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An Iterative Method for Solving Linear Inequalities

G. W. Stewart*

ABSTRACT

This paper describes and analyzes a method for finding nontrivial solutions of the inequality $Ax \geq 0$, where A is an $m \times n$ matrix of rank n. The method is based on the observation that a certain function f has a unique minimum if and only if the inequality *fails to have* a nontrivial solution. Moreover, if there is a solution, an attempt to minimize f will produce a sequence that will diverge in a direction that converges to a solution of the inequality. The technique can also be used to solve inhomogeneous inequalities and hence linear programming problems, although no claims are made about competitiveness with existing methods.

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1 Introduction

In this paper we will describe an iterative method for finding nontrivial solutions of the homogeneous inequality

$$Ax \geq 0, \tag{1.1}$$

where A is an $m \times n$ matrix of rank n. The underlying idea is simple. Consider the function

$$f(x) = \mathbf{1}^{\mathrm{T}} \exp(-Ax), \tag{1.2}$$

where $\mathbf{1} = (1\ 1\ \ldots\ 1)^{\mathrm{T}}$ and for any vector y

$$\exp(y) = (e^{y_1}, e^{y_2}, \ldots, e^{y_m})^{\mathrm{T}}.$$

We shall show that one of two things must happen if f is minimized iteratively. If (1.1) has no nontrivial solution, then f has a unique minimum, to which the iteration must converge. On the other hand, if (1.1) has a

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nontrivial solution, then the iterates will grow unboundedly in such a way that a solution can be computed from them.

Unfortunately, in at least one application it is not enough to compute just any solution. Specifically, any linear programming problem can be reduced to a sequence of inhomogeneous inequalities of the form

$$Ax \geq b. \tag{1.3}$$

This inequality has a solution if and only if the homogeneous inequality

$$\begin{pmatrix} A & -b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y \\ \eta \end{pmatrix} \geq 0 \tag{1.4}$$

has a solution with $\eta > 0$. If we apply the technique sketched in the last paragraph to (1.4), there is a possibility that it might return a solution with $\eta = 0$, in which case we cannot say whether (1.3) has a solution or not. Thus if one wants to apply our method to linear programming, it is necessary to show that it not only computes a solution of (1.1) but that it computes one for which as many components of Ax as possible are positive. It is this necessity that accounts for most of the technical detail in the paper.

In 1952, Motzkin [3] proposed finding solutions of (1.1) by minimizing (1.2). The author [5] rediscovered the method independently in connection with problems in the statistical analysis of categorical data (for the connection see [1, Ch.2]), and the algorithm has actually been incorporate into a set of programs for solving such problems.

In the next section we shall introduce some preliminary notation and definitions. In §3 we will show that if f does not have a minimum, any diverging sequence that drives f to its infimum will produce the required solution. The problem then reduces to finding an iterative method that diverges properly, and we will show in §4 that one such is Newton's method with line searches. The paper concludes with some general observations.

2 Preliminaries

Since we must maximize the number of positive components of Ax, let us introduce the following notation. Let A be partitioned by rows in the form

$$\begin{pmatrix} a_1^{\mathrm{T}} \\ a_2^{\mathrm{T}} \\ \vdots \\ a_m^{\mathrm{T}} \end{pmatrix},$$

and for any vector x set

$$\begin{aligned}\mathcal{P}(x) &= \{i:\ a_i^{\mathrm{T}}x > 0\},\\ \mathcal{Z}(x) &= \{i:\ a_i^{\mathrm{T}}x = 0\},\\ \mathcal{N}(x) &= \{i:\ a_i^{\mathrm{T}}x < 0\}.\end{aligned}$$

Thus $\mathcal{P}$, $\mathcal{Z}$, and $\mathcal{N}$ comprise the indices for which the components Ax are positive, zero, and negative.

If x is a solution of (1.1), then $\mathcal{N}(x) = \emptyset$. If x^1 and x^2 are solutions, then $x^1 + x^2$ is a solution and

$$\begin{aligned}\mathcal{P}(x^1+x^2) &= \mathcal{P}(x^1) \cup \mathcal{P}(x^2),\\ \mathcal{Z}(x^1+x^2) &= \mathcal{Z}(x^1) \cap \mathcal{Z}(x^2).\end{aligned}$$

From this it follows that there is a solution x^* of (1.1) for which the cardinality of $\mathcal{P}(x^*)$ is greatest. We will call this a *maximally positive* (MP) solution. MP solutions are not unique, but they all have the same sets $\mathcal{P}^* = \mathcal{P}(x^*)$ and $\mathcal{Z}^* = \mathcal{Z}(x^*)$.

A transformed version of the inequality will be needed in the sequel. Let x^* be a MP solution of (1.1). Without loss of generality we may assume that the rows indexed by $\mathcal{Z}^*$ are the last rows of A; i.e.,

$$A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, \tag{2.1}$$

where $A_1x^* > 0$ and $A_2x^* = 0$. Let $V = (V_1\ V_2)$ be an orthogonal matrix with the columns of V_1 spanning the null space of A_2.[1] If we set

$$B_{ij} = A_iV_j$$

and

$$u_i = V_i^{\mathrm{T}}x, \tag{2.2}$$

then the inequality (1.1) becomes

$$Bu \equiv \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \geq 0. \tag{2.3}$$

Here both B_{11} and B_{22} have full column rank. Moreover, the MP solution u^* corresponding to x^* satisfies

$$B_{11}u_1^* > 0 \tag{2.4}$$

and

$$u_2^* = 0.$$

[1] If $\mathcal{P}^* = \emptyset$, both A_1 and V_1 will be void matrices.

3 Diverging to a solution

In this section we shall investigate the relation of the function f defined by (1.2) to the homogeneous inequality (1.1). Specifically, we will show that if (1.1) has a nontrivial solution then the members of any sequence $\{x^k\}$ satisfying

$$\lim_{k\to\infty} f(x^k) = \inf f(x)$$

must ultimately provide an MP solution.

We begin by stating some elementary facts about the function f. Clearly f is bounded below by zero. Its gradient and Hessian are given by

$$f'(x) = -A^{\mathrm{T}} \exp(-Ax) \tag{3.1}$$

and

$$f''(x) = A^{\mathrm{T}} D(x) A, \tag{3.2}$$

where

$$D(x) = \mathrm{diag}(e^{-a_1^{\mathrm{T}}x}, e^{-a_2^{\mathrm{T}}x}, \ldots, e^{-a_m^{\mathrm{T}}x}).$$

Since $D(x)$ is positive definite and A is of full column rank, $f''(x)$ is positive definite. If follows that f is strictly convex and can have at most one local minimum, which, when it exists, is also a global minimum. Necessary and sufficient conditions for the existence of a minimum are contained in the following theorem.

Theorem 1 *The function f has a minimum if and only if the inequality (1.1) has a nontrivial solution.*

Proof. First suppose that (1.1) does not have a nontrivial solution. To show that f has a minimum, it is sufficient to show [4, §4.3.3] that for any norm[2] $\lim_{\|x\|\to\infty} f(x) = +\infty$. For any x with $\|x\| = 1$ set

$$\phi(x) = \min\{a_i^{\mathrm{T}}x : \ i \in \mathcal{N}(x)\}.$$

Since (1.1) has no solution, $\mathcal{N}(x)$ is nonempty and $\phi(x) < 0$. Clearly ϕ is continuous. Hence

$$\varphi = \sup_{\|x\|=1} \phi(x) < 0.$$

[2]Throughout this paper, $\|\cdot\|$ will stand for both a vector norm and a submultiplicative matrix norm.

Now for any $x \neq 0$,

$$f(x) \geq \sum_{i \in \mathcal{N}(x)} e^{-a_i^{\mathrm{T}} x} \geq e^{-\varphi \|x\|},$$

which establishes the first part of the theorem.

Next suppose that (1.1) has a nontrivial solution x^*. Partition A as in (2.1) so that $A_1 x^* > 0$ and $A_2 x^* = 0$. Then for any point x,

$$\begin{aligned}\lim_{\alpha\to\infty} f(x+\alpha x^*) &= \lim_{\alpha\to\infty} \mathbf{1}^{\mathrm{T}} \exp(-A_1 x - \alpha A_1 x^*) + \mathbf{1}^{\mathrm{T}} \exp(-A_2 x - \alpha A_2 x^*) \\ &= \mathbf{1}^{\mathrm{T}} e^{-A_2 x} < f(x),\end{aligned}$$

which shows that x cannot be a minimum of f.

Theorem 3.1 shows that if we apply a globally convergent minimization algorithm to f it will converge whenever (1.1) does not have a nontrivial solution. The proof of the theorem also suggests that when (1.1) does have a unique solution the iterates may diverge along a direction that is itself a solution. In the next section we will show that this is actually true of Newton's method with line searches. However, since our goal is to find MP solutions, we must first establish the conditions under which a diverging sequence furnishes an MP solution.

For the rest of this section we assume that (1.1) has a MP solution x^* with a corresponding partition (2.1) of A. The problem is best approached through the transformed inequality (2.3). If we define u by (2.2) and set

$$\begin{aligned} g(u) &= \mathbf{1}^{\mathrm{T}} \exp(-Bu) \\ &= \mathbf{1}^{\mathrm{T}} \exp(-B_{11}u_1 - B_{12}u_2) + \mathbf{1}^{\mathrm{T}} \exp(-B_{22}u_2) \\ &\equiv g_1(u) + g_2(u_2), \end{aligned} \tag{3.3}$$

then $g(u) = f(x)$, so that g serves the same role in the transformed inequality (2.3) as does f in (1.1).

Lemma 2 *The system*

$$B_{22}u_2 \geq 0 \tag{3.4}$$

has no nontrivial solution. Hence g_2 has a unique minimum

$$\gamma = g_2(u_2'). \tag{3.5}$$

Proof. Suppose u_2 is a nontrivial solution of (3.4). Because B_{22} is of full rank, $B_{22}u_2$ is nonzero and hence has at least one positive component. From (2.4) it follows that there is a $\sigma > 0$ such that

$$\sigma B_{11}u_1^* + B_{12}u_2 > 0.$$

Hence the vector

$$\tilde{u} = \begin{pmatrix} \sigma u_1^* \\ u_2 \end{pmatrix}$$

is a solution of (2.3) with $\mathcal{P}(u^*)$ a proper subset of $\mathcal{P}(\tilde{u})$, which contradicts the fact that u^* is an MP solution. The existence of a unique minimum now follows from Theorem 3.1.

Since g_1 and g_2 are both positive, the number γ defined by (3.5) is a strict lower bound on $g(u)$. In the next theorem we will show that it is actually the infimum of $g(u)$. Moreover, any sequence of points that drives $g(u)$ to γ must have properties that enable us to extract an MP solution.

Theorem 3 *The function g satisfies*

$$g(u) > \inf g(v) = \gamma. \tag{3.6}$$

Moreover, if $\{u^k\}$ is any sequence with $g(u^k) \to \gamma$, then with u_2' defined by (3.5)

$$u_2^k \to u_2' \tag{3.7}$$

and

$$B_{11}u_1^k > O\{-\ln[g(u^k) - \gamma]\}. \tag{3.8}$$

Proof. We have already noted that γ is a lower bound for the values of g. If we set

$$u_\alpha = \begin{pmatrix} \alpha u_1^* \\ u_2' \end{pmatrix},$$

then

$$\begin{aligned} \lim_{\alpha\to\infty} g(u_\alpha) &= \lim_{\alpha\to\infty} g_1(u_\alpha) + g_2(u_2') \\ &= 0 + \gamma = \gamma, \end{aligned}$$

which establishes (3.6).

Now let $\{u^k\}$ be any sequence with $g(u^k) \to \gamma$. Then by (3.3) we must have $g_2(u_2^k) \to \gamma$, which in view of Lemma 3.2 establishes (3.7).

Finally, since $g_1(u^k) \le g(u^k) - \gamma$, it follows that

$$\exp(-B_{12}u_2^k) \odot \exp(-B_{11}u_1^k) \le g(u^k) - \gamma,$$

where $\odot$ denotes component-wise multiplication. Hence

$$\exp(-B_{11}u_1^k) \le [g(u^k) - \gamma]\exp(B_{12}u_2^k),$$

or

$$B_{11}u_1^k \ge -\ln[g(u^k) - \gamma]\mathbf{1} - B_{12}u_2^k.$$

Since u_2^k is converging, this is equivalent to (3.8).

Let us reinterpret this theorem in terms of the original inequality. Suppose we are given a sequence $\{x^k\}$ with the property that $\lim f(x^k) = \inf f(x)$. Then one of two things must happen. Either the x^k remain bounded, in which case they must converge to a local minimum of f, and by Theorem 3.1 the inequality (1.1) has no nontrivial solution. Or the x^k grow unboundedly. In this case, theorem 3.3 then says that the components of Ax^k divide into two classes: those which converge and those which grow unboundedly. The indices of the former make up the set $\mathcal{Z}^*$, while the indices of the latter make up the set $\mathcal{P}^*$. Once these sets have been recognized we may compute the transformation V to the u-coordinate system and hence trial solutions

$$\bar{x}^k = V_1V_1^{\mathrm{T}}x^k = V_1u_1^k$$

(i.e., the vectors obtained by setting $u_2^k = 0$ so that $A_2\bar{x}^k = 0$). Initially the $\bar{x}^k$ may fail to be solutions, owing to the suppression of the terms $B_{12}u_2^k$; but (3.8) insures that ultimately $A_1\bar{x}^k > 0$, and at that point $\bar{x}^k$ is a solution.

The above procedure gives no problems when (1.1) either has no solution or when it has a solution with $\mathcal{Z}^* = \emptyset$. In the former case the sequence x^k converges; in the latter it ultimately exhibits a solution. When there is a solution with $\mathcal{Z}^* \ne \emptyset$, we are faced with the problem of determining which of the components of Ax^k are converging and which are diverging, a decision which must be based on tolerances that are to some extent arbitrary. It should be stressed that this is not a failing of the method; the problem itself is intrinsically difficult, since a small perturbation of the matrix A can cause $\mathcal{Z}^*$ to become zero, on the one hand, or cause there to be no nontrivial solution on the other. How such a case should be treated will depend on the application.

4 The divergence of Newton's method

To complete our algorithm for solving the inequality (1.1), we must generate a sequence of vectors $\{x^k\}$ such that $\lim x^k = \inf f(x)$. In this section we will show that Newton's method with line search will produce such a sequence. This method generates an iterate x^{k+1} from a previous iterate x^k as follows.

1. Set $d^k = -f''(x^k)^{-1}f'(x^k)$.
2. Determine α_k so that the function $\varphi_k(\alpha) = f(x^k + \alpha d^k)$ is minimized.
3. Set $x_{k+1} = x^k + \alpha_k d^k$.

There are two reasons why Newton's method with line search is particularly well suited for this application. First, if f has a minimum, the sequence $\{x^k\}$ converges to it, ultimately quadratically. Second, the form of f makes it cheap to determine α_k. For if we precompute $y^k = \exp(-Ax^k)$ and $z^k = Ad^k$, then

$$\varphi_k(\alpha) = \sum_{i=1}^{m} y_i^k e^{-\alpha z_i^k},$$

a very simple function to work with.

The proof that Newton's method with line search forces f to its infimum depends on two results which are of independent interest. The first gets us started by showing that the method diverges when f does not have a minimum.

Theorem 4 *Let $f : \mathcal{R}^n \to \mathcal{R}$ be thrice continuously differentiable. Let $\mathcal{D}$ be any compact set with the property that $f'(x) \neq 0$ and $f''(x)$ is positive definite for all $x \in \mathcal{D}$. Then there are positive constants $\alpha < 1/4$, ϵ, and θ, such that if $x \in \mathcal{D}$ and*

$$d = -f''(x)^{-1}f(x) + e, \tag{4.1}$$

where $\|e\| \leq \epsilon$, then

$$f(x + \alpha d) \leq f(x) - \theta.$$

Proof. By Taylor's theorem

$$f(x + \alpha d) = f(x) + \alpha f'(x)^{\mathrm{T}} d + \frac{\alpha^2}{2} d^{\mathrm{T}} f''(x) d + r(x), \tag{4.2}$$

where for some fixed $M \geq 0$

$$|r(x)| \leq M\alpha^3\|d\|^3.$$

Substituting the definition (4.1) of d into (4.2), we get

$$\begin{aligned} f(x+\alpha d) &= f(x) - \left(\alpha - \frac{\alpha^2}{2}\right) f'(x)^{\mathrm{T}} f''(x) f'(x) + \\ &\qquad (\alpha - 2\alpha^2) f'(x)^{\mathrm{T}} e + \frac{\alpha^2}{2} e^{\mathrm{T}} f''(x) e + r(x). \end{aligned}$$

Since $f'(x)$ is nonzero and $f''(x)$ is positive definite on the compact set $\mathcal{D}$, the quantity $f'(x)^{\mathrm{T}} f''(x) f'(x)$ is positive and uniformly bounded below on $\mathcal{D}$, say

$$f'(x)^{\mathrm{T}} f''(x) f'(x) \geq \mu > 0.$$

The norms of the terms $f'(x)$ and $f''(x)$ are bounded above on $\mathcal{D}$, and if we stipulate that $\epsilon < 1$, so is the norm of d. Hence we may choose $\alpha < 1/4$ so that

$$M\alpha^3\|d\|^3 \leq \frac{\alpha^2}{2}\mu$$

on $\mathcal{D}$. For this value of α, we may choose $\epsilon < 1$ so that

$$\left|(\alpha - 2\alpha^2) f'(x)^{\mathrm{T}} e + \frac{\alpha^2}{2} e^{\mathrm{T}} f''(x) e\right| \leq \frac{\alpha}{2}\mu$$

on $\mathcal{D}$. It follows that for all $x \in \mathcal{D}$

$$f(x+\alpha d) \leq f(x) - \left(\frac{\alpha}{2} - \alpha^2\right)\mu,$$

which establishes the theorem with $\theta = (\frac{\alpha}{2} - \alpha^2)\mu$.

In application to our problem, Theorem 4.1 (with $e = 0$) shows that if f does not have a minimum, then the iterates generated by Newton's method with line search cannot remain in a compact set; for within that set each iteration must reduce f by at least θ. Consequently, the iterates x^k must diverge. However, the divergent sequence could possibly approach a contour that is greater than the infimum of f. Before we can prove that this does not happen, we need another technical result.

From (3.1) and (3.2), we see that the Newton step is given by

$$[A^{\mathrm{T}} D(x^k) A]^{-1} A^{\mathrm{T}} D(x^k) \mathbf{1}.$$

Although at first glance this formula seems uncomplicated, in our application the diagonal elements of D are the numbers $e^{-a_i^{\mathrm{T}}x}$, some of which are converging to zero while others remain finite. This means that the matrices $A^{\mathrm{T}}D(x^k)A$ will become increasingly ill conditioned. Fortunately, although $[A^{\mathrm{T}}D(x^k)A]^{-1}$ can become unbounded, the weighted pseudo-inverse $[A^{\mathrm{T}}D(x^k)A]^{-1}A^{\mathrm{T}}D(x^k)$ remains bounded, no matter what happens to $D(x^k)$. Specifically, we have the following surprising theorem.[3]

Theorem 5 *Let A be an $m \times n$ of rank n, and let $\mathcal{D}^+$ denote the space of diagonal matrices with positive diagonal elements. Then*

$$\sup_{D\in\mathcal{D}^+} \|(A^{\mathrm{T}}DA)^{-1}A^{\mathrm{T}}D\| < \infty. \tag{4.3}$$

Proof. The proof is by induction on m, assuming that the result holds for all matrices of full column rank having fewer than m rows. To start the induction, we observe that for an $m\times m$ matrix, the weighted pseudo inverse reduces to A^{-1}, whose norm is clearly independent of D.

For the induction step, let us fix D and suppose that the smallest diagonal element of D is the ith element δ_i. Let D_i be the matrix obtained from D by deleting its ith row and column, and let A_i be the matrix obtained from A by deleting its ith row. There are two cases to consider: $\mathrm{rank}(A_i) = n$ and $\mathrm{rank}(A_i) = n-1$.

First assume $\mathrm{rank}(A_i) = n$. Then by the induction hypothesis

$$\|(A_i^{\mathrm{T}}D_iA_i)^{-1}A_i^{\mathrm{T}}D_i\| \le \alpha_i,$$

where α_i is independent of D_i. Now the norm we seek to bound is

$$\begin{aligned}
\tau &= \|(\delta_i a_i a_i^{\mathrm{T}} + A_i^{\mathrm{T}}D_iA_i)^{-1}(\delta_i a_i \ \ A_i^{\mathrm{T}}D_i)\| \\
&= \left\|\left(I + \delta_i(A_i^{\mathrm{T}}D_iA_i)^{-1}a_ia_i^{\mathrm{T}}\right)^{-1}(A_i^{\mathrm{T}}D_iA_i)^{-1}(\delta_i a_i \ \ A_i^{\mathrm{T}}D_i)\right\| \qquad (4.4)\\
&\le \left\|\left(I + \delta_i(A_i^{\mathrm{T}}D_iA_i)^{-1}a_ia_i^{\mathrm{T}}\right)^{-1}\right\| \left(\|\delta_i(A_i^{\mathrm{T}}D_iA_i)^{-1}a_i\| + \alpha_i\right).
\end{aligned}$$

Let

$$\gamma_i^{-1} = \inf_{\|x\|=1} \|A_ix\|.$$

[3] The same kind of systems come up in connection with Karmarkar's algorithm for linear programming [2].

Then

$$\inf_{\|x\|=1} \|D_i^{\frac{1}{2}} A_i x\| \geq \gamma_i^{-1} \|D_i^{-\frac{1}{2}}\|^{-1}.$$

Since $\delta_i \|D_i^{-1}\| \leq 1$, it follows that

$$\|\delta_i (A_i^{\mathrm{T}} D_i A_i)^{-1}\| \leq \gamma_i^2. \tag{4.5}$$

Moreover

$$\left(I + \delta_i (A_i^{\mathrm{T}} D_i A_i)^{-1} a_i a_i^{\mathrm{T}}\right)^{-1} = I - \frac{\delta_i (A_i^{\mathrm{T}} D_i A_i)^{-1} a_i a_i^{\mathrm{T}}}{1 + \delta_i a_i^{\mathrm{T}} (A_i^{\mathrm{T}} D_i A_i)^{-1} a_i}.$$

Hence if $\beta_i = \|a_i\|$, then

$$\left\| \left(I + \delta_i (A_i^{\mathrm{T}} D_i A_i)^{-1} a_i a_i^{\mathrm{T}}\right)^{-1} \right\| \leq 1 + \beta_i^2 \gamma_i^2. \tag{4.6}$$

It then follows from (4.4), (4.5), and (4.6) that

$$\tau \leq (\alpha_i + \beta_i \gamma_i^2)(1 + \beta_i^2 \gamma_i^2). \tag{4.7}$$

In the second case, where $\operatorname{rank}(A_i) = n - 1$, we may assume that $n > 1$; for if $n = 1$, then $A_i = 0$ and the result follows by direct computation. Let $W = (w_1 \; W_2)$ be an orthogonal matrix such that $A_i w_1 = 0$. Set

$$\begin{pmatrix} a_i^{\mathrm{T}} \\ A_i \end{pmatrix} (w_1 \; W_2) = \begin{pmatrix} \eta_i & y_i^{\mathrm{T}} \\ 0 & Y_i^{\mathrm{T}} \end{pmatrix}. \tag{4.8}$$

Since postmultiplication by orthogonal transformations does not affect the norm of the weighted pseudo-inverse, we can equivalently bound the norm of the weighted pseudo-inverse of the right-hand side of (4.8), which by direct calculation is seen to be

$$Z = \begin{pmatrix} \eta_i^{-1} & -\eta_i^{-1} y_i^{\mathrm{T}} (Y_i^{\mathrm{T}} D_i Y_i)^{-1} Y_i^{\mathrm{T}} D_i \\ 0 & (Y_i^{\mathrm{T}} D_i Y_i)^{-1} Y_i^{\mathrm{T}} D_i \end{pmatrix}.$$

Now $(Y_i^{\mathrm{T}} D_i Y_i)^{-1} Y_i^{\mathrm{T}} D_i$ is a weighted pseudo-inverse of the $(m-1) \times (n-1)$ matrix Y_i. Hence by the induction hypothesis, there is a constant $\bar{\alpha}_i$ independent of D_i such that

$$\|(Y_i^{\mathrm{T}} D_i Y_i)^{-1} Y_i^{\mathrm{T}} D_i\| \leq \bar{\alpha}_i.$$

Hence if we set $\bar{\beta}_i = \|y_i\|$ and $\bar{\gamma}_i = |\eta_i^{-1}|$, then

$$\tau = \|Z\| \leq \bar{\alpha}_i + (1 + \bar{\alpha}_i\bar{\beta}_i)\bar{\gamma}_i. \tag{4.9}$$

The theorem now follows from the observation that the bounds (4.7) and (4.9) depend only on the row index i and not on D. Maximizing over i gives the required bound.

It is natural to ask if the theorem remains true when D is replaced by an arbitrary positive definite matrix. The following example shows that it does not. Let A be the vector $(0\ 1)^{\mathrm{T}}$ and let

$$D = \begin{pmatrix} 1 & -\epsilon \\ \epsilon & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix} \begin{pmatrix} 1 & \epsilon \\ -\epsilon & 1 \end{pmatrix}.$$

Then it is easily verified by direct computation that

$$\lim_{\delta \to 0} \|(A^{\mathrm{T}}DA)^{-1}A^{\mathrm{T}}D\| = (\epsilon^{-1}\ 1)$$

which becomes unbounded as $\epsilon \to 0$. It is probably significant that in this example the nearer D is to a diagonal matrix the larger $\|(A^{\mathrm{T}}DA)^{-1}A^{\mathrm{T}}D\|$.

We are now in a position to establish the main result of this section.

Theorem 6 *Let Newton's method with line search be applied to the function f producing a sequence of iterates $\{x^k\}$. Then*

$$f(x^k) \to \inf f(x^k).$$

Proof. If f has a local minimum, then the theorem follows from the global convergence of Newton's method with line searches. Hence we may assume that f has no global minimum. By Theorem 4.1, the vectors x^k grow unboundedly.

Let us look at the components of Ax^k. Some of these must remain bounded; otherwise, $f(x^k)$ will approach zero, which is a lower bound on f and hence its infimum. By passing to a subsequence $\{y^k\}$ we may divide the components of Ay^k into to sets: those that remain bounded and those that approach $+\infty$. Let us assume that the latter are grouped at the top of A, which we then partition as in (2.1).

We claim that A_2 has a nontrivial null space. Indeed, since A_2y^k remains bounded while $\|y^k\| \to \infty$, any accumulation point of the sequence

$\{y^k/\|y^k\|\}$ is a null vector. Thus as in §2, we may transform the matrix A to the form

$$\begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix}$$

by an orthogonal change of variables $u = V^{\mathrm{T}}x$. Let w^k denote the subsequence of the $u^k = V^{\mathrm{T}}x^k$ corresponding to the subsequence y^k. Finally, let g, g_1, and g_2 be the functions defined by (3.3).

The idea of the proof is the following. Since $B_{11}w_1^k + B_{12}w_2^k$ grows unboundedly, we must have $g_1(w^k) \to 0$. Thus the only way the sequence $g(w^k)$ can fail to approach its infimum is for $g_2(w^k)$ to fail to approach its infimum. Let s^k denote the Newton step corresponding to w^k and partition it in the form

$$s^k = \begin{pmatrix} s_1^k \\ s_2^k \end{pmatrix}$$

We shall show asymptotically s_2^k approaches the Newton step for g_2 at u_2^k and that s_1^k has negligible effect on the value of g. Thus as far as g_2 is concerned, we are taking Newton steps, and Theorem 4.1 will yield a contradiction if g_2 does not achieve a minimum.

Let us write $D^k = \mathrm{diag}(D_1^k, D_2^k)$ for $D(y^k)$. Then $\lim D_1^k = 0$, while both $\|D_2^k\|$ and $\|(D_2^k)^{-1}\|$ are bounded. Now the Newton equations for s^k are

$$\begin{pmatrix} B_{11}^{\mathrm{T}}D_1^kB_{11} & B_{11}^{\mathrm{T}}D_1^kB_{12} \\ B_{12}^{\mathrm{T}}D_1^kB_{11} & B_{22}^{\mathrm{T}}D_2^kB_{22} + B_{12}^{\mathrm{T}}D_1^kB_{22} \end{pmatrix} \begin{pmatrix} s_1^k \\ s_2^k \end{pmatrix} = \begin{pmatrix} B_{11}^{\mathrm{T}}D_1^k\mathbf{1} \\ B_{22}^{\mathrm{T}}D_2^k\mathbf{1} + B_{12}^{\mathrm{T}}D_1^k\mathbf{1} \end{pmatrix}$$

or

$$\begin{pmatrix} I & (B_{11}^{\mathrm{T}}D_1^kB_{11})^{-1}B_{11}^{\mathrm{T}}D_1^kB_{12} \\ (B_{22}^{\mathrm{T}}D_2^kB_{22})^{-1}B_{12}^{\mathrm{T}}D_1^kB_{11} & I + (B_{22}^{\mathrm{T}}D_2^kB_{22})^{-1}B_{12}^{\mathrm{T}}D_1^kB_{22} \end{pmatrix} \begin{pmatrix} s_1^k \\ s_2^k \end{pmatrix} =$$

$$\begin{pmatrix} (B_{11}^{\mathrm{T}}D_1^kB_{11})^{-1}B_{11}^{\mathrm{T}}D_1^k\mathbf{1} \\ (B_{22}^{\mathrm{T}}D_2^kB_{22})^{-1}B_{22}^{\mathrm{T}}D_2^k\mathbf{1} + (B_{22}^{\mathrm{T}}D_2^kB_{22})^{-1}B_{12}^{\mathrm{T}}D_1^k\mathbf{1} \end{pmatrix}.$$

By Theorem 4.2, the matrices $(B_{11}^{\mathrm{T}}D_1^kB_{11})^{-1}B_{11}^{\mathrm{T}}D_1^k$ remain bounded. Moreover, the matrices $(B_{22}^{\mathrm{T}}D_2^kB_{22})^{-1}B_{12}^{\mathrm{T}}D_1^k$ approach zero. Consequently, if we define $\bar{s}_1^k$ and $\bar{s}_2^k$ as the solution of

$$\begin{pmatrix} I & (B_{11}^{\mathrm{T}}D_1^kB_{11})^{-1}B_{11}^{\mathrm{T}}D_1^kB_{12} \\ 0 & I \end{pmatrix} \begin{pmatrix} \bar{s}_1^k \\ \bar{s}_2^k \end{pmatrix} = \begin{pmatrix} (B_{11}^{\mathrm{T}}D_1^kB_{11})^{-1}B_{11}^{\mathrm{T}}D_1^k\mathbf{1} \\ (B_{22}^{\mathrm{T}}D_2^kB_{22})^{-1}B_{22}^{\mathrm{T}}D_2^k\mathbf{1} \end{pmatrix}.$$

then $s_1^k \to \bar{s}_1^k$ and $s_2^k \to \bar{s}_2^k$. From this we see that s_2^k approaches the Newton step for g_2 and is uniformly bounded in k. Hence s_1^k and s^k are also uniformly bounded.

The result now follows from Theorem 4.1 as follows. Suppose that $\lim g(w^k) = \gamma' > \inf g(u)$. Let the Newton steps satisfy $\|s_k\| \leq \sigma$. Let $\mathcal{D}$ be a compact set that includes the w_2^k and excludes the minimum of g_2, if it has one. Let $\alpha < 1/4$, ϵ, and θ be the positive constants from Theorem 4.1. Choose k so large that

1. $\|s_2^k - \bar{s}_2^k\| \leq \epsilon$,
2. $g(w^k) \leq \gamma' + \theta/3$,
3. $\sup\{g_1(u) : \|u - w^k\| \ \leq \sigma\} \leq \theta/3$.

Then since $\alpha < 1/4$,

$$\begin{aligned} g(w^k + \alpha s^k) &= g_1(w^k + \alpha s^k) + g_2(w_2^k + \alpha s_2^k) \\ &\leq \sup\{g_1(u) : \|u - y^k\| \ \leq \sigma\} + g_2(w_2^k) - \theta \\ &\leq \theta/3 + (\gamma' + \theta/3) - \theta \\ &= \gamma' - \theta/3. \end{aligned}$$

Thus the iterate following w^k gives g a value less than γ'—which is a contradiction.

5 Conclusions

As was mentioned in the introduction, the algorithm proposed here is being used in the statistical analysis of categorical data, where it performs quite well. What makes it a particularly attractive choice is that the underlying problem is to maximize a convex function, so that an optimizer is already at hand. However, the problems are of low dimension.

Although we have given a complete mathematical analysis of the algorithm, there are still open questions. For example, scaling the rows of A does not affect the existence of a solution; but it can be expected to have a profound effect on the behavior of the algorithm, since exponentials are such rapidly varying functions. More generally, the exponentials could be replaced by any sufficiently smooth function with the property that they increase monotonically from 0 at $-\infty$ to ∞ at $+\infty$. What effect this will have on the algorithm is unclear.

Nor is it clear how the algorithm will perform for large problems. Here Newton's method becomes less desirable, since it involves forming and solving the large Newton equations. A possibility is some method based on the conjugate gradient algorithm, especially since line searches remain comparatively cheap. However, this is a subject for further analysis and experimentation.

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