ABSTRACT

Title of Dissertation:	AN ANALYTIC CONSTRUCTION OF THE MODULI SPACE OF HIGGS BUNDLES OVER RIEMANN SURFACES
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The moduli space of Higgs bundles over Riemann surfaces can be defined as a quotient of an infinite-dimensional space by an infinite-dimensional Lie group. In this thesis, we use the Kuranishi slice method to endow this quotient with the structure of a normal complex space. We also give a direct proof that the moduli space is locally modeled on an affine geometric invariant theory quotient of a quadratic cone by a complex reductive group. Moreover, we show that the moduli space admits an orbit type decomposition such that the decomposition is a Whitney stratification, and each stratum has a complex symplectic structure and a Kähler structure. The complex symplectic structures glue to a complex Poisson bracket on the structure sheaf, and the Kähler structures glue to a singular weak Kähler metric on the moduli space. Finally, we use the symplectic cut to show that the moduli space admits a projective compactification and hence is quasi-projective.

AN ANALYTIC CONSTRUCTION OF THE MODULI SPACE OF HIGGS BUNDLES OVER RIEMANN SURFACES

by

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Dedication

谨以此文献给我的妻子，你的爱与鼓励是我前进的动力，让我穿越黑暗到达光明；也感谢我的父母，你们的支持和理解让这一切成为可能。

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Chapter 1: Introduction

Let M be a closed Riemann surface with genus ≥ 2. Introduced by Hitchin in the seminal paper [33], a Higgs bundle $(\mathcal{E}, \Phi)$ is a pair of a holomorphic vector bundle $\mathcal{E} \to M$ and a holomorphic section $\Phi \in H^0(\operatorname{End}\mathcal{E} \otimes \mathcal{K}_M)$, where $\mathcal{K}_M$ is the canonical bundle of M. Their moduli space has rich geometric structures and therefore is interesting in its own right, let alone its roles in many other different areas including gauge theory, Kähler and hyperKähler geometry, non-abelian Hodge theory and integrable systems (see [66] for more discussions).

To obtain a "nice" moduli space, some stability conditions have to be introduced, since the space of Higgs bundles up to isomorphism is not even Hausdorff. We recall that a Higgs bundle $(\mathcal{E}, \Phi)$ is semistable if $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$ for every Φ-invariant holomorphic subbundle $0 \subsetneq \mathcal{F} \subsetneq \mathcal{E}$, where $\mu(\mathcal{F}) = \deg(\mathcal{F})/\operatorname{rank}(\mathcal{F})$ is the slope of $\mathcal{F}$. If the equality $\mu(\mathcal{F}) = \mu(\mathcal{E})$ cannot occur, then $(\mathcal{E}, \Phi)$ is stable. Finally, $(\mathcal{E}, \Phi)$ is polystable if it is a direct sum of stable Higgs bundles with the same slope. We observe that the various stability conditions are "ordered" in the following way: "stable" $\Longrightarrow$ "polystable" $\Longrightarrow$ "semistable", and these implications can be reversed only when the rank and the degree of $\mathcal{E}$ is coprime.

In [33], Hitchin used the Kuranishi slice method to construct the moduli space of stable Higgs bundles first as a smooth manifold and then as a hyperKähler manifold. Such a method was first introduced by Kuranishi in [42] and has been used in several papers to construct moduli spaces in different contexts (for example, see [4, 5, 40, 43] and [39, Chapter 7]). On the other hand, the moduli space of semistable Higgs bundles was constructed by Nitsure in [48] where M is a smooth projective curve and by Simpson in [57] where M is a smooth projective variety. They both used Geometric Invariant Theory (GIT for short), and the method is entirely algebro-geometric. The resulting moduli space is a quasi-projective variety. Note that it is no longer smooth and contains the moduli space of stable Higgs bundles as an smooth open dense subset.

Given a successful algebraic construction of the moduli space, we may wonder if an analytic construction is possible. More specifically, it is natural to ask if the Kuranishi slice method can be used to construct the moduli space of semistable Higgs bundles as a complex space. In fact, this seems to be a folklore theorem (for example, see [6,66]). In this thesis, we will give a positive answer and provide a proof in detail. Other motivation for an analytic construction comes from the definition of the moduli space in the analytic settings. Roughly speaking, the moduli space will be defined as the quotient of an infinite-dimensional space by an infinite-dimensional Lie group. By the Hitchin-Kobayashi correspondence, it is homeomorphic to a singular hyperKähler quotient. As a consequence, the open smooth subspace consisting of stable Higgs bundles acquires a hyperKähler metric. This makes it possible to study the hyperKähler geometry on the moduli space. From the perspective of the algebraic construction, it is hard to see how the hyperKähler geometry comes into the picture. Moreover, since the moduli space is defined as a quotient by a Lie group, we may consider its orbit type decomposition. By generalizing Mayrand's results in [44] to infinite-dimensional settings, we will show that the decomposition is a Whitney stratification, and each stratum is a hyperKähler manifold. With respect to the complex structure of the moduli space, the hyperKähler structure on each stratum can be decomposed into a Kähler structure and a complex symplectic structure. The complex Poisson brackets induced by the complex symplectic structures on the strata glue to a Poisson bracket on the structure sheaf of the moduli space. Moreover, the Kähler structures on the strata glue to a weak Kähler (singular) metric on the moduli space. Following Mayrand, the moduli space is a stratified complex symplectic space (also see Sjamaar-Lerman [59] for the real case). Moreover, following Moishezon [45], the moduli space is a Kähler space. All of these structures are hard to obtain from the algebraic construction.

By our analytic construction, the moduli space is so far just a normal complex space. It is also natural to ask if analytic methods can be used to show that this complex space is a quasi-projective variety, the end result from the algebraic construction. The first step toward this goal is to compactify the moduli space. In [25], Hausel used the symplectic cut to compactify the moduli space when it is smooth and the underlying smooth bundle of $(\mathcal{E}, \Phi)$ is of rank 2. He further showed that the compactification is projective and thus the quasi-projectivity of the moduli space follows. In this thesis, we will follow the same method to compactify the moduli space and prove the projectivity of the compactification. However, we will not impose the smoothness conditions as Hausel did. Therefore, this thesis is a generalization of

Hausel's results. It should be noted that Simpson compactified the moduli space using algebro-geometric methods in [53]. However, the projectivity of the compactification was not proved. In a recent paper [10], de Cataldo followed Simpson's method and constructed a projective compactification of the moduli space. It can be shown that our compactification is isomorphic to de Cataldo's compactification.

1.1 Main results

In this section, we will make the above discussion precise and rigorous and state the main results in this thesis. The entire thesis is based on author's papers [16–18]. Before stating the main results, we first set up the notations and introduce the general settings.

Fix a smooth Hermitian vector bundle $E \to M$ and let $\mathfrak{g}_E \to M$ be the bundle of skew-Hermitian endomorphisms of E. For convenience, we assume that the degree of E is zero. This condition is not essential. By the Newlander-Nirenberg theorem, a holomorphic structure on E (described by holomorphic transition functions) is equivalent to an integrable Dolbeault operator $\overline{\partial}_E \colon \Omega^0(E) \to \Omega^{0,1}(E)$. Since $\dim_{\mathbb{C}} M = 1$, the integrability condition, $\overline{\partial}_E^2 = 0$, is vacuous. Therefore, via the Chern correspondence, the space of holomorphic structures on E can be identified with the space $\mathcal{A}$ of unitary connections on E, which is an infinite-dimensional affine space modeled on $\Omega^1(\mathfrak{g}_E)$. Let $\mathcal{C} = \mathcal{A} \times \Omega^{1,0}(\mathfrak{g}_E^{\mathbb{C}})$. Then, the configuration space of

Higgs bundles (with a fixed underlying smooth bundle E) is defined as

$$\mathcal{B} = \{(A, \Phi) \in \mathcal{C} \colon \overline{\partial}_A \Phi = 0\} \tag{1.1}$$

(see [66] for more details). The complex gauge group $\mathcal{G}^{\mathbb{C}} = \mathrm{Aut}(E)$ acts on $\mathcal{B}$ by

$$(\overline{\partial}_A, \Phi) \cdot g = (g^{-1} \circ \overline{\partial}_A \circ g, g^{-1}\Phi g), \qquad g \in \mathcal{G}^{\mathbb{C}}, (A, \Phi) \in \mathcal{B}. \tag{1.2}$$

Then, two Higgs bundles are isomorphic if and only if they are in the same $\mathcal{G}^{\mathbb{C}}$-orbit. Let $\mathcal{B}^{ss}$, $\mathcal{B}^{s}$ and $\mathcal{B}^{ps}$ be the subspaces of $\mathcal{B}$ consisting of semistable, stable and polystable Higgs bundles, respectively. They are $\mathcal{G}^{\mathbb{C}}$-invariant. The moduli space of semistable Higgs bundles is defined as the quotient $\mathcal{M} = \mathcal{B}^{ps}/\mathcal{G}^{\mathbb{C}}$ equipped with the C^∞-topology.

The moduli space $\mathcal{M}$ can be realized as an infinite-dimensional singular Kähler quotient of the singular space $\mathcal{B}$ as follows. Recall that $\mathcal{C}$ is an infinite-dimensional affine hyperKähler manifold that is modeled on $\Omega^1(\mathfrak{g}_E) \oplus \Omega^{1,0}(\mathfrak{g}_E^{\mathbb{C}})$ (see [33, §6]). If we identify $\Omega^1(\mathfrak{g}_E)$ with $\Omega^{0,1}(\mathfrak{g}_E^{\mathbb{C}})$ using the map $\alpha \mapsto \alpha''$, where $\alpha'' \in \Omega^{0,1}(\mathfrak{g}_E^{\mathbb{C}})$ is the $(0,1)$-component of α, then an L^2-metric on $\mathcal{C}$ is given by

$$g(\alpha'', \eta; \alpha'', \eta) = \frac{2\sqrt{-1}}{4\pi^2} \int_M \mathrm{tr}\Big((\alpha'')^*\alpha'' + \eta\eta^*\Big), \qquad (\alpha'', \eta) \in \Omega^{0,1}(\mathfrak{g}_E^{\mathbb{C}}) \oplus \Omega^{1,0}(\mathfrak{g}_E^{\mathbb{C}}). \tag{1.3}$$

(Here, the constant is needed for only §5 and will be omitted in §3 and §4. The same applies to (1.4)). Let I be the complex structure given by the multiplication by $\sqrt{-1}$, Ω_I its associated Kähler form, and $\mathcal{G}$ the subgroup of $\mathcal{G}^{\mathbb{C}}$ consisting of unitary gauge transformations. The $\mathcal{G}$-action on $\mathcal{C}$ is Hamiltonian with respect to the Kähler form Ω_I, and the moment map is given by

$$\mu(A,\Phi) = \frac{1}{4\pi^2}(F_A + [\Phi,\Phi^*])\colon \mathcal{C} \to \Omega^2(\mathfrak{g}_E). \tag{1.4}$$

Then, the Hitchin-Kobayashi correspondence, a gauge-theoretic interpretation of the polystability condition, states that a Higgs bundle $(A,\Phi) \in \mathcal{B}$ is polystable if and only if $(A,\Phi)\cdot g$ satisfies Hitchin's equation $\mu = 0$ for some $g \in \mathcal{G}^{\mathbb{C}}$. A stronger version of this result states that the inclusion $\mu^{-1}(0)\cap\mathcal{B} \hookrightarrow \mathcal{B}^{ps}$ induces a homeomorphism

$$i\colon (\mu^{-1}(0)\cap\mathcal{B})/\mathcal{G} \xrightarrow{\sim} \mathcal{B}^{ps}/\mathcal{G}^{\mathbb{C}} = \mathcal{M}, \tag{1.5}$$

where the inverse is induced by the retraction $r\colon \mathcal{B}^{ss} \to \mu^{-1}(0)\cap\mathcal{B}$ defined by the Yang-Mills-Higgs flow (for more details, see [66] and [67]).

Another point of view is that the moduli space can be realized as an infinite-dimensional singular hyperKähler quotient of $\mathcal{C}$ as follows. The holomorphicity condition $\mu_{\mathbb{C}}(A,\Phi) = \overline{\partial}_A\Phi$ can be regarded as a complex moment map for the $\mathcal{G}^{\mathbb{C}}$-action with respect to the complex symplectic form induced by the other two complex structures J and K. As a consequence, the map $\mathbf{m} = (\mu,\mu_{\mathbb{C}})$ can be regarded as a hyperKähler moment map on $\mathcal{C}$. Therefore, $\mu^{-1}(0)\cap\mathcal{B} = \mathbf{m}^{-1}(0)$, and the homeomorphism i can be rephrased as

$$i\colon \mathbf{m}^{-1}(0)/\mathcal{G} \xrightarrow{\sim} \mathcal{B}^{ps}/\mathcal{G}^{\mathbb{C}} = \mathcal{M}, \tag{1.6}$$

where $\mathbf{m}^{-1}(0)/\mathcal{G}$ is an infinite-dimensional singular hyperKähler quotient.

The main result in §3 is the following.

Theorem A. *The moduli space $\mathcal{M}$ is a normal complex space.*

More can be said about the local structure of $\mathcal{M}$. We recall that the deformation complex for a Higgs bundle (A, Φ) satisfying Hitchin's equation $\mu(A, \Phi) = 0$ is given by

$$C_{\mu_{\mathbb{C}}}: \qquad \Omega^0(\mathfrak{g}_E^{\mathbb{C}}) \xrightarrow{D''} \Omega^{0,1}(\mathfrak{g}_E^{\mathbb{C}}) \oplus \Omega^{1,0}(\mathfrak{g}_E^{\mathbb{C}}) \xrightarrow{D''} \Omega^{1,1}(\mathfrak{g}_E^{\mathbb{C}}), \tag{1.7}$$

where $D'' = \overline{\partial}_A + \Phi$. It is an elliptic complex. Let H be the $\mathcal{G}$-stabilizer at (A, Φ). Since the $\mathcal{G}$-action is proper, H is a compact Lie group. Moreover, its complexification $H^{\mathbb{C}}$ is precisely the $\mathcal{G}^{\mathbb{C}}$-stabilizer at (A, Φ) (see §3.2) and acts on $\mathbf{H}^1$ linearly. Then, the local structure of $\mathcal{M}$ is described as follows.

Theorem B. *Let $[A, \Phi] \in \mathcal{M}$ be a point such that $\mu(A, \Phi) = 0$ and $\mathbf{H}^1$ its deformation space, the harmonic space $\mathbf{H}^1(C_{\mu_{\mathbb{C}}})$ defined in $C_{\mu_{\mathbb{C}}}$. Then, the following hold:*

1. *$\mathbf{H}^1$ is a complex symplectic vector space.*

2. *The $H^{\mathbb{C}}$-action on $\mathbf{H}^1$ is complex Hamiltonian with a complex moment map given by*

$$\nu_{0,\mathbb{C}}(x) = \frac{1}{2} P[x, x], \tag{1.8}$$

where P is the harmonic projection in $C_{\mu_{\mathbb{C}}}$.

3. *Around $[A, \Phi]$, the moduli space $\mathcal{M}$ is locally biholomorphic to an open neighborhood of $[0]$ in the complex symplectic quotient $\nu_{0,\mathbb{C}}^{-1}(0) /\!\!/ H^{\mathbb{C}}$, which is an*

affine GIT quotient.

This result is not surprising for two reasons. The first reason is that Simpson proved in [57, §10] that the differential graded Lie algebra $C_{\mu_{\mathbb{C}}}$ is formal. As a consequence, the moduli space is locally biholomorphic to a GIT quotient of a quadratic cone in $H^1(C_{\mu_{\mathbb{C}}})$ by a complex reductive group. The second reason is that Theorem B is an infinite-dimensional generalization of [44, Theorem 1.4 (iv)] to Higgs bundles.

After the construction of the moduli space $\mathcal{M}$, it is natural to compare the analytic and the algebraic moduli spaces. More precisely, let us also use $\mathcal{M}_{an}$ to mean the quotient $\mathcal{B}^{ps}/\mathcal{G}^{\mathbb{C}}$ and $\mathcal{M}_{alg}$ the moduli space of semistable Higgs bundles of rank r and degree 0 in the category of schemes, where r is the rank of E. By construction, $\mathcal{M}_{alg}$ parametrizes S-equivalence classes of Higgs bundles. Let us recall the definition of the S-equivalence relation. Every semistable Higgs bundle $(\mathcal{E}, \Phi)$ admits a filtration,

$$0 = (\mathcal{E}_0, \Phi_0) \subset (\mathcal{E}_1, \Phi_1) \subset \cdots \subset (\mathcal{E}_\ell, \Phi_\ell) = (\mathcal{E}, \Phi), \tag{1.9}$$

called the Seshadri filtration, whose successive quotients are stable, all with slope $\mu(E)$. Let $\mathrm{Gr}(\mathcal{E}, \Phi) = \bigoplus_{i=1}^{\ell}(\mathcal{E}_i/\mathcal{E}_{i-1}, \Phi_i)$ be the graded object associated with the Seshadri filtration of $(\mathcal{E}, \Phi)$. It is uniquely determined by the isomorphism class of $(\mathcal{E}, \Phi)$. Then, two Higgs bundles $(\mathcal{E}_1, \Phi_1)$ and $(\mathcal{E}_2, \Phi_2)$ are S-equivalent if $\mathrm{Gr}(\mathcal{E}_1, \Phi_1)$ and $\mathrm{Gr}(\mathcal{E}_2, \Phi_2)$ are isomorphic as Higgs bundles. As a consequence, there is a natural comparison map $i\colon \mathcal{M}_{an} \to \mathcal{M}_{alg}$ of the underlying sets that sends each $\mathcal{G}^{\mathbb{C}}$-orbit of a point (A, Φ) in $\mathcal{B}^{ps}$ to the S-equivalence class of the Higgs bundle $(\mathcal{E}_A, \Phi)$ defined by (A, Φ).

Theorem C. *The comparison map $i \colon \mathcal{M}_{an} \to \mathcal{M}_{alg}$ is a biholomorphism.*

To state the main results in §4, we now define the orbit type decompositions of $\mathbf{m}^{-1}(0)/\mathcal{G}$ and $\mathcal{M}$. Let H be a $\mathcal{G}$-stabilizer at some Higgs bundle in $\mathbf{m}^{-1}(0)$ and (H) the conjugacy class of H in $\mathcal{G}$. Consider the subspace

$$\mathbf{m}^{-1}(0)_{(H)} = \{(A, \Phi) \in \mathbf{m}^{-1}(0) \colon \mathcal{G}_{(A,\Phi)} \in (H)\}. \tag{1.10}$$

It is $\mathcal{G}$-invariant, and the orbit type decomposition of the singular hyperKähler quotient $\mathbf{m}^{-1}(0)/\mathcal{G}$ is defined as

$$\mathbf{m}^{-1}(0)/\mathcal{G} = \coprod_{(H)} \text{components of } \mathbf{m}^{-1}(0)_{(H)}/\mathcal{G}. \tag{1.11}$$

By abusing the notation, we generally use π to denote the quotient map $\mathcal{B}^{ps} \to \mathcal{M}$ or $\mathbf{m}^{-1}(0) \to \mathbf{m}^{-1}(0)/\mathcal{G}$. The following is a slight generalization of Hitchin's construction of the moduli space of stable Higgs bundles in [33, §5, §6] (cf. [61, Proposition 2.21]).

Theorem D. *Every stratum Q in the orbit type decomposition of the hyperKähler quotient $\mathbf{m}^{-1}(0)/\mathcal{G}$ is a locally closed smooth manifold, and $\pi^{-1}(Q)$ is a smooth submanifold of $\mathcal{C}$ such that the restriction $\pi \colon \pi^{-1}(Q) \to Q$ is a smooth submersion. Moreover, the restriction of the hyperKähler structure from $\mathcal{C}$ to $\pi^{-1}(Q)$ descends to Q.*

Similarly, if L is a $\mathcal{G}^{\mathbb{C}}$-stabilizer at some Higgs bundle in $\mathcal{B}^{ps}$, and (L) denotes the conjugacy class of L in $\mathcal{G}^{\mathbb{C}}$, then we consider the subspace

$$\mathcal{B}^{ps}_{(L)} = \{(A, \Phi) \in \mathcal{B}^{ps} \colon (\mathcal{G}^{\mathbb{C}})_{(A,\Phi)} \in (L)\}. \tag{1.12}$$

It is $\mathcal{G}^{\mathbb{C}}$-invariant, and the orbit type decomposition of the moduli space $\mathcal{M}$ is defined as

$$\mathcal{M} = \coprod_{(L)} \text{components of } \mathcal{B}^{ps}_{(L)}/\mathcal{G}^{\mathbb{C}}. \tag{1.13}$$

Then, we will prove the following that is similar to Theorem D.

Theorem E. *Every stratum Q in the orbit type decomposition of the moduli space $\mathcal{M}$ is a locally closed complex submanifold of $\mathcal{M}$, and $\pi^{-1}(Q)$ is a complex submanifold of $\mathcal{C}$ with respect to the complex structure I such that the restriction $\pi\colon \pi^{-1}(Q) \to Q$ is a holomorphic submersion. This decomposition is a complex Whitney stratification.*

Here, by complex Whitney stratification, we mean that the orbit type decomposition of $\mathcal{M}$ is a disjoint union of locally closed complex submanifolds such that if $Q_1 \cap \overline{Q_2} \neq \emptyset$ then $Q_1 \subset \overline{Q_2}$ for any strata Q_1 and Q_2 in the decomposition. This is called the *frontier condition*. Moreover, the strata are required to satisfy Whitney conditions A and B. Although Whitney conditions A and B are conditions for submanifolds in an Euclidean space, they make sense for complex spaces, since they are local conditions and invariant under diffeomorphisms (see [44, Definition 2.2, 2.5, 2.7] for more details).

Moreover, the Hitchin-Kobayashi correspondence i preserves the orbit type decompositions in the following way.

Theorem F. *If Q is a stratum in the orbit type decomposition of $\mathbf{m}^{-1}(0)/\mathcal{G}$, then $i(Q)$ is a stratum in the orbit type decomposition of $\mathcal{M}$, and the restriction $i\colon Q \to i(Q)$ is a biholomorphism with respect to the complex structure I_Q on Q coming from $\mathcal{C}$ and the natural complex structure on $i(Q)$.*

Therefore, each stratum Q in the orbit type decomposition of $\mathcal{M}$ acquires a complex symplectic structure from the corresponding stratum in the orbit type decomposition of $\mathbf{m}^{-1}(0)/\mathcal{G}$. As a consequence, each Q admits a complex Poisson bracket. We will show that these Poisson brackets glue to a complex Poisson bracket on the structure sheaf of $\mathcal{M}$. By Theorem B, around $[A, \Phi]$, the moduli space $\mathcal{M}$ is locally biholomorphic to an open neighborhood of $[0]$ in the complex symplectic quotient $\nu_{0,\mathbb{C}}^{-1}(0) /\!\!/ H^{\mathbb{C}}$, which is an affine GIT quotient. Note that $\nu_{0,\mathbb{C}}^{-1}(0) /\!\!/ H^{\mathbb{C}}$ also has an orbit type decomposition, since every point in $\nu_{0,\mathbb{C}}^{-1}(0) /\!\!/ H^{\mathbb{C}}$ has a unique closed orbit, and this orbit has an orbit type. By Mayrand [44], the orbit type decomposition of $\nu_{0,\mathbb{C}}^{-1}(0) /\!\!/ H^{\mathbb{C}}$ is a Whitney stratification, and each stratum is a complex symplectic submanifold and hence admits a complex Poisson bracket. Moreover, these Poisson brackets glue to a Poisson bracket on the structure sheaf such that the inclusion from each stratum to $\nu_{0,\mathbb{C}}^{-1}(0) /\!\!/ H^{\mathbb{C}}$ is a Poisson map. Then, we will prove the following.

Theorem G. *There is a unique complex Poisson bracket on the structure sheaf of $\mathcal{M}$ such that the inclusion $Q \hookrightarrow \mathcal{M}$ is a Poisson map for each stratum Q in $\mathcal{M}$. Moreover, we have the following.*

1. *The local biholomorphism between $\mathcal{M}$ and $\nu_{0,\mathbb{C}}^{-1}(0) /\!\!/ H^{\mathbb{C}}$ preserves the orbit type stratifications and is a Poisson map.*

2. *Its restriction to each stratum Q in $\mathcal{M}$ is a complex symplectomorphism, and hence serves as complex Darboux coordinates on Q.*

Following Mayrand [44] and Sjamaar-Lerman [59], a complex space is called a *stratified complex symplectic space* if it admits a complex Whitney stratification, a complex symplectic structure on each stratum, and a complex Poisson bracket on the structure sheaf such that the inclusion from each stratum to the complex space is a holomorphic Poisson map. As a consequence of the main theorems proved in this thesis, we conclude the following.

Corollary. *The moduli space $\mathcal{M}$ of Higgs bundles is a stratified complex symplectic space with the orbit type decomposition as the complex Whitney stratification.*

One of the main results in §5 is that the moduli space $\mathcal{M}$ admits a compactification.

Theorem H. *There is a normal compact complex space $\overline{\mathcal{M}}$ in which the moduli space $\mathcal{M}$ embeds as an open dense subset. Moreover, the complement $Z = \overline{\mathcal{M}} \setminus \mathcal{M}$ is a closed complex subspace of pure codimension 1.*

As a consequence, the quasi-projectivity of $\mathcal{M}$ follows if we can show that the compactification $\overline{\mathcal{M}}$ is projective. Therefore, we need to construct an ample line bundle on $\overline{\mathcal{M}}$. To construct such a line bundle, we need a descent lemma for vector bundles. In [13], Drezet and Narasimhan proved a descent lemma for good quotients of algebraic varieties. A natural analogue of good quotients in our settings is the quotient map $\pi\colon \mathcal{B}^{ss} \to \mathcal{B}^{ss} /\!\!/ \mathcal{G}^{\mathbb{C}}$, where $\mathcal{B}^{ss} /\!\!/ \mathcal{G}^{\mathbb{C}}$ is the quotient space of $\mathcal{B}^{ss}$ by the S-equivalence relation of Higgs bundles. Heuristically, we think of $\pi\colon \mathcal{B}^{ss} \to \mathcal{B}^{ss} /\!\!/ \mathcal{G}^{\mathbb{C}}$ as an infinite-dimensional GIT quotient and naturally expect that its properties are similar to those of good quotients of algebraic varieties. To justify this heuristic thinking, we will first prove in §5.1.1 that the inclusion $\mathcal{B}^{ps} \hookrightarrow \mathcal{B}^{ss}$ induces a homeomorphism $\mathcal{M} \to \mathcal{B}^{ss} /\!\!/ \mathcal{G}^{\mathbb{C}}$, and hence will routinely identify $\mathcal{M}$ with $\mathcal{B}^{ss} /\!\!/ \mathcal{G}^{\mathbb{C}}$. Then, we will show the following.

Theorem I. *The quotient map $\pi\colon \mathcal{B}^{ss} \to \mathcal{M}$ satisfies the following properties:*

1. *π identifies $\mathcal{G}^{\mathbb{C}}$-orbits whose closures in $\mathcal{B}^{ss}$ intersect.*

2. *Every fiber of π contains a unique $\mathcal{G}^{\mathbb{C}}$-orbit that is closed in $\mathcal{B}^{ss}$. Moreover, a $\mathcal{G}^{\mathbb{C}}$-orbit is closed in $\mathcal{B}^{ss}$ if and only if it contains a polystable Higgs bundle.*

3. *$\mathcal{O}_{\mathcal{M}} = \pi_* \mathcal{O}_{\mathcal{B}^{ss}}^{\mathcal{G}^{\mathbb{C}}}$. In other words, if U is an open subset of $\mathcal{M}$, the map $\mathcal{O}_{\mathcal{M}}(U) \to \mathcal{O}_{\mathcal{B}^{ss}}(\pi^{-1}(U))^{\mathcal{G}^{\mathbb{C}}}$ given by $f \mapsto \pi^* f$ is a bijection.*

4. *π is a categorical quotient in the sense that every $\mathcal{G}^{\mathbb{C}}$-invariant holomorphic map from $\mathcal{B}^{ss}$ into a complex space factors through the quotient map $\pi\colon \mathcal{B}^{ss} \to \mathcal{B}^{ss} /\!\!/ \mathcal{G}^{\mathbb{C}}$.*

To make sense of (3) in Theorem I, we equip the space $\mathcal{B}$ with a naive structure sheaf by restricting the sheaf of I-holomorphic functions on $\mathcal{C}$ to $\mathcal{B}$. Moreover, $\mathcal{O}_{\mathcal{M}}$ denotes the structure sheaf of the moduli space $\mathcal{M}$.

Generalizing the descent lemma for vector bundles in [13], we will prove the following (cf. [58, Lemma 2.13]).

Theorem J. *Let $\mathbb{E} \to \mathcal{B}^{ss}$ be a holomorphic $\mathcal{G}^{\mathbb{C}}$-bundle. Suppose that the stabilizer $\mathcal{G}^{\mathbb{C}}_{(A,\Phi)}$ acts trivially on the fiber $\mathbb{E}_{(A,\Phi)}$ for every $(A, \Phi) \in \mu^{-1}(0)$. Then, there is a holomorphic vector bundle E over $\mathcal{M}$ such that $\pi^* E = \mathbb{E}$. Moreover, $\mathcal{O}(E) = \pi_* \mathcal{O}(\mathbb{E})^{\mathcal{G}^{\mathbb{C}}}$, where $\mathcal{O}(E)$ and $\mathcal{O}(\mathbb{E})$ are sheaves of holomorphic sections of E and $\mathbb{E}$, respectively.*

Now we are able to construct an ample line bundle on $\mathcal{M}$ as follows. Recall that $\mathcal{A}$ is an infinite-dimensional Kähler manifold that is modeled on $\Omega^1(\mathfrak{g}_E)$ (see [3, p.587]). In [12], Donaldson constructed a holomorphic line bundle on $\mathcal{A}$ together with a Hermitian metric whose curvature is a multiple of Kähler form on $\mathcal{A}$ (also see [50]). Moreover, the $\mathcal{G}^{\mathbb{C}}$-action on $\mathcal{A}$ lifts to this line bundle. By pulling back this line bundle to $\mathcal{C}$ by the projection map $\mathcal{C} \to \mathcal{A}$, we obtain an I-holomorphic line bundle $\mathbb{L} \to \mathcal{C}$, and the $\mathcal{G}^{\mathbb{C}}$-action on $\mathcal{C}$ lifts to $\mathbb{L}$. By slightly modifying the pullback Hermitian metric, we are able to show that the curvature of the resulting Hermitian metric h on $\mathbb{L}$ is $-2\pi\sqrt{-1}\Omega_I$. Then, the projectivity of the compactification $\overline{\mathcal{M}}$ is shown in the following result.

Theorem K.

1. *The restriction of the line bundle $\mathbb{L} \to \mathcal{C}$ to $\mathcal{B}^{ss}$ descends to $\mathcal{M}$ and defines a line bundle $\mathcal{L} \to \mathcal{M}$.*

2. *$\mathcal{L}$ extends to a line bundle $\overline{\mathcal{L}}$ on $\overline{\mathcal{M}}$.*

3. $\overline{\mathcal{L}}$ *is ample.*

Therefore, $\overline{\mathcal{M}}$ *is projective, and hence* $\mathcal{M}$ *is quasi-projective.*

In the proof of (1) in Theorem K, a byproduct is that the moduli space $\mathcal{M}$ has a weak Kähler metric. More precisely, by Theorem D, E and F, $\mathcal{M}$ admits an orbit type stratification such that each stratum Q is a complex submanifold of $\mathcal{M}$ together with a Kähler form ω_Q. A weak Kähler metric on $\mathcal{M}$ is a family of continuous stratum-wise strictly plurisubharmonic functions $\rho_i \colon U_i \to \mathbb{R}$ such that $\{U_i\}$ is an open covering of $\mathcal{M}$ and that $\rho_i - \rho_j = \Re(f_{ij})$ for some holomorphic function $f_{ij} \in \mathcal{O}_{\mathcal{M}}(U_i \cap U_j)$. Here, a continuous stratum-wise strictly plurisubharmonic function is a continuous function that is smooth and strictly plurisubharmonic along every stratum Q in the orbit type stratification of $\mathcal{M}$. Note that stratum-wise strictly plurisubharmonic functions are not necessarily strictly plurisubharmonic. If each ρ_i can be chosen to be strictly plurisubharmonic, then $\{\rho_i \colon U_i \to \mathbb{R}\}$ defines a (strong) Kähler metric on $\mathcal{M}$. (see [29] for more details on strictly plurisubharmonic functions). Finally, since $\sqrt{-1}\partial\overline{\partial}(\rho_i|_Q)$ patches together, the Kähler metric on $\mathcal{M}$ restricts to Q. Then, our last result is the following.

Theorem L. *The moduli space* $\mathcal{M}$ *admits a weak Kähler metric whose restriction to each stratum* Q *in the orbit type stratification of* $\mathcal{M}$ *is the Kähler form* ω_Q.

We are unable to prove that the weak Kähler metric on $\mathcal{M}$ is strong, although it is highly likely. Moreover, it should be noted that we only work with reduced complex spaces in this thesis.

1.2 Guidance for readers

In this section, we give the outline of this thesis and the ideas of proofs. In §2, we will review the notions of analytic Hilbert quotients, stratified complex symplectic spaces and the Yang-Mills-Higgs flow, and their useful properties.

In §3, we will prove Theorem A, B and C. The major step in the proof of Theorem A and B is to construct a Kuranishi local model for $\mathcal{M}$ at every Higgs bundle (A,Φ) that satisfies Hitchin's equation. This is done in §3.2. Here, a Kuranishi local model is the analytic Hilbert quotient (developed by Heinzner and Loose in [30]) of a Kuranishi space in $\mathbf{H}^1$ by the $\mathcal{G}^{\mathbb{C}}$-stabilizer at (A,Φ), and is homeomorphic to an open neighborhood of (A,Φ) in $\mathcal{M}$. After that, we will show that the transition functions associated with Kuranishi local models are holomorphic so that $\mathcal{M}$ is a complex space. This is done in §3.3. To prove Theorem B, we adapt Huebschmann's argument in [34, Corollary 2.20] which is further based on Arms-Marsden-Moncrief [2]. This is done in §3.4.

The techniques in the construction of Kuranishi local models mainly come from [62], [11] and [37]. Let H be the $\mathcal{G}$-stabilizer at (A,Φ) with $\mu(A,\Phi)=0$ so that $H^{\mathbb{C}}$ is the $\mathcal{G}^{\mathbb{C}}$-stabilizer. We will construct a H-equivariant perturbed Kuranishi map Θ (following Székelyhidi's argument in [62, Proposition 7]) that is defined on a Kuranishi space in $\mathbf{H}^1$ and takes values in $\mathcal{B}^{ss}$ such that the pullback moment map $\Theta^*\mu$ is a moment map for the H-action on $\mathbf{H}^1$ with respect to the pullback symplectic form $\Theta^*\Omega_I$. Then, roughly speaking, an $H^{\mathbb{C}}$-orbit is closed in $\mathbf{H}^1$ if and only if it contains a zero of the pullback moment map $\Theta^*\mu$. The precise statement is given in Theorem 3.2.6 (cf. [11, Theorem 2.9], [37, Proposition 3.8], [7, Proposition 2.4] and [65, Proposition 3.3.2]). Since the perturbed Kuranishi map Θ is no longer holomorphic, $\Theta^*\Omega_I$ is not a Kähler form on $\mathbf{H}^1$, which causes some trouble. To remedy this problem, in the proof of Theorem 3.2.6, the Yang-Mills-Higgs flow will be used to detect polystable orbits in $\mathcal{B}^{ss}$. Since Kuranishi spaces are locally complete, every Yang-Mills-Higgs flow near (A, Φ) induces a "reduced flow" in $\mathbf{H}^1$ that stays in a single $H^{\mathbb{C}}$-orbit and converges to a zero of Θ. Therefore, if a $H^{\mathbb{C}}$-orbit is closed, it contains a zero of Θ. Hence, Θ maps polystable $H^{\mathbb{C}}$-orbits in $\mathbf{H}^1$ to polystable orbits in $\mathcal{B}^{ss}$ so that Θ induces a map from a Kuranishi local model to $\mathcal{M}$. The rest of the proof is to show that this map is an open embedding.

The idea to prove Theorem C is the following. It is easy to see that i is a bijection. To show that it is continuous, recall that Nitsure constructed a scheme F^{ss} in [48] that parameterizes semistable Higgs bundles on M, and $\mathcal{M}_{alg}$ is a good quotient of F^{ss}. We show that the comparison map i can be locally lifted to a map σ, called a *classifying map*, that is defined locally on $\mathcal{B}^{ss}$ and takes values in F^{ss}. Here, the terminology comes from Sibley and Wentworth's paper [52], and we adapt the proof of Theorem 6.1 in this paper to show that σ is continuous with respect to the C^∞-topology on $\mathcal{B}^{ss}$ and the analytic topology on F^{ss}. Therefore, i is continuous. By the properness of the Hitchin fibration defined on $\mathcal{M}_{an}$, we see that i is proper and hence a homeomorphism. Then, by constructing Kuranishi families of stable Higgs bundles, we show that the restriction $i\colon \mathcal{M}^s_{an} \to \mathcal{M}^s_{alg}$ is a biholomorphism, where $\mathcal{M}^s_{an}$ and $\mathcal{M}^s_{alg}$ are the open subsets of $\mathcal{M}_{an}$ and $\mathcal{M}_{alg}$ consisting of stable Higgs bundles, respectively. By the normality of $\mathcal{M}_{alg}$, the holomorphicity of $i^{-1}|_{\mathcal{M}^s_{alg}}$ can be extended to i^{-1}. Then, we use Theorem B to prove that $\mathcal{M}_{an}$ is normal. The rest of the proof follows from the fact that a holomorphic bijection between normal, reduced and irreducible complex spaces of the same dimension is a biholomorphism.

In §4, we will prove Theorem D, E, F, and G. To prove Theorem D and the first part of Theorem E, the basic tools are local slice theorems for the $\mathcal{G}$-action and the $\mathcal{G}^{\mathbb{C}}$-action. Since the $\mathcal{G}$-action is proper, its local slice theorem is available. To obtain a local slice theorem for the $\mathcal{G}^{\mathbb{C}}$-action around Higgs bundles satisfying Hitchin's equation, we adapt Buchdahl and Schumacher's argument in [8, Proposition 4.5]. To prove the second part of Theorem E, we simply follow Mayrand's arguments in [44, §4.6, §4.7]. The idea is that the Whitney conditions and the frontier condition are local conditions and therefore can be checked on an open neighborhood of $[0]$ in $\nu_{0,\mathbb{C}}^{-1}(0) /\!\!/ H^{\mathbb{C}}$, provided that the biholomorphism between $\mathcal{M}$ and a local model $\nu_{0,\mathbb{C}}^{-1}(0) /\!\!/ H^{\mathbb{C}}$ preserves the orbit type decompositions. We will prove that this is the case. These results are contained in §4.2 and §4.3.

To prove Theorem F, the major obstacle is to show that the Hitchin-Kobayashi correspondence preserves orbit types. We will follow Sjamaar's argument in [58, Theorem 2.10]. However, this argument crucially relies on Mostow's decomposition for complex reductive Lie groups. Since $\mathcal{G}^{\mathbb{C}}$ is infinite-dimensional, we need to extend Mostow's decomposition to $\mathcal{G}^{\mathbb{C}}$ in the following way.

Theorem M (Mostow's decomposition)**.** *Let H be a compact subgroup of $\mathcal{G}$ and $\mathfrak{h}$ its Lie algebra. The map*

$$\mathfrak{h}^{\perp} \times_H \mathcal{G} \to \mathcal{G}^{\mathbb{C}}/H^{\mathbb{C}}, \qquad [s,u] \mapsto H^{\mathbb{C}} \exp(is)u, \tag{1.14}$$

is a $\mathcal{G}$-equivariant bijection, where $\mathcal{G}$ acts on both sides by right multiplication, and $\mathfrak{h}^\perp$ is the L^2-orthogonal complement of $\mathfrak{h}$ in the Lie algebra $\Omega^0(\mathfrak{g}_E)$ of $\mathcal{G}$.

It is likely that the map mentioned in Theorem M is not only a bijection but also a diffeomorphism. That said, for the purpose of this thesis, a bijection is all we need. Once Mostow's decomposition for $\mathcal{G}^{\mathbb{C}}$ is established, the rest of the proof follows easily. To prove Theorem M, we will instead prove that the map $H^{\mathbb{C}} \times_H (\mathfrak{h}^\perp \times \mathcal{G}) \to \mathcal{G}^{\mathbb{C}}$ is a bijection (see Theorem 4.1.1 for a more precise statement). To this end, following the Heinzner and Schwarz's idea in [32, §9], we will realize $\mathfrak{h}^\perp \times \mathcal{G}$ as a zero set of some moment map on $\mathcal{G}^{\mathbb{C}}$. Therefore, we need to show that $\mathcal{G}^{\mathbb{C}}$ is a weak Kähler manifold and that the left H-action on $\mathcal{G}^{\mathbb{C}}$ is Hamiltonian. In [35], Huebschmann and Leicht provided a framework to deal with this problem. Although their results are in finite-dimensional settings, they can be carried out for $\mathcal{G}^{\mathbb{C}}$ without any problems. For the sake of completeness, we provide the details in §4.6, and the proofs are taken or adapted from [35]. Then, it will be shown that every $H^{\mathbb{C}}$-orbit in $\mathcal{G}^{\mathbb{C}}$ intersects $\mathfrak{h}^\perp \times \mathcal{G}$, and the intersection is a single H-orbit. Here, we will use the framework laid out in Mundet I Riera's paper [47]. All these results will be proved in §4.1.

To prove Theorem G, we need to define a complex Poisson bracket on the structure sheaf of $\mathcal{M}$. Since every stratum in the orbit type decomposition has a complex Poisson bracket, and $\mathcal{M}$ is a disjoint union of these strata, we may pointwise define the complex Poisson bracket of any two holomorphic functions on $\mathcal{M}$. Therefore, the real question is to answer whether the resulting function is still holomorphic. We will show that the local biholomorphism between $\mathcal{M}$ and a local model $\nu_{0,\mathbb{C}}^{-1}(0) /\!\!/ H^{\mathbb{C}}$ is a Poisson map. Then, Theorem G follows from this. Now the key observation to see that the local biholomorphism is a Poisson map is that the Kuranishi map θ induces the local biholomorphism and preserves the complex symplectic structures on $\mathbf{H}^1$ and $\mathcal{C}$. Moreover, all the complex symplectic structures on the strata in the orbit type decompositions of $\mathcal{M}$ and $\nu_{0,\mathbb{C}}^{-1}(0) /\!\!/ H^{\mathbb{C}}$ come from those on $\mathcal{C}$ and $\mathbf{H}^1$.

In §5, we will prove Theorem I, J, K and L. The key tools in the proof of Theorem I are a local slice theorem for the $\mathcal{G}^{\mathbb{C}}$-action and the retraction $r\colon \mathcal{B}^{ss} \to \mu^{-1}(0) \cap \mathcal{B}$ defined by the Yang-Mills-Higgs flow. We will prove Theorem I in §5.1. To prove Theorem J, we will first prove a descent lemma for analytic Hilbert quotients of complex spaces. The proof of Theorem J is an adaptation of Drezet and Narasimhan's argument in [13]. Then, this result will be applied to Kuranishi local models that are used to construct the moduli space $\mathcal{M}$, since Kuranishi local models are analytic Hilbert quotients of Kuranishi spaces. In this way, we can show that every point in $\mathcal{B}^{ss}$ admits an open neighborhood that is saturated with respect to the quotient map $\pi\colon \mathcal{B}^{ss} \to \mathcal{M}$ and in which the vector bundle $\mathbb{E}$ in question is trivial. This shows that $\mathbb{E}$ descends to $\mathcal{M}$. These results will be proved in §5.2.

After Theorem I and J are proved, we are ready to prove Theorem L and (1) in Theorem K. By verifying the hypothesis in Theorem J for the line bundle $\mathbb{L}|_{\mathcal{B}^{ss}}$, we can easily show that it defines a line bundle $\mathcal{L} \to \mathcal{M}$. To show Theorem L, we may choose an open covering $\{U_i\}$ of $\mathcal{M}$ such that $\mathcal{L}$ is trivial over each $\pi^{-1}(U_i)$. Then, we choose a holomorphic section s_i of $\mathbb{L}$ over each $\pi^{-1}(U_i)$ that is $\mathcal{G}^{\mathbb{C}}$-equivariant and nowhere vanishing. Then, we consider the functions

$$u_i = -\frac{1}{2\pi}\log|s_i|_h^2, \tag{1.15}$$

where h is the Hermitian metric on $\mathbb{L}$. Since it is $\mathcal{G}$-invariant, its restriction to $\pi^{-1}(U_i)\cap\mu^{-1}(0)$ defines a continuous map $u_{i,0}\colon U_i\to\mathbb{R}$. It will be shown that the restriction of each $u_{i,0}$ to each stratum Q is smooth and a Kähler potential for the Kähler form ω_Q on Q. In this way, we obtain a family of continuous stratum-wise strictly plurisubharmonic functions $u_{i,0}\colon U_i\to\mathbb{R}$ such that $\{U_i\}$ covers $\mathcal{M}$. Then, the normality of $\mathcal{M}$ and the fact that $\operatorname{codim}_x(\mathcal{M}\setminus\mathcal{M}^s)\geq 2$ for all $x\in\mathcal{M}\setminus\mathcal{M}^s$ show that $\{u_{i,0}\colon U_i\to\mathbb{R}\}$ defines a weak Kähler metric on $\mathcal{M}$. These results will be proved in §5.3.

Then, we will prove Theorem H and the rest of the statements in Theorem K in §5.4. Following Hausel's strategy in [25], we will use the symplectic cut to compactify $\mathcal{M}$. Recall that $\mathcal{M}$ admits a holomorphic $\mathbb{C}^*$-action. Moreover, the induced $U(1)$-action is stratum-wise Hamiltonian. More precisely, the restriction of the $\mathcal{G}$-invariant map

$$f(A,\Phi) = -\frac{1}{4\pi^2}\frac{1}{2}\|\Phi\|_{L^2}^2\colon \mathcal{C}\to\mathbb{R} \tag{1.16}$$

to $\mu^{-1}(0)$ defines a continuous map $f\colon\mathcal{M}\to\mathbb{R}$. When restricted to a stratum Q, $f|_Q$ is smooth and a moment map for the induced $U(1)$-action with respect to the Kähler form ω_Q on Q. In this sense, f is a stratum-wise moment map on $\mathcal{M}$. Then, we consider the direct product $\mathcal{M}\times\mathbb{C}$. If we let $\mathbb{C}^*$ act on $\mathbb{C}$ by multiplication,

$\mathcal{M} \times \mathbb{C}$ admits a diagonal $\mathbb{C}^*$-action. The induced $U(1)$-action is also stratum-wise Hamiltonian. Here, the stratification of $\mathcal{M}\times\mathbb{C}$ is given by the disjoint union of $Q\times\mathbb{C}$, where Q ranges in the orbit type stratification of $\mathcal{M}$. Moreover, the stratum-wise moment map on $\mathcal{M} \times \mathbb{C}$ is given by

$$\widetilde{f} = f - \frac{1}{2}\|\cdot\|^2. \tag{1.17}$$

By [33, Theorem 8.1] or [66, Theorem 2.15], the Hitchin fibration h is proper, and hence the nilpotent cone $h^{-1}(0)$ is compact. Therefore, we are able to choose a level $c < 0$ such that $h^{-1}(0) \subset f^{-1}[0, c)$. Then the symplectic cut of $\mathcal{M}$ at the level c is defined as the singular symplectic quotient $\widetilde{f}^{-1}(c)/U(1)$, and it should be a compactification of $\mathcal{M}$. Here, the rough idea is that the subspace $f^{-1}[0, c]$ is compact by the properness of f (see [33, Proposition 7.1]). Moreover, if a Higgs bundle is away from $f^{-1}[0, c]$, following its $\mathbb{C}^*$-orbit, it "flows" into $f^{-1}[0, c]$, since the 0-limit of the $\mathbb{C}^*$-action on a Higgs bundle always exists, and hence the limiting point is a $\mathbb{C}^*$-fixed point and is contained in the nilpotent cone. Therefore, the moduli space $\mathcal{M}$ should be "contained in" $\widetilde{f}^{-1}(c)/U(1)$, which is compact because of the properness of f.

To carry out this idea rigorously, we first need to equip $\widetilde{f}^{-1}(c)/U(1)$ with the structure of a complex space. Let $(\mathcal{M} \times \mathbb{C})^{ss}$ be the subspace of semistable points in $\mathcal{M} \times \mathbb{C}$ determined by the stratum-wise moment map $\widetilde{f} - c$. More precisely, it consists of points in $\mathcal{M} \times \mathbb{C}$ whose $\mathbb{C}^*$-orbit closures intersect $\widetilde{f}^{-1}(c)$. To show that the analytic Hilbert quotient of $(\mathcal{M} \times \mathbb{C})^{ss}$ by $\mathbb{C}^*$ exists, we run into a technical difficulty. Since we are unable to prove that the Kähler metric on $\mathcal{M}$ is a strong one, we cannot directly apply the analytic GIT developed by Heinzner and Loose in [30] and must take a detour. To motivate the following detour, let us recall that a complex reductive Lie group acts properly at a point if and only if its stabilizer at that point is finite, provided that a local slice theorem is available around that point. Since the $\mathbb{C}^*$-stabilizers are finite away from the nilpotent cone $h^{-1}(0)$, it is reasonable to expect that the $\mathbb{C}^*$-action acts properly away from the nilpotent cone. Hence, we consider the $\mathbb{C}^*$-invariant open subset $W = (\mathcal{M} \times \mathbb{C}) \setminus (h^{-1}(0) \times \{0\})$. By the properness of the Hitchin fibration h, we can show that the $\mathbb{C}^*$-action on W is proper, and hence the analytic Hilbert quotient of W by $\mathbb{C}^*$ exists. Moreover, $W/\mathbb{C}^*$ is a geometric quotient. Then, we use the properness of h and f to show that $W = (\mathcal{M} \times \mathbb{C})^{ss} = \mathbb{C}^* \widetilde{f}^{-1}(c)$. It then follows that the inclusion $\widetilde{f}^{-1}(c) \hookrightarrow W$ induces a homeomorphism $\widetilde{f}^{-1}(c)/U(1) \to W/\mathbb{C}^*$. Now, note that W can be written as a disjoint union

$$W = (\mathcal{M} \setminus h^{-1}(0) \times \{0\}) \cup (\mathcal{M} \times \mathbb{C}^*). \tag{1.18}$$

We will show that the quotient $(\mathcal{M} \times \mathbb{C}^*)/\mathbb{C}^*$ is biholomorphic to the moduli space $\mathcal{M}$, and therefore $\overline{\mathcal{M}} = W/\mathbb{C}^*$ is a compactification of $\mathcal{M}$.

To show the rest of the statements in Theorem K, we pullback the line bundle $\mathcal{L} \to \mathcal{M}$ to $\mathcal{M} \times \mathbb{C}$ by the projection map $\mathcal{M} \times \mathbb{C} \to \mathcal{M}$ to obtain a line bundle $\mathcal{L}_{\mathbb{C}} \to \mathcal{M} \times \mathbb{C}$. By slightly modifying the Hermitian metric h on $\mathcal{L}_{\mathbb{C}}$, we can easily show that the resulting Hermitian metric, again denoted by h, is smooth along each stratum $Q \times \mathbb{C}$, and the curvature is $-2\pi\sqrt{-1}\omega_{Q\times\mathbb{C}}$, where $\omega_{Q\times\mathbb{C}}$ is the product

Kähler metric on $Q \times \mathbb{C}$. By the descent lemma for the analytic Hilbert quotients, the restriction of $\mathcal{L}_{\mathbb{C}}$ to W induces a line bundle $\overline{\mathcal{L}} \to \overline{\mathcal{M}}$ such that the restriction of $\overline{\mathcal{L}}$ to $(\mathcal{M} \times \mathbb{C}^*)/\mathbb{C}^*$ is isomorphic to $\mathcal{L} \to \mathcal{M}$. In this sense, the line bundle $\mathcal{L} \to \mathcal{M}$ extends to the line bundle $\overline{\mathcal{L}} \to \overline{\mathcal{M}}$. Moreover, the Hermitian metric h on $\mathcal{L}_{\mathbb{C}}$ also induces a Hermitian metric $\overline{h}$ on $\overline{\mathcal{L}}$. Then, we will use Popovici's bigness criterion (see [49, Theorem 1.3]) to show that the restriction of $\overline{\mathcal{L}}$ to any irreducible closed complex subspace (not reduced to a point) of $\overline{\mathcal{M}}$ is big. Then, the ampleness of $\overline{\mathcal{L}}$ follows from a theorem of Grauert (see [21]): a line bundle over a compact complex space is ample if its restriction to any irreducible closed complex subspace (not reduced to a point) admits a nontrivial holomorphic section that vanishes somewhere on that subspace. These results will be proved in §5.4.

Chapter 2: Preliminaries

2.1 Stratified complex symplectic spaces

In this section, we review the notion of stratifications in complex spaces (see [24] and [44] for more details).

Let X be a topological space. A *stratification* of X is a countable and locally finite covering of X by disjoint locally closed subspaces $\{Q_i\}$ such that the following hold:

1. Each stratum Q_i is a topological manifold.

2. For every strata Q_i and Q_j with $Q_i \cap \overline{Q_j} \neq \emptyset$, $Q_i \subset \overline{Q_j}$.

The *Whitney conditions A and B* specify how the strata fit together. Let Q_1 and Q_2 be two disjoint smooth submanifolds of $\mathbb{R}^n$. Q_1 is said to be regular over Q_2 if the following conditions hold for all $y \in \overline{Q_1} \cap Q_2$:

1. (Whitney condition A) If $x_i \in Q_1$ is a sequence converging to y and the sequence of subspaces $T_{x_i}S \subset \mathbb{R}^n$ converges (in the Grassmannian) to some $V \subset \mathbb{R}^n$, then $T_yQ_2 \subset V$.

2. (Whitney condition B) If $x_i \in Q_1$ and $y_i \in Q_2$ are two sequences converging

to y in such a way that the sequence of lines $\mathbb{R}(x_i - y_i) \subset \mathbb{R}^n$ converges to some $l \in \mathbb{RP}^{n-1}$ and the subspaces $T_{x_i}S$ to some $V \subset \mathbb{R}^n$, then $l \subset V$.

A *Whitney stratification* of a subspace X of $\mathbb{R}^n$ is a stratification in which Q_1 is regular over Q_2 for every strata Q_1 and Q_2. Although this definition is defined for subspaces in $\mathbb{R}^n$, it is invariant under diffeomorphisms and hence makes sense for complex spaces. Therefore, we may define a *complex Whitney stratified space* as a complex space X with a Whitney stratification such that each stratum is a complex submanifold of X.

Finally, we define a *stratified complex symplectic space* as a complex space X together with a complex Whitney stratification, a complex symplectic structure on each stratum, and a sheaf of Poisson brackets on $\mathcal{O}_X$ such that the embedding $Q \hookrightarrow X$ is a holomorphic Poisson map for every stratum Q. It is the complex analogue of stratified symplectic spaces introduced in [59].

2.2 Analytic Hilbert quotients

Since the ultimate goal in this thesis is to construct the moduli space of Higgs bundles using analytic methods, quotients of complex spaces by complex reductive Lie groups will play a major role. In this section, we review the notion of analytic Hilbert quotients and their useful results. Roughly speaking, analytic Hilbert quotients are an analogue of good quotients in the category of algebraic varieties. More details can be found in [28] and [31].

Let G be a complex reductive Lie group acting holomorphically on a complex space X. An *analytic Hilbert quotient* of X is a complex space $X /\!\!/ G$ together with a surjective G-invariant holomorphic map $\pi\colon X \to X /\!\!/ G$ such that the following hold:

1. π is Stein in the sense that inverse images of Stein subspaces are Stein.

2. $\mathcal{O}_{X/\!\!/G} = \pi_* \mathcal{O}_X^G$. In other words, for every open subset U of $X /\!\!/ G$, the map $\pi^*\colon \mathcal{O}_{X/\!\!/G}(U) \to \mathcal{O}_X(\pi^{-1}U)^G$ is an isomorphism.

Sometimes, they are also called semistable quotients or analytic GIT quotients. Note that analytic Hilbert quotients are categorical quotients in the category of (reduced) complex spaces. Therefore, $X /\!\!/ G$ is unique up to isomorphism. The complex space $X /\!\!/ G$ is said to be the *geometric quotient* of X if fibers of π are exactly G-orbits. The following useful properties are proved in [31, §1, (ii) and §3, Corollary 3]

Proposition 2.2.1. *Let $\pi\colon X \to X /\!\!/ G$ be an analytic Hilbert quotient.*

1. *If U is an π-saturated open subset of X, then $\pi\colon U \to \pi(U)$ is an analytic Hilbert quotient.*

2. *If $Y \subset X$ is a G-invariant closed complex subspace, then $\pi(Y)$ is a closed complex subspace of $X /\!\!/ G$, and $\pi\colon Y \to Y /\!\!/ G$ is an analytic Hilbert quotient.*

3. *Every fiber of π contains a unique G-orbit that is closed in X.*

One sufficient condition that ensures the existence of the analytic Hilbert quotient of X is the properness of the G-action. The following is proved in [31, §4, Corollary 1, 2].

Proposition 2.2.2. *If G acts properly on X, then the set-theoretic quotient X/G is an analytic Hilbert quotient of X. Moreover, for every $x \in X$, there exists a locally closed G_x-invariant subspace S of X such that $G \cdot S$ is open in X and*

$$G \times_{G_x} S \to G \cdot S, \qquad [g, s] \mapsto g \cdot s, \tag{2.1}$$

is biholomorphic.

Note that the second statement is the local slice theorem for the G-action on X.

2.3 Kähler spaces and Kähler quotients

In this section, we review another sufficient condition that ensures the existence of the analytic Hilbert quotient of X. This is where Hamiltonian actions come in. Let us briefly review the smooth case and then the general (singular) case (see [58] and [28, 30] for more details on smooth case and singular case, respectively).

Let (X, ω) be a Kähler manifold and $K^{\mathbb{C}}$ a complex reductive Lie group acting holomorphically on X. Here, K is a maximal compact subgroup of $K^{\mathbb{C}}$. The K-action on X is said to be Hamiltonian if the following hold:

1. The K-action preserves the Kähler form ω.

2. There exists a moment map μ for the K-action with respect to the Kähler

form ω. More precisely, μ is a K-equivariant map $\mu\colon X \to \mathfrak{g}^*$ such that

$$d\langle \mu, \xi \rangle = i_{\xi^\#}\omega \tag{2.2}$$

for every $x \in \mathfrak{g}$, where $\xi^\#$ is the infinitesimal action of ξ on X, and i_v is the contraction operator. Here, K acts on the dual space of the Lie algebra $\mathfrak{g}$ of K by the coadjoint action.

Then, we are able to define the μ-semistable points in X as follows:

$$X^{ss} = \{x \in X\colon \overline{K^{\mathbb{C}} \cdot x} \cap \mu^{-1}(0) \neq \emptyset\}. \tag{2.3}$$

The following is shown in [58, Proposition 2.4, Theorem 2.5].

Proposition 2.3.1.

1. *X^{ss} is open in X.*

2. *The analytic Hilbert quotient $X^{ss} /\!\!/ K^{\mathbb{C}}$ exists such that the inclusion $\mu^{-1}(0) \hookrightarrow X^{ss}$ induces a homeomorphism*

$$\mu^{-1}(0)/K \xrightarrow{\sim} X^{ss} /\!\!/ K^{\mathbb{C}}. \tag{2.4}$$

3. *A $K^{\mathbb{C}}$-orbit is closed in X^{ss} if and only if it intersects $\mu^{-1}(0)$.*

Here, the analytic Hilbert quotient $X^{ss} /\!\!/ K^{\mathbb{C}}$ is called the *Kähler quotient* of X, and $\mu^{-1}(0)/K$ is called the *symplectic quotient* of X. As a consequence of (3) in

Proposition 2.3.1, we may define the subset X^{ps} of polystable points in X as follows:

$$X^{ps} = \{x \in X\colon K^{\mathbb{C}} \cdot x \cap \mu^{-1}(0) \neq \emptyset\}. \tag{2.5}$$

Then, the inclusion $X^{ps} \hookrightarrow X^{ss}$ induces a homeomorphism

$$X^{ps}/K^{\mathbb{C}} \xrightarrow{\sim} X^{ss} /\!\!/ K^{\mathbb{C}}. \tag{2.6}$$

To generalize the above result to the singular case, we need to make sense of Kähler metrics on complex spaces and then Hamiltonian actions on complex spaces. We start with the definition of plurisubharmonic functions. A holomorphic disc in a complex space X is a holomorphic map $\varphi\colon D \to X$, where $D = \{z \in \mathbb{C}\colon |z| < 1\}$ is the open unit disc in $\mathbb{C}$. If U is an open subset of X, a continuous map $\rho\colon U \to \mathbb{R}$ is *plurisubharmonic* if the pullback $\varphi^*\rho$ is subharmonic in D for every holomorphic disc φ in U. Moreover, ρ is *strictly plurisubharmonic* if for every $x \in U$ and a smooth function f defined in an open neighborhood of x there exists $\epsilon > 0$ such that $\rho + \epsilon f$ is plurisubharmonic in an open neighborhood $V \subset U$ of x.

Now, a *Kähler metric* on a complex space X is a family $\{\rho_i\colon U_i \to \mathbb{R}\}$ of continuous strictly plurisubharmonic functions such that the following hold:

1. The open subsets U_i cover X.

2. If $U_i \cap U_j \neq \emptyset$, then $\rho_i - \rho_j = \Re(f_{ij})$ for some $f_{ij} \in \mathcal{O}_X(U_i \cap U_j)$.

Note that if X is a smooth manifold, then a family $\{\rho_i\colon U_i \to \mathbb{R}\}$ of continuous strictly plurisubharmonic functions satisfying the above conditions naturally defines a Kähler form ω on X, since we may simply define ω on each U_i by $\frac{\sqrt{-1}}{2}\partial\overline{\partial}\rho_i$. As a consequence, the above notion does generalize the classical notion of Kähler metrics on complex manifolds. Hence, we define a *Kähler space* as a complex space together with a Kähler metric in the above sense. Those ρ_i are called the Kähler potentials for the Kähler metric. Moreover, the restriction of the Kähler metric to the smooth locus X_{reg} of X defines a Kähler form on X_{reg}. Therefore, the smooth locus of a Kähler space is a Kähler manifold.

Sometimes, the notion above is too strong to work with. More specifically, we would like to replace the strict plurisubharmonicity by a weaker condition. Assume that X admits a complex stratification such that each stratum is a complex submanifold of X. A continuous plurisubharmonic function ρ is said to be stratum-wise strictly plurisubharmonic if the restriction $\rho|_Q$ to each stratum Q is smooth and strictly plurisubharmonic (in the usual sense) on Q. In general, "strictly plurisubharmonic" $\implies$ "stratum-wise strictly plurisubharmonic" $\implies$ "plurisubharmonic", and none of these implications can be reversed.

Then, by replacing "strictly plurisubharmonic functions" by "stratum-wise strictly plurisubharmonic functions" in the definition of Kähler metrics, we obtain the notion of *weak Kähler metrics* on a complex space X. Finally, note that every complex space admits a stratification as follows. The singular locus X_{sing} of X is a closed complex subspace of smaller dimension, and so is the singular locus of X_{sing}. By repeating this procedure, we obtain a natural stratification of X. Moreover, the strata are necessarily $K^{\mathbb{C}}$-invariant. Therefore, every Kähler metric is also a weak Kähler metric.

Now we are able to generalize the notion of Hamiltonian actions to Kähler spaces. A slightly more general definition can be found in [30]. The definition presented here is enough for our purposes. Let X be a complex space and $K^{\mathbb{C}}$ a complex reductive Lie group acting holomorphically on X, where K is a maximal compact subgroup of $K^{\mathbb{C}}$. Suppose that X admits a stratification such that each stratum is a complex submanifold and also $K^{\mathbb{C}}$-invariant. Let $\{\rho_i \colon U_i \to \mathbb{R}\}$ be a Kähler metric on X. Note that each stratum Q thus acquires a Kähler form from the restriction of $\{\rho_i \colon U_i \to \mathbb{R}\}$ to Q. The K-action is said to preserve the Kähler metric if every $k \in K$ defines a pullback Kähler metric $\{k^*\rho_i \colon k^{-1}U_i \to \mathbb{R}\}$ that is equivalent to the original Kähler metric. This means that $k^*\rho_i - \rho_j = \Re(f)$ for some $f \in \mathcal{O}_X(k^{-1}(U_i) \cap U_j)$ whenever $k^{-1}(U_i) \cap U_j \neq \emptyset$. The K-action on X is said to be Hamiltonian if the following hold:

1. The K-action preserves the Kähler metric on X.

2. There exists a stratum-wise moment map μ for the K-action on X, a K-equivariant continuous map $\mu \colon X \to \mathfrak{g}^*$ such that the restriction $\mu|_Q$ to each stratum Q is smooth and defines a moment map for the K-action on Q with respect to the Kähler form ω_Q.

Then, we have the following, which is proved in [30, (1.3), Theorem, (2.7), Theorem, (3.4)]

Proposition 2.3.2.

1. *Proposition 2.3.1 holds for X.*

2. *The local slice theorem for $K^{\mathbb{C}}$-action holds for every $x \in \mu^{-1}(0)$.*

3. *The analytic Hilbert quotient $X^{ss} /\!\!/ K^{\mathbb{C}}$ carries a natural Kähler metric. More precisely, every K-invariant Kähler potential ρ in an open neighborhood of a point in $\mu^{-1}(0)$ descends to a Kähler potential ρ_0 via the map*

$$\pi\colon \mu^{-1}(0) \to \mu^{-1}(0)/K \xrightarrow{\sim} X^{ss} /\!\!/ K^{\mathbb{C}}. \tag{2.7}$$

2.4 Higgs Bundles

In this section, we review basics of Higgs bundles and explain some notions that appear in §1 frequently and are used without explanations. More details can be found in [66] and [39].

2.4.1 Configuration space

We start with some general definitions and then specialize them to the situations we are interested in. Let M be a complex manifold. A *holomorphic vector bundle* $\mathcal{E} \to M$ is a smooth vector bundle over M such that the transition functions are holomorphic. A *Higgs bundle* $(\mathcal{E}, \Phi)$ over M consists of a holomorphic vector bundle $\mathcal{E} \to M$ and a holomorphic 1-form Φ, called the *Higgs field*, taking values in the endomorphism bundle $\operatorname{End}\mathcal{E}$ of $\mathcal{E}$ such that $\Phi \wedge \Phi = 0$.

The first step to construct the moduli space of Higgs bundles is to parameterize the holomorphic vector bundles and then Higgs bundles. To this end, we regard a holomorphic vector bundle as a smooth vector bundle together with an integrable

Dolbeault operator $\overline{\partial}_E$, where E denotes the underlying smooth vector bundle of $\mathcal{E}$. By definition, a *Dolbeault operator* is a $\mathbb{C}$-linear map,

$$\overline{\partial}_E \colon \Omega^0(E) \to \Omega^{0,1}(E), \tag{2.8}$$

satisfying the Leibniz rule, i.e.,

$$\overline{\partial}_E(fs) = \overline{\partial} f \otimes s + f\overline{\partial}_E s \tag{2.9}$$

for all sections s of E and smooth functions f on M. Note that the operator $\overline{\partial}_E$ can be naturally extended to a unique map

$$\overline{\partial}_E \colon \Omega^{p,q}(E) \to \Omega^{p,q+1}(E) \tag{2.10}$$

by the formula

$$\overline{\partial}_E(\alpha \otimes s) = \overline{\partial}\alpha \otimes s + (-1)^{p+q}\alpha \wedge \overline{\partial}_E s. \tag{2.11}$$

An *integrable Dolbeault operator* is a Dolbeault operator $\overline{\partial}_E$ such that $\overline{\partial}_E^2 = 0$. The requirement $\overline{\partial}_E^2 = 0$ makes sense by the above extension.

Every holomorphic vector bundle $\mathcal{E}$ defines an integrable Dolbeault operator $\overline{\partial}_E$ as follows. Let $\{s_i\}$ be a holomorphic local frame for $\mathcal{E}$. Let σ be a smooth section of $\mathcal{E}$. Then locally we may write $\sigma = \sigma^i s_i$ for some smooth functions σ^i on

M. (Here, we are using the Einstein summation convention.) Then, we define

$$\overline{\partial}_E \sigma = \overline{\partial} \sigma^i \otimes s_i. \tag{2.12}$$

Since the transition functions of $\mathcal{E}$ are holomorphic, this definition is independent of the choices of the holomorphic local frames for $\mathcal{E}$ and defines an integrable Dolbeault operator. Note that σ is holomorphic if and only if $\overline{\partial}_E \sigma = 0$. Conversely, we have the following result (see [3, §5]).

Proposition 2.4.1. *Let E be a smooth vector bundle and $\overline{\partial}_E$ an integrable Dolbeault operator on E. Then, there is a unique holomorphic structure on E such that a local section s is holomorphic if and only if $\overline{\partial}_E s = 0$.*

Therefore, parameterizing the holomorphic structures on E is equivalent to parameterizing integrable Dolbeault operators on E.

Another closely related notion is the notion of connections on E. A *connection* A on E is a $\mathbb{C}$-linear map

$$d_A \colon \Omega^0(E) \to \Omega^1(E) \tag{2.13}$$

satisfying the Leibniz rule. Since $\Omega^1(E) = \Omega^{1,0}(E) \oplus \Omega^{0,1}(E)$, the connection d_A can be decomposed as

$$d_A = \partial_A + \overline{\partial}_A, \tag{2.14}$$

where ∂_A and $\overline{\partial}_A$ are the $(1,0)$-component and $(0,1)$-component of d_A, respectively. It is easy to see that the $(0,1)$-component $\overline{\partial}_A$ of A is a Dolbeault operator on E. Conversely, we have the following (see [39, Proposition 1.4.9]).

Proposition 2.4.2. *Let h be a Hermitian metric on E. For every integrable Dolbeault operator $\overline{\partial}_E$, there is a unique unitary connection A, i.e.,*

$$d_A h(s,t) = h(d_A s, t) + h(s, d_A t) \tag{2.15}$$

for every sections s and t of E, such that $\overline{\partial}_A = \overline{\partial}_E$. Such a connection is called the Chern connection associated with $\overline{\partial}_E$ and h.

As a consequence, if a Hermitian metric on E if fixed, then parameterizing the holomorphic structures on E is equivalent to parameterizing unitary connections on E.

Now let us specialize the above discussion to the situation we are interested in. Let M be a Riemann surface and E a Hermitian vector bundle on M. Since $\dim_{\mathbb{C}} M = 1$, the integrability condition for Dolbeault operators on E is vacuous. Therefore, from the discussion above, the space of holomorphic structures on E can be identified with the space of Dolbeault operators on E, which can be further identified with the space $\mathcal{A}$ of unitary connections on E. Note that $\mathcal{A}$ is an infinite-dimensional affine space modeled on $\Omega^1(\mathfrak{g}_E)$, where $\mathfrak{g}_E$ is the bundle of skew-Hermitian endomorphisms. This is because $d_A + \alpha$ with $\alpha \in \Omega^1(\mathfrak{g}_E^{\mathbb{C}})$ is a unitary connection if and only if $\alpha \in \Omega^1(\mathfrak{g}_E)$. Define $\mathcal{C} = \mathcal{A} \times \Omega^{1,0}(\mathfrak{g}_E^{\mathbb{C}})$. Then, the configuration space of Higgs bundles (with underlying smooth bundle E) is defined as

$$\mathcal{B} = \{(A, \Phi) \colon \overline{\partial}_A \Phi = 0\}. \tag{2.16}$$

Note that the holomorphicity condition $\overline{\partial}_A \Phi = 0$ means precisely that Φ is a $\mathfrak{g}_E^{\mathbb{C}}$-valued $(1,0)$-form that is holomorphic with respect to the holomorphic structure determined by the Dolbeault operator $\overline{\partial}_A$. Moreover, we also note that the condition $\Phi \wedge \Phi = 0$ is vacuous, since $\dim_{\mathbb{C}} M = 1$.

Finally, in the category of Higgs bundles, there is a natural notion of homomorphisms. A homomorphism f from a Higgs bundle $(\mathcal{E}, \Phi)$ to another $(\mathcal{F}, \Psi)$ is simply a holomorphic bundle map $f \colon \mathcal{E} \to \mathcal{F}$ that commutes with the Higgs fields, i.e., $f \circ \Phi = \Psi \circ f$. In the language of Dolbeault operators, a homomorphism f from a Higgs bundle $(E, \overline{\partial}_E, \Phi)$ to another $(F, \overline{\partial}_F, \Psi)$ is a smooth bundle map $f \colon E \to F$ such that $f \circ \overline{\partial}_E = \overline{\partial}_F \circ f$ and $f \circ \Phi = \Psi \circ f$.

As a consequence, if g is a group element from the automorphism bundle $\mathcal{G}^{\mathbb{C}} = \mathrm{Aut}(E)$ of E, it acts on a Higgs bundle $(\overline{\partial}_E, \Phi)$ by the formula

$$(\overline{\partial}_E, \Phi) \cdot g = (g^{-1} \circ \overline{\partial}_E \circ g, g^{-1} \Phi g). \tag{2.17}$$

The group $\mathcal{G}^{\mathbb{C}}$ is called the *complex gauge group* of E. Let A be the Chern connection associated with $\overline{\partial}_E$. Since $g^{-1} \circ \overline{\partial}_E \circ g$ is also a Dolbeault operator on E, it has a Chern connection $A \cdot g$. The map $A \mapsto A \cdot g$ defines the action of g on the connection A. In this way, the complex gauge group $\mathcal{G}^{\mathbb{C}}$ acts on the configuration space $\mathcal{B}$. Clearly, two Higgs bundles (A_1, Φ_1) and (A_2, Φ_2) are isomorphic as Higgs bundles if and only if they are on the same $\mathcal{G}^{\mathbb{C}}$-orbit.

2.4.2 Stability conditions

Naively, one may define the moduli space of Higgs bundles as the quotient $\mathcal{B}/\mathcal{G}^{\mathbb{C}}$ equipped with the C^∞-topology. The problem is that this quotient is not even Hausdorff. Therefore, stability conditions must be introduced to rule out some "ill-behaved" Higgs bundles.

Let us first review the definition of the degrees of holomorphic vector bundles. Let E be a smooth vector bundle and $\overline{\partial}_E$ a Dolbeault operator on E. Let us fix a Hermitian metric h on E. Then, the Chern connection $A = (\overline{\partial}_E, h)$ associated with $\overline{\partial}_E$ and h exists. Note that the map $d_A \colon \Omega^0(E) \to \Omega^1(E)$ can be uniquely extended to a map

$$d_A \colon \Omega^m(E) \to \Omega^{m+1}(E) \tag{2.18}$$

by the formula

$$d_A(\alpha \otimes s) = d\alpha \otimes s + (-1)^m \alpha \wedge d_A s. \tag{2.19}$$

As a consequence, $F_A = d_A^2$ is well-defined. Since $F_A(fs) = fF_A(s)$ for every section s of E and smooth function f on M, F_A defines a 2-form taking values in $\mathfrak{g}_E$. This 2-form $F_A \in \Omega^2(\mathfrak{g}_E)$ is called the *curvature* of the connection A. By the Chern-Weil theory, the integer

$$\deg(E) = \frac{\sqrt{-1}}{2\pi} \int_M \operatorname{tr} F_A \in \mathbb{Z} \tag{2.20}$$

is independent of the choice of the Hermitian metric h. Moreover, it actually does not depend on the holomorphic structure $\overline{\partial}_E$ and hence a purely topological invariant of E. This integer is called the *degree* of E. Then, the *slope* of E is defined as $\mu(E) = \deg(E)/\operatorname{rank}(E)$.

Now we are able to define the notions of stability, semistability and polystability of Higgs bundles. A Higgs bundle $(\mathcal{E}, \Phi)$ is semistable if $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$ for every Φ-invariant holomorphic subbundle $0 \subsetneq \mathcal{F} \subsetneq \mathcal{E}$. If the equality $\mu(\mathcal{F}) = \mu(\mathcal{E})$ cannot occur, then $(\mathcal{E}, \Phi)$ is stable. Finally, $(\mathcal{E}, \Phi)$ is polystable if it is a direct sum of stable Higgs bundles with the same slope. Here, a Φ-invariant holomorphic subbundle $\mathcal{F}$ is a holomorphic subbundle $\mathcal{F}$ of $\mathcal{E}$ such that $\Phi(\mathcal{F}) \subset \mathcal{F} \otimes \mathcal{K}_M$, where $\mathcal{K}_M$ is the canonical bundle of M. This makes sense since the Higgs field Φ can be regarded as a map $\Phi \colon \mathcal{E} \to \mathcal{E} \otimes \mathcal{K}_M$. In general, these stability conditions are "ordered" in the following way:

$$\text{stable} \implies \text{polystable} \implies \text{semistable} \tag{2.21}$$

When the degree $\deg(E)$ and the rank $\operatorname{rank}(E)$ are coprime, every semistable Higgs bundle is also stable.

A simple consequence of the stability conditions is the following (see [66, Lemma 2.8]).

Proposition 2.4.3. *Let $f \colon (\mathcal{E}, \Phi) \to (\mathcal{F}, \Psi)$ be a homomorphism of Higgs bundles.*

1. *If $(\mathcal{E}, \Phi)$ and $(\mathcal{F}, \Psi)$ are semistable with $\mu(\mathcal{E}) > \mu(\mathcal{F})$, then $f \equiv 0$.*
2. *If $\mu(\mathcal{E}) = \mu(\mathcal{F})$ and one of $(\mathcal{E}, \Phi)$ and $(\mathcal{F}, \Psi)$ is stable, then either $f \equiv 0$ or f is an isomorphism.*

As a consequence, if f is an automorphism of a stable Higgs bundle $(\mathcal{E}, \Phi)$, there exists some $c \neq 0$ such that $f = cI_{\mathcal{E}}$, where $I_{\mathcal{E}}$ is the identity map of $\mathcal{E}$. In other words, the $\mathcal{G}^{\mathbb{C}}$-stabilizer at each stable Higgs bundle must be $\mathbb{C}^*$. This is certainly not true for strictly semistable Higgs bundles.

Finally, every semistable Higgs bundle $(\mathcal{E}, \Phi)$ admits a filtration,

$$0 = (\mathcal{E}_0, \Phi_0) \subset (\mathcal{E}_1, \Phi_1) \subset \cdots \subset (\mathcal{E}_\ell, \Phi_\ell) = (\mathcal{E}, \Phi), \tag{2.22}$$

called the *Seshadri filtration*, whose successive quotients $(\mathcal{E}_i/\mathcal{E}_{i-1}, \Phi_i)$ are stable, all with slope $\mu(E)$. Let $\mathrm{Gr}(\mathcal{E}, \Phi) = \bigoplus_{i=1}^{\ell}(\mathcal{E}_i/\mathcal{E}_{i-1}, \Phi_i)$ be the graded object associated with the Seshadri filtration of $(\mathcal{E}, \Phi)$. It is uniquely determined by the isomorphism class of $(\mathcal{E}, \Phi)$. Moreover, it is necessarily a polystable Higgs bundle. As a consequence, a Higgs bundle $(\mathcal{E}, \Phi)$ is polystable if and only if it is isomorphic to $\mathrm{Gr}(\mathcal{E}, \Phi)$ as Higgs bundles.

2.4.3 Hitchin's equation

In this section, we review Hitchin's equation and the Hitchin-Kobayashi correspondence, a gauge-theoretic interpretation of the polystability of Higgs bundles.

We fix a Hermitian vector bundle E of slope μ. Hitchin's equation for a pair $(A, \Phi) \in \mathcal{B}$ is defined as

$$F_A + [\Phi, \Phi^*] = -\sqrt{-1}\mu\omega_M \otimes I_E, \tag{2.23}$$

where I_E is the identity map of E and ω_M is a fixed Kählaer form on M such that

$\mathrm{vol}(M) = 2\pi$. Here, Φ^* is the conjugate transpose of Φ with respect to the fixed Hermitian metric on E.

The Hitchin-Kobayashi correspondence provides a gauge-theoretic interpretation of the polystability of Higgs bundles (see [33, 54]).

Proposition 2.4.4. *A Higgs bundle $(A, \Phi) \in \mathcal{B}$ is polystable if and only if there is a complex gauge transformation $g \in \mathcal{G}^{\mathbb{C}}$, unique up to real gauge transformations, such that $(A, \Phi) \cdot g$ satisfies Hitchin's equation.*

Hitchin's equation is more than just an equation. It is involved in a Hamiltonian action and can be interpreted as a moment map. (see (2.2) for the definition of moment maps.) Define the *real gauge group* $\mathcal{G}$ as

$$\mathcal{G} = \{g \in \mathcal{G}^{\mathbb{C}} \colon gg^* = I_E\}. \tag{2.24}$$

In other words, $\mathcal{G}$ is the group of unitary gauge transformations. The infinite-dimensional space $\mathcal{C} = \mathcal{A} \times \Omega^{1,0}(\mathfrak{g}_E^{\mathbb{C}})$ turns out to be a hyperKähler manifold as follows. (see [33, §6].) At any point of $\mathcal{C}$, the tangent space of $\mathcal{C}$ is $\Omega^1(\mathfrak{g}_E) \oplus \Omega^{1,0}(\mathfrak{g}_E^{\mathbb{C}})$. Moreover, the space $\Omega^1(\mathfrak{g}_E)$ can be canonically identified with $\Omega^{0,1}(\mathfrak{g}_E^{\mathbb{C}})$ by the map $\alpha \mapsto \alpha''$, where α'' is the $(0,1)$-component of α. Therefore, the L^2-metric on $\mathcal{C}$ is defined as

$$g(\alpha'', \eta; \alpha'', \eta) = 2\sqrt{-1} \int_M \mathrm{tr}\Big((\alpha'')^* \alpha'' + \eta\eta^*\Big), \quad (\alpha'', \eta) \in \Omega^{0,1}(\mathfrak{g}_E^{\mathbb{C}}) \oplus \Omega^{1,0}(\mathfrak{g}_E^{\mathbb{C}}). \tag{2.25}$$

To give three complex structures, we recall the following natural isomorphisms

$$\begin{aligned} &\Omega^{0,1}(\mathfrak{g}_E) \xrightarrow{\sim} \Omega^1(\mathfrak{g}_E), \qquad \alpha'' \mapsto \alpha'' - (\alpha'')^*, \\ &\Omega^{1,0}(\mathfrak{g}_E^{\mathbb{C}}) \xrightarrow{\sim} \Omega^1(\sqrt{-1}\mathfrak{g}_E), \qquad \eta \mapsto \eta + \eta^*. \end{aligned} \tag{2.26}$$

Hence,

$$\Omega^{0,1}(\mathfrak{g}_E^{\mathbb{C}}) \oplus \Omega^{1,0}(\mathfrak{g}_E^{\mathbb{C}}) = \Omega^1(\mathfrak{g}_E) \oplus \Omega^1(\sqrt{-1}\mathfrak{g}_E). \tag{2.27}$$

On the left hand side, a natural complex structure I is the multiplication by $\sqrt{-1}$. On the right hand side, there is another natural complex structure given by the direct sum. More precisely,

$$J(a, \sqrt{-1}b) = (-b, \sqrt{-1}a), \qquad (a, b) \in \Omega^1(\mathfrak{g}_E) \oplus \Omega^1(\sqrt{-1}\mathfrak{g}_E). \tag{2.28}$$

Since (2.27) is only $\mathbb{R}$-linear, we may transfer the complex structure on the right to the space on the left. Still letting J denote the resulting complex structure, we have

$$J(\alpha'', \eta) = \left(\sqrt{-1}\eta^*, -\sqrt{-1}(\alpha'')^*\right) \tag{2.29}$$

Then, the third complex structure on $\Omega^{0,1}(\mathfrak{g}_E^{\mathbb{C}}) \oplus \Omega^{1,0}(\mathfrak{g}_E^{\mathbb{C}})$ is defined as $K = IJ$. Therefore, we have obtained the following.

Proposition 2.4.5. *There are three complex structures on $\mathcal{C} = \mathcal{A} \times \Omega^{1,0}(\mathfrak{g}_E^{\mathbb{C}})$ defined* *as follows:*

$$\begin{aligned} I(\alpha'', \eta) &= (\sqrt{-1}\alpha'', \sqrt{-1}\eta), \\ J(\alpha'', \eta) &= \Big(\sqrt{-1}\eta^*, -\sqrt{-1}(\alpha'')^*\Big), \\ K(\alpha'', \eta) &= \Big(-\eta^*, (\alpha'')^*\Big) \end{aligned} \tag{2.30}$$

Therefore, there are three associated Kähler forms

$$\Omega_I = g(I\cdot, \cdot), \quad \Omega_J = g(J\cdot, \cdot), \quad \Omega_K = g(K\cdot, \cdot). \tag{2.31}$$

Moreover, the complex symplectic form $\Omega_{\mathbb{C}} = \Omega_J + \sqrt{-1}\Omega_K$ is given by

$$\Omega_{\mathbb{C}}(\alpha_1'', \eta_1; \alpha_2'', \eta_2) = \int_M \operatorname{tr}(\eta_2\alpha_1'' - \eta_1\alpha_2'') \tag{2.32}$$

Then, we have the following.

Proposition 2.4.6.

1. *The group $\mathcal{G}$ preserves Ω_I, Ω_J and Ω_K, and the group $\mathcal{G}^{\mathbb{C}}$ preserves $\Omega_{\mathbb{C}}$.*

2. *Via the natural inclusion $\Omega^{1,1}(\mathfrak{g}_E^{\mathbb{C}}) \hookrightarrow \Omega^0(\mathfrak{g}_E^{\mathbb{C}})^*$, the map*

$$\mu_{\mathbb{C}}(A, \Phi) = \overline{\partial}_A\Phi \colon \mathcal{C} \to \Omega^{1,1}(\mathfrak{g}_E^{\mathbb{C}}) \tag{2.33}$$

is a complex moment map for the $\mathcal{G}^{\mathbb{C}}$-action on $\mathcal{C}$ with respect to the complex symplectic form $\Omega_{\mathbb{C}}$.

3. *Via the natural inclusion* $\Omega^2(\mathfrak{g}_E) \hookrightarrow \Omega^0(\mathfrak{g}_E)^*$*, the map*

$$\mu(A,\Phi) = F_A + [\Phi,\Phi^*] \colon \mathcal{C} \to \Omega^2(\mathfrak{g}_E) \tag{2.34}$$

is a moment map for the $\mathcal{G}$*-action on* $\mathcal{C}$ *with respect to the Kähler form* Ω_I*.*

2.4.4 Yang-Mills-Higgs flow

The Yang-Mills-Higgs flow will play a major role in the construction of the Kuranishi local models in this thesis. In this section, we gather some of its useful properties. We still fix a Hermitian vector bundle E.

The *Yang-Mills-Higgs functional* on $\mathcal{B}$ is defined as

$$\mathrm{YMH}(A,\Phi) = \|\mu(A,\Phi)\|^2_{L^2} = \|F_A + [\Phi,\Phi^*]\|^2_{L^2}. \tag{2.35}$$

where $\|\cdot\|_{L^2}$ is the L^2-norm on $\Omega^2(\mathfrak{g}_E)$. The (negative) gradient flow of YMH (regarded as a function on $\mathcal{C}$) is the *Yang-Mills-Higgs flow.* The following results, contained in [67, Theorem 1.1], will be used later.

Proposition 2.4.7.

1. *The gradient flow of* YMH *exists for all time and preserves the* $\mathcal{G}^{\mathbb{C}}$*-orbits.*

2. *The (negative) gradient flow of* YMH *converges in the* C^∞*-topology to the critical points of* YMH.

3. *For every semistable Higgs bundle* (A,Φ)*, define* $r(A,\Phi) = (A_\infty,\Phi_\infty)$ *as the*

limiting point of the gradient flow (A_t, Φ_t) *starting at* (A, Φ)*. Then, the map*

$$r\colon \mathcal{B}^{ss} \to \mu^{-1}(\gamma) \cap \mathcal{B} \tag{2.36}$$

is a continuous retraction, where $\gamma = 2\pi\mu^2 \operatorname{rank}(E)$*.*

4. *For a semistable Higgs bundle* (A, Φ)*, the Higgs bundle* $r(A, \Phi)$ *is isomorphic to the graded object* $\mathrm{Gr}(A, \Phi)$ *associated with the Seshadri filtration of* (A, Φ)*.*

5. *In particular, a semistable Higgs bundle* (A, Φ) *is polystable if and only if* $r(A, \Phi)$ *and* (A, Φ) *are in the same* $\mathcal{G}^{\mathbb{C}}$*-orbit.*

An essential ingredient in the proof of the convergence property of the Yang-Mills-Higgs flow is the Łojasiewicz inequality, [67, Proposition 3.5]. Another consequence of this inequality is the following, which is extracted from [67, Proposition 3.7] and will be used in the proof of Theorem 3.2.6.

Proposition 2.4.8. *Let* (A, Φ) *be a Higgs bundle with* $\mathrm{YMH}(A, \Phi) = 0$*. Let* $k > 0$*. Then, there exists open neighborhoods* $V \subset U$ *of* (A, Φ) *in the* L^2_k*-topology such that every Yang-Mills-Higgs flow starting in* V *stays and converges in* U*.*

Chapter 3: The moduli space as a normal complex space

This chapter is based on the author's paper [17].

3.1 Deformation complexes

In this section, after reviewing the deformation complex for Higgs bundles, we introduce another useful Fredholm complex that will be used later. Let $(A, \Phi) \in \mathcal{B}$ such that $\mu(A, \Phi) = 0$. Then, consider the deformation complex

$$C_{\mu_{\mathbb{C}}}\colon \qquad \Omega^0(\mathfrak{g}_E^{\mathbb{C}}) \xrightarrow{D''} \Omega^{0,1}(\mathfrak{g}_E^{\mathbb{C}}) \oplus \Omega^{1,0}(\mathfrak{g}_E^{\mathbb{C}}) \xrightarrow{D''} \Omega^{1,1}(\mathfrak{g}_E^{\mathbb{C}}), \tag{3.1}$$

where $D'' = \overline{\partial}_A + \Phi$. Recall that $C_{\mu_{\mathbb{C}}}$ is obtained by linearizing the equation $\overline{\partial}_A \Phi = 0$ and the $\mathcal{G}^{\mathbb{C}}$-action.

Proposition 3.1.1 ([55, §1] and [57, §10]). *$C_{\mu_{\mathbb{C}}}$ is an elliptic complex and a differential graded Lie algebra. Moreover, the Kähler identities,*

$$(D'')^* = -i[*, D'], \qquad (D')^* = +i[*, D''], \tag{3.2}$$

hold, where $D' = \partial_A + \Phi^*$ and $*$ is the Hodge star.

There is another useful sequence

$$C_\mu\colon \qquad \Omega^0(\mathfrak{g}_E) \xrightarrow{d_1} \ker D'' \xrightarrow{d_2} \Omega^2(\mathfrak{g}_E), \tag{3.3}$$

where d_2 is the derivative of μ from (1.4) at (A, Φ), and $d_1(u) = (d_A u, [\Phi, u])$. The operator d_2, viewed as a map $\Omega^1(\mathfrak{g}_E) \oplus \Omega^{1,0}(\mathfrak{g}_E^{\mathbb{C}}) \to \Omega^2(\mathfrak{g}_E)$, has a surjective symbol. Hence, $d_2 d_2^*\colon \Omega^2(\mathfrak{g}_E) \to \Omega^2(\mathfrak{g}_E)$ is a self-adjoint elliptic operator. As a consequence, the Hodge decomposition

$$\Omega^2(\mathfrak{g}_E^{\mathbb{C}}) = \operatorname{im} d_2 d_2^* \oplus \ker d_2 d_2^*, \tag{3.4}$$

holds. Moreover, since $d_2 (D'')^* = 0$ and

$$\Omega^{0,1}(\mathfrak{g}_E^{\mathbb{C}}) \oplus \Omega^{1,0}(\mathfrak{g}_E^{\mathbb{C}}) = \ker D'' \oplus \operatorname{im}(D'')^*, \tag{3.5}$$

we have

$$d_2(\ker D'') = d_2(\Omega^1(\mathfrak{g}_E) \oplus \Omega^{1,0}(\mathfrak{g}_E^{\mathbb{C}})) \tag{3.6}$$

(In this thesis, we routinely identify $\Omega^1(\mathfrak{g}_E)$ with $\Omega^{0,1}(\mathfrak{g}_E^{\mathbb{C}})$ using the map $\alpha \mapsto \alpha''$, where α'' is the $(0,1)$-component of α). As a consequence, the natural map $\ker d_2^* \to H^2(C_\mu)$ is an isomorphism. We denote $\ker d_2^*$ by $\mathbf{H}^2(C_\mu)$. Finally, we note that $H^1(C_\mu)$ is equal to the first cohomology of the following elliptic complex that is used by Hitchin in [33, p. 85]

$$C_{Hit}\colon \qquad \Omega^0(\mathfrak{g}_E) \xrightarrow{d_1} \Omega^1(\mathfrak{g}_E) \oplus \Omega^{1,0}(\mathfrak{g}_E^{\mathbb{C}}) \xrightarrow{d_2 \oplus D''} \Omega^2(\mathfrak{g}_E) \oplus \Omega^{1,1}(\mathfrak{g}_E^{\mathbb{C}}). \tag{3.7}$$

In fact, by direct computation, the identification $\Omega^1(\mathfrak{g}_E) \xrightarrow{\sim} \Omega^{0,1}(\mathfrak{g}_E^{\mathbb{C}})$ induces an isomorphism $\mathbf{H}^1(C_{Hit}) \xrightarrow{\sim} \mathbf{H}^1(C_{\mu_{\mathbb{C}}})$. Therefore, in the rest of the paper, if no confusion can appear, we will simply use $\mathbf{H}^1$ to mean the harmonic space $\mathbf{H}^1(C_{\mu_{\mathbb{C}}})$. In summary, we have obtained

Proposition 3.1.2. *The sequence C_μ is a Fredholm complex with Hodge decomposition*

$$\Omega^2(\mathfrak{g}_E) = \mathbf{H}^2(C_\mu) \oplus \operatorname{im} d_2. \tag{3.8}$$

Lastly, note that the natural non-degenerate pairing $\Omega^0(\mathfrak{g}_E) \times \Omega^2(\mathfrak{g}_E) \to \mathbb{R}$ restricts to a non-degenerate pairing $\mathbf{H}^0(C_\mu) \times \mathbf{H}^2(C_\mu) \to \mathbb{R}$ so that $\mathbf{H}^2(C_\mu)$ can be identified with the dual space $\mathbf{H}^0(C_\mu)^*$ of $\mathbf{H}^0(C_\mu)$.

3.2 Kuranishi local models

3.2.1 Kuranishi maps

A crucial ingredient in the Kuranishi slice method is the Kuranishi maps. They relate polystable orbits in $\mathbf{H}^1$ and polystable orbits in $\mathcal{B}$. Moreover, they eventually induce local charts for the moduli space. To construct Kuranishi maps, we need to use the implicit function theorem, and it is a standard practice to work with the Sobolev completions of relevant spaces.

More precisely, for any smooth Hermitian bundle $F \to M$, we equip the space $\Omega^0(F)$ of smooth sections of F with the Sobolev L^2_k-norm as follows. Fix a unitary connection ∇^F on F and let ∇^M be the Levi-Civita connection on M. For $k \geq 0$, and any smooth section s of F, we define

$$\|s\|^2_{L^2_k} = \sum_{i=0}^{k} \int_B |\nabla^i s|^2 d\text{vol}, \tag{3.9}$$

where $\nabla^i \colon \Omega^0(F) \to \Omega^0(\otimes^i T^*M \otimes F)$ is the composition

$$\begin{aligned} \Omega^0(F) \xrightarrow{\nabla^F} \Omega^0(T^*M \otimes F) &\xrightarrow{\nabla^M \otimes \nabla^F} \Omega^0(\otimes^2 T^*M \otimes F) \\ &\xrightarrow{(\nabla^M)^{\otimes 2} \otimes \nabla^F} \cdots \to \Omega^0(\otimes^i T^*M \otimes F), \end{aligned} \tag{3.10}$$

and $|\cdot|$ is the pointwise norm on $\Omega^0(\otimes^i T^*M \otimes F)$ induced by the Hermitian metric on F and the Riemannian metric on M. Let $\Omega^0(F)_k$ be the completion of $\Omega^0(F)$ with respect to the L^2_k-norm. As a result, $\Omega^0(F)_k$ is a Banach space. In fact, it is a Hilbert space. In this thesis, if Y is a space on which the Sobolev L^2_k-norm is well-defined, we will use Y_k to denote the completion of Y with respect to the L^2_k-norm. Otherwise, Y is equipped with C^∞-topology. From now on, we fix $k > 1$.

Now, we describe the Kuranishi maps. Let $(A, \Phi) \in \mathcal{B}$ with $\mu(A, \Phi) = 0$. Recall that $\mathcal{G}^{\mathbb{C}}_{k+1}$ and $\mathcal{G}_{k+1}$ are Hilbert Lie groups and act smoothly on the Hilbert affine manifold $\mathcal{C}_k$. Moreover, the $\mathcal{G}_{k+1}$-action on $\mathcal{C}_k$ is proper (see [20, Section 4.4]). Therefore, if H is the $\mathcal{G}_{k+1}$-stabilizer at (A, Φ), then H is a compact Lie group with Lie algebra $\mathbf{H}^0(C_\mu)$. The following result relates the $\mathcal{G}^{\mathbb{C}}_{k+1}$-stabilizer to the $\mathcal{G}_{k+1}$-stabilizer at (A, Φ).

Proposition 3.2.1. *The $\mathcal{G}_{k+1}^{\mathbb{C}}$-stabilizer at (A, Φ) is the complexification of H and acts on $\mathbf{H}^1$.*

Proof. This follows from [58, Proposition 1.6]. The rest follows from direct computation. □

If $\mathbf{H}^2(C_{\mu_{\mathbb{C}}}) = 0$, then the implicit function theorem implies that $\mathcal{B}_k$ is locally a complex manifold around (A, Φ). In general, following Lyapunov-Schmidt reduction, we consider

$$\widetilde{\mathcal{B}}_k = [(1 - P)\mu_{\mathbb{C}}]^{-1}(0) \subset \mathcal{C}_k, \tag{3.11}$$

where P is the harmonic projection defined in the elliptic complex $C_{\mu_{\mathbb{C}}}$. By construction, the derivative of $(1 - P)\mu_{\mathbb{C}}$ at (A, Φ) is surjective. Hence, $\widetilde{\mathcal{B}}_k$ is locally a complex manifold around (A, Φ). To parameterize $\widetilde{\mathcal{B}}_k$, consider the map

$$\begin{gathered} F \colon \Omega^{0,1}(\mathfrak{g}_E^{\mathbb{C}})_k \oplus \Omega^{1,0}(\mathfrak{g}_E^{\mathbb{C}})_k \to \Omega^{0,1}(\mathfrak{g}_E^{\mathbb{C}})_k \oplus \Omega^{1,0}(\mathfrak{g}_E^{\mathbb{C}})_k, \\ F(\alpha, \eta) = (\alpha, \eta) + (D'')^* G[\alpha'', \eta], \end{gathered} \tag{3.12}$$

where α'' is the $(0,1)$-part of α, and G is the Green operator defined in the deformation complex $C_{\mu_{\mathbb{C}}}$. It has the following properties.

Lemma 3.2.2.

1. *F is $H^{\mathbb{C}}$-equivariant.*

2. *F is a local biholomorphism around 0.*

3. *$D''F(\alpha, \eta) = (1 - P)\mu_{\mathbb{C}}(A + \alpha, \Phi + \eta)$.*

4. $(D'')^*F(\alpha,\eta) = (D'')^*(\alpha,\eta)$.

Proof. (1) follows from the fact that the $H^{\mathbb{C}}$-action commutes with $(D'')^*$ and G. Since the derivative of F at 0 is the identity map, the inverse function theorem implies (2). Since $(D'')^*(D'')^* = 0$, (4) follows. To prove (3), we compute

$$\begin{aligned}
&(1-P)\mu_{\mathbb{C}}(A+\alpha,\Phi+\eta)\\
&= D''(D'')^*G(D''(\alpha,\eta)+[\alpha'',\eta])\\
&= D''((\alpha,\eta)-H(\alpha,\eta)-D''(D'')^*G(\alpha,\eta)+(D'')^*G[\alpha'',\eta])\\
&= D''((\alpha,\eta)+(D'')^*G[\alpha,\eta])\\
&= D''F(\alpha,\eta).
\end{aligned} \tag{3.13}$$

□

As a consequence, F induces a well-defined map,

$$F\colon \widetilde{\mathcal{B}}_k \cap [(A,\Phi)+\ker(D'')^*] \to \ker D'' \cap \ker(D'')^* = \mathbf{H}^1. \tag{3.14}$$

Since $\widetilde{\mathcal{B}}_k$ and $(A,\Phi)+\ker(D'')^*$ intersect transversely at (A,Φ), their intersection is locally a complex manifold around (A,Φ). Hence, there are an open ball $U \subset \mathbf{H}^1$ in the L^2-norm around 0 and an open neighborhood $\widetilde{U}$ of (A,Φ) in $\widetilde{\mathcal{B}}_k \cap [(A,\Phi)+\ker(D'')^*]$ such that $F\colon \widetilde{U} \to U$ is a biholomorphism. The *Kuranishi map* θ is defined as its inverse viewed as a map $\theta\colon U \hookrightarrow \mathcal{C}_k$, and the *Kuranishi space* is defined as

$Z := \theta^{-1}(\mathcal{B} \cap \widetilde{U})$. More concretely, by the construction of $\widetilde{\mathcal{B}}_k$,

$$Z := \{x \in U \colon P[\theta(x), \theta(x)] = 0\}. \tag{3.15}$$

Here, (A, Φ) serves as the origin in the affine manifold $\mathcal{C}_k$. Clearly, Z is a closed complex subspace of U. Moreover, since $\mathcal{B}_k^{ss}$ is open in $\mathcal{B}_k$ (see [67, Theorem 4.1]), by shrinking U and hence Z if necessary, we may assume that $\theta(Z) \subset \mathcal{B}_k^{ss}$.

The next result shows that the Kuranishi space Z is locally complete.

Proposition 3.2.3. *The map*

$$\begin{gathered} T \colon \mathbf{H}^0(C_\mu)^\perp_{k+1} \times \mathbf{H}^2(C_\mu)^\perp_{k+1} \times [((A, \Phi) + \ker(D'')^*) \cap \mathcal{B}_k^{ss}] \to \mathcal{B}_k^{ss}, \\ T(u, \beta, B, \Psi) = (B, \Psi) \cdot \exp(-i * \beta) \exp(u), \end{gathered} \tag{3.16}$$

is a local homeomorphism around $(0, 0, A, \Phi)$. As a consequence, there exists an open neighborhood W of (A, Φ) in $\mathcal{B}_k^{ss}$ such that the $\mathcal{G}_{k+1}^{\mathbb{C}}$-orbit of every $(B, \Psi) \in W$ intersects the image $\theta(Z)$.

Proof. Consider the map

$$\begin{gathered} T \colon \mathbf{H}^0(C_\mu)^\perp_{k+1} \times \mathbf{H}^2(C_\mu)^\perp_{k+1} \times ((A, \Phi) + \ker(D'')^*) \to \mathcal{C}_k, \\ T(u, \beta, B, \Psi) = (B, \Psi) \cdot \exp(-i * \beta) \exp(u), \end{gathered} \tag{3.17}$$

where $\mathbf{H}^0(C_\mu)^\perp$ and $\mathbf{H}^2(C_\mu)^\perp$ are the L^2-orthogonal complements of $\mathbf{H}^0(C_\mu)$ and

$\mathbf{H}^2(C_\mu)$ in $\Omega^0(\mathfrak{g}_E)$ and $\Omega^2(\mathfrak{g}_E)$, respectively. Its derivative at $(0,0,A,\Phi)$ is given by

$$\begin{aligned} d_{(0,0,A,\Phi)}T(u,\beta,x) &= \frac{d}{dt}\bigg|_{t=0} T(tu,t\beta,(A,\Phi)+tx) \\ &= \frac{d}{dt}\bigg|_{t=0} (A,\Phi)+tx)\cdot\exp(-i*t\beta)\cdot\exp(tu) \\ &= x + \frac{d}{dt}\bigg|_{t=0}(A,\Phi)\cdot\exp(-i*t\beta) + \frac{d}{dt}\bigg|_{t=0}(A,\Phi)\cdot\exp(tu) \\ &= x + D''(-i*\beta) + D''u \\ &= D''(u-i*\beta)+x. \end{aligned} \tag{3.18}$$

Note that

$$\mathbf{H}^0(C_\mu)^\perp \oplus i*\mathbf{H}^2(C_\mu)^\perp = \mathbf{H}^0(C_\mu)^\perp \oplus i\mathbf{H}^0(C_\mu)^\perp = \mathbf{H}^0(C_{\mu_{\mathbb{C}}})^\perp. \tag{3.19}$$

Since

$$\Omega^{0,1}(\mathfrak{g}_E^{\mathbb{C}})_k \oplus \Omega^{1,0}(\mathfrak{g}_E^{\mathbb{C}})_k = \ker(D'')^* \oplus \operatorname{im} D'', \tag{3.20}$$

we conclude that $d_{(0,A,\Phi)}T$ is an isomorphism. Hence, the inverse function theorem implies that there are open neighborhoods $N_1\times N_2\times V$ of $(0,0,A,\Phi)$ and W of (A,Φ) such that $T\colon N_1\times N_2\times V\to W$ is a diffeomorphism. Since $\mathcal{B}_k^{ss}$ is $\mathcal{G}_{k+1}^{\mathbb{C}}$-invariant, we conclude that

$$T\colon N_1\times N_2\times(V\cap\mathcal{B}_k^{ss})\to W\cap\mathcal{B}_k^{ss} \tag{3.21}$$

is a homeomorphism. Finally, if $U\subset\mathbf{H}^1$ is sufficiently small, then θ is a homeomorphism from Z to $V\cap\mathcal{B}_k^{ss}$. $\square$

Moreover, θ maps $H^{\mathbb{C}}$-orbits to $\mathcal{G}^{\mathbb{C}}$-orbits in the following way.

Proposition 3.2.4 (cf. [9, Lemma 6.1]). *If U is sufficiently small, then the following hold:*

1. *If $x_1, x_2 \in U$ are such that $x_1 = x_2 g$ for some $g \in H^{\mathbb{C}}$, then $\theta(x_1) = \theta(x_2)g$. Hence, if $x_1 \in Z$, then $x_2 \in Z$.*

2. *Conversely, if $d_x\theta(v) = u^{\#}_{\theta(x)}$ for some $u \in \Omega^0(\mathfrak{g}_E^{\mathbb{C}})_{k+1}$, then $u \in \mathbf{H}^0(C_{\mu_{\mathbb{C}}})$, and $v = u^{\#}_x$, where $u^{\#}$ is the infinitesimal action of u.*

Proof. Since U is an open ball around 0, it is orbit-convex by [58, Lemma 1.14]. Hence, the holomorphicity of θ and [58, Proposition 1.4] imply that $\theta(x_1) = \theta(x_2)g$. Since $\mathcal{B}_k^{ss}$ is $\mathcal{G}_{k+1}^{\mathbb{C}}$-invariant, if $\theta(x_1) \in \mathcal{B}_k^{ss}$, then $\theta(x_2) \in \mathcal{B}_k^{ss}$ so that $x_2 \in Z$. To prove (2), we claim that $u \in \mathbf{H}^0(C_{\mu_{\mathbb{C}}})$. Then, the claim implies that

$$v = d_{\theta(x)}F(d_x\theta(v)) = d_{\theta(x)}F(u^{\#}_{\theta(x)}) = \frac{d}{dt}\bigg|_{t=0} F(\theta(x)e^{tu}) = \frac{d}{dt}\bigg|_{t=0} xe^{tu} = u^{\#}_x. \quad (3.22)$$

To prove the claim, write $u = u' + u''$ for some $u' \in \mathbf{H}^0(C_{\mu_{\mathbb{C}}})$ and $u'' \in \mathbf{H}^0(C_{\mu_{\mathbb{C}}})^{\perp}_{k+1}$. Since θ takes values in $(A, \Phi) + \ker(D'')^*$, $(u'')^{\#}_{\theta(x)} \in \ker(D'')^*$. In the proof of Proposition 3.2.3, we see that the map

$$T \colon \mathbf{H}^0(C_\mu)^{\perp}_{k+1} \times \mathbf{H}^2(C_\mu)^{\perp}_{k+1} \times ((A, \Phi) + \ker(D'')^*) \to \mathcal{C}_k \quad (3.23)$$

is a local diffeomorphism around $(0, 0, A, \Phi)$. Hence, there are open neighborhoods $N_1 \times N_2 \times V$ of $(0, 0, A, \Phi)$ and W of (A, Φ) such that $T \colon N_1 \times N_2 \times V \to W$ is a diffeomorphism. If U is sufficiently small, $\theta\colon Z \to V \cap \mathcal{B}_k^{ss}$ is a homeomorphism. Therefore, the derivative $d_{(0,0,\theta(x))}T$ of T is injective. Note that

$$\mathbf{H}^0(C_\mu)^\perp \oplus i * \mathbf{H}^2(C_\mu)^\perp = \mathbf{H}^0(C_{\mu_\mathbb{C}})^\perp. \tag{3.24}$$

Then, we see that

$$d_{(0,0,\theta(x))}T(u'', 0) = D''_{\theta(x)}u'' = d_{(0,0,\theta(x))}T(0, (u'')^\#_{\theta(x)}) \tag{3.25}$$

so that $u'' = 0$. □

3.2.2 Perturbed Kuranishi maps

The Hitchin-Kobayashi correspondence characterizes polystable orbits in $\mathcal{B}^{ss}$ via the moment map μ. Since θ should eventually induce a local chart for the moduli space, we should be able to relate the polystable orbits in $\mathbf{H}^1$ with respect to the complex reductive group $H^\mathbb{C}$ to the polystable orbits in $\mathcal{B}$. Therefore, we would like to pullback the moment map μ to $U \subset \mathbf{H}^1$ by θ and then use the pullback moment map $\theta^*\mu$ to characterize polystable orbits in U. However, $\theta^*\mu$ takes values in $\Omega^2(\mathfrak{g}_E)_{k-1}$ instead of $\mathbf{H}^2(C_\mu) \cong \mathbf{H}^0(C_\mu)^*$. To fix this issue, we will perturb the Kuranishi map along $\mathcal{G}^\mathbb{C}$-orbits in the following way.

Lemma 3.2.5. *If $U \subset \mathbf{H}^1$ is sufficiently small, then there is a unique smooth function β defined on U and taking values in an open neighborhood of 0 in $\mathbf{H}^2(C_\mu)^\perp_{k+1}$ such that the perturbed Kuranishi map $\Theta := \theta e^{-i*\beta}$ is smooth and H-equivariant,* *and* $\nu := \Theta^*\mu$ *takes values in* $\mathbf{H}^2(C_\mu)$ *and hence is a moment map for the* H*-action on* U *with respect to the symplectic form* $\Theta^*\Omega_I$*. Moreover, the derivative of* Θ *at* 0 *is the inclusion map.*

Before giving the proof, we remark that the perturbed Kuranshi map Θ is no longer holomorphic and hence the form $\Theta^*\Omega_I$ is no longer Kähler.

Proof. We follow the proof of [62, Proposition 7]. Consider the map

$$\begin{gathered} L\colon U \times \mathbf{H}^2(C_\mu)^\perp_{k+1} \to \mathbf{H}^2(C_\mu)^\perp_{k-1}, \\ L(x,\beta) = (1-P)\mu(\theta(x)e^{-i*\beta}), \end{gathered} \tag{3.26}$$

where P is the harmonic projection defined in C_μ. Then, the derivative of L at $(0,0)$ along the direction $(0,\beta)$ is given by

$$d_{(0,0)}L(0,\beta) = (1-P)d_2(-Id_1 * \beta) = d_2 d_2^* \beta, \tag{3.27}$$

where the second equality follows from the formula $d_2^* = -Id_1*$. Since

$$d_2 d_2^*\colon \mathbf{H}^2(C_\mu)^\perp_{k+1} \to \mathbf{H}^2(C_\mu)^\perp_{k-1} \tag{3.28}$$

is an isomorphism, the implicit function theorem guarantees the existence of the desired function β. Since L is H-equivariant, the uniqueness of β implies that Θ is also H-equivariant. A direct computation shows that $d_0\Theta$ is the inclusion map. □

The following result relates the polystability of Higgs bundles to that of points in $\mathbf{H}^1$ with respect to the $H^{\mathbb{C}}$-action.

Theorem 3.2.6. *If U is sufficiently small, then the induced map*

$$U \times_H \mathcal{G}_{k+1} \to \mathcal{C}_k, \qquad [x, g] \mapsto \Theta(x)g, \tag{3.29}$$

is injective. Moreover, there is an open ball $B \subset U$ around 0 in the L^2-norm such that the following are equivalent for every $x \in B \cap Z$:

1. *$xH^{\mathbb{C}}$ is closed in $\mathbf{H}^1$.*

2. *$xH^{\mathbb{C}} \cap \nu^{-1}(0) \neq \emptyset$.*

Proof. The derivative of the induced map at $[0, 1]$ is given by

$$\mathbf{H}^1 \oplus \mathbf{H}^0(C_\mu)^{\perp}_{k+1} \to \Omega^{0,1}(\mathfrak{g}_E^{\mathbb{C}})_k \oplus \Omega^{1,0}(\mathfrak{g}_E^{\mathbb{C}})_k, \qquad (x, u) \mapsto x + D''u. \tag{3.30}$$

Since it is injective, we see that the induced map is locally injective around $[0, 1]$. Then, we assume to the contrary that such U does not exist. Therefore, there are sequences $[x_n, g_n]$ and $[x'_n, g'_n]$ such that

1. x_n, x'_n converge to 0 in $\mathbf{H}^1$.

2. $\Theta(x_n)g_n = \Theta(x'_n)g'_n$.

3. $[x_n, g_n] \neq [x'_n, g'_n]$ for all n.

Since the $\mathcal{G}_{k+1}$-action is proper, by passing to a subsequence, we may assume that $g'_n g_n^{-1}$ converges to some $g \in \mathcal{G}_{k+1}$. Letting $n \to \infty$, we see that $\Theta(0) = \Theta(0)g$ so that

$g \in K$. Now, on the one hand, $[x'_n, g'_n g_n^{-1}] \neq [x_n, 1]$ for any n. On the other hand, both $[x'_n, g'_n g_n^{-1}]$ and $[x_n, 1]$ converge to $[0, 1]$ so that they are equal when $n \gg 0$, since the induced map is locally injective around $[0, 1]$. This is a contradiction.

Now, we prove the second part of the proposition. By Proposition 3.2.3, there are open neighborhoods $N_1 \times N_2 \times V$ of $(0, 0, A, \Phi)$ and W of (A, Φ) such that $T\colon N_1 \times N_2 \times V \to W$ is a homeomorphism. Here, V and W are open subsets in $\mathcal{B}_k^{ss}$. If U is sufficiently small, $\theta\colon Z \to V$ is a homeomorphism so that Proposition 3.2.4 holds. Let O be an open neighborhood of 0 in $\mathbf{H}^2(C_\mu)_{k+1}^{\perp}$ such that the smooth function $\beta\colon U \to O$ and hence $\Theta := \theta e^{-i*\beta}$ are defined. By shrinking N_2 if necessary, we may assume that $N_2 \subset O$. Then, by [67, Proposition 3.7], there is an open neighborhood $W' \subset W$ of (A, Φ) in $\mathcal{B}_k^{ss}$ such that the Yang-Mills-Higgs flow starting at any Higgs bundle inside W' stays and converges in W. Moreover, we may assume that $T(N'_1 \times N'_2 \times V') = W'$ for some open neighborhood $N'_1 \times N'_2 \times V' \subset N_1 \times N_2 \times V$ of $(0, 0, A, \Phi)$ such that $\theta\colon Z \cap B \to V'$ for some open ball $B \subset U$ around 0.

Now, suppose $x \in B \cap Z$ is such that $xH^{\mathbb{C}}$ is closed in $\mathbf{H}^1$. Let (B_t, Ψ_t) be the gradient flow starting at $\theta(x)$. By the previous setup, $\theta(x) \in V' \subset W'$ so that (B_t, Ψ_t) stays in W. Therefore, we may write $(B_t, \Psi_t) = \theta(x_t)e^{-i*\beta_t}e^{u_t}$ for some $x_t \in Z$ and $(u_t, \beta_t) \in N_1 \times N_2$. We claim that x_t stays in the $H^{\mathbb{C}}$-orbit of x. Since the gradient of $\|\mu\|^2$ is tangent to $\mathcal{G}_{k+1}^{\mathbb{C}}$-orbits, we may write $d_x\theta(\dot{x}_t) = (u_t)^{\#}_{\theta(x_t)}$ for some $u_t \in \Omega^0(\mathfrak{g}_E^{\mathbb{C}})_{k+1}$ that depends on t smoothly. Here, $u_t^{\#}$ is the infinitesimal action of u_t. Then, Proposition 3.2.4 implies that $u_t \in \mathbf{H}^0(C_{\mu_{\mathbb{C}}})$ and $\dot{x}_t = (u_t)^{\#}_{x_t}$. On the other hand, the ordinary differential equation in $H^{\mathbb{C}}$,

$$g_t^{-1}\dot{g}_t = u_t, \qquad g_0 = 1, \tag{3.31}$$

has a unique solution $g_t \in H^{\mathbb{C}}$. By the uniqueness, we see that $x_t = xg_t$. Therefore, the claim follows. Then, the fact that T is a homeomorphism implies that both x_t, β_t and u_t converge. Therefore, letting $t \to \infty$, we have $\theta(x_\infty)e^{-i*\beta_\infty}e^{u_\infty} = (B_\infty, \Psi_\infty)$ and $\mu(B_\infty, \Psi_\infty) = 0$. Since $e^{u_\infty} \in \mathcal{G}_{k+1}$, $\theta(x_\infty)e^{-i*\beta_\infty} \in \mu^{-1}(0)$. Since $N_2 \subset O$, the uniqueness of β in Lemma 3.2.5 implies that $\beta(x_\infty) = \beta_\infty$. Hence,

$$\Theta(x_\infty) = \theta(x_\infty)e^{-i*\beta_\infty} \in \mu^{-1}(0). \tag{3.32}$$

Finally, since $xH^{\mathbb{C}}$ is closed in $\mathbf{H}^1$, we see that $x_\infty \in xH^{\mathbb{C}}$. Again, by the previous setup, $x_\infty \in Z \subset U$.

Conversely, suppose $xH^{\mathbb{C}}$ is not closed in $\mathbf{H}^1$. Note that the complex structure I (the one given by multiplication by $\sqrt{-1}$) on $\mathcal{C}$ restricts to $\mathbf{H}^1$. Since the H-action on $\mathbf{H}^1$ is linear, I-holomorphic and preserves the L^2-metric, it admits a standard moment map ν_0 such that $\nu_0(0) = 0$. Since $(\operatorname{grad} \| \cdot \|^2_{L^2}, \operatorname{grad} \|\nu_0\|^2)_{L^2} = 8\|\nu_0\|^2$ (see [58, Example 2.3]), the gradient flow of $\|\nu_0\|^2$ starting at x stays in B and converges to some $y \in B \cap Z$ such that $\nu_0(y) = 0$. By the Kempf-Ness theorem, $yH^{\mathbb{C}}$ is closed in $\mathbf{H}^1$. Of course, $y \in \overline{xH^{\mathbb{C}}} \setminus xH^{\mathbb{C}}$. Hence, by the previous paragraph, we can find $y_\infty \in yH^{\mathbb{C}} \cap U$ such that $\mu(\Theta(y_\infty)) = 0$. Hence, we have

$$\Theta(y_\infty) \sim_{\mathcal{G}^{\mathbb{C}}_{k+1}} \Theta(y) \in \overline{\Theta(x)\mathcal{G}^{\mathbb{C}}_{k+1}}, \tag{3.33}$$

where $\sim_{\mathcal{G}^{\mathbb{C}}_{k+1}}$ is the equivalence relation generated by the $\mathcal{G}^{\mathbb{C}}_{k+1}$-action. Now, since $xH^{\mathbb{C}}$ contains a zero of ν in U, we may assume that $\mu(\Theta(x)) = 0$. Then, the following Lemma 3.2.7 implies that $\Theta(y_\infty) \sim_{\mathcal{G}^{\mathbb{C}}_{k+1}} \Theta(x)$ so that $\Theta(y_\infty) \sim_{\mathcal{G}_{k+1}} \Theta(x)$ by the Hitchin-Kobayashi correspondence. Then, the injectivity of $[x, g] \mapsto \Theta(x)g$ implies that $y_\infty \sim_H x$. This is a contradiction. $\square$

The following result is nothing but the fact that the closure of the $\mathcal{G}^{\mathbb{C}}_{k+1}$-orbit of a semistable Higgs bundle contains a unique polystable orbit. Since we cannot find a proof in the literature, we provide one here:

Lemma 3.2.7. *Let (B, Ψ) be a semistable Higgs bundle. If $(B_i, \Psi_i) \in \overline{(B, \Psi)\mathcal{G}^{\mathbb{C}}_{k+1}}$ $(i = 1, 2)$ are polystable Higgs bundles, then $(B_1, \Psi_1) \sim_{\mathcal{G}^{\mathbb{C}}_{k+1}} (B_2, \Psi_2)$.*

Proof. We may assume that $\mu(B_i, \Psi_i) = 0$ for $i = 1, 2$. Let $r\colon \mathcal{B}^{ss}_k \to \mu^{-1}(0)$ be the retraction (see [67, Theorem 1.1]) given by the Yang-Mills-Higgs flow. Suppose there are sequences $(B^j_i, \Psi^j_i) \in (B, \Psi)\mathcal{G}^{\mathbb{C}}_{k+1}$ such that $(B^j_i, \Psi^j_i) \xrightarrow{j\to\infty} (B_i, \Psi_i)$. By the openness of $\mathcal{B}^{ss}_k$, each (B^j_i, Ψ^j_i) is semistable if $j \gg 0$. By the continuity of r, we have

$$r(B^j_i, \Psi^j_i) \xrightarrow{j\to\infty} r(B_i, \Psi_i) = (B_i, \Psi_i). \tag{3.34}$$

By [67, Theorem 1.4], we see that each $r(B^j_i, \Psi^j_i)$ is the graded object of the Seshadri filtration of (B^j_i, Ψ^j_i). Since graded objects are determined by $\mathcal{G}^{\mathbb{C}}_{k+1}$-orbits, we conclude that

$$r(B_1^j, \Psi_1^j) \sim_{\mathcal{G}_{k+1}^{\mathbb{C}}} \mathrm{Gr}(B, \Psi) \sim_{\mathcal{G}_{k+1}^{\mathbb{C}}} r(B_2^l, \Psi_2^l) \tag{3.35}$$

for each j, l so that $r(B_1^j, \Psi_1^j) \sim_{\mathcal{G}_{k+1}} r(B_2^l, \Psi_2^l)$. Since the $\mathcal{G}_{k+1}$-action is proper, $\mathcal{G}_{k+1}$-orbits are closed. Letting $j \to \infty$, we see that $(B_1, \Psi_1) \in r(B_2^l, \Psi_2^l)\mathcal{G}_{k+1}$. Now, letting $l \to \infty$, we see that $(B_1, \Psi_1) \sim_{\mathcal{G}_{k+1}} (B_2, \Psi_2)$. □

3.2.3 Open embeddings into the moduli space

Let $\mathcal{Z} := Z \cap B$ which is a closed complex subspace of B. Note that $\mathcal{Z}$ is H-invariant but not $H^{\mathbb{C}}$-invariant. To fix this issue, recall that every open ball around 0 (in the L^2-norm) in $\mathbf{H}^1$ is H-invariant and orbit-convex (see [58, Definition 1.2 and Lemma 1.14]). By [26, §3.3, Proposition], $\mathcal{Z}H^{\mathbb{C}}$ is a closed complex subspace of $BH^{\mathbb{C}}$, and $\mathcal{Z}$ is open in $\mathcal{Z}H^{\mathbb{C}}$. Recall the standard moment map $\nu_0\colon \mathbf{H}^1 \to \mathbf{H}^2(C_\mu)$ used in the proof of Theorem 3.2.6. This is the moment map for the H-action on $\mathbf{H}^1$ with respect to the L^2-metric and the restricted complex structure I. Then, by the analytic GIT developed in [30] or [27, §0], there is a categorical quotient $\pi\colon \mathcal{Z}H^{\mathbb{C}} \to \mathcal{Z}H^{\mathbb{C}} /\!\!/ H^{\mathbb{C}}$ in the category of reduced complex spaces such that every fiber of π contains a unique closed $H^{\mathbb{C}}$-orbit, and the inclusion $\nu_0^{-1}(0) \cap \mathcal{Z}H^{\mathbb{C}} \hookrightarrow \mathcal{Z}H^{\mathbb{C}}$ induces a homeomorphism

$$(\nu_0^{-1}(0) \cap \mathcal{Z}H^{\mathbb{C}})/K \xrightarrow{\sim} \mathcal{Z}H^{\mathbb{C}} /\!\!/ H^{\mathbb{C}}. \tag{3.36}$$

Moreover, as a topological space, $\mathcal{Z}H^{\mathbb{C}} /\!\!/ H^{\mathbb{C}}$ is the quotient space defined by the equivalence relation that $x \sim y$ if and only if $\overline{xH^{\mathbb{C}}} \cap \overline{yH^{\mathbb{C}}} \neq \emptyset$.

A corollary of Theorem 3.2.6 is that $\mathcal{Z}H^{\mathbb{C}} /\!\!/ H^{\mathbb{C}}$ can be realized as a singular symplectic quotient with respect to the pullback moment map $\nu = \Theta^*\mu$ instead of ν_0.

Corollary 3.2.8. *The inclusion $j\colon \nu^{-1}(0)\cap\mathcal{Z}H^{\mathbb{C}} \hookrightarrow \mathcal{Z}H^{\mathbb{C}}$ induces a homeomorphism*

$$\overline{j}\colon (\nu^{-1}(0) \cap \mathcal{Z}H^{\mathbb{C}})/K \xrightarrow{\sim} \mathcal{Z}H^{\mathbb{C}} /\!\!/ H^{\mathbb{C}}. \tag{3.37}$$

As a consequence, the perturbed Kuranishi map Θ induces well-defined continuous maps $\overline{\Theta}$ and φ in the following commutative diagram

$$\begin{tikzcd} \mathcal{Z}H^{\mathbb{C}} /\!\!/ H^{\mathbb{C}} \arrow[r, "\varphi"] & \mathcal{B}_k^{ps}/\mathcal{G}_{k+1}^{\mathbb{C}} \\ (\nu^{-1}(0) \cap \mathcal{Z}H^{\mathbb{C}})/K \arrow[u, "\sim"] \arrow[r, "\overline{\Theta}"] & (\mu^{-1}(0) \cap \mathcal{B}_k)/\mathcal{G}_{k+1} \arrow[u, "\sim"] \end{tikzcd} \tag{3.38}$$

More explicitly, φ is given by the formula

$$\varphi[x] = [r\theta(x)], \qquad x \in \mathcal{Z}, \tag{3.39}$$

where $r\colon \mathcal{B}_k^{ss} \to \mu^{-1}(0)$ is the retraction defined by the Yang-Mills-Higgs flow.

Proof. Clearly, $\overline{\Theta}$ is a well-defined continuous map. To define φ, it suffices to show that $\overline{j}$ is a homeomorphism. Therefore, we show that it has a continuous inverse and follow the notations and the setup in the proof of Theorem 3.2.6. Let $\pi\colon \mathcal{Z}H^{\mathbb{C}} \to$

$\mathcal{Z}H^{\mathbb{C}} /\!\!/ H^{\mathbb{C}}$ be the quotient map. If $xg \in \mathcal{Z}H^{\mathbb{C}}$ with $x \in \mathcal{Z}$, by using the gradient flow of $\|\nu_0\|^2$, we see that there is a closed $H^{\mathbb{C}}$-orbit $\widetilde{x}H^{\mathbb{C}} \subset \overline{xH^{\mathbb{C}}}$ with $\widetilde{x} \in \mathcal{Z}$. Then, Theorem 3.2.6 implies that there exists

$$x_\infty \in \nu^{-1}(0) \cap \widetilde{x}H^{\mathbb{C}} \subset \nu^{-1}(0) \cap \overline{xH^{\mathbb{C}}}. \tag{3.40}$$

Therefore, if $\pi(xg) = \pi(yh)$, then $\pi(x_\infty) = \pi(y_\infty)$ so that

$$\overline{\Theta(x_\infty)\mathcal{G}_{k+1}^{\mathbb{C}}} \cap \overline{\Theta(y_\infty)\mathcal{G}_{k+1}^{\mathbb{C}}} \neq \emptyset. \tag{3.41}$$

If we can show that $x_\infty \sim_H y_\infty$, then the map

$$\bar{j}^{-1} \colon \mathcal{Z}H^{\mathbb{C}} /\!\!/ H^{\mathbb{C}} \to (\nu^{-1}(0) \cap \mathcal{Z}H^{\mathbb{C}})/K, \qquad [xg] \mapsto [x_\infty], \tag{3.42}$$

is well-defined. Now, $x_\infty \sim_H y_\infty$ follows from the following Lemma.

Lemma 3.2.9. *If (A_i, Φ_i) $(i = 1, 2)$ are Higgs bundles such that $\mu(A_i, \Phi_i) = 0$ and $\overline{(A_1, \Phi_1)\mathcal{G}_{k+1}^{\mathbb{C}}} \cap \overline{(A_2, \Phi_2)\mathcal{G}_{k+1}^{\mathbb{C}}} \neq \emptyset$, then $(A_1, \Phi_1) \sim_{\mathcal{G}_{k+1}} (A_2, \Phi_2)$.*

Proof. Let (B, Ψ) be a Higgs bundle in the intersection of the closures. Hence, there is a sequence $(A_i^j, \Phi_i^j) \in (A_i, \Phi_i)\mathcal{G}_{k+1}^{\mathbb{C}}$ converging to (B, Ψ). The continuity of r implies that $r(A_i^j, \Phi_i^j) \xrightarrow{j\to\infty} r(B, \Psi)$. By [67, Theorem 1.4],

$$r(A_i^j, \Phi_i^j) \sim_{\mathcal{G}_{k+1}^{\mathbb{C}}} Gr(A_i, \Phi_i) = (A_i, \Phi_i) \tag{3.43}$$

so that $r(A_i^j, \Phi_i^j) \sim_{\mathcal{G}_{k+1}} (A_i, \Phi_i)$. Hence, there is a sequence of $g_i^j \in \mathcal{G}$ such that $(A_i, \Phi_i)g_i^j \xrightarrow{j\to\infty} r(B, \Psi)$. Since the $\mathcal{G}_{k+1}$-action is proper, by passing to a subsequence, we may assume that $g_i^j \xrightarrow{j\to\infty} g_i$ for some $g_i \in \mathcal{G}_{k+1}$. Hence, $(A_i, \Phi_i)g_i = r(B, \Psi)$. $\square$

Continuing with the proof of Corollary 3.2.8, we show that $\overline{j}^{-1}$ is continuous. Recall that x_∞ is determined by the equation $\theta(x_\infty)e^{-i*\beta_\infty}e^{u_\infty} = r(\theta(\widetilde{x}))$. By the continuity of r, T^{-1} and θ^{-1}, we see that the map $\mathcal{Z} \ni \widetilde{x} \mapsto x_\infty$ is continuous. Moreover, $\mathcal{Z} \ni x \mapsto \widetilde{x}$ is also continuous, which is a general property of the gradient flow of $\|\nu_0\|^2$. Since $\mathcal{Z}$ is open in $\mathcal{Z}H^{\mathbb{C}}$, we conclude that $\overline{j}^{-1}$ is continuous.

It remains to show that $\overline{j}^{-1}$ is indeed the inverse of $\overline{j}$. If $xg \in \nu^{-1}(0) \cap \mathcal{Z}H^{\mathbb{C}}$ with $x \in \mathcal{Z}$, then $xH^{\mathbb{C}}$ is closed in $\mathbf{H}^1$ (Theorem 3.2.6). Since $\overline{j}^{-1}$ is well-defined, we see that

$$(xg)_\infty \sim_H x_\infty \sim_{H^{\mathbb{C}}} \widetilde{x} \sim_{H^{\mathbb{C}}} x \sim_{H^{\mathbb{C}}} xg. \tag{3.44}$$

Then, $\nu((xg)_\infty) = \nu(xg) = 0$ implies that $(xg)_\infty \sim_H xg$. Conversely, if $xg \in \mathcal{Z}H^{\mathbb{C}}$ with $x \in \mathcal{Z}$, then $x_\infty \in \overline{xH^{\mathbb{C}}}$ so that $\pi(xg) = \pi(x_\infty)$.

Finally, to obtain a formula for φ, note that

$$\Theta(x_\infty) \in \overline{\Theta(x)\mathcal{G}_{k+1}^{\mathbb{C}}} = \overline{\theta(x)\mathcal{G}_{k+1}^{\mathbb{C}}}. \tag{3.45}$$

Moreover, $r(\theta(x)) \in \overline{\theta(x)\mathcal{G}_{k+1}^{\mathbb{C}}}$. Hence, by Lemma 3.2.7, $\Theta(x_\infty) \sim_{\mathcal{G}_{k+1}^{\mathbb{C}}} r(\theta(x))$. $\square$

The next result shows that $\mathcal{Z}H^{\mathbb{C}} /\!\!/ H^{\mathbb{C}}$ is a local model for the quotient $\mathcal{M}_k = \mathcal{B}_k^{ps}/\mathcal{G}_{k+1}^{\mathbb{C}}$. Strictly speaking, $\mathcal{M}_k$ is not the moduli space $\mathcal{M}$. That said, there is a natural map $\mathcal{M} \to \mathcal{M}_k$. Note that [3, Lemma 14.8] and the elliptic regularity for $\overline{\partial}_A$ with $A \in \mathcal{A}$ imply that every point in $\mathcal{M}_k$ has a C^∞ representative. As a consequence, the natural map $\mathcal{M} \to \mathcal{M}_k$ is surjective. Its injectivity follows from [3, Lemma 14.9]. Later, as a consequence of Theorem 3.2.10, we will show that $\mathcal{M} \to \mathcal{M}_k$ is a homeomorphism, which justifies our use of Sobolev completions.

Theorem 3.2.10. *If B is sufficiently small, $\varphi\colon \mathcal{Z}H^{\mathbb{C}} /\!\!/ H^{\mathbb{C}} \to \mathcal{M}_k$ is an open embedding.*

Proof. We will follow the notations and the setup in the proof of Theorem 3.2.6. Since $\overline{\Theta}$ is injective, φ is injective. Let $\Pi\colon \mathcal{B}_k^{ps} \to \mathcal{M}_k$ be the quotient map, and consider the open set $O = \Pi(W' \cap \mathcal{B}_k^{ps})$. If $(B, \Psi) \in W' \cap \mathcal{B}_k^{ps}$, then $(B, \Psi) = \theta(x)e^{-i*\beta}e^u$ for some $x \in \mathcal{Z}$. We claim that $\varphi[x] = [B, \Psi]$. By the construction of φ in the proof of Corollary 3.2.8, we see that $\varphi[x] = [\Theta(x_\infty)]$ for some $x_\infty \in \nu^{-1}(0) \cap \mathcal{Z}H^{\mathbb{C}} \cap \overline{xH^{\mathbb{C}}}$ so that

$$\Theta(x_\infty) \in \overline{\theta(x)\mathcal{G}_{k+1}^{\mathbb{C}}} = \overline{(B, \Psi)\mathcal{G}_{k+1}^{\mathbb{C}}}. \tag{3.46}$$

By Lemma 3.2.7, we have $\Theta(x_\infty) \sim_{\mathcal{G}_{k+1}^{\mathbb{C}}} (B, \Psi)$. As a consequence, the open set O is contained in the image of φ. Hence, we obtain a bijective continuous map $\varphi\colon \widetilde{O} \to O$, where $\widetilde{O} = \varphi^{-1}(O)$.

To show that $\varphi|_{\widetilde{O}}$ is a homeomorphism, we will show that its inverse is continuous. From the previous paragraph, we see that its inverse should be $[B, \Psi] \mapsto [x]$. The continuity follows from the continuity of θ^{-1} and T^{-1}. Therefore, it remains to prove that it is well-defined. If $(B', \Psi') \in W' \cap \mathcal{B}_k^{ps}$ lies in the $\mathcal{G}_{k+1}^{\mathbb{C}}$-orbit of (B, Ψ), then

$$\Theta(x_\infty) \sim_{\mathcal{G}^{\mathbb{C}}_{k+1}} (B, \Psi) \sim_{\mathcal{G}^{\mathbb{C}}_{k+1}} (B', \Psi') \sim_{\mathcal{G}^{\mathbb{C}}_{k+1}} \Theta(x'_\infty) \tag{3.47}$$

so that

$$\overline{xH^{\mathbb{C}}} \ni x_\infty \sim_H x'_\infty \in \overline{x'H^{\mathbb{C}}}. \tag{3.48}$$

Hence, $\overline{xH^{\mathbb{C}}} \cap \overline{x'H^{\mathbb{C}}} \neq \emptyset$.

Finally, we show that if B is sufficiently small, then φ is an open embedding. Write $\pi^{-1}(\widetilde{O}) = \mathcal{Z}H^{\mathbb{C}} \cap Q$ for some open set Q in $\mathbf{H}^1$, where $\pi\colon \mathcal{Z}H^{\mathbb{C}} \to \mathcal{Z}H^{\mathbb{C}} /\!\!/ H^{\mathbb{C}}$ is the quotient map. Since $0 \in Q$, choose some open ball $B' \subset Q \cap B$ around 0. By [58, Lemma 1.14], we know that B and B' are ν_0-convex (see [30, (2.6), Definition]). Hence, by definition of $\mathcal{Z}$, $\mathcal{Z}$ is also ν_0-convex. Hence, by [30, (3.1), Lemma], we see that $\mathcal{Z}H^{\mathbb{C}} \cap B'H^{\mathbb{C}} = (\mathcal{Z} \cap B')H^{\mathbb{C}}$. Then, we claim that $(\mathcal{Z} \cap B')H^{\mathbb{C}} \subset \pi^{-1}(\widetilde{O})$. In fact, if $xg \in (\mathcal{Z} \cap B')H^{\mathbb{C}}$ with $x \in \mathcal{Z} \cap B'$, then $x \in ZH^{\mathbb{C}} \cap Q$. Since $ZH^{\mathbb{C}} \cap Q$ is $H^{\mathbb{C}}$-invariant, $xg \in ZH^{\mathbb{C}} \cap Q$. Finally, we claim that $(\mathcal{Z} \cap B')H^{\mathbb{C}}$ is also π-saturated so that $(\mathcal{Z} \cap B')H^{\mathbb{C}} /\!\!/ H^{\mathbb{C}}$ is an open neighborhood of $[0]$ in $\mathcal{Z}H^{\mathbb{C}} /\!\!/ H^{\mathbb{C}}$. Therefore, if B is shrunk to B', and $\mathcal{Z}$ is shrunk to $\mathcal{Z} \cap B'$, we see that φ is an open embedding.

Suppose $\pi(xg) = \pi(yh)$ for some $x \in \mathcal{Z}$ and $y \in \mathcal{Z} \cap B'$. We want to show that $xg \in (\mathcal{Z} \cap B')H^{\mathbb{C}}$. By using the gradient flow of $\|\nu_0\|^2$, we can find a closed orbit $y'H^{\mathbb{C}} \subset \overline{yH^{\mathbb{C}}}$ with $y' \in \mathcal{Z} \cap B'$. Since every fiber of π contains a unique closed orbit, $y'H^{\mathbb{C}} \subset \overline{xH^{\mathbb{C}}}$. Since B' is open, $xH^{\mathbb{C}} \cap B' \neq \emptyset$. Hence, $x \in B'H^{\mathbb{C}} \cap \mathcal{Z}H^{\mathbb{C}} = (\mathcal{Z} \cap B')H^{\mathbb{C}}$. □

To show that $\mathcal{M} \to \mathcal{M}_k$ is a homeomorphism, we need the following lemma.

Lemma 3.2.11. *Elements in $\mathcal{B}_k \cap [(A, \Phi) + \ker(D'')^*]$ are of class C^∞.*

Proof. Suppose $(D'')^*(\alpha'', \eta) = 0$ and $(\overline{\partial}_A + \alpha'')(\Phi + \eta) = 0$, where α'' is the $(0,1)$-part of α. The second equation is also equivalent to $D''(\alpha'', \eta) + [\alpha'', \eta] = 0$. Hence, $\Delta(\alpha, \eta) = -(D'')^*[\alpha'', \eta]$ where $\Delta = D''(D'')^* + (D'')^* D''$ is the Laplacian defined in $C_{\mu_{\mathbb{C}}}$. Since $k > 1$, the Sobolev multiplication theorem (see [20, Theorem 4.4.1]) implies that $[\alpha'', \eta]$ is in L^2_k and hence $(D'')^*[\alpha'', \eta]$ is in L^2_{k-1}. By the elliptic regularity, (α'', η) is hence in L^2_{k+1}. By induction, (α'', η) is in C^∞. $\square$

Lemma 3.2.12. *The map φ in Corollary 3.2.8 factors through the natural map $\mathcal{M} \to \mathcal{M}_k$.*

Proof. Recall that the formula for φ is given by $\varphi[x] = [r\theta(x)]$ where $x \in \mathcal{Z}$. By Lemma 3.2.11, θ restricts to a continuous map $Z \to \mathcal{B}^{ss} \cap ((A, \Phi) + \ker(D'')^*)$. Since $r \colon \mathcal{B}^{ss} \to \mu^{-1}(0)$ is continuous, $\mathcal{Z} \ni x \mapsto [r\theta(x)] \in \mathcal{M}$ is continuous. Finally, [3, Lemma 14.9] and the fact that φ is well-defined imply that φ factors through $\mathcal{M} \to \mathcal{M}_k$. $\square$

Corollary 3.2.13. *The natural map $\mathcal{M} \to \mathcal{M}_k$ is a homeomorphism. Therefore, the map $\varphi \colon \mathcal{Z}H^{\mathbb{C}} /\!\!/ H^{\mathbb{C}} \to \mathcal{M}$ is an open embedding.*

Proof. By Lemma 3.2.12 and Theorem 3.2.10, $\mathcal{M} \to \mathcal{M}_k$ is locally an open map and hence open. $\square$

3.3 Gluing local models

For the rest of the paper, we will drop the subscripts that indicate Sobolev completions for notational convenience. By Lemma 3.2.11, 3.2.12 and Corollary 3.2.13, this should not cause any confusion. The main result in this section is the following, which is part of Theorem A. The normality of $\mathcal{M}$ will be proved in Lemma 3.5.7.

Theorem 3.3.1. *The moduli space $\mathcal{M}$ is a complex space locally biholomorphic to a Kuranishi local model $\mathcal{Z}H^{\mathbb{C}} /\!\!/ H^{\mathbb{C}}$.*

Let (A_i, Φ_i) $(i = 1, 2)$ be Higgs bundles such that $\mu(A_i, \Phi_i) = 0$. We will use subscript i to denote relevant objects associated with (A_i, Φ_i). Let $\mathcal{Z}_i$ be their Kuranishi spaces and $\mathcal{Z}_i H_i^{\mathbb{C}} /\!\!/ H_i^{\mathbb{C}}$ Kuranishi local models, where H_i is the $\mathcal{G}$-stabilizer of (A_i, Φ_i). Let

$$\varphi_i \colon \mathcal{Z}_i H_i^{\mathbb{C}} /\!\!/ H_i^{\mathbb{C}} \xrightarrow{\sim} O_i \subset \mathcal{M} \tag{3.49}$$

be the map constructed in Theorem 3.2.10 such that $O_1 \cap O_2 \neq \emptyset$. Hence, the transition function is given by

$$\varphi_2^{-1}\varphi_1 \colon \varphi_1^{-1}(O_1 \cap O_2) \to \varphi_2^{-1}(O_1 \cap O_2). \tag{3.50}$$

Our goal is to show that $\varphi_2^{-1}\varphi_1$ is holomorphic so that $\mathcal{M}$ is a complex space. Since holomorphicity is a local condition, the idea is that the transition function $\varphi_2^{-1}\varphi_1$ should be locally induced by a holomorphic $H_1^{\mathbb{C}}$-invariant map from an open set in $\mathcal{Z}_1 H_1^{\mathbb{C}}$ to $\mathcal{Z}_2 H_2^{\mathbb{C}} /\!\!/ H_2^{\mathbb{C}}$. Then, the rest of the argument follows from the universal property of the quotient map $\pi_i \colon \mathcal{Z}_i H_i^{\mathbb{C}} \to \mathcal{Z}_i H_i^{\mathbb{C}} /\!\!/ H_i^{\mathbb{C}}$. Here, the technical difficulty is to find an appropriate open set in $\mathcal{Z}_1 H_1^{\mathbb{C}}$ that is also π_1-saturated. This will be overcome in the following Lemma 3.3.2.

To proceed, we follow the notations and the setup in the proof of Theorem 3.2.6. Let $[x] \in \varphi_1^{-1}(O_1 \cap O_2)$. Using the gradient flow of $\|\nu_0\|^2$, we may assume that $x \in \mathcal{Z}_1$ has a closed $H_1^{\mathbb{C}}$-orbit. Hence, $\theta_1(x)$ is polystable (Theorem 3.2.6), and $\varphi_1[x] = [r\theta_1(x)] = [\theta_1(x)]$. Similarly, there is some $x' \in \mathcal{Z}_2$ with closed $H_2^{\mathbb{C}}$-orbit such that $\varphi_2[x'] = \varphi_1[x]$ so that $\theta_1(x) \sim_{\mathcal{G}^{\mathbb{C}}} \theta_2(x')$. Since $\theta_i \colon \mathcal{Z}_i \to V_i' \subset W_i'$ is a homeomorphism, $\theta_1(x) \in W_1' \cap W_2' h^{-1}$ for some $h \in \mathcal{G}^{\mathbb{C}}$.

Lemma 3.3.2. *There is an open neighborhood C of x in $\mathcal{Z}_1$ such that*

1. *$CH_1^{\mathbb{C}}$ is π_1-saturated,*
2. *$\theta_1(C) \subset W_1' \cap W_2' h^{-1}$, and*
3. *$[x] \in \pi_1(C) \subset \varphi_1^{-1}(O_1 \cap O_2)$.*

Proof. Since $T_1 \colon N_1' \times V_1' \to W_1'$ and $\theta_1 \colon \mathcal{Z}_1 \to V_1'$ are homeomorphisms, there is an open ball Q around x such that

$$\theta_1(\mathcal{Z}_1 \cap Q) \subset W_1' \cap W_2' h^{-1}. \tag{3.51}$$

Since $\mathcal{Z}_1$ is open in $\mathcal{Z}_1 H_1^{\mathbb{C}}$, $(\mathcal{Z}_1 \cap Q)H_1^{\mathbb{C}}$ is open in $\mathcal{Z}_1 H_1^{\mathbb{C}}$. Then, set

$$C = \pi_1^{-1}\pi_1(\nu_1^{-1}(0) \cap (\mathcal{Z}_1 \cap Q)H_1^{\mathbb{C}}) \cap (\mathcal{Z}_1 \cap Q). \tag{3.52}$$

By Corollary 3.2.8, C is open in $\mathcal{Z}_1$. Clearly, (2) follows and $x \in C$.

To show that $CH_1^{\mathbb{C}}$ is π_1-saturated, let $y \in \mathcal{Z}_1 H_1^{\mathbb{C}}$ be such that $\pi_1(y) = \pi_1(y')$ for some $y' \in C$. By definition of C, $\pi_1(y') = \pi_1(y'')$ for some $y'' \in \nu_1^{-1}(0) \cap (\mathcal{Z}_1 \cap Q)H_1^{\mathbb{C}}$. Since $y''H_1^{\mathbb{C}}$ is closed, $y''H_1^{\mathbb{C}} \subset \overline{yH_1^{\mathbb{C}}}$. Since $y''H_1^{\mathbb{C}} \cap C \neq \emptyset$, and C is open, we conclude that $yH_1^{\mathbb{C}} \cap C \neq \emptyset$. This shows (1). If $y \in C$, then $\pi_1(y) = \pi_1(y'g)$ for some $y'g \in \nu_1^{-1}(0) \cap (\mathcal{Z}_1 \cap Q)H_1^{\mathbb{C}}$ with $y' \in \mathcal{Z}_1 \cap Q$. Therefore, $\varphi_1[y] = [\theta_1(y')]$. By the construction of φ_i in Corollary 3.2.8 and Theorem 3.2.10, we see that

$$O_i = \Pi r\theta_i(\mathcal{Z}_i) = \Pi r(V_i') = \Pi r(W_i'), \tag{3.53}$$

where $\Pi\colon \mathcal{B}^{ps} \to \mathcal{M}$ is the quotient map. Since $\theta_1(y') \in W_1' \cap W_2'h^{-1}$ is polystable, it is easy to see that $[\theta_1(y')] \in O_1 \cap O_2$. This proves (3). $\square$

Now, for $y \in C$, $\theta_1(y)h \in W_2'$. Since T_2 is a homeomorphism, there is $g(y) \in \mathcal{G}^{\mathbb{C}}$, as a function of $y \in C$, such that $\theta_1(y)hg(y) \in V_2'$. Hence, we have obtained a map

$$\psi_{21}\colon C \to \mathcal{Z}_2H_2^{\mathbb{C}} /\!\!/ H_2^{\mathbb{C}}, \qquad \psi_{21}(y) = \pi_2\theta_2^{-1}(\theta_1(y)hg(y)). \tag{3.54}$$

Lemma 3.3.3.

1. *ψ_{21} is holomorphic.*

2. *If $y, y' \in C$ are in the same $H_1^{\mathbb{C}}$-orbit, then $\psi_{21}(y) = \psi_{21}(y')$.*

Proof. Explicitly, we have

$$g(y) = \exp(-p_1T_2^{-1}(\theta_1(y)h)), \tag{3.55}$$

where p_1 is the projection onto the first factor. Since

$$T_2\colon \mathbf{H}^0(C^2_{\mu_{\mathbb{C}}})^{\perp} \times ((A_2, \Phi_2) + \ker D_2''^*) \to \mathcal{C} \tag{3.56}$$

is holomorphic, its inverse, when restricted to appropriate open neighborhoods, is also holomorphic. Moreover, since the Kuranishi map is holomorphic, θ_1 is also holomorphic when the codomain is appropriately extended. Therefore, we conclude that $g\colon C \to \mathcal{G}^{\mathbb{C}}$ is holomorphic. Finally, since the $\mathcal{G}^{\mathbb{C}}$-action is holomorphic, we conclude that ψ_{21} is holomorphic.

To show (2), suppose there are $z, z' \in \mathcal{Z}_2$ such that

$$\begin{aligned} \theta_2(z) &= \theta_1(y)hg(y), \\ \theta_2(z') &= \theta_1(y')hg(y'). \end{aligned} \tag{3.57}$$

We want to show that $\pi_2(z) = \pi_2(z')$. Since y and y' are in the same $H_1^{\mathbb{C}}$-orbit,

$$\theta_2(z) \sim_{\mathcal{G}^{\mathbb{C}}} \theta_1(y) \sim_{\mathcal{G}^{\mathbb{C}}} \theta_1(y') \sim_{\mathcal{G}^{\mathbb{C}}} \theta_2(z') \tag{3.58}$$

so that $r\theta_2(z) \sim_{\mathcal{G}} r\theta_2(z')$. This means that $\varphi_2[z] = \varphi_2[z']$. Since φ_2 is injective, $[z] = [z']$. $\square$

Lemma 3.3.4. *The transition function $\varphi_2^{-1}\varphi_1$ is holomorphic.*

Proof. By Lemma 3.3.3, ψ_{21} extends to a $H_1^{\mathbb{C}}$-invariant holomorphic map

$$\psi_{21}\colon CH_1^{\mathbb{C}} \to \mathcal{Z}_2 H_2^{\mathbb{C}} /\!\!/ H_2^{\mathbb{C}}. \tag{3.59}$$

Since $CH_1^{\mathbb{C}}$ is a π_1-saturated open set (Lemma 3.3.2),

$$\pi_2\colon CH_1^{\mathbb{C}} \to \pi_1(CH_1^{\mathbb{C}}) =: CH_1^{\mathbb{C}} /\!\!/ H_1^{\mathbb{C}} \tag{3.60}$$

is also a categorical quotient. As a consequence, ψ_{21} descends to a holomorphic map

$$\overline{\psi}_{21}\colon CH_1^{\mathbb{C}} /\!\!/ H_1^{\mathbb{C}} \to \mathcal{Z}_2 H_2^{\mathbb{C}} /\!\!/ H_2^{\mathbb{C}}. \tag{3.61}$$

Let $[c] \in CH_1^{\mathbb{C}} /\!\!/ H_1^{\mathbb{C}}$ with $c \in C$ and $z = \theta_2^{-1}(\theta_1(c)hg(c))$. Hence, $\theta_2(z) \sim_{\mathcal{G}^{\mathbb{C}}} \theta_1(c)$. Therefore,

$$\varphi_2\overline{\psi}_{21}[c] = \varphi_2\psi_{21}(c) = \varphi_2\pi_2(z) = \Pi(r\theta_2(z)) = \Pi(r\theta_1(z)) = \varphi_1[c]. \tag{3.62}$$

This shows that the transition function $\varphi_2^{-1}\varphi_1$ coincides with a holomorphic map $\overline{\psi}_{21}$ on an open neighborhood $CH_1^{\mathbb{C}} /\!\!/ H_1^{\mathbb{C}}$ of $[x]$ in $\varphi_1^{-1}(O_1 \cap O_2)$. This completes the proof. □

Proof of Theorem 3.3.1. By the properness of the $\mathcal{G}$-action, $(\mu^{-1}(0) \cap \mathcal{B})/\mathcal{G}$ is Hausdorff. The Hitchin-Kobayashi correspondence implies that $\mathcal{M}$ is Hausdorff. The Kuranishi local models are constructed in Corollary 3.2.8 and Theorem 3.2.10. By Lemma 3.3.4, the transition functions are holomorphic. □

3.4 Singularities in Kuranishi spaces

In this section, we will show that Kuranishi spaces have only cone singularities. We will use the same notations as in Section 3.2. The main result in this section is the following (cf. [34, Theorem 2.24] and [2, Theorem 3]).

Theorem 3.4.1. *The following diagram commutes:*

$$\begin{tikzcd} \widetilde{\mathcal{B}} \cap ((A,\Phi) + \ker(D'')^*) \arrow[r, "F"] \arrow[d, "\mu_{\mathbb{C}}"] & \mathbf{H}^1 \arrow[dl, "\frac{1}{2}P[\cdot,\cdot]"] \\ \mathbf{H}^2(C_{\mu_{\mathbb{C}}}) & \end{tikzcd} \tag{3.63}$$

Proof. By construction of $\widetilde{\mathcal{B}}$, the restriction of $\mu_{\mathbb{C}}$ to $\widetilde{\mathcal{B}}$ is given by

$$\mu_{\mathbb{C}}(A+\alpha, \Phi+\eta) = P\mu_{\mathbb{C}}(A+\alpha, \Phi+\eta) = \frac{1}{2}P[\alpha'', \eta; \alpha'', \eta] = P[\alpha'', \eta], \tag{3.64}$$

where $(A+\alpha'', \Phi+\eta) \in \widetilde{\mathcal{B}}$. By definition of the Kuranishi space Z, it suffices to prove

1. $P[(\alpha'', \eta), (D'')^* G[\alpha'', \eta; \alpha'', \eta]] = 0$, and

2. $P[(D'')^* G[\alpha'', \eta; \alpha'', \eta], (D'')^* G[\alpha'', \eta; \alpha'', \eta]] = 0$

for any $(\alpha'', \eta) \in \ker(D'')^*$. By Kähler identities,

$$P[(\alpha'', \eta), (D'')^* G[\alpha'', \eta; \alpha'', \eta]] = \pm i P[(\alpha'', \eta), D' * G[\alpha'', \eta; \alpha'', \eta]] \tag{3.65}$$

and $(\alpha'', \eta) \in \ker D'$. Since D' is a derivation with respect to $[\cdot,\cdot]$, we see that

$$P[(\alpha'', \eta), D' * G[\alpha'', \eta; \alpha'', \eta]] = \pm P D'[(\alpha'', \eta), *G[\alpha'', \eta; \alpha'', \eta]] = 0. \tag{3.66}$$

This proves (1). The same argument shows (2). This completes the proof. $\square$

As a corollary, we obtain a description of singularities in the Kuranishi spaces.

Corollary 3.4.2. *The Kuranishi space Z is an open neighborhood of* 0 *in the quadratic cone*

$$Q = \left\{ x \in \mathbf{H}^1 \colon \frac{1}{2} P[x, x] = 0 \right\}. \tag{3.67}$$

Proof. This is clear by definition of Kuranishi spaces and Theorem 3.4.1. $\square$

It is easy to see that the complex structures on $\mathcal{C}$ restrict to $\mathbf{H}^1$ so that $\mathbf{H}^1$ has a linear hyperKähler structure. In particular, the complex symplectic form $\Omega_{\mathbb{C}}$ on $\mathcal{C}$ restricts to $\mathbf{H}^1$. Hence, there is a standard complex moment map $\nu_{0,\mathbb{C}} \colon \mathbf{H}^1 \to \mathbf{H}^2(C_{\mu_{\mathbb{C}}})$ for the $H^{\mathbb{C}}$-action with respect to the linear complex symplectic structure. More precisely, $\nu_{0,\mathbb{C}}$ is defined by

$$\langle \nu_{0,\mathbb{C}}(x), \xi \rangle = \frac{1}{2} \Omega_{\mathbb{C}}(x \cdot \xi, x), \qquad \xi \in \mathbf{H}^0(C_{\mu_{\mathbb{C}}}). \tag{3.68}$$

Since $i \colon \mathbf{H}^1 \hookrightarrow \mathcal{C}$ is $H^{\mathbb{C}}$-equivariant, and $\mu_{\mathbb{C}}$ is a complex moment map, $Hi^*\mu_{\mathbb{C}}$ is a complex moment map for the $H^{\mathbb{C}}$-action on $\mathbf{H}^1$, where H is the harmonic projection onto $\mathbf{H}^2(C_{\mu_{\mathbb{C}}})$. Since $Pi^*\mu_{\mathbb{C}}(0) = 0$, we see that $Pi^*\mu_{\mathbb{C}} = \nu_{0,\mathbb{C}}$. On the other hand, $Pi^*\mu_{\mathbb{C}} = \frac{1}{2}P[\cdot, \cdot]$. Hence, Q is the zero set of the standard complex moment map $\nu_{0,\mathbb{C}}$.

Obviously, $\nu_{0,\mathbb{C}}^{-1}(0)$ is a closed complex subspace of $\mathbf{H}^1$. In fact, it is an affine variety. Therefore, the affine GIT quotient $\nu_{0,\mathbb{C}}^{-1}(0) /\!\!/ H^{\mathbb{C}}$ exists such that the inclusion $\nu_0^{-1}(0) \cap \nu_{0,\mathbb{C}}^{-1}(0) \hookrightarrow \nu_{0,\mathbb{C}}^{-1}(0)$ induces a homeomorphism (see [30, (1.4)])

$$(\nu_0^{-1}(0) \cap \nu_{0,\mathbb{C}}^{-1}(0))/H \xrightarrow{\sim} \nu_{0,\mathbb{C}}^{-1}(0) /\!\!/ H^{\mathbb{C}}. \tag{3.69}$$

Note that $(\nu_0^{-1}(0) \cap \nu_{0,\mathbb{C}}^{-1}(0))/H$ is precisely the hyperKähler quotient with respect to the standard hyperKähler moment maps on $\mathbf{H}^1$.

Theorem 3.4.3 (=Theorem B)**.** *Let $[A, \Phi] \in \mathcal{M}$ be a point such that $\mu(A, \Phi) = 0$ and $\mathbf{H}^1$ its deformation space, a harmonic space defined in $C_{\mu_\mathbb{C}}$. Then, the following hold:*

1. *$\mathbf{H}^1$ is a complex-symplectic vector space.*

2. *The $\mathcal{G}^\mathbb{C}$-stabilizer $H^\mathbb{C}$ at (A, Φ) is a complex reductive group, acts on $\mathbf{H}^1$ linearly and preserves the complex-symplectic structure on $\mathbf{H}^1$. Moreover, the $H^\mathbb{C}$-action on $\mathbf{H}^1$ admits a canonical complex moment map $\nu_{0,\mathbb{C}}$ such that $\nu_{0,\mathbb{C}}(0) = 0$.*

3. *Around $[A, \Phi]$, the moduli space $\mathcal{M}$ is locally biholomorphic to an open neighborhood of $[0]$ in the complex symplectic quotient $\nu_{0,\mathbb{C}}^{-1}(0) /\!\!/ H^\mathbb{C}$ which is an affine GIT quotient.*

Proof. It remains to show (3). Since $\mathcal{Z}$ is open in Z which is also open in Q, we have $\mathcal{Z}H^\mathbb{C}$ is open in Q. Since $\mathcal{Z}H^\mathbb{C}$ is saturated with respect to the quotient $Q \to Q /\!\!/ H^\mathbb{C}$, $\mathcal{Z}H^\mathbb{C} /\!\!/ H^\mathbb{C}$ is an open neighborhood of $[0]$ in $Q /\!\!/ H^\mathbb{C}$. The rest follows from Theorem 3.2.10 and 3.3.1. ∎

3.5 The Isomorphism between the analytic and the algebraic constructions

Let $\mathcal{M}_{an}$ be the moduli space $\mathcal{B}^{ps}/\mathcal{G}^{\mathbb{C}}$ and $\mathcal{M}_{alg}$ the coarse moduli space of the semistable Higgs bundles of rank r and degree 0, where r is the rank of E. By [57, Theorem 4.7, Theorem 11.1], $\mathcal{M}_{alg}$ is a normal irreducible quasi-projective variety. By abusing the notation, we also use $\mathcal{M}_{alg}$ to mean its analytification. Then, there is a natural comparison map

$$i\colon \mathcal{M}_{an} \to \mathcal{M}_{alg}, \qquad [A, \Phi] \mapsto [\mathcal{E}_A, \Phi]_S. \tag{3.70}$$

Here, $(\mathcal{E}_A, \Phi)$ is the Higgs bundle determined by (A, Φ), and $[\mathcal{E}_A, \Phi]_S$ means the S-equivalence class of $(\mathcal{E}_A, \Phi)$. We will prove Theorem C in this section. By [67, Proposition 5.1], we see that i is a bijection of sets.

3.5.1 Continuity

The first step towards our goal is to show that i is a homeomorphism. To this end, we need some preparations. First, we may assume that the degree of E is sufficiently large. This can be arranged as follows. Fix a holomorphic line bundle $\mathcal{L} = (L, \overline{\partial}_L)$ of degree $d > 0$. Here, L is the underlying smooth line bundle of $\mathcal{L}$, and $\overline{\partial}_L$ is the $\overline{\partial}$-operator defined by the holomorphic structure on $\mathcal{L}$. We may also fix a Hermitian metric on L so that the Chern connection of $\overline{\partial}_L$ is d_L. Then, there is a map

$$\mathcal{B}(E) \to \mathcal{B}(E \otimes L), \qquad (A, \Phi) \mapsto (A \otimes 1 + 1 \otimes d_L, \Phi \otimes 1). \tag{3.71}$$

Here, $\mathcal{B}(E)$ and $\mathcal{B}(E \otimes L)$ are the configuration spaces of Higgs bundles with underlying smooth bundles E and $E \otimes L$, respectively. Since $(\mathcal{E}, \Phi)$ is (semi)stable if and only if $(\mathcal{E} \otimes \mathcal{L}, \Phi)$ is (semi)stable, this map restricts to a map

$$\mathcal{B}(E)^{ps} \to \mathcal{B}(E \otimes L)^{ps} \tag{3.72}$$

and eventually descends to a homeomorphism (in the C^∞-topology)

$$\mathcal{M}_{an} \xrightarrow{\otimes \mathcal{L}} \mathcal{M}_{an}(rd), \tag{3.73}$$

where $\mathcal{M}_{an}(rd) = \mathcal{B}^{ps}(E \otimes L)^{ps} / \operatorname{Aut}(E \otimes L)$, and rd is the degree of $E \otimes L$. On the other hand, there is a homeomorphism (in the analytic topology) $\mathcal{M}_{alg} \to \mathcal{M}_{alg}(rd)$ given by tensoring by $\mathcal{L}$. Here, $\mathcal{M}_{alg}(rd)$ is the moduli space of the semistable Higgs bundles of rank r and degree rd in the category of schemes. Finally, these maps fit into the following commutative diagram

$$\begin{tikzcd} \mathcal{M}_{an} \arrow[r, "i"] \arrow[d, "\otimes \mathcal{L}"] & \mathcal{M}_{alg} \arrow[d, "\otimes \mathcal{L}"] \\ \mathcal{M}_{an}(rd) \arrow[r, "i"] & \mathcal{M}_{alg}(rd) \end{tikzcd} \tag{3.74}$$

Therefore, the bottom map is a homeomorphism if and only if the top one is a homeomorphism.

Now, let us recall Nitsure's construction of $\mathcal{M}_{alg}$ in [48]. By the previous paragraph, we may assume that the degree d of E is sufficiently large so that if $(\mathcal{E}_A, \Phi)$ is a semistable Higgs bundle defined by $(A, \Phi) \in \mathcal{B}$ then $\mathcal{E}_A$ is generated by global sections and $H^1(M, \mathcal{E}_A) = 0$. Let $p = d + r(1-g)$ and Q be the Quot scheme parameterizing isomorphism classes of quotients $\mathcal{O}_M^p \to \mathcal{E} \to 0$, where $\mathcal{E}$ is a coherent sheaf on M with rank r and degree d, and $\mathcal{O}_M$ is the structure sheaf of M. Let $\mathcal{O}^p_{M\times Q} \to \mathscr{U} \to 0$ be the universal quotient sheaf on $M \times Q$, and $R \subset Q$ be the subset of all $q \in Q$ such that

1. the sheaf $\mathscr{U}_q$ is locally free, and
2. the map $H^0(M, \mathcal{O}_M^p) \to H^0(M, \mathscr{U}_q)$ is an isomorphism.

It is shown that R is open in Q. Moreover, Nitsure constructed a linear scheme F over R such that closed points in F_q correspond to Higgs fields on $\mathscr{U}_q$ for any $q \in Q$. Let F^{ss} denote the subset of F consisting of semistable Higgs bundles $(\mathcal{O}_M^p \to \mathcal{E} \to 0, \Phi)$. It is open in F. Moreover, the group $PGL(p)$ acts on Q, and the action lifts to F. Finally, Nitsure showed that the good quotient of F^{ss} by the group $PGL(p)$ exists and is the moduli space $\mathcal{M}_{alg}$.

Following [52], if U is an open subset of $\mathcal{B}^{ss}$ (in the C^∞-topology), a map $\sigma\colon U \to F^{ss}$ is called a *classifying map* if $\sigma(A, \Phi)$ is a Higgs bundle isomorphic to $(\mathcal{E}_A, \Phi)$.

Lemma 3.5.1. *Fix $(A_0, \Phi_0) \in \mathcal{B}^{ss}$. There exists an open neighborhood U of (A_0, Φ_0) in $\mathcal{B}^{ss}$ in the C^∞-topology such that a classifying map $\sigma\colon U \to F^{ss}$ exists and is continuous with respect to the analytic topology on F^{ss}.*

Before giving the proof, we first show how it implies the continuity of i.

Corollary 3.5.2. *The comparison map $i\colon \mathcal{M}_{an} \to \mathcal{M}_{alg}$ is a homeomorphism.*

Proof. Fix $[A_0, \Phi_0] \in \mathcal{M}_{an}$ such that $(A_0, \Phi_0) \in \mathcal{B}^{ps}$. By Lemma 3.5.1, there exists an open neighborhood U of (A_0, Φ_0) such that a continuous classifying map $\sigma\colon U \to F^{ss}$ exists. Composed with the categorical quotient $F^{ss} \to \mathcal{M}_{alg}$, which is continuous in the analytic topology, we obtain a continuous map $U \to \mathcal{M}_{alg}$. By construction, it descends to the restriction of i to the open set $\pi(U)$, where $\pi\colon \mathcal{B}^{ps} \to \mathcal{M}_{an}$ is the quotient map.

To see that i is a homeomorphism, we show that it is proper. Since $\mathcal{M}_{alg}$ is locally compact in the analytic topology, if i is proper, then it is a closed map and hence a homeomorphism. Let us recall the definitions of Hitchin fibrations in the analytic and algebraic settings. Given a Higgs bundle $(\mathcal{E}, \Phi)$, the coefficient of λ^{n-i} in the characteristic polynomial $\det(\lambda + \Phi)$ is a holomorphic section of $\mathcal{K}_M^i$, where n is the rank of $\mathcal{E}$, $i = 1, \cdots, n$, and $\mathcal{K}_M$ is the canonical bundle on the Riemann surface M. Since these sections are clearly $\mathcal{G}^{\mathbb{C}}$-invariant, we have obtained a well-defined map

$$h_{an}\colon \mathcal{M}_{an} \to \bigoplus_{i=1}^{n} H^0(M, \mathcal{K}_M^i). \tag{3.75}$$

It is known that h_{an} is a proper map (see [33, Theorem 8.1] or [66, Theorem 2.15]). On the other hand, let $(\mathscr{V}, \mathbf{\Phi})$ be the local universal family of semistable Higgs bundles parameterized by the scheme F^{ss}. Therefore, $(\mathscr{V}, \mathbf{\Phi})$ is a pair of a vector bundle $\mathscr{V} \to M \times F^{ss}$ and a section $\mathbf{\Phi} \in H^0(M \times F^{ss}, p_M^* \mathcal{K}_M \otimes \operatorname{End} \mathscr{V})$, where $p_M \colon M \times F^{ss} \to M$ is the projection onto the first factor. Moreover, if $q = (\mathcal{E}, \Phi) \in F^{ss}$ is a semistable Higgs bundle then the restriction $(\mathscr{V}_q, \mathbf{\Phi}_q)$ of $(\mathscr{V}, \mathbf{\Phi})$ to $M \times \{q\}$ is isomorphic to $(\mathcal{E}, \Phi)$. Hence, there is a map $\widetilde{h}_{alg} \colon F^{ss} \to \oplus_{i=1}^n H^0(M, \mathcal{K}_M^i)$ sending a closed point $q \in F^{ss}$ to the coefficients of the characteristic polynomial $\det(\lambda + \mathbf{\Phi}_q)$. Since the Higgs fields of two S-equivalent Higgs bundles have the same characteristic polynomial, $\widetilde{h}_{alg}$ induces a well-defined map $h_{alg} \colon \mathcal{M}_{alg} \to \oplus_{i=1}^n H^0(M, \mathcal{K}_M^i)$ (see [48, §6] for more details). The maps h_{an} and h_{alg} are called Hitchin fibrations. Therefore, if $[A, \Phi] \in \mathcal{M}_{an}$ and $q = (\mathcal{E}_A, \Phi) \in F^{ss}$ is the Higgs bundle determined by (A, Φ), then

$$h_{alg} \circ i[A, \Phi] = h_{alg}([\mathcal{E}_A, \Phi]_S) = \widetilde{h}_{alg}(q). \tag{3.76}$$

By definition, $\widetilde{h}_{alg}(q) \in \oplus_{i=1}^n H^0(M, \mathcal{K}_M^i)$ is the coefficients of the characteristic polynomial $\det(\lambda + \mathbf{\Phi}_q)$. Since $(\mathscr{V}_q, \mathbf{\Phi}_q)$ is isomorphic to $(\mathcal{E}_A, \Phi)$, $\widetilde{h}_{alg}(q) = h_{an}(A, \Phi)$, and we have proved that $h_{alg} \circ i = h_{an}$.

As a consequence, if K is a compact subset in $\mathcal{M}_{alg}$ in the analytic topology, then $i^{-1}(K) \subset h_{an}^{-1} h_{alg}(K)$. Since h_{alg} is continuous, $h_{alg}(K)$ is compact and hence $h_{an}^{-1} h_{alg}(K)$ is compact by the properness of h_{an}. Since $\mathcal{M}_{alg}$ is a separated scheme, $\mathcal{M}_{alg}$ is Hausdorff in the analytic topology. Hence, K is closed and $i^{-1}(K)$ is also closed and contained in a compact set. Therefore, $i^{-1}(K)$ is compact. □

Proof of Lemma 3.5.1. The proof is essentially taken from that of [52, Theorem 6.1]. We first show that a classifying map σ exists and then prove its continuity.

Let $V_0 = \ker \overline{\partial}_{A_0} \subset \Omega^0(E)$. By definition of $\overline{\partial}$-operators, $V_0 = H^0(M, \mathcal{E}_A)$. Since $H^1(\mathcal{E}_A) = 0$, the Riemann-Roch theorem implies that $\dim V_0 = p$. Hence, by choosing a basis for V_0, we may identify V_0 with $\mathbb{C}^p$. Moreover, since $\mathcal{E}_{A_0}$ is generated by global sections, the evaluation map

$$M \times V_0 \to \mathcal{E}_A, \qquad (x, s) \mapsto s(x), \tag{3.77}$$

realizes $\mathcal{E}_{A_0}$ as a quotient of $V_0 \otimes \mathcal{O}_M \cong \mathcal{O}_M^p$. Let (A, Φ) be another point in $\mathcal{B}^{ss}$, and consider the map defined by the composition

$$\pi_A \colon V_A = \overline{\partial}_A \hookrightarrow \Omega^0(E) \to V_0, \tag{3.78}$$

where $\Omega^0(E) \to V_0$ is given by the harmonic projection defined in the following elliptic complex

$$C(A_0) \colon \Omega^0(E) \xrightarrow{\overline{\partial}_{A_0}} \Omega^{0,1}(E). \tag{3.79}$$

We claim that there exists an open neighborhood U of (A_0, Φ_0) such that π_A is an isomorphism for every $(A, \Phi) \in U$. Write $\pi_A(s) = s + u_s$ for some $u_s \in V_0^\perp$ and $\overline{\partial}_A = \overline{\partial}_{A_0} + a$ for some $a \in \Omega^{0,1}(\mathfrak{g}_E^{\mathbb{C}})$. Let G_0 be the Green operator in the elliptic complex $C(A_0)$. Since $u_s \in V_0^\perp$,

$$u_s = \overline{\partial}^*_{A_0} \overline{\partial}_{A_0} G_0 u_s = \overline{\partial}^*_{A_0} G_0 \overline{\partial}_{A_0} u_s = \overline{\partial}^*_{A_0} G_0(-\overline{\partial}_{A_0} s) = \overline{\partial}^*_{A_0} G_0(as). \tag{3.80}$$

Hence, π_A has a natural extension

$$\widetilde{\pi}_A \colon \Omega^0(E) \to \Omega^0(E), \qquad s \mapsto s + \overline{\partial}^*_{A_0} G_0(as), \tag{3.81}$$

satisfying the following estimate

$$\|\overline{\partial}^*_{A_0} G_0(as)\|_{L^2_k} \le C\|as\|_{L^2_{k-1}} \le C\|as\|_{L^2_k} \le C\|a\|_{L^2_k}\|s\|_{L^2_k} \le C\|a\|_{C^\infty}\|s\|_{L^2_k}, \tag{3.82}$$

where we have used the Sobolev multiplication theorem (see [20, Theorem 4.4.1]). Therefore, if $A_1, A_2 \in \mathcal{B}^{ss}$ and $\overline{\partial}_{A_i} = \overline{\partial}_{A_0} + a_i$ for some $a_i \in \Omega^{0,1}(E)$, we have

$$\|(\widetilde{\pi}_{A_2} - \widetilde{\pi}_{A_1})s\|_{L^2_k} = \|\overline{\partial}^*_{A_0} G_0(a_2 - a_1)s\|_{L^2_k} \le C\|a_2 - a_1\|_{C^\infty}\|s\|_{L^2_k}. \tag{3.83}$$

Now if U is sufficiently small, we may assume that

$$\|\overline{\partial}^*_{A_0} G_0(as)\|_{L^2_k} \le (1/2)\|s\|_{L^2_k} \tag{3.84}$$

so that

$$\|\widetilde{\pi}_A s\|_{L^2_k} \ge (1/2)\|s\|_{L^2_k}. \tag{3.85}$$

This shows that $\widetilde{\pi}_A$ is injective. Since $H^1(\mathcal{E}_A) = 0$, $\dim V_A = \dim V_0 = p$, π_A is an isomorphism. Therefore, the map

$$M \times V_0 \xrightarrow{1\times\pi_A^{-1}} M \times V_A \xrightarrow{(x,s)\mapsto s(x)} \mathcal{E}_A \tag{3.86}$$

realizes $\mathcal{E}_A$ as a quotient of $V_0 \otimes \mathcal{O}_M \cong \mathcal{O}_M^p$, since $\mathcal{E}_A$ is generated by global sections. As a consequence, the classifying map

$$\sigma\colon U \to F^{ss}, \qquad (A, \Phi) \mapsto (\mathcal{O}_M^p \to \mathcal{E}_A \to 0, \Phi), \tag{3.87}$$

is well-defined.

Now, we show that σ is continuous. Let $G(p, r)$ be the Grassmannian parameterizing isomorphism classes of quotients $\mathbb{C}^p \to V \to 0$, where V is a vector space of dimension r. Over $G(p, r)$, there is a universal quotient bundle $H \to G(p, r)$. Fix $x \in M$ and choose a basis for the fiber $(\mathcal{K}_M)_x$ of the canonical bundle $\mathcal{K}_M$ over x. Therefore, any Higgs field $\Phi \in H^0(\operatorname{End} \mathcal{E} \otimes \mathcal{K}_M)$ induces an endomorphism $\Phi_x\colon E_x \to E_x \otimes (\mathcal{K}_M)_x \cong E_x$. Then, Nitsure showed in [48] that there is a morphism

$$\tau_x\colon F \to \operatorname{End} H, \qquad (\mathcal{O}_M^p \to \mathcal{E}_A \to 0, \Phi) \mapsto (\mathbb{C}^p \to E_x, \Phi_x\colon E_x \to E_x), \tag{3.88}$$

where $\mathbb{C}^p \to E_x$ is obtained by evaluating the map $\mathcal{O}_M^p \to \mathcal{E}_A$ at x. Moreover, [48, Proposition 5.7] states that there are N points $x_1, \cdots, x_N \in M$ such that $\{\tau_{x_i}\}$ induces an injective and proper morphism (in the category of schemes) $\tau\colon F^{ss} \to W$ for some open subset W of $(\operatorname{End} H)^N$. Therefore, the underlying continuous map of τ is a closed embedding with respect to the analytic topology. Hence, σ is continuous if the composition

$$\sigma_x\colon U \xrightarrow{\sigma} F^{ss} \xrightarrow{\tau_x} \operatorname{End} H \tag{3.89}$$

is continuous for any $x \in M$. More explicitly, σ_x is given by

$$(A, \Phi) \mapsto (V_0 \to E_x \to 0, \Phi_x \colon E_x \to E_x), \tag{3.90}$$

where $V_0 \to E_x$ is defined by

$$V_0 \xrightarrow{\pi_A^{-1}} V_A \xrightarrow{s \mapsto s(x)} E_x. \tag{3.91}$$

Clearly, the map $\Phi \mapsto \Phi_x$ is continuous. It suffices to show that

$$A \mapsto (V_0 \to E_x \to 0) \tag{3.92}$$

is continuous. Fix $s \in V_0$ and $A_1, A_2 \in U$. Write $\overline{\partial}_{A_i} = \overline{\partial}_{A_0} + a_i$ for some $a_i \in \Omega^{0,1}(E)$ $(i = 1, 2)$. Then, the following estimate follows from (3.83), (3.85), and Sobolev embedding $L_k^2 \hookrightarrow C^0$,

$$\begin{aligned}
|(\pi_{A_1}^{-1} - \pi_{A_2}^{-1})s(x)| &\leq \|(\pi_{A_1}^{-1} - \pi_{A_2}^{-1})s\|_{C^0} \\
&\leq C\|(\pi_{A_1}^{-1} - \pi_{A_2}^{-1})s\|_{L_k^2} \\
&\leq C\|\widetilde{\pi}_{A_1}^{-1}(s - \widetilde{\pi}_{A_1}\widetilde{\pi}_{A_2}^{-1}s)\|_{L_k^2} \\
&\leq C\|s - \widetilde{\pi}_{A_1}\widetilde{\pi}_{A_2}^{-1}s\|_{L_k^2} \\
&= C\|(\widetilde{\pi}_{A_2} - \widetilde{\pi}_{A_1})\pi_{A_2}^{-1}s\|_{L_k^2} \\
&\leq C\|a_2 - a_1\|_{C^\infty}\|\widetilde{\pi}_{A_2}^{-1}s\|_{L_k^2} \\
&\leq C\|a_2 - a_1\|_{C^\infty}\|s\|_{L_k^2}.
\end{aligned} \tag{3.93}$$

Hence, $A \mapsto (V_0 \to E_x \to 0)$ is continuous. $\square$

3.5.2 Holomorphicity

We continue to show that the comparison map i is a biholomorphism. Let $\mathcal{M}_{an}^s$ and $\mathcal{M}_{alg}^s$ be the subsets of $\mathcal{M}_{an}$ and $\mathcal{M}_{alg}$ consisting of stable Higgs bundles, respectively. We first show that the restriction $i\colon \mathcal{M}_{an}^s \to \mathcal{M}_{alg}^s$ is a biholomorphism. By [56, Theorem 4.7], $\mathcal{M}_{alg}^s$ is open in $\mathcal{M}_{alg}$. By [56, Corollary 11.7] and [48, Proposition 7.1], we see that $\mathcal{M}_{alg}^s$ is smooth. On the other hand, a polystable Higgs bundle (A, Φ) is stable if and only if its $\mathcal{G}^{\mathbb{C}}$-stabilizer is equal to $\mathbb{C}^*$ or equivalently $\dim \mathbf{H}^0(C_{\mu_{\mathbb{C}}}(A, \Phi)) = 1$. Since $\mathbb{C}^*$ is contained in every $\mathcal{G}^{\mathbb{C}}$-stabilizer, by the upper semicontinuity of dimensions of cohomology (see [39, Chapter VII, (2.37)]), $\mathcal{B}^s$ is open in $\mathcal{B}^{ps}$. Therefore, we conclude that $\mathcal{M}_{an}^s$ is open in $\mathcal{M}_{an}$.

Proposition 3.5.3. *$\mathcal{M}_{an}^s$ is a smooth submanifold of $\mathcal{M}_{an}$.*

Proof. Fix $(A, \Phi) \in \mathcal{B}^s$ that satisfies Hitchin's equation. Let H be its $\mathcal{G}$-stabilizer so that $H^{\mathbb{C}}$ is its $\mathcal{G}^{\mathbb{C}}$-stabilizer. To show that $\mathcal{M}_{an}^s$ is smooth, we will use Theorem B. It is enough to show that $\nu_{0,\mathbb{C}}^{-1}(0) /\!\!/ H^{\mathbb{C}} = \mathbf{H}^1$. In fact, since $H^{\mathbb{C}} = \mathbb{C}^*$, $H^{\mathbb{C}}$ acts on $\mathbf{H}^1$ trivially. Moreover, $\nu_{0,\mathbb{C}}(x) = \frac{1}{2}P[x, x]$ is trace-free for every $x \in \mathbf{H}^1$. Since $\mathbf{H}^2(C_{\mu_{\mathbb{C}}}) = \mathbb{C}^*\omega_M$, we conclude that $P[x, x] = 0$ for every $x \in \mathbf{H}^1$, where ω_M is a fixed Kähler form on M. $\square$

Fix $[A, \Phi] \in \mathcal{M}_{an}^s$ such that $(A, \Phi) \in \mathcal{B}^s$ satisfies Hitchin's equation. By Corollary 3.2.13 and Proposition 3.5.3, we see that $\varphi\colon \mathcal{Z} \to \mathcal{M}_{an}^s$ is a biholomorphism onto an open neighborhood of $[A, \Phi]$ in $\mathcal{M}_{an}^s$, where $\mathcal{Z}$ is an open neighborhood of 0 in $\mathbf{H}^1$ and φ the map induced by the Kuranishi map $\theta\colon \mathcal{Z} \to \mathcal{B}^s$ (see Section 3.2). Therefore, to show that $i|_{\mathcal{M}^s_{an}}$ is holomorphic, it is enough to show that $i\varphi\colon \mathcal{Z} \to \mathcal{M}^s_{alg}$ is holomorphic. By the remark after the proof of [56, Corollary 5.6], we see that the analytification of $\mathcal{M}_{alg}$ is the coarse moduli space of semistable Higgs bundles in the category of complex spaces. Therefore, to show that $i\varphi$ is holomorphic, we need to construct a family $(\mathscr{V}, \mathbf{\Phi})$, called the *Kuranishi family* associated with θ, of stable Higgs bundles over $\mathcal{Z}$ such that $(\mathscr{V}_t, \mathbf{\Phi}_t)$ is isomorphic to $(\mathcal{E}_{A_t}, \Phi_t)$ for every $t \in \mathcal{Z}$, where $(A_t, \Phi_t) = \theta(t)$. In general, a family $(\mathscr{V}, \mathbf{\Phi})$ of Higgs bundles over a complex space T is a holomorphic vector bundle $\mathscr{V} \to M \times T$ together with a holomorphic section $\mathbf{\Phi} \in H^0(M \times T, p_M^*\mathcal{K}_M \otimes \operatorname{End} \mathscr{V})$, where $p_M\colon M \times T \to M$ is the projection onto the first factor.

Proposition 3.5.4. *For any $(A, \Phi) \in \mathcal{B}^s$, let $\theta\colon \mathcal{Z} \to \mathcal{B}^s$ be the Kuranishi map defined by (A, Φ). Then, there exists a Kuranishi family $(\mathscr{V}, \mathbf{\Phi})$ of stable Higgs bundles over $\mathcal{Z}$ such that $(\mathscr{V}_t, \mathbf{\Phi}_t)$ is isomorphic to $(\mathcal{E}_{A_t}, \Phi_t)$ for every $t \in \mathcal{Z}$, where $(A_t, \Phi_t) = \theta(t)$.*

Proof. We adapt the proof of [19, Proposition 2.6]. Let $V = p_M^* E$ be the smooth vector bundle over $M \times \mathcal{Z}$, and $\mathbf{\Phi}(x, t) := \Phi_t(x)$ can be regarded as a smooth section of $p_M^*\Lambda^{1,0}M \otimes \operatorname{End}(U) \subset \Omega^{1,0}(M \times \mathcal{Z}, \operatorname{End} U)$. Then, we need to put a holomorphic structure on V so that $\mathbf{\Phi}$ is a holomorphic section.

Let $\{s_i\}$ be a smooth local frame for E. Then $\{p_M^* s_i\}$ is a smooth local frame for V. Then, we define a $\overline{\partial}$-operator $\overline{\partial}_V\colon \Omega^0(V) \to \Omega^{0,1}(V)$ by the requirement that

$$\overline{\partial}_V(p_M^* s_i) = \overline{\partial}_{A_t} s_i. \tag{3.94}$$

Here, $\overline{\partial}_{A_t} s_i$ is regarded as a local section of $\Lambda^{0,1}(M \times \mathcal{Z}) \otimes V$. It is easy to show that $\overline{\partial}_V$ is independent of the choices of smooth local frames $\{s_i\}$. Therefore, $\overline{\partial}_V$ is a well-defined $\overline{\partial}$-operator on V.

Then, we show that $\overline{\partial}_V$ is integrable so that $\mathscr{V} = (V, \overline{\partial}_V)$ is a holomorphic vector bundle over $M \times \mathcal{Z}$. Write $\overline{\partial}_{A_t} s_i = f_i^j s_j$ for some smooth local function f_i^j on $M \times \mathcal{Z}$. Since θ is holomorphic, each f_i^j is holomorphic in the direction of $\mathcal{Z}$. As a consequence,

$$\overline{\partial}_V^2(p_M^* s_i) = \overline{\partial}_{M\times\mathcal{Z}} f_i^j \wedge s_j + f_i^j \overline{\partial}_{A_t} s_j = \overline{\partial}_M f_i^j \wedge s_j + f_i^j \overline{\partial}_{A_t} s_j, \tag{3.95}$$

where $\overline{\partial}_{M\times\mathcal{Z}}$ and $\overline{\partial}_M$ are usual $\overline{\partial}$-operators on the complex manifolds $M \times \mathcal{Z}$ and M, respectively. On the other hand,

$$0 = \overline{\partial}_{A_t}^2 s_i = \overline{\partial}_M f_i^j \wedge s_j + f_i^j \overline{\partial}_{A_t} s_j. \tag{3.96}$$

Then, we show that $\overline{\partial}_V \mathbf{\Phi} = 0$. Write $\Phi_s = \phi^i s_i$ for some smooth local function ϕ^i on $M \times \mathcal{Z}$. Since θ is holomorphic, ϕ^i is holomorphic in the direction of $\mathcal{Z}$. As a consequence,

$$\overline{\partial}_V \mathbf{\Phi} = \overline{\partial}_{M\times\mathcal{Z}} \phi^i \wedge s_i + \phi^i \overline{\partial}_{A_t} s_i = \overline{\partial}_M \phi^i \wedge s_i + \phi^i \overline{\partial}_{A_t} s_i = \overline{\partial}_{A_t} \Phi_t = 0. \tag{3.97}$$

Finally, we need to show that if $(\mathscr{V}_t, \mathbf{\Phi}_t)$ is isomorphic to $(\mathcal{E}_{A_t}, \Phi_t)$ for any $t \in \mathcal{Z}$. If $i_t(x) = (x,t)$ is the holomorphic map $M \to M \times \mathcal{Z}$, then the holomorphic structure on $i_t^*\mathscr{V}$ is given by the pullback $\overline{\partial}$-operator $i_t^*\overline{\partial}_V$. Since

$$[i_t^*(\overline{\partial}_V)](i_t^* p_M^* s) = i_t^*(\overline{\partial}_V s) = \overline{\partial}_{A_t} s \tag{3.98}$$

for any smooth local section s of E, we see that $i_t^*\mathscr{V}$ is isomorphic to $\mathcal{E}_{A_t}$. Moreover, $i_t^*\mathbf{\Phi} = \Phi_t = \Phi$. □

Corollary 3.5.5. *The comparison map $i\colon \mathcal{M}_{an}^s \to \mathcal{M}_{alg}^s$ is a biholomorphism.*

Proof. Since the analytification of $\mathcal{M}_{alg}$ is the coarse moduli space of semistable Higgs bundles in the category of complex spaces, the family $(\mathscr{V}, \mathbf{\Phi})$ constructed in Proposition 3.5.4 induces a holomorphic map

$$\mathcal{Z} \to \mathcal{M}_{alg}^s, \qquad t \mapsto [\mathscr{V}_t, \mathbf{\Phi}_t]. \tag{3.99}$$

On the other hand, the map $i\varphi\colon \mathcal{Z} \to \mathcal{M}_{alg}^s$ is given by

$$i\varphi(t) = i[A_t, \Phi_t] = [\mathcal{E}_{A_t}, \Phi_t] = [\mathscr{V}_t, \mathbf{\Phi}_t]. \tag{3.100}$$

Hence, $i\varphi$ is holomorphic. Since both $\mathcal{M}_{an}^s$ and $\mathcal{M}_{alg}^s$ are smooth complex manifolds, and i is a holomorphic bijection, i is a biholomorphism. □

Then, we extend the holomorphicity of i^{-1} on $\mathcal{M}_{alg}^s$ to the full moduli space $\mathcal{M}_{alg}$.

Corollary 3.5.6. *The map $i^{-1}\colon \mathcal{M}_{alg} \to \mathcal{M}_{an}$ is holomorphic.*

Proof. Recall that $\mathcal{M}_{an}$ is assumed to be reduced, and $\mathcal{M}_{alg}$ is reduced. Take a holomorphic $f\colon U \to \mathbb{C}$ where U is an open subset of $\mathcal{M}_{an}$. Then, the pullback $(i^{-1})^*f$ is continuous on the open set $i(U)$ and holomorphic on $i(U) \cap \mathcal{M}^s_{alg}$. By [41], the normality of $\mathcal{M}_{alg}$ implies the normality of its analytification. Since $\mathcal{M}^s_{alg}$ is open in the Zariski topology, $\mathcal{M}_{alg} \setminus \mathcal{M}^s_{alg}$ is a closed analytic subset of $\mathcal{M}_{alg}$ in the analytic topology. Since $(i^{-1})^*f$ is already continuous on $i(U)$, the Riemann extension theorem for normal complex spaces implies that the restriction $(i^{-1})^*f\colon \mathcal{M}^s_{alg} \cap i(U) \to \mathbb{C}$ can be extended to a holomorphic function g on $i(U)$. Since $\mathcal{M}_{alg}$ is irreducible, the open set $\mathcal{M}^s_{alg}$ is dense in the Zariski topology and hence in the analytic topology ([46, §10, Theorem 1]). Since both $(i^{-1})^*f$ and g are continuous and agree on an open dense subset $\mathcal{M}^s_{alg} \cap i(U)$ of $i(U)$, $(i^{-1})^*f = g$. This shows that i^{-1} is holomorphic. □

The final ingredient is the normality of $\mathcal{M}_{an}$.

Lemma 3.5.7. *$\mathcal{M}_{an}$ is a normal complex space.*

Proof. Let us temporarily use Q to mean $\nu^{-1}_{0,\mathbb{C}}(0)$ viewed as an affine variety in $\mathbf{H}^1$ and Q^{an} to mean the analytification of Q. By Theorem 3.4.3, it suffices to prove that $Q^{an} /\!\!/ H^{\mathbb{C}}$ is normal at the origin $[0]$. Here, $Q^{an} /\!\!/ H^{\mathbb{C}}$ is the analytic GIT quotient of Q^{an} by $H^{\mathbb{C}}$. By [30], the analytification of the affine GIT quotient $Q /\!\!/ H^{\mathbb{C}}$ is $Q^{an} /\!\!/ H^{\mathbb{C}}$.

Now, we fix a Higgs bundle (A, Φ) such that $\mu(A, \Phi) = 0$. By choosing a point $x \in M$, the holomorphic bundle $(\mathcal{E}_A, \Phi, x)$ defines a point in the moduli space

$\mathbf{R}_{Dol}(M,x,n)$ of the semistable Higgs bundles of rank n and degree 0 and with a frame at x. In [57, Corollary 11.7], it is shown that $\mathbf{R}_{Dol}(M,x,n)$ is normal. Moreover, in the proof of [57, Proposition 10.5], it is shown that the formal completion of Q (regarded as an affine variety in $\mathbf{H}^1$) at 0 is isomorphic to the formal completion of a subscheme Y at $(\mathcal{E}_A, \Phi, x)$. Here, Y is a local slice, provided by Luna's slice theorem (see [36, Theorem 4.2.12]) at $(\mathcal{E}_A, \Phi, x)$ for the $GL_n(\mathbb{C})$ action on $\mathbf{R}_{Dol}(M,x,n)$. Moreover, since $\mathbf{R}_{Dol}(M,x,n)$ is normal at $(\mathcal{E}_A, \Phi, x)$, Y can be taken to be normal at $(\mathcal{E}_A, \Phi, x)$. As a consequence, the formal completion of Q is normal at 0. By [63, Tag 0FIZ], Q is normal at 0. Since taking invariants commutes with localizations and preserves the normality, we conclude that $Q /\!\!/ H^{\mathbb{C}}$ is normal at $[0]$. Since normality is preserved by the analytification (see [41]), we see that $Q^{an} /\!\!/ H^{\mathbb{C}}$ is normal at $[0]$. □

The proof of Theorem C rests on the following theorem.

Theorem 3.5.8 ([23, Theorem, p.166])**.** *Let $f\colon X \to Y$ be an injective holomorphic map between reduced and pure dimensional complex spaces. Assume that Y is normal and that $\dim X = \dim Y$. Then f is open, and f maps X biholomorphically onto $f(X)$. In particular, the space X is normal.*

Proof of Theorem C. Now the map

$$i^{-1}\colon \mathcal{M}_{alg} \to \mathcal{M}_{an} \tag{3.101}$$

is a holomorphic homeomorphism. To use Theorem 3.5.8, we verify that $\mathcal{M}_{an}$ is pure dimensional, normal and $\dim \mathcal{M}_{an} = \dim \mathcal{M}_{alg}$. By Lemma 3.5.7, $\mathcal{M}_{an}$ is normal. Since $\mathcal{M}_{alg}$ is connected in the analytic topology, $\mathcal{M}_{an}$ is connected. Then, the normality and connectedness of $\mathcal{M}_{an}$ implies that $\mathcal{M}_{an}$ is irreducible and hence pure dimensional (see [23, Theorem, p.168]). Finally, by Corollary 3.5.5, $\dim \mathcal{M}_{an} = \dim \mathcal{M}_{alg}$. $\square$

Chapter 4: The moduli space as a stratified complex symplectic space

This chapter is based on the author's paper [18].

4.1 Mostow's decomposition

In this section, we will prove Mostow's decomposition for $\mathcal{G}^{\mathbb{C}}$, Theorem M. In fact, we will prove the following Theorem 4.1.1, and Theorem M follows as a corollary.

Let H be a compact subgroup of $\mathcal{G}$ and $\mathfrak{h}$ its Lie algebra. The compactness of H implies that $\mathfrak{h}$ is a finite-dimensional subspace of $\Omega^0(\mathfrak{g}_E)$ and hence closed. Therefore, $\mathfrak{h}$ has a L^2-orthogonal complement $\mathfrak{h}^\perp$ in $\Omega^0(\mathfrak{g}_E)$ so that $\Omega^0(\mathfrak{g}_E) = \mathfrak{h} \oplus \mathfrak{h}^\perp$. Moreover, let $H^{\mathbb{C}}$ be the complexification of H.

Theorem 4.1.1 (cf. [32, Corollary 9.5])**.** *The map*

$$H^{\mathbb{C}} \times_H (\mathfrak{h}^\perp \times \mathcal{G}) \to \mathcal{G}^{\mathbb{C}}, \qquad [h, s, u] \mapsto h \exp(is)u, \tag{4.1}$$

is a bijection, where H acts on $H^{\mathbb{C}} \times (\mathfrak{h}^\perp \times \mathcal{G})$ by

$$h_0 \cdot (h, s, u) = (hh_0^{-1}, h_0 s h_0^{-1}, h_0 u). \tag{4.2}$$

To prove Theorem 4.1.1, we adapt the proof of [32, Corollary 9.5]. Recall that the polar decomposition $\mathfrak{u}(n) \times U(n) \to GL_n(\mathbb{C})$ induces a polar decomposition

$$\Omega^0(\mathfrak{g}_E) \times \mathcal{G} \to \mathcal{G}^{\mathbb{C}}, \qquad (s, u) \mapsto \exp(is)u. \tag{4.3}$$

Via the polar decomposition, the left multiplication of $H^{\mathbb{C}}$ on $\mathcal{G}^{\mathbb{C}}$ induces a left $H^{\mathbb{C}}$-action on $\Omega^0(\mathfrak{g}_E) \times \mathcal{G}$. In particular, H acts on $\Omega^0(\mathfrak{g}_E) \times \mathcal{G}$ by $h_0 \cdot (s, u) = (h_0 s h_0^{-1}, h_0 u)$. In §4.6, we will show that both $\Omega^0(\mathfrak{g}_E) \times \mathcal{G}$ and $\mathcal{G}^{\mathbb{C}}$ are weak Kähler manifolds such that the polar decomposition is an isomorphism of Kähler manifolds. Moreover, the left H-action is Hamiltonian with a moment map given by

$$\kappa\colon \Omega^0(\mathfrak{g}_E) \times \mathcal{G} \to \mathfrak{h}, \qquad (s, u) \mapsto Ps, \tag{4.4}$$

where $P\colon \Omega^0(\mathfrak{g}_E) \to \mathfrak{h}$ is the projection. Now, we routinely identify $\Omega^0(\mathfrak{g}_E) \times \mathcal{G}$ with $\mathcal{G}^{\mathbb{C}}$ using the polar decomposition. Then, Theorem 4.1.1 follows from the following.

Lemma 4.1.2.

1. *Every $H^{\mathbb{C}}$-orbit in $\mathcal{G}^{\mathbb{C}}$ intersect $\kappa^{-1}(0)$.*

2. *$\kappa^{-1}(0) \cap H^{\mathbb{C}} g = Hg$ for every $g \in \mathcal{G}^{\mathbb{C}}$.*

Proof. Since H is a compact Lie group (hence finite-dimensional), [47, Lemma 5.2 and Theorem 5.4] apply. Therefore, it suffices to show that

$$\lim_{t\to\infty} (\kappa(\exp(its)g), s)_{L^2} > 0 \tag{4.5}$$

for any $s \in \mathfrak{h}$ and $g \in \mathcal{G}^{\mathbb{C}}$. Using the polar decomposition, we may write

$$\exp(its)g = \exp(it\eta(t))u(t) \tag{4.6}$$

for some $\eta(t) \in \Omega^0(\mathfrak{g}_E)$ and $u(t) \in \mathcal{G}$. Hence,

$$(\kappa(\exp(its)g), s)_{L^2} = (P\eta(t), s)_{L^2} = (\eta(t), s)_{L^2}. \tag{4.7}$$

Since $H^{\mathbb{C}}$ acts on $\mathcal{G}^{\mathbb{C}}$ freely, by [47, Lemma 2.2], $(\eta(t), s)_{L^2}$ is a strictly increasing function of t. Therefore, it suffices to prove that if $t \gg 0$, $(\eta(t), s)_{L^2} \geq 0$. Hence, we may assume that $\eta(t) \neq 0$ for any t. By the proof of [64, Theorem 5.12], we see that

$$\lim_{t\to\infty} \frac{\eta(t)}{\|\eta(t)\|_{L^2}} = \frac{s}{\|s\|_{L^2}} \tag{4.8}$$

in L^2-norm so that

$$\lim_{t\to\infty} \left(\frac{\eta(t)}{\|\eta(t)\|_{L^2}}, \frac{s}{\|s\|_{L^2}} \right)_{L^2} = 1. \tag{4.9}$$

Therefore, if $t \gg 0$, $(\eta(t), s)_{L^2} > 0$. □

Proof of Theorem 4.1.1. Consider the map

$$H^{\mathbb{C}} \times_H \kappa^{-1}(0) \to \mathcal{G}^{\mathbb{C}}, \qquad [h, s, u] \mapsto h \exp(is)u. \tag{4.10}$$

The surjectivity and the injectivity follow from (1) and (2) in Lemma 4.1.2, respectively. Moreover, $\kappa^{-1}(0) = \mathfrak{h}^{\perp} \times \mathcal{G}$. □

As a corollary of Mostow's decomposition, Theorem M, we obtain the following that will be used often in this paper.

Corollary 4.1.3. *Let H and K be compact subgroups of $\mathcal{G}$. Then, $H^{\mathbb{C}}$ and $K^{\mathbb{C}}$ are conjugate in $\mathcal{G}^{\mathbb{C}}$ if and only if H and K are conjugate in $\mathcal{G}$.*

Proof. This follows from Mostow's decomposition (Theorem M) and the first paragraph in the proof of [58, Theorem 2.10]. Note that all we need is the fact that the map in Theorem M is a $\mathcal{G}$-equivariant bijection. $\square$

4.2 The orbit type decompositions

4.2.1 Orbit types in the hyperKähler quotient

In this section, we will prove Theorem D.

Proof of Theorem D. Fix $[A, \Phi] \in Q$ such that (A, Φ) is of class C^∞. Hence, gauge transformations in its $\mathcal{G}$-stabilizer H are of class C^∞. By definition, Q is a component of $\mathbf{m}^{-1}(0)_{(H)}/\mathcal{G}$. Since the $\mathcal{G}$-action is proper, a standard argument (e.g. [20, Proposition 4.4.5]) shows that there is an H-invariant open neighborhood S of (A, Φ) in $(A, \Phi) + \ker d_1^*$ such that the natural map $f\colon S \times_H \mathcal{G} \to \mathcal{C}$ is a $\mathcal{G}$-equivariant diffeomorphism onto an open neighborhood of (A, Φ), where d_1 is defined in the complex C_{Hit} (see §3.1). Therefore, the restriction

$$f\colon ((\mathbf{m}^{-1}(0) \cap S) \times_H \mathcal{G})_{(H)} \to \mathbf{m}^{-1}(0)_{(H)} \tag{4.11}$$

is a $\mathcal{G}$-equivariant homeomorphism onto an open neighborhood of (A, Φ) in $\mathbf{m}^{-1}(0)_{(H)}$.

Since Q is open in $\mathbf{m}^{-1}(0)_{(H)}/\mathcal{G}$, $\pi^{-1}(Q)$ is open in $\mathbf{m}^{-1}(0)_{(H)}$ and contains (A, Φ). By shrinking S, we may further assume that f takes values in $\pi^{-1}(Q)$. We claim that

$$((\mathbf{m}^{-1}(0) \cap S) \times_H \mathcal{G})_{(H)} = (\mathbf{m}^{-1}(0) \cap S^H) \times_H \mathcal{G} = (\mathbf{m}^{-1}(0) \cap S^H) \times (\mathcal{G}/H), \quad (4.12)$$

where S^H consists of elements in S that are fixed by H. The second equality is obvious. To show the first one, let $[B, \Psi, g]$ be a point in $(\mathbf{m}^{-1}(0) \cap S) \times_H \mathcal{G}$ with $\mathcal{G}$-stabilizer conjugate to H in $\mathcal{G}$. As a consequence,

$$\mathcal{G}_{[B,\Psi,1]} = g\mathcal{G}_{[B,\Psi,g]}g^{-1} \in (H). \quad (4.13)$$

Since S is a local slice for the $\mathcal{G}$-action on $\mathcal{C}$, $\mathcal{G}_{[B,\Psi,1]} \subset H$. Since $\mathcal{G}_{[B,\Psi,1]}$ and H have the same dimension and the same number of components, we see that $\mathcal{G}_{[B,\Psi,1]} = H$. Hence, H fixes (B, Ψ), and the claim follows. Therefore, the map $\pi^{-1}(Q) \to Q$ can be locally identified with the projection

$$(\mathbf{m}^{-1}(0) \cap S^H) \times (\mathcal{G}/H) \to \mathbf{m}^{-1}(0) \cap S^H. \quad (4.14)$$

Moreover, since H is compact, the quotient map $\mathbf{m}^{-1}(0) \cap S \to (\mathbf{m}^{-1}(0) \cap S)/H$ is closed. Since S^H is closed in S, we conclude that $(\mathbf{m}^{-1}(0) \cap S^H)/H$ is closed in $(\mathbf{m}^{-1}(0) \cap S)/H$. Since $\mathbf{m}^{-1}(0) \cap S^H = (\mathbf{m}^{-1}(0) \cap S^H)/H$ is homeomorphic to an open neighborhood of $[A, \Phi]$ in Q, Q is a locally closed subset of $\mathbf{m}^{-1}(0)/\mathcal{G}$. Then, we prove that $\mathbf{m}^{-1}(0) \cap S^H$ is a submanifold of S^H. As a consequence, $\pi^{-1}(Q)$

is a submanifold of $\mathcal{C}$, Q is a smooth manifold, and $\pi\colon \pi^{-1}(Q) \to Q$ is a smooth submersion.

To show that $\mathbf{m}^{-1}(0) \cap S^H$ is a submanifold of S^H, we adapt the proof of [61, Theorem 2.24]. Let μ_i be a component of the hyperKähler moment map $\mathbf{m}$. We first show that the restriction $\mu_i|_{S^H}$ has a constant finite corank so that $S^H \cap \mu_i^{-1}(0)$ is a submanifold of S^H. After that, we show that $S^H \cap \mathbf{m}^{-1}(0) = \cap_{i=1}^3 S^H \cap \mu_i^{-1}(0)$ is a submanifold of S^H. Fix $(B, \Psi) \in S^H$. Note that $T_{(B,\Psi)} S^H = (\ker d_1^*)^H$. Consider the sequence

$$\Omega^0(\mathfrak{g}_E) \xrightarrow{d_1} T_{(B,\Psi)}\mathcal{C} \xrightarrow{d\mu_i} \Omega^2(\mathfrak{g}_E), \tag{4.15}$$

where $d\mu_i$ is the derivative of μ_i at (B, Ψ). Note that H acts on each term by conjugation, and both d_1 and $d\mu_i$ are H-equivariant. Since the complex C_{Hit} is elliptic, the symbol of $d\mu_i$ is surjective so that the Hodge decomposition

$$\Omega^2(\mathfrak{g}_E) = \ker(d\mu_i)^* \oplus \operatorname{im} d\mu_i \tag{4.16}$$

holds, where $(d\mu_i)^*$ is the L^2-formal adjoint of $d\mu_i$. We claim that

$$d\mu_i\Big((T_{(B,\Psi)}\mathcal{C})^H\Big) = (\operatorname{im} d\mu_i)^H. \tag{4.17}$$

Since $d\mu_i$ is H-equivariant, the inclusion "$\subset$" is obvious. Conversely, suppose $y = d\mu_i(x)$ is fixed by H for some $x \in T_{(B,\Psi)}\mathcal{C}$. Since H is compact, $\int_H (x \cdot h) dh$ is well-defined and fixed by H. Therefore, the inclusion "$\supset$" follows from

$$y = \int_H (y \cdot h) dh = \int_H d\mu_i(x \cdot h) dh = d\mu_i\Big(\int_H (x \cdot h) dh\Big). \tag{4.18}$$

Then, the H-equivariance of $d\mu_i$ implies that

$$\Omega^2(\mathfrak{g}_E)^H = (\ker(d\mu_i)^*)^H \oplus (\operatorname{im} d\mu_i)^H = (\ker(d\mu_i)^*)^H \oplus d\mu_i\Big((T_{(B,\Psi)}\mathcal{C})^H\Big). \tag{4.19}$$

Moreover, the formula $(d\mu_i)^* = -I_i d_1 *$ implies that

$$\dim \ker(d\mu_i)^* = \dim \ker d_1 = \dim \mathcal{G}_{(B,\Psi)}. \tag{4.20}$$

Since S is a local slice for the $\mathcal{G}$-action on $\mathcal{C}$, $\mathcal{G}_{(B,\Psi)} = H$. Finally, since

$$(T_{(B,\Psi)}\mathcal{C})^H = (\operatorname{im} d_1)^H \oplus (\ker d_1^*)^H = (\operatorname{im} d_1)^H \oplus (T_{(B,\Psi)}S^H) \tag{4.21}$$

and $d\mu_i d_1 = 0$, we conclude that $\mu_i|_{S^H} \colon S^H \to \Omega^2(\mathfrak{g}_E)^H$ has a constant finite corank so that $\mu_i^{-1}(0) \cap S^H$ is a smooth submanifold of S^H.

Now, we show that $\cap_{i=1}^3 S^H \cap \mu_i^{-1}(0)$ is a submanifold of S^H. Let $\Delta \colon S^H \to (S^H)^3$ be the diagonal map. If we can show that Δ is transversal to

$$W = (\mu_1^{-1}(0) \cap S^H) \times (\mu_2^{-1}(0) \cap S^H) \times (\mu_3^{-1}(0) \cap S^H) \subset (S^H)^3, \tag{4.22}$$

then $\Delta^{-1}(W) = \cap_{i=1}^3 S^H \cap \mu^{-1}(0)$ is a smooth submanifold of S^H. So, we fix $(B, \Psi) \in$

$\Delta^{-1}(W)$. Then, we have

$$T_{\Delta(B,\Psi)}W = \bigoplus_{i=1}^{3}\Big(\ker d\mu_i \cap (\ker d_1^*)^H\Big) \tag{4.23}$$

and

$$\Delta_* T_{(B,\Psi)}S^H = \{(v,v,v)\colon v \in (\ker d_1^*)^H\}. \tag{4.24}$$

Note that $\ker d\mu_i = (I_i \operatorname{im} d_1)^\perp$. Therefore, if $u_i \in (\ker d_1^*)^H$ $(i = 1,2,3)$, then we may write $u_i = u_i' + u_i''$ for $u_i' \in I_i \operatorname{im} d_1$ and $u_i'' \in (I_i \operatorname{im} d_1)^\perp$. Since $\mu_i(B,\Psi) = 0$, it is not hard to check that $u_i'' \in (\ker d_1^*)^H$. Moreover, since $I_i \operatorname{im} d_1$ are orthogonal to each other, we may further write

$$u_i = (u_i' - \sum_{j\neq i} u_j'') + u_1'' + u_2'' + u_3''. \tag{4.25}$$

Therefore, $T_{\Delta(B,\Psi)}W$ and $\Delta_* T_{(B,\Psi)}S^H$ generate $(T_{(B,\Psi)}S^H)^3$, and we are done.

By construction, we see that $\mathbf{m}^{-1}(0) \cap S^H$ is a smooth manifold with tangent space

$$\ker d\mathbf{m} \cap (\ker d_1^*)^H = (\ker d_2 \cap \ker d_1^*)^H = \mathbf{H}^1(C_{Hit})^H = (\mathbf{H}^1)^H \tag{4.26}$$

at $[A,\Phi]$. Since the L^2-metric on $\mathcal{C}$ is preserved by the $\mathcal{G}$-action, it descends to a metric on Q. By the proof of [33, Theorem 6.7], we see that I, J and K restrict to $\mathbf{H}^1$. Since they are preserved by the H-action on $\mathbf{H}^1$, they further restrict to $(\mathbf{H}^1)^H$. Moreover, since they are preserved by the $\mathcal{G}$-action, they, together with the

L^2-metric on $(\mathbf{H}^1)^H$, define an almost hyperKähler structure on Q.

Let Ω_I, Ω_J and Ω_K be the Kähler forms on $\mathcal{C}$ associated with complex structures I, J and K, respectively. By the proof of [33, Theorem 6.7], we see that if $v \in \operatorname{im} d_1 \oplus \mathbf{H}^1$ then $\Omega_i(v, \xi) = 0$ for any $i \in \{I, J, K\}$ and any vector ξ tangent to the $\mathcal{G}$-orbit. This also holds for $v \in \operatorname{im} d_1 \oplus (\mathbf{H}^1)^H = T_{(A,\Phi)}\pi^{-1}(Q)$. Therefore, there are unique Kähler forms ω_i, $i \in \{I, J, K\}$, on Q such that $\pi^*\omega_i = \Omega_i|_{\pi^{-1}(Q)}$ for $i \in \{I, J, K\}$. Therefore, each ω_i is closed. Then, the integrability of complex structures on Q follows from [33, Lemma 6.8].

Finally, by the elliptic regularity, it is easy to see that elements in $\ker d_1^* \cap \mathbf{m}^{-1}(0)$ are of class C^∞. Since $S^H \cap \mathbf{m}^{-1}(0) \subset \ker d_1^* \cap \mathbf{m}^{-1}(0)$, our heuristic use of infinite-dimensional manifolds can be justified by working with Sobolev completions.

$\square$

4.2.2 A local slice theorem

Now we study the strata in the orbit type decomposition of $\mathcal{M}$. Therefore, we need a local slice theorem for the $\mathcal{G}^{\mathbb{C}}$-action on $\mathcal{C}$.

Theorem 4.2.1. *Let (A, Φ) be a Higgs bundle in $\mathbf{m}^{-1}(0)$ with $\mathcal{G}$-stabilizer H. Then, there exists an open neighborhood O of (A, Φ) in $(A, \Phi) + \ker(D'')^*$ such that the natural map $OH^{\mathbb{C}} \times_{H^{\mathbb{C}}} \mathcal{G}^{\mathbb{C}} \to \mathcal{C}$ is a biholomorphism onto an open neighborhood of (A, Φ).*

Proof. Note that the $\mathcal{G}^{\mathbb{C}}$-stabilizer of (A, Φ) is $H^{\mathbb{C}}$ and acts on $(D'')^*$. Consider the natural map

$$f\colon ((A,\Phi)+\ker(D'')^*)\times_{H^{\mathbb{C}}}\mathcal{G}^{\mathbb{C}}\to\mathcal{C}. \tag{4.27}$$

Its derivative at $[A,\Phi,1]$ is given by

$$\ker(D'')^*\oplus\mathbf{H}^0(C_{\mu_{\mathbb{C}}})^{\perp}\to\Omega^{0,1}(\mathfrak{g}_E^{\mathbb{C}})\oplus\Omega^{1,0}(\mathfrak{g}_E^{\mathbb{C}}),\qquad (x,u)\mapsto x+D''u, \tag{4.28}$$

and hence an isomorphism, since $T_{(A,\Phi)}\mathcal{C}=\ker(D'')^*\oplus\operatorname{im}D''$. Therefore, there are open neighborhoods $O\times N$ of $(A,\Phi,1)$ and W of (A,Φ) such that $f\colon\pi(O\times N)\to W$ is a biholomorphism, where π is the quotient map. Then, we consider the restriction

$$f\colon OH^{\mathbb{C}}\times_{H^{\mathbb{C}}}\mathcal{G}^{\mathbb{C}}\to\mathcal{C}. \tag{4.29}$$

Since $OH^{\mathbb{C}}\mathcal{G}^{\mathbb{C}}=W\mathcal{G}^{\mathbb{C}}$, its image is $W\mathcal{G}^{\mathbb{C}}$ which is an open neighborhood of (A,Φ) in $\mathcal{C}$. Since $\mathcal{G}^{\mathbb{C}}=\cup_{g\in\mathcal{G}^{\mathbb{C}}}Ng$ and f is $\mathcal{G}^{\mathbb{C}}$-equivariant, f is a local biholomorphism. Therefore, it remains to show that f is injective provided that O is small enough. We will follow the proof of [8, Proposition 4.5]. Suppose that

$$((A,\Phi)+(\alpha_1,\eta_1))g=(A,\Phi)+(\alpha_2,\eta_2) \tag{4.30}$$

for some $g\in\mathcal{G}^{\mathbb{C}}$ and $(D'')^*(\alpha_i,\eta_i)=0$. Equivalently,

$$D''g+(\alpha_1''g-g\alpha_2'',\eta_1g-g\eta_2)=0, \tag{4.31}$$

where α_i'' is the $(0,1)$-component of α_i. We show that if each $\|(\alpha_i,\eta_i)\|_{L_k^2}$ is small enough, then $g \in H^{\mathbb{C}}$. Write $g = g_0 + g_1$ for some $g_0 \in \mathbf{H}^0(C_{\mu_{\mathbb{C}}})$ and $g_1 \in \mathbf{H}^0(C_{\mu_{\mathbb{C}}})^{\perp}$. The idea is to show that $D''g_1 = 0$ so that $g = g_0 \in H^{\mathbb{C}}$. Applying $(D'')^*$, we obtain

$$(D'')^* D'' g + (D'')^* (\alpha_1'' g - g\alpha_2'', \eta_1 g - g\eta_2) = 0. \tag{4.32}$$

We first claim that $D'g_0 = 0$, where $D' = \partial_A + \Phi^*$. In fact, using the Kähler identity, $D'^* = +i[*, D'']$, we have

$$\|D'g_0\|_{L^2}^2 = (D'^* D' g_0, g_0)_{L^2} = i(D'' D' g_0, g_0)_{L^2} = -i(D' D'' g_0, g_0)_{L^2} = 0. \tag{4.33}$$

Here, we have used the fact that $D''D' + D'D'' = 0$, since $\mu(A, \Phi) = F_A + [\Phi, \Phi^*] = 0$. Then, using the Kähler's identity, $(D'')^* = -i[*, D']$, we see that $D'(\alpha_i, \eta_i) = 0$ for each i, and

$$(D'')^* (\alpha_1'' g - g\alpha_2'', \eta_1 g - g\eta_2) = (D'')^* (\alpha_1'' g_1 - g_1\alpha_2'', \eta_1 g_1 - g_1\eta_2). \tag{4.34}$$

As a consequence,

$$\begin{aligned}
\|D''g_1\|_{L^2}^2 &= -((\alpha_1'' g_1 - g_1\alpha_2'', \eta_1 g_1 - g_1\eta_2), D''g_1)_{L^2} \\
&\leq \|(\alpha_1'' g_1 - g_1\alpha_2'', \eta_1 g_1 - g_1\eta_2)\|_{L^2} \|D''g_1\|_{L^2} \\
&\leq (\|\alpha_1''\|_{C^0} + \|\alpha_2''\|_{C^0} + \|\eta_1\|_{C^0} + \|\eta_2\|_{C^0}) \|g_1\|_{L^2} \|D''g_1\|_{L^2} \\
&\leq C(\|\alpha_1''\|_{L_k^2} + \|\alpha_2''\|_{L_k^2} + \|\eta_1\|_{L_k^2} + \|\eta_2\|_{L_k^2}) \|g_1\|_{L^2} \|D''g_1\|_{L^2},
\end{aligned} \tag{4.35}$$

where we have used the Sobolev embedding $L_k^2 \hookrightarrow C^0$. Moreover, since $g_1 \in$

$\mathbf{H}^0(C_{\mu_\mathbb{C}})^\perp$,

$$\|g_1\|_{L^2} \le \|g_1\|_{L^2_1} = \|(D'')^* G D'' g_1\|_{L^2_1} \le C\|D'' g_1\|_{L^2}. \tag{4.36}$$

Therefore,

$$\|D''g_1\|^2_{L^2} \le C(\|\alpha_1''\|_{L^2_k} + \|\alpha_2''\|_{L^2_k} + \|\eta_1\|_{L^2_k} + \|\eta_2\|_{L^2_k})\|D''g_1\|^2_{L^2}. \tag{4.37}$$

Since the isomorphism $\Omega^1(\mathfrak{g}_E) \to \Omega^{1,0}(\mathfrak{g}_E^\mathbb{C})$ given by $\alpha \mapsto \alpha''$ is a homeomorphism in the L^2_k-topology, we conclude that that if $\|(\alpha_i, \eta_i)\|_{L^2_k}$ is small enough for every i, then $D''g_1 = 0$. □

As a corollary of Proposition 4.2.1, every Kuranishi map $\theta \colon B \to \mathcal{C}$ (see § 3.2.1) preserves stabilizers in the following sense.

Proposition 4.2.2. *If B is sufficiently small, then $(H^\mathbb{C})_x = (\mathcal{G}^\mathbb{C})_{\theta(x)}$ for every $x \in \mathcal{Z}$.*

Proof. By [17, Proposition 3.4], if $x_1 = x_2 g$ for some $x_1, x_2 \in B$ and $g \in H^\mathbb{C}$ then $\theta(x_1) = \theta(x_2)g$. This proves the inclusion "$\subset$". To prove the inclusion "$\supset$", we shrink B so that $\theta(\mathcal{Z}) \subset O$ where O is obtained in Proposition 4.2.1. As a consequence, if $\theta(x)g = \theta(x)$ for some $g \in \mathcal{G}^\mathbb{C}$ then $g \in H^\mathbb{C}$. Since F is $H^\mathbb{C}$-equivariant and $F(\theta(x)) = x$ for every $x \in B$, we conclude that $xg = x$. □

4.2.3 Orbit types in the moduli space

Now we are able to prove Theorem E. Before giving the proof, we first show that there is a one-to-one correspondence between the conjugacy classes appearing in the orbit type decompositions of $\mathcal{M}$ and $\mathbf{m}^{-1}(0)/\mathcal{G}$.

Proposition 4.2.3. *Every conjugacy class (L) appearing in the orbit type decomposition of $\mathcal{M}$ is equal to a conjugacy class $(H^{\mathbb{C}})$ for some $\mathcal{G}$-stabilizer H at some Higgs bundle in $\mathbf{m}^{-1}(0)$.*

Proof. Let (A, Φ) be a polystable Higgs bundle whose $\mathcal{G}^{\mathbb{C}}$-stabilizer is conjugate to L in $\mathcal{G}^{\mathbb{C}}$. By the Hitchin-Kobayashi correspondence, $\mu((A, \Phi)g) = 0$ for some $g \in \mathcal{G}^{\mathbb{C}}$. Therefore,

$$(\mathcal{G}_{(A,\Phi)g})^{\mathbb{C}} = (\mathcal{G}^{\mathbb{C}})_{(A,\Phi)g} = g^{-1}(\mathcal{G}^{\mathbb{C}})_{(A,\Phi)}g. \tag{4.38}$$

Let $H = \mathcal{G}_{(A,\Phi)g}$ and the $\mathcal{G}^{\mathbb{C}}$-stabilizer of (A, Φ) is conjugate to $H^{\mathbb{C}}$ in $\mathcal{G}^{\mathbb{C}}$. □

Proof of Theorem E. Fix $[A, \Phi] \in Q$ such that $(A, \Phi) \in \mathbf{m}^{-1}(0)$ is of class C^∞. Therefore, gauge transformations in the $\mathcal{G}$-stabilizer H at (A, Φ) are of class C^∞. Since $H^{\mathbb{C}}$ is the $\mathcal{G}^{\mathbb{C}}$-stabilizer at (A, Φ), Q is a component of $\mathcal{B}^{ps}_{(H^{\mathbb{C}})}/\mathcal{G}^{\mathbb{C}}$. By Theorem 4.2.1, there is an open neighborhood O of (A, Φ) in $(A, \Phi) + \ker(D'')^*$ such that the natural map $OH^{\mathbb{C}} \times_{H^{\mathbb{C}}} \mathcal{G}^{\mathbb{C}} \to \mathcal{C}$ is a diffeomorphism onto an open neighborhood of (A, Φ). The $\mathcal{G}^{\mathbb{C}}$-equivariance implies that

$$(OH^{\mathbb{C}} \times_{H^{\mathbb{C}}} \mathcal{G}^{\mathbb{C}})_{(H^{\mathbb{C}})} \to \mathcal{C}_{(H^{\mathbb{C}})}. \tag{4.39}$$

Since $H^{\mathbb{C}}$ has finitely many components and is finite-dimensional, following the proof of Theorem D, we see that

$$(OH^{\mathbb{C}} \times_{H^{\mathbb{C}}} \mathcal{G}^{\mathbb{C}})_{(H^{\mathbb{C}})} = O^{H^{\mathbb{C}}} \times (\mathcal{G}^{\mathbb{C}}/H^{\mathbb{C}}). \tag{4.40}$$

As a consequence, the natural map

$$(O^{H^{\mathbb{C}}} \cap \mathcal{B}^{ss}) \times (\mathcal{G}^{\mathbb{C}}/H^{\mathbb{C}}) \to \mathcal{B}^{ss}_{(H^{\mathbb{C}})} \tag{4.41}$$

is a diffeomorphism onto an open image. On the other hand, the Kuranishi map $\theta\colon \mathcal{Z} \to O \cap \mathcal{B}^{ss}$ is a homeomorphism if B is sufficiently small. Since θ preserves stabilizers (Proposition 4.2.2), we see that $\theta\colon \mathcal{Z}^{H^{\mathbb{C}}} \to O^{H^{\mathbb{C}}} \cap \mathcal{B}^{ss}$ is a homeomorphism. Since the $H^{\mathbb{C}}$-action on $\mathbf{H}^1$ is holomorphic with respect to I, $(\mathbf{H}^1)^{H^{\mathbb{C}}}$ is a complex symplectic subspace of $\mathbf{H}^1$ so that $\mathbf{H}^1 = F \oplus (\mathbf{H}^1)^{H^{\mathbb{C}}}$ where F is the $\omega_{\mathbb{C}}$-complement of $(\mathbf{H}^1)^{H^{\mathbb{C}}}$. As a consequence,

$$\nu_{0,\mathbb{C}}^{-1}(0) = (\nu_{0,\mathbb{C}}|_F)^{-1}(0) \times (\mathbf{H}^1)^{H^{\mathbb{C}}} \tag{4.42}$$

so that $\mathcal{Z}^{H^{\mathbb{C}}}$ is an open subset of $\nu_{0,\mathbb{C}}^{-1}(0)^{H^{\mathbb{C}}} = (\mathbf{H}^1)^{H^{\mathbb{C}}}$. Since every point in $(\mathbf{H}^1)^{H^{\mathbb{C}}}$ has closed $H^{\mathbb{C}}$-orbits, $\theta(\mathcal{Z}^{H^{\mathbb{C}}}) \subset O^{H^{\mathbb{C}}} \cap \mathcal{B}^{ps}$. Moreover, since Q is open in $\mathcal{B}^{ps}_{(H^{\mathbb{C}})}/\mathcal{G}^{\mathbb{C}}$, $\pi^{-1}(Q)$ is open in $\mathcal{B}^{ps}_{(H^{\mathbb{C}})}$. Therefore, if O and B are sufficiently small, we obtain a well-defined map

$$f\colon \mathcal{Z}^{H^{\mathbb{C}}} \times (\mathcal{G}^{\mathbb{C}}/H^{\mathbb{C}}) \to \pi^{-1}(Q), \qquad (x,[g]) \mapsto \theta(x)g, \tag{4.43}$$

which is a homeomorphism onto its open image. This already shows that $\pi^{-1}(Q)$ is a complex submanifold of $\mathcal{C}$. Moreover, f induces a well-defined map $\mathcal{Z}^{H^{\mathbb{C}}} \to Q$ given by $[x] \mapsto [\theta(x)]$. It is exactly the restriction of the local chart (see §3.2.3)

$\varphi\colon \mathcal{Z}H^{\mathbb{C}} /\!\!/ H^{\mathbb{C}} \to \mathcal{M}$, since

$$\nu_{0,\mathbb{C}}^{-1}(0) /\!\!/ H^{\mathbb{C}} = (\nu_{0,\mathbb{C}}|_F)^{-1}(0) /\!\!/ H^{\mathbb{C}} \times (\mathbf{H}^1)^{H^{\mathbb{C}}}, \tag{4.44}$$

and $\theta(x)$ is polystable for every $x \in \mathcal{Z}^{H^{\mathbb{C}}}$. Therefore, we see that Q is a complex submanifold of $\mathcal{M}$. Moreover, the quotient map $\pi\colon \pi^{-1}(Q) \to Q$ can be locally identified with the projection

$$\mathcal{Z}^{H^{\mathbb{C}}} \times (\mathcal{G}^{\mathbb{C}}/H^{\mathbb{C}}) \to \mathcal{Z}^{H^{\mathbb{C}}}, \tag{4.45}$$

which is clearly a holomorphic submersion. Finally, by elliptic regularity, elements in $\mathbf{H}^1$ are of class C^∞. Since $\mathcal{Z}^{H^{\mathbb{C}}} \subset \mathbf{H}^1$, our heuristic use of infinite-dimensional manifolds can be justified by working with Sobolev completions. □

4.3 Complex Whitney stratification

To show that the orbit type decomposition of $\mathcal{M}$ is a complex Whitney stratification, we follow Mayrand's arguments in [44, §4.6 and §4.7]. The idea is that the problem can be reduced to a local model $\nu_{0,\mathbb{C}}^{-1}(0) /\!\!/ H^{\mathbb{C}}$ near $[0]$, once we show that the Kuranishi map $\varphi\colon \widetilde{U} \to U$ preserves the orbit type decompositions, where $\widetilde{U} = \mathcal{Z}H^{\mathbb{C}} /\!\!/ H^{\mathbb{C}}$ and $U = \varphi(\widetilde{U})$ (see §3.2.3). To clarify different possible partitions on $\nu_{0,\mathbb{C}}^{-1}(0) /\!\!/ H^{\mathbb{C}}$, we adopt the following notation. By [30], the natural map

$\nu_{0,\mathbb{C}}^{-1}(0)^{ps} \hookrightarrow \nu_{0,\mathbb{C}}^{-1}(0)$ induces a bijection

$$\nu_{0,\mathbb{C}}^{-1}(0)^{ps}/H^C \xrightarrow{\sim} \nu_{0,\mathbb{C}}^{-1}(0) /\!\!/ H^{\mathbb{C}}, \tag{4.46}$$

where $\nu_{0,\mathbb{C}}^{-1}(0)^{ps}$ is the subspace of $\nu_{0,\mathbb{C}}^{-1}(0)$ consisting of polystable points, or equivalently points whose $H^{\mathbb{C}}$-stabilizers are closed in $\mathbf{H}^1$. Let L be a $H^{\mathbb{C}}$-stabilizer at some point in $\nu_{0,\mathbb{C}}^{-1}(0)$ and $(L)_{H^{\mathbb{C}}}$ the conjugacy class of L in $H^{\mathbb{C}}$. Then, we may define

$$\nu_{0,\mathbb{C}}^{-1}(0)^{ps}_{(L)_{H^{\mathbb{C}}}} = \{x \in \nu_{0,\mathbb{C}}^{-1}(0)^{ps} \colon (H^{\mathbb{C}})_x \in (L)_{H^{\mathbb{C}}}\}. \tag{4.47}$$

As a consequence, $\nu_{0,\mathbb{C}}^{-1}(0) /\!\!/ H^{\mathbb{C}}$ has a partition

$$\widetilde{\mathcal{P}}_{H^{\mathbb{C}}} = \left\{\nu_{0,\mathbb{C}}^{-1}(0)^{ps}_{(L)_{H^{\mathbb{C}}}}/H^{\mathbb{C}} \colon L = (H^{\mathbb{C}})_x \text{ for some } x \in \nu_{0,\mathbb{C}}^{-1}(0)\right\}. \tag{4.48}$$

Here, we have identified $\nu_{0,\mathbb{C}}^{-1}(0)^{ps}_{(L)_{H^{\mathbb{C}}}}$ with its image in $\nu_{0,\mathbb{C}}^{-1}(0) /\!\!/ H^{\mathbb{C}}$. The orbit type decomposition of $\nu_{0,\mathbb{C}}^{-1}(0) /\!\!/ H^{\mathbb{C}}$ is defined as the refinement $\widetilde{\mathcal{P}}^{\circ}_{H^{\mathbb{C}}}$ of $\widetilde{\mathcal{P}}_{H^{\mathbb{C}}}$ into connected components. If $(L)_{\mathcal{G}^{\mathbb{C}}}$ is the conjugacy class of $L \subset H^{\mathbb{C}}$ in $\mathcal{G}^{\mathbb{C}}$, then we may similarly define the partition

$$\widetilde{\mathcal{P}}_{\mathcal{G}^{\mathbb{C}}} = \left\{\nu_{0,\mathbb{C}}^{-1}(0)^{ps}_{(L)_{\mathcal{G}^{\mathbb{C}}}}/H^{\mathbb{C}} \colon L = (H^{\mathbb{C}})_x \text{ for some } x \in \nu_{0,\mathbb{C}}^{-1}(0)\right\}. \tag{4.49}$$

Finally, note that $\nu_{0,\mathbb{C}}^{-1}(0) /\!\!/ H^{\mathbb{C}}$ can be realized as a hyperKähler quotient as follows. Since H acts linearly on $\mathbf{H}^1$ and preserves the Kähler form ω_I, there is a moment map ν_0 associated with the Kähler form ω_I such that $\nu_0(0) = 0$. Then, $\mathbf{n}_0 = (\nu_0, \nu_{0,\mathbb{C}})$ is a hyperKähler moment map for the H-action. By [30], the inclusion $\mathbf{n}_0^{-1}(0) \hookrightarrow \nu_{0,\mathbb{C}}^{-1}(0)$ induces a homeomorphism

$$\mathbf{n}_0^{-1}(0)/H \xrightarrow{\sim} \nu_{0,\mathbb{C}}^{-1}(0) /\!\!/ H^{\mathbb{C}}. \tag{4.50}$$

Then, the same proof of Proposition 4.2.3 shows that every conjugacy class (L) appearing in $\widetilde{\mathcal{P}}_{H^{\mathbb{C}}}$ or $\widetilde{\mathcal{P}}_{\mathcal{G}^{\mathbb{C}}}$ is equal to a conjugacy class $(H_1^{\mathbb{C}})$ for some H-stabilizer H_1 at some point in $\mathbf{n}_0^{-1}(0)$. Then, the relation between $\widetilde{\mathcal{P}}_{H^{\mathbb{C}}}$ and $\widetilde{\mathcal{P}}_{\mathcal{G}^{\mathbb{C}}}$ is stated below.

Lemma 4.3.1 ([44, Lemma 4.2])**.** $\widetilde{\mathcal{P}}^{\circ}_{H^{\mathbb{C}}} = \widetilde{\mathcal{P}}^{\circ}_{\mathcal{G}^{\mathbb{C}}}$.

Proof. This follows from the same proof of [44, Lemma 4.2]. The only difference is that Mostow's decomposition in that proof must be replaced by Theorem M. □

Now, let $\mathcal{P}$ be the partition of $\mathcal{M}$ defined as

$$\mathcal{P} = \{\mathcal{B}^{ps}_{(L)_{\mathcal{G}^{\mathbb{C}}}}/\mathcal{G}^{\mathbb{C}} \colon L = (\mathcal{G}^{\mathbb{C}})_{(A,\Phi)} \text{ for some } (A,\Phi) \in \mathcal{B}^{ps}\}. \tag{4.51}$$

Note that the orbit type decomposition of $\mathcal{M}$ is simply $\mathcal{P}^{\circ}$. Then, we have the following.

Proposition 4.3.2. *The Kuranishi map* $\varphi\colon \big(\widetilde{U}, (\widetilde{\mathcal{P}}^{\circ}_{H^{\mathbb{C}}}|_{\widetilde{U}})^{\circ}\big) \to \big(U, (\mathcal{P}^{\circ}|_U)^{\circ}\big)$ *is an isomorphism of partitioned spaces.*

Proof. We first record a simple fact without a proof.

Lemma 4.3.3. *Let X be a space and $\mathcal{P}$ a partition of X. If U is an open subset of X, then* $(\mathcal{P}^{\circ}|_U)^{\circ} = (\mathcal{P}|_U)^{\circ}$.

Then, by Lemma 4.3.1 and 4.3.3,

$$(\widetilde{\mathcal{P}}^{\circ}_{H^{\mathbb{C}}}|_{\widetilde{U}})^{\circ} = (\widetilde{\mathcal{P}}^{\circ}_{\mathcal{G}^{\mathbb{C}}}|_{\widetilde{U}})^{\circ} = (\widetilde{\mathcal{P}}_{\mathcal{G}^{\mathbb{C}}}|_{\widetilde{U}})^{\circ} \tag{4.52}$$

and $(\mathcal{P}^{\circ}|_U)^{\circ} = (\mathcal{P}|_U)^{\circ}$. Then, it suffices to show that

$$\varphi\colon (\widetilde{U}, \widetilde{\mathcal{P}}_{\mathcal{G}^{\mathbb{C}}}|_{\widetilde{U}}) \to (U, \mathcal{P}|_U) \tag{4.53}$$

is an isomorphism of partitioned spaces. If $[x] \in \widetilde{U}$ is such that the $H^{\mathbb{C}}$-orbit of x is closed in $\mathbf{H}^1$, then $\varphi[x] = [\theta(x)]$. The rest follows from $(\mathcal{G}^{\mathbb{C}})_{\theta(x)} = (H^{\mathbb{C}})_x$ (Proposition 4.2.2). □

Theorem 4.3.4. *The orbit type decomposition of $\mathcal{M}$ is a complex Whitney stratification.*

Proof. We first show that $\mathcal{P}^{\circ}$ satisfies the frontier condition. In other words, we need to show that if $Q_1, Q_2 \in \mathcal{P}^{\circ}$ and $Q_1 \cap \overline{Q_2} \neq \emptyset$, then $Q_1 \subset \overline{Q_2}$. Fix $[A, \Phi] \in \mathcal{M}$ such that $\mu(A, \Phi) = 0$ and $\mathcal{G}_{(A,\Phi)} = H$. Let $\varphi\colon \widetilde{U} \to U$ be the Kuranishi map. Let Q be the component of $\mathcal{B}^{ps}_{(H^{\mathbb{C}})}/\mathcal{G}^{\mathbb{C}}$ containing $[A, \Phi]$. If $[x] \in \varphi^{-1}(Q \cap U)$ such that its $H^{\mathbb{C}}$-orbit is closed in $\mathbf{H}^1$, then $\varphi[x] = [\theta(x)] \in Q \cap U$. By Proposition 4.2.2, $(\mathcal{G}^{\mathbb{C}})_{\theta(x)} = (H^{\mathbb{C}})_x$. Since $[\theta(x)] \in Q$, we conclude that $(H^{\mathbb{C}})_x = H^{\mathbb{C}}$. From the proof of Theorem E in Section 4.2.3, we see that

$$\varphi^{-1}(Q \cap U) = \mathcal{Z}^{H^{\mathbb{C}}} = B \cap (\mathbf{H}^1)^{H^{\mathbb{C}}}. \tag{4.54}$$

This shows that $Q \cap U$ is connected so that $Q \cap U \in (\mathcal{P}^\circ|_U)^\circ$. By [44, Lemma 4.7], it suffices to show that $(\mathcal{P}^\circ|_U)^\circ$ is conical at $Q \cap U$ (see [44, p.18] for the definition). Since $B \cap (\mathbf{H}^1)^{H^{\mathbb{C}}} \in (\widetilde{\mathcal{P}}^\circ_{(H^{\mathbb{C}})})^\circ$, Proposition 4.3.2 implies that it suffices to show that $(\widetilde{\mathcal{P}}^\circ_{(H^{\mathbb{C}})}|_{\widetilde{U}})^\circ$ is conical at $B \cap (\mathbf{H}^1)^{H^{\mathbb{C}}}$. Moreover, by Lemma 4.3.3, it suffices to show that $(\widetilde{\mathcal{P}}_{(H^{\mathbb{C}})}|_{\widetilde{U}})^\circ$ is conical at $B \cap (\mathbf{H}^1)^{H^{\mathbb{C}}}$. This follows from the proof of [44, Proposition 4.8].

Now we show that $\mathcal{P}^\circ$ satisfies the Whitney conditions at every point of $\mathcal{M}$. Fix $[A, \Phi] \in \mathcal{M}$ and let $\varphi \colon \widetilde{U} \to U$ be a Kuranishi map such that $[A, \Phi] \in U$. Then, it suffices to check that $(\mathcal{P}^\circ|_U)^\circ$ satisfies the Whitney conditions at $[A, \Phi]$. By Proposition 4.3.2, it suffices to check that $(\widetilde{\mathcal{P}}^\circ_{H^{\mathbb{C}}}|_{\widetilde{U}})^\circ = (\widetilde{\mathcal{P}}_{H^{\mathbb{C}}}|_{\widetilde{U}})^\circ$ satisfies the Whitney conditions at $[0]$. This follows from [44, Proposition 4.12]. □

4.4 The Hitchin-Kobayashi correspondence

Finally, we show that the Hitchin-Kobayashi correspondence preserves the orbit type decompositions, Theorem F (cf. [44, Proposition 4.6]).

Proof of Theorem F. Suppose Q is a component of $\mathbf{m}^{-1}(0)_{(H)}/\mathcal{G}$ for some $\mathcal{G}$-stabilizer at a Higgs bundle in $\mathbf{m}^{-1}(0)$. We first show that the restriction

$$i \colon \mathbf{m}^{-1}(0)_{(H)}/\mathcal{G} \to \mathcal{B}^{ps}_{(H^{\mathbb{C}})}/\mathcal{G}^{\mathbb{C}} \tag{4.55}$$

is a bijection and hence a homeomorphism. As a consequence, $i(Q)$ is a stratum in the orbit type decomposition of $\mathcal{M}$. The injectivity is obvious. To show the surjectivity, let $[A,\Phi] \in \mathcal{B}^{ps}_{(H^{\mathbb{C}})}/\mathcal{G}^{\mathbb{C}}$. We may further assume that $(\mathcal{G}^{\mathbb{C}})_{(A,\Phi)} = H^{\mathbb{C}}$. The Hitchin-Kobayashi correspondence provides some $g \in \mathcal{G}^{\mathbb{C}}$ such that $(A,\Phi)g \in \mathbf{m}^{-1}(0)$. Hence,

$$(\mathcal{G}_{(A,\Phi)g})^{\mathbb{C}} = (\mathcal{G}^{\mathbb{C}})_{(A,\Phi)g} = g^{-1}(\mathcal{G}^{\mathbb{C}})_{(A,\Phi)}g = g^{-1}H^{\mathbb{C}}g. \tag{4.56}$$

Then, Corollary 4.1.3 implies that $\mathcal{G}_{(A,\Phi)g}$ is conjugate to H in $\mathcal{G}$.

Then, we show that the restriction $i\colon Q \to i(Q)$ is holomorphic. Consequently, since Q and $i(Q)$ are smooth, $i|_Q$ is a biholomorphism. By the proofs of Theorem D and E, we see that $i|_Q$ can be locally identified with a map $\mathbf{m}^{-1}(0) \cap S^H \to (\mathbf{H}^1)^{H^{\mathbb{C}}}$. More precisely, this map is given by $(B,\Psi) \mapsto x$ where x is determined by the equation $(B,\Psi) = \theta(x)g$ for a unique $g \in \mathcal{G}^{\mathbb{C}}$. It is holomorphic, since x and g depend on (B,Ψ) holomorphically, which can be seen by Proposition 4.2.1 and the holomorphicity of $\theta\colon B \to \mathcal{C}$. □

4.5 Poisson structure

Let Q be a stratum in the orbit type stratification of $\mathcal{M}$. As a consequence, $i^{-1}(Q)$ is a stratum in the orbit type stratification of $\mathbf{m}^{-1}(0)/\mathcal{G}$, where i is the Hitchin-Kobayashi correspondence. We have shown in Theorem D that $i^{-1}(Q)$ is a hyperKähler manifold and hence has a complex symplectic form $\omega_{\mathbb{C}} = \omega_J + \sqrt{-1}\omega_K$. Using the Hitchin-Kobayashi correspondence i, we may transport $\omega_{\mathbb{C}}$ to Q so that Q is also a complex symplectic manifold. We will still use $\omega_{\mathbb{C}}$ to denote the resulting complex symplectic form on Q. Alternatively, $\omega_{\mathbb{C}}$ can be defined as follows. Let

$\pi\colon \mathcal{B}^{ps} \to \mathcal{M}$ be the quotient map. It is shown in Theorem E that $\pi^{-1}(Q)$ is a complex submanifold of $\mathcal{C}$, and $\pi\colon \pi^{-1}(Q) \to Q$ is a holomorphic submersion. Then, it follows that $\pi^*\omega_{\mathbb{C}} = \Omega_{\mathbb{C}}|_{\pi^{-1}(Q)}$, where $\Omega_{\mathbb{C}} = \Omega_J + \sqrt{-1}\Omega_K$ is the complex symplectic form on $\mathcal{C}$. This can be seen by Theorem D and the definition of i. Then, a complex Poisson bracket can be defined on the structure sheaf of $\mathcal{M}$ as follows. Let U be an open subset of $\mathcal{M}$ and $f, g\colon U \to \mathbb{C}$ holomorphic functions. Let Q be a stratum in the orbit type stratification of $\mathcal{M}$. Therefore, the restrictions $f|_{U\cap Q}$ and $g|_{U\cap Q}$ are holomorphic so that the Poisson bracket $\{f|_{U\cap Q}, g|_{U\cap Q}\}_Q$ is well-defined using the complex symplectic form $\omega_{\mathbb{C}}$ on Q. Consequently, there is a unique function $\{f, g\}\colon U \to \mathbb{C}$ such that

$$\{f, g\}|_{U\cap Q} = \{f|_{U\cap Q}, g|_{U\cap Q}\}_Q \tag{4.57}$$

for every stratum Q. Then, it remains to show that $\{f, g\}\colon U \to \mathbb{C}$ is holomorphic. Since the complex space $\mathcal{M}$ is constructed by gluing Kuranishi local models, $\{f, g\}$ is holomorphic if and only if its pullback along any Kuranishi map is holomorphic. On the other hand, from Section 4.3, we see that $\nu_{0,\mathbb{C}}^{-1}(0) /\!\!/ H^{\mathbb{C}}$ can be realized as a singular hyperKähler quotient. Hence, the structure sheaf of $\nu_{0,\mathbb{C}}^{-1}(0) /\!\!/ H^{\mathbb{C}}$ has a Poisson structure by [44, Theorem 1.4]. Then, the holomorphicity of the Poisson bracket on $\mathcal{M}$ follows from the following result (cf. [44, Proposition 4.18]).

Theorem 4.5.1 (=Theorem G)**.** *Every Kuranishi map $\varphi\colon \widetilde{U} \to U$ is a Poisson map. In other words, if $f, g\colon U \to \mathbb{C}$ are holomorphic functions, then $\varphi^*\{f, g\} = \{\varphi^* f, \varphi^* g\}$.*

Before giving the proof, we need the following lemmas.

Lemma 4.5.2. *The Kuranishi map $\theta\colon B \to \mathcal{C}$ preserves the complex symplectic forms.*

Proof. By construction of θ (see Section 4.2.2), it suffices to show that the map

$$\begin{aligned} &F\colon \widetilde{B} \cap ((A,\Phi) + \ker(D'')^*) \to \mathbf{H}^1, \\ &F(\alpha'',\eta) = (\alpha'',\eta) + \frac{1}{2}(D'')^* G[\alpha'',\eta;\alpha'',\eta], \end{aligned} \tag{4.58}$$

preserves the restrictions of Kähler forms Ω_J and Ω_K. For notational convenience, let $X = \widetilde{\mathcal{B}} \cap ((A,\Phi) + \ker(D'')^*)$. If $(B,\Psi) = (A,\Phi) + (\alpha_0'',\eta_0) \in X$, then

$$d_{(B,\Psi)}F(\alpha,\eta) = (\alpha,\eta) + (D'')^* G[\alpha_0'',\eta_0;\alpha,\eta], \qquad (\alpha,\eta) \in T_{(B,\Psi)}X. \tag{4.59}$$

Then, we need to show that

$$\Omega_i(d_{(B,\Psi)}F(\alpha_1'',\eta_1), d_{(B,\Psi)}F(\alpha_2'',\eta_2)) = \Omega_i(\alpha_1'',\eta_1;\alpha_2'',\eta_2) \tag{4.60}$$

for any $i \in \{J,K\}$, $(B,\Psi) \in X$, and $(\alpha_j'',\eta_j) \in T_{(B,\Psi)}X$. This amounts to show that

1. $\Omega_i(\alpha_1'',\eta_1;(D'')^* G[\alpha_0'',\eta_0;\alpha_2'',\eta_2]) = 0$, and
2. $\Omega_i((D'')^* G[\alpha_0,\eta_0,\alpha_1'',\eta_1];(D'')^* G[\alpha_0'',\eta_0;\alpha_2'',\eta_2) = 0$

for any $i \in \{J,K\}$ and $(\alpha_j'',\eta_j) \in \ker(D'')^*$ $(j = 0,1,2)$. To show (1), we compute

$$\Omega_J(\alpha_1'',\eta_1;(D'')^* G[\alpha_0'',\eta_0;\alpha_2'',\eta_2]) = g(D'' J(\alpha_1'',\eta_1); G[\alpha_0'',\eta_0;\alpha_2'',\eta_2]), \tag{4.61}$$

where g is the L^2-metric. Moreover,

$$D''J(\alpha_1'', \eta_1) = D''(i\eta^*, -i\alpha''^*) = -i\overline{\partial}_A\alpha''^* + i[\Phi, \eta^*] = (iD'(\alpha_1'', \eta_1))^* = 0, \quad (4.62)$$

where the last equality follows from the Kähler's identity, $(D'')^* = -i[*, D']$ and the assumption that $(D'')^*(\alpha_1'', \eta_1) = 0$. Similarly,

$$\Omega_K(\alpha_1'', \eta_1; (D'')^* G[\alpha_0'', \eta_0; \alpha_2'', \eta_2]) = g(D''K(\alpha_1'', \eta_1); G[\alpha_0'', \eta_0; \alpha_2'', \eta_2]) \qquad (4.63)$$

and

$$D''K(\alpha_1'', \eta_1) = D''(-\eta^*, \alpha''^*) = \overline{\partial}_A\alpha''^* - [\Phi, \eta^*] = D'(\alpha_1'', \eta_1)^* = 0. \qquad (4.64)$$

Finally, the same argument shows (2). $\square$

Now, let $\widetilde{C}$ be a connected component of $\widetilde{Q} \cap \widetilde{U}$, where $\widetilde{Q}$ is a stratum in the orbit type stratification of $\nu_{0,\mathbb{C}}^{-1}(0) /\!\!/ H^{\mathbb{C}}$. In other words, $\widetilde{C} \in (\mathcal{P}^\circ_{H^{\mathbb{C}}}|_{\widetilde{U}})^\circ$. By Proposition 4.3.2, there is some connected component C of $Q \cap U$ for some stratum Q in the orbit type stratification of $\mathcal{M}$ such that the restriction $\varphi\colon \widetilde{C} \to C$ is a biholomorphism. Let π denote the projections $\mathcal{B}^{ps} \to \mathcal{M}$ and $\nu_{0,\mathbb{C}}^{-1}(0)^{ps} \to \nu_{0,\mathbb{C}}^{-1}(0) /\!\!/ H^{\mathbb{C}}$. By Theorem E and [44, Lemma 4.14], $\pi^{-1}(C)$ and $\pi^{-1}(\widetilde{C})$ are complex submanifolds of

$\mathcal{C}$ and $\mathbf{H}^1$, respectively. Moreover, the following diagram commutes

$$\begin{tikzcd} \pi^{-1}(\widetilde{C}) \cap B \arrow[r, "\theta"] \arrow[d, "\pi"] & \pi^{-1}(C) \arrow[d, "\pi"] \\ \widetilde{C} \arrow[r, "\varphi"] & C \end{tikzcd} \tag{4.65}$$

Now, by Lemma 4.5.2, θ preserves the restrictions of the complex symplectic forms $\Omega_{\mathbb{C}}$ on $\mathcal{C}$ and $\omega_{\mathbb{C}}$ on $\mathbf{H}^1$. By Theorem E and [44, Lemma 4.14] again, we see that these restrictions of complex symplectic forms descend to $\widetilde{C}$ and C. As a consequence, we obtain the following.

Lemma 4.5.3. *The Kuranishi map $\varphi\colon \widetilde{C} \to C$ is a complex symplectomorphism. In particular, it preserves the complex Poisson brackets.*

Proof of Theorem 4.5.1. Let $f, g\colon U \to \mathbb{C}$ be holomorphic functions. Then, we compute

$$\begin{aligned}
(\varphi^*\{f,g\})|_{\widetilde{C}} &= (\varphi|_{\widetilde{C}})^*(\{f,g\}|_C) \\
&= (\varphi|_{\widetilde{C}})^*(\{f|_C, g|_C\}_Q) \\
&= \{(\varphi|_{\widetilde{C}})^*(f|_C), (\varphi|_{\widetilde{C}})^*(g|_C)\}_{\widetilde{Q}} \\
&= \{(\varphi^* f)|_{\widetilde{C}}, (\varphi^* g)|_{\widetilde{C}}\}_{\widetilde{Q}} \\
&= \{\varphi^* f, \varphi^* g\}|_{\widetilde{C}}.
\end{aligned} \tag{4.66}$$

for any connected component $\widetilde{C}$ of $\widetilde{Q} \cap \widetilde{U}$ for some stratum $\widetilde{Q}$ in the orbit type stratification of $\nu_{0,\mathbb{C}}^{-1}(0) /\!\!/ H^{\mathbb{C}}$. This completes the proof. $\square$

4.6 Appendix: Kähler structure on the complex gauge group

In this section, we will prove that $\mathcal{G}^{\mathbb{C}}$ is a weak Kähler manifold such that the left $\mathcal{G}$-action on $\mathcal{G}^{\mathbb{C}}$ is Hamiltonian with a moment map $\kappa\colon \mathcal{G}^{\mathbb{C}} \to \Omega^0(\mathfrak{g}_E)$ given by $\kappa(\exp(is)u) = s$. Most of the proofs are taken or adapted from [35].

We first describe a weak symplectic form on $\Omega^0(\mathcal{G}_E) \times \mathcal{G}$. Define the 1-form τ on $\Omega^0(\mathfrak{g}_E) \times \mathcal{G}$ by

$$\tau_{(s,u)}(z, w) = (s, wu^{-1})_{L^2}, \qquad (z, w) \in \Omega^0(\mathfrak{g}_E) \oplus T_u\mathcal{G}, \tag{4.67}$$

where $(\cdot,\cdot)_{L^2}$ is the L^2-metric on $\Omega^0(\mathfrak{g}_E)$. Note that if $\mathcal{G}$ were finite-dimensional, then τ would be exactly the tautological 1-form on the cotangent bundle $T^*\mathcal{G} = \mathrm{Lie}(\mathcal{G}) \times \mathcal{G}$. There are left and right actions of $\mathcal{G}$ on $\Omega^0(\mathfrak{g}_E) \times \mathcal{G}$ given by

$$u_0 \cdot (s, u) = (u_0 s u_0^{-1}, u_0 u) \text{ and } (s, u) \cdot u_0 = (s, uu_0). \tag{4.68}$$

By direct computation, we see that both the left and right $\mathcal{G}$-actions preserve τ and hence the 2-form $\omega := -d\tau$. In this section, a map is said to be left (resp. right) $\mathcal{G}$-equivariant if it is $\mathcal{G}$-equivariant with respect to the left (resp. right) $\mathcal{G}$-action, and $\mathcal{G}$-equivariant if it is $\mathcal{G}$-equivariant with respect to both the left and right $\mathcal{G}$-actions.

Proposition 4.6.1. *If the 2-form ω is non-degenerate, then the projection $\kappa\colon \Omega^0(\mathfrak{g}_E) \times \mathcal{G} \to \Omega^0(\mathfrak{g}_E)$ onto the first factor is a moment map for the left $\mathcal{G}$-action on $\big(\Omega^0(\mathfrak{g}_E) \times \mathcal{G}, \omega\big)$.*

Proof. To verify that κ is a moment map for the left $\mathcal{G}$-action on $\Omega^0(\mathfrak{g}_E) \times \mathcal{G}$, fix $\xi \in \Omega^0(\mathfrak{g}_E)$ and let ξ^* denote the vector field generated by the left $\mathcal{G}$-action on $\Omega^0(\mathfrak{g}_E) \times \mathcal{G}$. Since the left $\mathcal{G}$-action preserves ω, the Lie derivative of ω along ξ^* vanishes. Therefore, $-i_{\xi^*} d\tau = d i_{\xi^*}\tau$. Moreover,

$$\xi^*_{(s,u)} = \left.\frac{d}{dt}\right|_{t=0} (\mathrm{Ad}(e^{t\xi})s, e^{t\xi}u) = ([\xi, s], \xi u), \tag{4.69}$$

and hence $\tau(\xi^*)(s,u) = (s,\xi)_{L^2}$. Finally, it is easy to verify that κ is left $\mathcal{G}$-equivariant. $\square$

Now, we describe a complex structure J on $\Omega^0(\mathfrak{g}_E) \times \mathcal{G}$ and later verify that ω is compatible with J and positive so that ω is a Kähler form on $\Omega^0(\mathfrak{g}_E) \times \mathcal{G}$. Let $\psi\colon \Omega^0(\mathfrak{g}_E) \times \mathcal{G} \to \mathcal{G}^{\mathbb{C}}$ be the polar decomposition given by $\psi(s,u) = \exp(is)u$. It is clear that ψ is $\mathcal{G}$-equivariant. Then, there is a unique complex structure J on $\Omega^0(\mathfrak{g}_E) \times \mathcal{G}$ such that ψ is a biholomorphism. To see the relation between the symplectic form ω and the complex structure J, we also view $P = \Omega^0(\mathfrak{g}_E) \times \mathcal{G}$ as a principal $\mathcal{G}$-bundle over $\Omega^0(\mathfrak{g}_E)$, and show that P has a connection induced by the complex structure J.

Proposition 4.6.2. *Every tangent vector of P at a point $(s,u) \in P$ can be uniquely written as $\xi^\#_{(s,u)} + J\eta^\#_{(s,u)}$ for some $\xi, \eta \in \Omega^0(\mathfrak{g}_E)$, where $\xi^\#_{(s,u)}$ is the tangent vector of P at (s,u) generated by the right $\mathcal{G}$-action.*

Proof. Note that any tangent vector of $\mathcal{G}^{\mathbb{C}}$ at $\psi(s,u)$ can be uniquely written as $Z^\#_{\psi(s,u)}$ for some $Z \in \Omega^0(\mathfrak{g}_E^{\mathbb{C}})$, where $Z^\#_{\psi(s,u)}$ is the tangent vector on $\mathcal{G}^{\mathbb{C}}$ generated by the right translations. Then, write $Z = \xi + i\eta$ for some $\xi, \eta \in \Omega^0(\mathfrak{g}_E)$. Since the right $\mathcal{G}$-action and ψ are holomorphic, and ψ is $\mathcal{G}$-equivariant, we obtain

$$\begin{aligned} Z^{\#}_{\psi(s,u)} &= (\xi + i\eta)^{\#}_{\psi(s,u)} \\ &= \xi^{\#}_{\psi(s,u)} + i\eta^{\#}_{\psi(s,u)} \\ &= d_{(s,u)}\psi(\xi^{\#}_{(s,u)} + J\eta^{\#}_{(s,u)}), \end{aligned} \tag{4.70}$$

where i in the second equality also denotes the complex structure on $\mathcal{G}^{\mathbb{C}}$. The rest follows from the fact that the derivative $d_{(s,u)}\psi$ is an isomorphism. $\square$

By Proposition 4.6.2, we are able to define a $\Omega^0(\mathfrak{g}_E)$-valued 1-form γ by

$$\gamma(\xi^{\#}_{(s,u)} + J\eta^{\#}_{(s,u)}) = \xi. \tag{4.71}$$

Another $\Omega^0(\mathfrak{g}_E)$-valued 1-form χ on P is given by

$$\chi(\xi^{\#}_{(s,u)} + J\eta^{\#}_{(s,u)}) = \eta. \tag{4.72}$$

It is clear that $\chi = -\gamma J$. Since ψ is $\mathcal{G}$-equivariant, the right $\mathcal{G}$-action on P is holomorphic. Therefore, it is easy to verify that both γ and χ are $\mathcal{G}$-equivariant in the sense that $R(u_0)^*\gamma u_0 = \mathrm{Ad}(u_0^{-1})\gamma$ and $R(u_0)^*\chi = \mathrm{Ad}(u_0^{-1})\chi$, where $R(u_0)$ is the right $\mathcal{G}$-action on P given by u_0. The following are some useful formulas.

Proposition 4.6.3. *For any $(s,e) \in P$ and $(z,w) \in T_{(s,e)}P = \Omega^0(\mathfrak{g}_E) \oplus \Omega^0(\mathfrak{g}_E)$, the* *formulas for χ and γ are given by*

$$\begin{aligned}\gamma_{(s,e)}(z,w) &= \frac{1-\cos \operatorname{ad} s}{\operatorname{ad} s} z + w,\\ \chi_{(s,e)}(z,w) &= \frac{\sin \operatorname{ad} s}{\operatorname{ad} s} z.\end{aligned} \tag{4.73}$$

Proof. The derivative $d_{(s,e)}\psi$ is given by

$$\begin{aligned} d_{(s,e)}\psi(z,w) &= \frac{d}{dt}\bigg|_{t=0} \exp(is+itz)\exp(tw)\\ &= \frac{d}{dt}\bigg|_{t=0} \exp(is+itz) + \frac{d}{dt}\bigg|_{t=0} \exp(is)\exp(tw). \end{aligned} \tag{4.74}$$

Moreover, by the formula for the derivative of the exponential map (e.g. [14, Theorem 1.5.3]),

$$\begin{aligned} \exp(is)^{-1}\frac{d}{dt}\bigg|_{t=0} \exp(is+itz) &= \frac{1-\exp(-\operatorname{ad}(is))}{\operatorname{ad}(is)}(iz)\\ &= \Big(\frac{1-\cos \operatorname{ad} s}{\operatorname{ad} s} + i\frac{\sin \operatorname{ad} s}{\operatorname{ad} s}\Big) z. \end{aligned} \tag{4.75}$$

As a consequence,

$$\psi(s,e)^{-1} d_{(s,e)}\psi(z,w) = \frac{1-\cos \operatorname{ad} s}{\operatorname{ad} s} z + w + i\frac{\sin \operatorname{ad} s}{\operatorname{ad} s} z. \tag{4.76}$$

The rest follows from the proof of Proposition 4.6.2. $\square$

Consider a right $\mathcal{G}$-equivariant map $\overline{\kappa}$ given by $\overline{\kappa}(s,e) = \kappa(s,e)$. In other words, $\overline{\kappa}(s,u) = u^{-1}su$.

Proposition 4.6.4.

1. $\tau = (\overline{\kappa}, \gamma)_{L^2}$.

2. If $f\colon P \to \mathbb{R}$ *is a function given by* $f(s,u) = \frac{1}{2}\|s\|_{L^2}$*, then* $(\overline{\kappa}, \chi)_{L^2} = df$.

3. There is a unique right $\mathcal{G}$*-equivariant* $\mathrm{Hom}(\Omega^0(\mathfrak{g}_E), \Omega^0(\mathfrak{g}_E))$*-valued 1-form* Ψ *on* P *such that* $d_\gamma \overline{\kappa}_{(s,u)} = \Psi_{(s,u)}\chi_{(s,u)}$ *for any* $(s,u) \in P$*, where* $d_\gamma \overline{\kappa}$ *is the covariant derivative of* $\overline{\kappa}$*. More explicitly,*

$$\begin{aligned} d_\gamma \overline{\kappa}_{(s,e)}(z,w) &= \cos \mathrm{ad}\, s(z), \\ \Psi_{(s,e)} &= \cos \mathrm{ad}\, s \frac{\mathrm{ad}\, s}{\sin \mathrm{ad}\, s}, \end{aligned} \tag{4.77}$$

for any $(z,w) \in T_{(s,e)}P$.

Before giving the proof, we claim that $\frac{\sin \mathrm{ad}\, s}{\mathrm{ad}\, s}$ is invertible so that $\frac{\mathrm{ad}\, s}{\sin \mathrm{ad}\, s}$ simply means its inverse. In fact, if $\xi \in \Omega^0(\mathfrak{g}_E)$, then, by definition, $\chi_{(s,e)}(J\xi^\#_{(s,e)}) = \xi$. Moreover, if $J\xi^\#_{(s,e)} = (z,w)$ for some $(z,w) \in T_{(s,e)}P$, then

$$\xi = \chi_{(s,e)}(J\xi^\#_{(s,e)}) = \chi_{(s,e)}(z,w) = \chi_{(s,e)}(z,0). \tag{4.78}$$

Therefore, the map

$$\Omega^0(\mathfrak{g}_E) \to \Omega^0(\mathfrak{g}_E), \qquad z \mapsto \chi_{(s,e)}(z,0) = \frac{\sin \mathrm{ad}\, s}{\mathrm{ad}\, s} z, \tag{4.79}$$

is invertible.

Proof of Proposition 4.6.4. If $(z,w) \in T_{(s,e)}P$, then Proposition 4.6.3 implies that

$$(\overline{\kappa}, \gamma)_{L^2}(z,w) = \Big(s, \frac{1-\cos \operatorname{ad} s}{\operatorname{ad} s} z + w\Big)_{L^2}. \tag{4.80}$$

Since $(s, [s,z])_{L^2} = 0$,

$$\Big(s, \frac{1-\cos \operatorname{ad} s}{\operatorname{ad} s} z\Big)_{L^2} = \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{(2j)!} \Big(s, (\operatorname{ad} s)^{2j-1} z\Big)_{L^2} = 0. \tag{4.81}$$

Similarly,

$$\begin{aligned}(\overline{\kappa}, \chi)_{L^2}(z,w) &= \Big(s, \frac{\sin \operatorname{ad} s}{\operatorname{ad} s} z\Big)_{L^2} \\ &= \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} (s, (\operatorname{ad} s)^{2j} z)_{L^2} \\ &= (s,z)_{L^2}.\end{aligned} \tag{4.82}$$

Therefore, the identities (1) and (2) hold at (s,e). By the right $\mathcal{G}$-equivariance, they hold everywhere.

Then, we prove the formula for the covariant derivative $d_\gamma \overline{\kappa}_{(s,e)}$. Note that $d_\gamma \overline{\kappa} = d\overline{\kappa} + [\gamma, \overline{\kappa}]$. Therefore, if $(z,w) \in T_{(s,e)}P$, we have

$$\begin{aligned}d_\gamma \overline{\kappa}_{(s,e)}(z,w) &= d\overline{\kappa}_{(s,e)}(z,w) + \Big[\frac{1-\cos \operatorname{ad} s}{\operatorname{ad} s} z + w, s\Big] \\ &= \operatorname{ad} s(w) + z + \operatorname{ad} s\Big(\frac{\cos \operatorname{ad} s - 1}{\operatorname{ad} s} z - w\Big) \\ &= \cos \operatorname{ad} s(z).\end{aligned} \tag{4.83}$$

To define Ψ, it is enough to define $\Psi_{(s,e)}$ which needs to satisfy

$$\cos \operatorname{ad} s(z) = \Psi_{(s,e)} \frac{\sin \operatorname{ad} s}{\operatorname{ad} s}(z). \tag{4.84}$$

As a consequence,

$$\Psi_{(s,e)} = \cos \operatorname{ad} s \frac{\operatorname{ad} s}{\sin \operatorname{ad} s}. \tag{4.85}$$

$\square$

The following results verify that ω is compatible with J. Then, we will verify that ω is positive so that ω is a Kähler form on $\Omega^0(\mathfrak{g}_E) \times \mathcal{G}$.

Proposition 4.6.5. *The following hold for any $\xi, \eta \in \Omega^0(\mathfrak{g}_E)$:*

1. $\omega(\xi^\#, J\eta^\#) = (\xi, \Psi(\eta))_{L^2}$.

2. $\omega(J\xi^\#, J\eta^\#) = \omega(\xi^\#, \eta^\#)$.

3. $(\xi, \Psi(\eta))_{L^2} = (\eta, \Psi(\xi))_{L^2}$.

As a consequence, $\omega(J\cdot, J\cdot) = \omega(\cdot, \cdot)$.

Proof. We compute

$$\begin{aligned}
\omega(\xi^\#, J\eta^\#) &= -d(\overline{\kappa}, \lambda)_{L^2}(\xi^\#, J\eta^\#) \\
&= -\xi^\#(\overline{\kappa}, \lambda(J\eta^\#))_{L^2} + J\eta^\#(\overline{\kappa}, \lambda(\xi^\#)) + (\overline{\kappa}, \lambda([\xi^\#, J\eta^\#])_{L^2} \\
&= (\overline{\kappa}(J\eta^\#), \xi)_{L^2}) + (\overline{\kappa}, \lambda(J[\xi^\#, \eta^\#]))_{L^2} \\
&= (d_\gamma \overline{\kappa}(J\eta^\#), \xi)_{L^2} \\
&= (\Psi(\eta), \xi)_{L^2}.
\end{aligned} \tag{4.86}$$

Here, we have used the formula that $[\xi^\#, J\eta^\#] = J[\xi^\#, \eta^\#]$, since J commutes with the right $\mathcal{G}$-action. Moreover, $d\overline{\kappa}(J\eta^\#) = d_\gamma\overline{\kappa}(J\eta^\#)$, since $J\eta^\#$ is horizontal. This proves (1). To prove (2), we compute

$$\begin{aligned}
\omega(J\xi^\#, J\eta^\#) &= (\overline{\kappa}, \gamma)_{L^2}([J\xi^\#, J\eta^\#])_{L^2} \\
&= (\overline{\kappa}, \gamma)_{L^2}(-[\xi^\#, \eta^\#])_{L^2} \\
&= (\overline{\kappa}, \gamma)_{L^2}(-[\xi, \eta]^\#)_{L^2} \\
&= -(\overline{\kappa}, [\xi, \eta])_{L^2}.
\end{aligned} \tag{4.87}$$

On the other hand,

$$\begin{aligned}
\omega(\xi^\#, \eta^\#) &= -\xi^\#(\overline{\kappa}, \gamma(\eta^\#))_{L^2} + \eta^\#(\overline{\kappa}, \gamma(\xi^\#))_{L^2} + (\overline{\kappa}, \gamma([\xi^\#, \eta^\#]) \\
&= -([\overline{\kappa}, \xi], \eta)_{L^2} + ([\overline{\kappa}, \eta], \xi)_{L^2} + (\overline{\kappa}, [\xi, \eta])_{L^2} \\
&= (\overline{\kappa}, [\eta, \xi])_{L^2}.
\end{aligned} \tag{4.88}$$

Finally, to prove (3), we first compute

$$\begin{aligned}
d(\overline{\kappa}, \chi)_{L^2}(J\xi^\#, J\eta^\#) &= J\xi^\#(\overline{\kappa}, \chi(J\eta^\#))_{L^2} - J\eta^\#(\overline{\kappa}, \chi(J\xi^\#))_{L^2} - (\overline{\kappa}, \chi([J\xi^\#, J\eta^\#])_{L^2} \\
&= (d\overline{\kappa}(J\xi^\#), \eta)_{L^2} - (d\overline{\kappa}(J\eta^\#), \xi)_{L^2} \\
&= (d_\gamma\overline{\kappa}(J\xi^\#), \eta)_{L^2} - (d_\gamma\overline{\kappa}(J\eta^\#), \xi)_{L^2} \\
&= (\Psi\chi(J\xi^\#), \eta)_{L^2} - (\Psi\chi(J\eta^\#), \xi)_{L^2} \\
&= (\Psi(\xi), \eta)_{L^2} - (\Psi(\eta), \xi)_{L^2},
\end{aligned} \tag{4.89}$$

and then note that $d(\overline{\kappa}, \chi)_{L^2} = 0$ by Proposition 4.6.4. $\square$

Proposition 4.6.6. *The metric $g = \omega(\cdot, J\cdot)$ is positive-definite. In particular, ω is non-degenerate.*

Proof. If $\xi, \eta \in \Omega^0(\mathfrak{g}_E)$, then

$$\begin{aligned} g(\xi^\# + J\eta^\#, \xi^\# + J\eta^\#) &= \omega(\xi^\# + J\eta^\#, J\xi^\# - \eta^\#) \\ &= \omega(\xi^\#, J\xi^\#) - \omega(\xi^\#, \eta^\#) + \omega(J\eta^\#, J\xi^\#) - \omega(J\eta^\#, \eta^\#) \\ &= (\xi, \Psi(\xi))_{L^2} + (\eta, \Psi(\eta))_{L^2} - 2(\overline{\kappa}, [\xi, \eta])_{L^2}. \end{aligned} \tag{4.90}$$

Since g is right $\mathcal{G}$-invariant, it is enough to show the positive-definiteness at (s, e). Hence, we need to show that

$$\begin{aligned} &(\xi, \Psi_{(s,e)}(\xi))_{L^2} + (\eta, \Psi_{(s,e)}(\eta))_{L^2} - 2(s, [\xi, \eta])_{L^2} \\ &= \int_X \langle \xi, \Psi_{(s,e)}(\xi)\rangle_{L^2} + \langle \eta, \Psi_{(s,e)}(\eta)\rangle_{L^2} - 2\langle s, [\xi, \eta]\rangle_{L^2} > 0 \end{aligned} \tag{4.91}$$

for every $s \in \Omega^0(\mathfrak{g}_E)$ and nonzero $\xi + i\eta \in \Omega^0(\mathfrak{g}_E^{\mathbb{C}})$. Note that the integrand is positive pointwise. In fact, after fixing a base point $x \in X$, the polar decomposition ψ becomes the usual polar decomposition $\psi_x \colon \mathfrak{u}(n) \times U(n) \to GL_n(\mathbb{C})$. The unique complex structure on $\mathfrak{u}(n) \times U(n)$ making ψ_x a biholomorphism is compatible with the tautological 1-form on $\mathfrak{u} \times U(n) = T^*U(n)$ so that the tautological 1-form is a Kähler form (see [35, Theorem 5.1 and Remark 5.2]). The resulting Kähler metric on $\mathfrak{u}(n) \times U(n)$ evaluated at $(\xi^\# + J\eta^\#)(x)$ is exactly the integrand and hence positive. Here, $(\xi^\# + J\eta^\#)(x)$ is the value of the section $\xi^\# + J\eta^\# \in \Omega^0(\mathfrak{g}_E) \oplus \Omega^0(\mathfrak{g}_E)$ at

x. $\square$

Chapter 5: The moduli space as a quasi-projective variety

This chapter is based on the author's paper [16].

5.1 Infinite dimensional GIT quotient

5.1.1 S-equivalence classes and closures of orbits

In Section 5.1, we will prove Theorem I. We start with a simple lemma in point-set topology.

Lemma 5.1.1. *Let Y be a first countable topological space on which a topological group G acts. Let x_n be a sequence in Y such that $[x_n]$ converges to $[x]$ for some $x \in Y$ in Y/G. Then, there exists a subsequence x_{n_k} and a sequence $g_k \in G$ such that $x_{n_k} g_k$ converges to x in Y.*

Proof. Let $\pi \colon Y \to Y/G$ be the quotient map. Since Y is first countable, we can find nested open neighborhoods

$$\cdots \subset U_k \subset U_{k-1} \subset \cdots \subset U_1 \tag{5.1}$$

of x that form a neighborhood basis. Now, for each U_k, $\pi(U_k)$ is open and contains

$[x]$. Hence, there exists some $[x_{n_k}] \in \pi(U_k)$. Therefore, there exists some $g_k \in G$ such that $x_{n_k} g_k \in U_k$. Now we claim that $x_{n_k} g_k$ converges to x. Let V be an open neighborhood of x. Then, $U_j \subset V$ for some j. If $k > j$, then $x_{n_k} g_k \in U_k \subset U_j \subset V$. $\square$

Then, among other properties, we first show that π identify $\mathcal{G}^{\mathbb{C}}$-orbits whose closures in $\mathcal{B}^{ss}$ intersect.

Proposition 5.1.2. *Two semistable Higgs bundles are S-equivalent if and only if the closures of their $\mathcal{G}^{\mathbb{C}}$-orbits in $\mathcal{B}^{ss}$ intersect.*

Proof. Let $r\colon \mathcal{B}^{ss} \to \mathbf{m}^{-1}(0)$ be the retraction defined by the Yang-Mills-Higgs flow. Since the Yang-Mills-Higgs flow preserves $\mathcal{G}^{\mathbb{C}}$-orbits, if (A, Φ) is a semistable Higgs bundle, then $r(A, \Phi)$ is contained in $\overline{(A, \Phi)\mathcal{G}^{\mathbb{C}}}$. Moreover, by [67, Theorem 1.4], $r(A, \Phi)$ is isomorphic to $Gr(A, \Phi)$. Therefore, if (A_1, Φ_1) and (A_2, Φ_2) are two semistable Higgs bundles that are S-equivalent, then $r(A_1, \Phi_1)$ is isomorphic to $r(A_2, \Phi_2)$. Therefore,

$$\overline{(A_1, \Phi_1)\mathcal{G}^{\mathbb{C}}} \ni r(A_1, \Phi_1) \sim_{\mathcal{G}^{\mathbb{C}}} r(A_2, \Phi_2) \in \overline{(A_2, \Phi_2)\mathcal{G}^{\mathbb{C}}}, \tag{5.2}$$

where $\sim_{\mathcal{G}^{\mathbb{C}}}$ means the equivalence relation induced by the $\mathcal{G}^{\mathbb{C}}$-action.

Conversely, suppose that

$$(B, \Psi) \in \overline{(A_1, \Phi_1)\mathcal{G}^{\mathbb{C}}} \cap \overline{(A_2, \Phi_2)\mathcal{G}^{\mathbb{C}}} \cap \mathcal{B}^{ss}. \tag{5.3}$$

By replacing (B, Ψ) by $r(B, \Psi)$, we may assume that (B, Ψ) is polystable. Now,

$r(A_1,\Phi_1)$ is also polystable and contained in $\overline{(A_1,\Phi_1)\mathcal{G}^{\mathbb{C}}}$. Since $\overline{(A_1,\Phi_1)\mathcal{G}^{\mathbb{C}}}$ contains a unique polystable orbit (see [17, Lemma 3.7]), $r(A_1,\Phi_1)$ is isomorphic to (B,Ψ). Similar argument shows that $r(A_2,\Phi_2)$ is isomorphic to (B,Ψ). Since $r(A_i,\Phi_i)$ is further isomorphic to $Gr(A_i,\Phi_i)$ for $i=1,2$. We see that (A_1,Φ_1) and (A_2,Φ_2) are S-equivalent. $\square$

Using the local slice theorem, Theorem 4.2.1, for the $\mathcal{G}^{\mathbb{C}}$-action, we are able to prove that polystable orbits in $\mathcal{B}^{ss}$ are exactly closed orbits.

Proposition 5.1.3. *A semistable Higgs bundle is polystable if and only if its $\mathcal{G}^{\mathbb{C}}$-orbit is closed in $\mathcal{B}^{ss}$.*

Proof. The same proof of [58, Proposition 2.4(ii)] works. For the sake of completeness, we spell out the details. Let (A,Φ) be a semistable Higgs bundle and $r\colon \mathcal{B}^{ss}\to \mathbf{m}^{-1}(0)$ be the retraction defined by the Yang-Mills-Higgs flow. Since the Yang-Mills-Higgs flow preserves the $\mathcal{G}^{\mathbb{C}}$-orbits, if $(A,\Phi)\mathcal{G}^{\mathbb{C}}$ is closed in $\mathcal{B}^{ss}$, then obviously $r(A,\Phi)\in(A,\Phi)\mathcal{G}^{\mathbb{C}}$. This means that (A,Φ) is polystable. Conversely, assume that (A,Φ) is polystable. By the Hitchin-Kobayashi correspondence, we may assume that (A,Φ) lies in $\mathbf{m}^{-1}(0)$. Let $(A,\Phi)g_i$ be a sequence converging to some $(B,\Psi)\in\mathcal{B}^{ss}$. Since (A,Φ) is polystable, $r[(A,\Phi)g_i]$ is isomorphic to (A,Φ). By the Hitchin-Kobayashi correspondence, $r[(A,\Phi)g_i]\in(A,\Phi)\mathcal{G}$. Moreover, by continuity, $r[(A,\Phi)g_i]$ converges to $r(B,\Psi)$. Since the $\mathcal{G}$-action is proper, $(A,\Phi)\mathcal{G}$ is closed, and hence $r(B,\Psi)\in(A,\Phi)\mathcal{G}$. On the other hand, $r(B,\Psi)\in\overline{(B,\Psi)\mathcal{G}^{\mathbb{C}}}$, and hence $(A,\Phi)\in\overline{(B,\Psi)\mathcal{G}^{\mathbb{C}}}$. By Theorem 4.2.1, there is an $\mathcal{G}^{\mathbb{C}}$-invariant open neighborhood U of (A,Φ) such that $\mathcal{Z}H^{\mathbb{C}}\times_{H^{\mathbb{C}}}\mathcal{G}^{\mathbb{C}}\to U$ is a homeomorphism. As a consequence,

$(B, \Psi) \in U$.

Then, it suffices to show that $(A, \Phi)\mathcal{G}^{\mathbb{C}}$ is closed in U. By the homeomorphism $\mathcal{Z}H^{\mathbb{C}} \times_{H^{\mathbb{C}}} \mathcal{G}^{\mathbb{C}} \to U$, it suffices to prove that if $[0, g_i]$ converges to $[x, g]$, then $x = 0$. By Lemma 5.1.1, there is a subsequence g_{i_k} and a sequence $h_k \in H^{\mathbb{C}}$ such that $(0 \cdot h_k^{-1}, h_k g_{i_k})$ converges to (x, g). This immediately shows that $x = 0$. $\square$

The following result allows us to identify $\mathcal{M}$ with $\mathcal{B}^{ss} /\!\!/ \mathcal{G}^{\mathbb{C}}$. From now on, we will use $[\cdot]_S$ and $[\cdot]$ to denote S-equivalence classes and isomorphism classes, respectively.

Proposition 5.1.4. *The inclusion $\mathcal{B}^{ps} \hookrightarrow \mathcal{B}^{ss}$ induces a homeomorphism*

$$\mathcal{B}^{ps}/\mathcal{G}^{\mathbb{C}} \xrightarrow{\sim} \mathcal{B}^{ss} /\!\!/ \mathcal{G}^{\mathbb{C}}. \tag{5.4}$$

Proof. Let $r \colon \mathcal{B}^{ss} \to \mathbf{m}^{-1}(0)$ be the retraction defined by the Yang-Mills-Higgs flow. We claim that r induces the inverse of the map

$$\bar{j} \colon \mathcal{B}^{ps}/\mathcal{G}^{\mathbb{C}} \xrightarrow{\sim} \mathcal{B}^{ss} /\!\!/ \mathcal{G}^{\mathbb{C}}, \tag{5.5}$$

where $j \colon \mathcal{B}^{ps} \hookrightarrow \mathcal{B}^{ss}$ is the inclusion. By definition of the S-equivalence, r induces a well-defined continuous map

$$\bar{r} \colon \mathcal{B}^{ss} /\!\!/ \mathcal{G}^{\mathbb{C}} \to \mathcal{B}^{ps}/\mathcal{G}^{\mathbb{C}}, \qquad [A, \Phi]_S \mapsto [r(A, \Phi)]. \tag{5.6}$$

Then, if (A, Φ) is a polystable Higgs bundle,

$$\overline{r}\overline{j}[A, \Phi] = \overline{r}[A, \Phi]_S = [r(A, \Phi)]. \tag{5.7}$$

Since (A, Φ) is polystable, it is isomorphic to the graded object $Gr(A, \Phi)$ and hence to $r(A, \Phi)$. Therefore, $[r(A, \Phi)] = [A, \Phi]$. Conversely, if (A, Φ) is semistable, then

$$\overline{j}\overline{r}[A, \Phi]_S = \overline{j}[r(A, \Phi)] = [r(A, \Phi)]_S. \tag{5.8}$$

By definition of the S-equivalence, (A, Φ) is S-equivalent to $r(A, \Phi)$, since $r(A, \Phi)$ is isomorphic to $Gr(A, \Phi)$. Hence, $[r(A, \Phi)]_S = [A, \Phi]_S$. $\square$

Corollary 5.1.5. *Every fiber of $\pi \colon \mathcal{B}^{ss} \to \mathcal{B}^{ss} /\!\!/ \mathcal{G}^{\mathbb{C}}$ contains a unique $\mathcal{G}^{\mathbb{C}}$-orbit that is closed in $\mathcal{B}^{ss}$.*

Proof. This follows from Proposition 5.1.3 and 5.1.4. $\square$

5.1.2 π-saturated open neighborhoods

To further study the quotient map $\pi \colon \mathcal{B}^{ss} \to \mathcal{M}$, we will improve the local slice theorem, Theorem 4.2.1, so that the open neighborhood in $\mathcal{B}^{ss}$ provided by the theorem is not only $\mathcal{G}^{\mathbb{C}}$-invariant but also saturated with respect to π.

Lemma 5.1.6. *Let U be a $\mathcal{G}^{\mathbb{C}}$-invariant open subset of $\mathcal{B}^{ss}$. Then the following are equivalent.*

1. *If $(A, \Phi) \in U$, then the closure of its $\mathcal{G}^{\mathbb{C}}$-orbit in $\mathcal{B}^{ss}$ is contained in U.*

2. U is π-saturated.

Proof. By Proposition 5.1.2, (2) implies (1). To show that (1) implies (2), suppose that $(B, \Phi) \in U$ and $(B', \Phi') \in B^{ss}$ such that $\pi(B, \Psi) = \pi(B', \Psi')$. We need to show that $(B', \Psi') \in U$. By Corollary 5.1.5, there exists a polystable Higgs bundle (B'', Ψ'') such that

$$(B'', \Psi'')\mathcal{G}^{\mathbb{C}} \subset \overline{(B, \Psi)\mathcal{G}^{\mathbb{C}}} \cap \overline{(B', \Psi')\mathcal{G}^{\mathbb{C}}} \cap \mathcal{B}^{ss}. \tag{5.9}$$

By assumption (1), $(B'', \Psi'') \in U$. If $(B', \Psi') \notin U$, then

$$\overline{(B', \Psi')\mathcal{G}^{\mathbb{C}}} \cap U \cap \mathcal{B}^{ss} = \emptyset. \tag{5.10}$$

This is a contradiction. □

Proposition 5.1.7. *Let (A, Φ) be a polystable Higgs bundle. Then, every $\mathcal{G}^{\mathbb{C}}$-invariant open neighborhood of (A, Φ) in $\mathcal{B}^{ss}$ contains a π-saturated open neighborhood.*

Proof. Let U be an $\mathcal{G}^{\mathbb{C}}$-invariant open neighborhood of (A, Φ) in $\mathcal{B}^{ss}$. Take a neighborhood basis V_n of $[A, \Phi]_S$ in $\mathcal{B}^{ss} /\!\!/ \mathcal{G}^{\mathbb{C}}$ such that $V_n \subset V_{n-1}$ for all $n \geq 1$. We claim that $\pi^{-1}(V_n)$ is contained in U for some n, where $\pi\colon \mathcal{B}^{ss} \to \mathcal{B}^{ss} /\!\!/ \mathcal{G}^{\mathbb{C}}$is the quotient map. Assuming the contrary, we can choose a sequence (A_n, Φ_n) such that

1. $(A_n, \Phi_n) \notin U$, and

2. $[A_n, \Phi_n]_S$ converges to $[A, \Phi]_S$ in $\mathcal{M}$.

Since U is $\mathcal{G}^{\mathbb{C}}$-invariant, the closure of $(A_n, \Phi_n)\mathcal{G}^{\mathbb{C}}$ in $\mathcal{B}^{ss}$ is contained in $\mathcal{B}^{ss} \setminus U$. Then, since the closure of $(A_n, \Phi_n)\mathcal{G}^{\mathbb{C}}$ in $\mathcal{B}^{ss}$ contains a unique polystable orbit (Proposition 5.1.5), we may assume that each (A_n, Φ_n) is polystable. As a consequence, $[A_n, \Phi_n]$ converges to $[A, \Phi]$ in $\mathcal{B}^{ps}/\mathcal{G}^{\mathbb{C}}$. By Lemma 5.1.1, there is a subsequence (A_{n_k}, Φ_{n_k}) and a sequence $g_k \in \mathcal{G}^{\mathbb{C}}$ such that $(A_{n_k}, \Phi_{n_k}) \cdot g_k$ converges to (A, Φ). This is impossible, since $(A_{n_k}, \Phi_{n_k}) \cdot g_k \notin U$. $\square$

Theorem 5.1.8. *Let* $(A, \Phi) \in \mathbf{m}^{-1}(0)$. *If* B *is sufficiently small, then the map*

$$\overline{\theta}\colon \mathcal{Z}H^{\mathbb{C}} \times_{H^{\mathbb{C}}} \mathcal{G}^{\mathbb{C}} \to \mathcal{B}^{ss}, \qquad [x, g] \mapsto \theta(x)g \tag{5.11}$$

is a homeomorphism onto an π*-saturated open neighborhood of* (A, Φ) *in* $\mathcal{B}^{ss}$.

Proof. By the local slice theorem, Theorem 4.2.1, there exists some open neighborhood U of (A, Φ) in $\mathcal{B}^{ss}$ such that the map

$$\mathcal{Z}H^{\mathbb{C}} \times_{H^{\mathbb{C}}} \mathcal{G}^{\mathbb{C}} \to U, \qquad [x, g] \mapsto \theta(x)g, \tag{5.12}$$

is a homeomorphism, where θ is the Kuranishi map (see §3.2.1). By Proposition 5.1.7, let U' be an open neighborhood of (A, Φ) in $\mathcal{B}^{ss}$ that is π-saturated and contained in U. Then, there is a $H^{\mathbb{C}}$-invariant open neighborhood Q of 0 in $\mathbf{H}^1$ such that $\overline{\theta}$ maps $(\mathcal{Z}H^{\mathbb{C}} \cap Q) \times_{H^{\mathbb{C}}} \mathcal{G}^{\mathbb{C}}$ into U'. Now, let B' be an open ball around 0 in $\mathbf{H}^1$ such that $B' \subset B \cap Q$. Then, we see that

$$(B' \cap \mathcal{Z})H^{\mathbb{C}} \subset (\mathcal{Z} \cap Q)H^{\mathbb{C}} \subset \mathcal{Z}H^{\mathbb{C}} \cap Q. \tag{5.13}$$

Let $\mathcal{Z}' = B' \cap \nu_{0,\mathbb{C}}^{-1}(0) = B' \cap \mathcal{Z}$, and we have $\mathcal{Z}' \subset \mathcal{Z}$. Let U'' be the image of $\mathcal{Z}' H^{\mathbb{C}} \times_{H^{\mathbb{C}}} \mathcal{G}^{\mathbb{C}}$ under $\overline{\theta}$. Then

$$\overline{\theta} \colon \mathcal{Z}' H^{\mathbb{C}} \times_{H^{\mathbb{C}}} \mathcal{G}^{\mathbb{C}} \to U'' \tag{5.14}$$

is a homeomorphism, and U'' is contained in U'.

We prove that U'' is π-saturated. Let $\theta(x)g \in U''$ for some $x \in \mathcal{Z}'$ and $g \in \mathcal{G}^{\mathbb{C}}$. By Lemma 5.1.6, we need to show that the closure of $\theta(x)g\mathcal{G}^{\mathbb{C}} = \theta(x)\mathcal{G}^{\mathbb{C}}$ in $\mathcal{B}^{ss}$ is contained in U''. Let g_n be a sequence in $\mathcal{G}^{\mathbb{C}}$ such that $\theta(x)g_n$ converges in $\mathcal{B}^{ss}$. Since U' is π-saturated, the limiting point is in U', and we may assume that it is $\theta(y)h$ for some $y \in \mathcal{Z}$ and $h \in \mathcal{G}^{\mathbb{C}}$. Since $\overline{\theta}$ is a homeomorphism, we see that $[x, g_n]$ converges to $[y, h]$ in $\mathcal{Z}H^{\mathbb{C}} \times_{H^{\mathbb{C}}} \mathcal{G}^{\mathbb{C}}$. By Lemma 5.1.1, there is a subsequence $g_{n_j} \in \mathcal{G}^{\mathbb{C}}$ and a sequence $k_j \in H^{\mathbb{C}}$ such that $(xk_j^{-1}, k_j g_{n_j})$ converges to (y, h) in $\mathcal{Z}H^{\mathbb{C}} \times \mathcal{G}^{\mathbb{C}}$. By [60, Corollary 4.9], $\mathcal{Z}'H^{\mathbb{C}}$ is saturated with respect to the quotient map $\mathcal{Z}H^{\mathbb{C}} \to \mathcal{Z}H^{\mathbb{C}} /\!\!/ H^{\mathbb{C}}$. Hence $y \in \mathcal{Z}'$ so that $[y, h] \in \mathcal{Z}'H^{\mathbb{C}} \times_{H^{\mathbb{C}}} \mathcal{G}^{\mathbb{C}}$ and $\theta(y)h \in U''$. $\square$

Now we can obtain Kuranishi local models for $\mathcal{B}^{ss} /\!\!/ \mathcal{G}^{\mathbb{C}}$ in the following way. Fix $[A, \Phi]_S \in \mathcal{B}^{ss} /\!\!/ \mathcal{G}^{\mathbb{C}}$ such that $(A, \Phi) \in \mathbf{m}^{-1}(0)$. By Theorem 5.1.8, the natural map

$$\overline{\theta} \colon \mathcal{Z}H^{\mathbb{C}} \times_{H^{\mathbb{C}}} \mathcal{G}^{\mathbb{C}} \to U \tag{5.15}$$

is a homeomorphism onto an π-saturated open neighborhood U of (A, Φ). By the results in §3.2.3, θ induces a well-defined map

$$\varphi\colon \mathcal{Z}H^{\mathbb{C}} /\!\!/ H^{\mathbb{C}} \to \pi(U) \subset \mathcal{B}^{ss} /\!\!/ \mathcal{G}^{\mathbb{C}}, \qquad [x] \mapsto [\theta(x)]_S, \tag{5.16}$$

and $\pi(U)$ is an open neighborhood of $[A,\Phi]_S$ in $\mathcal{B}^{ss} /\!\!/ \mathcal{G}^{\mathbb{C}}$. By Proposition 5.1.4, we see that it is a biholomorphism.

Moreover, we can also describe the structure sheaf of $\mathcal{M}$ in the following way.

Proposition 5.1.9. *The structure sheaf of $\mathcal{M}$ is equal to $\pi_*\mathcal{O}_{\mathcal{B}}^{\mathcal{G}^{\mathbb{C}}}$. In other words, for any open subset V in $\mathcal{M}$, the natural map $\pi^*\colon \mathcal{O}(V) \mapsto \mathcal{O}(\pi^{-1}V)^{\mathcal{G}^{\mathbb{C}}}$ is a bijection.*

Proof. It suffices to prove the following. Let $(A,\Phi) \in \mathbf{m}^{-1}(0)$. By Theorem 5.1.8 and the remark after it, we see that the natural map

$$\overline{\theta}\colon \mathcal{Z}H^{\mathbb{C}} \times_{H^{\mathbb{C}}} \mathcal{G}^{\mathbb{C}} \to U \tag{5.17}$$

is a homeomorphism onto an π-saturated open neighborhood of (A,Φ) in $\mathcal{B}^{ss}$. Moreover, it induces a biholomorphic map $\varphi\colon \mathcal{Z}H^{\mathbb{C}} /\!\!/ H^{\mathbb{C}} \to \pi(U)$. By the definition of the structure sheaf of $\mathcal{B}$, we easily see that $\overline{\theta}$ is actually a biholomorphism. As a consequence, there is a chain of isomorphisms

$$\mathcal{O}(\pi(U)) \xrightarrow{\varphi^*} \mathcal{O}(\mathcal{Z}H^{\mathbb{C}} /\!\!/ H^{\mathbb{C}}) \xrightarrow{\pi^*} \mathcal{O}(\mathcal{Z}H^{\mathbb{C}})^{H^{\mathbb{C}}} \xrightarrow{\sim} \mathcal{O}(\mathcal{Z}H^{\mathbb{C}} \times_{H^{\mathbb{C}}} \mathcal{G}^{\mathbb{C}})^{\mathcal{G}^{\mathbb{C}}} \xrightarrow{\overline{\theta}^{-1}} \mathcal{O}(U)^{\mathcal{G}^{\mathbb{C}}}. \tag{5.18}$$

Moreover, the composition is exactly $\pi^*\colon \mathcal{O}(\pi(U)) \to \mathcal{O}(U)^{\mathcal{G}^{\mathbb{C}}}$. $\square$

As a corollary, the quotient map $\pi\colon \mathcal{B}^{ss} \to \mathcal{B}^{ss} /\!\!/ \mathcal{G}^{\mathbb{C}}$ is a categorical quotient in the following sense.

Corollary 5.1.10. *Let Z be a complex space and $g\colon \mathcal{B}^{ss} \to Z$ a $\mathcal{G}^{\mathbb{C}}$-invariant holomorphic map. Then, g induces a unique holomorphic map $\overline{g}\colon \mathcal{M} \to Z$.*

Proof. Define $\overline{g}[A,\Phi]_S = g(A,\Phi)$. By Proposition 5.1.2, it is well-defined. The holomorphicity of $\overline{f}$ follows from Proposition 5.1.9. □

Proof of Theorem I. This follows from Proposition 5.1.2, 5.1.3, 5.1.5, 5.1.9 and Corollary 5.1.10. □

5.2 Descent lemmas for vector bundles

In this section, we will first generalize the descent lemma for vector bundles in [13, Theorem 2.3] to analytic Hilbert quotients, and then prove a similar descent lemma for the quotient map $\pi\colon \mathcal{B}^{ss} \to \mathcal{B}^{ss} /\!\!/ \mathcal{G}^{\mathbb{C}}$.

Let G be a complex reductive Lie group acting holomorphically on a complex space X. Suppose that X admits an analytic Hilbert quotient. In other words, there is a surjective G-invariant holomorphic map $\pi\colon X \to X /\!\!/ G$ such that the following hold.

1. π is Stein in the sense that inverse images of Stein subspaces are Stein.
2. $\mathcal{O}_{X/\!\!/G} = \pi_*\mathcal{O}_X^G$. In other words, for every open subset U of $X /\!\!/ G$, the map $\pi^*\colon \mathcal{O}_{X/\!\!/G}(U) \to \mathcal{O}_X(\pi^{-1}U)^G$ is an isomorphism.

Proposition 5.2.1. *Let $E \to X$ be a holomorphic G-bundle over X. If G_x acts trivially on the fiber E_x for every $x \in X$ whose G-orbit is closed, then there is a* *vector bundle $F \to X /\!\!/ G$ such that $\pi^*F = E$. Moreover, $\mathcal{O}(F) = \pi_*\mathcal{O}(E)^G$, where $\mathcal{O}(F)$ and $\mathcal{O}(E)$ are the sheaves of holomorphic sections of F and E, respectively.*

Proof. We closely follow the proof of [13, Theorem 2.3]. Fix $x \in X$ such that Gx is closed in X. Choose a basis $\sigma_1, \cdots, \sigma_r$ for E_x. Then, we may consider the map

$$s_i \colon G/G_x \to E, \qquad s_i(gG_x) = g\sigma_i. \tag{5.19}$$

Since G_x acts trivially on E_x, s_i is well-defined and holomorphic. Now, choose an open Stein neighborhood U of $\pi(x)$. Since π is an analytic Hilbert quotient, $\pi^{-1}(U)$ is an open Stein neighborhood containing Gx. Since Gx is closed in $\pi^{-1}(U)$, by [28, Proposition 3.1.1], Gx is a closed complex subspace of $\pi^{-1}(U)$. Since G acts transitively on Gx, Gx is smooth. Therefore, the natural map $G/G_x \to Gx$ is a biholomorphism. As a consequence, we obtain a G-equivariant map

$$s_i \colon Gx \to E, \qquad s_i(gx) = g\sigma_i, \tag{5.20}$$

which is a holomorphic section of E over Gx. Since Gx is a closed complex subspace of $\pi^{-1}(U)$, and $\pi^{-1}(U)$ is Stein, s_i can be extended to a holomorphic section of E over $\pi^{-1}(U)$. By averaging over a maximal compact subgroup K of G, we may assume that each s_i is K-equivariant. Since G is the complexification of K, the argument in the proof of [51, Theorem 1.1] shows that each s_i is also G-equivariant. Since G_x acts trivially on E_x, $s_i(x) = \sigma_i$, and hence $\{s_i\}$ is linearly independent over an open neighborhood V of x in $\pi^{-1}(U)$. Since each s_i is G-equivariant, V can be chosen to be G-invariant. By [44, Proposition 3.10], we may further assume that $V = \pi^{-1}(U')$ for some smaller open neighborhood $U' \subset U$ of $\pi(x)$ in $\mathcal{M}$.

Now, note that every fiber of π contains a unique closed orbit ([31, §3, Corollary 3]). Therefore, the argument in the above paragraph provides an open covering U_i of $X /\!\!/ G$ such that E is trivial over $\pi^{-1}(U_i)$, and the transition functions $g_{ij} \colon \pi^{-1}(U_i) \cap \pi^{-1}(U_j) \to GL_n(\mathbb{C})$ are G-invariant. As a consequence, by the definition of the structure sheaf of $X /\!\!/ G$, they descend to holomorphic functions $\widetilde{g}_{ij} \colon \pi(\pi^{-1}(U_i) \cap \pi^{-1}(U_j)) \to GL_n(\mathbb{C})$. Since every fiber of π contains a unique closed orbit, $\pi(\pi^{-1}(U_i) \cap \pi^{-1}(U_j)) = U_i \cap U_j$. Then, the data $\{\widetilde{g}_{ij}, U_i\}$ defines a holomorphic vector bundle F over $X /\!\!/ G$. It is easy to see that $\pi^* F = E$ and $\mathcal{O}(F) = \pi_* \mathcal{O}(E)^G$. $\square$

Proof of Theorem J. Fix $(A, \Phi) \in \mathbf{m}^{-1}(0)$. By Theorem 5.1.8, the map

$$\overline{\theta} \colon \mathcal{Z}H^{\mathbb{C}} \times_{H^{\mathbb{C}}} \mathcal{G}^{\mathbb{C}} \to \mathcal{B}^{ss} \tag{5.21}$$

induced by the Kuranishi map $\theta \colon \mathcal{Z}H^{\mathbb{C}} \to \mathcal{B}^{ss}$ for (A, Φ) is a $\mathcal{G}^{\mathbb{C}}$-equivariant homeomorphism onto a π-saturated open neighborhood of (A, Φ) in $\mathcal{B}^{ss}$. Then, we consider the pullback bundle $\overline{\theta}^* \mathbb{L}$ on $\mathcal{Z}H^{\mathbb{C}} \times_{H^{\mathbb{C}}} \mathcal{G}^{\mathbb{C}}$. Clearly, $\theta^* \mathbb{L}$ is the restriction of $\overline{\theta}^* \mathbb{L}$ to $\mathcal{Z}H^{\mathbb{C}}$. By Corollary 3.2.8 and Proposition 4.2.2, if $x \in \mathcal{Z}$ has a closed $H^{\mathbb{C}}$-orbit, then $\theta(x)$ is polystable, and $(H^{\mathbb{C}})_x = (\mathcal{G}^{\mathbb{C}})_{\theta(x)}$. Therefore, $(H^{\mathbb{C}})_x$ acts trivially on the fiber $(\theta^* \mathbb{L})_x = \mathbb{L}_{\theta(x)}$ for every $x \in \mathcal{Z}$ that has a closed $H^{\mathbb{C}}$-orbit. By Proposition 5.2.1, the bundle $\theta^* \mathbb{L}$ descends to $\mathcal{Z}H^{\mathbb{C}} /\!\!/ H^{\mathbb{C}}$. By shrinking $\mathcal{Z}$ if necessary, we may assume that the descended bundle is trivial over $\mathcal{Z}H^{\mathbb{C}} /\!\!/ H^{\mathbb{C}}$. As a consequence, there is a holomorphic frame $\{\sigma_i\}$ for $\theta^*\mathbb{L}$ over $\mathcal{Z}H^{\mathbb{C}}$ such that each σ_i is $H^{\mathbb{C}}$-equivariant. Hence, each section σ_i extends to a $\mathcal{G}^{\mathbb{C}}$-equivariant holomorphic section of $\overline{\theta}^*\mathbb{L}$ over $\mathcal{Z}H^{\mathbb{C}} \times_{H^{\mathbb{C}}} \mathcal{G}^{\mathbb{C}}$. Transported back to $\mathcal{B}^{ss}$ by $\overline{\theta}$, we obtain a local frame $\{\sigma_i\}$ for $\mathbb{L} \to \mathcal{B}^{ss}$ such that each σ_i is a $\mathcal{G}^{\mathbb{C}}$-equivariant holomorphic section over a π-saturated open neighborhood of (A, Φ) in $\mathcal{B}^{ss}$. The rest follows from the second paragraph in the proof of Proposition 5.2.1. □

5.3 The Kähler metric on the moduli space

In this section, we will prove Theorem L. Let us start with the construction of a line bundle on the moduli space $\mathcal{M}$. By [12], there is a holomorphic Hermitian line bundle $\mathbb{L}$ over $\mathcal{A}$ such that the curvature of the Hermitian metric is precisely $-2\pi\sqrt{-1}\Omega_1$, where

$$\Omega_1(\alpha_1, \alpha_2) = \frac{1}{4\pi^2}\int_X \operatorname{tr}(\alpha_1 \wedge \alpha_2), \qquad \alpha_1, \alpha_2 \in \Omega^1(\mathfrak{g}_E). \tag{5.22}$$

Moreover, the $\mathcal{G}^{\mathbb{C}}$-action on $\mathcal{A}$ lifts to $\mathbb{L}$, and the $\mathcal{G}$-action preserves the Hermitian metric. The vertical part of the infinitesimal action of $\xi \in \Omega^0(\mathfrak{g}_E)$ on a smooth section s of $\mathbb{L}$ is given by $2\pi\sqrt{-1}\langle\mu_1, \xi\rangle s$, where $\xi^\#$ is the vector field on $\mathcal{A}$ generated by ξ, and

$$\mu_1(A) = \frac{1}{4\pi^2}F_A. \tag{5.23}$$

On the other hand, we consider the trivial line bundle $\Omega^{1,0}(\mathfrak{g}_E^{\mathbb{C}}) \times \mathbb{C}$ over $\Omega^{1,0}(\mathfrak{g}_E^{\mathbb{C}})$. A Kähler potential on $\Omega^{1,0}(\mathfrak{g}_E^{\mathbb{C}})$ is given by

$$\rho(\Phi) = \frac{1}{8\pi^2}\|\Phi\|_{L^2}^2. \tag{5.24}$$

Letting $\Omega_2 = \sqrt{-1}\partial\overline{\partial}\rho$, we see that $\Omega_1 + \Omega_2 = \Omega_I$ on $\mathcal{C}$. Let $s(\Phi) = (\Phi, 1)$ be the canonical section of the trivial line bundle $\Omega^{1,0}(\mathfrak{g}_E^{\mathbb{C}}) \times \mathbb{C}$. Setting

$$|s|^2 = \exp(-2\pi\rho), \tag{5.25}$$

we obtain a Hermitian metric on the trivial line bundle $\Omega^{1,0}(\mathfrak{g}_E^{\mathbb{C}}) \times \mathbb{C}$ such that its curvature is $-2\pi\sqrt{-1}\Omega_2$. Letting $\mathcal{G}^{\mathbb{C}}$ act on $\mathcal{C}$ trivially, we see that the $\mathcal{G}^{\mathbb{C}}$-action on $\Omega^{1,0}(\mathfrak{g}_E^{\mathbb{C}})$ lifts to the trivial line bundle $\Omega^{1,0}(\mathfrak{g}_E^{\mathbb{C}}) \times \mathbb{C}$. Moreover, the induced $\mathcal{G}$-action preserves the Hermitian metric, since ρ is $\mathcal{G}$-invariant. Finally, the vertical part of the infinitesimal action of $\xi \in \Omega^0(\mathfrak{g}_E)$ on s is given by $2\pi\sqrt{-1}\langle\mu_2, \xi\rangle s$, where

$$\mu_2(\Phi) = \frac{1}{4\pi^2}[\Phi, \Phi^*]. \tag{5.26}$$

Now, we pullback the line bundle $\mathbb{L}$ on $\mathcal{A}$ and the trivial line bundle on $\Omega^{1,0}(\mathfrak{g}_E^{\mathbb{C}})$ to $\mathcal{C} = \mathcal{A} \times \Omega^{1,0}(\mathfrak{g}_E^{\mathbb{C}})$, and denote the resulting line bundle still by $\mathbb{L}$. We equip $\mathbb{L}$ with the product of the pullback Hermitian metrics. As a consequence, the $\mathcal{G}^{\mathbb{C}}$-action on $\mathcal{C}$ lifts to $\mathbb{L}$, and the $\mathcal{G}$-action still preserves the resulting Hermitian metric. The curvature of this Hermitian metric is precisely

$$-2\pi\sqrt{-1}(\Omega_1+\Omega_2) = -2\pi\sqrt{-1}\Omega_I. \tag{5.27}$$

Moreover, the vertical part of the infinitesimal action of $\xi \in \Omega^0(\mathfrak{g}_E)$ on a smooth section s of $\mathbb{L}$ is given by $2\pi\sqrt{-1}\langle \mu_1+\mu_2, \xi\rangle s$, and $\mu_1+\mu_2=\mu$ on $\mathcal{C}$. The following shows that the restriction of the line bundle $\mathbb{L}$ to $\mathcal{B}^{ss}$ descends to the moduli space $\mathcal{M}$.

Proposition 5.3.1. *There is a line bundle $\mathcal{L} \to \mathcal{M}$ such that $\pi^*\mathcal{L} = \mathbb{L}|_{B^{ss}}$, and $\mathcal{O}(\mathcal{L}) = \pi_*\mathcal{O}(\mathbb{L}|_{\mathcal{B}^{ss}})^{\mathcal{G}^{\mathbb{C}}}$.*

Proof. We follow the proof of [58, Proposition 2.14]. By Theorem J, it suffices to prove that $\mathcal{G}^{\mathbb{C}}_{(A,\Phi)}$ acts on $\mathbb{L}_{(A,\Phi)}$ trivially for every $(A,\Phi) \in \mathbf{m}^{-1}(0)$. If $\xi \in \mathrm{Lie}(\mathcal{G}_{(A,\Phi)})$ and $s \in \mathbb{L}_{(A,\Phi)}$, then

$$\xi \cdot s = 2\pi\sqrt{-1}\langle \mu, \xi\rangle s. \tag{5.28}$$

Therefore, if $\mu(A,\Phi)=0$, then $\xi \cdot s = 0$. Since $\mathcal{G}$-stabilizers are connected (Proposition 5.5.1), we conclude that $\mathcal{G}_{(A,\Phi)}$ acts trivially on $\mathbb{L}_{(A,\Phi)}$. Since $(\mathcal{G}^{\mathbb{C}})_{(A,\Phi)}$ is the complexification of $\mathcal{G}_{(A,\Phi)}$, we conclude that $\mathcal{G}^{\mathbb{C}}_{(A,\Phi)}$ acts trivially on $\mathbb{L}_{(A,\Phi)}$. $\square$

Now, we show that the moduli space $\mathcal{M}$ admits a weak Kähler metric. Let Q be a stratum in the orbit type stratification of $\mathcal{M}$. By Theorem F, $i^{-1}(Q)$ is a stratum in the orbit type stratification of $\mathbf{m}^{-1}(0)/\mathcal{G}$, where $i\colon \mathbf{m}^{-1}(0)/\mathcal{G} \xrightarrow{\sim} \mathcal{B}^{ps}/\mathcal{G}^{\mathbb{C}}$ is the Hitchin-Kobayashi correspondence. By Theorem D, $i^{-1}(Q)$ is a hyperKähler manifold. Let $\omega_{i^{-1}Q}$ be the Kähler form on $i^{-1}(Q)$ induced by the Kähler form Ω_I

on $\mathcal{C}$. Then, $(i^{-1})^*\omega_{i^{-1}Q}$ is a Kähler form on Q. Therefore, every stratum in $\mathcal{M}$ is a Kähler manifold.

Let $[A,\Phi]_S \in \mathcal{M}$. By Proposition 5.3.1, we may choose a $\mathcal{G}^{\mathbb{C}}$-equivariant holomorphic section s of $\mathbb{L}$ over an π-saturated open neighborhood $\pi^{-1}(U)$ of (A,Φ) in $\mathcal{B}^{ss}$ such that s vanishes nowhere in $\pi^{-1}(U)$, where U is an open neighborhood of $[A,\Phi]_S$ in $\mathcal{M}$. Then, we define

$$u = -\frac{1}{2\pi}\log|s|_h^2, \tag{5.29}$$

where h is the Hermitian metric on $\mathbb{L}$. Since h is preserved by the $\mathcal{G}$-action, u is $\mathcal{G}$-invariant. As a consequence, the restriction of u to $\pi^{-1}(U)\cap\mathbf{m}^{-1}(0)$ induces a well-defined continuous function $u_0\colon U \to \mathbb{R}$.

Proposition 5.3.2. *The function u_0 is continuous and smooth along each stratum Q. Moreover, $u_0|_Q$ is a Kähler potential for the Kähler form on each stratum Q in $\mathcal{M}$. In particular, u_0 is a continuous plurisubharmonic function.*

Proof. By Theorem D, $\pi^{-1}(Q)\cap\mathbf{m}^{-1}(0) \to Q$ is a submersion. Hence, u_0 is smooth along Q. By construction of u and the Hermitian metric h on $\mathbb{L}$,

$$\sqrt{-1}\partial\overline{\partial}(u|_{\pi^{-1}(Q)\cap\mathbf{m}^{-1}(0)}) = \Omega_I|_{\pi^{-1}(Q)\cap\mathbf{m}^{-1}(0)}. \tag{5.30}$$

Therefore, the second statement follows from the construction of the Kähler form on Q. Now we have shown that the restriction of u_0 to $\mathcal{M}^s$ is strictly plurisubharmonic. Since it is continuous, by the normality of $\mathcal{M}$ and the extension theorem of plurisubharmonic functions (see [22]), we conclude that u_0 is plurisubharmonic. $\square$

Proof of Theorem L. By Proposition 5.3.2, there is an open covering U_i of $\mathcal{M}$, and a stratum-wise strictly plurisubharmonic function $\rho_i \colon U_i \to \mathbb{R}$ on each U_i such that $\rho_i|_{\mathcal{M}^s} - \rho_j|_{\mathcal{M}^s}$ is pluriharmonic on $\mathcal{M}^s \cap U_i \cap U_j$. Hence, we may write

$$\rho_i|_{\mathcal{M}^s} - \rho_j|_{\mathcal{M}^s} = \Re(f_{ij}) \tag{5.31}$$

for some holomorphic function $f_{ij} \colon U_i \cap U_j \cap \mathcal{M}^s \to \mathbb{C}$. By Corollary 5.5.4 and the normality of $\mathcal{M}$, f_{ij} has a unique holomorphic extension to $U_i \cap U_j$. Then, we have

$$\rho_i - \rho_j = \Re(f_{ij}) \text{ on } U_i \cap U_j. \tag{5.32}$$

Hence, $\{U_i, \rho_i\}$ determines a weak Kähler metric on $\mathcal{M}$. $\square$

5.4 Projective compactification

5.4.1 Symplectic cuts

In this section, we will use the symplectic cut to compactify the moduli space and thus prove Theorem H. Recall that there is a holomorphic $\mathbb{C}^*$-action on $\mathcal{C}$ given by

$$t \cdot (A, \Phi) = (A, t\Phi), \qquad t \in \mathbb{C}^*, (A, \Phi) \in \mathcal{C}. \tag{5.33}$$

Clearly, $\mathcal{B}^{ss}$ is $\mathbb{C}^*$-invariant. Then, it is easy to verify that the natural map $\mathbb{C}^* \times \mathcal{B}^{ss} \to \mathbb{C}^* \times \mathcal{M}$ satisfies (4) in Theorem I, where $\mathcal{G}^{\mathbb{C}}$ acts on $\mathbb{C}^*$ trivially. Since the

$\mathcal{G}^{\mathbb{C}}$-action and the $\mathbb{C}^*$-action on $\mathcal{C}$ commute, we see that the holomorphic action $\mathbb{C}^* \times \mathcal{B}^{ss} \to \mathcal{B}^{ss}$ descends to a holomorphic action $\mathbb{C}^* \times \mathcal{M} \to \mathcal{M}$. Moreover, each stratum in the orbit type stratification of $\mathcal{M}$ is $\mathbb{C}^*$-invariant.

Furthermore, the induced $U(1)$-action on $\mathcal{M}$ is stratum-wise Hamiltonian. To see this, we first note that $U(1)$ preserves the Kähler form Ω_I on $\mathcal{C}$. Then, consider the function $f\colon \mathcal{C} \to \mathbb{R}$ given by

$$f(A, \Phi) = -\frac{1}{4\pi^2}\frac{1}{2}\|\Phi\|^2_{L^2}. \tag{5.34}$$

Proposition 5.4.1. *The restriction of f to $\mathbf{m}^{-1}(0)$ defines a continuous function, denoted by the same letter f, on $\mathcal{M}$ that is smooth along each stratum Q of $\mathcal{M}$. Moreover, the restriction $f|_Q$ is a moment map for the $U(1)$-action on Q with respect to the Kähler form on Q, the one induced by the Kähler form Ω_I on $\mathcal{C}$.*

Proof. It is shown in [33, p.92] that f is a moment map for the $U(1)$-action on $\mathcal{C}$ with respect to the Kähler form Ω_I. Since f is $\mathcal{G}$-invariant, its restriction to $\mathbf{m}^{-1}(0)$ descends to $\mathbf{m}^{-1}(0)/\mathcal{G}$ and hence defines a continuous function on $\mathcal{M}$, which we denote by the same letter f. Let Q be a stratum in the moduli space. By Theorem D, the restriction of f to $\pi^{-1}(Q) \cap \mathbf{m}^{-1}(0)$ descends to a smooth function on Q which is precisely the restriction of $f\colon \mathcal{M} \to \mathbb{R}$ to Q. Since the quotient map $\pi^{-1}(Q) \cap \mathbf{m}^{-1}(0) \to Q$ is $U(1)$-equivariant, we conclude that $f|_Q$ is a moment map for the $U(1)$-action on Q. □

To perform the symplectic cut of $\mathcal{M}$, we consider the direct product $\mathcal{M} \times \mathbb{C}$ and let $\mathbb{C}^*$ act on $\mathbb{C}$ by multiplication. Hence, $\mathbb{C}^*$ acts diagonally on $\mathcal{M} \times \mathbb{C}$. Moreover, $\mathcal{M} \times \mathbb{C}$ admits a stratification such that each stratum $Q \times \mathbb{C}$ is equipped with the product Kähler form. The next result implies that the induced $U(1)$ action on $\mathcal{M} \times \mathbb{C}$ is also stratum-wise Hamiltonian.

Proposition 5.4.2. *The continuous map*

$$\widetilde{f}([A, \Phi]_S, z) = f([A, \Phi]_S) - \frac{1}{2}\|z\|^2 \tag{5.35}$$

is smooth along each stratum $Q \times \mathbb{C}$, and its restriction to $Q \times \mathbb{C}$ is a moment map for the induced $U(1)$-action on $Q \times \mathbb{C}$ with respect to the product Kähler form on $Q \times \mathbb{C}$.

Proof. It is clear that $\widetilde{f}$ is continuous on $\mathcal{M} \times \mathbb{C}$. For each stratum Q, Proposition 5.4.1 implies that $f|_Q$ is a smooth moment map on Q. Since $U(1)$ acts diagonally on $Q \times \mathbb{C}$, it is easy to see that $\widetilde{f}|_{Q\times\mathbb{C}}$ is a moment map for the $U(1)$-action with respect to the product Kähler form on $Q \times \mathbb{C}$. Therefore, $\widetilde{f}$ is a stratum-wise moment map. □

Now we recall the definition of the Hitchin fibration. Given a Higgs bundle (A, Φ), the coefficient of λ^{n-i} in the characteristic polynomial $\det(\lambda + \Phi)$ is a holomorphic section of $\mathcal{K}_M^i$, where n is the rank of $\mathcal{E}$, $i = 1, \cdots, n$, and $\mathcal{K}_M$ is the canonical bundle on the Riemann surface M. Since these sections are clearly $\mathcal{G}^{\mathbb{C}}$-invariant, by Theorem I, we have obtained a well-defined holomorphic map, called the Hitchin fibration,

$$h\colon \mathcal{M} \to \bigoplus_{i=1}^{n} H^0(M, \mathcal{K}_M^i). \tag{5.36}$$

It is known that h is proper (see [66, Theorem 2.15] or [33, Theorem 8.1]). Therefore, the nilpotent cone $h^{-1}(0)$ is compact so that f has a lower bound on $h^{-1}(0)$. We choose a constant $c < 0$ such that $h^{-1}(0) \subset f^{-1}(c, 0]$. In other words, $f^{-1}(-\infty, c]$ does not contain the nilpotent cone. Then, we perform the symplectic cut of $\mathcal{M}$ at the level c. By definition, it is the singular symplectic quotient

$$\widetilde{f}^{-1}(c)/U(1) = \Big\{([A,\Phi]_S, z) \in \mathcal{M} \times \mathbb{C}\colon f([A,\Phi]_S) - \frac{1}{2}\|z\|^2 = c\Big\}/U(1). \tag{5.37}$$

If $\mathcal{M} \times \mathbb{C}$ admits a (strong) Kähler metric, then we may directly apply the analytic GIT developed in [30]. Since we are unable to prove this, we will have to take a detour to prove that the symplectic cut of $\mathcal{M}$ at the level c is a compact complex space.

Let $W = (\mathcal{M} \times \mathbb{C}) \setminus (h^{-1}(0) \times \{0\})$. It is clear that W is $\mathbb{C}^*$-invariant and open. We first show that the analytic Hilbert quotient $W/\mathbb{C}^*$ exists.

Lemma 5.4.3. *The $\mathbb{C}^*$-action on $\mathcal{M} \setminus h^{-1}(0)$ is proper.*

Proof. Clearly, $h^{-1}(0)$ is $\mathbb{C}^*$-invariant. Suppose that

1. x_i converges to $x' \notin h^{-1}(0)$, and

2. $t_i \cdot x_i$ converges to $y \notin h^{-1}(0)$.

We first claim that $|t_i|$ cannot be unbounded. If not, we may assume that $|t_i| \to \infty$ and let $t_i' = t_i/|t_i|$. By passing to a subsequence, we may assume that t_i' converges to t'_∞, and $|t'_\infty| = 1$. As a consequence, since $x' \notin h^{-1}(0)$,

$$\lim_{i\to\infty} t'_i \cdot x_i = t'_\infty \cdot x' \notin h^{-1}(0). \tag{5.38}$$

On the other hand, since $t_i \cdot x_i$ converges,

$$\lim_{i\to\infty} h\left(\frac{1}{|t_i|} t_i \cdot x_i\right) = 0. \tag{5.39}$$

Since h is proper, by passing to a subsequence, we may assume that

$$\frac{1}{|t_i|} t_i \cdot x_i = t'_i \cdot x_i \tag{5.40}$$

converges to an element in $h^{-1}(0)$. This is a contradiction.

Since t_i is bounded, it has a subsequence that is convergent. We claim that such a sequence cannot converge to 0. If not, suppose that $t_i \to 0$. Then,

$$\lim_{i\to\infty} h(t_i \cdot x_i) = 0, \tag{5.41}$$

and hence $t_i \cdot x_i$ has a subsequence converging to an element in $h^{-1}(0)$. Therefore, $y \in h^{-1}(0)$. This is a contradiction. $\square$

Corollary 5.4.4. *The $\mathbb{C}^*$-action on W is proper.*

Proof. Note that

$$W = (\mathcal{M} \setminus h^{-1}(0) \times \mathbb{C}) \cup (\mathcal{M} \times \mathbb{C}^*). \tag{5.42}$$

Suppose there are sequences

1. $(x_i, a_i) \in W$ converging to $(x', a') \in W$, and

2. $t_i \cdot (x_i, a_i) \in W$ converging to $(y, b) \in W$.

We need to show that t_i has a subsequence that converges in $\mathbb{C}^*$. We prove this by considering the following cases:

1. Suppose $(x', a') \in \mathcal{M} \times \mathbb{C}^*$. Then, $(x_i, a_i) \in \mathcal{M} \times \mathbb{C}^*$ if $i \gg 0$. Hence, $t_i = (t_i a_i) a_i^{-1}$ converges to ba'^{-1}. If $b \neq 0$, then $ba'^{-1} \in \mathbb{C}^*$, and we are done with this case. If $b = 0$, then $y \notin h^{-1}(0)$. Moreover, $\lim_{i\to\infty} h(t_i x_i) = 0$. Then, $t_i x_i$ has a subsequence converging to an element in $h^{-1}(0)$. Since this element has to be y, we have shown that $b = 0$ is impossible.

2. Suppose that $(x', a') \in (\mathcal{M} \setminus h^{-1}(0)) \times \mathbb{C}$. Then, both (x_i, a_i) and $t_i \cdot (x_i, a_i)$ lie in $(\mathcal{M} \setminus h^{-1}(0)) \times \mathbb{C}$ if $i \gg 0$. If $y \notin h^{-1}(0)$, then Lemma 5.4.3 applies. Hence, we may assume that $y \in h^{-1}(0)$ and hence $b \neq 0$. If $a' \neq 0$, then $t_i = (t_i a_i) a_i^{-1}$ converges to ba'^{-1}, and we are done. Hence, we may assume that $a' = 0$, and therefore

$$\begin{aligned} t_i a_i &\to b \neq 0, \\ a_i &\to a' = 0. \end{aligned} \tag{5.43}$$

We claim that t_i is bounded, so that $t_i a_i$ converges to 0, which is a contradiction. If not, we may assume that $|t_i| \to \infty$ and let $t_i' = t_i / |t_i|$. By passing to a subsequence, we may further assume that t_i' converges to t_∞' with $|t_\infty'| = 1$.

As a consequence, $t_i'x_i$ converges to $t_\infty' x' \notin h^{-1}(0)$. On the other hand,

$$\lim_{i\to\infty} h(t_i'x_i) = \lim_{i\to\infty} h\left(\frac{1}{|t_i|}t_ix_i\right) = 0. \tag{5.44}$$

Hence, the properness of h implies that $t_i'x_i$ contains a subsequence converging to an element in $h^{-1}(0)$, which is a contradiction.

□

Corollary 5.4.5. *The analytic Hilbert quotient of W by $\mathbb{C}^*$ exists. Moreover, $W/\mathbb{C}^*$ is a geometric quotient.*

Proof. This follows from Corollary 5.4.4 and [31, §4, Corollary 2]. □

Then, we study the relationship between the symplectic cut $\widetilde{f}^{-1}(c)/U(1)$ and the analytic Hilbert quotient $W/\mathbb{C}^*$. Let $(\mathcal{M}\times\mathbb{C})^{ss}$ be the semistable points in $\mathcal{M}\times\mathbb{C}$ determined by $\widetilde{f}-c$. In other words, $([A,\Phi]_S, z)$ lies in $(\mathcal{M}\times\mathbb{C})^{ss}$ if and only if the closure of its $\mathbb{C}^*$-orbit in $\mathcal{M}\times\mathbb{C}$ intersects $\widetilde{f}^{-1}(c)$.

Lemma 5.4.6. $W = (\mathcal{M}\times\mathbb{C})^{ss} = \mathbb{C}^*\cdot\widetilde{f}^{-1}(c)$.

Proof. We first show that $\widetilde{f}^{-1}(c) \subset W$. Suppose that this is not true, and we choose some $([A,\Phi]_S, z) \in \widetilde{f}^{-1}(c)$ such that $([A,\Phi]_S, z) \notin W$. In other words, $[A,\Phi]_S \in h^{-1}(0)$ and $z = 0$. Hence, $\widetilde{f}([A,\Phi]_S, z) = f([A,\Phi]_S) = c$. This cannot happen by the choice of the level c.

Then, we show that $(\mathcal{M}\times\mathbb{C})^{ss} \subset W$. If the closure of the $\mathbb{C}^*$-orbit of a point $([A,\Phi]_S, z)$ in $\mathcal{M}\times\mathbb{C}$ meets $\widetilde{f}^{-1}(c)$, then it must meet W, since W is open. Since W is also $\mathbb{C}^*$-invariant, W contains $([A,\Phi]_S, z)$.

Finally, we show that $W \subset (\mathcal{M} \times \mathbb{C})^{ss}$. For every $([A, \Phi]_S, z) \in W$, consider the function

$$q(t) = f([A, t\Phi]) - \frac{1}{2}t^2\|z\|^2 - c, \qquad t > 0. \tag{5.45}$$

We show that $q(t_0) = 0$ for some $t_0 > 0$ so that $t \cdot ([A, \Phi]_S, z)$ lies in $\widetilde{f}^{-1}(c)$ for some $t > 0$. Since $h([A, t\Phi]) \to 0$ as $t \to 0$, the properness of h implies that there exists a sequence $t_n \subset \mathbb{C}^*$ such that $t_n \to 0$ and $[A, t_n\Phi]$ converges to some $[B, \Psi]_S \in h^{-1}(0)$. Hence, letting $n \to \infty$, we see that

$$\lim_{n\to\infty} q(t_n) = f([B, \Psi]_S) - c > 0. \tag{5.46}$$

On the other hand, since $f \leq 0$, we have

$$q(t) \leq -\frac{1}{2}t^2\|z\|^2 - c. \tag{5.47}$$

If $z \neq 0$, then $t \gg 0$ implies that $q(t) < 0$. Hence, $q(t_0) = 0$ for some $t_0 > 0$.

Now, we assume that $z = 0$ so that $[A, \Phi]_S \notin h^{-1}(0)$. We claim that the function $t \mapsto f([A, t\Phi]_S)$ is unbounded below as $t \to \infty$. If this claim is true, then $q(t) < 0$ if $t \gg 0$, and hence $q(t_0) = 0$ for some $t_0 > 0$. Now, we prove the claim. Assuming the contrary, we may choose a sequence of $\{t_n\} \subset \mathbb{C}^*$ such that $t_n \to \infty$ and $f([A, t_n\Phi]_S)$ is bounded. By the properness of f, by passing to a subsequence, we may assume that $[A, t_n\Phi]_S$ converges to some $[B, \Psi]_S$. Hence, $h([A, t_n\Phi]_S)$ also converges as $t_n \to \infty$. This implies that $h([A, \Phi]_S) = 0$, which is a contradiction.

Finally, note that the proof has already shown that $(\mathcal{M}\times\mathbb{C})^{ss} = \mathbb{C}^*\cdot\widetilde{f}^{-1}(0)$. $\square$

Corollary 5.4.7. *The inclusion $\widetilde{f}^{-1}(c) \hookrightarrow (\mathcal{M} \times \mathbb{C})^{ss}$ induces a homeomorphism*

$$\widetilde{f}^{-1}(c)/U(1) \xrightarrow{\sim} (\mathcal{M} \times \mathbb{C})^{ss}/\mathbb{C}^* = W/\mathbb{C}^*. \tag{5.48}$$

Moreover, $W/\mathbb{C}^*$ is compact.

Proof. Since f and the norm $\|\cdot\|$ on $\mathbb{C}$ are proper, $f^{-1}(c)$ is compact. Therefore, $\widetilde{f}^{-1}(c)/U(1)$ is also compact. Moreover, since $(\mathcal{M} \times \mathbb{C})^{ss}/\mathbb{C}^*$ is Hausdorff, to show that the map is a homeomorphism, it suffices to show that it is a continuous bijection. The continuity is obvious. By Lemma 5.4.6, the surjectivity is clear.

To show the injectivity, suppose that $([A_1, \Phi_1], z_1)$ and $([A_2, \Phi_2], z_2)$ lie in $\widetilde{f}^{-1}(c)$ and the same $\mathbb{C}^*$-orbit. Since each orbit type stratum in $\mathcal{M}$ is $\mathbb{C}^*$-invariant, they lie in $Q \times \mathbb{C}$ for some stratum Q in $\mathcal{M}$. By Proposition 5.4.2, $\widetilde{f}|_{Q \times \mathbb{C}}$ is a moment map for the $U(1)$-action on $Q \times \mathbb{C}$ with respect to the product Kähler form on $Q \times \mathbb{C}$. Hence, $([A_1, \Phi_1], z_1)$ and $([A_2, \Phi_2], z_2)$ must lie in the same $U(1)$-orbit by general properties of moment maps (see [38, Lemma 7.2]). $\square$

Proof of Theorem H. Write $W = (\mathcal{M} \setminus h^{-1}(0) \times \{0\}) \cup (\mathcal{M} \times \mathbb{C}^*)$. Note that it is a disjoint union. Let $W^* = \mathcal{M} \times \mathbb{C}^*$ and consider the map

$$W^* \to \mathcal{M}, \qquad ([A, \Phi]_S, z) \mapsto z^{-1}[A, \Phi]_S. \tag{5.49}$$

Since it is $\mathbb{C}^*$-invariant, it induces a well-defined map $(\mathcal{M} \times \mathbb{C}^*)/\mathbb{C}^* \to \mathcal{M}$. The injectivity is clear. Its inverse is given by $[A, \Phi]_S \mapsto ([A, \Phi]_S, 1)$.

Then, we show that it is a biholomorphism. Since $\mathcal{M}$ is normal, $\mathcal{M} \times \mathbb{C}$ is also normal. Therefore, both W and W^* are normal. As categorical quotients of normal spaces, $W^*/\mathbb{C}^*$ and $W/\mathbb{C}^*$ are also normal. Moreover, fibers of $W^* \to W^*/\mathbb{C}^*$ have pure dimension 1. Since $\mathcal{M}^s$ is pure dimensional, Proposition 5.5.3 implies that $\mathcal{M}$ and hence W^* are pure dimensional. Therefore, by Remmert's rank theorem (see [1, Proposition 1.21]), we conclude that $W^*/\mathbb{C}^*$ is pure dimensional, and

$$\dim W^*/\mathbb{C}^* = \dim W^* - \dim \mathbb{C}^* = \dim \mathcal{M}. \tag{5.50}$$

Then, by [23, p.166, Theorem], the map $W^*/\mathbb{C}^* \to \mathcal{M}$ is a biholomorphism.

Since $W/\mathbb{C}^*$ is compact, we have shown that $\mathcal{M}$ admits a compactification

$$W/\mathbb{C}^* = \mathcal{M} \cup Z, \tag{5.51}$$

where $Z = (\mathcal{M} \setminus h^{-1}(0) \times \{0\})/\mathbb{C}^*$ is of pure codimension 1. $\square$

Finally, we prove the following result that will be used later. Note that W inherits a stratification from $\mathcal{M} \times \mathbb{C}$. More precisely, W is a disjoint union of $Q_W = W \cap (Q \times \mathbb{C})$ where Q ranges in the stratification of $\mathcal{M}$. Moreover, if necessary, we may also refine this stratification into connected components. The following shows that how $Q_W/\mathbb{C}^*$ fits together in $\overline{\mathcal{M}}$.

Proposition 5.4.8. *Let $\pi\colon W \to \overline{\mathcal{M}}$ be the quotient map.*

1. *Each $\pi(\overline{Q_W})$ is a closed complex subspace of $\overline{\mathcal{M}}$, where the closure is taken in W.*

2. *Each $\pi(Q_W)$ is a locally closed complex subspace of $\overline{\mathcal{M}}$, and its closure is precisely $\pi(\overline{Q_W})$.*

3. *If $\pi(Q_W) \cap \overline{\pi(S_W)} \neq \emptyset$, then $\pi(Q_W) \subset \overline{\pi(S_W)}$.*

4. *$\overline{\mathcal{M}}$ is a disjoint union of $\pi(Q_W)$.*

5. *The restriction $\pi\colon Q_W \to \pi(Q_W)$ is the analytic Hilbert quotient of Q_W by $\mathbb{C}^*$. Moreover, the inclusion $(\widetilde{f}|_{Q_W})^{-1}(c) \hookrightarrow Q_W$ induces a homeomorphism*

$$(\widetilde{f}|_{Q_W})^{-1}(c)/U(1) \xrightarrow{\sim} Q_W/\mathbb{C}^*. \tag{5.52}$$

Proof. Fix a stratum Q in $\mathcal{M}$. By Proposition 5.5.3, $\overline{Q_W}$ is a closed complex subspace of W that is also $\mathbb{C}^*$-invariant. Therefore, [31, §1(ii)] implies that $\pi(\overline{Q_W})$ is a closed complex subspace of $\overline{M}$. Moreover, the restriction $\pi\colon \overline{Q_W} \to \pi(\overline{Q_W})$ is also an analytic Hilbert quotient. This proves (1).

Since Q_W is open in $\overline{Q_W}$, and $\pi\colon \overline{Q_W} \to \pi(\overline{Q_W})$ is an open map, $\pi(Q_W)$ is open in $\pi(\overline{Q_W})$. Moreover, the continuity of π shows that $\pi(\overline{Q_W}) \subset \overline{\pi(Q_W)}$. Since $\pi(\overline{Q_W})$ is closed in $\overline{M}$, we have $\pi(\overline{Q_W}) = \overline{\pi(Q_W)}$. This proves (2).

If $\pi(Q_W) \cap \overline{\pi(S_W)} \neq \emptyset$ for some stratum S in $\mathcal{M}$, then $\pi(Q_W) \cap \pi(\overline{S_W}) \neq \emptyset$. Since $\pi\colon W \to \overline{M}$ is a geometric quotient, and both Q_W and $\overline{S_W}$ are $\mathbb{C}^*$-invariant, we conclude that $Q_W \cap \overline{S_W} \neq \emptyset$. Therefore, $Q_W \subset \overline{S_W}$, and hence $\pi(Q_W) \subset \overline{\pi(S_W)}$. This shows (3).

Obviously, $\overline{\mathcal{M}}$ is a union of $\pi(Q_W)$ as Q ranges in the stratification of $\mathcal{M}$. Since each Q_W is $\mathbb{C}^*$-invariant, and $\pi\colon W \to \overline{\mathcal{M}}$ is a geometric quotient, it is a disjoint union. This proves (4).

Finally, (5) immediately follows from Corollary 5.4.7 and the fact that Q_W is $\mathbb{C}^*$-invariant. $\square$

5.4.2 Projectivity

In this section, we will prove (2) and (3) in Theorem K. Let us start with the construction of a line bundle on $\overline{\mathcal{M}} = W/\mathbb{C}^*$. Note that the $\mathbb{C}^*$-action on $\mathcal{M}$ lifts to the line bundle $\mathcal{L} \to \mathcal{M}$. This can be seen as follows. The $\mathbb{C}^*$-action on $\Omega^{1,0}(\mathfrak{g}_E^{\mathbb{C}})$ lifts to the trivial line bundle by letting $\mathbb{C}^*$ act on the fiber trivially. By construction of the line bundle $\mathbb{L} \to \mathcal{C}$, the $\mathbb{C}^*$-action on $\mathcal{C}$ lifts to $\mathbb{L}$. Since the $\mathcal{G}^{\mathbb{C}}$-action and the $\mathbb{C}^*$-action commutes, we see that the $\mathbb{C}^*$-action on $\mathbb{L}$ descends to $\mathcal{L}$ which covers the $\mathbb{C}^*$-action on $\mathcal{M}$. Then, consider the trivial line bundle over $\mathbb{C}$. The $\mathbb{C}^*$-action on $\mathbb{C}$ lifts to the trivial line bundle by letting $\mathbb{C}^*$ act on the fiber trivially. Moreover, we equip the trivial line bundle with a Hermitian metric determined by

$$|s|^2 = \exp(-2\pi\chi), \tag{5.53}$$

where $s(z) = (z, 1)$ is a section of the trivial line bundle, and $\chi(z) = \frac{1}{2}\|z\|^2$ is a Kähler potential for the standard Kähler form on $\mathbb{C}$.

Now, we pullback the trivial line bundle on $\mathbb{C}$ and the line bundle $\mathcal{L} \to \mathcal{M}$ to $\mathcal{M} \times \mathbb{C}$, and denote the resulting line bundle by $\mathcal{L}_{\mathbb{C}}$. Moreover, we equip the line bundle $\mathcal{L}_{\mathbb{C}} \to \mathcal{M} \times \mathbb{C}$ with the product of the pullback Hermitian metrics, and the $\mathbb{C}^*$-action on $\mathcal{M} \times \mathbb{C}$ lifts to $\mathcal{L}_{\mathbb{C}}$. We will still use the letter h to denote the resulting

Hermitian metric on $\mathcal{L}_{\mathbb{C}}$.

Proposition 5.4.9. *The Hermitian metric h on $\mathcal{L}_{\mathbb{C}}$ is smooth along $Q \times \mathbb{C}$ for every stratum Q in $\mathcal{M}$. Moreover, its curvature on $Q \times \mathbb{C}$ is precisely $-2\pi\sqrt{-1}\omega_{Q\times\mathbb{C}}$.*

Proof. By Proposition 5.3.2, the curvature of the Hermitian metric on $\mathcal{L}$ along Q is $-2\pi\sqrt{-1}\omega_Q$. By the construction of h on $\mathcal{L}_{\mathbb{C}}$, we see that the curvature of h along $Q \times \mathbb{C}$ must be $-2\pi\sqrt{-1}\omega_{Q\times\mathbb{C}}$. □

By construction, if $p = ([A, \Phi]_S, z) \in \mathcal{M}\times\mathbb{C}$, then $(\mathbb{C}^*)_p$ acts trivially on $(\mathcal{L}_{\mathbb{C}})_p$. As a consequence, we obtain the following.

Proposition 5.4.10. *The canonical line bundle $\mathcal{L}_{\mathbb{C}} \to \mathcal{M} \times \mathbb{C}$ descends to $\overline{\mathcal{M}}$. In other words, there is a line bundle $\overline{\mathcal{L}} \to \overline{\mathcal{M}}$ such that $\pi^*\overline{\mathcal{L}} = \mathcal{L}_{\mathbb{C}}|_W$, where $\pi\colon W \to W/\mathbb{C}^* = \overline{\mathcal{M}}$ is the quotient map.*

Proof. This follows from Proposition 5.2.1. □

Since the Hermitian metric h on $\mathcal{L}_{\mathbb{C}}$ is preserved by the $U(1)$-action, Corollary 5.4.7 implies that h induces a continuous Hermitian metric $\overline{h}$ on $\overline{\mathcal{L}} \to \overline{\mathcal{M}}$. Although h is smooth along Q_W for each stratum Q in $\mathcal{M}$, $\overline{h}$ may not be smooth along $\pi(Q_W)$. That said, we note that Q_W is a Kähler manifold with a $\mathbb{C}^*$-action such that the induced $U(1)$-action is Hamiltonian with respect to the Kähler form ω_{Q_W} on Q_W. Hence, by [58, Theorem 2.10] and Proposition 5.4.8, we may further stratify $\pi(Q_W)$ by $\mathbb{C}^*$-orbit types. Since the curvature of $(\mathcal{L}_{\mathbb{C}}, h)$ on Q_W is $-2\pi\sqrt{-1}\omega_{Q_W}$, by [58, Lemma 2.16], we conclude the following.

Proposition 5.4.11. *For each stratum Q in $\mathcal{M}$, $\pi(Q_W)$ admits a $\mathbb{C}^*$-orbit type stratification such that each stratum S is a locally closed Kähler submanifold of $\pi(Q_W)$ with Kähler form ω_S. Moreover, the Hermitian metric $\overline{h}$ on $\overline{\mathcal{L}}$ is smooth along S, and the its curvature on S is precisely $-2\pi\sqrt{-1}\omega_S$.*

Now we are ready to prove that the line bundle $\overline{\mathcal{L}} \to \overline{\mathcal{M}}$ is ample. The first step is the following.

Lemma 5.4.12. *The Chern current $c_1(\overline{\mathcal{L}}, \overline{h})$ of $(\overline{\mathcal{L}}, \overline{h})$ is positive, where*

$$c_1(\overline{\mathcal{L}}, \overline{h}) = \frac{\sqrt{-1}}{2\pi}\overline{\partial}\partial \log |s|^2_{\overline{h}}, \tag{5.54}$$

and s is any local holomorphic section of $\overline{\mathcal{L}}$ that is nowhere vanishing.

Proof. By Proposition 5.4.10, for every open subset U of $\overline{\mathcal{M}}$, we may choose a $\mathbb{C}^*$-equivariant holomorphic section s of $\mathcal{L}_{\mathbb{C}}$ over $\pi^{-1}(U)$ that is nowhere vanishing, where $\pi \colon W \to \overline{\mathcal{M}}$ is the quotient map. Then, we define

$$v = -\frac{1}{2\pi} \log |s|^2_h. \tag{5.55}$$

Since v is $U(1)$-invariant, by Corollary 5.4.7, the restriction of v to $\pi^{-1}(U) \cap \widetilde{f}^{-1}(c)$ induces a well-defined continuous function $v_0 \colon U \to \mathbb{R}$. If $Q = \mathcal{M}^s$, then Proposition 5.4.8 and 5.4.11 imply that $\pi(Q_W)$ is open in $\overline{\mathcal{M}}$ and that $\pi(Q_W)$ admits a $\mathbb{C}^*$-orbit type stratification. Moreover, if S is the top-dimensional stratum, then S is open and dense in $\pi(Q_W)$. By Proposition 5.4.11 again, the restriction of v_0 to S is a Kähler potential for the Kähler form on S so that $v_0|_S$ is strictly plurisubharmonic. Since v_0 is already continuous, and $\overline{\mathcal{M}}$ is normal, the extension theorem of plurisubharmonic functions (see [22]) implies that $v_0\colon U \to \mathbb{R}$ is plurisubharmonic. Since $c_1(\overline{\mathcal{L}}, \overline{h}) = \sqrt{-1}\partial\overline{\partial}v_0$, we see that $c_1(\overline{\mathcal{L}}, \overline{h})$ is positive. □

Then, the key result to show that $\overline{\mathcal{L}}$ is ample is the following.

Lemma 5.4.13. *For every closed irreducible complex subspace Y of $\overline{\mathcal{M}}$ with $\dim Y > 0$, the restriction of the line bundle $\overline{\mathcal{L}} \to \overline{\mathcal{M}}$ to Y is big.*

Proof. By (3) in Proposition 5.4.8, there is a natural partial order among $\pi(Q_W)$, where Q ranges in the stratification of $\mathcal{M}$. We define $\pi(Q_W) \leq \pi(S_W)$ if $\pi(Q_W) \subset \overline{\pi(S_W)}$. If Y is a closed irreducible complex subspace, Y must intersect some $\pi(Q_W)$. We choose $\pi(Q_W)$ to be the largest one with respect to the partial order $\leq$ just mentioned. By (3) in Proposition 5.4.8 again, $\pi(Q_W)$ is open in $\overline{\mathcal{M}} \backslash \cup_{\pi(S_W) > \pi(Q_W)} \pi(S_W)$. Therefore, $\pi(Q_W) \cap Y$ is open in Y. By Proposition 5.4.11, $\pi(Q_W)$ admits a $\mathbb{C}^*$-orbit type stratification. Similarly, we may further choose a stratum S in $\pi(Q_W)$ such that $Y \cap \pi(Q_W) \cap S = Y \cap S$ is open in $Y \cap \pi(Q_W)$. Therefore, $Y_{reg} \cap S$ is also open in Y, where Y_{reg} is the smooth locus of Y.

Now we consider the restriction of $\overline{\mathcal{L}}$ to Y. We will use [49, Theorem 1.3] to show that $\overline{\mathcal{L}}|_Y$ is big. By taking a desingularization of Y, we may assume that Y is a compact complex manifold. Clearly, the Chern current $c_1(\overline{\mathcal{L}}, \overline{h})$ is still positive. Therefore, by Lebesgue's decomposition theorem, the absolutely continuous part $c_1(\overline{\mathcal{L}}, \overline{h})_{ac}$ of $c_1(\overline{\mathcal{L}}, \overline{h})$ is also positive. Hence,

$$\int_Y c_1(\overline{\mathcal{L}}, \overline{h})_{ac}^{\dim Y} \geq \int_{Y_{reg} \cap S} c_1(\overline{\mathcal{L}}, \overline{h})_{ac}^{\dim Y} > 0 \tag{5.56}$$

To justify the last inequality, we note that $Y_{reg} \cap S$ is a complex submanifold of S and hence Kähler. By Proposition 5.4.11, $c_1(\overline{\mathcal{L}}, \overline{h})$ is the Kähler form on S. Therefore, the restriction of $c_1(\overline{\mathcal{L}}, \overline{h})_{ac}^{\dim Y}$ to $Y_{reg} \cap S$ is precisely the volume form on $Y_{reg} \cap S$. Hence, the last inequality in Equation (5.56) holds. $\square$

Proof of Theorem K. Note that (1) in Theorem K is already proved in Proposition 5.3.1. To prove (3), we use Grauert's criterion of ampleness for a line bundle over a compact complex space (see [21]). Therefore, we need to show that the restriction of $\overline{\mathcal{L}}$ to any irreducible closed complex subspace Y with $\dim Y > 0$ admits a nontrivial holomorphic section that vanishes somewhere on Y. Let Y be an irreducible closed complex subspace of $\overline{\mathcal{M}}$ with $\dim Y > 0$. By Lemma 5.4.13, $\overline{\mathcal{L}}|_Y$ is big. Hence, it admits a nontrivial holomorphic section. Such a section must vanish somewhere on Y. Otherwise, $\overline{\mathcal{L}}|_Y$ is holomorphically trivial and cannot be big. Therefore, $\overline{\mathcal{L}}$ is ample, and $\overline{\mathcal{M}}$ is projective.

To see that $\mathcal{M}$ is quasi-projective, let us recall that $\overline{\mathcal{M}} = \mathcal{M} \cup Z$, where Z is a closed complex subspace of $\overline{\mathcal{M}}$. Moreover, let $i\colon \overline{\mathcal{M}} \to \mathbb{P}^N$ be a projective embedding. By Remmert's proper mapping theorem, $i(Z)$ is a closed complex subspace of $\mathbb{P}^N$. By Chow's theorem, both $i(\overline{\mathcal{M}})$ and $i(Z)$ are Zariski closed in $\mathbb{P}^N$ so that $i(\mathcal{M})$ is Zariski open in $i(\overline{\mathcal{M}})$. By definition, $\mathcal{M}$ is quasi-projective.

Finally, we show (2). It suffices to show that the line bundle $\overline{\mathcal{L}} \to W^*/\mathbb{C}^*$ is isomorphic to $\mathcal{L} \to \mathcal{M}$ via the biholomorphism $W^*/\mathbb{C}^* \to \mathcal{M}$ described in the proof of Theorem H. By definition, the total space of the line bundle $\mathcal{L}_{\mathbb{C}} \to \mathcal{M} \times \mathbb{C}$ is $\mathcal{L} \times \mathbb{C}$. If we restrict $\mathcal{L}_{\mathbb{C}}$ to $W^* = \mathcal{M} \times \mathbb{C}^*$, we obtain the following commutative diagram

$$
\begin{tikzcd}
\mathcal{L} \times \mathbb{C}^* \arrow[r] \arrow[d] & \mathcal{L} \arrow[d] \\
\mathcal{M} \times \mathbb{C}^* \arrow[r] & \mathcal{M}
\end{tikzcd}, \tag{5.57}
$$

where the top horizontal map is given by $(v, z) \mapsto z^{-1} \cdot v$. Therefore, the diagram (5.57) defines a map $(\mathcal{L}_{\mathbb{C}}|_{W^*})/\mathbb{C}^* \to \mathcal{L}$ covering the biholomorphism $W^*/\mathbb{C}^* \to \mathcal{M}$. Finally, by the proof of Proposition 5.2.1, it is easy to verify that the total space $\overline{\mathcal{L}}$ of the line bundle $\overline{\mathcal{L}} \to W^*/\mathbb{C}^*$ is precisely $(\mathcal{L}_{\mathbb{C}}|_{W^*})/\mathbb{C}^*$. Therefore, we have obtained a bundle map $\overline{\mathcal{L}} \to \mathcal{L}$ that is an isomorphism on each fiber. $\square$

5.5 Appendix: codimension estimate of the stable locus

In this section, we provide an estimate the codimension of $\mathcal{M} \setminus \mathcal{M}^s$ that is used in §5.

Proposition 5.5.1.

1. *Every $\mathcal{G}^{\mathbb{C}}$-stabilizer of a polystable Higgs bundle is connected.*

2. *There are finitely many strata in $\mathcal{M}$.*

Proof. Let (A, Φ) be a polystable Higgs bundle. By definition, we may write

$$
(\mathcal{E}_A, \Phi) = (\mathcal{E}_1, \Phi_1)^{\oplus m_1} \oplus \cdots \oplus (\mathcal{E}_r, \Phi_r)^{\oplus m_r}, \qquad m_i \geq 0, \tag{5.58}
$$

where $(\mathcal{E}_1, \Phi_1), \cdots, (\mathcal{E}_r, \Phi_r)$ are pairwise non-isomorphic stable Higgs bundles that have the same slope as $(\mathcal{E}_A, \Phi)$, and $(\mathcal{E}_A, \Phi)$ is the Higgs bundle determined by

(A, Φ). As a consequence,

$$(\mathcal{G}^{\mathbb{C}})_{(A,\Phi)} = \prod_{i=1}^{r} GL(m_i, \mathbb{C}). \tag{5.59}$$

This proves (1) and (2). Here, we have used the fact that if f is a morphism between two stable Higgs bundles of the same slope, then either $f \equiv 0$ or f is an isomorphism. Moreover, every endomorphism of a stable Higgs bundle must be a scalar. □

Proposition 5.5.2. *Let Q_1 and Q_2 be two strata in $\mathcal{M}$. If $Q_1 \subset \overline{Q_2}$, then* $\dim Q_2 > \dim Q_1$*.*

Proof. Since Kuranishi maps preserve the orbit type stratifications, this problem can be transferred to $\nu_{0,\mathbb{C}}^{-1}(0) /\!\!/ H^{\mathbb{C}}$. Write

$$\mathbf{H}^1 = F \oplus (\mathbf{H}^1)^{H^{\mathbb{C}}}, \tag{5.60}$$

where F is the $\omega_{0,\mathbb{C}}$-orthogonal complement of $(\mathbf{H}^1)^{H^{\mathbb{C}}}$. By definition of $\nu_{0,\mathbb{C}}$,

$$\nu_{0,\mathbb{C}}^{-1}(0) = (\nu_{0,\mathbb{C}}|_F)^{-1}(0) \times (\mathbf{H}^1)^{H^{\mathbb{C}}} \tag{5.61}$$

so that

$$\nu_{0,\mathbb{C}}^{-1}(0) /\!\!/ H^{\mathbb{C}} = (\nu_{0,\mathbb{C}}|_F)^{-1}(0) /\!\!/ H^{\mathbb{C}} \times (\mathbf{H}^1)^{H^{\mathbb{C}}}. \tag{5.62}$$

Therefore, it is clear that the unique stratum containing $[0]$ is $(\mathbf{H}^1)^{H^{\mathbb{C}}}$. If L is a proper subgroup of $H^{\mathbb{C}}$, then

$$(\nu_{0,\mathbb{C}}^{-1}(0) /\!\!/ H^{\mathbb{C}})_{(L)} = ((\nu_{0,\mathbb{C}}|_F)^{-1}(0) /\!\!/ H^{\mathbb{C}})_{(L)} \times (\mathbf{H}^1)^{H^{\mathbb{C}}}, \tag{5.63}$$

where the subscript (L) denote the orbit type stratum determined by (L). As a consequence,

$$\dim(\nu_{0,\mathbb{C}}^{-1}(0) /\!\!/ H^{\mathbb{C}})_{(L)} = \dim((\nu_{0,\mathbb{C}}|_F)^{-1}(0) /\!\!/ H^{\mathbb{C}})_{(L)} + \dim(\mathbf{H}^1)^{H^{\mathbb{C}}}. \tag{5.64}$$

Now, we claim that if $F \neq 0$ and $((\nu_{0,\mathbb{C}}|_F)^{-1}(0) /\!\!/ H^{\mathbb{C}})_{(L)} \neq \emptyset$, then

$$\dim((\nu_{0,\mathbb{C}}|_F)^{-1}(0) /\!\!/ H^{\mathbb{C}})_{(L)} > 0. \tag{5.65}$$

Suppose that this dimension is 0 and pick a connected component Q. Hence, Q is a singleton, and its preimage in $(\nu_{0,\mathbb{C}}|_F)^{-1}(0)^{ps}$ is a single $H^{\mathbb{C}}$-orbit $xH^{\mathbb{C}}$ for some $x \neq 0$. By Kempf-Ness theorem, the restriction of the L^2-norm $\|\cdot\|_{L^2}$ to the orbit $xH^{\mathbb{C}}$ attains a minimum value $r > 0$. Therefore, we may assume that $\|x\|_{L^2} = r$. Now, we show that if $t_1 x$ and $t_2 x$ are in the same $H^{\mathbb{C}}$-orbit for some $t_1, t_2 > 0$, then $t_1 = t_2$. In fact, if $t_1 x = g t_2 x$ for some $g \in H^{\mathbb{C}}$, then $t_1 r = t_2 \|gx\|_{L^2} \geq t_2 r$ so that $t_1 \geq t_2$. Applying the same argument to $g^{-1} t_1 x = t_2 x$, we obtain that $t_1 \leq t_2$. Since the $H^{\mathbb{C}}$-action is linear, tx is also polystable and has the same orbit type of x for every $t \in (0, 1]$. Therefore, Q contains a subspace $\{[tx] \colon t \in (0, 1]\}$, which is a contradiction. $\square$

Using Proposition 5.5.2, we can show that closures of strata are closed complex subspaces of $\mathcal{M}$.

Proposition 5.5.3. *If Q is a stratum in $\mathcal{M}$, then $\overline{Q}$ is a closed complex subspace of $\mathcal{M}$, and $\dim_x \overline{Q} = \dim Q$ for every $x \in \overline{Q}$.*

Proof. We prove by induction. Note that every stratum is pure dimensional. By Proposition 5.5.1, let $d_1 < d_2 < \cdots < d_k$ be possible values of dimensions among all the strata. By Proposition 5.5.2, every stratum Q of dimension d_1 is closed, and hence $\dim_x \overline{Q} = \dim Q$ for all $x \in \overline{Q}$. Now, suppose that the statement is true for all the strata of dimensions smaller than d_i. Let Q be a stratum of dimension d_i. Therefore, Q is a closed complex subspace of $\mathcal{M} \setminus \partial Q$. Write

$$\partial Q = Q_{l_1} \cup \cdots \cup Q_{l_k} = \overline{Q_{l_1}} \cup \cdots \cup \overline{Q_{l_k}}, \tag{5.66}$$

where each Q_{l_i} is a stratum of dimension smaller than d_i. By induction, each $\overline{Q_{l_i}}$ is a closed complex subspace and $\dim_x \overline{Q_{l_i}} = \dim Q_{l_i}$ for all $x \in \overline{Q_{l_i}}$. Hence, $\dim Q > \dim \partial Q$. By the Remmert-Stein theorem, $\overline{Q}$ is a closed complex subspace. Now we show that $\dim_x \overline{Q} = \dim Q$ for every $x \in \overline{Q}$ to finish the proof. If $x \in Q$, then the openness of Q in $\overline{Q}$ implies that $\dim_x \overline{Q} = \dim_x Q$. Therefore, we may assume that $x \in \partial Q$. Since Q is open and dense in $\overline{Q}$, ∂Q is nowhere dense. Hence, by [23, Lemma of Ritt], $\dim_x \partial Q < \dim_x \overline{Q}$. Let S be an irreducible component of $\overline{Q}$ containing x such that $\dim S = \dim_x \overline{Q}$. If $S \subset \partial Q$, then $\dim S \leq \dim_x \partial Q$, which is a contradiction. Hence, $S \cap Q \neq \emptyset$. As a consequence, since Q is open in $\overline{Q}$,

$$\dim_x \overline{Q} = \dim S = \dim_x(Q \cap S) \leq \dim Q. \tag{5.67}$$

Now, by the upper semicontinuity of the function $x \mapsto \dim_x \overline{Q}$ (see [23, p.94]), there is an open neighborhood U of x in $\overline{Q}$ such that $\dim_y \overline{Q} \leq \dim_x \overline{Q}$ for all $y \in U$. Since Q is open and dense in $\overline{Q}$, we may choose $y \in U \cap Q$. Hence,

$$\dim Q = \dim_y(U \cap Q) = \dim_y \overline{Q} \leq \dim_x \overline{Q}. \tag{5.68}$$

Hence, $\dim Q = \dim_x \overline{Q}$. □

As a corollary, we obtain a codimension estimate of $\mathcal{M} \setminus \mathcal{M}^s$, where $\mathcal{M}^s$ is the moduli space of stable Higgs bundles. Although this is a well-known result (see [15, Theorem II.6] and [57, Lemma 11.2]), we couldn't find an analytic proof in the literature.

Corollary 5.5.4. *$\mathcal{M}^s$ is open and dense in $\mathcal{M}$, and $\operatorname{codim}_x(\mathcal{M} \setminus \mathcal{M}^s) \geq 4g - 6$ for every $x \in \mathcal{M} \setminus \mathcal{M}^s$, where g is the genus of the Riemann surface M.*

Proof. The first statement follows from [61, Corollary 3.24] and Theorem F. To show the second statement, we write

$$\mathcal{M} \setminus \mathcal{M}^s = Q_1 \cup Q_2 \cup \cdots \cup Q_k = \overline{Q_1} \cup \cdots \cup \overline{Q_k}, \tag{5.69}$$

where each Q_i is a stratum. As a consequence,

$$\dim_x(\mathcal{M} \setminus \mathcal{M}^s) = \dim_x \overline{Q_j} = \dim Q_j \tag{5.70}$$

for some j (depending on x). Therefore, by Proposition 5.5.3, we obtain

$$\begin{aligned} \operatorname{codim}_x(\mathcal{M} \setminus \mathcal{M}^s) &= \dim_x \mathcal{M} - \dim_x(\mathcal{M} \setminus \mathcal{M}^s) \\ &= \dim \mathcal{M}^s - \dim Q_j. \end{aligned} \tag{5.71}$$

By [61, Corollary 3.24] again, $\dim \mathcal{M}^s - \dim Q_j \geq 4g - 6$. $\square$

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