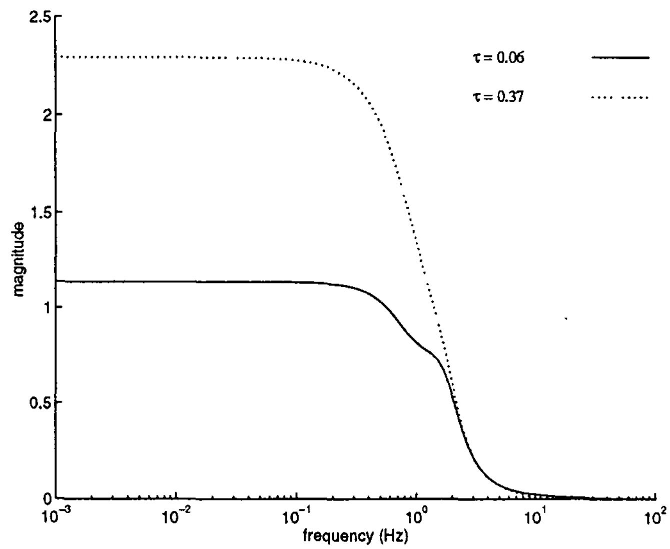

Figure 8: $|W_2(iy)C(iy)S(iy)|$.

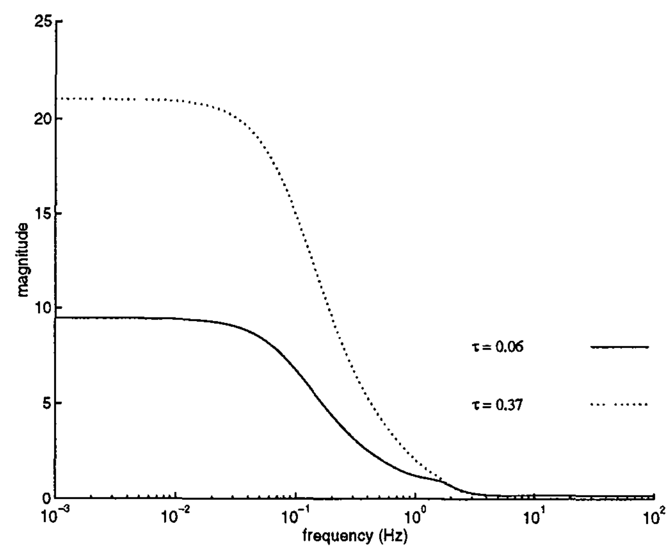

Figure 9: $\sqrt{|W_1 S(iy)|^2 + |W_2(iy)C(iy)S(iy)|^2}$.

5 General underdetermined systems of linear Diophantine equations

So far this paper has dealt only with controller design for a single-input single-output plant. It turns out that the same basic methods extend naturally to the multiple-input multiple-output case at the cost of an increase in the algebraic complexity of the formulation. This section shows how the standard, general formulation for an H^∞ control problem can be reduced to two general, underdetermined systems of linear Diophantine equations. After extending the algebraic setting from Section 3, Theorem 1 is used to give expressions for the solutions to these general systems of equations. A practical implementation of the methods of this section would take the same form and encounter the same problems as the computations described in Section 4.

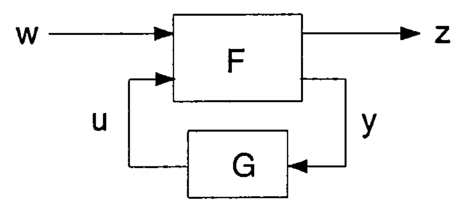

Figure 10: Configuration of Multi-input Multi-output Controller

Figure 10 depicts a feedback controller for a general class of multiple-input multiple-output robust control problems [Fra87]. The plant has a block transfer function matrix

$$F = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix},$$

and the equations for the closed loop system are:

$$\begin{aligned} z &= F_{11}w + F_{12}u \\ y &= F_{21}w + F_{22}u \\ u &= Gy \end{aligned}$$

The robust stabilization problem is the problem of choosing a controller with transfer function matrix G in a way that minimizes the operator norm of the matrix transfer function that maps the input signal w on to the output signal z. For a stable linear system the operator norm is the L^∞ norm of the largest singular value of the transfer function matrix which, in the single-input single-output case is just the H^∞ norm of the transfer function. The following theorem combines a number of results from Chapter 4 of [Fra87] including the Youla parameterization of stabilizing controllers.

Theorem 8 [Fra87]
Assume that F is stabilizable, then:

(i) G stabilizes F if and only if G stabilizes F_{22}.

(ii) Suppose $F_{22} = F_1 F_2^{-1} = \widetilde{F}_2^{-1}\widetilde{F}_1$ are co-prime factorizations of F_{22}, then there exist X_1, X_2, and $\widetilde{X}_1$, $\widetilde{X}_2$ such that[5]

$$\begin{bmatrix} \widehat{X}_2 & -\widehat{X}_1 \\ -\widehat{F}_1 & \widehat{F}_2 \end{bmatrix} \begin{bmatrix} F_2 & X_1 \\ F_1 & X_2 \end{bmatrix} = \mathbf{1}. \tag{42}$$

and the set of all G stabilizing F_{22} is parameterized by the formulae

$$\begin{aligned} G &= (-X_1 + F_2 Q)(X_2 - F_1 Q)^{-1} \\ &= (\widehat{X}_2 - Q\widehat{F}_1)^{-1}(-\widehat{X}_1 + Q\widehat{F}_2) \\ Q &\in \mathrm{H}^\infty. \end{aligned}$$

(iii) With G given by the parameterization in (ii), and with the transfer functions T_1, T_2, T_3 given by

$$\begin{aligned} T_1 &= F_{11} - F_{12}F_2\widehat{X}_1 F_{21} \\ T_2 &= F_{12}F_2 \\ T_3 &= \widehat{F}_2 F_{21}, \end{aligned}$$

the transfer function from w to z in Figure 10 equals $T_1 - T_2 Q T_3$.

Although this theorem was proved by Francis in the setting of rational H^∞ functions, a suitable definition of co-primeness broadens its scope to more general rings. An appropriate notion of co-primeness for the ring H^∞ is one based on the condition in the premise of Lemma 2; this link between the co-prime factorizations for transfer functions and the corona theorem

[5] $\mathbf{1}$ is used to denote the identity matrix which, in this context, is a matrix of constant H^∞ functions

was stated by Baras [Bar80]. Also related to the theorem is the work of Smith [Smi89] who shows that any plant that has a factorization over H^∞ and is feedback stabilizable has a doubly co-prime factorization.

Given a plant transfer function matrix F with left and right co-prime factorizations, the design of an optimal controller requires two steps: the first is to find eight H^∞ matrices that satisfy Equation (42), and the second is the search for the H^∞ matrix that minimizes the norm of the closed loop transfer function

$$P = T_1 - T_2 Q T_3. \tag{43}$$

Both steps can be reformulated as the solution to an underdetermined system of equations of the form

$$Ax = b. \tag{44}$$

Finding matrices that satisfy Equation (42) is equivalent to finding a left factorization $F = \widetilde{F}_2^{-1}\widetilde{F}_1$ a right factorization $F = F_1 F_2^{-1}$, and H^∞ matrices X_1, X_2, $\widetilde{X}_1$ and $\widetilde{X}_2$ that satisfy the four equations

$$\begin{aligned} \widetilde{X}_2 F_2 - \widetilde{X}_1 F_1 &= 1 && (45)\\ \widetilde{F}_2 X_2 - \widetilde{F}_1 X_1 &= 1 && (46)\\ \widetilde{F}_2 F_1 - \widetilde{F}_1 F_2 &= 0 && (47)\\ \widetilde{X}_2 X_1 - \widetilde{X}_1 X_2 &= 0. && (48) \end{aligned}$$

Equations (45) and (46) are matrix Bezout equations, and Equation (47) is automatically satisfied since the left and right factorizations are factorizations of the same transfer function matrix. Given left and right co-prime factorizations of F, and arbitrary solutions $\widetilde{Y}_1$ and $\widetilde{Y}_2$ to Equation (46), and Y_1, Y_2 to Equation (45), a little algebraic manipulation yields the following parameterization of all doubly co-prime factorizations that satisfy equations (45 – 48):

$$\begin{aligned} X_1 &= Y_1 + F_2 A\\ X_2 &= Y_2 + F_1 A\\ \widetilde{X}_1 &= \widetilde{Y}_1 + (A - \widetilde{Y}_1 Y_2 + \widetilde{Y}_1 Y_2)\widetilde{F}_2\\ \widetilde{X}_2 &= \widetilde{Y}_2 + (A - \widetilde{Y}_1 Y_2 + \widetilde{Y}_1 Y_2)\widetilde{F}_2. \end{aligned}$$

The parameter A is a matrix with entries in H^∞. With this result the computation of a doubly co-prime factorization reduces to the solution of the two matrix Bezout equations (45) and (46). The form of these equations is similar to the form of (43), and with suitable substitutions the solution of each of the three equations is subsumed by the following problem: given A^1, B^1, A^2, B^2, and C, find X^1 and X^2 that solve

$$A^1X^1B^1 + A^2X^2B^2 = C. \tag{49}$$

This equation is the matrix analog of (8) for multiple-input multiple-output systems.

Equation (49) has the form of a general linear equation in the entries of the matrices X^1 and X^2. This equation is recast in the form of (44), $Ax = b$ by stacking the columns of the matrices X^1 and X^2 to form a long vector x, stacking the columns of C to form the vector b, and replacing the left and right multiplying matrices by one left multiplying matrix A. If the dimension of A is $m \times n$ then $m < n$, and A represents a module homomorphism with domain $\mathrm{H}^\infty \times \overset{n}{\dots} \times \mathrm{H}^\infty$, and image $\mathrm{H}^\infty \times \overset{m}{\dots} \times \mathrm{H}^\infty$.

As in Section 3.1 the solution to (44) is based on Theorem 1, but the definition of the spaces and the operators in (10) and Figure 2 need to be changed. Define the following modules over a ring R

$$\begin{aligned}
\Lambda^0(R) &= R \times \overset{m}{\dots} \times R \\
\Lambda^1(R) &= R \times \overset{n}{\dots} \times R \\
\Lambda^2(R) &= \wedge^{n-m-1}(R \times \overset{m}{\dots} \times R).
\end{aligned}$$

The three rings of interest are the same as those in Section 3, H^∞, $\mathcal{B}$, and $\mathcal{C}$. Denote the rows of the matrix A by $A_1 \dots A_m$ and the columns by $a_1 \dots a_n$, then $A_i \in \Lambda^1(\mathrm{H}^\infty)$, and $a_j \in \Lambda^0(\mathrm{H}^\infty)$. Define the homomorphism $P_A : \Lambda^1(R) \to \Lambda^0(R)$ by $P_A x = \sum x_i a_i$, then Equation (44) can be written $P_A x = b$. let $\{e_\beta\}$ be a basis for $\wedge^{n-m-1} \mathbb{C}^n$, and let $y = y_\beta e_\beta$ be an element of $\Lambda^2(R)$. Define a second homomorphism $P_A : \Lambda^2(R) \to \Lambda^1(R)$ by

$$(P_A y) = \sum_\beta y_\beta(\star(A_1 \wedge \dots \wedge A_m \wedge e_\beta)) \tag{50}$$

in which the star homomorphism[6] is taken relative to the normal Euclidean scalar product on $\mathbb{C}^n$. With these definitions the sequence (10) may be rewritten as the sequence

$$\Lambda^2(R) \xrightarrow{P_A} \Lambda^1(R) \xrightarrow{P_A} \Lambda^0(R) \longrightarrow 0, \tag{51}$$

[6]The star homomorphism is determined by its action on homogeneous forms. On these it satisfies the equation $\star(e_{i_1} \wedge \dots \wedge e_{i_k}) \wedge (e_{i_1} \wedge \dots \wedge e_{i_k}) = e_1 \wedge \dots \wedge e_n$, where the vectors e_i are the unit vectors in $\mathbb{C}^n$

which is exact at $\Lambda^1(R)$. In fact the sequence in (51) may be extended leftward to form a complete sequence, but the definitions given are already enough for an application of Theorem 1.

Theorem 2 provides an algorithm that produces a solution to (44) as follows. First find $x^1 \in \Lambda^1(\mathcal{B})$ that solves the equation

$$P_A x^1 = b. \tag{52}$$

The ring $\mathcal{B}$ is the ring of distributions with boundary values in L^∞ that was introduced in Section 3.1. The solution needs to be chosen so that $\partial x^1/\partial \bar{z} \in \Lambda^1(\mathcal{C})$, where $\mathcal{C}$ is the ring of Carleson measures that have support on the right half plane. With this in mind choose $x^2 \in \Lambda^2(\mathcal{C})$ to be a solution of

$$P_A x^2 = \frac{\partial x^1}{\partial \bar{z}} \tag{53}$$

and $x^3 \in \Lambda^2(\mathcal{B})$ to be a solution of

$$\frac{\partial x^3}{\partial \bar{z}} = x^2. \tag{54}$$

It follows from Theorem 1 that a solution to (44) is given by

$$x = x^1 - P_A x^3. \tag{55}$$

As in the scalar case, the real computational problems lie in inverting the three operators $\partial/\partial\bar{z} : \Lambda^2(\mathcal{B}) \to \Lambda^2(\mathcal{C})$, $P_A : \Lambda^1(\mathcal{B}) \to \Lambda^0(\mathcal{B})$ and $P_A : \Lambda^2(\mathcal{C}) \to \Lambda^1(\mathcal{C})$. Fortunately though, the same approach that was used in the scalar case may be applied to systems of equations with some extra algebra. The first operator, $\partial/\partial\bar{z}$ is the easiest to deal with, Equation (54) may be solved by applying the method of Section 3.2 to calculate each component of x^3 from the corresponding component of x^2. The remaining operators are inverted by using a method due to [Rao83] to construct a left inverse, and some algebraic constructions that are similar to those presented in [BS86] and [Str83].

First consider Equation (52). Denote by A_γ the rank m minors of A, then the index γ can take one of $n!/(n-m)!m!$ values that correspond to the choices of m columns from the n columns of A. Provided that the functions A_γ satisfy the condition of Lemma 2, there exists a partition of the plane into sets Ω_γ such that if ϕ_γ is the characteristic function of the set Ω_γ, that is,

$$\phi_\gamma(z) = \begin{cases} 0 & z \in \Omega_\gamma \\ 1 & z \notin \Omega_\gamma \end{cases},$$

then A_γ is bounded away from zero outside the set Ω_γ. The distributional derivatives $\partial\phi_\gamma/\partial\bar{z}$ are Carleson measures supported on the boundaries $\partial\Omega_\gamma$. Choose $G_\gamma = \phi_\gamma/A_\gamma$, then each G_γ is a bounded analytic function on the interior of Ω_γ, is identically zero outside Ω_γ, and has a distributional derivative $\partial G_\gamma/\partial\bar{z}$ that is a Carleson measure supported on the boundary $\partial\Omega_\gamma$. Further, the functions G_γ solve the equation

$$\sum_\gamma A_\gamma G_\gamma = 1.$$

[Rao83] uses the Cauchy Binet theorem to show that if

$$g_{jk} = \sum_\gamma G_\gamma \frac{\partial A_\gamma}{\partial a_{kj}}$$

then the matrix $G = [g_{jk}]$ is a right inverse of A with rank m minors G_γ. It follows that a solution to equation (52) is given by

$$\begin{aligned} x_j^1 &= \sum_k \sum_\gamma G_\gamma \frac{\partial A_\gamma}{\partial a_{kj}} b_k \\ &= \sum_k \sum_\gamma \frac{\phi_\gamma}{A_\gamma} \frac{\partial A_\gamma}{\partial a_{kj}} b_k \end{aligned} \tag{56}$$

The final equation that needs to be solved is Equation (53) $P_A x^2 = \partial x^1/\partial\bar{z}$. Let x^2 have components y_β with respect to the canonical basis for $\Lambda^2(C)$. When the solution from (56) is substituted for x^1, and the expression for the operator P_A from (50) is expanded in coordinates, the j'th component of Equation (53) becomes

$$\sum_{\beta,\alpha} y_\beta A_\alpha = \sum_{\substack{\gamma \\ j\in\gamma}} \sum_k \frac{\partial}{\partial\bar{z}} \left(G_\gamma \left(\frac{\partial A_\gamma}{\partial a_{kj}} \right) b_k \right)$$

The summation on the left hand side in this formula is taken over all multi-indices α and β such that $j \notin \alpha \cup \beta$, $\alpha \cap \beta = \emptyset$, and $|\alpha| = m$. Substituting the solution for G_γ gives

$$\sum_{\beta,\alpha} y_\beta A_\alpha = \sum_{\substack{\gamma \\ j\in\gamma}} \sum_k \frac{b_k}{A_\gamma} \frac{\partial A_\gamma}{\partial a_{kj}} \frac{\partial \phi_\gamma}{\partial\bar{z}}.$$

It follows from the choice of ϕ that the sum on the right hand side is supported entirely on the curve segments $\partial\Omega_{\gamma_p} \cap \partial\Omega_{\gamma_p}$. So the components y_β of the solution x^2 are measures supported on the boundaries $\partial\Omega_{\gamma_p}$, and at any point on these boundaries there are n equations for the $n!/m!(n-m-1)!$ variables y_β of the form

$$\sum_{\beta,\alpha} y_\beta A_\alpha = \begin{cases} 0 & j \notin \gamma_p \cup \gamma_q \\ \displaystyle\sum_k \pm \frac{b_k}{A_{\gamma_p}} \frac{\partial A_{\gamma_p}}{\partial a_{jk}} & j \in \gamma_p - \gamma_q \\ \displaystyle\sum_k \left(\pm \frac{1}{A_{\gamma_p}} \frac{\partial A_{\gamma_p}}{\partial a_{jk}} \pm \frac{1}{A_{\gamma_q}} \frac{\partial A_{\gamma_q}}{\partial a_{jk}} \right) & j \in \gamma_p \cap \gamma_q \end{cases}$$

The arbitrary signs are determined by the sense of integration inherent in the measures $\partial\psi_p/\partial\bar{z}$ and $\partial\psi_q/\partial\bar{z}$

Although the algebra associated with the inversion of the operators $P_A : \Lambda^1(\mathcal{B}) \to \Lambda^0(\mathcal{B})$ and $P_A : \Lambda^2(\mathcal{C}) \to \Lambda^1(\mathcal{C})$ seems complicated, the real computational difficulties are the same as those experienced with the single-input single-output system, namely, choosing a partition Ω_γ and computing minimal norm solutions of $\partial b/\partial\bar{z} = \mu$ for a Carleson measure μ. The requirement that Lemma 2 places on the minors A_γ of the matrix A induces the appropriate co-primeness conditions on the left and right factorizations of the transfer function matrix F for the multiple-input multiple-output system.

6 Conclusion

This paper has presented a new computational method for H^∞ controller design. The method places two requirements on the systems to which it applies: an explicitly computable co-prime factorization of the system over H^∞ functions should exist, and sufficient information about the location of the zeros of the factors is needed to construct the partition of unity in Lemma 2. These requirements are very close to necessary conditions for a linear plant to be stabilizable, a fact that indicates that the techniques presented are potentially widely applicable.

The method has been demonstrated on a simple example drawn from the literature. For this example independent methods may be used to construct a controller that is optimal in the sense of H^∞ control; for this reason it provides a good indication of how close the controller computed for the example is to an optimal controller. For a controller design to be truly practical it is important that it produce near optimal controllers. This is particularly true for infinite dimensional systems which often have transfer functions that are sensitive to small parameter variations. Under this criterion for practicality, the conclusion to be drawn from the result in Section 4 is that more work is needed.

Two areas in need of further work stand out. The first concerns the selection of the partition of unity that is postulated in in Lemma 2, and is used in the construction of the Carleson measures. The particular selection made for a given problem affects the quality of the solution through the norm of the inverse in Equation (18), and through the Carleson constant associated with the Blaschke product in the inequality (19). The intricate construction that is required in the Carleson's proof of the Corona Theorem would indicate that in the most general case choosing an optimal selection is a difficult problem. however, many of the situations that are of interest in engineering are described by boundary value problems and delay differential equations of the type presented in this paper; in these cases the additional structure provided by the problem description can be exploited to provide partitions of unity without recourse to elaborate constructions.

The second area for future work is the problem of constructing bounded solutions to the Cauchy Riemann equations. In particular, attention should be paid to the interpolating function that is used in the computation of the solutions. Although the function given in Theorem 3 is optimal in the sense that it satisfies the bound given in Equation (22) independent of the measure, a particular choice of interpolating function tailored to a particular measure could produce a lower bound.

Finally, a note on the calculations. The method described is computationally intensive, however, with careful programming, it certainly is feasible. The computations for the example presented took minutes, rather than hours, on a workstation with a RISC processor and floating-point coprocessor[7]. The predominant computation involves evaluating a small set of functions over a large set of data points with no interdependencies in the evaluations. This type of calculation is trivially parallelizable on massively parallel architectures.

[7] The calculations were made in compiled C code on a SPARC Station 10

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