COMPLETIONS

by

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ABSTRACT

Title of Thesis: Completions

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This paper presents a new approach to the theory of completions. The treatment is based on the concept of convergence on filters and related topologies [3]. For a given uniform Hausdorff space $X_{\mathcal{U}}$ and a collection $\mathcal{S}$ of Cauchy filters in $X_{\mathcal{U}}$, the basic result is the construction of a uniform Hausdorff space $\tilde{X}_{\mathfrak{U}}$ having the properties that $X_{\mathcal{U}}$ is isomorphic to a dense subspace of $\tilde{X}_{\mathfrak{U}}$ and every filter in $\mathcal{S}$ converges to a point in S. As a special case, the completion $\hat{X}_{\mathcal{U}}$ of $X_{\mathcal{U}}$ is obtained. The construction is so given as to prove the existence of the space $\tilde{X}_{\mathfrak{U}}$.

The technique involves embedding the object X to be "completed" in a space of functions F which has as its domain a space of continuous functions C(X) defined on X. The procedure is analogous to the process of taking the bidual E'' of a locally convex topological vector space. Indeed, E'' is obtained as a special case. In the absence of sufficient structure on X, the space $\tilde{X}_{\mathfrak{U}}$ is obtained as the closure of X in F. In a locally convex space or an abelian topological group having enough characters to separate points, $\tilde{X}_{\mathfrak{U}}$ is obtained as a bidual or a second character group of the object X.

This dissertation is dedicated to my wife Joan.

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TABLE OF CONTENTS

INTRODUCTION

This paper presents a unified theory of completions which is applicable to linear topological spaces, topological groups and in general, to any uniform Hausdorff space. The development is based on the concept of convergence on filters and related topologies [3]. The central idea of this paper is to use the topology obtained from the uniformity of convergence on a class of Cauchy filters to obtain a representation of the completion or partial completion of a uniform Hausdorff space $X_{\mathcal{U}}$. The construction is so given as to yield a proof of the existence of the completion. The completion of the space $X_{\mathcal{U}}$ is understood to be a complete uniform Hausdorff space $\hat{X}_{\mathcal{U}}$ with the property that $X_{\mathcal{U}}$ is uniformly isomorphic to a dense subspace X_o of $\hat{X}_{\mathcal{U}}$. The partial completion of the space $X_{\mathcal{U}}$ is the smallest uniform Hausdorff space $\widetilde{X}_{\tilde{\mathcal{U}}}$ having the property that $X_{\mathcal{U}}$ is uniformly isomorphic to a dense subspace of $\widetilde{X}_{\tilde{\mathcal{U}}}$ and a prescribed family of Cauchy filters in $X_{\mathcal{U}}$ converge to limits in $\widetilde{X}_{\tilde{\mathcal{U}}}$.

Section I introduces the concept of convergence on filters and contains some elementary results that are not only important to this development but also to obtaining characterizations of compact subsets of function spaces in both the pointwise and uniform convergence topologies [3].

Section II gives the theory of completion and partial completion for a locally convex topological space along with some applications.

This case is treated first for two reasons: first, because the theory was initially conceived in this setting and secondly because the details of the construction are more transparent in a locally convex space.

Section III deals with an arbitrary uniform Hausdorff space and gives the Stone-Čech Compactification as an application.

Section IV treats topological groups and is divided into two parts, the first devoted to an abelian Hausdorff topological group having enough characters to separate points and the second to an arbitrary Hausdorff topological group.

Throughout this paper the notation $\sigma(S, C)$ denotes the weak topology on an abstract set S induced by a space of functions C defined on S. That is, $\sigma(S, C)$ is the topology on S of pointwise convergence on C. For a locally convex space E, $\epsilon(E, E')$ denotes the topology on E of uniform convergence on equicontinuous subsets of E' (the dual of E). It should be noted that if $\mathcal{J}$ is the original topology on E, then $\epsilon(E, E') = \mathcal{J}$ [8, 4]. The strong topology on E' will be written as $\beta(E', E)$ (uniform convergence on all bounded subsets of E).

SECTION I

CONVERGENCE ON FILTERS

Let $G(S, \mathbb{R})$ denote a space of functions from an abstract set S into the reals.

1.1 Definition [3,1]: A filter $\mathcal{G}$ composed of subsets of $G(S, \mathbb{R})$ converges to a function f_o on a filter $\mathcal{F}$ of subsets of S if for every $\epsilon > 0$ there is a D in $\mathcal{G}$ such that for each $f \in D$ there is an F_f in $\mathcal{F}$ with the property that

$$|f(s) - f_o(s)| < \epsilon \qquad \text{for all } s \in F_f.$$

Note, that if $\mathcal{G}$ converges to f_o on $\mathcal{F}$, then $\mathcal{G}$ converges to f_o on every refinement of $\mathcal{F}$. The concept of convergence on filters is related to the continuity properties of pointwise limits of continuous functions. This is of fundamental importance to the whole developement. The precise relationship is given in the following theorem.

1.2 THEOREM [3,1]: Let $\mathcal{G}$ be a filter in $G(S, \mathbb{R})$ where S is a topological space and every g in a member of $\mathcal{G}$ is continuous at a point s_o in S. Then $\mathcal{G}$ converges at s_o to a function f_o which is continuous at s_o if and only if $\mathcal{G}$ converges to f_o on the filter of neighborhoods of s_o.

1.3 THEOREM [3,1]: In $G(S, \mathbb{R})$, the class of all filters which converge on a filter $\mathcal{F}$ in S has an associated topology obtained from the uniformity having as a base for its entourages, sets of the form:

$$U(\epsilon, \mathcal{F}) = \left\{ (f, g) \mid \text{ there is an } F_{fg} \text{ in } \mathcal{F} \text{ such that } |f(s) - g(x)| < \epsilon \text{ for all } s \in F_{fg} \right\}.$$

The topology is Hausdorff if and only if for each pair f and g in $G(S, \mathbb{R})$ there exists an $\epsilon > 0$ such that for every F in $\mathcal{F}$ there is an $s \in F$ such that $|f(s) - g(s)| \geq \epsilon$.

The above theorem extends easily to convergence on a family of filters in S and it is in this form that it will be used throughout this paper.

A duality exists between the topology on $G(S, \mathbb{R})$ of convergence on a class of filters $\mathcal{D}$ in S and the convergence of the members of $\mathcal{D}$ and is the content of the following theorem

1.4 THEOREM [3,4]: Let φ be a single function having as its domain $G(S, \mathbb{R}) \times S$ and range in $\mathbb{R}$. Consider filters $\mathcal{G}$ and $\mathcal{F}$ respectively, with the property that $\varphi(\mathcal{G}, s)$ converges to $\varphi(g_o, s)$ for all $s \in S$ and $\varphi(g, \mathcal{F})$ converges to $\varphi(g, s_o)$ for all $g \in G(S, \mathbb{R})$. Then the filter $\mathcal{G}$ converges to g_o on $\mathcal{F}$ if and only if $\mathcal{F}$ converges to s_o on $\mathcal{G}$.

There is a close relationship between the topologies, on a space of continuous functions, of pointwise convergence, and the topology of convergence on convergent filters in the domain of the functions. This relationship is given in the next two propositions.

1.5 PROPOSITION: If $C(S)$ is a space of continuous real valued functions on a topological space S, then the topology on $C(S)$ of convergence on all convergent filters in S is the same as the $\sigma(C(S), S)$-topology.

PROOF: Let $\mathcal{F}$ be a filter in $C(S)$ converging to f_o in the $\sigma(C,S)$-topology. Then $\mathcal{F}$ converges on every filter of neighborhoods $\mathcal{N}(x)$ in S (Theorem 1.2). Since every convergent filter refines some $\mathcal{N}(x)$, $\mathcal{F}$ converges on every convergent filter by a remark following Def. 1.1. Conversely, suppose $\mathcal{F}$ converges to f_o in the topology of convergence on all convergent filters. Then, in particular, $\mathcal{F}$ converges to f_o on every $\mathcal{N}(x)$ and thus by Theorem 1.2, $\mathcal{F}$ converges to f_o in the $\sigma(C, S)$-topology.

1.6 PROPOSITION: Let A be a dense subspace of S, $\mathcal{C}$ a filter in A with limit $\hat{x} \in S$ and $\mathcal{F}$ a filter in $C(A)$ converging to a function $f_o \in C(S)$ on $\mathcal{C}$. Then, if every f in $C(A)$ has a unique continuous extension $\tilde{f}$ to all of S, the filter $\tilde{\mathcal{F}}$ of extended functions converges to f_o at $\hat{x}$.

PROOF: Since $\mathcal{F}$ converges to f_o on $\mathcal{C}$ so does $\tilde{\mathcal{F}}$. Thus for every $\epsilon > 0$ there is a D in $\tilde{\mathcal{F}}$ such that for each $f \in D$ there exists a C_f in $\mathcal{C}$ such that $|f(x) - f_o(x)| < \epsilon/3$ for all $x \in C_f$.

Continuity of f and f_o implies the existence of sets C_f' and C_f'' contained in C_f such that $|f(\hat{x}) - f(x)| < \epsilon/3$ for all $x \in C_f'$ and $|f_o(x) - f_o(\hat{x})| < \epsilon/3$ for all $x \in C_f''$. The proposition now follows from the triangle inequality.

1.7 REMARK: Brace has shown [3.2] that the topology on $G(S, \mathbb{R})$ obtained from the uniformity of convergence on all ultra filters in S is equivalent to the topology of almost uniform convergence on S. For a completely regular space S, this topology is then equivalent to pointwise convergence on the Stone-Čech Compactification of S [1.4]. This characterization, in conjunction with theorems 1.3, 1.4 and 1.5 has yielded characterizations of compact sets of functions in both the pointwise and uniform convergence topologies.

SECTION II

LINEAR SPACES

The following two known examples will indicate the point of view being adopted.

1) Let B be a Banach space. The bidual B'' can be thought of as the completion of the bounded subsets of B in the $\sigma(E, E')$-topology. This result also holds for E a locally convex space [8, 3]

2) Grothendiek has shown [8, 3] that the space $(E', \varphi)'$, equipped with the $\epsilon(E'_\varphi, E')$-topology is the completion of E, where φ denotes the finest locally convex topology which coincides with the $\sigma(E', E)$-topology on every equicontinuous subset of E'.

In the above cases the process of completing or partially completing a locally convex space can be viewed as a process of taking biduals. The primary goal of this paper is to establish that the completion or partial completion of any uniform Hausdorff space can be obtained in a similar manner. In a locally convex space, this process is exact, in the sense that precisely the desired completed space is obtained. In the absence of structure or structural maps the desired space is obtained as a closed subset of a certain "smallest" linear space.

In what follows, E will denote a locally convex Hausdorff space, E' its dual and E'^{*} the algebraic dual of E'. E will always be considered as a subspace of E'^* under the natural embedding $x \to \hat{x}$, where $\hat{x}(x') = x'(x)$. In order to minimize notational entanglements, it will be convenient to denote by $E''_{\mathcal{J}}$, the space $(E', \mathcal{J})'$. That is, the dual of the topological space $(E', \mathcal{J})$ is $E''_{\mathcal{J}}$.

2.1 Definition : Let $\mathcal{S}$ be an arbitrary collection of Cauchy filters in E. The γ-topology on E' is defined as the topology of convergence on a set $\mathcal{D}$ of filters in E, where a filter $\mathcal{F}$ is a member of $\mathcal{D}$ if and only if $\mathcal{F}$ is a member of $\mathcal{S}$ or $\mathcal{F}$ is a convergent filter. If $\mathcal{S}$ is the family of all Cauchy filters in E, this topology is referred to as the δ-topology.

2.2 PROPOSITION: $E \subset E''_{\gamma} \subset E''_{\delta}$

PROOF: Since the $\sigma(E', E)$-topology is the topology of convergence on all convergent filters (Prop. 1.5) we have,

$$\delta > \gamma > \sigma(E', E).$$

Thus, from the natural embedding of E in E'^*, the desired conclusion follows.

Since the topologies related to convergence on filters are closely related to pointwise convergence topologies it is natural to expect that convergence of the filters in $\mathcal{S}$ (Def. 2.1) will be obtained, initially in a pointwise convergence topology. Thus, the following lemma will be needed to obtain convergence of these filters in the topology of E.

2.3 LEMMA: If $\mathcal{F}$ is a Cauchy filter in E and g is a linear scalar valued function defined on E' as follows:

$$\mathcal{F}(x') \to g(x') \quad \text{for all } x' \varepsilon E',$$

then $\mathcal{F}$ (embedded in E'^{*}) converges to g uniformly on every equicontinuous subset of E'.

PROOF: See [11], lemma 8.5.

The next proposition establishes the fact that not only is E embedded in E''_{γ}, but every member of $\mathcal{S}$ converges to a limit which is in E''_{γ}.

2.4 PROPOSITION: Every Cauchy filter $\mathcal{F}$ in the collection $\mathcal{S}$ converges to a point $f \in E''_{\gamma}$ for the topology of uniform convergence on equicontinuous subsets of E'.

PROOF: Take an arbitrary $x'_0 \in E'$ and let $\mathcal{N}_{\gamma}$ denote the filter of neighborhoods of x'_0 for the γ-topology. The filter of $\mathcal{N}_{\gamma}$ converges to x'_0 on every member of $\mathcal{S}$. Let $\mathcal{F}$ be a filter in $\mathcal{S}$. There is a $\mu_0 \varepsilon E'^{*}$ such that $\mathcal{F}(x') \to \mu_0(x')$ for all $x' \varepsilon E'$. Also, $\mathcal{N}_{\gamma}(x) \to x'_0(x)$ for each $x \in E$. Thus by Theorem 1.4, $\mathcal{F}$ converges to μ_0 on $\mathcal{N}_{\gamma}$. Since each f in a member of $\mathcal{F}$ is continuous for the γ-topology, μ_0 is continuous on E' for the γ-topology. That is, by Theorem 1.3, $\mu_0 \varepsilon E''_{\gamma}$. By Proposition 2.3, $\mathcal{F}$ converges to μ_0 uniformly on every equicontinuous subset of E'.

If we denote by Q the point set $E \cup \{$limits of members of $\mathcal{S}$ in $E''_\gamma\}$, it has now been established that the linear span of Q is algebraically embedded in E''_γ. The next objective is to make this embedding a topological one. The following proposition and corollaries are intended to establish this fact. That is, it will be shown that the space E''_γ can be topologized in such a manner that the original topology on E can be induced on $E \subset E''_\gamma$. This is equivalent to showing that E''_γ can be equipped with the $\epsilon(E''_\gamma, E')$ topology.

2.5 PROPOSITION: The $\sigma(E', E)$-topology is the same as the γ-topology on every equicontinuous subset of E'.

PROOF: The γ-topology is finer than the $\sigma(E', E)$-topology. (See proof of prop. 2.2.) For the converse, consider an equicontinuous subset H of E' and let $\mathcal{C}$ be a member of $\mathcal{D}$ (Def. 2.1). By Proposition 2.4 there exists an $x_o \in E''_\gamma$ such that $\mathcal{C}$ converges to x_o uniformly on H. Consider a filter $\mathcal{F}$ on H converging to $f_o \in H$ for the $\sigma(E', E)$-topology. Given $\epsilon > 0$ there is a C in $\mathcal{C}$ such that

$$|f(x) - f(x_o)| < \epsilon/3 \qquad \text{for all } x \in C \text{ and all } f \in H.$$

(By uniform convergence of $\mathcal{C}$ on H.)

Let y be fixed in C. There is an F in $\mathcal{F}$ such that

$$|f(y) - f_o(y)| < \epsilon/3 \qquad \text{for all } f \in F. \quad \text{(By } \sigma(E', E)\text{-convergence of } \mathcal{F}.)$$

Thus,

$$|f(x_o) - f_o(x_o)| < |f(x_o) - f(y)| + |f(y) - f_o(y)| + |f_o(y) - f_o(x_o)|$$

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \quad \text{for all } f \in F.$$

Therefore, $\mathcal{F}(x_o) \to f_o(x_o)$. Since each $f \in H$ is continuous at x_o for the $\sigma(E''_\gamma, E')$ topology, the filter $\mathcal{F}$ converges to f_o on the filter of neighborhoods of x_o. Since $\mathcal{C}$ converges to x_o for a finer topology, $\mathcal{F}$ converges to f_o on $\mathcal{C}$.

2.6 COROLLARY: If x is an element of E''_γ, then the restriction of x to every equicontinuous subset H of E' is $\sigma(E', E)$-continuous.

2.7 COROLLARY: $\epsilon(E''_\gamma, E')$ is a locally convex topology.

PROOF: All that needs to be shown is that the neighborhoods of 0 for $\epsilon(E''_\gamma, E')$ are absorbing, i.e. that for every $x \in E''_\gamma$, $x|_H(H)$ is bounded in the scalars for every equicontinuous subset $H \subset E'$. Using Corollary 2.6 and the fact that H is $\sigma(E', E)$-relatively compact, it is concluded that $x|_H(H)$ is a relatively compact set of scalars, hence bounded.

2.8 COROLLARY: The locally convex space $\widetilde{E}$ exists and $\widetilde{E} \subset E''_\gamma$.

2.9 PROPOSITION: γ-topology $= \sigma(E', Q) = \sigma(E', \widetilde{E})$.

PROOF: Since $\widetilde{E}$ is the linear span of Q in E''_γ, the $\sigma(E', Q)$-topology is the same as the $\sigma(E', \widetilde{E})$-topology. Note that every filter $\mathcal{C}$ in the collection $\mathcal{S}$ is a convergent filter in $\widetilde{E}$, for $\epsilon(\widetilde{E}, E')$-topology. Thus the $\sigma(E', \widetilde{E})$-topology is finer than the γ-topology. Conversely, suppose $\mathcal{F}$ is a filter in E' converging to x' for the γ—topology and let $\hat{x}$ be an arbitrary point in Q The filter $\mathcal{F}$ converges to x' at $\hat{x}$, by Proposition 1.6. Thus the γ-topology is pointwise convergence on Q and since $\hat{x}$ is in the linear span of Q it follows that $\mathcal{F}$ converges to x' on $\widetilde{E}$.

2.10 COROLLARY: δ-topology = $\sigma(E', \widehat{E})$-topology.

It is now possible to give the two principal results of this section. They are contained in theorems 2.11 and 2.12.

2.11 THEOREM: $\widehat{E} = E''_\delta$

PROOF: As a result of linear space duality theory, the following relations are true [8, 3].

$$\widehat{E} = (\widehat{E}', \sigma(\widehat{E}', \widehat{E}))' = (E', \sigma(E', \widehat{E}))' = E''_\delta .$$

By Corollary 2.7 the space E''_δ can be equipped with the $\epsilon(E''_\delta, E')$-topology, which will induce the original topology on E.

2.12 THEOREM: <u>The space</u> E''_γ <u>equipped with the</u> $\epsilon(E''_\gamma, E')$-<u>topology is the smallest locally convex space</u> F <u>having the property</u>

<u>that E is a dense subspace and each $\mathcal{F}$ in $\mathcal{S}$ converges to a point in F.</u>

PROOF: E is dense in E''_γ by Theorem 2.11 and Proposition 2.2. Each $\mathcal{F}$ in $\mathcal{S}$ converges to an $x \in E''_\gamma$ by Proposition 2.4. Since $E \subset \widetilde{E} \subset \widehat{E}$ and $E' = \widehat{E}'$ it follows that $E' = \widetilde{E}'$. Thus, by the duality theorem for locally convex spaces and Proposition 2.9 ,

$$\widetilde{E} = (\widetilde{E}', \sigma(\widetilde{E}', \widetilde{E}))' = (E', \sigma(E', \widetilde{E}))' = E''_\gamma .$$

By Corollary 2.7, E''_γ can be equipped with the $\epsilon(E''_\gamma, E')$-topology. Since $\widetilde{E}$ is the linear span of Q , E''_γ is the smallest locally convex space containing E and the limit points of all the filters in the collection $\mathcal{S}$.

It should be noted that the γ- and δ- topologies are the weak topologies with respect to the dual systems $(\widetilde{E}, E')$ and $(\widehat{E}, E')$.

2.13 APPLICATIONS OF THEOREM 2.12

1) <u>Completions.</u> Let $\mathcal{S}$ be the collection of all Cauchy filters.

2) <u>Quasi-completions.</u> Let $\mathcal{S}$ be the collection of all bounded Cauchy filters.

3) <u>Biduals.</u> Let $\mathcal{S}$ be the collection of all bounded weak Cauchy filters. In this situation the γ -topology is the coarsest topology $\mathcal{F}$ such that $(E', \mathcal{F})' = E''$. That is, γ is the weak topology with respect to the $\beta(E', E)$-topology.

4) If $\mathcal{S}$ is taken to be the collection of all precompact Cauchy filters ($\mathcal{F}$ is precompact if it contains a precompact set), then E''_{γ} has the property that every precompact subset of E has a compact closure in E''_{γ}.

5) By taking $\mathcal{S}$ to be the Fréchet filters associated with a family of sequences, arbitrary families of sequences can be "completed."

6) A set $A \subset E$ is said to be $\aleph_{\alpha}$-complete, if every Cauchy filter in A which contains a set of cardinality $\aleph_{\alpha}$ converges to a point in A. [1 2]. Such a filter is called an $\aleph_{\alpha}$-Cauchy filter. By taking $\mathcal{S}$ to be a set of $\aleph_{\alpha}$-Cauchy filters an $\aleph_{\alpha}$-partial completion of E can be obtained.

SECTION III

UNIFORM SPACES

Throughout this section $X_{\mathcal{U}}$ will denote a point set X with a uniform structure $\mathcal{U}$. The family C^u (or $C^u(X_{\mathcal{U}})$) of all uniformly continuous real valued functions defined on $X_{\mathcal{U}}$ will play a role analogous to that of E' in Section II. The set X will always be considered as a subset of $(C^u)^*$ under the natural embedding $x \to \hat{x}$ where $\hat{x}(f) = f(x)$. As in Section II, the main concern will be the proper topologizing of the space C^u. Thus, the notation $C^{u'}_{\gamma}$ will be convenient for the dual of the linear topological space (C^u, γ) where γ will be some locally convex topology. The C^u-uniformity on X will be understood to mean the uniform structure on X which has a base consisting of sets of the form $U(f,\epsilon) = \{(x, y) \mid |f(x) - f(y)| < \epsilon\}$.

3.1 PROPOSITION: X can be embedded as an algebraically free subset of $C^{u'}_{\mathcal{J}}$ for any topology $\mathcal{J}$ which is finer than the $\sigma(C^u, X)$-topology (the topology of pointwise convergence).

PROOF: X can be embedded in $(C^u)^*$ by the remarks above. Let $\mathcal{F}$ be a filter in C^u converging to f_o in the $\sigma(C^u, X)$-topology. That is, $\mathcal{F}(x)$ converges to $f_o(x)$ for each x in X. Then $\hat{x}(\mathcal{F}) = \mathcal{F}(x)$ converges to $\hat{x}(f_o)$. Thus $\hat{x} \in (C^u, \sigma(C^u, X))'$.

To prove that X is algebraically free in $C^{u'}_{\mathcal{J}}$, suppose that the finite subset $\{x_i\}_{1 \le i \le k}$ of X is given and that there exists a finite set of scalars $\{a_i\}_{1 \le i \le k}$ with the property that at least one a_i is nonzero and

$$\sum_{1}^{k} a_i \hat{x}_i = 0.$$

That is, $(\sum_{1}^{k} a_i \hat{x}_i)(f) = \sum_{1}^{k} a_i \hat{x}_i(f) = 0$ for all $f \in C^u$. Since the $\mathcal{U}$-topology is regular, finite sets in $X_{\mathcal{U}}$ are closed. Hence for each x_i, there exists an $f_i \in C^u$ [9, 1] such that $f_i(x_i) = 1$ and $f_i(x_j) = 0$ for all $i \neq j$. Therefore,

$$0 = \sum_{1}^{k} a_i \hat{x}_i(f_i) = \sum_{1}^{k} a_i f_i(x_i) = a_i \quad \text{for } i = 1, \ldots, k,$$

in contradiction to the earlier assumption.

REMARK: In all that follows, X will always be considered as a subset of $C^{u'}_{\mathcal{J}}$, where $\mathcal{J}$ is some locally convex topology finer than the $\sigma(C^u, X)$-topology. As in the linear space case, it is desirable to have this embedding be a topological one. Thus, the next objective is to equip the space $C^{u'}_{\mathcal{J}}$ with a topology which will induce the $\mathcal{U}$-topology on X.

3.2 PROPOSITION: The $\sigma(X, C^u)$-topology on X is the same as the $\mathcal{U}$-topology.

PROOF: That the $\sigma(X, C^u)$-topology is coarser than the $\mathcal{U}$-topology follows from the continuity of each f in C^u. Conversely, let $U[x]$ be a neighborhood of x for the $\mathcal{U}$-topology. Since the $\mathcal{U}$-topology is completely regular and $X_{\mathcal{U}}$ is a uniform space, there exists a function f in C^u [9, 1] with the property that $f(x) = 0$ and $f(\widetilde{U[x]}) = 1$. Let $\epsilon > 0$ be given and denote by V_ϵ the open interval $(-\epsilon, \epsilon)$. Then $f^{-1}(V_\epsilon) \subset U[x]$. Since $f^{-1}(V_\epsilon)$ is a base set for the $\sigma(X, C^u)$-topology the desired conclusion is obtained.

3.3 COROLLARY: The $\sigma(C^{u'}_{\mathcal{J}}, C^u)$-topology on $C^{u'}_{\mathcal{J}}$ induces the $\mathcal{U}$-topology on X.

We note that the C^u-uniformity on the space $C^{u'}_{\mathcal{J}}$ does not necessarily induce the uniformity $\mathcal{U}$ on X. It is clear that the C^u-uniformity on X is coarser than the $\mathcal{U}$-uniformity, but it is evidently unknown as to whether or not the C^u-uniformity is identical with $\mathcal{U}$. It is identical in some cases. For example, the usual uniformity on the real line can be obtained from $C^u(\mathbb{R})$.

3.4 PROPOSITION: X is a closed subspace of $(C^u, \sigma(C^u, X))' = C^{u'}_{\sigma}$ for the $\sigma(C^{u'}_{\sigma}, C^u)$-topology.

PROOF: The proof is divided into two parts. First it is shown that X is a closed subspace of its linear span, spX, in $C^{u'}_{\sigma}$ for the $\sigma(\text{spX}, C^u)$-topology, and then that $\text{spX} = C^{u'}_{\sigma}$.

Let $y = \sum_{1}^{k} a_i\hat{x}_i$, $k > 1$, $a_i \neq 0$ for $i = 1, \ldots, k$, be an element of spX. Consider the neighborhood V of y,

$$V = V^{(y)}_{f_1,\ldots f_k,} = \left\{ \hat{x} \in C^{u'}_{\sigma} \mid |\hat{x}(f_j) - \sum_{1}^{k} a_i\hat{x}_i(f_j)| < \epsilon \quad 1 \leq i \leq k \right\},$$

where the f_j's are defined as follows: There exist neighborhoods $U[x_i]$ of the x_i's such that

$$U[x_i] \cap U[x_j] = \phi \quad \text{for } i \neq j.$$

Let f_j be the member of C^u satisfying $f_j(x_j) = \frac{2\epsilon}{a_j}$ and $f_j(\widetilde{U[x_j]}) = 0$. Thus, V can be written as

$$V = \left\{ \hat{x} \in C^{u'}_{\sigma} \mid |\hat{x}(f_j) - 2\epsilon| < \epsilon \quad 1 \leq i \leq k \right\}.$$

Consider an $x \in X$ such that $\hat{x}$ is in V. If $x \notin U[x_i]$ for $1 \leq i \leq k$, then $\hat{x}(f_j) = 0$ for $1 \leq i \leq k$ and $x \notin V$. If $x \in U[x_i]$ for some index i, then $x \notin U[x_j]$ for all $j \neq i$ and again $|\hat{x}(f_j) - y| = 2\epsilon > \epsilon$ for all $j \neq i$ and thus $x \notin V$. Therefore it is concluded that $V \cap X = \phi$ Thus X is $\sigma(\text{spX}, C^u)$-closed in spX.

For the final part of the proof, note that the locally convex spaces $(spX, \sigma(spX, C^u))$ and $(C^u, \sigma(C^u, spX))$ are in duality [8,3]. Since the $\sigma(C^u, spX)$-topology is the same as the $\sigma(C^u,X)$-topology it follows that

$$C^{u'}_{\sigma} = (C^u, \sigma(C^u, X))' = (C^u, \sigma(C^u, spX))' = spX.$$

The importance of Proposition 3.4 in this development is to show that the operation of closing X in $C^{u'}_{\sigma}$ does not add any new points to X. The technique for obtaining the completion or partial completion of the uniform space $X_{\mathcal{U}}$ will be to put a suitable topology $\mathcal{J}$ on C^u and show that closing X in $C^{u'}_{\mathcal{J}}$ adjoins precisely the desired limit points to X.

The γ- and δ-topologies are defined on C^u in precisely the same manner as they were in Definition 2.1. Again, it is seen that the γ-topology is finer than the $\sigma(C^u,X)$-topology and as a result, X can be embedded in $C^{u'}_{\gamma}$. In all that follows, X will be identified with its image in $C^{u'}_{\gamma}$.

By Corollary 3.3, the $\sigma(C^{u'}_{\gamma}, C^u)$-topology induces the $\mathcal{U}$-topology on X (since $\mathcal{U}$-topology = $\sigma(X, C^u)$-topology). It is now necessary to construct a uniformity $\mathcal{V}$ for the $\sigma(C^{u'}_{\gamma}, C^u)$-closure X of X in $C^{u'}_{\gamma}$ which will induce the uniform structure $\mathcal{U}$ on X. This can be done by simply closing each set U in $\mathcal{U}$ in the product topology $\sigma(C^{u'}_{\gamma}, C^u) \times \sigma(C^{u'}_{\gamma}, C^u)$ on $C^{u'}_{\gamma} \times C^{u'}_{\gamma}$. Then V is a member of $\mathcal{V}$

if and only if $V = \overline{U}$ for some U in $\mathcal{U}$. The uniformity thus obtained will be denoted by $\overline{\mathcal{U}}$.

3.5 PROPOSITION: The $\overline{\mathcal{U}}$ uniformity, defined above, is a uniformity on the $\sigma(C^{u'}_{\gamma}, C^u)$-closure $\overline{X}$ of X in $C^{u'}_{\gamma}$ and induces the $\mathcal{U}$-uniformity on X.

PROOF: First, note that the $\sigma(C^{u'}_{\gamma}, C^u) X \sigma(C^{u'}_{\gamma}, C^u)$-topology induces the product uniformity topology ($\mathcal{U} X \mathcal{U}$-topology) on $X \times X$. That $\overline{\mathcal{U}}$ is a uniformity on $\overline{X}$ then follows upon examination of the base sets for the product topology. Since $\mathcal{U}$ has a base of sets which are closed in the product uniform topology it follows that $\overline{U} \cap X \times X = U$ for any U in that base for the uniformity $\mathcal{U}$. Thus, $\overline{\mathcal{U}}$ induces $\mathcal{U}$ on X [10, 6].

3.6 PROPOSITION: The $\overline{\mathcal{U}}$-topology on $\overline{X}$ is the relative topology generated on $\overline{X}$ by the $\sigma(C^{u'}_{\gamma}, C^u)$-topology on $C^{u'}_{\gamma}$.

PROOF: The space $C^u(\overline{X}_{\overline{\mathcal{U}}})$ consists of the unique extensions of the functions in $C^u(X)$ [10, 6]. Thus $C^u = C^u(\overline{X}_{\overline{\mathcal{U}}})$ and the proposition follows from Proposition 3.2.

The following theorem is the uniform space analogue of Proposition 2.4.

3.7 PROPOSITION: Every Cauchy filter in the collection $\mathcal{S}$ converges to a point in $C_\gamma^{u'}$ for the $\sigma(C_\gamma^{u'}, C^u)$-topology.

PROOF: By the definition of the γ-topology, every filter of neighborhoods in (C^u, γ) converges on every member of $\mathcal{S}$ (Def. 2.1). Let g be an arbitrary function in C^u and $\mathcal{N}_\gamma$ be the filter of neighborhoods of g for the γ-topology. Let $\mathcal{F}$ be any member of $\mathcal{S}$. Since $\mathcal{F}$ is Cauchy and every $g \varepsilon C^u$ is uniformly continuous it follows that $\mathcal{F}(g)$ is a base for a Cauchy filter of real numbers. Thus, there exists a function $h \varepsilon (C^u)^*$ such that $\mathcal{F}(g) \to h(g)$ for each $g \varepsilon C^u$. Also, $\mathcal{N}_\gamma(x) \to g(x)$ for each $x \epsilon X$. Applying Theorem 1.4 to the filters $\mathcal{F}$ and $\mathcal{N}_\gamma$ it is concluded that $\mathcal{F}$ converges to h on $\mathcal{N}_\gamma$. Since each x in a member of $\mathcal{F}$ is a continuous function on C^u for the γ-topology, Theorem 1.2 implies that h is continuous for the γ-topology. Thus it is concluded that h is in $C_\gamma^{u'}$.

It should be pointed out that, in the uniform space theory, there is no need for an analogue to Lemma 2.3 since the $\sigma(C_\gamma^{u'}, C^u)$-topology induces the $\mathcal{U}$-topology on X.

As in Section II, Q will denote the point set $X \cup \{$limits in $C_\gamma^{u'}$ of members of $\mathcal{S}\}$. Proposition 3.7 establishes that the set Q exists and thus, $X \subset Q \subset \bar{X} \subset C_\gamma^{u'}$, where all of these spaces are equipped with the topology induced on them by the $\sigma(C_\gamma^{u'}, C^u)$-topology. The uniformity $\bar{\mathcal{U}}$ on $\bar{X}$ induces a uniformity $\mathcal{U}'$ on Q ($\mathcal{U}' = \{\bar{U} \cap Q \times Q$ for all $\bar{U} \varepsilon \bar{\mathcal{U}}\}$) which in turn induces the uniformity $\mathcal{U}$ on X. Hence the following inclusions are valid: $X_{\mathcal{U}} \subset Q_{\mathcal{U}'} \subset \overline{X}_{\overline{\mathcal{U}}}$ and it can be concluded that the space $\widetilde{X}_{\widetilde{\mathcal{U}}}$ (defined in the introduction) exists and is contained in $C^{u'}_{\gamma}$. Since each function in C^u has a unique uniformly continuous extension to $\overline{X}_{\overline{\mathcal{U}}}$ it can be concluded that $C^u = C^u(X_{\mathcal{U}}) = C^u(Q_{\mathcal{U}'}) = C^u(\overline{X}_{\overline{\mathcal{U}}})$ [10,6]. The uniform space analogue to Proposition 2.9 can now be stated.

3.8 PROPOSITION: On C^u, the γ-topology = $\sigma(C^u, Q)$-topology = $\sigma(C^u, \text{sp } Q)$-topology.

PROOF: See proof of Proposition 2.9.

Letting $\hat{X}$ denote the point set $X \cup \{$limits in $C^{u'}_{\gamma}$ of all Cauchy filters in $X_{\mathcal{U}}\}$, the following corollary is also valid.

3.9 COROLLARY: The δ-topology on C^u is the same as the $\sigma(C^u, \hat{X})$-topology.

3.10 THEOREM: <u>The $\sigma(C^{u'}_{\gamma} - C^u)$-closure $\overline{X}$ of X in $C^{u'}_{\gamma}$, equipped with the $\overline{\mathcal{U}}$-uniformity, is the partial completion $\widetilde{X}_{\widetilde{\mathcal{U}}}$ of $X_{\mathcal{U}}$.</u>

PROOF: All the sets X, Q, and $\overline{X}$ are equipped with the relative topology generated on them by the $\sigma(C^{u'}_{\gamma}, C^u)$-topology on C^u. Each Cauchy filter in $\mathcal{S}$ converges to a point in Q by Proposition 3.7. By the remarks following Proposition 3.7, the uniform space $Q_{\mathcal{U}'}$ contains $X_{\mathcal{U}}$ as a dense subspace. Thus, all that remains to be proven is that

$\overline{X} = Q$. This is done in the same manner as in Proposition 3.4. That is, by Proposition 3.8 and linear space duality theory, the linear span of Q is $C^{u'}_{\gamma}$. Hence, by the same argument used in Proposition 3.4, Q is a closed subspace of $C^{u'}_{\gamma}$. Since $X \subset Q \subset \overline{X}$ it is concluded that $\overline{X} = Q$.

3.11 THEOREM: <u>The</u> $\sigma(C^{u'}_{\delta}, C^{u})$-<u>closure</u> $\overline{X}$ <u>of</u> X <u>in</u> $C^{u'}_{\delta}$ <u>equipped with the</u> $\overline{\mathcal{U}}$-<u>uniformity is the completion</u> $X_{\hat{\mathcal{U}}}$ <u>of</u> $X_{\mathcal{U}}$.

PROOF: All that needs to be proven is the completeness of $\overline{X}_{\overline{\mathcal{U}}}$. This follows from the fact that every Cauchy filter in $X_{\mathcal{U}}$ converges to a point in $\overline{X}_{\overline{\mathcal{U}}}$, and the density of $X_{\mathcal{U}}$ in $\overline{X}_{\overline{\mathcal{U}}}$.

An interesting application of Theorem 3.10 and Corollary 3.11 is the obtaining of another representation of the Stone-Cěch Compactification of a completely regular topological space.

3.12 STONE-CĔCH COMPACTIFICATION

Let X be a completely regular topological space, with topology $\mathcal{J}$, in which points are closed (a Tychonoff space) and $C^* = C^*(X)$ be the family of all bounded continuous real valued functions on X. The functions in C^* induce a Hausdorff uniform structure C^* on X in a natural manner. A base for the C^*-uniformity consists of sets of the form $U(f,\epsilon) = \{(x, y) \mid |f(x) - f(y)| < \epsilon, \quad f \in C^*\}$. The C^*-topology on X is simply the topology of pointwise convergence on C^* (i.e. the $\sigma(X, C^*)$-topology). Furthermore, the C^*-topology is the same as $\mathcal{J}$.

It is easy to see that the uniform Hausdorff space $X_{\mathcal{C}^*}$ is precompact (totally bounded) [6,6]. Thus the problem of compactifying X can be interpreted as one of completing $X_{\mathcal{C}^*}$. That is, the Stone-Čech Compactification βX of X is the completion $\hat{X}_{\mathcal{C}^*}$ of $X_{\mathcal{C}^*}$. It should be noted that the space C^* is the same as $C^u(X_{\mathcal{C}^*})$. Hence it is sufficient to use the space C^* in the role of $C^u(X_{\mathcal{C}^*})$ in Theorem 3.10. Thus Corollary 3.11 implies that $\hat{X}_{\mathcal{C}^*}$ is the $\sigma(C_\delta^{*'}, C^*)$-closure of X in the topological space $(C_\delta^{*'}, \sigma(C_\delta^{*'}, C^*))$

Corollary 3.9 combines with a theorem of Brace [1] to characterize the δ-topology on C^* as the topology of almost uniform convergence on X.

SECTION IV

TOPOLOGICAL GROUPS

In order to maintain uniformity of notation and to make the analogy between topological groups and linear spaces clearer, the character group of a topological group G will be denoted by G' and will be called the dual of G.

It is not always possible to find a group completion of an arbitrary Hausdorff topological group. Thus it will be convenient to begin with a case for which it is, namely an abelian topological group having enough characters to separate points. For such groups there is a relationship between G and G' which is exactly the same as that enjoyed by a locally convex linear space E and its dual E'. The precise relationship is the content of the following proposition.

4.1 PROPOSITION: Let G be an abelian topological group with enough characters to separate points. Then G and G' equipped with the $\sigma(G, G')$ and $\sigma(G', G)$-topologies respectively are in duality. That is, $(G, \sigma(G, G'))' = G'$ and $(G', \sigma(G', G)) = G$.

PROOF: From the definition of the $\sigma(G, G')$-topology, G' is the dual of $(G, \sigma(G, G'))$. For the converse, let G_d be the group G equipped with the discrete topology and denote by G_d' the dual of G_d. Equip G_d' with the $\sigma(G_d', G)$-topology (note that this is the topology of uniform convergence on compact subsets of G_d). Since the discrete topology is finer than the original topology on G, G' can be embedded in G_d', i.e. $G' \subset G_d'$. It will now be shown that G' is dense in G_d' for the $\sigma(G_d', G)$-topology. The topological group $(G_d', \sigma(G_d', G))$ is a locally compact group, since it is the dual of the locally compact group G_d equipped with the topology of compact convergence on G_d. It can be assumed that $(G', \sigma(G', (G))$ is a closed subgroup of $(G_d', \sigma(G_d', G))$; if not, close it. Thus, by a corollary to the Pontrjagin Duality Theorem [7, 6], G' is $\sigma(G_d', G)$-dense in G_d'. It can now be concluded that any character on $(G', \sigma(G', G))$ can be uniquely extended to a character on $(G_d', \sigma(G_d', G))$ [10, 6]. Thus, the dual of $(G', \sigma(G', G))$ can be identified with the dual of $(G_d', \sigma(G_d', G))$. Applying the Pontrjagin Duality Theorem to the group G_d it is seen that $(G_d', \sigma(G_d', G))' = G$. Thus $(G', \sigma(G', G))' = G$, which was to be proven.

In what follows, G will denote a topological group, with topology $\mathcal{J}$ and having enough characters to separate points. The γ- and δ-topologies on G' will be defined exactly as in Definition 2.1. Both γ and δ are Hausdorff group topologies for G' [3, 2] and it is clear that γ and δ are finer than $\sigma(G', G)$. Thus, it can be concluded that $G_\gamma' = (G', \gamma)$ is a topological group and $G \subset G_\gamma''$. Furthermore, $(G, \sigma(G, G'))$ is a subgroup of $(G_\gamma'', \sigma(G_\gamma'', G'))$.

As in previous cases, the next objective is to topologize G_γ'' with a topology $\overline{\mathcal{J}}$ such that $\overline{\mathcal{J}}$ induces $\mathcal{J}$ on G. The next two propositions establish that such a topology can be constructed.

4.2 PROPOSITION: G is $\sigma(G''_\gamma, G')$-dense in G''_γ.

PROOF: Since $G \subset G''_\gamma \subset (G', d)'$ and G is dense in $(G', d)'$ for the topology of pointwise convergence on G', the $\sigma(G''_\gamma, G')$-density of G in G''_γ follows.

Denote by $\mathcal{N}$ the filter of neighborhoods of the identity e for the $\mathcal{J}$-topology and by $\overline{\mathcal{N}}$ the collection of $\sigma(G''_\gamma, G')$-closures in G''_γ of the members of $\mathcal{N}$.

4.3 PROPOSITION: The collection $\overline{\mathcal{N}}$ is a basis of neighborhoods of e for a Hausdorff group topology $\overline{\mathcal{J}}$ in G''_γ which induces $\mathcal{J}$ on G.

PROOF: From the properties of the sets in $\mathcal{N}$, it is clear that $\overline{\mathcal{N}}$ satisfies the neighborhood axioms for a topology. Since $\mathcal{J}$ is Hausdorff, it follows that $\overline{\mathcal{J}}$ is Hausdorff. It remains to show that $\overline{\mathcal{J}}$ is a group topology for G''_γ. (a). Let $\overline{W}$ be a member of $\overline{\mathcal{N}}$. There is a V in $\mathcal{N}$ such that $V \cdot V \subset W$ and hence $\overline{V \cdot V} \subset \overline{W}$. Thus in order to show that there exists a $\overline{V}$ in $\overline{\mathcal{N}}$ such that $\overline{V} \cdot \overline{V} \subset \overline{W}$ it is sufficient to show that $\overline{V} \cdot \overline{V} \subset \overline{V \cdot V}$. Let $xy \in \overline{V} \cdot \overline{V}$. There are filters $\mathcal{F}$ and $\mathcal{G}$ in V such that $\mathcal{F}(h) \to x(h)$ and $\mathcal{G}(h) \to y(h)$ for all h in G'. Now, consider the filter $\mathcal{F} \cdot \mathcal{G}$ in $V \cdot V$. Since $\mathcal{F} \cdot \mathcal{G}(h) = \mathcal{F}(h)\mathcal{G}(h) \to x(h)y(h)$ $= xy(h), \forall\, h \in G'$, it is concluded that $xy \in \overline{V \cdot V}$ and hence that

$\bar{V} \cdot \bar{V} \subset \bar{W}$. (b). Let $\bar{W}$ be a member of $\bar{\mathcal{N}}$. It must be shown that there exists a $\bar{U}$ in $\bar{\mathcal{N}}$ such that $\bar{U}^{-1} \subset \bar{W}$. There is a U in $\mathcal{N}$ such that $\overline{U^{-1}} \subset \bar{W}$. Thus it is sufficient to show that $\bar{U}^{-1} \subset \overline{U^{-1}}$. Let $x \in \bar{U}^{-1}$. Then $x^{-1} \in \bar{U}$ and there is a filter $\mathcal{F}$ in U such that $\mathcal{F}(h) \to x^{-1}(h)$ for all $h \in G'$. But this implies that $\mathcal{F}^{-1}(h) \to x(h)$ and it is concluded that $x \in \overline{U^{-1}}$.

It is noted that the $\bar{\mathcal{J}}$-topology has a base consisting of sets which are $\sigma(G''_\gamma, G')$-closed.

4.4 LEMMA [4, II, 3]. Let $\mathcal{J}$ and $\mathcal{J}'$ be two uniformities on a set X such that $\mathcal{J}'$ is coarser than $\mathcal{J}$ and $\mathcal{J}$ has a basis of sets which are closed in the $\mathcal{J}' \times \mathcal{J}'$-product topology. Then, if $\mathcal{F}$ is a filter on X which is $\mathcal{J}$-Cauchy and $\mathcal{J}'$-convergent to a point x, $\mathcal{F}$ is $\mathcal{J}$-convergent to x.

4.5 PROPOSITION: Every member of the collection $\mathcal{S}$ converges to a point in G''_γ for the $\bar{\mathcal{J}}$-topology.

PROOF: By the same argument used in Proposition 2.4 every filter in $\mathcal{S}$ converges to a point in G''_γ for the $\sigma(G''_\gamma, G')$-topology. Applying the above lemma to the topologies $\bar{\mathcal{J}}$ and $\sigma(G''_\gamma, G')$, it is concluded that these filters converge in the $\bar{\mathcal{J}}$-topology.

Now, denote by $\mathcal{Q}$ and P, the point sets $G \cup \{$limits in G''_γ of filters in $\mathcal{S}\}$ and $G \cup \{$ limits in G''_δ of all Cauchy filters in $G\}$. The smallest subgroups of G''_γ containing $\mathcal{Q}$ and P will be denoted by $[\mathcal{Q}]$ and $[P]$. Propositions 4.3 and 4.5 have established the fact that the partial completion $\widetilde{G}$ and completion $\widehat{G}$ of G exist and are in fact the groups $[\mathcal{Q}]$ and $[P]$ equipped with the relative topology generated on them by the $\bar{\mathcal{J}}$-topology.

4.6 PROPOSITION: γ-topology = $\sigma(G', \mathcal{Q})$-topology = $\sigma(G', [\mathcal{Q}])$-topology.

PROOF: See Proposition 2.9.

4.7 PROPOSITION $[\mathcal{Q}]' = G'$.

PROOF: It has been established that $G \subset [\mathcal{Q}] \subset \widehat{G}$ and since $G' = \widehat{G}'$ the proposition follows.

4.8 THEOREM: <u>The group G''_γ, equipped with the $\bar{\mathcal{J}}$-topology, is the smallest Hausdorff topological group F containing G as a dense subgroup and having the property that every Cauchy filter in $\mathcal{S}$ converges to a point in F.</u>

PROOF: The group G can be embedded in $(G''_\gamma, \bar{\mathcal{J}})$ by the remarks preceding Proposition 4.2 and by Proposition 4.3. By Proposition 4.6, every filter in $\mathcal{S}$ converges to a point in G_γ'' for the $\bar{\mathcal{J}}$-topology. Applying Proposition 4.1 to the group $[Q]$ in conjunction with Propositions 4.6 and 4.7, it is seen that

$$[Q] = ([Q]', \sigma([Q]', [Q]))' = (G', \sigma(G', [Q])) = G_\gamma'' .$$

Thus, by the definition of $[Q]$, G_γ'' is the smallest group containing G and the desired limit points, and the theorem follows.

4.9 COROLLARY: The Group G_δ'' equipped with the $\bar{\mathcal{J}}$-topology is the completion $\hat{G}$ of G.

The remainder of this section is devoted to the problem of "completing" an arbitrary Hausdorff topological group G.

Now, let $(G, \cdot, \mathcal{J})$ be an arbitrary Hausdorff topological group and consider the uniform space $G_{\mathcal{R}}$, where the $\mathcal{R}$-uniformity has a base consisting of subsets of GxG of the form:

$$R(V) = \left\{(x, y) \mid xy^{-1} \varepsilon V, \quad \text{for } V \text{ a neighborhood of the identity } e \text{ in } G\right\}.$$

The object of what follows is not necessarily to construct a group completion, but rather to construct a topological group $(\tilde{G}, \cdot, \tilde{\mathcal{J}})$ with the property that an arbitrary collection of $\mathcal{R}$-Cauchy filters converge to points in $\tilde{G}$ and G is a dense subgroup of $\tilde{G}$. When a group completion exists (e.g., when G is abelian) it can be obtained by the same methods used here. The space of all uniformly continuous real valued functions defined on $G_{\mathcal{R}}$ will be denoted by C^r. The γ- and δ-topologies will be defined in the same manner as in Section II.

4.10 PROPOSITION: The $\mathcal{R}$-topology and the $\sigma(G, C^r)$-topology are the same as the group topology $\mathcal{J}$.

PROOF: That the $\mathcal{R}$-topology is the same as the $\sigma(G, C^r)$-topology, see [10, 6]. The $\mathcal{J}$-topology and the $\sigma(G, C^r)$-topology are the same by Proposition 3.2.

By the results of Section III it is seen that the completion [partial completion] of the uniform space $G_{\mathcal{R}}$ can be obtained by closing G in the space $C_\delta^{r'}$ $[C_\delta^{r'}]$ and equipping $\bar{G}$ with the $\bar{\mathcal{U}}$-uniformity. The only question remaining is whether or not the group operation can be extended to $\bar{G}$, i.e. is $\bar{G}$ a topological group? In order that $\bar{G}$ be a topological group, the following maps must be extended:

1) $(x, y) \rightarrow xy$

2) $x \rightarrow x^{-1}$

Since $(x, y) \rightarrow xy$ is not uniformly continuous in general the usual extension theorem will not work. However the following proposition shows that this extension can always be made.

4.11 PROPOSITION [5]: Let $\mathcal{F}$ and $\mathcal{G}$ be two Cauchy filters on $G_{\mathcal{R}}$. Then the image, under the mapping $(x, y) \to xy$, of $\mathcal{F} \times \mathcal{G}$ is a base for a Cauchy filter on $G_{\mathcal{R}}$.

To be able to extend the map $x \to x^{-1}$ to $\overline{G}$ it is necessary and sufficient that the image of a Cauchy filter on $G_{\mathcal{R}}$ by this map be a Cauchy filter base on $G_{\mathcal{R}}$. This is not always the case. There are examples of topological groups for which this condition is not satisfied [5]. If this condition is assumed, then it is easy to see that both the partial completion and the completion of $G_{\mathcal{R}}$ become topological groups.

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