ABSTRACT

Title of dissertation: SMALL MASS ASYMPTOTICS FOR PROBLEMS IN STOCHASTIC DIFFERENTIAL EQUATIONS

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Small mass asymptotics of the motion of a particle moving in a force field (Smoluchowski-Kramers approximation) was first studied in Smoluchowski [13] and Kramers [9]. Freidlin summarized the results and considered various asymptotic problems related to it in 2004 [4]. Recently, there have been papers from various authors on small mass asymptotics [1,5,7] after Freidlin's work. Cerrai and Freidlin showed in 2011 [1] that a type of the Smoluchowski-Kramers approximation works in the case of the motion of a charged particle moving in a constant magnetic field and Freidlin and Hu showed in 2011 [5] that a type of the Smoluchowski-Kramers approximation works in the case of the motion of a particle moving in a space with friction coefficient dependent upon position. We summarize these results in Chapter 1.

We consider generalizations of the works by Freidlin [4], Cerrai and Freidlin [1], Freidlin and Hu [5], and Gitterman [6]. We study the problem of the motion of a charged particle moving in a variable magnetic field dependent upon position

[10] in Chapter 2, the Smoluchowkski-Kramers approximation in the case of linear differential operators in Chapter 3, and the small mass asymptotics in the case of random mass in Chapter 4.

SMALL MASS ASYMPTOTICS FOR PROBLEMS IN STOCHASTIC DIFFERENTIAL EQUATIONS

by

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It is impossible to remember all, and I apologize to those I've inadvertently left out.

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List of Abbreviations and Symbols

SDE	Stochastic Differential Equation
$\dot{f}(t)$	the derivative of f in t
$\ddot{f}(t)$	the second derivative of f in t
$M_d(\mathbb{R})$	the space of $d \times d$ matrices
$\mathbb{R}^d$	d dimensional Euclidean space of real numbers
$\mathbb{R}^+$	the set of positive real numbers
$\mathbb{E}(X)$	the expectation of X
$\mathrm{Var}(X)$	the variance of X
w_t	d dimensional Wiener process for some $d \in \mathbb{N}$
w_t^i	the ith component of the Wiener process w_t
$\int_0^T f(t) \circ dw_t$	a stochastic integral in the Stratonovich sense
$\lvert \cdot \rvert$	the standard Euclidean norm in $\mathbb{R}^d$
$\lvert \cdot \rvert_\infty$	the supremum norm in spaces of functions
C	a constant
C_i	a constant indexed by i
$C(\cdot)$	a constant depending on the arguments
σ^*	the matrix transpose of the matrix σ

Chapter 1: Introduction

1.1 Smoluchowski-Kramers Approximation

1.1.1 Introduction

Suppose a particle of mass $\mu > 0$ is moving in $\mathbb{R}^d$ in a force field with random noise and friction proportional to the velocity of the particle in the system. We can model this system using the Newton's second law of motion:

$$\begin{cases} \mu\, \ddot{q}_t^\mu = b(q_t^\mu) - \alpha \dot{q}_t^\mu + \sigma(q_t^\mu)\, \dot{w}_t \\ q_0^\mu = q_0 \in \mathbb{R}^d, \quad \dot{q}_0^\mu = p_0 \in \mathbb{R}^d, \end{cases} \tag{1.1.1}$$

where q_t^μ is the position of the particle at time t, $b : \mathbb{R}^d \mapsto \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \mapsto M_d(\mathbb{R})$ are functions such that $b(q_t^\mu)$ and $\sigma(q_t^\mu)\, \dot{w}_t$ are the deterministic and random part of the force respectively, w_t is a d-dimensional Wiener process, and $\alpha > 0$ is a positive constant such that $\alpha \dot{q}_t^\mu$ is the friction with the surrounding medium. It is clear that, in general, the process q_t^μ is not Markovian. To consider the Markov process and to use the machinery developed for Markov processes, one should consider process $(q_t^\mu, \dot{q}_t^\mu)$ in $\mathbb{R}^{2d}$. The generator of $(q_t, \dot{q}_t)$ is degenerate, which makes the study of this process more difficult.

Now, suppose that the mass μ is small. It is tempting to replace the solution

q_t^μ of equation (1.1.1) with the solution q_t of the following equation

$$\begin{cases} 0 = b(q_t) - \alpha \dot{q}_t + \sigma(q_t)\,\dot{w}_t \\ q_0 \in \mathbb{R}^d, \end{cases} \tag{1.1.2}$$

which is equation (1.1.1) with $\mu = 0$. The process q_t in $\mathbb{R}^d$ is Markovian. Moreover, if the diffusion matrix $a(q) = \sigma(q)\sigma^*(q)$ is non-degenerate, the process q_t has a non-degenerate generator. If we can justify that q_t^μ is close to q_t in some sense, then we can replace q_t^μ with q_t, which reduces the computation efforts significantly. The Smoluchowski-Kramers approximation tells us that q_t^μ can be approximated by q_t, because q_t^μ converges to q_t in probability in the space of continuous functions $C([0,T];\mathbb{R}^d)$ with usual maximum norm and with the measure given by the Wiener process. We will state this theorem in the next subsection.

1.1.2 Main Result

For the proof of the Smoluchowski-Kramers approximation, we require some conditions on $b(q)$ and $\sigma(q)$. It is enough that $b : \mathbb{R}^d \mapsto \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \mapsto M_d(\mathbb{R})$ are Lipschitz continuous.

Theorem 1.1.1. *Assume thatb* $: \mathbb{R}^d \mapsto \mathbb{R}^d$ *and* $\sigma : \mathbb{R}^d \mapsto M_d(\mathbb{R})$ *are Lipschitz continuous. Then,*

$$\lim_{\mu \downarrow 0} \mathbb{E} \max_{0 \le t \le T} |q_t^\mu - q_t|^k = 0,$$

for $k \ge 1$*, so* q_t^μ *converges to* q_t *in probability in* $C([0,T];\mathbb{R}^d)$.

We skip the proof of this theorem, since we prove a more general statement in Chapter 3. In the original paper by Freidlin [4], he proved the same result in the case that b and σ were bounded. The case of unbounded b and σ was considered in Cerrai and Freidlin [1].

1.1.3 Asymptotic Problems

Freidlin considered various asymptotic problems related to the Smoluchowski-Kramers approximation in his paper [4]. In this section, we summarize the ideas of approximation of noise and homogenization. In Section 1.1.3.1, we approximate the Wiener process w_t with a δ-correlated smooth process V_t^δ in equation (1.1.1) and describe the two-parameter asymptotic problem as μ and δ tend to 0. In Section 1.1.3.2, we consider a homogenization problem related to the Smoluchowski-Kramers approximation.

1.1.3.1 Approximation of Noise

Let

$$V_t^\delta := \frac{1}{\sqrt{\delta}} \int_0^t \xi_{\frac{s}{\delta}} ds,$$

where ξ_s is a mean zero stationary Gaussian process with a fast enough decreasing smooth correlation function $R(|t|)$ such that

$$\lim_{\delta \downarrow 0} \max_{0 \leq t \leq T} |w_t - V_t^\delta| = 0.$$

It is known that if we replace w_t with V_t^δ in a stochastic differential equation, then as $\delta \downarrow 0$, the solution of the SDE will converge to the solution of the SDE with the stochastic term understood in the Stratonovich sense [14]. We may replace w_t

in equation (1.1.1) with V_t^δ and consider two-parameter asymptotic problem as μ and δ go to 0.

Note that equation (1.1.1) can be rewritten as a system of first order equations:

$$\begin{cases} \dot{q}_t^\mu = p_t^\mu \\ \dot{p}_t^\mu = \frac{1}{\mu} b(q_t^\mu) - \frac{\alpha}{\mu} p_t^\mu + \frac{1}{\mu} \sigma(q_t^\mu)\, \dot{w}_t \\ q_0^\mu = q_0 \in \mathbb{R}^d, \quad p_0^\mu = p_0 \in \mathbb{R}^d. \end{cases}$$

From this expression, we can easily check that q_t^μ is continuously differentiable and so

$$\int_0^T \sigma(q_t^\mu)\, dw_t = \int_0^T \sigma(q_t^\mu) \circ dw_t,$$

where the second integral is understood in the Stratonovich sense; however, note that the solution q_t of equation (1.1.2) is not continuously differentiable, and in general,

$$\int_0^T \sigma(q_t)\, dw_t \neq \int_0^T \sigma(q_t) \circ dw_t.$$

From this observation, we can expect that if $\mu \downarrow 0$ first and $\delta \downarrow 0$, then the solution $q_t^{\mu,\delta}$ of equation (1.1.1) with V_t^δ in place of w_t will converge to the solution $\hat{q}_t$ of equation (1.1.2) with the stochastic integral in the Stratonovich sense. On the other hand, if $\delta \downarrow 0$ first and $\mu \downarrow 0$, then $q_t^{\mu,\delta}$ will converge to the solution q_t of equation (1.1.2) with Itô's integral. We state a sharper result in the following theorem, the proof of which can be found in [4].

Theorem 1.1.2. *Let b and σ be Lipschitz continuous and bounded. The solution $q_t^{\mu,\delta}$ of equation* (1.1.1) *with V_t^δ in place of w_t converges in probability in $C([0,T];\mathbb{R})$ to the solution q_t of equation* (1.1.2) *as $\mu \downarrow 0$ and $\delta \downarrow 0$ so that $\delta < f(\mu)$ for some* *positive function f. If $\mu \downarrow 0$ and $\delta \downarrow 0$ so that $\mu e^{\frac{1}{\delta}} \downarrow 0$, then $q_t^{\mu,\delta}$ converges to the solution $\hat{q}_t$ of equation* (1.1.2) *with the Stratonovich's stochastic term.*

1.1.3.2 Homogenization

Consider the following variant of equation (1.1.1):

$$\begin{cases} \dot{q}_t^{\mu} = p_t^{\mu} \\ \dot{p}_t^{\mu,\epsilon} = \frac{1}{\mu} b\left(\frac{q_t^{\mu,\epsilon}}{\epsilon}\right) - \frac{\alpha}{\mu} p_t^{\mu,\epsilon} + \frac{1}{\mu}\sigma\left(\frac{q_t^{\mu,\epsilon}}{\epsilon}\right) \dot{w}_t \\ q_0^{\mu,\epsilon} = q_0 \in \mathbb{R}^d, \quad p_0^{\mu,\epsilon} = p_0 \in \mathbb{R}^d, \end{cases}$$

where b and σ are 1-periodic and $a(q) := \sigma(q)\sigma^*(q)$ is uniformly nondegenerate. This equation models the motion of a particle moving in a periodic force field with period ϵ.

We can think of weak limits of the solution $q_t^{\mu,\epsilon}$ in the space $C([0,T];\mathbb{R}^d)$ as $\mu \downarrow 0$ and $\epsilon \downarrow 0$. It turns out that depending on the relative speed of convergence of μ and ϵ to 0, there exist different weak limits. If $\epsilon \downarrow 0$ first and $\mu \downarrow 0$, then we can expect that due to the continuously differentiability of $q_t^{\mu,\epsilon}$, homogenization with respect to Lebesgue measure holds first and then the Smoluchowksi-Kramers approximation holds. If $\mu \downarrow 0$ first and $\epsilon \downarrow 0$, then we can expect that the Smoluchowski-Kramers approximation holds first and then homogenization result for Stochastic differential equations holds [2, 11, 12]. We state this result in the following theorem, the proof of which can be found in [4].

Theorem 1.1.3. *Assume that the functions $b(q)$ and $\sigma(q)$ are 1-periodic in each variable, twice continuously differentiable, and the matrix $a(q) = \sigma(q)\sigma^*(q)$ is non-*

degenerate. Let $\mathbb{T}^d$ be the d-dimensional unit torus.

1. *Suppose $\mu \downarrow 0$ and $\epsilon \downarrow 0$ so that for any $C > 0$,*

$$\mu \exp\left(\frac{C}{\epsilon^2}\right) \downarrow 0.$$

Then, for any $T > 0$, process $q_t^{\mu,\epsilon}$ converges weakly in $C([0,T];\mathbb{R}^d)$ to the Gaussian Markov process

$$\bar{q}_t = q + \bar{b}t + \bar{\sigma}w_t.$$

Here,

$$\bar{b} = \int_{\mathbb{T}^d} b(q)m(q)dq,$$

$$\bar{\sigma} = \bar{a}^{\frac{1}{2}},$$

and

$$\bar{a} = \int_{\mathbb{T}^d} a(q)m(q)dq,$$

where $m(q)$ is the unique solution of the problem

$$\sum_{i,j=1}^{d} \frac{\partial^2}{\partial q^i \partial q^j}\left(a^{ij}(q)m(q)\right) = 0, \qquad \int_{\mathbb{T}^d} m(q)dq = 1.$$

2. *Suppose that $\mu \downarrow 0$ and $\epsilon \downarrow 0$ so that*

$$\epsilon\left(\frac{\ln \mu}{\mu}\right)^2 \downarrow 0.$$

Then the process $q_t^{\mu,\epsilon}$ converges weakly to the Gaussian Markov process

$$\hat{q}_t = q + \hat{b}t + \hat{\sigma}w_t,$$

where

$$\hat{b} = \int_{\mathbb{T}^d} b(q)dq,$$

$$\hat{\sigma} = \hat{a}^{\frac{1}{2}},$$

and

$$\hat{a} = \int_{\mathbb{T}^d} a(q)dq.$$

1.2 Small Mass Asymptotics in the Case of a Constant Magnetic Field

1.2.1 Introduction

Consider a charged particle of mass $\mu > 0$ moving on a plane. Let the position of this particle at time t be $q_t^\mu \in \mathbb{R}^2$. We may express the force field with random noise on the plane as

$$b(q_t^\mu) + \sigma(q_t^\mu)\dot{w}_t,$$

where $b : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is a vector-valued function, $\sigma : \mathbb{R}^2 \mapsto M_2(\mathbb{R})$ is a matrix-valued function, and $w_t \in \mathbb{R}^2$ is a two dimensional Wiener process.

Now, suppose that the motion of the particle is subject to a constant magnetic field perpendicular to the plane. The force on the particle due to this magnetic field can be expressed as

$$A\,\dot{q}_t^\mu = \alpha A_0\,\dot{q}_t^\mu, \tag{1.2.1}$$

where $\alpha > 0$ is a constant and

$$A_0 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The motion of this particle is governed by the Newton law, so that

$$\begin{cases} \mu\, \ddot{q}_t^\mu = b(q_t^\mu) + A\dot{q}_t^\mu + \sigma(q_t^\mu)\, \dot{w}_t \\ q_0^\mu = q_0 \in \mathbb{R}^2, \quad \dot{q}_0^\mu = p_0 \in \mathbb{R}^2. \end{cases} \tag{1.2.2}$$

Let q_t be the solution of the following first order SDE with $\mu = 0$ from equation (1.2.2):

$$\begin{cases} \dot{q}_t = -A^{-1}\, b(q_t) - A^{-1}\, \sigma(q_t)\dot{w}_t \\ q_0 \in \mathbb{R}^2. \end{cases} \tag{1.2.3}$$

On the first glance one can expect the convergence of q_t^μ to q_t in probability as $\mu \downarrow 0$; however, this is not true in general. From the fact that the real parts of the eigenvalues of A are zero, we will have the stochastic integral terms as

$$\int_0^t \sin\left(\frac{s}{\mu}\right) dw_s^1$$

from (1.2.2), where w_s^1 is the first component of the two dimensional Wiener process w_t. We require this to go to 0 in probability as $\mu \downarrow 0$ for the convergence, but this is not the case due to the property of stochastic integrals:

$$\mathrm{Var}\left(\int_0^t \sin\left(\frac{s}{\mu}\right) dw_s^1\right) = \mathbb{E}\left|\int_0^t \sin\left(\frac{s}{\mu}\right) dw_s^1\right|^2 = \int_0^t \sin^2\left(\frac{s}{\mu}\right) ds = \frac{t}{2}.$$

Nonetheless, we may regularize the problem in different ways and check if the convergence similar to the Smoluchowski-Kramers approximation holds.

Firstly, it is physically reasonable to introduce small friction proportional to the velocity. We may write $A_\epsilon = A - \epsilon\, I$ and approximate q_t^μ and q_t with $q_t^{\mu,\epsilon}$ and q_t^ϵ, the solutions of (1.2.2) and (1.2.3) with A_ϵ in place of A. This small friction term makes the real parts of the eigenvalues of A_ϵ negative and gives us the exponential decay of the terms

$$\frac{1}{\mu}\exp\left(\frac{1}{\mu}A_\epsilon t\right)=\frac{1}{\mu}\exp\left(-\frac{\epsilon}{\mu}t\right)\begin{pmatrix}\cos\left(\frac{t}{\mu}\right) & -\sin\left(\frac{t}{\mu}\right)\\ \sin\left(\frac{t}{\mu}\right) & \cos\left(\frac{t}{\mu}\right)\end{pmatrix}$$

as $\mu\downarrow 0$.

As another regularization method, we may approximate the Wiener process w_t with a δ-correlated smooth process w_t^δ such that w_t^δ converges to w_t in probability. From the fact that w_t^δ is a smooth function, we now have

$$\lim_{\mu\downarrow 0}\int_0^t \sin\left(\frac{s}{\mu}\right)dw_s^\delta=\lim_{\mu\downarrow 0}\int_0^t \sin\left(\frac{s}{\mu}\right)\dot{w}_s^\delta ds=0$$

almost surely thanks to the Riemann-Lebesgue lemma.

We state the results of these approximations in the next subsection.

1.2.2 Main Result

First, we state the results of the first approximation: we include a friction term with the friction coefficient ϵ in the system.

Theorem 1.2.1. *Assume that $b:\mathbb{R}^2\mapsto\mathbb{R}^2$ and $\sigma:\mathbb{R}^2\mapsto M_2(\mathbb{R})$ are Lipschitz continuous. Then, for any $T>0$ and $k>2$,*

$$\mathbb{E}\max_{0\le t\le T}|q_t^{\mu,\epsilon}-q_t^\epsilon|^k\le C(k,T,p_0,q_0)\left(\frac{\mu}{\epsilon}\wedge 1\right)^{\frac{k}{2}-1},$$

where $q_t^{\mu,\epsilon}$ and q_t^ϵ are the solutions of equations (1.2.2) and (1.2.3) respectively with A_ϵ in place of A.

In particular, for any $\epsilon_0>0$ and $k\ge 1$,

$$\lim_{\mu\downarrow 0}\sup_{\epsilon\ge\epsilon_0}|q_t^{\mu,\epsilon}-q_t^\epsilon|^k=0.$$

Theorem 1.2.2. *Assume that $b : \mathbb{R}^2 \mapsto \mathbb{R}^2$ and $\sigma : \mathbb{R}^2 \mapsto M_2(\mathbb{R})$ are Lipschitz continuous. Then,*

$$\lim_{\epsilon\downarrow 0} \mathbb{E} \max_{0\le t\le T} |q_t^\epsilon - q_t|^k = 0$$

for $k \ge 1$.

The proofs of the above theorems can be found in Cerrai and Freidlin [1]. Thanks to the above two theorems, we have the small mass asymptotics result:

Corollary 1.2.3. *Assume that $b : \mathbb{R}^2 \mapsto \mathbb{R}^2$ and $\sigma : \mathbb{R}^2 \mapsto M_2(\mathbb{R})$ are Lipschitz continuous. Then,*

$$\lim_{\mu\downarrow 0, \epsilon\downarrow 0, \frac{\mu}{\epsilon}\downarrow 0} \mathbb{E} \max_{0\le t\le T} |q_t^{\mu,\epsilon} - q_t|^k = 0$$

for $k \ge 1$.

Next, we consider the second approximation: we replace the Wiener process w_t with a δ-correlated smooth process w_t^δ. There can be many different ways of choosing such w_t^δ. We follow the method introduced in Ikeda and Watanabe [8, Example 7.3 Chapter VI]: define w_t^δ as a mollification of the Wiener process w_t.

Definition 1.2.4.

$$w_t^\delta := \frac{1}{\delta} \int_0^\infty w(s)\rho\left(\frac{s-t}{\delta}\right) ds,$$

where $\rho : \mathbb{R} \mapsto \mathbb{R}^+ \cup \{0\}$ is smooth, has the support in $[0,1]$, and satisfies

$$\int_0^1 \rho(s)ds = 1.$$

w_t^δ is a smooth approximation of w_t satisfying

$$\lim_{\delta\downarrow 0} \mathbb{E} \max_{0\le t\le T} |w_t^\delta - w_t|^2 = 0.$$

To give enough regularity for the problem, we assume the following conditions on $b(q)$ and $\sigma(q)$.

Hypothesis 1. *$b : \mathbb{R}^2 \to \mathbb{R}^2$ and $\sigma : \mathbb{R}^2 \to M_2(\mathbb{R})$ are differentiable and bounded with their derivatives.*

Under Hypothesis 1, Ikeda and Watanabe [8, Theorem 7.2 Chapter VI] tells us that we have the relation

$$\lim_{\delta \downarrow 0} \mathbb{E} \max_{0 \le t \le T} |q_t^{\mu,\delta} - \hat{q}_t^{\mu}|^2 = 0,$$

where $\hat{q}_t^{\mu}$ is the solution of equation (1.2.2) with the stochastic integral in the Stratonovich sense. From the continuous differentiability of q_t^{μ} in t, the stochastic integral in the Itô sense and the Stratonovich sense coincide in equation (1.2.2):

$$\int_0^T \sigma(q_t^{\mu})\, dw_t = \int_0^T \sigma(q_t^{\mu}) \circ dw_t,$$

where the integral on the right is understood in the Stratonovich sense. So, we can conclude $q_t^{\mu} = q_t$ and

$$\lim_{\delta \downarrow 0} \mathbb{E} \max_{0 \le t \le T} |q_t^{\mu,\delta} - q_t^{\mu}|^2 = 0,$$

The following theorem can be proved using the methods in Cerrai and Freidlin [1] and Lee [10]. We state the theorem without proof since we will see the proof of a more general case in Chapter 2.

Theorem 1.2.5. *Under Hypothesis 1, there exists a constant $C > 0$ depending on T, q_0, p_0, $|b|_\infty$, $|Db|_\infty$, $|\sigma|_\infty$, and $|D\sigma|_\infty$ such that for any $0 < \mu \le 1$, $0 < \delta \le 1$, and $k \ge 1$,*

$$E \max_{0 \le t \le T} |q_t^{\mu,\delta} - q_t^{\delta}|^k \le \exp\left(\frac{C}{\delta^2}\right) \mu^k,$$

where $q_t^{\mu,\delta}$ *and* q_t^δ *are the solutions of equations* (1.2.2) *and* (1.2.3) *respectively with* w_t^δ *in place of* w_t.

In particular, for any fixed $0 < \delta \leq 1$,

$$\lim_{\mu \downarrow 0} \mathbb{E} \max_{0 \leq t \leq T} |q_t^{\mu,\delta} - q_t^\delta|^k = 0.$$

The limit of q_t^δ as $\delta \downarrow 0$ can be found as an application of Ikeda and Watanabe [8, Theorem 7.2 Chapter VI]. We state the result without proof:

Theorem 1.2.6. *Under Hypothesis 1,*

$$\lim_{\delta \downarrow 0} \mathbb{E} \max_{0 \leq t \leq T} |q_t^\delta - \hat{q}_t|^2 = 0,$$

where $\hat{q}_t$ *is the solution of the first order stochastic differential equation*

$$\begin{cases} \dot{\hat{q}}_t = -A^{-1} b(\hat{q}_t) - (A^{-1}\sigma(\hat{q}_t)) \circ \dot{w}_t \\ \hat{q}_0 = q_0 \in \mathbb{R}^2. \end{cases}$$

We state the combination of the above two theorems in the following corollary.

Corollary 1.2.7. *Under Hypothesis 1,* $q_t^{\mu,\delta}$ *converges to* $\hat{q}_t$ *in probability in* $C([0,T];\mathbb{R}^2)$ *as* $\mu \downarrow 0$ *and* $\delta \downarrow 0$ *so that* $\mu e^{\frac{C}{\delta^2}} \downarrow 0$ *for each constant* $C > 0$.

1.3 Smoluchowski-Kramers Approximation in the Case of Variable Friction

1.3.1 Introduction

In this section, we generalize the results of the Smoluchowski-Kramers approximation in Section 1.2 by allowing the friction coefficient to be dependent upon the position of the particle. We replace (1.1.1) and (1.1.2) with the following equations by replacing the constant α with a function $\alpha : \mathbb{R}^d \mapsto \mathbb{R}^+$:

$$\begin{cases} \mu\, \ddot{q}_t^\mu = b(q_t^\mu) - \alpha(q_t^\mu)\dot{q}_t^\mu + \sigma(q_t^\mu)\, \dot{w}_t \\ q_0^\mu = q_0 \in \mathbb{R}^d, \quad \dot{q}_0^\mu = p_0 \in \mathbb{R}^d \end{cases} \tag{1.3.1}$$

and

$$\begin{cases} 0 = b(q_t) - \alpha(q_t)\dot{q}_t + \sigma(q_t)\, \dot{w}_t \\ q_0 \in \mathbb{R}^d. \end{cases} \tag{1.3.2}$$

Freidlin and Hu [5] showed that in this case, the Smoluchowski-Kramers approximation does not hold: q_t^μ does not converge to q_t in probability in $C([0,T];\mathbb{R}^d)$ as $\mu \downarrow 0$. A way to overcome this difficulty was introduced in [5] (See also Cerrai and Freidlin [1]). We may replace w_t with w_t^δ from equations (1.3.1) and (1.3.2), where w_t^δ is a smooth δ-correlated process which converges to w_t in probability as $\delta \downarrow 0$. The solutions $q_t^{\mu,\delta}$ and q_t^δ of the new equations now don't have the erratic behavior inherent from the Wiener process w_t. It can be shown that $q_t^{\mu,\delta}$ converges to q_t^δ in probability in $C([0,T];\mathbb{R}^d)$ as $\mu \downarrow 0$. The limit of q_t^δ in probability in $C([0,T];\mathbb{R}^d)$ as $\delta \downarrow 0$ is calculated in Ikeda and Watanabe [8], so that we can find the limit of $q_t^{\mu,\delta}$ as $\mu \downarrow 0$ and $\delta \downarrow 0$ in order [1, 5, 10].

1.3.2 Main Result

We first state the required conditions on the coefficients for the proof of convergence:

Hypothesis 2.

1. *$b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to M_d(\mathbb{R})$ are differentiable and bounded with their derivatives.*

2. *$\alpha : \mathbb{R}^d \to \mathbb{R}$ is differentiable and bounded with its derivative. Moreover,*

$$\inf_{q \in \mathbb{R}^d} \alpha(q) = \alpha_0 > 0.$$

Now, we are ready to state the main theorems.

Theorem 1.3.1. *Under Hypothesis 2, there exists a constant $C > 0$ depending on T, q_0, p_0, α_0, $|\alpha|_\infty$, $|\nabla\alpha|_\infty$, $|b|_\infty$, $|Db|_\infty$, $|\sigma|_\infty$, and $|D\sigma|_\infty$ such that for any $0 < \mu \leq 1$, $0 < \delta \leq 1$, and $k \geq 1$,*

$$\mathbb{E} \max_{0 \leq t \leq T} |q_t^{\mu,\delta} - q_t^\delta|^k \leq \exp\left(\frac{C}{\delta^2}\right) \mu^k.$$

In particular, for any fixed $0 < \delta \leq 1$,

$$\lim_{\mu \downarrow 0} \mathbb{E} \max_{0 \leq t \leq T} |q_t^{\mu,\delta} - q_t^\delta|^k = 0.$$

Theorem 1.3.1 can be proved by a combination of methods in Freidlin and Hu [5] and Lee [10]. We skip the proof of this theorem since we will give a proof of a more general problem in Chapter 2. In combination with the following theorem, which is an application of [8, Theorem 7.2 Chapter VI], we can find a limit of $q_t^{\mu,\delta}$ as $\mu \downarrow 0$ first and $\delta \downarrow 0$.

Theorem 1.3.2. *Under Hypothesis 2,*

$$\lim_{\delta \downarrow 0} \mathbb{E} \max_{0 \leq t \leq T} |q_t^\delta - \hat{q}_t|^2 = 0,$$

where $\hat{q}_t$ is the solution of the first order stochastic differential equation

$$\begin{cases} \dot{\hat{q}}_t = -b(\hat{q}_t) - \sigma(\hat{q}_t) \circ \dot{w}_t \\ \\ \hat{q}_0 = q_0 \in \mathbb{R}^2. \end{cases} \tag{1.3.3}$$

The following corollary is the main result of this section.

Corollary 1.3.3. *Under Hypothesis 2, $q_t^{\mu,\delta}$ converges to $\hat{q}_t$ in probability in $C([0,T];\mathbb{R}^d)$ as $\mu \downarrow 0$ and $\delta \downarrow 0$ so that $\mu e^{\frac{C}{\delta^2}} \downarrow 0$ for each constant $C > 0$.*

Chapter 2: Small Mass Asymptotics of a Charged Particle in a Magnetic Field

Small mass asymptotics of the motion of a charged particle moving in a force field combined with a constant magnetic field was considered by Cerrai and Freidlin in 2011 [1]. Later in 2014, Lee [10] showed that this result can be generalized in the case of the magnetic field which varies depending on the position on a plane. We summarize this result in this chapter.

2.1 Introduction

Recall that we consider the motion of a charged particle of mass $\mu > 0$ moving on a plane as in section 1.3. We define q_t^μ as the position of this particle on the plane at time t,

$$b(q_t^\mu) + \sigma(q_t^\mu)\,\dot{w}_t$$

as the force on the particle due to the force field with random noise on the plane, and

$$A(q_t^\mu)\dot{q}_t^\mu$$

as the force on the particle due to the magnetic field. Dependence of the magnetic field on the position can be expressed as

$$A(q) := \alpha(q) A_0, \tag{2.1.1}$$

where $\alpha : \mathbb{R}^2 \mapsto \mathbb{R}^+$ and

$$A_0 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We replace (1.2.2) and (1.2.3) with the following equations by changing the constant α with a function $\alpha : \mathbb{R}^d \mapsto \mathbb{R}^+$:

$$\begin{cases} \mu\, \ddot{q}_t^\mu = b(q_t^\mu) + A(q_t^\mu)\dot{q}_t^\mu + \sigma(q_t^\mu)\, \dot{w}_t \\ q_0^\mu = q_0 \in \mathbb{R}^2, \quad \dot{q}_0^\mu = p_0 \in \mathbb{R}^2 \end{cases} \tag{2.1.2}$$

and

$$\begin{cases} \dot{q}_t = -A^{-1}(q_t) b(q_t) - A^{-1}(q_t)\sigma(q_t)\dot{w}_t \\ q_0 \in \mathbb{R}^2. \end{cases} \tag{2.1.3}$$

It is shown in Section 1.3 that even in the case of constant A, the Smoluchowski-Kramers approximation does not work. We don't have the convergence of q_t^μ to q_t in probability in $C([0,T];\mathbb{R}^2)$. Nonetheless, we may regularize the problem and check a convergence similar to the Smoluchowski-Kramers approximation as in Sections 1.3 and 1.4.

Firstly, we may regularize the problem by introducing small friction proportional to the velocity. We may write $A_\epsilon(q) = A(q) - \epsilon\, I$ and approximate q_t^μ with $q_t^{\mu,\epsilon}$, the solution of the following SDE, which is equation (2.1.2) with A_ϵ in place of

A:

$$\begin{cases} \mu\, \ddot{q}_t^\mu = b(q_t^\mu) + A_\epsilon(q_t^\mu)\dot{q}_t^\mu + \sigma(q_t^\mu)\, \dot{w}_t \\ q_0^\mu = q_0 \in \mathbb{R}^2, \quad \dot{q}_0^\mu = p_0 \in \mathbb{R}^2. \end{cases}$$

This small friction term makes the real parts of the eigenvalues of $A_\epsilon(q)$ negative and gives us an exponential decay of the term

$$\frac{1}{\mu} \exp\left(\frac{1}{\mu}\int_0^t A_\epsilon(q_s^{\mu,\epsilon})\, ds\right) = \frac{1}{\mu}\exp\left(-\frac{\epsilon}{\mu}t\right) \begin{pmatrix} \cos\left(\frac{\beta_t^{\mu,\epsilon}}{\mu}\right) & -\sin\left(\frac{\beta_t^{\mu,\epsilon}}{\mu}\right) \\ \sin\left(\frac{\beta_t^{\mu,\epsilon}}{\mu}\right) & \cos\left(\frac{\beta_t^{\mu,\epsilon}}{\mu}\right) \end{pmatrix},$$

where

$$\beta_t^{\mu,\epsilon} := \int_0^t \alpha(q_s^{\mu,\epsilon}) ds,$$

as $\mu \downarrow 0$; however, it turns out that this approximation does not support us with enough regularity for the convergence of the system. This follows from the fact that

$$\int_0^t \frac{1}{\mu}\exp\left(-\frac{2\epsilon}{\mu}s\right)\left(\int_0^s \exp\left(\frac{\epsilon}{\mu}r\right) dw_r^1\right)^2 ds,$$

where w_r^1 is the first component of the two dimensional Wiener process w_r, does not converge to 0 in probability as $\mu \downarrow 0$. Details of the proof can be found in Freidlin and Hu [5].

As another regularization method, we may approximate the Wiener process w_t with the δ-correlated smooth process w_t^δ as defined in Definition 1.2.4 of section 1.3. We substitute w_t^δ for w_t in equations (2.1.2) and (2.1.3) and let the solutions of the new equations be $q_t^{\mu,\delta}$ and q_t^δ. The strength of this approximation is that it makes $q_t^{\mu,\delta}$ differentiable as many times as we want in t depending on the regularity conditions on b and σ. For instance, if b and σ are differentiable, $q_t^{\mu,\delta}$ is twice differentiable in t. It allows us to solve the problem for each realization using the usual techniques of differential equations and analysis; however, a new difficulty arises from the fact that we cannot use the properties of martingales, especially the Burkholder-Davis-Gundy inequality. We resolve this difficulty by finding a bound of each realization as a function of the Wiener process w_t and showing that the expectation of this expression is bounded. In the following Sections 2.2 and 2.3, we will state and prove that as $\mu \downarrow 0$ and $\delta \downarrow 0$ in the way that $\mu e^{\frac{C}{\delta^2}} \downarrow 0$ for each constant $C > 0$, the solution $q_t^{\mu,\delta}$ of approximated second order equation (2.2.1) converges to the solution $\hat{q}_t$ of first order SDE (2.2.4) in the sense that

$$\lim_{\mu\downarrow 0,\delta\downarrow 0,\mu e^{\frac{C}{\delta^2}}\downarrow 0} \mathbb{E} \max_{0\le t\le T} \left| q_t^{\mu,\delta} - \hat{q}_t \right|^k = 0$$

for all $k \geq 1$. In Section 2.4, we consider an application of this approximation, a homogenization problem.

2.2 Main Result

Recall the definition of w_t^δ in Definition 1.2.4. We rewrite (2.1.2) and (2.1.3) with w_t^δ in place of w_t:

$$\begin{cases} \mu\, \ddot{q}_t^{\mu,\delta} = b(q_t^{\mu,\delta}) + A(q_t^{\mu,\delta})\dot{q}_t^{\mu,\delta} + \sigma(q_t^{\mu,\delta})\, \dot{w}_t^\delta \\ q_0^{\mu,\delta} = q_0 \in \mathbb{R}^2, \quad \dot{q}_0^{\mu,\delta} = p_0 \in \mathbb{R}^2 \end{cases} \tag{2.2.1}$$

and

$$\begin{cases} \dot{q}_t^\delta = -A^{-1}(q_t^\delta)b(q_t^\delta) - A^{-1}(q_t^\delta)\sigma(q_t^\delta)\dot{w}_t^\delta \\ q_0^\delta = q_0 \in \mathbb{R}^2. \end{cases} \tag{2.2.2}$$

Note that (2.2.1) can be rewritten as the system of first order differential equations:

$$\begin{cases} \dot{q}_t^{\mu,\delta} = p_t^{\mu,\delta} \\ \mu\, \dot{p}_t^{\mu,\delta} = b(q_t^{\mu,\delta}) + A(q_t^{\mu,\delta})\, p_t^{\mu,\delta} + \sigma(q_t^{\mu,\delta})\, \dot{w}_t^{\delta} \\ q_0^{\mu,\delta} = q_0 \in \mathbb{R}^2, \quad p_0^{\mu,\delta} = p_0 \in \mathbb{R}^2. \end{cases} \tag{2.2.3}$$

Now, we consider the case that $\mu \downarrow 0$ from (2.2.1):

Theorem 2.2.1. *Under Hypothesis 2, there exists a constant $C > 0$ depending on q_0, p_0, α_0, $|\alpha|_\infty$, $|\nabla\alpha|_\infty$, $|b|_\infty$, $|Db|_\infty$, $|\sigma|_\infty$, and $|D\sigma|_\infty$ such that for any $0 < \mu \leq 1$, $0 < \delta \leq 1$, and $k \geq 1$,*

$$\mathbb{E} \max_{0 \leq t \leq T} |q_t^{\mu,\delta} - q_t^{\delta}|^k \leq \exp\left(\frac{C}{\delta^2}(1+T)^3\right) \mu^k.$$

In particular, for any fixed $0 < \delta \leq 1$,

$$\lim_{\mu \downarrow 0} \mathbb{E} \max_{0 \leq t \leq T} |q_t^{\mu,\delta} - q_t^{\delta}|^k = 0.$$

We postpone the proof of Theorem 2.2.1 to the next section. Now, we tend δ to 0 from q_t^{δ}. By [8, Theorem 7.2 Chapter VI], we have the following result.

Theorem 2.2.2. *Under Hypothesis 2,*

$$\lim_{\delta \downarrow 0} \mathbb{E} \max_{0 \leq t \leq T} |q_t^{\delta} - \hat{q}_t|^2 = 0,$$

where $\hat{q}_t$ is the solution of the first order stochastic differential equation

$$\begin{cases} \dot{\hat{q}}_t = -A^{-1}(\hat{q}_t) b(\hat{q}_t) - (A^{-1}(\hat{q}_t)\sigma(\hat{q}_t)) \circ \dot{w}_t \\ \hat{q}_0 = q_0 \in \mathbb{R}^2. \end{cases} \tag{2.2.4}$$

We state the combination of the above two theorems in the following corollary.

Corollary 2.2.3. *Under Hypothesis 2, $q_t^{\mu,\delta}$ converges to $\hat{q}_t$ in probability in $C([0,T];\mathbb{R}^2)$ as $\mu \downarrow 0$ and $\delta \downarrow 0$ so that $\mu e^{\frac{C}{\delta^2}} \downarrow 0$ for each constant $C > 0$.*

2.3 Proof of Theorem 2.2.1

For the proof of Theorem 2.2.1, it is necessary to find some auxiliary bounds. In the following three lemmas, we find these bounds. First of all, in Lemma 2.3.1, we find a uniform bound of

$$\max_{0\le t\le T} |p_t^{\mu,\delta}|$$

in $C([0,T];\mathbb{R}^2)$ independent of μ.

Lemma 2.3.1. *Under Hypothesis 2, there exists a constant $C > 0$ depending on p_0, α_0, $|\alpha|_\infty$, $|b|_\infty$, $|Db|_\infty$, $|\sigma|_\infty$, and $|D\sigma|_\infty$ such that for any $0 < \delta \le 1$,*

$$\sup_{\mu>0}\left[\max_{0\le t\le T}|p_t^{\mu,\delta}|\right] \le \exp\left(\frac{C}{\delta}(1+T)(1+X_T)\right) \qquad \mathbb{P}-a.s.,$$

where

$$X_T := \max_{0\le t\le T+1}|w_t|. \tag{2.3.1}$$

Proof. Suppose $0 \le t \le T$. From equation (2.2.3),

$$\dot{p}_t^{\mu,\delta} - \frac{1}{\mu}A(q_t^{\mu,\delta})\,p_t^{\mu,\delta} = \frac{1}{\mu}\,b(q_t^{\mu,\delta}) + \frac{1}{\mu}\,\sigma(q_t^{\mu,\delta})\,\dot{w}_t^{\delta}.$$

Multiplying both sides by

$$\exp\left(-\frac{1}{\mu}\int_0^t A(q_s^{\mu,\delta})\,ds\right),$$

we get

$$\left(\exp\left(-\frac{1}{\mu}\int_0^t A(q_s^{\mu,\delta})\,ds\right)p_t^{\mu,\delta}\right)' = \frac{1}{\mu}\exp\left(-\frac{1}{\mu}\int_0^t A(q_s^{\mu,\delta})\,ds\right)b(q_t^{\mu,\delta})$$
$$+\frac{1}{\mu}\exp\left(-\frac{1}{\mu}\int_0^t A(q_s^{\mu,\delta})\,ds\right)\sigma(q_t^{\mu,\delta})\,\dot{w}_t^{\delta}.$$

Define $\beta_t^{\mu,\delta}$ as

$$\beta_t^{\mu,\delta} := \int_0^t \alpha(q_s^{\mu,\delta})\, ds.$$

Considering the definition of $A(q_t^{\mu,\delta})$ in (2.1.1), we have

$$\int_0^t A(q_s^{\mu,\delta})\, ds = \int_0^t \alpha(q_s^{\mu,\delta})\, ds\, A_0 = \beta_t^{\mu,\delta} A_0.$$

So, we may rewrite the above equation as

$$\begin{aligned}\left(\exp\left(-\frac{\beta_t^{\mu,\delta}}{\mu}A_0\right) p_t^{\mu,\delta}\right)' = \frac{1}{\mu}\exp\left(-\frac{\beta_t^{\mu,\delta}}{\mu}A_0\right) b(q_t^{\mu,\delta}) \\ + \frac{1}{\mu}\exp\left(-\frac{\beta_t^{\mu,\delta}}{\mu}A_0\right) \sigma(q_t^{\mu,\delta})\, \dot{w}_t^{\delta}.\end{aligned}$$

Integrating both sides with respect to t, we get

$$\begin{aligned}p_t^{\mu,\delta} &= \exp\left(\frac{\beta_t^{\mu,\delta}}{\mu}A_0\right) p_0 + \frac{1}{\mu}\exp\left(\frac{\beta_t^{\mu,\delta}}{\mu}A_0\right)\int_0^t \exp\left(-\frac{\beta_s^{\mu,\delta}}{\mu}A_0\right) b(q_s^{\mu,\delta})\, ds \\ &\quad + \frac{1}{\mu}\exp\left(\frac{\beta_t^{\mu,\delta}}{\mu}A_0\right)\int_0^t \exp\left(-\frac{\beta_s^{\mu,\delta}}{\mu}A_0\right) \sigma(q_s^{\mu,\delta})\, dw_s^{\delta} \\ &=: I_1(t) + I_2(t) + I_3(t). \end{aligned} \tag{2.3.2}$$

By the definition of A_0 in (2.1.1), we can calculate the matrix exponentials

$$\exp\left(\pm\frac{\beta_t^{\mu,\delta}}{\mu}A_0\right) = \begin{pmatrix} \cos\left(\frac{\beta_t^{\mu,\delta}}{\mu}\right) & \mp\sin\left(\frac{\beta_t^{\mu,\delta}}{\mu}\right) \\ \pm\sin\left(\frac{\beta_t^{\mu,\delta}}{\mu}\right) & \cos\left(\frac{\beta_t^{\mu,\delta}}{\mu}\right) \end{pmatrix}. \tag{2.3.3}$$

Since (2.3.3) is an orthogonal matrix, for any $v \in \mathbb{R}^2$,

$$\left|\exp\left(\pm\frac{\beta_t^{\mu,\delta}}{\mu}A_0\right) v\right| = |v|, \tag{2.3.4}$$

so that

$$|I_1(t)| \leq |p_0|.$$

As A_0 and A_0^{-1} commute, we have

$$
\begin{aligned}
I_2(t) &= \exp\left(\frac{\beta_t^{\mu,\delta}}{\mu}A_0\right)\int_0^t \left(-\frac{\alpha(q_s^{\mu,\delta})}{\mu}A_0\exp\left(-\frac{\beta_s^{\mu,\delta}}{\mu}A_0\right)\right)\left(-\frac{1}{\alpha(q_s^{\mu,\delta})}A_0^{-1}b(q_s^{\mu,\delta})\right)ds \\
&= \exp\left(\frac{\beta_t^{\mu,\delta}}{\mu}A_0\right)\left(\left[\exp\left(-\frac{\beta_s^{\mu,\delta}}{\mu}A_0\right)\left(-\frac{1}{\alpha(q_s^{\mu,\delta})}A_0^{-1}b(q_s^{\mu,\delta})\right)\right]_0^t\right. \\
&\quad - \int_0^t \exp\left(\frac{\beta_s^{\mu,\delta}}{\mu}A_0\right) \\
&\qquad \left.\left(\frac{\nabla\alpha(q_s^{\mu,\delta})\cdot p_s^{\mu,\delta}}{\alpha(q_s^{\mu,\delta})^2}A_0^{-1}b(q_s^{\mu,\delta}) - \frac{1}{\alpha(q_s^{\mu,\delta})}A_0^{-1}Db(q_s^{\mu,\delta})p_s^{\mu,\delta}\right)ds\right) \\
&= \exp\left(\frac{\beta_t^{\mu,\delta}}{\mu}A_0\right)\left(-\frac{1}{\alpha(q_t^{\mu,\delta})}A_0^{-1}\exp\left(-\frac{\beta_t^{\mu,\delta}}{\mu}A_0\right)b(q_t^{\mu,\delta}) + \frac{1}{\alpha(q_0)}A_0^{-1}b(q_0)\right. \\
&\quad - \int_0^t \exp\left(-\frac{\beta_s^{\mu,\delta}}{\mu}A_0\right) \\
&\qquad \left.\left(\frac{\nabla\alpha(q_s^{\mu,\delta})\cdot p_s^{\mu,\delta}}{\alpha(q_s^{\mu,\delta})^2}A_0^{-1}b(q_s^{\mu,\delta}) - \frac{1}{\alpha(q_s^{\mu,\delta})}A_0^{-1}Db(q_s^{\mu,\delta})p_s^{\mu,\delta}\right)ds\right) \\
&= -\frac{1}{\alpha(q_t^{\mu,\delta})}A_0^{-1}b(q_t^{\mu,\delta}) + \frac{1}{\alpha(q_0)}A_0^{-1}\exp\left(\frac{\beta_t^{\mu,\delta}}{\mu}A_0\right)b(q_0) \\
&\quad - \int_0^t \exp\left(\frac{\beta_t^{\mu,\delta}-\beta_s^{\mu,\delta}}{\mu}A_0\right) \\
&\qquad \left(\frac{\nabla\alpha(q_s^{\mu,\delta})\cdot p_s^{\mu,\delta}}{\alpha(q_s^{\mu,\delta})^2}A_0^{-1}b(q_s^{\mu,\delta}) - \frac{1}{\alpha(q_s^{\mu,\delta})}A_0^{-1}Db(q_s^{\mu,\delta})p_s^{\mu,\delta}\right)ds. \qquad (2.3.5)
\end{aligned}
$$

Considering Hypothesis 2,

$$
\begin{aligned}
|I_2(t)| &\leq \left|\frac{1}{\alpha(q_t^{\mu,\delta})}A_0^{-1}b(q_t^{\mu,\delta})\right| + \left|\frac{1}{\alpha(q_0)}A_0^{-1}\exp\left(\frac{\beta_t^{\mu,\delta}}{\mu}A_0\right)b(q_0)\right| \\
&\quad + \int_0^t \left|\exp\left(\frac{\beta_t^{\mu,\delta}-\beta_s^{\mu,\delta}}{\mu}A_0\right)\right. \\
&\qquad \left.\left(\frac{\nabla\alpha(q_s^{\mu,\delta})\cdot p_s^{\mu,\delta}}{\alpha(q_s^{\mu,\delta})^2}A_0^{-1}b(q_s^{\mu,\delta}) - \frac{1}{\alpha(q_s^{\mu,\delta})}A_0^{-1}Db(q_s^{\mu,\delta})p_s^{\mu,\delta}\right)\right| ds \\
&\leq \frac{|b|_\infty}{\alpha_0} + \frac{|b|_\infty}{\alpha_0} + \int_0^t \frac{|\nabla\alpha|_\infty|b|_\infty}{\alpha_0^2}|p_s^{\mu,\delta}| + \frac{|Db|_\infty}{\alpha_0}|p_s^{\mu,\delta}|ds \\
&\leq C_1 + C_2\int_0^t |p_s^{\mu,\delta}|ds.
\end{aligned}
$$

The same method can be used for $I_3(t)$ and we get

$$\begin{aligned}
I_3(t) &= \frac{1}{\mu}\exp\left(\frac{\beta_t^{\mu,\delta}}{\mu}A_0\right)\int_0^t \exp\left(-\frac{\beta_s^{\mu,\delta}}{\mu}A_0\right)\sigma(q_s^{\mu,\delta})\dot{w}_s^\delta ds \\
&= -\frac{1}{\alpha(q_t^{\mu,\delta})}A_0^{-1}\sigma(q_t^{\mu,\delta})\dot{w}_t^\delta + \frac{1}{\alpha(q_0)}A_0^{-1}\exp\left(\frac{\beta_t^{\mu,\delta}}{\mu}A_0\right)\sigma(q_0)\dot{w}_0^\delta \\
&\quad - \int_0^t \exp\left(\frac{\beta_t^{\mu,\delta}-\beta_s^{\mu,\delta}}{\mu}A_0\right)\left(\frac{\nabla\alpha(q_s^{\mu,\delta})\cdot p_s^{\mu,\delta}}{\alpha(q_s^{\mu,\delta})^2}A_0^{-1}\sigma(q_s^{\mu,\delta})\dot{w}_s^\delta\right. \\
&\qquad \left. -\frac{1}{\alpha(q_s^{\mu,\delta})}A_0^{-1}D\sigma(q_s^{\mu,\delta})p_s^{\mu,\delta}\dot{w}_s^\delta - \frac{1}{\alpha(q_s^{\mu,\delta})}A_0^{-1}\sigma(q_s^{\mu,\delta})\ddot{w}_s^\delta\right) ds.
\end{aligned}$$

To find a bound of $I_3(t)$, we need bounds of $\dot{w}_t^\delta$ and $\ddot{w}_t^\delta$. In view of Definition 1.2.4, we note that $(w_t^\delta)^{(n)}$, the nth derivative of w_t^δ with respect to t, satisfies

$$(w_t^\delta)^{(n)} = \frac{(-1)^n}{\delta^n}\int_0^\infty w(s)\rho^{(n)}\left(\frac{s-t}{\delta}\right)ds = \frac{(-1)^n}{\delta^n}\int_0^1 w(t+\delta r)\rho^{(n)}(r)dr.$$

Hence, for any $0 \le t \le T$,

$$\begin{aligned}
|(w_t^\delta)^{(n)}| &\le \frac{1}{\delta^n}\int_0^1 |w(t+\delta s)||\rho^{(n)}(s)|ds \\
&\le \frac{1}{\delta^n}\max_{0\le t\le T+\delta}|w(t)|\int_0^1 |\rho^{(n)}(s)|ds = \frac{C(n,\rho)}{\delta^n}\max_{0\le t\le T+1}|w(t)|,
\end{aligned}$$

where $C(n,\rho)$ is a constant depending on n and ρ.

Letting

$$X_T := \max_{0\le t\le T+1}|w_t|,$$

we have

$$\max_{0\le t\le T}|(w_t^\delta)^{(n)}| \le \frac{C(n,\rho)}{\delta^n}X_T.$$

In particular, we can find a constant $C > 0$ such that

$$\max_{0\le t\le T}|\dot{w}_t^\delta| \le \frac{C}{\delta}X_T \tag{2.3.6}$$

and

$$\max_{0\le t\le T}|\ddot{w}_t^\delta| \le \frac{C}{\delta^2}X_T.$$

Now, we are ready to find a bound of $I_3(t)$. Applying Hypothesis 2, (2.3.4) and (2.3.6) to (2.3.5), we get

$$\begin{aligned}
|I_3(t)| &\le \left|\frac{1}{\alpha(q_t^{\mu,\delta})}A_0^{-1}\sigma(q_t^{\mu,\delta})\dot{w}_t^\delta\right| + \left|\frac{1}{\alpha(q)}A_0^{-1}\exp\left(\frac{\beta_t^{\mu,\delta}}{\mu}A_0\right)\sigma(q)\dot{w}_0^\delta\right| \\
&\quad + \int_0^t \left|\exp\left(\frac{\beta_t^{\mu,\delta}-\beta_s^{\mu,\delta}}{\mu}A_0\right)\left(\frac{\nabla\alpha(q_s^{\mu,\delta})\cdot p_s^{\mu,\delta}}{\alpha(q_s^{\mu,\delta})^2}A_0^{-1}\sigma(q_s^{\mu,\delta})\dot{w}_s^\delta\right.\right. \\
&\qquad \left.\left. -\frac{1}{\alpha(q_s^{\mu,\delta})}A_0^{-1}D\sigma(q_s^{\mu,\delta})p_s^{\mu,\delta}\dot{w}_s^\delta + \frac{1}{\alpha(q_s^{\mu,\delta})}A_0^{-1}\sigma(q_s^{\mu,\delta})\ddot{w}_s^\delta\right)\right| ds \\
&\le \frac{C(\alpha_0,|\sigma|_\infty)}{\delta}X_T + \frac{C(\alpha_0,|\sigma|_\infty)}{\delta}X_T + \frac{C(\alpha_0,|\nabla\alpha|_\infty,|\sigma|_\infty)}{\delta}X_T\int_0^t |p_s^{\mu,\delta}|ds \\
&\quad + \frac{C(\alpha_0,|D\sigma|_\infty)}{\delta}X_T\int_0^t|p_s^{\mu,\delta}|ds + \frac{C(\alpha_0,|\sigma|_\infty)t}{\delta^2}X_T \\
&\le \frac{C_3}{\delta^2}(1+t)X_T + \frac{C_4}{\delta}X_T\int_0^t|p_s^{\mu,\delta}|ds.
\end{aligned}$$

Applying the bounds of $I_1(t)$, $I_2(t)$, and $I_3(t)$ to (2.3.2), we get a bound of $p_t^{\mu,\delta}$:

$$\begin{aligned}
|p_t^{\mu,\delta}| &\le |I_1(t)| + |I_2(t)| + |I_3(t)| \\
&\le |p_0| + C_1 + C_2\int_0^t |p_s^{\mu,\delta}|ds + \frac{C_3}{\delta^2}(1+t)X_T + \frac{C_4}{\delta}X_T\int_0^t|p_s^{\mu,\delta}|ds \\
&\le \frac{C_5}{\delta^2}(1+T)(1+X_T) + \frac{C_6}{\delta}(1+X_T)\int_0^t|p_s^{\mu,\delta}|ds.
\end{aligned}$$

By Gronwall's lemma,

$$\begin{aligned}
|p_t^{\mu,\delta}| &\le \frac{C_5}{\delta^2}(1+T)(1+X_T)\exp\left(\frac{C_6}{\delta}(1+X_T)t\right) \\
&\le \exp\left(\frac{C_5}{\delta^2}(1+T)(1+X_T) + \frac{C_6}{\delta}(1+X_T)t\right) \\
&\le \exp\left(\frac{C}{\delta}(1+T)(1+X_T)\right)
\end{aligned}$$

for sufficiently large $C > 0$.

So, we have

$$\max_{0\le t\le T} |p_t^{\mu,\delta}| \le \exp\left(\frac{C}{\delta}(1+T)(1+X_T)\right).$$

□

Remark 1. *Note that by Lemma 2.3.1, $q_t^{\mu,\delta}$ is Lipschitz continuous with its Lipschitz constant independent of μ on the interval $[0,T]$. That is, for $0 \le t_1 \le t_2 \le T$,*

$$|q_{t_2}^{\mu,\delta} - q_{t_1}^{\mu,\delta}| \le C(T,\delta,X_T)|t_2 - t_1| \qquad \mathbb{P} - a.s.$$

Next, we find a bound of the integral of a highly oscillating function. The result is similar to that of the Riemann-Lebesgue lemma. This result guarantees that $q_t^{\mu,\delta}$ converges to q_t^{δ} in $C([0,T];\mathbb{R}^2)$ for each realization.

Lemma 2.3.2. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a bounded Lipschitz continuous function with the Lipschitz constant K_f. Under Hypothesis 2, there exists a constant $C > 0$ depending on K_f, $|f|_\infty$, p_0, α_0, $|\alpha|_\infty$, $|\nabla\alpha|_\infty$, $|b|_\infty$, $|Db|_\infty$, $|\sigma|_\infty$, and $|D\sigma|_\infty$ such that for any $\mu > 0$ and $0 < \delta \le 1$,*

$$\left|\int_0^t \cos\left(\frac{\beta_s^{\mu,\delta}}{\mu}\right) f(q_s^{\mu,\delta})ds\right| + \left|\int_0^t \sin\left(\frac{\beta_s^{\mu,\delta}}{\mu}\right) f(q_s^{\mu,\delta})ds\right| \le C(t,\delta,X_t)\mu \tag{2.3.7}$$

and

$$\left|\int_0^t \cos\left(\frac{\beta_s^{\mu,\delta}}{\mu}\right) f(q_s^{\mu,\delta})\dot{w}_s^{\delta}ds\right| + \left|\int_0^t \sin\left(\frac{\beta_s^{\mu,\delta}}{\mu}\right) f(q_s^{\mu,\delta})\dot{w}_s^{\delta}ds\right| \le C(t,\delta,X_t)\mu \tag{2.3.8}$$

$\mathbb{P} - a.s.$, where

$$C(t,\delta,X_t) = \exp\left(\frac{C}{\delta}(1+t)(1+X_t)\right).$$

Proof. Since $\alpha(q_s^{\mu,\delta})$ is strictly positive,

$$\beta_t^{\mu,\delta} = \int_0^t \alpha(q_s^{\mu,\delta})ds$$

is strictly increasing, so that

$$u = \frac{\beta_s^{\mu,\delta}}{\mu}$$

provides a good change of variables.

Then, as

$$du = \frac{\alpha(q_s^{\mu,\delta})}{\mu}ds, \tag{2.3.9}$$

we have

$$\int_0^t \cos\left(\frac{\beta_s^{\mu,\delta}}{\mu}\right) f(q_s^{\mu,\delta})ds = \mu \int_0^{\frac{\beta_t^{\mu,\delta}}{\mu}} \cos(u)\, \frac{f(q_{s(u)}^{\mu,\delta})}{\alpha(q_{s(u)}^{\mu,\delta})}du.$$

If we define

$$g^{\mu,\delta}(u) := \frac{f(q_{s(u)}^{\mu,\delta})}{\alpha(q_{s(u)}^{\mu,\delta})}, \tag{2.3.10}$$

we get

$$
\begin{aligned}
\left|\int_0^t \cos\left(\frac{\beta_s^{\mu,\delta}}{\mu}\right)\, f(q_s^{\mu,\delta})ds\right| &= \left|\mu\int_0^{\frac{\beta_t^{\mu,\delta}}{\mu}} \cos(u)\, g^{\mu,\delta}(u)du\right| \\
&= \left|\mu \sum_{k=0}^{\left\lfloor \frac{\beta_t^{\mu,\delta}}{2\pi\mu}\right\rfloor -1} \int_{2\pi k}^{2\pi(k+1)} \cos(u)\, g^{\mu,\delta}(u)du + \mu\int_{2\pi\left\lfloor \frac{\beta_t^{\mu,\delta}}{2\pi\mu}\right\rfloor}^{\frac{\beta_t^{\mu,\delta}}{\mu}} \cos(u) g^{\mu,\delta}(u)du\right| \\
&\leq \left|\mu \sum_{k=0}^{\left\lfloor \frac{\beta_t^{\mu,\delta}}{2\pi\mu}\right\rfloor -1} \int_{2\pi k}^{2\pi(k+1)} \cos(u)\, \big(\big(g^{\mu,\delta}(u) - g^{\mu,\delta}(2\pi k)\big) + g^{\mu,\delta}(2\pi k)\big)\, du\right| \\
&\quad + \left|\mu\int_{2\pi\left\lfloor \frac{\beta_t^{\mu,\delta}}{2\pi\mu}\right\rfloor}^{\frac{\beta_t^{\mu,\delta}}{\mu}} \cos(u) g^{\mu,\delta}(u)du\right| \\
&= \left|\mu \sum_{k=0}^{\left\lfloor \frac{\beta_t^{\mu,\delta}}{2\pi\mu}\right\rfloor -1} \int_{2\pi k}^{2\pi(k+1)} \cos(u)\, \big(g^{\mu,\delta}(u) - g^{\mu,\delta}(2\pi k)\big)\, du\right| \\
&\quad + \left|\mu\int_{2\pi\left\lfloor \frac{\beta_t^{\mu,\delta}}{2\pi\mu}\right\rfloor}^{\frac{\beta_t^{\mu,\delta}}{\mu}} \cos(u) g^{\mu,\delta}(u)du\right| \\
&=: |I_1(t)| + |I_2(t)|. \qquad (2.3.11)
\end{aligned}
$$

We first find a bound of $|I_1(t)|$.

$$
\begin{aligned}
\left|g^{\mu,\delta}(u) - g^{\mu,\delta}(2\pi k)\right| &= \left|\frac{f(q^{\mu,\delta}_{s(u)})}{\alpha(q^{\mu,\delta}_{s(u)})} - \frac{f(q^{\mu,\delta}_{s(2\pi k)})}{\alpha(q^{\mu,\delta}_{s(2\pi k)})}\right| \\
&= \frac{1}{\alpha(q^{\mu,\delta}_{s(u)})\alpha(q^{\mu,\delta}_{s(2\pi k)})}\left|f(q^{\mu,\delta}_{s(u)})\alpha(q^{\mu,\delta}_{s(2\pi k)}) - f(q^{\mu,\delta}_{s(2\pi k)})\alpha(q^{\mu,\delta}_{s(u)})\right| \\
&\le \frac{1}{\alpha_0^2}\left|f(q^{\mu,\delta}_{s(u)})\alpha(q^{\mu,\delta}_{s(2\pi k)}) - f(q^{\mu,\delta}_{s(2\pi k)})\alpha(q^{\mu,\delta}_{s(u)})\right| \\
&\le \frac{1}{\alpha_0^2}\Big(\left|f(q^{\mu,\delta}_{s(u)})\alpha(q^{\mu,\delta}_{s(2\pi k)}) - f(q^{\mu,\delta}_{s(u)})\alpha(q^{\mu,\delta}_{s(u)})\right| \\
&\qquad + \left|f(q^{\mu,\delta}_{s(u)})\alpha(q^{\mu,\delta}_{s(u)}) - f(q^{\mu,\delta}_{s(2\pi k)})\alpha(q^{\mu,\delta}_{s(u)})\right|\Big) \\
&\le \frac{1}{\alpha_0^2}\left(|f|_\infty|\nabla\alpha|_\infty\left|q^{\mu,\delta}_{s(u)} - q^{\mu,\delta}_{s(2\pi k)}\right| + |\alpha|_\infty K_f\left|q^{\mu,\delta}_{s(u)} - q^{\mu,\delta}_{s(2\pi k)}\right|\right) \\
&\le C_1\left|q^{\mu,\delta}_{s(u)} - q^{\mu,\delta}_{s(2\pi k)}\right|. \qquad (2.3.12)
\end{aligned}
$$

From Lemma 2.3.1 and Remark 1, we have

$$
\left|\frac{d}{du}q^{\mu,\delta}_{s(u)}\right| = \left|p^{\mu,\delta}_{s(u)}\frac{ds(u)}{du}\right| = \left|p^{\mu,\delta}_{s(u)}\frac{\mu}{\alpha(q^{\mu,\delta}_{s(u)})}\right| \le C(t,\delta,X_t)\frac{\mu}{\alpha_0}
$$

for $0 \le s(u) \le t$, where

$$
C(t,\delta,X_t) = \exp\left(\frac{C}{\delta}(1+t)(1+X_t)\right). \qquad (2.3.13)
$$

So, from (2.3.12),

$$
|g^{\mu,\delta}(u) - g^{\mu,\delta}(2\pi k)| \le C_2 C(t,\delta,X_t)\mu|u - 2\pi k|.
$$

This implies

$$
\begin{aligned}
|I_1(t)| &\le \mu \sum_{k=0}^{\left\lfloor \frac{\beta_t^{\mu,\delta}}{2\pi\mu} \right\rfloor - 1} \int_{2\pi k}^{2\pi(k+1)} |\cos(u)||g^{\mu,\delta}(u) - g^{\mu,\delta}(2\pi k)| du \\
&\le \mu \sum_{k=0}^{\left\lfloor \frac{\beta_t^{\mu,\delta}}{2\pi\mu} \right\rfloor - 1} \int_{2\pi k}^{2\pi(k+1)} C_2 C(t, \delta, X_t)\mu(u - 2\pi k) du \\
&= \mu \sum_{k=0}^{\left\lfloor \frac{\beta_t^{\mu,\delta}}{2\pi\mu} \right\rfloor - 1} C_2 C(t, \delta, X_t)\mu 2\pi^2 \\
&= C_2 C(t, \delta, X_t)\mu^2 2\pi^2 \left\lfloor \frac{\beta_t^{\mu,\delta}}{2\pi\mu} \right\rfloor .
\end{aligned}
$$

Since

$$
\beta_t^{\mu,\delta} \le |\alpha|_\infty t,
$$

we get

$$
\begin{aligned}
|I_1(t)| &\le C_2 C(t, \delta, X_t)\mu\pi|\alpha|_\infty t \\
&= C_3 C(t, \delta, X_t) t\mu. \qquad (2.3.14)
\end{aligned}
$$

A bound of $|I_2(t)|$ can be found relatively easily. From (2.3.11),

$$
\begin{aligned}
|I_2(t)| &\le \mu \int_{2\pi\left\lfloor \frac{\beta_t^{\mu,\delta}}{2\pi\mu} \right\rfloor}^{\frac{\beta_t^{\mu,\delta}}{\mu}} \left| \frac{f(q_{s(u)}^{\mu,\delta})}{\alpha(q_{s(u)}^{\mu,\delta})} \right| du \\
&\le \mu \int_{2\pi\left\lfloor \frac{\beta_t^{\mu,\delta}}{2\pi\mu} \right\rfloor}^{\frac{\beta_t^{\mu,\delta}}{\mu}} \frac{|f|_\infty}{\alpha_0} du \\
&\le \mu 2\pi \frac{|f|_\infty}{\alpha_0} = C_4\mu. \qquad (2.3.15)
\end{aligned}
$$

Applying the bounds of $I_1(t)$ and $I_2(t)$ to (2.3.11),

$$\begin{aligned}\left|\int_0^t \cos\left(\frac{\beta_s^{\mu,\delta}}{\mu}\right) f(q_s^{\mu,\delta})ds\right| &\le C_3 C(t,\delta,X_t)t\mu + C_4\mu \\ &\le C_5(1+t)\exp\left(\frac{C_6}{\delta}(1+t)(1+X_t)\right)\mu \\ &\le \exp\left(\frac{C_7}{\delta}(1+t)(1+X_t)\right)\mu.\end{aligned}$$

This proves inequality (2.3.7) for the cosine part. The sine part can be treated analogously. Now consider inequality (2.3.8). As in (2.3.11),

$$\begin{aligned}\left|\int_0^t \cos\left(\frac{\beta_s^{\mu,\delta}}{\mu}\right) f(q_s^{\mu,\delta})\dot{w}_s^{\delta}ds\right| &\le \left|\mu\sum_{k=0}^{\left\lfloor\frac{\beta_t^{\mu,\delta}}{2\pi\mu}\right\rfloor-1}\int_{2\pi k}^{2\pi(k+1)}\cos(u)\,\left(g_1^{\mu,\delta}(u)-g_1^{\mu,\delta}(2\pi k)\right)du\right| \\ &\quad+\left|\mu\int_{2\pi\left\lfloor\frac{\beta_t^{\mu,\delta}}{2\pi\mu}\right\rfloor}^{\frac{\beta_t^{\mu,\delta}}{\mu}} g_1^{\mu,\delta}(u)du\right| \\ &=: |I_1(t)|+|I_2(t)|, \qquad (2.3.16)\end{aligned}$$

where

$$g_1^{\mu,\delta}(u) := \frac{f(q_{s(u)}^{\mu,\delta})}{\alpha(q_{s(u)}^{\mu,\delta})}\dot{w}_{s(u)}^{\delta}.$$

By a similar argument as in (2.3.12), we obtain

$$\begin{aligned}\left|g_1^{\mu,\delta}(u)-g_1^{\mu,\delta}(2\pi k)\right| &= \left|\frac{f(q_{s(u)}^{\mu,\delta})}{\alpha(q_{s(u)}^{\mu,\delta})}\dot{w}_{s(u)}^{\delta} - \frac{f(q_{s(2\pi k)}^{\mu,\delta})}{\alpha(q_{s(2\pi k)}^{\mu,\delta})}\dot{w}_{s(2\pi k)}^{\delta}\right| \\ &\le \frac{1}{\alpha_0^2}\Bigg(|f|_\infty \max_{0\le s\le t}\left\{|\dot{w}_s^{\delta}|\right\}|\nabla\alpha|_\infty\left|q_{s(u)}^{\mu,\delta}-q_{s(2\pi k)}^{\mu,\delta}\right| \\ &\quad+|\alpha|_\infty\max_{0\le s\le t}\left\{|\dot{w}_s^{\delta}|\right\}K_f\left|q_{s(u)}^{\mu,\delta}-q_{s(2\pi k)}^{\mu,\delta}\right| \\ &\quad+|\alpha|_\infty|f|_\infty\max_{0\le s\le t}\left\{|\ddot{w}_s^{\delta}|\right\}|s(u)-s(2\pi k)|\Bigg).\end{aligned}$$

Considering inequalities in (2.3.6) and Remark 1,

$$\begin{aligned}
\left|g_1^{\mu,\delta}(u) - g_1^{\mu,\delta}(2\pi k)\right| &\le \frac{C_8}{\delta} X_t \left|q^{\mu,\delta}_{s(u)} - q^{\mu,\delta}_{s(2\pi k)}\right| + \frac{C_9}{\delta^2} X_t |s(u) - s(2\pi k)| \\
&\le \frac{C_8}{\delta} X_t C(t,\delta,X_t)|s(u) - s(2\pi k)| + \frac{C_9}{\delta^2} X_t |s(u) - s(2\pi k)| \\
&\le C(t,\delta,X_t) X_t |s(u) - s(2\pi k)|.
\end{aligned}$$

The last inequality was from the fact that $\frac{C_8}{\delta}$ or $\frac{C_9}{\delta^2}$ can be absorbed in the term $C(t,\delta,X_t)$ by possibly changing the constant inside $C(t,\delta,X_t)$.

Note that from (2.3.9),

$$\left|\frac{d}{du} s(u)\right| \le \frac{\mu}{\alpha_0}.$$

Therefore,

$$\left|g_1^{\mu,\delta}(u) - g_1^{\mu,\delta}(2\pi k)\right| \le C_{10} C(t,\delta,X_t) X_t \mu |u - 2\pi k|.$$

By the same procedures as in (2.3.14) and (2.3.15),

$$|I_1(t)| \le \frac{C_{11}}{\delta^2} C(t,\delta,X_t) X_t t \mu$$

and

$$|I_2(t)| \le \frac{C_{12}}{\delta} X_t \mu.$$

Now from (2.3.16), we get

$$\begin{aligned}
\left|\int_0^t \cos \frac{\beta_s^{\mu,\delta}}{\mu} f(q_s^{\mu,\delta}) \dot{w}_s^\delta ds\right| &\le \frac{C_{11}}{\delta^2} C(t,\delta,X_t) X_t t \mu + \frac{C_{12}}{\delta} X_t \mu \\
&\le C_{13} C(t,\delta,X_t) X_t \mu \\
&\le \exp\left(\frac{C_{14}}{\delta}(1+t)(1+X_t)\right)\mu.
\end{aligned}$$

□

In the next lemma, we show that the expectation of the exponential of the uniform norm of the two dimensional Wiener process in $C([0,t+1];\mathbb{R}^2)$ is finite. This property guarantees that the approach of analyzing the problem for each realization is working.

Lemma 2.3.3. *For $a \in \mathbb{R}^+$,*

$$\mathbb{E}\left(e^{(1+X_t)a}\right) \leq 16 \exp\left((t+1)a^2 + a\right),$$

where

$$X_t = \max_{0 \leq s \leq t+1} |w_s|.$$

Proof. Since $w_s = (w_s^1, w_s^2)$, where w_s^1 and w_s^2 are independent one dimensional Wiener processes, defining

$$X_{i,t} := \max_{0 \leq s \leq t+1} |w_s^i|$$

for $i = 1, 2$, we have

$$X_t \leq X_{1,t} + X_{2,t}.$$

By independence of $X_{1,t}$ and $X_{2,t}$,

$$\mathbb{E}\left(e^{(1+X_t)a}\right) \leq e^a \mathbb{E}\left(e^{(X_{1,t}+X_{2,t})a}\right) \leq e^a \mathbb{E}\left(e^{X_{1,t}a}\right)^2. \tag{2.3.17}$$

To find the bound of $\mathbb{E}\left(e^{X_{1,t}a}\right)$, we may use the symmetry of the Wiener process and the reflection principle. For $x \geq 0$,

$$\begin{aligned} P(\max_{0\leq s\leq T} |w_s^1| > x) &= P(\{\max_{0\leq s\leq T}\{w_s^1\} > x\} \cup \{\min_{0\leq s\leq T}\{w_s^1\} < -x\}) \\ &\leq P(\max_{0\leq s\leq T}\{w_s^1\} > x) + P(\min_{0\leq s\leq T}\{w_s^1\} < -x) \\ &= 2P(\max_{0\leq s\leq T}\{w_s^1\} > x). \end{aligned}$$

By the reflection principle,

$$P(\max_{0\le s\le T}|w_s^1|>x)\le 4P(w_T^1>x)=4\int_x^\infty \frac{1}{\sqrt{2\pi T}}e^{-\frac{y^2}{2T}}dy.$$

So, for $T=t+1$,

$$P(X_{1,t}>x)\le 4\int_x^\infty \frac{1}{\sqrt{2\pi(t+1)}}e^{-\frac{y^2}{2(t+1)}}dy.$$

Using this inequality,

$$\begin{aligned}
E(e^{X_{1,t}a}) &= \int_0^\infty P(e^{X_{1,t}a}>x)dx=\int_0^\infty P(X_{1,t}>\frac{1}{a}\ln x)dx\\
&\le 4\int_0^\infty\int_{\frac{1}{a}\ln x}^\infty \frac{1}{\sqrt{2\pi(t+1)}}e^{-\frac{y^2}{2(t+1)}}dy\,dx\\
&\le 4\exp\left(\frac{t+1}{2}a^2\right).
\end{aligned}$$

Applying these bounds to (2.3.17),

$$\begin{aligned}
E\left(e^{(1+X_t)a}\right) &\le e^a 16\exp\left((t+1)a^2\right)\\
&= 16\exp\left((t+1)a^2+a\right).
\end{aligned}$$

□

Remark 2. *Note that the boundedness of*

$$\mathbb{E}e^{aX_t}$$

is a simple consequence of the Fernique's theorem. We did calculations in Lemma 2.3.3 for the necessity of finding the relationship of the bound with a.

Finally, we are ready to prove the main theorem, Theorem 2.2.1.

Proof of Theorem 2.2.1. Consider $0 \le t \le T$. First, we find representations of $q_t^{\mu,\delta}$ and q_t^{δ}. Integrating equations (2.3.2) and (2.2.2),

$$\begin{aligned} q_t^{\mu,\delta} = q_0 &+ \int_0^t \exp\left(\frac{\beta_s^{\mu,\delta}}{\mu} A_0\right) p_0 \, ds \\ &+ \frac{1}{\mu} \int_0^t \exp\left(\frac{\beta_s^{\mu,\delta}}{\mu} A_0\right) \int_0^s \exp\left(-\frac{\beta_r^{\mu,\delta}}{\mu} A_0\right) b(q_r^{\mu,\delta}) \, dr \, ds \\ &+ \frac{1}{\mu} \int_0^t \exp\left(\frac{\beta_s^{\mu,\delta}}{\mu} A_0\right) \int_0^s \exp\left(-\frac{\beta_r^{\mu,\delta}}{\mu} A_0\right) \sigma(q_r^{\mu,\delta}) \, dw_r^{\delta} \, ds \end{aligned}$$

and

$$\begin{aligned} q_t^{\delta} &= q_0 - \int_0^t A^{-1}(q_s^{\delta}) \, b(q_s^{\delta}) \, ds - \int_0^t A^{-1}(q_s^{\delta}) \, \sigma(q_s^{\delta}) \, dw_s^{\delta} \\ &= q_0 - \int_0^t \frac{1}{\alpha(q_s^{\delta})} A_0^{-1} \, b(q_s^{\delta}) \, ds - \int_0^t \frac{1}{\alpha(q_s^{\delta})} A_0^{-1} \, \sigma(q_s^{\delta}) \, dw_s^{\delta}. \end{aligned}$$

Subtracting q_t^{δ} from $q_t^{\mu,\delta}$,

$$\begin{aligned} q_t^{\mu,\delta} - q_t^{\delta} = &\int_0^t \exp\left(\frac{\beta_s^{\mu,\delta}}{\mu} A_0\right) p_0 \, ds \\ &+ \left(\frac{1}{\mu} \int_0^t \exp\left(\frac{\beta_s^{\mu,\delta}}{\mu} A_0\right) \int_0^s \exp\left(-\frac{\beta_r^{\mu,\delta}}{\mu} A_0\right) b(q_r^{\mu,\delta}) \, dr \, ds \right.\\ &\quad \left. + \int_0^t \frac{1}{\alpha(q_s^{\mu,\delta})} A_0^{-1} \, b(q_s^{\mu,\delta}) \, ds \right) \\ &+ \left(\frac{1}{\mu} \int_0^t \exp\left(\frac{\beta_s^{\mu,\delta}}{\mu} A_0\right) \int_0^s \exp\left(-\frac{\beta_r^{\mu,\delta}}{\mu} A_0\right) \sigma(q_r^{\mu,\delta}) \, dw_r^{\delta} \, ds \right.\\ &\quad \left. + \int_0^t \frac{1}{\alpha(q_s^{\mu,\delta})} A_0^{-1} \, \sigma(q_s^{\mu,\delta}) \, dw_s^{\delta} \right) \\ &- \left(\int_0^t \frac{1}{\alpha(q_s^{\mu,\delta})} A_0^{-1} \, b(q_s^{\mu,\delta}) \, ds - \int_0^t \frac{1}{\alpha(q_s^{\delta})} A_0^{-1} \, b(q_s^{\delta}) \, ds \right) \\ &- \left(\int_0^t \frac{1}{\alpha(q_s^{\mu,\delta})} A_0^{-1} \, \sigma(q_s^{\mu,\delta}) \, dw_s^{\delta} - \int_0^t \frac{1}{\alpha(q_s^{\delta})} A_0^{-1} \, \sigma(q_s^{\delta}) \, dw_s^{\delta} \right) \\ &=: I_1(t) + I_2(t) + I_3(t) + I_4(t) + I_5(t). \qquad (2.3.18) \end{aligned}$$

To get a bound of $q_t^{\mu,\delta} - q_t^{\delta}$, we will find bounds of the terms from $I_1(t)$ to $I_5(t)$. First, consider $I_1(t)$.

From (2.3.3), expressing $p_0 = \begin{pmatrix} p_0^1 \\ p_0^2 \end{pmatrix}$, we have

$$
\begin{aligned}
|I_1(t)| &= \left| \begin{pmatrix} \int_0^t \cos(\frac{\beta_s^{\mu,\delta}}{\mu}) ds\, p_0^1 - \int_0^t \sin(\frac{\beta_s^{\mu,\delta}}{\mu}) ds\, p_0^2 \\ \int_0^t \sin(\frac{\beta_s^{\mu,\delta}}{\mu}) ds\, p_0^1 + \int_0^t \cos(\frac{\beta_s^{\mu,\delta}}{\mu}) ds\, p_0^2 \end{pmatrix} \right| \\
&\le |p_0| \left(\left| \int_0^t \cos(\frac{\beta_s^{\mu,\delta}}{\mu}) ds \right| + \left| \int_0^t \sin(\frac{\beta_s^{\mu,\delta}}{\mu}) ds \right| \right) \\
&\le |p_0| \exp\left(\frac{C_1}{\delta}(1+t)(1+X_t) \right) \mu \\
&= \exp\left(\frac{C_2}{\delta}(1+t)(1+X_t) \right) \mu.
\end{aligned}
$$

In the last inequality, we used Lemma 2.3.2.

Now, let's consider $I_2(t)$. Note that the commutativity of A_0 and A_0^{-1} justifies the commutativity of matrix exponentials. Applying integration by parts,

$$
\begin{aligned}
&\frac{1}{\mu} \int_0^t \exp\left(\frac{\beta_s^{\mu,\delta}}{\mu} A_0 \right) \int_0^s \exp\left(-\frac{\beta_r^{\mu,\delta}}{\mu} A_0 \right) b(q_r^{\mu,\delta})\, dr\, ds \\
&= \int_0^t \frac{\alpha(q_s^{\mu,\delta})}{\mu} A_0 \exp\left(\frac{\beta_s^{\mu,\delta}}{\mu} A_0 \right) \frac{1}{\alpha(q_s^{\mu,\delta})} A_0^{-1} \int_0^s \exp\left(-\frac{\beta_r^{\mu,\delta}}{\mu} A_0 \right) b(q_r^{\mu,\delta})\, dr\, ds \\
&= \left[\exp\left(\frac{\beta_s^{\mu,\delta}}{\mu} A_0 \right) \frac{1}{\alpha(q_s^{\mu,\delta})} A_0^{-1} \int_0^s \exp\left(-\frac{\beta_r^{\mu,\delta}}{\mu} A_0 \right) b(q_r^{\mu,\delta})\, dr \right]_0^t \\
&\quad - \int_0^t \exp\left(\frac{\beta_s^{\mu,\delta}}{\mu} A_0 \right) \left(\frac{1}{\alpha(q_s^{\mu,\delta})} \right)' A_0^{-1} \int_0^s \exp\left(-\frac{\beta_r^{\mu,\delta}}{\mu} A_0 \right) b(q_r^{\mu,\delta})\, dr\, ds \\
&\quad - \int_0^t \frac{1}{\alpha(q_s^{\mu,\delta})} A_0^{-1} b(q_s^{\mu,\delta})\, ds \\
&= \exp\left(\frac{\beta_t^{\mu,\delta}}{\mu} A_0 \right) \frac{1}{\alpha(q_t^{\mu,\delta})} A_0^{-1} \int_0^t \exp\left(-\frac{\beta_s^{\mu,\delta}}{\mu} A_0 \right) b(q_s^{\mu,\delta})\, ds \\
&\quad - \int_0^t \exp\left(\frac{\beta_s^{\mu,\delta}}{\mu} A_0 \right) \frac{\nabla \alpha(q_s^{\mu,\delta}) \cdot p_s^{\mu,\delta}}{\alpha(q_s^{\mu,\delta})^2} A_0^{-1} \int_0^s \exp\left(-\frac{\beta_r^{\mu,\delta}}{\mu} A_0 \right) b(q_r^{\mu,\delta})\, dr\, ds \\
&\quad - \int_0^t \frac{1}{\alpha(q_s^{\mu,\delta})} A_0^{-1} b(q_s^{\mu,\delta})\, ds. \qquad (2.3.19)
\end{aligned}
$$

This yields

$$
\begin{aligned}
|I_2(t)| &= \Bigg| \exp\left(\frac{\beta_t^{\mu,\delta}}{\mu} A_0\right) \frac{1}{\alpha(q_t^{\mu,\delta})} A_0^{-1} \int_0^t \exp\left(-\frac{\beta_s^{\mu,\delta}}{\mu} A_0\right) b(q_s^{\mu,\delta})\, ds \\
&\qquad - \int_0^t \exp\left(\frac{\beta_s^{\mu,\delta}}{\mu} A_0\right) \frac{\nabla\alpha(q_s^{\mu,\delta}) \cdot p_s^{\mu,\delta}}{\alpha(q_s^{\mu,\delta})^2} A_0^{-1} \int_0^s \exp\left(-\frac{\beta_r^{\mu,\delta}}{\mu} A_0\right) b(q_r^{\mu,\delta})\, dr\, ds \Bigg| \\
&\leq \left| \exp\left(\frac{\beta_t^{\mu,\delta}}{\mu} A_0\right) \frac{1}{\alpha(q_t^{\mu,\delta})} A_0^{-1} \int_0^t \exp\left(-\frac{\beta_s^{\mu,\delta}}{\mu} A_0\right) b(q_s^{\mu,\delta})\, ds \right| \\
&\qquad + \left| \int_0^t \exp\left(\frac{\beta_s^{\mu,\delta}}{\mu} A_0\right) \frac{\nabla\alpha(q_s^{\mu,\delta}) \cdot p_s^{\mu,\delta}}{\alpha(q_s^{\mu,\delta})^2} A_0^{-1} \int_0^s \exp\left(-\frac{\beta_r^{\mu,\delta}}{\mu} A_0\right) b(q_r^{\mu,\delta})\, dr\, ds \right|.
\end{aligned}
$$

Considering (2.3.4) and Hypothesis 2,

$$
\begin{aligned}
|I_2(t)| &\leq \frac{1}{\alpha_0} \left| \int_0^t \exp\left(-\frac{\beta_s^{\mu,\delta}}{\mu} A_0\right) b(q_s^{\mu,\delta})\, ds \right| \\
&\qquad + \int_0^t \frac{|\nabla\alpha|_\infty}{\alpha_0^2} |p_s^{\mu,\delta}| \left| \int_0^s \exp\left(-\frac{\beta_r^{\mu,\delta}}{\mu} A_0\right) b(q_r^{\mu,\delta})\, dr \right| ds.
\end{aligned}
$$

Applying Lemma 2.3.2,

$$
\begin{aligned}
|I_2(t)| &\leq C_3 \exp\left(\frac{C_4}{\delta}(1+t)(1+X_t)\right) \mu \\
&\qquad + C_5 \int_0^t |p_s^{\mu,\delta}| \exp\left(\frac{C_4}{\delta}(1+s)(1+X_s)\right) \mu\, ds \\
&\leq \exp\left(\frac{C_6}{\delta}(1+t)(1+X_t)\right) \mu + \exp\left(\frac{C_6}{\delta}(1+t)(1+X_t)\right) \mu \int_0^t |p_s^{\mu,\delta}|\, ds.
\end{aligned}
$$

Note that by Lemma 2.3.1,

$$
\begin{aligned}
\int_0^t |p_s^{\mu,\delta}|\, ds &\leq \exp\left(\frac{C_7}{\delta}(1+t)(1+X_t)\right) t \\
&\leq \exp\left(\frac{C_8}{\delta}(1+t)(1+X_t)\right)
\end{aligned}
$$

and so,

$$
|I_2(t)| \leq \exp\left(\frac{C_9}{\delta}(1+t)(1+X_t)\right) \mu.
$$

We can apply a similar procedure as in getting the bound for $I_2(t)$ in the case of $I_3(t)$ and get the bound

$$|I_3(t)| \leq \exp\left(\frac{C_{10}}{\delta}(1+t)(1+X_t)\right)\mu.$$

Now, we find a bound of $I_4(t)$. From the expression of $I_4(t)$ in (2.3.18),

$$\begin{aligned}
|I_4(t)| &= \left|\int_0^t A_0^{-1}\left(\frac{1}{\alpha(q_s^{\mu,\delta})}b(q_s^{\mu,\delta}) - \frac{1}{\alpha(q_s^{\delta})}b(q_s^{\delta})\right)ds\right| \\
&= \left|\int_0^t A_0^{-1}\left(\frac{b(q_s^{\mu,\delta})\alpha(q_s^{\delta}) - b(q_s^{\delta})\alpha(q_s^{\mu,\delta})}{\alpha(q_s^{\delta})\alpha(q_s^{\mu,\delta})}\right)ds\right| \\
&\leq \int_0^t \left|\frac{b(q_s^{\mu,\delta})\alpha(q_s^{\delta}) - b(q_s^{\delta})\alpha(q_s^{\mu,\delta})}{\alpha(q_s^{\delta})\alpha(q_s^{\mu,\delta})}\right| ds \\
&\leq \frac{1}{\alpha_0^2}\int_0^t |b(q_s^{\mu,\delta})\alpha(q_s^{\delta}) - b(q_s^{\delta})\alpha(q_s^{\delta})| + |b(q_s^{\delta})\alpha(q_s^{\delta}) - b(q_s^{\delta})\alpha(q_s^{\mu,\delta})|ds \\
&\leq \frac{1}{\alpha_0^2}\int_0^t |\alpha|_\infty|b(q_s^{\mu,\delta}) - b(q_s^{\delta})| + |b|_\infty|\alpha(q_s^{\delta}) - \alpha(q_s^{\mu,\delta})|ds \\
&\leq \frac{1}{\alpha_0^2}\int_0^t |\alpha|_\infty|Db|_\infty|q_s^{\mu,\delta} - q_s^{\delta}| + |b|_\infty|\nabla\alpha|_\infty|q_s^{\delta} - q_s^{\mu,\delta}|ds \\
&= C_{11}\int_0^t |q_s^{\mu,\delta} - q_s^{\delta}|ds.
\end{aligned}$$

By a similar method, a bound for $I_5(t)$ can also be found. We have

$$|I_5(t)| \leq \frac{C_{12}}{\delta}X_t\int_0^t |q_s^{\mu,\delta} - q_s^{\delta}|ds.$$

Combining these results and applying the bounds of $I_1(t)$ to $I_5(t)$ to (2.3.18), we obtain

$$\begin{aligned}
|q_t^{\mu,\delta} - q_t^{\delta}| &\leq \exp\left(\frac{C}{\delta}(1+t)(1+X_t)\right)\mu + \exp\left(\frac{C}{\delta}(1+t)(1+X_t)\right)\mu \\
&\qquad + \exp\left(\frac{C}{\delta}(1+t)(1+X_t)\right)\mu + C\int_0^t |q_s^{\mu,\delta} - q_s^{\delta}|ds \\
&\qquad + \frac{C}{\delta}X_t\int_0^t |q_s^{\mu,\delta} - q_s^{\delta}|ds \\
&\leq \exp\left(\frac{C}{\delta}(1+t)(1+X_t)\right)\mu + \frac{C}{\delta}(1+X_t)\int_0^t |q_s^{\mu,\delta} - q_s^{\delta}|ds.
\end{aligned}$$

Then, from the Gronwall's lemma, we can conclude

$$
\begin{aligned}
|q_t^{\mu,\delta} - q_t^{\delta}| &\le \exp\left(\frac{C}{\delta}(1+t)(1+X_t)\right)\mu \exp\left(\frac{C}{\delta}(1+X_t)t\right) \\
&\le \exp\left(\frac{C}{\delta}(1+t)(1+X_t)\right)\mu.
\end{aligned}
$$

This gives

$$
\max_{0\le t\le T} |q_t^{\mu,\delta} - q_t^{\delta}| \le \exp\left(\frac{C}{\delta}(1+T)(1+X_T)\right)\mu,
$$

so that

$$
\max_{0\le t\le T} |q_t^{\mu,\delta} - q_t^{\delta}|^k \le \exp\left(\frac{C}{\delta}(1+T)(1+X_T)\right)\mu^k.
$$

By taking expectation and applying Lemma 2.3.3,

$$
\begin{aligned}
E \max_{0\le t\le T} |q_t^{\mu,\delta} - q_t^{\delta}|^k &\le E\left[\exp\left(\frac{C}{\delta}(1+T)(1+X_T)\right)\right]\mu^k \\
&\le 16\exp\left((1+T)\left(\frac{C}{\delta}(1+T)\right)^2 + \frac{C}{\delta}(1+T)\right)\mu^k \\
&\le \exp\left(\frac{C}{\delta^2}(1+T)^3\right)\mu^k.
\end{aligned}
$$

$\square$

2.4 Homogenization

In this section, we consider the case of a fast oscillating periodic magnetic field. Consider the solution $q_t^{\mu,\delta,\epsilon}$ of

$$
\begin{cases}
\mu\, \ddot{q}_t^{\mu,\delta,\epsilon} = b(q_t^{\mu,\delta,\epsilon}) + \alpha\left(\frac{q_t^{\mu,\delta,\epsilon}}{\epsilon}\right) A_0 \dot{q}_t^{\mu,\delta,\epsilon} + \dot{w}_t^{\delta} \\
q_0^{\mu,\delta,\epsilon} = q_0 \in \mathbb{R}^2, \quad \dot{q}_0^{\mu,\delta,\epsilon} = p_0 \in \mathbb{R}^2,
\end{cases}
$$

where $\alpha : \mathbb{R}^2 \to \mathbb{R}$ is a 1-periodic function and $\epsilon > 0$ is a constant. By periodicity of α, we can consider the domain of α as $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, the two dimensional unit torus. In this case, a unique weak limit of the process $q_t^{\mu,\delta,\epsilon}$ as $\mu \downarrow 0$, $\delta \downarrow 0$, and $\epsilon \downarrow 0$ in order exists and we find this limit by applying homogenization results in the literature [2–5, 11, 12] to our system. Note that we solve for $\sigma(q) \equiv I$ for computational convenience. In general, if $\sigma(q)\sigma(q)^*$ is positive definite for all $q \in \mathbb{R}^2$, we can find a weak limit. For the proof of homogenization results, we need more restrictive assumptions than Hypothesis 2.

Hypothesis 3.

1. *$b : \mathbb{R}^2 \to \mathbb{R}^2$ is twice continuously differentiable and bounded with its derivatives.*

2. *$\alpha : \mathbb{R}^2 \to \mathbb{R}$ is twice continuously differentiable and bounded with its derivatives. Moreover,*

$$\inf_{q\in\mathbb{R}^2} \alpha(q) = \alpha_0 > 0.$$

Proposition 2.4.1. *Under Hypothesis 3, $q_t^{\mu,\delta,\epsilon}$ converges to $\hat{q}_t$ weakly as $\mu \downarrow 0$, $\delta \downarrow 0$, and $\epsilon \downarrow 0$ in order, where $\hat{q}_t$ solves*

$$\begin{cases} \dot{\hat{q}}_t = \hat{b}(\hat{q}_t) + \hat{\sigma}\dot{w}_t \\ \hat{q}_0 = q_0 \in \mathbb{R}^2. \end{cases}$$

Here,

$$\hat{b}(q) = \left(\frac{1}{\int_{\mathbb{T}^2} \alpha(q)dq} \int_{\mathbb{T}^2} (I - D\chi(q))\, dq\, A_0 \right) b(q)$$

and

$$\hat{\sigma}\hat{\sigma}^* = \frac{1}{\int_{\mathbb{T}^2} \alpha(q)dq} \int_{\mathbb{T}^2} \frac{1}{\alpha(q)} (I - D\chi(q)) (I - D\chi(q))^* dq$$

with $\chi(q) = (\chi^1(q), \chi^2(q))$ *solving*

$$L\chi^i(q) = -\frac{1}{2\alpha^3(q)}\frac{\partial \alpha}{\partial q^i}(q),$$

where L *is the operator*

$$L = \frac{1}{2}\frac{1}{\alpha^2(q)}\Delta_q - \frac{1}{2}\frac{\nabla\alpha(q)}{\alpha^3(q)} \cdot \nabla_q.$$

Proof. By Corollary 2.2.3, as $\mu \downarrow 0$ first and $\delta \downarrow 0$, $q_t^{\mu,\delta,\epsilon} \to \hat{q}_t^\epsilon$ in probability in $C([0,T];\mathbb{R}^2)$, where $\hat{q}_t^\epsilon$ solves

$$\begin{cases} \dot{\hat{q}}_t^\epsilon = -\frac{1}{\alpha\left(\frac{\hat{q}_t^\epsilon}{\epsilon}\right)} A_0^{-1} b(\hat{q}_t^\epsilon) - \frac{1}{\alpha\left(\frac{\hat{q}_t^\epsilon}{\epsilon}\right)} A_0^{-1} \circ \dot{w}_t \\ \hat{q}_0^\epsilon = q_0 \in \mathbb{R}^2. \end{cases} \tag{2.4.1}$$

Considering

$$A_0^{-1} = -A_0$$

from the definition of A_0, writing (2.4.1) in Itô integral, we get

$$\dot{\hat{q}}_t^\epsilon = \frac{1}{\alpha\left(\frac{\hat{q}_t^\epsilon}{\epsilon}\right)}\tilde{b}(\hat{q}_t^\epsilon) - \frac{1}{2\epsilon}\frac{\nabla\alpha\left(\frac{\hat{q}_t^\epsilon}{\epsilon}\right)}{\alpha^3\left(\frac{\hat{q}_t^\epsilon}{\epsilon}\right)} + \frac{1}{\alpha\left(\frac{\hat{q}_t^\epsilon}{\epsilon}\right)}\dot{\tilde{w}}_t,$$

where

$$\tilde{b}(q) := A_0 b(q)$$

and

$$\tilde{w}_t := A_0 w_t.$$

Note that $\tilde{w}_t$ is also a Wiener process in $\mathbb{R}^2$.

Under Hypothesis 3, we can apply [12, Theorem 6.1, Chapter 3] to $\hat{q}_t^\epsilon$.

The normalized solution $m(q)$ of the adjoint equation $L^* m(q) = 0$ can be found as

$$m(q) = \frac{1}{\int_{\mathbb{T}^2}\alpha(q)dq}\alpha(q)$$

as in [5] and the statement of the proposition follows. $\square$

Chapter 3: A Generalization of the Smoluchowski-Kramers Approximation in the Case of Linear Differential Operators with Constant Coefficients

3.1 Introduction

In this chapter, we consider another generalization of the results in Chapter 1. We may consider a Smoluchowski-Kramers type approximation for general differential operators. Then we can reduce the problem to a simpler one.

Consider the equation

$$\mu A q_t^\mu + B q_t^\mu = b(q_t^\mu) + \sigma(q_t^\mu)\dot{w}_t$$

with A and B differential operators. Our ultimate goal is to find conditions on A and B such that the solution q_t^μ of the equation converges to the solution q_t of

$$B q_t = b(q_t) + \sigma(q_t)\dot{w}_t$$

as $\mu \downarrow 0$. In this chapter, we show that in the case of A and B linear differential operators with constant coefficients, the Smoluchowski-Kramers approximation works.

3.2 Main Result

We consider the case $A = \sum_{i=0}^{n} a_i \frac{d^i}{dt^i}$ and $B = \sum_{i=0}^{n-1} b_i \frac{d^i}{dt^i}$, where $n \geq 2$ and a_i's and b_i's are real numbers such that a_n and b_{n-1} are nonzero and of the same sign.

Let q_t^μ and q_t be the solutions of the following equations respectively :

$$\begin{cases} \mu \sum_{i=0}^{n} a_i \frac{d^i}{dt^i} q_t^\mu + \sum_{i=0}^{n-1} b_i \frac{d^i}{dt^i} q_t^\mu = b(q_t^\mu) + \sigma(q_t^\mu)\dot{w}_t \\ q_0^{\mu,(i)} = q_0^{(i)} \in \mathbb{R}^d, \qquad 0 \leq i \leq n-1 \end{cases} \tag{3.2.1}$$

and

$$\begin{cases} \sum_{i=0}^{n-1} b_i \frac{d^i}{dt^i} q_t = b(q_t) + \sigma(q_t)\dot{w}_i \\ q_0^{(i)} \in \mathbb{R}^d, \qquad 0 \leq i \leq n-2, \end{cases} \tag{3.2.2}$$

where $\mu > 0$, $q_\cdot^\mu : \mathbb{R} \to \mathbb{R}^d$, $b : \mathbb{R}^d \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \to M_d(\mathbb{R})$, and w_t is a d-dimensional Wiener process. Then, the following theorem holds.

Theorem 3.2.1. *Suppose $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to M_d(\mathbb{R})$ are Lipschitz continuous and $0 < \mu \leq 1$. Then,*

$$\lim_{\mu \downarrow 0} \sum_{i=0}^{n-2} \mathbb{E}\left(\max_{0 \leq t \leq T} \left| q_t^{\mu,(i)} - q_t^{(i)} \right|^k \right) = 0$$

for $k \geq 1$.

3.3 Proof of Theorem 3.2.1

We show the identity for $k > 2$. Then, it is a trivial consequence that it holds for $k \geq 1$. Note that we may assume that $a_n \equiv 1$ and $b_{n-1} \equiv 1$ without loss of generality: we may divide (3.2.1) by b_{n-1} and redefine $\mu' = \mu\frac{a_n}{b_{n-1}}$, $b'(q) = \frac{b(q)}{b_{n-1}}$, and $\sigma'(q) = \frac{\sigma(q)}{b_{n-1}}$.

Let $q_t^{\mu,n}$ and q_t^n be the nth derivatives of q_t^μ and q_t in time and $q_t^{\mu,-n}$ and q_t^{-n} be the nth integrals of q_t^μ and q_t in time. Then, we may rewrite (3.2.1) and (3.2.2) as the systems of n first order differential equations and $n-1$ first order differential equations :

$$\begin{cases} \dot{q}_t^{\mu,0} = q_t^{\mu,1} \\ \dot{q}_t^{\mu,1} = q_t^{\mu,2} \\ \dots \\ \dot{q}_t^{\mu,n-2} = q_t^{\mu,n-1} \\ \dot{q}_t^{\mu,n-1} + \left(\frac{1}{\mu} + a_{n-1}\right) q_t^{\mu,n-1} = -\sum_{i=0}^{n-2} a_i q_t^{\mu,i} - \frac{1}{\mu}\sum_{i=0}^{n-2} b_i q_t^{\mu,i} + \frac{1}{\mu} b(q_t^{\mu,0}) + \frac{1}{\mu}\sigma(q_t^{\mu,0})\dot{w}_t \end{cases} \tag{3.3.1}$$

and

$$\begin{cases} \dot{q}_t^0 = q_t^1 \\ \dot{q}_t^1 = q_t^2 \\ \dots \\ \dot{q}_t^{n-3} = q_t^{n-2} \\ \dot{q}_t^{n-2} = -\sum_{i=0}^{n-2} b_i q_t^i + b(q_t^0) + \sigma(q_t^0)\dot{w}_t. \end{cases} \tag{3.3.2}$$

Define μ_1 as

$$\frac{1}{\mu_1} := \frac{1}{\mu} + a_{n-1}.$$

Then, since

$$\lim_{\mu\downarrow 0} \frac{\mu_1}{\mu} = 1,$$

without loss of generality, we may assume that $\mu_1 > 0$ and

$$\mu_1 < C\mu$$

for some $C > 0$.

By the definition of μ_1, we also note that there exists a constant $C > 0$ such that

$$|\mu_1 - \mu| \le C\mu^2 \tag{3.3.3}$$

for small μ.

Multiplying the last equation of (3.3.1) by $e^{\frac{t}{\mu_1}}$,

$$(e^{\frac{t}{\mu_1}} q_t^{\mu,n-1})' = -\sum_{i=0}^{n-2} a_i e^{\frac{t}{\mu_1}} q_t^{\mu,i} - \frac{1}{\mu}\sum_{i=0}^{n-2} b_i e^{\frac{t}{\mu_1}} q_t^{\mu,i} + \frac{1}{\mu} e^{\frac{t}{\mu_1}} b(q_t^{\mu,0}) + \frac{1}{\mu} e^{\frac{t}{\mu_1}} \sigma(q_t^{\mu,0})\dot{w}_t.$$

Integrating with respect to t and multiplying by $e^{-\frac{t}{\mu_1}}$,

$$\begin{aligned} q_t^{\mu,n-1} = e^{-\frac{t}{\mu_1}} q_0^{n-1} &- \sum_{i=0}^{n-2} a_i e^{-\frac{t}{\mu_1}} \int_0^t e^{\frac{s}{\mu_1}} q_s^{\mu,i} ds - \frac{1}{\mu}\sum_{i=0}^{n-2} b_i e^{-\frac{t}{\mu_1}} \int_0^t e^{\frac{s}{\mu_1}} q_s^{\mu,i} ds \\ &+ \frac{1}{\mu} e^{-\frac{t}{\mu_1}} \int_0^t e^{\frac{s}{\mu_1}} b(q_s^{\mu,0}) ds + \frac{1}{\mu} e^{-\frac{t}{\mu_1}} \int_0^t e^{\frac{s}{\mu_1}} \sigma(q_s^{\mu,0}) dw_s. \end{aligned}$$

Integrating with respect to t one more time,

$$\begin{aligned} q_t^{\mu,n-2} = q_0^{n-2} &+ \int_0^t e^{-\frac{s}{\mu_1}} ds\, q_0^{n-1} - \sum_{i=0}^{n-2} a_i \int_0^t e^{-\frac{s}{\mu_1}} \int_0^s e^{\frac{r}{\mu_1}} q_r^{\mu,i} dr\, ds \\ &- \frac{1}{\mu}\sum_{i=0}^{n-2} b_i \int_0^t e^{-\frac{s}{\mu_1}} \int_0^s e^{\frac{r}{\mu_1}} q_r^{\mu,i} dr\, ds + \frac{1}{\mu}\int_0^t e^{-\frac{s}{\mu_1}} \int_0^s e^{\frac{r}{\mu_1}} b(q_r^{\mu,0}) dr\, ds \\ &+ \frac{1}{\mu}\int_0^t e^{-\frac{s}{\mu_1}} \int_0^s e^{\frac{r}{\mu_1}} \sigma(q_r^{\mu,0}) dw_r ds. \end{aligned} \tag{3.3.4}$$

Integrating the last equation of (3.3.2),

$$q_t^{n-2} = q_0^{n-2} - \sum_{i=0}^{n-2} b_i \int_0^t q_s^i ds + \int_0^t b(q_s^0) ds + \int_0^t \sigma(q_s^0) dw_s. \tag{3.3.5}$$

Subtracting (3.3.5) from (3.3.4), we get

$$
\begin{aligned}
q_t^{\mu,n-2} - q_t^{n-2} = & \int_0^t e^{-\frac{s}{\mu_1}} ds \, q_0^{n-1} \\
& - \sum_{i=0}^{n-2} a_i \int_0^t e^{-\frac{s}{\mu_1}} \int_0^s e^{\frac{r}{\mu_1}} q_r^{\mu,i} dr \, ds \\
& - \left(\frac{1}{\mu} \sum_{i=0}^{n-2} b_i \int_0^t e^{-\frac{s}{\mu_1}} \int_0^s e^{\frac{r}{\mu_1}} q_r^{\mu,i} dr \, ds - \sum_{i=0}^{n-2} b_i \int_0^t q_s^i ds \right) \\
& + \left(\frac{1}{\mu} \int_0^t e^{-\frac{s}{\mu_1}} \int_0^s e^{\frac{r}{\mu_1}} b(q_r^{\mu,0}) dr \, ds - \int_0^t b(q_s^0) ds \right) \\
& + \left(\frac{1}{\mu} \int_0^t e^{-\frac{s}{\mu_1}} \int_0^s e^{\frac{r}{\mu_1}} \sigma(q_r^{\mu,0}) dw_r ds - \int_0^t \sigma(q_s^0) dw_s \right) \\
=: & \, I_t^1 + I_t^2 + I_t^3 + I_t^4 + I_t^5.
\end{aligned} \tag{3.3.6}
$$

A bound for I_t^1 can easily be found:

$$
\begin{aligned}
\max_{0 \le s \le t} |I_s^1|^k &= \max_{0 \le s \le t} \left| \mu_1 \left(1 - e^{-\frac{s}{\mu_1}} \right) q_0^{n-1} \right|^k \\
&\le \mu_1^k \left| q_0^{n-1} \right|^k \\
&\le C_1 \mu^k.
\end{aligned}
$$

Considering I_t^2, by integration by parts,

$$
\begin{aligned}
|I_t^2| &\le C_2 \sum_{i=0}^{n-2} \left| \int_0^t e^{-\frac{s}{\mu_1}} \int_0^s e^{\frac{r}{\mu_1}} q_r^{\mu,i} dr \, ds \right| \\
&= C_2 \sum_{i=0}^{n-2} \left| -\mu_1 e^{-\frac{t}{\mu_1}} \int_0^t e^{\frac{s}{\mu_1}} q_s^{\mu,i} ds + \mu_1 \int_0^t q_s^{\mu,i} ds \right| \\
&\le C_3 \mu \sum_{i=0}^{n-2} \left| \int_0^t \left(1 - e^{-\frac{t-s}{\mu_1}} \right) q_s^{\mu,i} ds \right| \\
&\le C_3 \mu \sum_{i=0}^{n-2} \int_0^t \left| q_s^{\mu,i} \right| ds \\
&\le C_3 t \mu \sum_{i=0}^{n-2} \max_{0 \le s \le t} \left| q_s^{\mu,i} \right|.
\end{aligned}
$$

So,

$$\mathbb{E}\left(\max_{0\le s\le t}|I_s^2|^k\right) \le C_4 t^k \mu^k \sum_{i=0}^{n-2} \mathbb{E}\left(\max_{0\le s\le t}\left|q_s^{\mu,i}\right|^k\right).$$

A bound of I_t^3 can also be found after applying integration by parts:

$$\begin{aligned}
|I_t^3| &= \left|\sum_{i=0}^{n-2} b_i \left(\frac{1}{\mu}\int_0^t e^{-\frac{s}{\mu_1}}\int_0^s e^{\frac{r}{\mu_1}} q_r^{\mu,i} dr\, ds - \int_0^t q_s^i ds\right)\right| \\
&= \left|\sum_{i=0}^{n-2} b_i \left(-\frac{\mu_1}{\mu} e^{-\frac{t}{\mu_1}}\int_0^t e^{\frac{s}{\mu_1}} q_s^{\mu,i} ds + \frac{\mu_1}{\mu}\int_0^t q_s^{\mu,i} ds - \int_0^t q_s^i ds\right)\right| \\
&= \left|\sum_{i=0}^{n-2} b_i \left(-\frac{\mu_1}{\mu} e^{-\frac{t}{\mu_1}}\int_0^t e^{\frac{s}{\mu_1}} q_s^{\mu,i} ds + \frac{\mu_1-\mu}{\mu}\int_0^t q_s^{\mu,i} ds + \int_0^t q_s^{\mu,i} - q_s^i ds\right)\right| \\
&\le C_5 \sum_{i=0}^{n-2}\int_0^t e^{-\frac{t-s}{\mu_1}}\left|q_s^{\mu,i}\right| ds + C_5\mu \sum_{i=0}^{n-2}\int_0^t \left|q_s^{\mu,i}\right| ds + C_5 \sum_{i=0}^{n-2}\int_0^t \left|q_s^{\mu,i} - q_s^i\right| ds.
\end{aligned}$$

The last inequality follows from (3.3.3).

So,

$$\begin{aligned}
\mathbb{E}\left(\max_{0\le s\le t}|I_s^3|^k\right) &\le C_6 \sum_{i=0}^{n-2}\left(\int_0^t e^{-\frac{t-s}{\mu_1}} ds\right)^k \mathbb{E}\left(\max_{0\le s\le t}\left|q_s^{\mu,i}\right|^k\right) \\
&\quad + C_6 t\mu^k \sum_{i=0}^{n-2}\mathbb{E}\left(\max_{0\le s\le t}\left|q_s^{\mu,i}\right|^k\right) \\
&\quad + C_6 t \sum_{i=0}^{n-2}\int_0^t \mathbb{E}\left(\max_{0\le r\le s}\left|q_r^{\mu,i} - q_r^i\right|^k\right) ds \\
&\le C_7\mu^k \sum_{i=0}^{n-2}\mathbb{E}\left(\max_{0\le s\le t}\left|q_s^{\mu,i}\right|^k\right) \\
&\quad + C_7 \sum_{i=0}^{n-2}\int_0^t \mathbb{E}\left(\max_{0\le r\le s}\left|q_r^{\mu,i} - q_r^i\right|^k\right) ds.
\end{aligned}$$

Considering I_t^4,

$$
\begin{aligned}
|I_t^4| &= \left| \frac{1}{\mu} \int_0^t e^{-\frac{s}{\mu_1}} \int_0^s e^{\frac{r}{\mu_1}} b(q_r^{\mu,0}) dr\, ds - \int_0^t b(q_s^0) ds \right| \\
&= \left| -\frac{\mu_1}{\mu} e^{-\frac{t}{\mu_1}} \int_0^t e^{\frac{s}{\mu_1}} b(q_s^{\mu,0}) ds + \frac{\mu_1 - \mu}{\mu} \int_0^t b(q_s^{\mu,0}) ds + \int_0^t b(q_s^{\mu,0}) - b(q_s^0) ds \right| \\
&\leq C_8 \left| \int_0^t e^{-\frac{t-s}{\mu_1}} b(q_s^{\mu,0}) ds \right| + C_8 \mu \left| \int_0^t b(q_s^{\mu,0}) ds \right| + C_8 \left| \int_0^t b(q_s^{\mu,0}) - b(q_s^0) ds \right| \\
&\leq C_8 \int_0^t e^{-\frac{t-s}{\mu_1}} \left| b(q_s^{\mu,0}) \right| ds + C_8 \mu \int_0^t \left| b(q_s^{\mu,0}) \right| ds + C_8 \int_0^t \left| b(q_s^{\mu,0}) - b(q_s^0) \right| ds \\
&\leq C_9 \int_0^t e^{-\frac{t-s}{\mu_1}} \left(1 + \left| q_s^{\mu,0} \right| \right) ds + C_9 \mu \int_0^t 1 + \left| q_s^{\mu,0} \right| ds \\
&\quad + C_9 \int_0^t \left| q_s^{\mu,0} - q_s^0 \right| ds.
\end{aligned}
$$

This leads to

$$
\begin{aligned}
\mathbb{E} \left(\max_{0 \leq s \leq t} |I_s^4|^k \right) \leq C_{10} \mu^k + C_{10} \mu^k \mathbb{E} \left(\max_{0 \leq s \leq t} \left| q_s^{\mu,0} \right|^k \right) \\
+ C_{10} \int_0^t \mathbb{E} \left(\max_{0 \leq r \leq s} \left| q_r^{\mu,0} - q_r^0 \right|^k \right) ds.
\end{aligned}
$$

The bound for I_t^5 can also be found similarly.

$$
\begin{aligned}
|I_t^5| &= \left| \frac{1}{\mu} \int_0^t e^{-\frac{s}{\mu_1}} \int_0^s e^{\frac{r}{\mu_1}} \sigma(q_r^{\mu,0}) dw_r\, ds - \int_0^t \sigma(q_s^0) dw_s \right| \\
&= \left| -\frac{\mu_1}{\mu} e^{-\frac{t}{\mu_1}} \int_0^t e^{\frac{s}{\mu_1}} \sigma(q_s^{\mu,0}) dw_s + \frac{\mu_1 - \mu}{\mu} \int_0^t \sigma(q_s^{\mu,0}) dw_s + \int_0^t \sigma(q_s^{\mu,0}) - \sigma(q_s^0) dw_s \right| \\
&\leq C_{11} \left| \int_0^t e^{-\frac{t-s}{\mu_1}} \sigma(q_s^{\mu,0}) dw_s \right| + C_{11} \mu \left| \int_0^t \sigma(q_s^{\mu,0}) dw_s \right| \\
&\quad + C_{11} \left| \int_0^t \sigma(q_s^{\mu,0}) - \sigma(q_s^0) dw_s \right|.
\end{aligned}
$$

Using the Burkholder-Davis-Gundy inequality in addition to the above technique,

$$
\begin{aligned}
\mathbb{E}\left(\max_{0\le s\le t}|I_s^5|^k\right) &\le C_{12}\mathbb{E}\left(\max_{0\le s\le t}\left|\int_0^s e^{-\frac{s-r}{\mu_1}}\sigma(q_r^{\mu,0})dw_r\right|^k\right)\\
&\quad + C_{12}\mu^k\mathbb{E}\left(\max_{0\le s\le t}\left|\int_0^s \sigma(q_r^{\mu,0})dw_r\right|^k\right)\\
&\quad + C_{12}\mathbb{E}\left(\max_{0\le s\le t}\left|\int_0^s \sigma(q_r^{\mu,0})-\sigma(q_r^{0})dw_r\right|^k\right)\\
&\le C_{13}\mathbb{E}\left(\int_0^t\left|e^{-\frac{t-s}{\mu_1}}\sigma(q_s^{\mu,0})\right|^2 ds\right)^{\frac{k}{2}} + C_{13}\mu^k\mathbb{E}\left(\int_0^t\left|\sigma(q_s^{\mu,0})\right|^2 ds\right)^{\frac{k}{2}}\\
&\quad + C_{13}\mathbb{E}\left(\int_0^t\left|\sigma(q_s^{\mu,0})-\sigma(q_s^{0})\right|^2 ds\right)^{\frac{k}{2}}\\
&\le C_{14}\mathbb{E}\left[\left(\int_0^t\left|e^{-\frac{t-s}{\mu_1}}\right|^{\frac{2k}{k-2}} ds\right)^{k-2}\left(\int_0^t\left|\sigma(q_s^{\mu,0})\right|^k ds\right)\right]\\
&\quad + C_{14}\mu^k\mathbb{E}\left(\int_0^t\left|\sigma(q_s^{\mu,0})\right|^k ds\right)\\
&\quad + C_{14}\mathbb{E}\left(\int_0^t\left|\sigma(q_s^{\mu,0})-\sigma(q_s^{0})\right|^k ds\right)\\
&\le C_{15}\mu^{k-2}\mathbb{E}\left(\int_0^t 1+\left|q_s^{\mu,0}\right|^k ds\right) + C_{15}\mu^k\mathbb{E}\left(\int_0^t 1+\left|q_s^{\mu,0}\right|^k ds\right)\\
&\quad + C\mathbb{E}\left(\int_0^t\left|q_s^{\mu,0}-q_s^{0}\right|^k ds\right)\\
&\le C_{16}\mu^{k-2} + C_{16}\mu^{k-2}\mathbb{E}\left(\max_{0\le s\le t}\left|q_s^{\mu,0}\right|^k\right)\\
&\quad + C_{16}\int_0^t\mathbb{E}\left(\max_{0\le r\le s}\left|q_r^{\mu,0}-q_r^{0}\right|^k\right)ds.
\end{aligned}
$$

Applying these bounds to (3.3.6), we get

$$
\begin{aligned}
E\left(\max_{0\le s\le t}\left|q_s^{\mu,n-2}-q_s^{n-2}\right|^k\right) &\le C_{17}\mu^{k-2}\left(1+\sum_{i=0}^{n-2}E\left(\max_{0\le s\le t}\left|q_s^{\mu,i}\right|^k\right)\right)\\
&\quad + C_{17}\sum_{i=0}^{n-2}\int_0^t E\left(\max_{0\le r\le s}\left|q_r^{\mu,i}-q_r^{i}\right|^k\right)ds. \qquad (3.3.7)
\end{aligned}
$$

Our next goal will be finding a bound of

$$
\sum_{i=0}^{n-2}E\left(\max_{0\le s\le t}\left|q_s^{\mu,i}\right|^k\right)
$$

independent of μ.

For $0 \leq i \leq n-3$, we can easily see that

$$\begin{aligned}
\left|q_t^{\mu,i}\right|^k &= \left|q_0^i + \int_0^t q_s^{\mu,i+1} ds\right|^k \\
&\leq C_{18} \left|q_0^i\right|^k + C_{18} \left|\int_0^t q_s^{\mu,i+1} ds\right|^k \\
&\leq C_{19} + C_{19} \int_0^t \left|q_s^{\mu,i+1}\right|^k ds.
\end{aligned}$$

So,

$$E\left(\max_{0\leq s\leq t} \left|q_s^{\mu,i}\right|^k\right) \leq C_{19} + C_{19} \int_0^t E\left(\max_{0\leq r\leq s} \left|q_r^{\mu,i+1}\right|^k\right) ds. \tag{3.3.8}$$

For $i = n-2$, from (3.3.4),

$$
\begin{aligned}
\left|q_t^{\mu,n-2}\right| \leq & \left|q_0^{n-2}\right| + C_{20}\left|\int_0^t e^{-\frac{s}{\mu_1}} ds\, q_0^{n-1}\right| + \sum_{i=0}^{n-2}\left|\int_0^t e^{-\frac{s}{\mu_1}} \int_0^s e^{\frac{r}{\mu_1}} q_r^{\mu,i} dr\, ds\right| \\
& + C_{20}\sum_{i=0}^{n-2}\left|\frac{1}{\mu}\int_0^t e^{-\frac{s}{\mu_1}} \int_0^s e^{\frac{r}{\mu_1}} q_r^{\mu,i} dr\, ds\right| + \left|\frac{1}{\mu}\int_0^t e^{-\frac{s}{\mu_1}} \int_0^s e^{\frac{r}{\mu_1}} b(q_r^{\mu,0}) dr\, ds\right| \\
& + \left|\frac{1}{\mu}\int_0^t e^{-\frac{s}{\mu_1}} \int_0^s e^{\frac{r}{\mu_1}} \sigma(q_r^{\mu,0}) dw_r ds\right| \\
\leq & C_{21} + C_{21}\mu + C_{21}\sum_{i=0}^{n-2}\left|-\mu_1 e^{-\frac{t}{\mu_1}} \int_0^t e^{\frac{s}{\mu_1}} q_s^{\mu,i} ds + \mu_1 \int_0^t q_s^{\mu,i} ds\right| \\
& + C_{21}\sum_{i=0}^{n-2}\left|-\frac{\mu_1}{\mu} e^{-\frac{t}{\mu_1}} \int_0^t e^{\frac{s}{\mu_1}} q_s^{\mu,i} ds + \frac{\mu_1}{\mu}\int_0^t q_s^{\mu,i} ds\right| \\
& + \left|-\frac{\mu_1}{\mu} e^{-\frac{t}{\mu_1}} \int_0^t e^{\frac{s}{\mu_1}} b(q_s^{\mu,0}) ds + \frac{\mu_1}{\mu}\int_0^t b(q_s^{\mu,0}) ds\right| \\
& + \left|-\frac{\mu_1}{\mu} e^{-\frac{t}{\mu_1}} \int_0^t e^{\frac{s}{\mu_1}} \sigma(q_s^{\mu,0}) dw_s + \frac{\mu_1}{\mu}\int_0^t \sigma(q_s^{\mu,0}) dw_s\right| \\
\leq & C_{21} + C_{21}\mu + C_{22}\mu\sum_{i=0}^{n-2}\left(\int_0^t \left(e^{-\frac{t-s}{\mu_1}} + 1\right)\left|q_s^{\mu,i}\right| ds\right) \\
& + C_{22}\sum_{i=0}^{n-2}\left(\int_0^t \left(e^{-\frac{t-s}{\mu_1}} + 1\right)\left|q_s^{\mu,i}\right| ds\right) \\
& + C_{22}\left(\int_0^t \left(e^{-\frac{t-s}{\mu_1}} + 1\right)\left|b(q_s^{\mu,0})\right| ds\right) \\
& + C_{22}\left(\left|\int_0^t \left(e^{-\frac{t-s}{\mu_1}} + 1\right)\sigma(q_s^{\mu,0}) dw_s\right|\right) \\
\leq & C_{23} + C_{23}\sum_{i=0}^{n-2}\left(\int_0^t \left(e^{-\frac{t-s}{\mu_1}} + 1\right)\left|q_s^{\mu,i}\right| ds\right) \\
& + C_{22}\left(\int_0^t \left(e^{-\frac{t-s}{\mu_1}} + 1\right)\left|b(q_s^{\mu,0})\right| ds\right) \\
& + C_{22}\left(\left|\int_0^t \left(e^{-\frac{t-s}{\mu_1}} + 1\right)\sigma(q_s^{\mu,0}) dw_s\right|\right).
\end{aligned}
$$

Thanks to Burkholder-Davis-Gundy inequality and the Lipschitz continuity of

b and σ,

$$\begin{aligned}
E\left(\max_{0\le s\le t}\left|q_s^{\mu,n-2}\right|^k\right) &\le C_{24} + C_{24}\sum_{i=0}^{n-2}\left(\int_0^t \max_{0\le r\le s}\left|q_r^{\mu,i}\right|^k ds\right)\\
&\quad + C_{24}\int_0^t \max_{0\le r\le s}\left|b(q_r^{\mu,0})\right|^k ds + C_{24}\int_0^t \left|\sigma(q_s^{\mu,0})\right|^k ds\\
&\le C_{24} + C_{24}\sum_{i=0}^{n-2}\left(\int_0^t \max_{0\le r\le s}\left|q_r^{\mu,i}\right|^k ds\right)\\
&\quad + C_{25}\int_0^t 1 + \max_{0\le r\le s}\left|q_r^{\mu,0}\right|^k ds + C_{25}\int_0^t 1 + \left|q_s^{\mu,0}\right|^k ds\\
&\le C_{26} + C_{26}\sum_{i=0}^{n-2}\left(\int_0^t \max_{0\le r\le s}\left|q_r^{\mu,i}\right|^k ds\right). \qquad (3.3.9)
\end{aligned}$$

Using the results of equations (3.3.8) and (3.3.9),

$$\sum_{i=0}^{n-2} E\left(\max_{0\le s\le t}\left|q_s^{\mu,k}\right|^k\right) \le C_{27} + C_{27}\int_0^t \sum_{i=0}^{n-2} E\left(\max_{0\le r\le s}\left|q_r^{\mu,i}\right|^k\right) ds.$$

By Gronwall's lemma,

$$\sum_{i=0}^{n-2} E\left(\max_{0\le s\le t}\left|q_s^{\mu,i}\right|^k\right) \le C_{27}e^{C_{27}t} \le C_{28}.$$

Applying this bound to (3.3.7),

$$E\left(\max_{0\le s\le t}\left|q_s^{\mu,n-2} - q_s^{n-2}\right|^k\right) \le C_{29}\mu^{k-2} + C_{29}\sum_{i=0}^{n-2}\int_0^t E\left(\max_{0\le r\le s}\left|q_r^{\mu,i} - q_r^i\right|^k\right) ds. \qquad (3.3.10)$$

Note that for $0 \le i \le n-3$,

$$E\left(\max_{0\le s\le t}\left|q_s^{\mu,i} - q_s^i\right|^k\right) \le C_{30}\int_0^t E\left(\max_{0\le r\le s}\left|q_r^{\mu,i+1} - q_r^{i+1}\right|^k\right) ds$$

as in (3.3.8).

Adding

$$\sum_{i=0}^{n-3} E\left(\max_{0\le s\le t}\left|q_s^{\mu,i} - q_s^i\right|^k\right)$$

to (3.3.10),

$$\sum_{i=0}^{n-2} E\left(\max_{0\le s\le t}\left|q_s^{\mu,i}-q_s^i\right|^k\right) \le C_{29}\mu^{k-2} + C_{31}\int_0^t \sum_{i=0}^{n-2} E\left(\max_{0\le r\le s}\left|q_r^{\mu,i}-q_r^i\right|^k\right) ds.$$

By Gronwall's lemma, we get

$$\sum_{i=0}^{n-2} E\left(\max_{0\le s\le t}\left|q_s^{\mu,i}-q_s^i\right|^k\right) \le C_{29}e^{C_{31}t}\mu^{k-2} \le C_{32}\mu^{k-2}.$$

Especially,

$$\lim_{\mu\downarrow 0}\sum_{i=0}^{n-2} E\left(\max_{0\le t\le T}\left|q_t^{\mu,i}-q_t^i\right|^k\right) = 0.$$

Chapter 4: Small Mass Asymptotics in the Case of a Random Mass

4.1 Introduction

Suppose a particle is moving and the particles of the surrounding medium randomly adhere to or detach from the moving particle after the collision. Then, the mass of the moving particle will randomly change and it will affect the whole system. This idea of randomly changing mass was considered by M. Gitterman for various physical problems recently [6]. In this chapter, we consider small mass asymptotics for the randomly changing mass problem.

Let q_t^μ be the solution of the following stochastic differential equation :

$$\begin{cases} \mu m_t \ddot{q}_t^\mu = b(\mu m_t, q_t^\mu) - \alpha(\mu m_t)\dot{q}_t^\mu + \sigma(q_t^\mu)\dot{w}_t \\ q_0^\mu = q_0 \in \mathbb{R}^d, \qquad \dot{q}_0^\mu = p_0 \in \mathbb{R}^d. \end{cases} \tag{4.1.1}$$

Here, $\mu m_t \in \mathbb{R}^+$ is the mass and $q_t^\mu \in \mathbb{R}^d$ is the position of the moving particle. We assume that m_t is a continuous time discrete Markov chain taking positive values such that it is independent of w_t. As before,

$$\alpha(\mu m_t)\dot{q}_t^\mu$$

is the friction term and

$$b(\mu m_t, q_t^\mu) + \sigma(q_t^\mu)\dot{w}_t$$

is the force field with random noise term. It is reasonable to assume that α and b are depending on the mass in addition to the position of the moving particle.

Now, suppose that $0 < \mu \ll 1$ so that μm_t is small. Then we may compare the solution q_t^μ of (4.1.1) with the solution q_t of the following equation, which is (4.1.1) with $\mu = 0$.

$$\begin{cases} 0 = b(0, q_t) - \alpha(0)\dot{q}_t + \sigma(q_t)\dot{w}_t \\ q_0 \in \mathbb{R}^d. \end{cases} \tag{4.1.2}$$

In Section 4.2, we will rigorously state the conditions and the result of this approach. In Section 4.3, we will prove the statement in Section 4.2.

4.2 Main Result

We first give conditions for the problem. Let $N(T)$ be the number of jumps of m_t in the time interval $[0, T]$.

Hypothesis 4.

1. *There exist constants α_*, m_*, m^* such that $0 < \alpha_* \le \alpha(x)$ and $0 < m_* \le m_t \le m^*$. α is continuous in a neighborhood of 0.*

2. *$b : \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \mapsto M_d(\mathbb{R})$ are Lipschitz continuous.*

3. *The continuous time discrete Markov chain m_t is independent of w_t.*

4. *$N(T) < \infty$ almost surely.*

Now, we are ready to state the main theorem:

Theorem 4.2.1. *Consider* $0 < \mu \le 1$. *Under Hypothesis 4,*

$$\lim_{\mu \downarrow 0} \mathbb{E} \sup_{0 \le t \le T} |q_t^\mu - q_t|^k = 0$$

for any $1 \le k$.

4.3 Proof of Theorem 4.2.1

We first consider the case $k > 2$. (4.1.1) can be written as

$$\ddot{q}_t^\mu + \frac{1}{\mu m_t}\alpha(\mu m_t)\dot{q}_t^\mu = \frac{1}{\mu m_t} b(\mu m_t, q_t^\mu) + \frac{1}{\mu m_t}\sigma(q_t^\mu)\dot{w}_t.$$

By multiplying by the integrating factor

$$e^{\frac{1}{\mu}\int_0^t \frac{\alpha(\mu m_s)}{m_s} ds},$$

$$\left(e^{\frac{1}{\mu}\int_0^t \frac{\alpha(\mu m_s)}{m_s} ds}\dot{q}_t^\mu\right)' = \frac{1}{\mu m_t} e^{\frac{1}{\mu}\int_0^t \frac{\alpha(\mu m_s)}{m_s} ds} b(\mu m_t, q_t^\mu) + \frac{1}{\mu m_t} e^{\frac{1}{\mu}\int_0^t \frac{\alpha(\mu m_s)}{m_s} ds}\sigma(q_t^\mu)\dot{w}_t.$$

Integrating in time,

$$\begin{aligned} \dot{q}_t^\mu - e^{-\frac{1}{\mu}\int_0^t \frac{\alpha(\mu m_s)}{m_s} ds}\dot{q}_0 = {} & \frac{1}{\mu} e^{-\frac{1}{\mu}\int_0^t \frac{\alpha(\mu m_s)}{m_s} ds} \int_0^t \frac{1}{m_s} e^{\frac{1}{\mu}\int_0^s \frac{\alpha(\mu m_r)}{m_r} dr} b(\mu m_s, q_s^\mu) ds \\ & + \frac{1}{\mu} e^{-\frac{1}{\mu}\int_0^t \frac{\alpha(\mu m_s)}{m_s} ds} \int_0^t \frac{1}{m_s} e^{\frac{1}{\mu}\int_0^s \frac{\alpha(\mu m_r)}{m_r} dr} \sigma(q_s^\mu) dw_s. \end{aligned}$$

Solving for $\dot{q}_t^\mu$ and integrating in time,

$$\begin{aligned} q_t^\mu = {} & q_0 + \int_0^t e^{-\frac{1}{\mu}\int_0^s \frac{\alpha(\mu m_r)}{m_r} dr} ds \dot{q}_0 \\ & + \frac{1}{\mu}\int_0^t e^{-\frac{1}{\mu}\int_0^s \frac{\alpha(\mu m_r)}{m_r} dr} \int_0^s \frac{1}{m_r} e^{\frac{1}{\mu}\int_0^r \frac{\alpha(\mu m_u)}{m_u} du} b(\mu m_r, q_r^\mu) dr ds \\ & + \frac{1}{\mu}\int_0^t e^{-\frac{1}{\mu}\int_0^s \frac{\alpha(\mu m_r)}{m_r} dr} \int_0^s \frac{1}{m_r} e^{\frac{1}{\mu}\int_0^r \frac{\alpha(\mu m_u)}{m_u} du} \sigma(q_r^\mu) dw_r ds \\ =: {} & q_0 + (I) + (II) + (III). \end{aligned} \tag{4.3.1}$$

Next, we take integration by parts to (II) and (III) to change them into more desirable form:

$$
\begin{aligned}
(II) = &\int_0^t \left(-\frac{1}{\mu}\frac{\alpha(\mu m_s)}{m_s}\right) e^{-\frac{1}{\mu}\int_0^s \frac{\alpha(\mu m_r)}{m_r}dr} \\
&\cdot \left(-\frac{m_s}{\alpha(\mu m_s)}\right)\int_0^s \frac{1}{m_r} e^{\frac{1}{\mu}\int_0^r \frac{\alpha(\mu m_u)}{m_u}du} b(\mu m_r, q_r^\mu) dr\, ds \\
= &\left[e^{-\frac{1}{\mu}\int_0^s \frac{\alpha(\mu m_r)}{m_r}dr}\left(-\frac{m_s}{\alpha(\mu m_s)}\right)\int_0^s \frac{1}{m_r} e^{\frac{1}{\mu}\int_0^r \frac{\alpha(\mu m_u)}{m_u}du} b(\mu m_r, q_r^\mu) dr\right]_0^t \\
&- \sum_{0\le s\le t} e^{-\frac{1}{\mu}\int_0^s \frac{\alpha(\mu m_r)}{m_r}dr} \Delta\left(-\frac{m_s}{\alpha(\mu m_s)}\right)\int_0^s \frac{1}{m_r} e^{\frac{1}{\mu}\int_0^r \frac{\alpha(\mu m_u)}{m_u}du} b(\mu m_r, q_r^\mu) dr \\
&- \int_0^t e^{-\frac{1}{\mu}\int_0^s \frac{\alpha(\mu m_r)}{m_r}dr}\left(-\frac{m_s}{\alpha(\mu m_s)}\right)\frac{1}{m_s} e^{\frac{1}{\mu}\int_0^s \frac{\alpha(\mu m_r)}{m_r}dr} b(\mu m_s, q_s^\mu) ds \\
= &-\frac{m_t}{\alpha(\mu m_t)} e^{-\frac{1}{\mu}\int_0^t \frac{\alpha(\mu m_s)}{m_s}ds}\int_0^t \frac{1}{m_s} e^{\frac{1}{\mu}\int_0^s \frac{\alpha(\mu m_r)}{m_r}dr} b(\mu m_s, q_s^\mu) ds \\
&+ \sum_{0\le s\le t} e^{-\frac{1}{\mu}\int_0^s \frac{\alpha(\mu m_r)}{m_r}dr}\int_0^s \frac{1}{m_r} e^{\frac{1}{\mu}\int_0^r \frac{\alpha(\mu m_u)}{m_u}du} b(\mu m_r, q_r^\mu) dr \Delta\left(\frac{m_s}{\alpha(\mu m_s)}\right) \\
&+ \int_0^t \frac{1}{\alpha(\mu m_s)} b(\mu m_s, q_s^\mu) ds
\end{aligned}
$$

and

$$
\begin{aligned}
(III) = &-\frac{m_t}{\alpha(\mu m_t)} e^{-\frac{1}{\mu}\int_0^t \frac{\alpha(\mu m_s)}{m_s}ds}\int_0^t \frac{1}{m_s} e^{\frac{1}{\mu}\int_0^s \frac{\alpha(\mu m_r)}{m_r}dr} \sigma(q_s^\mu) dw_s \\
&+ \sum_{0\le s\le t} e^{-\frac{1}{\mu}\int_0^s \frac{\alpha(\mu m_r)}{m_r}dr}\int_0^s \frac{1}{m_r} e^{\frac{1}{\mu}\int_0^r \frac{\alpha(\mu m_u)}{m_u}du} \sigma(q_r^\mu) dw_r \Delta\left(\frac{m_s}{\alpha(\mu m_s)}\right) \\
&+ \int_0^t \frac{1}{\alpha(\mu m_s)} \sigma(q_s^\mu) dw_s.
\end{aligned}
$$

So,

$$
q_t^\mu = q_0 + \int_0^t \frac{1}{\alpha(\mu m_s)} b(\mu m_s, q_s^\mu) ds + \int_0^t \frac{1}{\alpha(\mu m_s)} \sigma(q_s^\mu) dw_s + R_t^\mu, \tag{4.3.2}
$$

where

$$
\begin{aligned}
R_t^\mu = &\int_0^t e^{-\frac{1}{\mu}\int_0^s \frac{\alpha(\mu m_r)}{m_r}dr} ds\dot{q}_0 \\
&- \frac{m_t}{\alpha(\mu m_t)} e^{-\frac{1}{\mu}\int_0^t \frac{\alpha(\mu m_s)}{m_s}ds} \int_0^t \frac{1}{m_s} e^{\frac{1}{\mu}\int_0^s \frac{\alpha(\mu m_r)}{m_r}dr} b(\mu m_s, q_s^\mu) ds \\
&+ \sum_{0 \le s \le t} e^{-\frac{1}{\mu}\int_0^s \frac{\alpha(\mu m_r)}{m_r}dr} \int_0^s \frac{1}{m_r} e^{\frac{1}{\mu}\int_0^r \frac{\alpha(\mu m_u)}{m_u}du} b(\mu m_r, q_r^\mu) dr \Delta\left(\frac{m_s}{\alpha(\mu m_s)}\right) \\
&- \frac{m_t}{\alpha(\mu m_t)} e^{-\frac{1}{\mu}\int_0^t \frac{\alpha(\mu m_s)}{m_s}ds} \int_0^t \frac{1}{m_s} e^{\frac{1}{\mu}\int_0^s \frac{\alpha(\mu m_r)}{m_r}dr} \sigma(q_s^\mu) dw_s \\
&+ \sum_{0 \le s \le t} e^{-\frac{1}{\mu}\int_0^s \frac{\alpha(\mu m_r)}{m_r}dr} \int_0^s \frac{1}{m_r} e^{\frac{1}{\mu}\int_0^r \frac{\alpha(\mu m_u)}{m_u}du} \sigma(q_r^\mu) dw_r \Delta\left(\frac{m_s}{\alpha(\mu m_s)}\right) \\
=: & I_t^1 + I_t^2 + I_t^3 + I_t^4 + I_t^5.
\end{aligned}
$$

From (4.1.2),

$$
q_t = q_0 + \int_0^t \frac{1}{\alpha(0)} b(0, q_s) ds + \int_0^t \frac{1}{\alpha(0)} \sigma(q_s) dw_s. \tag{4.3.3}
$$

Comparing (4.3.2) and (4.3.3),

$$
\begin{aligned}
\mathbb{E}\left[\sup_{0\le t\le T} |q_t^\mu - q_t|^k\right] \le\; & C_1\,\mathbb{E}\left[\sup_{0\le t\le T} \left|\int_0^t \frac{1}{\alpha(\mu m_s)} b(\mu m_s, q_s^\mu) - \frac{1}{\alpha(0)} b(0, q_s) ds\right|^k\right] \\
&+ C_1\,\mathbb{E}\left[\sup_{0\le t\le T} \left|\int_0^t \frac{1}{\alpha(\mu m_s)} \sigma(q_s^\mu) - \frac{1}{\alpha(0)} \sigma(q_s) dw_s\right|^k\right] \\
&+ C_1\,\mathbb{E}\left[\sup_{0\le t\le T} |R_t^\mu|^k\right] \\
=: & A_1 + A_2 + A_3.
\end{aligned}
$$

Now, we find the bounds of A_1 to A_3. First we consider A_1.

$$
\begin{aligned}
A_1 &\le C_2\,\mathbb{E}\left[\sup_{0\le t\le T}\int_0^t \left|\frac{1}{\alpha(\mu m_s)}b(\mu m_s, q_s^\mu) - \frac{1}{\alpha(0)}b(0,q_s)\right|^k ds\right] \\
&= C_2\,\mathbb{E}\left[\int_0^T \left|\frac{1}{\alpha(\mu m_t)}b(\mu m_t, q_t^\mu) - \frac{1}{\alpha(0)}b(0,q_t)\right|^k dt\right] \\
&= C_2\,\mathbb{E}\left[\int_0^T \left|\frac{1}{\alpha(\mu m_t)}\left(b(\mu m_t, q_t^\mu) - b(0,q_t)\right)\right) + \left(\frac{1}{\alpha(\mu m_t)} - \frac{1}{\alpha(0)}\right) b(0,q_t)\right|^k dt\right] \\
&\le C_3\,\mathbb{E}\left[\int_0^T \left|\frac{1}{\alpha(\mu m_t)}\left(b(\mu m_t, q_t^\mu) - b(0,q_t)\right)\right)\right|^k \right. \\
&\qquad \left. + \left|\left(\frac{1}{\alpha(\mu m_t)} - \frac{1}{\alpha(0)}\right) b(0,q_t)\right|^k dt\right] \\
&\le C_4\,\mathbb{E}\left[\int_0^T \frac{1}{\alpha^k(\mu m_t)}\left(|\mu m_t|^k + |q_t^\mu - q_t|^k\right) + \left|\frac{1}{\alpha(\mu m_t)} - \frac{1}{\alpha(0)}\right|^k \left(1+|q_t|^k\right) dt\right] \\
&\le C_5\,\mathbb{E}\left[\int_0^T |q_t^\mu - q_t|^k\,dt\right] + C_5\mu^k\left(T + \mathbb{E}\left[\int_0^T |q_t|^k\,dt\right]\right) \\
&\le C_5 \int_0^T \mathbb{E}\left[\sup_{0\le s\le t}|q_s^\mu - q_s|^k\right] dt + C_5\mu^k\left(T + \int_0^T \mathbb{E}\,|q_t|^k\,dt\right).
\end{aligned}
$$

The second line from the last was from the property

$$
\left|\frac{1}{\alpha(\mu m_t)} - \frac{1}{\alpha(0)}\right| = \left|\frac{\alpha(0)-\alpha(\mu m_t)}{\alpha(\mu m_t)\alpha(0)}\right| \le \frac{K|\mu m_t - 0|}{\alpha_*^2} \le C(K,\alpha_*,m^*)\mu.
$$

Now, we find a bound of A_2.

$$
\begin{aligned}
A_2 &\le C_6\,\mathbb{E}\left(\int_0^T \left|\frac{1}{\alpha(\mu m_t)}\sigma(q_t^\mu) - \frac{1}{\alpha(0)}\sigma(q_t)\right|^2 dt\right)^{\frac{k}{2}} \\
&\le C_6 T^{\frac{k}{2}-1}\,\mathbb{E}\left[\int_0^T \left|\frac{1}{\alpha(\mu m_t)}\sigma(q_t^\mu) - \frac{1}{\alpha(0)}\sigma(q_t)\right|^k dt\right] \\
&\le C_7 T^{\frac{k}{2}-1}\,\mathbb{E}\left[\int_0^T |q_t^\mu - q_t|^k\,dt\right] + C_7 T^{\frac{k}{2}}\mu^k + C_7 T^{\frac{k}{2}-1}\mu^k\,\mathbb{E}\left[\int_0^T |q_t|^k\,dt\right] \\
&\le C_7 T^{\frac{k}{2}-1} \int_0^T \mathbb{E}\left[\sup_{0\le s\le t}|q_s^\mu - q_s|^k\right] dt + C_7 T^{\frac{k}{2}}\mu^k + C_7 T^{\frac{k}{2}-1}\mu^k \int_0^T \mathbb{E}\,|q_t|^k\,dt
\end{aligned}
$$

thanks to the Burkholder-Davis-Gundy inequality.

Now, we consider A_3. An upper bound of A_3 can be split into the following five terms:

$$\begin{aligned} A_3 \le C_8\,\mathbb{E}\left[\sup_{0\le t\le T}|I_t^1|^k\right] + C_8\,\mathbb{E}\left[\sup_{0\le t\le T}|I_t^2|^k\right] + C_8\,\mathbb{E}\left[\sup_{0\le t\le T}|I_t^3|^k\right] \\ + C_8\,\mathbb{E}\left[\sup_{0\le t\le T}|I_t^4|^k\right] + C_8\,\mathbb{E}\left[\sup_{0\le t\le T}|I_t^5|^k\right]. \end{aligned}$$

Before finding bounds of the terms above, we first note that for any $t \ge 0$,

$$0 < \beta_* := \frac{\alpha_*}{m^*} \le \frac{\alpha(\mu m_t)}{m_t} \le \beta^* \tag{4.3.4}$$

for some β_* and β^* for small $\mu > 0$.

So,

$$\begin{aligned} \mathbb{E}\left[\sup_{0\le t\le T}|I_t^1|^k\right] &= \mathbb{E}\left[\sup_{0\le t\le T}\left|\int_0^t e^{-\frac{1}{\mu}\int_0^s \frac{\alpha(\mu m_r)}{m_r}dr} ds\dot{q}_0\right|^k\right] \\ &\le \left|\int_0^T e^{-\frac{1}{\mu}\alpha_* t} ds\right|^k |\dot{q}_0|^k \\ &\le \frac{\mu^k}{\alpha_*^k}\left(1 - e^{-\frac{1}{\mu}\alpha_* T}\right)^k |\dot{q}_0|^k \\ &\le C_9\mu^k. \end{aligned}$$

Now,

$$
\begin{aligned}
\mathbb{E}\left[\sup_{0\le t\le T}|I_t^2|^k\right] &= \mathbb{E}\left[\sup_{0\le t\le T}\left|\frac{m_t}{\alpha(\mu m_t)}e^{-\frac{1}{\mu}\int_0^t\frac{\alpha(\mu m_s)}{m_s}ds}\int_0^t\frac{1}{m_s}e^{\frac{1}{\mu}\int_0^s\frac{\alpha(\mu m_r)}{m_r}dr}b(\mu m_s,q_s^\mu)ds\right|^k\right]\\
&\le \mathbb{E}\left[\sup_{0\le t\le T}\left|\frac{m_t}{\alpha(\mu m_t)}\right|^k\sup_{0\le t\le T}\left|\int_0^t\frac{1}{m_s}e^{-\frac{1}{\mu}\int_s^t\frac{\alpha(\mu m_r)}{m_r}dr}b(\mu m_s,q_s^\mu)ds\right|^k\right]\\
&\le \frac{1}{\alpha_*^k}\mathbb{E}\left(\int_0^T\frac{1}{m_t}e^{-\frac{1}{\mu}\int_t^T\frac{\alpha(\mu m_s)}{m_s}ds}|b(\mu m_t,q_t^\mu)|dt\right)^k\\
&\le \frac{1}{\alpha_*^k m_*^k}\mathbb{E}\left[\left(\int_0^T e^{-\frac{1}{\mu}\frac{k}{k-1}\int_t^T\frac{\alpha(\mu m_s)}{m_s}ds}dt\right)^{k-1}\left(\int_0^T|b(\mu m_t,q_t^\mu)|^k dt\right)\right]\\
&\le C_{10}\mathbb{E}\left[\left(\int_0^T e^{-\frac{1}{\mu}\frac{k}{k-1}\alpha_*(T-t)}dt\right)^{k-1}\int_0^T 1+\mu^k+|q_t^\mu|^k dt\right]\\
&\le C_{10}\left[\frac{\mu}{\alpha_*}\frac{k-1}{k}\left(1-e^{-\frac{1}{\mu}\frac{k}{k-1}\alpha_* T}\right)\right]^{k-1}\mathbb{E}\left[\int_0^T 1+\mu^k+|q_t^\mu|^k dt\right]\\
&\le C_{11}\mu^{k-1}\mathbb{E}\left[\int_0^T 1+|q_t^\mu|^k dt\right]\\
&= C_{11}T\mu^{k-1}+C_{11}\mu^{k-1}\int_0^T\mathbb{E}|q_t^\mu|^k dt
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{E}\left[\sup_{0\le t\le T}|I_t^4|^k\right] &= \mathbb{E}\left[\sup_{0\le t\le T}\left|\frac{m_t}{\alpha(\mu m_t)}e^{-\frac{1}{\mu}\int_0^t\frac{\alpha(\mu m_s)}{m_s}ds}\int_0^t\frac{1}{m_s}e^{\frac{1}{\mu}\int_0^s\frac{\alpha(\mu m_r)}{m_r}dr}\sigma(q_s^\mu)dw_s\right|^k\right]\\
&\le \mathbb{E}\left[\sup_{0\le t\le T}\left|\frac{m_t}{\alpha(\mu m_t)}\right|^k\sup_{0\le t\le T}\left|\int_0^t\frac{1}{m_s}e^{-\frac{1}{\mu}\int_s^t\frac{\alpha(\mu m_r)}{m_r}dr}\sigma(q_s^\mu)dw_s\right|^k\right]\\
&\le \frac{1}{\alpha_*^k}\mathbb{E}\left[\sup_{0\le t\le T}\left|\int_0^t\frac{1}{m_s}e^{-\frac{1}{\mu}\int_s^t\frac{\alpha(\mu m_r)}{m_r}dr}\sigma(q_s^\mu)dw_s\right|^k\right]\\
&\le C_{12}\mathbb{E}\left(\int_0^T\left|\frac{1}{m_t}e^{-\frac{1}{\mu}\int_t^T\frac{\alpha(\mu m_s)}{m_s}ds}\sigma(q_t^\mu)\right|^2dt\right)^{\frac{k}{2}}\\
&\le C_{13}\mathbb{E}\left[\left(\int_0^T\left(e^{-\frac{1}{\mu}\int_t^T\frac{\alpha(\mu m_s)}{m_s}ds}\right)^{\frac{2k}{k-2}}dt\right)^{\frac{k-2}{2}}\left(\int_0^T|\sigma(q_t^\mu)|^k dt\right)\right]\\
&\le C_{14}\left(\frac{\mu}{\alpha_*}\frac{k-2}{2k}\right)^{\frac{k-2}{2}}\mathbb{E}\left(\int_0^T 1+|q_t^\mu|^k dt\right)\\
&\le C_{15}T\mu^{\frac{k}{2}-1}+C_{15}\mu^{\frac{k}{2}-1}\int_0^T\mathbb{E}\left[|q_t^\mu|^k\right]dt
\end{aligned}
$$

thanks to the BDG inequality and Hölder inequality.

To find a bound of I_t^3, we first rearrange the terms inside the summation sign to change it into more desirable form. Let $\{\tau_i\}_{i=0,1,2,\dots}$ be the sequence of stopping times at which m has its ith jump. Note that $\tau_0 = 0$ almost surely. Define

$$T_i := \tau_i \wedge T.$$

Then, we may rewrite I_t^3 in the following way:

$$\begin{aligned}
|I_t^3| &= \left| \sum_{i=1}^{\infty} \left(\int_0^{T_i} \frac{1}{m_s} e^{-\frac{1}{\mu}\int_s^{T_i} \frac{\alpha(\mu m_r)}{m_r} dr} b(\mu m_s, q_s^\mu) ds \, \Delta \left(\frac{m_{T_i}}{\alpha(\mu m_{T_i})} \right) \right) \right| \\
&= \left| \sum_{i=1}^{\infty} \left(\left(\sum_{j=1}^{i} \int_{T_{j-1}}^{T_j} \frac{1}{m_s} e^{-\frac{1}{\mu}\int_s^{T_i} \frac{\alpha(\mu m_r)}{m_r} dr} b(\mu m_s, q_s^\mu) ds \right) \Delta \left(\frac{m_{T_i}}{\alpha(\mu m_{T_i})} \right) \right) \right| \\
&= \left| \sum_{i=1}^{\infty} \sum_{j=i}^{\infty} \left(\int_{T_{i-1}}^{T_i} \frac{1}{m_s} e^{-\frac{1}{\mu}\int_s^{T_j} \frac{\alpha(\mu m_r)}{m_r} dr} b(\mu m_s, q_s^\mu) ds \, \Delta \left(\frac{m_{T_j}}{\alpha(\mu m_{T_j})} \right) \right) \right| \\
&= \left| \sum_{i=1}^{\infty} \left(\int_{T_{i-1}}^{T_i} \frac{1}{m_s} e^{-\frac{1}{\mu}\int_s^{T_i} \frac{\alpha(\mu m_r)}{m_r} dr} b(\mu m_s, q_s^\mu) ds \right.\right. \\
&\qquad \left.\left. \cdot \sum_{j=i}^{\infty} \left(e^{-\frac{1}{\mu}\int_{T_i}^{T_j} \frac{\alpha(\mu m_r)}{m_r} dr} \Delta \left(\frac{m_{T_j}}{\alpha(\mu m_{T_j})} \right) \right) \right) \right| .
\end{aligned}$$

Note that the justification for changing the order of summations came from the fact that since $N(T) < \infty$ almost surely, the summation above is in fact finite summation almost surely.

It is easy to see that

$$0 \le \left| \sum_{j=i}^{\infty} \left(e^{-\frac{1}{\mu}\int_{T_i}^{T_j} \frac{\alpha(\mu m_r)}{m_r} dr} \Delta \left(\frac{m_{T_j}}{\alpha(\mu m_{T_j})} \right) \right) \right| \le 2\beta^*$$

from (4.3.4) and decreasing property of

$$e^{-\frac{1}{\mu}\int_{T_i}^{T_j} \frac{\alpha(\mu m_r)}{m_r} dr}$$

as j increases.

So,

$$
\begin{aligned}
|I_t^3| &\leq 2\beta^* \sum_{i=1}^{\infty} \left(\int_{T_{i-1}}^{T_i} \left| \frac{1}{m_s} e^{-\frac{1}{\mu}\int_s^{T_i} \frac{\alpha(\mu m_r)}{m_r} dr} b(\mu m_s, q_s^\mu) \right| ds \right) \\
&= 2\beta^* \int_0^T \left| \frac{1}{m_s} e^{-\frac{1}{\mu}\int_s^{T_i} \frac{\alpha(\mu m_r)}{m_r} dr} b(\mu m_s, q_s^\mu) \right| ds \\
&\leq \frac{2\beta^*}{m_*} \int_0^T \left| e^{-\frac{1}{\mu}\int_s^{T_i} \frac{\alpha(\mu m_r)}{m_r} dr} b(\mu m_s, q_s^\mu) \right| ds.
\end{aligned}
$$

The rest of the calculation follows exactly the same way as we did for I_t^2:

$$
\mathbb{E}\left[\sup_{0\leq t\leq T} |I_t^3|^k \right] \leq C_{16} T \mu^{k-1} + C_{16}\mu^{k-1} \int_0^T \mathbb{E}|q_t^\mu|^k dt.
$$

A bound of I_t^5 can be found using the methods used for finding bounds of I_t^3 and I_t^4:

$$
\mathbb{E}\left[\sup_{0\leq t\leq T} |I_t^5|^k \right] \leq C_{17} T \mu^{\frac{k}{2}-1} + C_{17}\mu^{\frac{k}{2}-1} \int_0^T \mathbb{E}|q_t^\mu|^k dt.
$$

Combining the bounds of I_t^1 to I_t^5, we get

$$
A_3 \leq C_{18}\mu^{\frac{k}{2}-1} \left(1 + \int_0^T \mathbb{E}|q_t^\mu|^k dt \right).
$$

Combining the bounds of A_1 to A_3, we get

$$
\begin{aligned}
\mathbb{E}\left[\sup_{0\leq t\leq T} |q_t^\mu - q_t|^k \right] \leq C_{19}\mu^{\frac{k}{2}-1} &\left(1 + \int_0^T \mathbb{E}|q_t^\mu|^k dt \right) \\
&+ C_{19} \int_0^T \mathbb{E}\left[\sup_{0\leq s\leq t} |q_s^\mu - q_s|^k \right] ds.
\end{aligned} \tag{4.3.5}
$$

If we can show that

$$
\sup_{0\leq t\leq T} \mathbb{E}|q_t^\mu|^k
$$

is bounded, we can apply Gronwall's lemma to find a bound of

$$
\mathbb{E}\left[\sup_{0\leq t\leq T} |q_t^\mu - q_t|^k \right].
$$

Applying almost the same technique as above to (4.3.2), we can conclude that

$$\sup_{0\le t\le T} \mathbb{E}|q_t^\mu|^k \le C_{20}\mu^{\frac{k}{2}-1}\left(1+\int_0^T \sup_{0\le s\le t} \mathbb{E}|q_s^\mu|^k dt\right).$$

Applying Gronwall's lemma, we conclude

$$\sup_{0\le t\le T} \mathbb{E}|q_t^\mu|^k \le C_{21}.$$

Enforcing this bound to equation (4.3.5) and applying Gronwall's lemma one more time, we get

$$\mathbb{E}\left[\sup_{0\le t\le T} |q_t^\mu - q_t|^k\right] \le C_{22}\mu^{\frac{k}{2}-1}e^{C_{22}T} = C_{23}\mu^{\frac{k}{2}-1}.$$

So,

$$\lim_{\mu\downarrow 0} \mathbb{E} \sup_{0\le t\le T} |q_t^\mu - q_t|^k = 0$$

for $k > 2$.

By Hölder inequality, this means that this holds for all $1 \le k$.

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