ABSTRACT

Title of Dissertation: Applications of Graph Theory and Logic in Computer Science

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In Chapter 2 we study the CSP of unary expansions of directed graphs. When we expand the simple structure ($\mathbb{Z}$,succ) with one unary predicate U, its CSP (Constraint Satisfaction Problem) may vary in complexity. We find some sufficient conditions for its tractability, prove bounds on its complexity, and then generalize our results to more complicated structures. We also give a Karp-equivalent characterization of CSP$(\mathbb{Z}, \text{succ}, U)$s.

Next, fixing $\alpha \in (0, 1]$, we generalized the axiomatizations in [1] to the class of hereditarily linearly sparse K_n-free graphs in Chapter 3, and made efforts towards connecting this with almost sure theories of Shelah-Spencer graphs, a type of random graphs, which may be of interest in theory of computer science.

In Chapter 4 we study the constraint satisfaction problems of selected infinite relational structures. For $\alpha \in (0, \frac{5}{6}]$, $\mathcal{K}_\alpha :=$ the class of hereditarily α-sparse graphs, $\mathfrak{M}_\alpha :=$ the generic structure of $\mathcal{K}_\alpha$; $\mathcal{K}_{\alpha,K_n-\text{free}}, \mathfrak{M}_{\alpha,K_n-\text{free}} :=$ the K_n-free subclass and its generic structure respectively, we prove multiple structural properties of graphs presenting the Hrushovski-Fraïssé class $\mathcal{K}_{\alpha,\text{TF}}$, strongly indicating that this class should have a finite homomorphic presentation when α is close to $\frac{5}{6}$ from below, which would imply NP-completeness. More general results are explored along. We also study the complexity-theoretic implications of our findings on a relevant decision problem, namely the constraint satisfaction problem of the corresponding generic structure.

APPLICATIONS OF GRAPH THEORY AND LOGIC IN COMPUTER SCIENCE

by

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[1] Not the book "life, the universe and everything" (which the author has not read).

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Chapter 1: Introduction

1.1 Constraint Satisfaction Problems

In 1990, Hell and Nešetřil [5] showed that, fixing a finite undirected graph H, the decision problem "given finite input G, does there exist a map $f : V(G) \to V(H)$, s.t. $\forall x, y\Big((x, y) \in E(G) \implies (f(x), f(y)) \in E(H)\Big)$?" is in $\mathtt{P}$ if H is bipartite, $\mathtt{NP}$-complete otherwise. Such edge-preserving maps are *graph homomorphisms*, and the problem is called H*-coloring*. The definition of *homomorphisms* naturally generalize to arbitrary relational structures by asserting each relation is unilaterally preserved by the map between domains. This gives rise to:

Problem 1.1 (CSP(H))**.**

Parameter (Fixed): $H := (D; R_1^D, \dots, R_k^D)$ a relational structure, with the symbol tuple $(R_1, \dots, R_k)$ its *signature*.

Input: a finite $(R_1, \dots, R_k)$-structure G.

Output: does G homomorphically map to H?

Such problems are called the *constraint satisfaction problems* (*CSP*s); we also use CSP(H) to denote the set of inputs that answers "Yes". Correspondingly, the *SEARCH-CSP*s admit the same description, except on the "Yes" instances they output a witnessing

$h \in \mathrm{Hom}(G, H)$.

Seeing Hell-Nešetřil's H-coloring dichotomy theorem, one naturally wonders if similar dichotomy exists for all finite relational structures. A dividing line was conjectured by Feder and Vardi [6]. Classification of many subclasses of finite domain CSPs (e.g. [7, 8, 9]) were studied over the years, but the conjecture remained open until independently proved by Bulatov and Zhuk ([10, 11, 12]) in 2017. A good reference for equivalent characterizations of the finite CSP dividing line can be found in [13].

For infinite domain CSPs, dichotomy has been established for many classes of "nice" structures; for example, fixing an infinite structure (A, τ), the class $\{\mathfrak{A}' := (A, \tau') : R^{\mathfrak{A}'}$ is first order definable in $\mathfrak{A}$ for each $R \in \tau'\}$ is called *first order reducts* of $\mathfrak{A}$; that is, all relational structures with the same domain A and whose finitely many relations are respectively definable in terms of $\mathfrak{A}$. Dichotomy has been established for the first order reducts of the Rado graph [14], *of* unary structures [15], and *of* $(\mathbb{Q}, <)$ [16]. These structures are "nice" in that their first order theory has (up to isomorphism) a unique countable model, a property called *ω-categoricity*. Another line of work focuses on dichotomizing the "numeric domains", such as first order reducts of $(\mathbb{Z}, \mathrm{succ})$ [17], more generally *of* $(\mathbb{Z}, <)$ [18], and *of* $(\mathbb{Z}, +, 1)$ containing 1 [19]. A long standing open problem is to characterize first order reducts of $(\mathbb{Z}, \leq, +)$. For thorough introduction to these topics, see [13, 20].

Instead of studying first order reducts, my Chapter 2 studies unary expansions (i.e. asserting"pre-coloring of vertices") of the infinite directed graphs $(\mathbb{Z}, \mathrm{succ})$ as well as unary expansions of general directed graphs $(\mathbb{Z}, E)$. We give equivalent characterizations of such CSPs, and discuss reductions upper bounding the hardness of certain classes of such unary expansions.

1.2 Hrushovski-Fraïssé Classes

Originally constructed by Hrushovski [21] in languages of ternary hypergraphs, Hrushovski-Fraïssé classes have inspired rich bodies of literature connecting various fields of mathematics. To name a few: Baldwin-Shi [22] proved that Th($\mathfrak{M}_\alpha$), the first order theory of the generic structure of $\mathcal{K}_\alpha$, a class of finite (undirected simple) graphs, coincide with the almost sure theory of $G(n, n^{-\alpha})$, the random graph with edge probability $n^{-\alpha}$ ($\alpha \in (0,1] \smallsetminus \mathbb{Q}$), which had been proved complete by Shelah-Spencer [23]. Laskowski [1] gave a Π_2 axiomatization of Th($\mathfrak{M}_\alpha$), inspiring subsequent work on model-theoretic approximation of $G(n, n^{-\alpha})$ [24, 25] and atomic models and regular types of Baldwin-Shi hypergraphs [26, 27, 28*]. Recently, Ghadernezhad, Khalilian, and Pourmahdianwas proved that the automorphism group of $\mathfrak{M}_\alpha$, at least for $\alpha \in \mathbb{Q}$, is not amenable. [29].

In Chapter 3 we focus on a specific type of Hrushovski-Fraïssé class of *relational structurals*, which one may think of as hypergraphs with multiple types of hyperedges. Such classes are defined by hereditary sparsity and forbidden cliques. Developping upon [1], we prove a simple axiomatization of the first order theory of the *generic structures* of such classes, which are infinite structures which “encodes” the original class of finite structures in a nice way. Focusing on the language of graphs, we also record some efforts towards connections with the *almost-sure theory* of Shelah-Spencer graphs, which is a specific type of random graphs of wide interest in computer science.

1.3 Constraint Satisfaction Problems Arising from Hrushovski-Fraïssé Classes

In Chapter 4 we "combine the two areas", in a manner of speaking. Namely, we focus on the language of (simple, undirected) graphs, and study the CSP of generic structures $\mathfrak{M}_{\alpha,*}$ of some Hrushovski-Fraïssé classes of hereditarily linearly sparse graphs forbidding substructures. It should be mentioned that behavior of graphs forbidding certain configurations is a heavily interested topic (e.g. [30, 31]). We have some conclusions for complexity of CSP($\mathfrak{M}_{\alpha,*}$) when the parameter $\alpha \in (0,1]$ is large, and we have some graph-theoretically interesting findings towards characterizing such CSPs for slightly smaller αs.

Work in Chapter 2 was done in Spring 2021 through Fall 2021, then collected in 2022 for publication in [32]; work in Chapter 3 was done in Fall 2020 and Spring 2021; work in Chapter 4 spanned across late 2021 to late 2022 and was tidied up in early 2023. Project-specific introductions and preliminaries are collected at the beginning of each chapter.

Chapter 2: $(\mathbb{Z}, \mathrm{succ}, U)$, $(\mathbb{Z}, E, U)$, and the hardness of their CSP's

2.1 Introduction

For general introduction to CSPs, see Chapter 1. In this chapter we temporarily shift our focus onto constraint satisfaction problems of infinite template, and, as mentioned in Chapter 1, aim at characterizing CSP of unary expansion of $(\mathbb{Z}, \mathrm{succ})$ and their computational complexity. The main contents of this section have been published in [32], of which the author of this dissertation was the correspondence author.

So why do we care about characterizing CSP$(\mathbb{Z}, \mathrm{succ} := \{(x, y) : y = x + 1\}, U)$? On one hand, an efficient greedy algorithm solves CSP$(\mathbb{Z}, \mathrm{succ})$ (Section 2.2); on the other hand, with an infinite $U \subseteq \mathbb{Z}$, CSP$(\mathbb{Z}, \mathrm{succ}, U, \mathbf{0})$ can be hard: take U to be the halting set, for example, then the reduction $n \mapsto$ the direct $(n + 1)$-path $\mathbf{0} \dots n$ witnesses the undecidability of CSP$(\mathbb{Z}, \mathrm{succ}, U, \mathbf{0})$, and likewise for U a computationally harder set, say in $\Pi_3 \setminus \Sigma_2$. One naturally wonders how wild CSP$(\mathbb{Z}, \mathrm{succ}, U)$s could be, $(\mathbb{Z}, \mathrm{succ}, U)$ being virtually "sandwiched" between $(\mathbb{Z}, \mathrm{succ})$ and $(\mathbb{Z}, \mathrm{succ}, U, \mathbf{0})$ for each $U \subseteq \mathbb{Z}$.

Moreover, $(\mathbb{Z}, \mathrm{succ})$ and $(\mathbb{Z}, \mathrm{succ}, U, \mathbf{0})$ are both non-ω-categorical: the former has 2-types $\{(x, x + k)\}_{k<\omega}$, while the latter being connected, "well-distanced", and pointed is rigid, i.e. has trivial automorphism group. This indicates that non-ω-categoricity, a model-theoretic "wildness" bears no overall impact on the tractability of the structure's

CSP. Hence $\{(\mathbb{Z}, \text{succ}, U)\}_{U\subseteq\mathbb{Z}}$ affords another entry point for classification of the CSPs of non-ω-categorical structures.[1]

As mentioned above, recent work ([17, 18]; see also[20]) established that for a structure $\mathfrak{A}$ over domain $\mathbb{Z}$ with *finite* relational *signature*, if each relation is first-order definable in $(\mathbb{Z}, <)$ without parameters, then CSP($\mathfrak{A}$) is either in `P` or `NP`-complete. A dichotomy has also been established when the relations are definable from unary structures [15]. $(\mathbb{Z}, succ, U)$ *however is in neither case*: succ is not first order definable from unary structures[2] and the only 0-definable unaries in $(\mathbb{Z}, <)$ are $\emptyset$ and $\mathbb{Z}$.[3] Therefore we prove a class of "nicely gapped" unary expansions of $(\mathbb{Z}, \text{succ})$ has efficient CSP (Section 2.3,Theorem 2.17), and proceed to characterize all CSP$(\mathbb{Z}, \text{succ}, U)$ by gaps in U (Section 2.4, Theorem 2.24). In particular, some CSP$(\mathbb{Z}, \text{succ}, U)$ might be candidates for relatively natural NP-intermediate problems (Section 2.4, Corollary 2.26 and discussion).

The characterizations in Sections (2.3)(2.4), understandably, highly relies on the arithmetic nature of $(\mathbb{Z}, \text{succ})$, so it is natural to ask if they could be generalized to unary expansions of undirected graphs over $\mathbb{Z}$, i.e. the $(\mathbb{Z}, E, U)$s with E just an irreflexive, symmetric binary relation. We record some progress in this direction in Section 2.5,

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[1] Aut$(\mathbb{Z}, \text{succ}) \supseteq$ Aut$(\mathbb{Z}, \text{succ}, U)$, so any Aut$(\mathbb{Z}, \text{succ}, U)$-orbit on $\mathbb{Z}^2$, which is of the form $\{g(a) : g \in$ Aut$(\mathbb{Z}, \text{succ}, U)\}$ for some $a \in \mathbb{Z}^2$, is a subset of an Aut$(\mathbb{Z}, \text{succ})$-orbit. Therefore if Aut$(\mathbb{Z}, \text{succ})$ fails to partition $\mathbb{Z}^2$ with finitely many orbits, so does Aut$(\mathbb{Z}, \text{succ}, U)$. Essentially we just argued that if $G_1 \subseteq G_2$ acts on S with the G_2-action non-oligomorphic, so is the G_1-action.

[2] Note again that $(\mathbb{Z}, \text{succ})$ is non-ω-categorical while purely unary structures on $\mathbb{Z}$ are ω-categorical, failing to record the distances; furthermore succ on $\mathbb{Z}$ is not 0-definable from any unary structures on $\mathbb{Z}$. See also Remark 2.34. We prove the latter. Assume for the sake of contradiction that succ is 0-definable from a unary structure $(\mathbb{Z}, U_1, \ldots, U_k)$ and let $\psi(x, y) \in Sn_{\{U_j\}_j}$ be the defining formula. Then (by taking the x-slice) every unary appearing in $\psi(x, y)$ with free variable x has to be $\emptyset$ or $\mathbb{Z}$, so is every unary appearing in $\psi(x, y)$ with free variable y. Thus $\psi(x, y)$ is $\{U_1, \ldots, U_k\}$-equivalent to a sentence, while succ(x, y) is not.

[3] Indeed, for unaries to be 0-definable in $(\mathbb{Z}, <)$ they have to be preserved by Aut$(\mathbb{Z}, <)$ actions, which contains all translations.

and give some lower bound of CSP complexity (up to direct limits) assuming reasonable structural properties in the underlying digraph. In Section 2.6 we discuss some possible future directions.

2.2 Preliminaries

Definition of *constraint satisfaction problems* and *homomorphisms* are introduced in Chapter 1. More generally, *homomorphisms between relational structures* with the same signatures $(A; R_1^A, \cdots, R_k^A) \to (B; R_1^B, \cdots, R_k^B)$ is a map $h : A \to B$ s.t. $\forall j \in [1, k]$, $(a_1, \cdots, a_{r_j}) \in R_j^A \implies (h(a_1), \cdots, h(a_{r_j})) \in R_j^B$, with r_j the arity of R_j. In e.g. Section 2.5, we adopt the alternative formulation:

Problem 2.1 (CSP$(D, \mathcal{S})$, alternative formulation)**.**

Parameters (Fixed): A structure $\mathfrak{A} = (A, \tau^{\mathfrak{A}})$ with τ a finite relational signature.

Input: a first order sentence $\psi :\equiv \exists x_1 \ldots x_r R(x_1, \ldots, x_r)$, where $R(x_1, \ldots, x_r)$ is a finite conjunction of equations ("$=$") and atomics in τ.

Output: does $\mathfrak{A} \models \psi$?

A walk through the definitions should convince one that these two formulations are equivalent up to the transformation $\psi :\equiv \exists x_1, \ldots, x_r R_1(\vec{x}) \wedge \ldots R_j(\vec{x}) \mapsto$ the structure G on vertices $\{x_1, \ldots, x_r\}$ with each $R_\bullet^G(\bullet \in [1, j])$ as positively prescribed in ψ,[4] and vice versa. [5] First order sentences (resp. formulae) in the form of inputs to Problem 2.1

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[4] That is, for the j-ary relation R_j, each $(y_1, \cdots, y_j) \in \{x_1, \cdots, x_r\}^j$ is in $R_j^G \iff R_j(y_1, \cdots, y_j)$ is a conjunct appearing in ψ.

[5] Background on the alternative formulation: In 1978, Schaefer [33] proved the following theorem: let $\mathcal{S}$ be a finite collection of relations over $D := \{0, 1\}$, which we call *atomics*. Then for any input formula $R(x_1, \ldots, x_r)$ over D formed by a conjunction of atomics and equalities, the decision problem: "does there exist $x_1^*, \ldots, x_r^* \in D$ s.t. $(D, \mathcal{S}) \models R(x_1^*, \ldots, x_r^*)$?" is always NP-complete except for six special cases. Thus begins theorists' quest for a characterization of hardness of the CSP problems.

are *pp-sentence* (resp. *pp-formulae*). For $\mathfrak{A} = (A, \tau^{\mathfrak{A}})$, a relation $R \subseteq A^r$ is *pp-definable* (resp. *fo-definable*) without parameters in $\mathfrak{A}$ if there exists a pp(resp. first order) τ-formula $\psi(x_1, \ldots, x_r)$ s.t. $(a_1, \ldots, a_r) \in R \iff \mathfrak{A} \vDash \psi(a_1, \ldots, a_r)$.

For $\mathcal{L}_1, \mathcal{L}_2 \subseteq \bigcup_{n<\omega}\{\mathtt{0}, \mathtt{1}\}^n \cong_{\textbf{Set}} \mathbb{Z}$, write $\mathcal{L}_1 \leq^p_m \mathcal{L}_2$, read as "$\mathcal{L}_1$ Karp reduces to $\mathcal{L}_2$", if there exists a polynomial-time computable function $f : \bigcup_{n<\omega}\{\mathtt{0}, \mathtt{1}\}^n \to \bigcup_{n<\omega}\{\mathtt{0}, \mathtt{1}\}^n$ s.t. $x \in \mathcal{L}_1 \iff f(x) \in \mathcal{L}_2$. In that case, call f a "polynomial-time many-one/ *Karp" reduction* from $\mathcal{L}_1$ to $\mathcal{L}_2$. In comparison, write $\mathcal{L}_1 \leq^p_T \mathcal{L}_2$, read as "$\mathcal{L}_1$ *Cook reduces* to $\mathcal{L}_2$", if given a $\mathcal{L}_2$ solver $M_{\mathcal{L}_2}$, $\mathcal{L}_1$ can be solved by calling $M_{\mathcal{L}_2}$ polynomially — again with respect to input length — many times with polynomial time overhead. Note that $\mathcal{L}_1 \leq^p_m \mathcal{L}_2 \implies \mathcal{L}_1 \leq^p_T \mathcal{L}_2$ via overhead (time cost) of f and a single call to the $\mathcal{L}_2$ solver; the converse is not in general true, for example `NP` is closed under $\leq^p_m$ but not $\leq^p_T$.[6] Write $\underline{\mathcal{L}_1 \equiv^p_m \mathcal{L}_2}$ (resp. $\underline{\mathcal{L}_1 \equiv^p_T \mathcal{L}_2}$) if $\mathcal{L}_1 \leq^p_m \mathcal{L}_2 \wedge \mathcal{L}_2 \leq^p_m \mathcal{L}_1$ (resp. $\mathcal{L}_1 \leq^p_T \mathcal{L}_2 \wedge \mathcal{L}_2 \leq^p_T \mathcal{L}_1$). The following is well known:

Fact 2.2. *(e.g. [13]) Let relational structures* $\mathfrak{A} := (A, \tau^{\mathfrak{A}}), \mathfrak{A}' := (A, \zeta^{\mathfrak{A}'})$ *be such that for each* $R \in \tau$, $R^{\mathfrak{A}}$ *is pp-definable in* $\mathfrak{A}'$. *Then* $\mathrm{CSP}(\mathfrak{A}) \leq^p_m \mathrm{CSP}(\mathfrak{A}')$.

When describing structures $(A, \tau^{\mathfrak{A}})$, we drop the superscript if $\mathfrak{A}$ is clear from context. E.g., $\mathfrak{A} = (\mathbb{Z}, \mathrm{succ})$ is technically $(\mathbb{Z}, \mathrm{succ}^{\mathfrak{A}} := \{(x, y) : y = x + 1\})$. Likewise for $U \subseteq \mathbb{Z}$, $\mathfrak{A} := (\mathbb{Z}, \mathrm{succ}, U)$ means the unary relation on in $\mathfrak{A}$ is interpreted as $U^{\mathfrak{A}} := U$.

Both $\mathrm{CSP}(\mathbb{Z}, \mathrm{succ})$ and $\text{SEARCH-CSP}(\mathbb{Z}, \mathrm{succ})$ can be solved by the same efficient greedy algorithm that, for each connected component $\mathfrak{D}_c$ of the input $\mathfrak{D}$, starts by

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[6] Any $\mathcal{L} \in$ `NP` poly-time Turing reduces to its complement in $\bigcup_{n<\omega}\{\mathtt{0}, \mathtt{1}\}^n$ using one oracle call and constant overhead. Since each $\mathcal{L}' \in$ `coNP` arises this way, if `NP` were closed under $\leq^p_T$ we would have `coNP` $\subseteq$ `NP`, which is not true.

This also serves as an example to show that not all Turing reductions with 1 oracle call and polynomial (in fact constant) time and space cost are many-one reductions!

assigning an arbitrary vertex $a \in D_c$ to 0 and then assigns each of a's neighbor to -1 (if $(*, a) \in \mathrm{succ}^{\mathfrak{D}}$) or 1 (if $(a, *) \in \mathrm{succ}^{\mathfrak{D}}$), then recurses on those neighbors. We refer to this algorithm as *the* CSP($\mathbb{Z}, \mathrm{succ}$) *decider / searcher*.

Let $\mathfrak{A}$ be a τ-structure; *Aut*($\mathfrak{A}$) (*resp. End*($\mathfrak{A}$)) denotes its group of τ-automorphisms (resp. monoid of endomorphisms). $\mathfrak{A}$ is ω-*categorical* if its first order theory Th($\mathfrak{A}$) (namely all sentences satisfied by $\mathfrak{A}$) is ω-*categorical*, i.e. there is exactly one countably infinite model of Th($\mathfrak{A}$) up to isomorphism. Another well known fact used in Section 2.5:

Fact 2.3 (e.g. [34], equivalent characterization of ω-categoricity)**.** *Let τ be a countable signature, T a complete τ-theory which has infinite models. Then the following are equivalent:*

1. *All countable models of T are isomorphic;*
2. *Let $\mathfrak{A} \models T$, then Aut($\mathfrak{A}$) has finitely many orbits in its action $(a_1, \ldots, a_n) \mapsto (ga_1, \ldots, ga_n)$ on A^n, for any $n \geq 1$.*
3. *Some countable model of T realizes only finitely many complete n-types for each $n < \omega$.*

2.3 A Sufficient Condition for Tractability of CSP($\mathbb{Z}, \mathrm{succ}, U$)

We collect the main results in this section in Figure 2.1, where the two (more-or-less) "horizontal" reductions are simple. $U \leq_m^{\exp}$ CSP($\mathbb{Z}, \mathrm{succ}, U, \mathbf{0}$) by constructing the directed path $\mathfrak{D} := 0 \to 1 \to \cdots \to n$ with $U^{\mathfrak{D}} := \{n\}, \mathbf{0}^{\mathfrak{D}} = 0$;[7] it is exponential time

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[7] If we view the constant symbol $\mathbf{0}$ as a unary $\mathbf{O} = \{0\}$ in order to meet the narrow definition of relational structures, we end up with a Karp-equivalent CSP($\mathbb{Z}, \mathrm{succ}, U, \mathbf{O}$), so for our purposes it does

$$\begin{tikzcd}
& U \arrow[r, "{\leq_m^{exp}\ ,\ n \longmapsto 0 \to 1 \to \cdots \to \underline{\underline{n}}}"] & \mathrm{CSP}\,(\mathbb{Z},\ \mathrm{succ},\ U, \mathbf{0}) \\
\begin{array}{c} 1 \rightsquigarrow\ k, \\ \text{ELG} \rightsquigarrow \text{ELMS}, \\ U \rightsquigarrow \lim_{\leftarrow j} U_j \end{array} & & \\
& \mathrm{CSP}(\mathbb{Z},\ \mathrm{succ},\ U) \arrow[uu, "{\leq_T^p}", "{\begin{array}{l}\text{when}\\ \text{ELG-SR}\end{array}}"'] \arrow[uur, "{\leq_m^p,\ id}"] &
\end{tikzcd}$$

Figure 2.1: Some reductions: U, CSP$(\mathbb{Z}, \mathrm{succ}, U)$, CSP$(\mathbb{Z}, \mathrm{succ}, U, \mathbf{0})$.

since the length$|\langle n\rangle| = \log(n)$. CSP$(\mathbb{Z}, \mathrm{succ}, U) \leq_m^p$ CSP$(\mathbb{Z}, \mathrm{succ}, U, \mathbf{0})$ using the same input pp-sentence. The "verticle" reduction is the main goal of this section.

We do not believe there is a many-one or Turing reduction from U to CSP$(\mathbb{Z}, \mathrm{succ}, U)$; intuitively, without a $\mathbf{0}$ (or any constant for that matter) to "anchor the origin", End$(\mathbb{Z}, \mathrm{succ})$ = Aut$(\mathbb{Z}, \mathrm{succ}) \cong \mathbb{Z}$ and only the gap patterns in U decide whether a particular translation of $\{\mathrm{succ}\}$-homomorphism makes a $\{\mathrm{succ}, U\}$-homomorphism. In Section 4 we shall formalize this intuition; here, we give some nice conditions on the gaps to guarantee the tractability of CSP$(\mathbb{Z}, \mathrm{succ}, U)$. First, things are nice if U is periodical:

Observation 2.4. Let $U = a\mathbb{Z} + b$ for some $a, b \in \mathbb{Z}$. Then CSP$(\mathbb{Z}, \mathrm{succ}, U) \in \mathtt{P}$.

Proof. Modify the greedy algorithm that determines if an input $G \in$ CSP$(\mathbb{Z}, \mathrm{succ})$. for each component $\mathfrak{D}_e \subseteq \mathfrak{D} = (D, \mathrm{succ}^{\mathfrak{D}}, U^{\mathfrak{D}})$, if $D_e \cap U^{\mathfrak{D}} = \emptyset$, run the CSP$(\mathbb{Z}, \mathrm{succ})$ decider on $\mathfrak{D}_e$; otherwise pick an arbitrary $d_e \in U^{\mathfrak{D}} \cap D_e$, then run the greedy CSP$(\mathbb{Z}, \mathrm{succ})$ algorithm on $\mathfrak{D}_e$ initiated by mapping d_e to b. When the algorithm accepts, it yields a $\{\mathrm{succ}\}$-homorphism $h_e : (D, \mathrm{succ}^{\mathfrak{D}}) \to (\mathbb{Z}, \mathrm{succ})$; accepts on this component only if

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not matter. Indeed, to see CSP$(\mathbb{Z}, \mathrm{succ}, U, \mathbf{0}) \leq_m^p$ CSP$(\mathbb{Z}, \mathrm{succ}, U, \mathbf{O} = \{0\})$, for each LHS input $\mathfrak{D}$ with $\mathbf{0}^{\mathfrak{D}} = d$ construct $\widetilde{\mathfrak{D}}$ with $\mathbf{O}^{\widetilde{\mathfrak{D}}} = \{d\}$; Conversely, collapse every member of $\mathbf{O}^{\mathfrak{D}}$ to one arbitrary member thereof, or create an isolated new vertex if $\mathbf{O}^{\mathfrak{D}} = \emptyset$.

$h_e(d'_e) \in U$ for each $d'_e \in U^{\mathfrak{D}} \cap D_e$. Accept $\mathfrak{D}$ only if all components accept.

If the algorithm above accepts, obviously each h_e is a $\{\text{succ}, U\}$-homomorphism; conversely if $\mathfrak{D} \in \text{CSP}(\mathbb{Z}, \text{succ}, U)$, so is each component $\mathfrak{D}_e$. Hence there exists some homomorphism h_e sending d_e to some $x \in b + a\mathbb{Z}$. Now $\text{Aut}(\mathbb{Z}, \text{succ}, U) \cong \mathbb{Z}$ composes of translations by ak ($k \in \mathbb{Z}$), so there exists $h'_e \in \text{Hom}_{\text{succ},U}(\mathfrak{D}_e, (\mathbb{Z}, \text{succ}, U))$ sending d_e to b, which is the one affording acceptance of $\mathfrak{D}_e$.

$\square$

$\text{CSP}(\mathbb{Z}, \text{succ}, U)$ also behaves tamely, with its hardness upper-bounded by U, if the gaps in U are eventually large. Let us generalize this observation.

Definition 2.5. Let $U = \{u(i)\}_{i<\omega} \subseteq \mathbb{Z}$. U is *eventually largely gapped (ELG)* if $\exists c > 0$, s.t. $\forall n > c$, $\forall n' \in \omega \setminus \{n\}$, $|u(n) - u(n')| > n$.

As the name suggests, ELG asserts that after certain point the gaps between elements are at least "linearly large" with respect to its index.

Example 2.6. *Let $f(x) \in \mathbb{Z}[x]$ with $\deg(f(x)) \geq 2$, e.g. $6x^3 + 4x + 7$. One may verify that $U_f := \{f(x) : x < \omega\}$ is* ELG. [8]

Example 2.7. *Likewise for any differentiable $g(x) : \mathbb{R} \to \mathbb{R}$, if $g(\omega) \subseteq \mathbb{Z}$ with superpolynomial growth rate and some more mild assumptions, we have $U_g := \{g(x) : x < \omega\}$ is* ELG. *E.g.* $g(x) = -x \cdot 2^x$. [9]

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[8] As $x \to \infty$, f is eventually monotonic, and the formal derivative $|f'(x)| \sim \Omega(x)$, and when $|f'(x)| \sim \Theta(x)$ the absolute value of coefficient of x-term is at least 2. For $c >> 0$ we have $|f(x) - f(x^*)| \geq \max\{|f(x) - f(x-1)|, |f(x) - f(x+1)|\} > x$ whenever $c < x \in \omega, x^* \in \omega \setminus \{x\}$. So indeed U_f is ELG.

[9] The two more mild assumptions are (1) g is eventually monotonic and (2) $|g'(x)| \sim \omega(|x|)$. Indeed, for $x >> 0$ we have $\frac{|g(x)-g(x-1)|}{1} \geq |g'(x-1)| \sim \omega(x-1)$ so *LHS* $> x$ eventually, and similarly, $|g(x+1) - g(x)| > x$ eventually.

Definition 2.8. Let $U = \{u(i)\}_{i<\omega} \subseteq \mathbb{Z}$. U has *small representation (SR)* if there exists a $p(x) \in \mathbb{N}[x]$ s.t. $|\langle u(i) \rangle| \sim O(p(i))$ for each $i < \omega$. [10]

We say U is *ELG-SR* if U is *ELG* and has *SR*.

Example 2.9. *In Examples 2.6 and 2.7, both U_f and U_g have* SR*, and are hence* ELG-SR*. Note $|\langle -2^n \rangle| \sim O(n)$. Also if $U = \{f(x)\}_{x\in\omega}$ for some $f(x) \in \mathbb{Z}[x]$, then U has small representation.* [11]

Proposition 2.10. *Let $U = \{u(i)\}_{i<\omega}$ be* ELG-SR. *Then* $\mathrm{CSP}(\mathbb{Z}, \mathrm{succ}, U) \leq^p_T U$.

Proof. For each connected component $\mathfrak{D}_c$ of the input $\mathfrak{D} = (D, \mathrm{succ}^{\mathfrak{D}}, U^{\mathfrak{D}})$, reject if the $\mathrm{CSP}(\mathbb{Z}, \mathrm{succ})$-solver rejects on $\mathfrak{D}_c$; otherwise the solver finds a succ-homomorphism $h : (\mathfrak{D}_c, \mathrm{succ}^{\mathfrak{D}} \cap D_c^2) \to (\mathbb{Z}, \mathrm{succ})$. Thanks to ELG, one needs to shift h *at most* $|D_c|$ *times* to decide if any translation makes h a (succ, U)-homomorphism. SR ensures the absence of exponential blowup in this algorithm.

We describe the procedure in more details: upon input $\mathfrak{D} = (D, \mathrm{succ}^{\mathfrak{D}}, U^{\mathfrak{D}})$, for each component $\mathfrak{D}_e$:

1. If $D_e \cap U^{\mathfrak{D}} = \emptyset$, run $\mathrm{CSP}(\mathbb{Z}, \mathrm{succ})$ decider on $(D_e, \mathrm{succ}^{\mathfrak{D}} \cap D_e^2)$; if $|D_e \cap U^{\mathfrak{D}}| = 1$, run the $\mathrm{CSP}(\mathbb{Z}, \mathrm{succ})$ algorithm beginning with assigning the unique $d_e \in D_e \cap U^{\mathfrak{D}}$ to an arbitrary $b \in U$.

   Correctness follows from the observation that, in these cases we are still free to translate a succ-homomorphism as long as one exists.

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[10] Conventionally, $\langle \bullet \rangle$ is the binary representation of an integer and $| \bullet |$ is the length.

[11] Use, e.g. $p(x) \in \mathbb{N}[x]$ such that each coefficient is the absolute value of the corresponding term of $f(x)$. The values of $p(x)$ and $f(x)$ are asymptotically of the same order, so the binary code of $f(x^*)$ is short compared to $p(x^*)$ for each x^*.

2. Now $|D_e \cap U^{\mathfrak{D}}| \geq 2$. Since U is ELG, by definition there exists $c \in \omega$ witnessing it, i.e. gaps in U are large after index c. Pick arbitrary $d_e \in D_e \cap U^{\mathfrak{D}}$. Run Algorithm 1:

```
while $i \leq \max\{c, |D_e|\}$ do
    run CSP$(\mathbb{Z}, \mathrm{succ})$ solver on $\mathfrak{D}_e$ initiated by sending $d_e$ to $u(i)$
    if CSP$(\mathbb{Z}, \mathrm{succ})$ reject$s$ then
        reject
    else
        //CSP$(\mathbb{Z}, \mathrm{succ})$ found $h \in \mathrm{Hom}(\mathfrak{D}_e, (\mathbb{Z}, \mathrm{succ}))$ with $h(d_e) = u(i)$
        for $d'_e \in D_e \cap U^{\mathfrak{D}}$ do
            Test if $h(d'_e) \in U$, reject if not
        end
        accept for $\mathfrak{D}_e$
    end
end
```

Algorithm 1: The reduction CSP$(\mathbb{Z}, \mathrm{succ}, U) \leq_T^p U$.

And $\mathfrak{D}$ is accepted if and only if each $\mathfrak{D}_e$ is accepted. To see efficiency: since U has SR, $|\langle u(i) \rangle|$ is polynomial in i, [12]and i is linear in $|D_e| \leq |D|$. If a $\{\mathrm{succ}\}$-homomorphism h is found, we have polynomially many calls to the U-oracle; note that each $h(d'_e) \in [u(i) - n, u(i) + n]$, so it remains polynomial size. The whole algorithm is therefore poly-calls to U with poly-overhead.

To see correctness, first observe the following fact: in a connected component of

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[12] Note we need SR here, not just $U \in$ P.

size n, for any vertices $x \neq y$ the longest "*signed path*" (vertices $x_0 := x, x_1, \dots, x_k := y$ s.t. for each i, either $(x_i, x_{i+1}) \in E$ or $(x_{i+1}, x_i) \in E$) has at most $n-1$ edges, and the *signed length* (computed by adding 1 if $(x_i, x_{i+1}) \in E$ and subtracting 1 if $(x_{i+1}, x_i) \in E$).[13] is at most $n-1$. Whenever the component homomorphically maps to $(\mathbb{Z}, \text{succ})$ via some h, the signed length of any signed path must be $h(y) - h(x)$.

Now for any $i > \max\{c, |D_e|\}$, $|u(i) - *| > i > |D_e|$ for any $* \in U \setminus \{u(i)\}$, and with $|D_e \cap U^{\mathfrak{D}}| \geq 2$, this gap is too big for there to exist a homomorphism sending d_e to $u(i)$. In other words, $\mathfrak{D}_e \in \text{CSP}(\mathbb{Z}, \text{succ}, U) \iff \exists h \in \text{Hom}_{\text{succ}}(\mathfrak{D}_e, (\mathbb{Z}, \text{succ}, U))$ s.t. $h(d_e) \in \{u(i)\}_{i \leq \max\{|D_e|, c\}}$, which is what the algorithm checks.

□

Corollary 2.11. *If U is* ELG-SR*,* CSP($\mathbb{Z}$, succ*,* U) *is in* `P`*.* [14]

Proof. The above reduced $\text{CSP}(\mathbb{Z}, \text{succ}, U)$ to $O(|D|)$-many tests of whether each $h(d'_e) \in U$. But this can be done by asking "is $h(d'_e) = u(j)$?" for each $j \in [0, \max\{c, |D_e|\} + 1]$". Both $h(d'_e)$ and each $u(j)$ are of size polynomial in $|D|$, thanks to SR and the connectedness of $\mathfrak{D}_e$. □

Remark 2.12. If the gaps are "eventually small" instead, we are requiring U to be finite, which puts $\text{CSP}(\mathbb{Z}, \text{succ}, U)$ in `P` by running the $\text{CSP}(\mathbb{Z}, \text{succ})$ decider $|U|$ times.

Proposition 2.10 generalizes to expansion of $(\mathbb{Z}, \text{succ})$ by finitely many unaries,

---

[13] Note: if for some i both (x_i, x_{i+1}) and (x_{i+1}, x_i) are in E then the graph cannot homomorphically map to $(\mathbb{Z}, \text{succ})$; if between x, y there exist two different signed paths with different signed lengths, the graph is also not in $\text{CSP}(\mathbb{Z}, \text{succ})$.

[14] Examples include all the unaries in Example 2.6.

modulo the following caveat: one needs to handle the "cross-set" gaps and work with the "least upper bound" of the finite collection of unaries in terms of hardness.

Definition 2.13. Let $k \geq 1$. A collection $\mathcal{U} := \big\{U_1 = \{u(i,1)\}_{i<\omega}, \ldots, U_k = \{u(i,k)\}_{i<\omega} \subseteq \mathbb{Z}\big\}$ is *eventually largely mutually separated (ELMS)* if

1. Each U_j *(*$j \in [k]$*)* is *ELG*, and
2. For each $(j,l) \in \binom{k}{2}$, $\exists c_{j,l} > 0$, s.t.
    - $\forall n_j > c_{j,l}, \forall n_l < \omega, |u(n_j, j) - u(n_l, l)| > n_j$, and
    - $\forall n_l > c_{j,l}, \forall n_j < \omega, |u(n_l, l) - u(n_j, j)| > n_l$.

Say $\mathcal{U}$ is *ELMS-SR* if it is *ELMS* and each U_j $(j \in [k])$ has *SR*.

Example 2.14. *For differentiable functions $f_1, f_2 \in \mathbb{R}^{\mathbb{R}}$ each as described in Example 2.7, if they have opposite signs when restricted to ω, $\{U_{f_1}, U_{f_2}\}$ is* ELMS*; in particular e.g. for $f_1(x) = x^3$, $f_2(x) = -2^x$, $\{U_{f_1}, U_{f_2}\}$ is* ELMS-SR.

Notation 2.15. Let $U_1, \ldots, U_k \subseteq \mathbb{Z}$. We denote by $\varinjlim_{j\in[k]} U_j$ the following set:

$$\varinjlim_{j\in[k]} U_j := \{n = mk + (r \mod k) : r \in [1,k], m \in U_r\} \tag{2.1}$$

Remark 2.16. We note that $\varinjlim_{j\in[k]} U_j$ is the least upper bound of $\{U_j\}_j$ in the $\leq_m^p$-order of Karp-equivalent classes.[15] To see this, observe that each $U_r \leq_m^p \varinjlim_j U_j$ via $m \mapsto mk + (r \mod k)$, and if $Q \subseteq \mathbb{Z}$ is s.t. $U_r \leq_m^p Q$ via f_r for each $r \in [k]$, then $\varinjlim_j U_j \leq_m^p Q$ via

---

[15] for similar constructions in a different setting, see e.g. [35].

$x \mapsto f_r(\frac{x-(x \mod k)}{k})$ where $r = \begin{cases} x \mod k & \text{if } k \nmid x \\ k & \text{o.w.} \end{cases}$ [16]

Now we generalize Proposition 2.10 and Corollary 2.11 below. Both proofs resemble and generalize the single-unary case.

Theorem 2.17. *Let* $\{U_j\}_{j\in[k]}$ *be* ELMS-SR. *Then* CSP$(\mathbb{Z}, \mathit{succ}, (U_j)_j) \leq_T^p \varinjlim_j U_j$.

Proof. Upon input $\mathfrak{D} = (D, \mathrm{succ}^{\mathfrak{D}}, U_1^{\mathfrak{D}}, \ldots, U_k^{\mathfrak{D}})$, for each component $\mathfrak{D}_e$:

1. If $|D_e \cap (\bigcup_j U_j^{\mathfrak{D}})| \leq 1$, like in Proposition 2.10 the gaps within and across U_js make no impact on existence of homomorphisms. One just needs to run the CSP$(\mathbb{Z}, \mathrm{succ})$ decider / searcher starting with assinging d_e to an arbitrary $b \in U_j$, where $U_j^{\mathfrak{D}}$ is the unique unary intersecting D_e with d_e the unique intersection.

2. Otherwise, let $c := \max$ of the $\leq k + \binom{k}{2}$ constants witnessing ELMS. Then for any $(j, l) \in k^2$, any $n > \max\{c, |D_e|\}$, any $n' \in \begin{cases} \mathbb{N} & \text{if } l \neq j \\ \mathbb{N} \smallsetminus \{n\} & \text{o.w.} \end{cases}$ we have $|u(n, j) - u(n', l)| > n > |D_e|$. It follows that one needs at most $O(|D_e|) \sim O(|D|)$ iterations, for $n \in [0, \max(c, |D_e|)]$.

   Fix arbitrary $d_e \in D_e \cap (\bigcup_j U_j^{\mathfrak{D}})$ before iterating. Within each iteration, run CSP$(\mathbb{Z}, \mathrm{succ})$ assigning d_e to some $u(n, y)$ where $d_e \in D_e \cap U_y^{\mathfrak{D}}$, n is the iteration number ($n \leq \max(c, |D_e|)$). Since U_y has SR, $u(n, y)$ has size polynonial in n, and n is linear in $|D|$. If a succ-homomophism h is found, verify whether $h(d'_e) \in U_{y'}$ for each $d'_e \in D_e \cap U_{y'}^{\mathfrak{D}}$. Via $U_{y'} \leq_m^p$ (hence $\leq_T^p$) $\varinjlim_j U_j$, the decision of $U_{y'}$-membership of $h(d'_e)$ reduces to calls to the $\varinjlim_j U_j$-oracle. Note each

---

[16] Note when $k \geq 2$, $x \mapsto \frac{x-(x \mod k)}{k}$ does not describe a many-one reduction from $\varinjlim_j U_j$ to any U_j as there are different congruent classes.

$h(d'_e) \in [u(n, y) - |D_e|, u(n, y) + |D_e|]$, so the reduction is of time polynomial with respect to $|D|$.

$\square$

Corollary 2.18. *If* $\{U_j\}_j$ *is* ELMS-SR*, Then* CSP$(\mathbb{Z}, \mathit{succ}, U_1, \cdots, U_k) \in \mathtt{P}$.

Proof. As in Corollary 2.11, for each $h(d'_e)$, ask if $h(d'_e) = u(s, t)$ for each $t \in [1, k]$ involved, $s \in [0, \max\{c, |D_e|\} + 1]$. By SR and connectedness, both $h(d'_e)$ and each $u(s, t)$ have size polynomial in $|D|$. $\square$

We exhibit an interesting application of the result in this section:

Corollary 2.19. *Let* $f(x), g(x) \in \mathbb{Z}[x]$ *with different degrees, both non-*0*, and different signs in the term of highest degree. Then CSP(*$\mathbb{Z}, \mathit{succ}, \{f(x)\}_{x\in\omega}, \{g(x)\}_{x\in\omega}$*)* $\in \mathtt{P}$.

Remark 2.20. With 3 or more polynomials, application of the Theorem may get issues in number theory.[17]

2.4 Bounds and Characterization of CSP$(\mathbb{Z}, \mathrm{succ}, U)$

We now proceed to more general cases, bearing in mind that less restrictions on the unitaries usually means less control over the behavior of expansion of $(\mathbb{Z}, \mathrm{succ})$ with those unaries.

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[17] Consider e.g. $U_3 = \{x^3\}_{x\in\omega}$. Then $\{U_1, U_2, U_3\}$ is not ELMS; in fact even $\{U_1, U_3\}$ are not. In particular, $x^2 - y^3 = 0$ has infinitely many solutions $(x, y) \in \mathbb{N}^2$: take any prime p, then for each $k > 0$, $(p^{3\cdot 2^k})^2 = (p^{2^{k+1}})^3$.

2.4.1 "Lower Bounds" of CSP$(\mathbb{Z}, \text{succ}, U)$

Let $U \subseteq \mathbb{Z}$ be arbitrary. Recall CSP$(\mathbb{Z}, \text{succ}, U) \leq_m^p$ CSP$(\mathbb{Z}, \text{succ}, U, \mathbf{0})$ as the upper bound. It would be nice if the converse held, but we believe it is unlikely due to similar "anchoring" issues (See e.g. Section 2.3 and *infra*). However,

Proposition 2.21. CSP$(\mathbb{Z}, \text{succ}, U, \mathbf{0}) \leq_T^p \varinjlim \{U, \text{CSP}(\mathbb{Z}, \text{succ}, U)\}$.

Proof. Test the linearly many non-$\mathbf{0}$ components of the input by CSP$(\mathbb{Z}, \text{succ}, U)$-oracle calls. For the $\mathbf{0}$-component, find if a $\{\text{succ},\mathbf{0}\}$-homomorphism h exists, and make $O(|D|)$-calls to the U-oracle to find if h is a $\{\text{succ}, U, \mathbf{0}\}$-homomorphism. In more details: upon each input $\mathfrak{D} = (D, \text{succ}^{\mathfrak{D}}, U^{\mathfrak{D}}, \mathbf{0}^{\mathfrak{D}})$, let $\mathfrak{D}_0 :=$ the unique component containing $\mathbf{0}^{\mathfrak{D}}$. Call CSP$(\mathbb{Z}, \text{succ}, U)$-oracle on $\mathfrak{D} \smallsetminus \mathfrak{D}_0$ and echo if rejects.

Call the CSP$(\mathbb{Z}, \text{succ})$-solver on $(D_0, \text{succ}^{\mathfrak{D}} \cap D_0^2)$ initiated by sending $\mathbf{0}^{\mathfrak{D}}$ to 0, and echo if rejects; otherwise, the unique succ-homomorphism $h : (D_0, \text{succ}^{\mathfrak{D}} \cap D_0^2) \to (\mathbb{Z}, \text{succ})$ that satisfies $h(\mathbf{0}^{\mathfrak{D}}) = 0$ has been found, and one just needs to verify if for each d we have $h(d) \in U$ by calling the U-oracle. Note each $h(d) \in [0 - |D_0|, 0 + |D_0|]$ so input size remains polynomial in $|D_0| \leq |D|$.

The only oracles we called above are the U-oracles and CSP$(\mathbb{Z}, \text{succ}, U)$-oracles, both of which Karp (hence Cook) reduces to the $\varinjlim$-oracle on RHS. □

Hence, up to taking direct limit with U, CSP$(\mathbb{Z}, \text{succ}, U)$ is "lower bounded" by CSP$(\mathbb{Z}, \text{succ}, U, \mathbf{0})$ in the $\leq_T^p$-order. A notable corollary:

Corollary 2.22. *If* $U \in \mathtt{P}$, *or* $U \leq_T^p$ CSP$(\mathbb{Z}, \text{succ}, U)$, *then* CSP$(\mathbb{Z}, \text{succ}, U) \equiv_T^p$ CSP$(\mathbb{Z}, \text{succ}, U, \mathbf{0})$.

Proof. If $U \in \mathtt{P}$, calls to the U-oracle counts as polynomial time overhead; if $U \leq_T^p$

CSP$(\mathbb{Z}, \text{succ}, U)$, note the direct limit can be computed as $\begin{cases} \chi_U(\frac{n-1}{2}) & \text{if } n = 2k+1 \\ \chi_{\text{CSP}(\mathbb{Z},\text{succ},U)}(\frac{n}{2}) & \text{o.w.} \end{cases}$,
where $\chi_S(\bullet)$ is the membership function of set S, and χ_U can be computed by computing $\chi_{\text{CSP}(\mathbb{Z},\text{succ},U)}$ now at polynomial cost. $\square$

2.4.2 Karp-Equivalent Characterizations of CSP$(\mathbb{Z}, \text{succ}, U)$

Recall (e.g. Section 2.3) that intuitively, for any $U \subseteq \mathbb{Z}$ its gap patterns determine the members of CSP$(\mathbb{Z}, \text{succ}, U)$, and conversely one may recover the gap patters of U from CSP$(\mathbb{Z}, \text{succ}, U)$. Our characterization below formalizes this intuition.

Definition 2.23. Let $U \subseteq \mathbb{Z}$ with $|U| \geq 2$. Define Gap(U) :=

$$\left\{(n, S_1, S_2) \in \left(\omega \setminus \{0,1\}\right) \times 2^{\binom{n}{2}} \times [-n, n]^{S_1} \;\middle|\; \exists t \in U^n, (\forall (i<j) \in S_1)(t(j) - t(i) = S_2(i,j))\right\}$$

Informally Gap(U) is the collection of gap patterns "partially realized in U" by some U-tuple. In particular, $S_2 : S_1 \to [-n, n]$ is a function assigning each pair of indices $(i, j) \in S_1$ to a distance $u_i - u_j \in [-n, n]$ with $u_i, u_j \in U$, i.e. u_i, u_j witnesses the fact that such a distance is realized in U.[18] Note, though, that having $S_1 \subseteq \binom{n}{2}$ is crucial: we need to include the cases where only *some* distances between the n vertices are specified. This corresponds to the fact that homomorphisms only remember adjacencies but tolerates non-adjacencies in the target. The machinery affords the following equivalent characterization:

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[18] In the definition it should really read that $n \in \omega \setminus \{0, 1\}$, $S_1 \subseteq \binom{n}{2}$, $S_2 \in \left([-n, n] \setminus 0\right)^{S_1}$, and $\exists t \ldots$. We write in the current way for a bit more succinctness of representation.

Theorem 2.24. *Let* $U \subseteq \mathbb{Z}$ *with* $|U| \geq 2$*. Then* $\mathrm{Gap}(U) \equiv_m^p \mathrm{CSP}(\mathbb{Z}, \mathrm{succ}, U)$.

Proof. The idea is as follows: on a $\mathrm{Gap}(U)$-instance (n, S_1, S_2), construct $\mathfrak{D}$ as follows: start with $D = U^{\mathfrak{D}} = \{1, \cdots, n\}$, then add paths of length $S_2(i, j)$ between vertices i, j for each $(i < j) \in S_1$. Conversely, on a $\mathrm{CSP}(\mathbb{Z}, \mathrm{succ}, U)$-instance $\mathfrak{D} = (D, \mathrm{succ}^{\mathfrak{D}}, U^{\mathfrak{D}})$, first let the subgraph $\mathfrak{D}'$ collect each $\mathfrak{D}$-component that intersects with $U^{\mathfrak{D}}$ at more than 1 vertices; let $n := |\mathfrak{D}' \cap U^{\mathfrak{D}}|$, and let (S_1, S_2) keep track of the positive finite distances of each $U^{\mathfrak{D}}$-pair that lies on the same component of $\mathfrak{D}'$.

Before going into the full details, we illustrate the reductions with the following example (Figure 2.2):

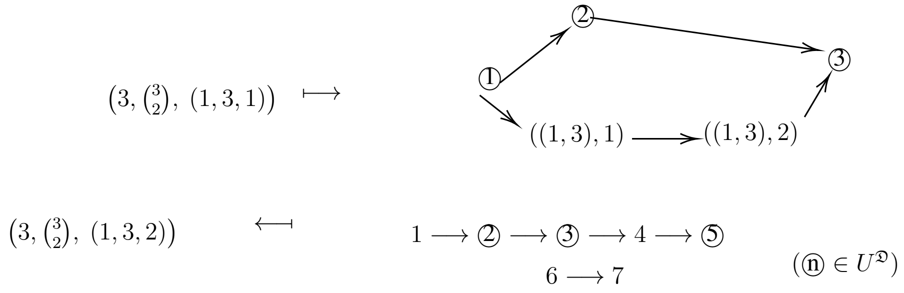

Figure 2.2: Example of Theorem 2.24 Reduction

1. $\leq_m^p$: upon (n, S_1, S_2), construct $D := \{1, \ldots, n\}, \mathrm{succ}^{\mathfrak{D}} = \emptyset, U^{\mathfrak{D}} := D$. Then for each $(i < j) \in S_1$:

   (a) If $S_2(i, j) > 0$, add intermediate vertices and edges $i = \big((i, j), 0\big) \overset{\mathrm{succ}^{\mathfrak{D}}}{\longrightarrow} \big((i, j), 1\big) \overset{\mathrm{succ}^{\mathfrak{D}}}{\longrightarrow} \ldots \overset{\mathrm{succ}^{\mathfrak{D}}}{\longrightarrow} \big((i, j), S_2(i, j)\big) = j$.

   (b) If $S_2(i, j) < 0$, add intermediate vertices and edges $i = \big((i, j), 0\big) \overset{\mathrm{succ}^{\mathfrak{D}}}{\longleftarrow} \big((i, j), 1\big) \overset{\mathrm{succ}^{\mathfrak{D}}}{\longleftarrow} \ldots \overset{\mathrm{succ}^{\mathfrak{D}}}{\longleftarrow} \big((i, j), |S_2(i, j)|\big) = j$.

   (c) if $S_2(i, j) = 0$, collapse vertices $i \sim j$ (so now the "vertex classes" $[i] = [j]$).

Now $D = \Big\{[1], \dots, [n]\Big\} \cup \Big\{\big((i,j),m\big)\Big\}_{S_2(i,j)\neq 0, m\in[1,|S_2(i,j)|-1]}$ and $U^{\mathfrak{D}} = \Big\{[1], \dots, [n]\Big\}$. Note $|D| \leq n + n \times \binom{n}{2}$, and as S_1 is part of the tuple, $|D|$ is polynomial in $|\langle(n, S_1, S_2)\rangle|$, so $(n, S_1, S_2) \mapsto \mathfrak{D} = (D, \mathrm{succ}^{\mathfrak{D}}, U^{\mathfrak{D}})$ is a polynomial-time construction. Further,

   (a) If $(n, S_1, S_2) \in \mathrm{Gap}(U)$ witnessed by $t \in U^n$, let $h : D \to \mathbb{Z}$, $[j] \mapsto t(j)$ ($j \in [1, n]$) and extend naturally to the intermediates, i.e. $\big((i,j),m\big) \mapsto t(i) + \mathrm{sign}(S_2(i,j)) \times m$. The map puts $\mathfrak{D} \in \mathrm{CSP}(\mathbb{Z}, \mathrm{succ}, U)$.

   (b) If $\mathfrak{D} \in \mathrm{CSP}(\mathbb{Z}, \mathrm{succ}, U)$ by h, then $\big(t(i) = h([i])\big)_{i\in[1,n]}$ witnesses $(n, S_1, S_2) \in \mathrm{Gap}(U)$.

2. ${}^{p}_{m}\geq$: Observe that $s_1 := \Big(3, \binom{3}{2}, (\underbrace{1}_{S_2(1,2)}, \underbrace{3}_{S_2(1,3)}, \underbrace{1}_{S_2(2,3)})\Big) \notin \mathrm{Gap}(U)$ due to triangle-inequality, and taking arbitrary $u_1 \neq u_2 \in U$ (not $U^{\mathfrak{D}}$), $s_2 := \Big(2, \big\{(1,2)\big\}, \big(u_2 - u_1\big)\Big) \in \mathrm{Gap}(U)$. s_1, s_2 are input-independent.

   Now let $\mathfrak{D} = (D, \mathrm{succ}^{\mathfrak{D}}, U^{\mathfrak{D}})$. We want to produce a short tuple $s_{\mathfrak{D}} = (n, S_1, S_2)$ in time polynomial in $|D|$. We present an algorithm for that.

   (a) Run $\mathrm{CSP}(\mathbb{Z}, \mathrm{succ})$ decider/searcher on $\mathfrak{D}$, and halt setting $s_{\mathfrak{D}} := s_1$ if `reject` was returned; else, we obtain an $h \in \mathrm{Hom}_{\mathrm{succ}}\big((D, \mathrm{succ}^{\mathfrak{D}}), (\mathbb{Z}, \mathrm{succ})\big)$ from the run.

   (b) Throw away the components $\mathfrak{D}_e$ with $|D_e \cap U^{\mathfrak{D}}| \leq 1$, keeping $\mathfrak{D}' := \coprod_{e:|D_e\cap U^{\mathfrak{D}}|\geq 2} \mathfrak{D}_e$. If $\mathfrak{D}' = \emptyset$, halt setting $s_{\mathfrak{D}} := s_2$.

   (c) Let $n := |U^{\mathfrak{D}} \cap D'|$; now $n \in [2, |D'|]$. Enumerate elements in $U^{\mathfrak{D}} \cap D'$ as $v_1, \dots, v_n$; let $S_1 := \{(i < j) : v_i, v_j$ are from the same component $\}$. Let

$S_2(i,j) := h(v_j) - h(v_i)$ with h from Step 2a. Let $s_{\mathfrak{D}} := \langle (n, S_1, S_2) \rangle$.

$s_{\mathfrak{D}}$ has size polynomial in $|D|$ and the algorithm is efficient. Correctness:

(a) Assume $\mathfrak{D} \in \mathrm{CSP}(\mathbb{Z}, \mathrm{succ}, U)$ witnessed by h', then $s_{\mathfrak{D}} \neq s_1$ for $h' \in \mathrm{Hom}_{\mathrm{succ}}\big((D, \mathrm{succ}^{\mathfrak{D}}), (\mathbb{Z}, \mathrm{succ})\big)$ and the CSP($\mathbb{Z}$, succ)-decider should've caught a homomorphism.

- If $\mathfrak{D}' \neq \emptyset$: for each $v_i \neq v_j$ from the same component in $\mathfrak{D}'$, $h'(v_j) - h'(v_i) = h(v_j) - h(v_i)$. Therefore setting $t_i = h'(v_i)$ for each $i \in [1, n]$, we get a witness for $(n, S_1, S_2) \in \mathrm{Gap}(U)$.
- If $\mathfrak{D}' = \emptyset$, we exited with an $s_2 \in \mathrm{Gap}(U)$.

(b) Assume $s_{\mathfrak{D}} \in \mathrm{Gap}(U)$, then again $s_{\mathfrak{D}} \neq s_1$.

- If $s_{\mathfrak{D}} = s_2$ then the h from Step 2a translates componentwise and then patches to a $\{\mathrm{succ}, U\}$-homomorphism.
- otherwise there exists $t \in U^n$ with $t(j) - t(i) = S_2(i,j)$ $(\forall (i < j) \in S_1)$. Each component in $\mathfrak{D}'$ contains a $v_{i_e} \in U^{\mathfrak{D}} \cap D'$; shift $h \restriction_{\mathfrak{D}_e}$ s.t. $h(v_{i_e}) = t(i_e)$. Now for each $v_{j_e} \neq v_{i_e} \in D_e \cap U^{\mathfrak{D}}$, $h(v_{j_e}) = h(v_{i_e}) + S_2(i_e, j_e) = t(j_e) \in U$. Do so for each component of $\mathfrak{D}'$.

  Lastly for each component in $\mathfrak{D} \setminus \mathfrak{D}'$, shift h if need be.

□

Remark 2.25. It may be worthy of mentioning that, letting $\mathcal{P}$:= the set of prime numbers, then the twin-prime conjecture is equivalent to a Π^0_2 statement about Gap($\mathcal{P}$). In particular, the twin-prime conjecture holds if and only if the following statement holds:

$$\forall k > 4, \exists k' > k, \text{ s.t. } \left(k', \big((1,2),(2,3),(3,4)\big), \begin{cases} (1,2) \mapsto & 1, \\ (2,3) \mapsto & k', \\ (3,4) \mapsto & 2 \end{cases} \right) \in \text{Gap}(\mathcal{P}).$$

Each witnessing k' is necessarily even. Indeed, if twin prime conjecture holds, for arbitrarily large $k > 4$ one may find $k' > k$, $p \in \mathcal{P}$, $p - 3 = k'$ and $p + 2 \in \mathcal{P}$. Then $(2, 3, p, p + 2)$ realizes the prescribed distance in $\mathcal{P}$. Conversely if the statement above holds, then for any $k > 4, k' > k$, we must have the witnessing $(u_1, u_2) = (2, 3)$. Furthermore (u_3, u_4) is a twin prime with $u_3 - u_2 = u_3 - 3 = k'$. Arbitrariness of k would ensure that there are infinitely many twin primes.

We close the section with an interesting question:

Question 2.26. Let $V \subseteq \omega$ be infinite. Is there a $U_V \subseteq \mathbb{Z}$ s.t. $V \leq_T$ *Gap*(U_V), and if not, what's an equivalent characterization?

That is, we would like for any infinite set V of naturals, there exists a set U_V whose "gap patterns" decide V-membership. Assuming that, an immediate corollary would be the unboundedness of computational complexity of Gap(U) and CSP$(\mathbb{Z}, \text{succ}, U)$s in general. If, in addition, $\leq_T$ in Question 2.26 could be refined to $\equiv^p_m$, then Ladner's theorem [36] and a recent construction [13, 37] would imply the existence of `NP`-intermediate CSPs *and* `CoNP`-intermediate CSPs respectively of the form CSP$(\mathbb{Z}, \text{succ}, U)$.

Combined with results from last section, we summarize our results in Figure 2.3.

$$\mathfrak{A} := (\mathbb{Z},\ \text{succ},\ U)$$

$$\begin{tikzcd}
\text{CSP}\,(\mathfrak{A},\mathbf{0}) \arrow[ddr, "\text{``}\leq_T^p\text{''}", "\text{up to } \lim_{\to}\{U,\bullet\}"'] & U \arrow[dr, "\leq_m^{exp}"] & \\
 & & \text{CSP}\,(\mathfrak{A},\mathbf{0}) \\
 & \text{CSP}(\mathfrak{A}) \arrow[uu, "\leq_T^p", "\text{when ELG-SR}"'] \arrow[ur, "\leq_m^p,"'] \arrow[d, shift right] & \\
 & \text{Gap}\,(U) \arrow[u, shift right, "\equiv_m^p"'] &
\end{tikzcd}$$

Figure 2.3: Bounds and Equivalences of CSP$(\mathbb{Z}, \text{succ}, U)$

2.5 From $(\mathbb{Z}, \text{succ}, U)$ to $(\mathbb{Z}, E, U)$

Now we generalize our results in Sections (2.3)(2.4) into arbitrary *partially*-1-*colored digraphs* on $\mathbb{Z}$, i.e. structures of the form $(\mathbb{Z}, E, U)$ with $E \subseteq \mathbb{Z}^2, U \subseteq \mathbb{Z}$. In particular, some primitive conditions for a Proposition 2.21-type reduction is presented (Theorem 2.31,Proposition 2.37), with the hope that some future research could yield further refined characterizations of $(\mathbb{Z}, E, U)$-structures. Note that:

1. The converses of Proposition 2.21-type reductions, i.e. $(\mathbb{Z}, E, U) \leq_m^p (\mathbb{Z}, E, U, \mathbf{0})$, always hold by the identity reduction (e.g. as pp-sentence).[19]

2. All arguments in Theorem 2.31 to Proposition 2.37 below generalize to $(\mathbb{Z}, \vec{E}, \vec{U})$, i.e. with finitely many binaries and finitely many unaries; $\mathbf{0}$ may be changed to $\mathbf{c}$ for any $\mathbf{c} \in \mathbb{Z}$. In these general cases we need a careful definition of connected

---

[19] Similarly by $(\mathbb{Z}, E) \leq_m^p (\mathbb{Z}, E, U)$, we know there are $(\mathbb{Z}, E, U)$s whose CSP is not in `NP`: just let the underlying $(\mathbb{Z}, E)$ be such that its CSP is CoNP-complete, which exists by Theorem 13.3.1 [13].Although that digraph is not originally defined on $\mathbb{Z}$, it is a Fraïssé limit hence countable and by a bijection to $\mathbb{Z}$ on the vertices, it can be made into the form $(\mathbb{Z}, E)$.

components , [20] and the arguments would be longer, but still completely analogous. For the ease of presentation, we assume our current format in the main text.

As a motivating example, fix any $d \geq 1$ and consider the relation $\text{Diff}_d(x, y) \iff y = x + d$. Note that Diff_d is pp-definable in $(\mathbb{Z}, \text{succ})$. The digraph $(\mathbb{Z}, \text{Diff}_d)$ consists of d pairwise isomorphic components, each on a coset of $d\mathbb{Z}$. Each component is also isomorphic to $(\mathbb{Z}, \text{succ})$, therefore $(\mathbb{Z}, \text{succ}) \to (\mathbb{Z}, \text{Diff}_d)$ by isomorphism with one component, while $(\mathbb{Z}, \text{Diff}_d) \to (\mathbb{Z}, \text{succ})$ by first collapsing all components to one and then applying the isomorphism with RHS. It follows that CSP$(\mathbb{Z}, \text{succ}) =$ CSP$(\mathbb{Z}, \text{Diff}_d)$, and SEARCH-CSP$(\mathbb{Z}, \text{Diff}_d) \in \mathtt{P}$ by running the SEARCH-CSP$(\mathbb{Z}, \text{succ})$ solver and then applying the isomorphism with a $(\mathbb{Z}, \text{Diff}_d)$-component. Moreover, just as fixing one vertex fixes a homomorphism – if any exists – to $(\mathbb{Z}, \text{succ})$, for finite *connected* $\{\text{Diff}_d, \mathbf{0}\}$-structure $\mathfrak{D}$ we have

$$\mathfrak{D} \in \text{CSP}(\mathbb{Z}, \text{Diff}_d, \mathbf{0}) \iff |\text{Hom}_{\text{Diff}_d, \mathbf{0}}(\mathfrak{D}, (\mathbb{Z}, \text{Diff}_d, \mathbf{0}))| = 1 \tag{2.2}$$

Indeed, LHS $\Longleftarrow$ RHS because $|\text{Hom}_{\text{Diff}_d, \mathbf{0}}(\mathfrak{D}, (\mathbb{Z}, \text{Diff}_d, \mathbf{0}))| > 0$, and $\Longrightarrow$ follows

---

[20] One may even have freedoms on the arities of the non unaries $\{E_j\}_j$. In these cases, call $x, y \in \mathbb{Z}$ "*connected*" if

(a) (Base) there exists some $j \in [1, k]$ s.t. $(\ldots, x, \ldots, y, \ldots)$ or $(\ldots, y, \ldots, x, \ldots) \in E_j^{\mathfrak{D}}$. OR

(b) $\exists z$ s.t. x, z are connected and z, y are connected.

Call $\mathfrak{D}' \subset_{\text{ind}} \mathfrak{D}$ a *connected component* of $\mathfrak{D}$ if for any $x \neq y \in \mathfrak{D}'$, x and y are connected; and $\forall x \in \mathfrak{D}'$, $\forall z \notin \mathfrak{D}'$, x, z are not connected. (This happens if and only if the Gaifman graph of $\mathfrak{D}'$ is a connected component of the Gaifman graph of $\mathfrak{D}$.) Vacuously, an isolated vertex (an $x \in D$ that does not appear in any $E_j^{\mathfrak{D}}$ with arity ≥ 2) is a connected component.

A $\{E_1, \ldots, E_k, \mathbf{c}\}$-structure $\mathfrak{D}$ is *connected* if $\mathfrak{D}$ itself is a connected component; equivalently, $\mathfrak{D}$ is not the disjoint union of two or more connected components. A notable property relevant to later discussion: when a finite, connected $\{E_1, \ldots, E_k\}$-structure $\mathfrak{D}$ has $E_j^{\mathfrak{D}} = \emptyset$ for all $j \in [1, k]$ then $|\text{Hom}_{E_1, \ldots, E_k, \mathbf{c}}(\mathfrak{D}, (\mathbb{Z}, E_1^{\mathfrak{A}}, \ldots, E_k^{\mathfrak{A}}, \mathbf{c}^{\mathfrak{A}}))| = 1$ for free, because $D = \{\mathbf{c}^{\mathfrak{D}}\}$ in this case: $\mathbf{c}^{\mathfrak{D}}$ is its own connected component and $\mathfrak{D}$ is connected.

from the fact that $\mathbf{0}^{\mathfrak{D}}$ must map to $\mathbf{0}$ and then the homomorphism must extend linearly as if in $(\mathbb{Z}, \mathrm{succ}, \mathbf{0})$.

Further expanding the structure by $U \subseteq \mathbb{Z}$, we no longer have $\mathrm{CSP}(\mathbb{Z}, \mathrm{succ}, U) = \mathrm{CSP}(\mathbb{Z}, \mathrm{Diff}_d, U)$: the gaps are not preserved by the homomorphic equivalence between base structures. However by an argument similar to Section 2.3 Proposition 2.21, one may obtain:

Observation 2.27. Let $U \subset \mathbb{Z}$. Then $\mathrm{CSP}(\mathbb{Z}, \mathrm{Diff}_d, U, \mathbf{0}) \leq_T^p \varinjlim \big\{U, \mathrm{CSP}(\mathbb{Z}, \mathrm{Diff}_d, U)\big\}$.

Proof. Similar to that of Proposition 2.21.

1. For each component $\mathfrak{D}_e$ s.t. $\mathbf{0}^{\mathfrak{D}} \notin \mathfrak{D}_e$, call the $\mathrm{CSP}(\mathbb{Z}, \mathrm{succ}, U)$-oracle, which reduces to the $\varinjlim \big\{U, \mathrm{CSP}(\mathbb{Z}, \mathrm{Diff}_d, U)\big\}$-oracle.

2. For the unique component $\mathfrak{D}_0 \ni \mathbf{0}^{\mathfrak{D}}$, run the greedy, efficient SEARCH-CSP$(\mathbb{Z}, \mathrm{Diff}_d)$ algorithm to decide if there is a (unique) $\{\mathrm{Diff}_d, \mathbf{0}\}$-homomorphism h. `Reject` if none found.

   Having found h, for each $u \in U^{\mathfrak{D}} \cap D_0$, call U-oracle on $h(u)$. Note that $h(u) \in [-|D|, |D|]$ and the oracle call once again reduces to calling $\varinjlim \big\{U, \mathrm{CSP}(\mathbb{Z}, \mathrm{Diff}_d, U)\big\}$.

□

Remark 2.28. Other interesting properties of $\mathrm{CSP}(\mathbb{Z}, \mathrm{Diff}_d, U)$ may be said. For example:

1. One may define *eventually d-largely gapped* $\big(ELG_d\big)$ $\Big($respectively *eventually d-largely mutually separated* $\big(ELMS_d\big)\Big)$ analogously to Definitions 2.5, 2.13, changing "$|u(n,*) - u(n',*)| > n$" into "$|u(n,*) - u(n',*)| > dn$". Keeping the defini-

tion of SR (2.8) unchanged, we have $\mathrm{CSP}(\mathbb{Z}, \mathrm{Diff}_d, U) \leq_T^p U$ when ELG$_d$-SR, and $\mathrm{CSP}(\mathbb{Z}, \mathrm{Diff}_d, \vec{U}) \leq_T^p \varinjlim \vec{U}$ when ELMS$_d$-SR.

2. Note that $\mathrm{Aut}(\mathbb{Z}, \mathrm{Diff}_d) \cong S_d \times (d\mathbb{Z})^d \cong S_d \times \mathbb{Z}^d$ as groups: indeed, an automorphism permutes the connected components and then shifts on each component by a distance in $d\mathbb{Z}$.

Remark 2.29. The above does not necessarily work for $(\mathbb{Z}, \mathrm{Dist}_d, U)$s where $\mathrm{Dist}_d(x, y) \iff |y - x| = d$, a relation defined in e.g. [17]; to begin with, $(\mathbb{Z}, \mathrm{Dist}_d) \not\to (\mathbb{Z}, \mathrm{succ})$, LHS failing to be directed acyclic.

Now consider any partially-1-colored digraph $\mathfrak{A} := (\mathbb{Z}, E, U)$. We are interested in generalizing Section 2.3 results to complexity lower bounds of $\mathrm{CSP}(\mathfrak{A})$.

Definition 2.30. Let $U \subseteq \mathbb{Z}, E \subseteq \mathbb{Z}^2$, and $\mathfrak{A} := (\mathbb{Z}, E^{\mathfrak{A}} := E, U^{\mathfrak{A}} := U)$. Consider its *reduct*[21] $\mathfrak{A}^{\downarrow} := (\mathbb{Z}, E)$. Let $\mathbf{0}$ be a constant.[22] We say $\mathfrak{A}^{\downarrow}$ is

1. *pointed homomorphically rigid (PHR)* if if for any finite, *connected* $\{E, \mathbf{0}\}$-structure $\mathfrak{D}$,

$$\mathfrak{D} \in \mathit{CSP}(\mathfrak{A}^{\downarrow}, \mathbf{0}) \iff |\mathrm{Hom}_{\{E,\mathbf{0}\}}\big(\mathfrak{D}, (\mathfrak{A}^{\downarrow}, \mathbf{0})\big)| = 1 \tag{2.3}$$

   That is, for any $\{E, \mathbf{0}\}$-structure $\mathfrak{D}$ there exists at most one $\{E, \mathbf{0}\}$-homomorphism to $(\mathfrak{A}^{\downarrow}, \mathbf{0})$.

2. *freely translational (FT)* if $\mathrm{Aut}_E(\mathfrak{A}^{\downarrow})$ contains all translations $f_z : x \mapsto x + z$ $(z \in \mathbb{Z})$.

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[21] I.e., forgetting U.

[22] Any interpretation of $\mathbf{0}$ works; without loss of generality we pick $\mathbf{0}^{(\mathfrak{A},\mathbf{0})} = 0 \in \mathbb{Z}$.

Note that the definition above does not "touch and concern" U. Therefore,

Theorem 2.31. *Let* $\mathfrak{A}^{\downarrow}$ *be* PHR *and* FT. *Then, for any* $U \subseteq \mathbb{Z}$,

$$\text{CSP}(\mathfrak{A}, \mathbf{0}) \leq_T^p \varinjlim \{U, \text{CSP}(\mathfrak{A}), \text{SEARCH-CSP}(\mathfrak{A}^{\downarrow})\} \tag{2.4}$$

Remark 2.32. When $E = succ$, we have SEARCH-CSP$(\mathbb{Z}, E) \leq_T^p$ *CSP*$(\mathbb{Z}, E) \in \mathtt{P}$ which gives back Proposition 2.21. In general we do not know if the Search to Decision reduction works for *infinite* structures.

Proof. [Proof of Theorem 2.31] Similar to Proposition 2.21. Decompose input $\mathfrak{D} = (\mathbb{Z}, E^{\mathfrak{D}}, U^{\mathfrak{D}}, \mathbf{0}^{\mathfrak{D}})$ into connected components.[23] For components $\mathfrak{D}_b$ avoiding $\mathbf{0}^{\mathfrak{D}}$, call the CSP$(\mathfrak{A})$ oracle.

For $\mathfrak{D}_{\mathbf{0}} \ni \mathbf{0}^{\mathfrak{D}}$, find an $\{E\}$-homomorphism $h : \mathfrak{D}_{\mathbf{0}} \to (\mathfrak{A}^{\downarrow})$ if it exists, by calling SEARCH-CSP$(\mathfrak{A}^{\downarrow})$. By FT, $s : x \mapsto x - h(\mathbf{0}^{\mathfrak{D}})$ is an E-endomorphism, hence $s \circ h$ is an "pointed" E-homomorphism where $\mathbf{0}^{\mathfrak{D}} \mapsto 0 = \mathbf{0}^{\mathfrak{A}}$, i.e. an $\{E, \mathbf{0}\}$-homomorphism. By PHR, $s \circ h$ is *the* unique $\{E, \mathbf{0}\}$-homomorphism $\mathfrak{D}_{\mathbf{0}} \to (\mathfrak{A}^{\downarrow}, \mathbf{0})$. Now to check if $s \circ h$ is further an $\{E, \mathbf{0}, U\}$-homomorphism, simply test if $s \circ h(u) \in U = U^{\mathfrak{A}}$ for each $u \in U^{\mathfrak{D}} \cap D_0$.[24] ☐

Theorem 2.31 says that when the underlying digraph $\mathfrak{A}^{\downarrow} = (\mathbb{Z}, E)$ has nice algebraic properties (PHR-FT), one obtains a generalized quasi-lowerbound of CSP$(\mathfrak{A})$ "up

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[23] When further generalizing to multiple non-unary relations, worst case this involves looking at each (x, y) pair and for each pair looking at each $E_j^{\mathfrak{D}}$, which is $O(|D|^2 \times |D|^k)$, polynomially bounded. Note that k is a constant.

[24] Note that the runtime of this SEARCH-CSP accounts for the time to write down the $h(x)$s for each $x \in \mathfrak{D}$. Later when we compute the shift $a - h(\mathbf{0}^{\mathfrak{D}})$, the overhead is dominated by the runtime of the SEARCH-CSP. The total overhead of computing $s \circ h$ is therefore at most that of running $O(|D|)$ times of the SEARCH-CSP cost.

to direct limit" , $\mathfrak{A}$ being the unary expansion of $\mathfrak{A}^{\downarrow}$, and the bound is given by CSP of the pointed structure $(\mathfrak{A}, \mathbf{0})$. In fact, this observation also has some structural implications:

Proposition 2.33. *For* $(\mathfrak{A}^{\downarrow}, \mathbf{0}) = (\mathbb{Z}, E, \mathbf{0})$*, let* ${\mathfrak{A}^{\downarrow}}_0$ *denote its* central component*, i.e. the pointed component of the underlying digraph containing* 0*.*

1. *If* $\mathfrak{A}^{\downarrow}$ *is* PHR*, then* $\mathrm{End}_{E,\mathbf{0}}({\mathfrak{A}^{\downarrow}}_0) = \{\mathbf{1}\}$*. The converse is false.*

2. *If* $\mathfrak{A}^{\downarrow}$ *is* FT*, then* ${\mathfrak{A}^{\downarrow}}_0$ *has infinite domain.*

3. *If* $\mathfrak{A}^{\downarrow}$ *is* PHR-FT*, then* $(\mathfrak{A}^{\downarrow}, \mathbf{0})$ *is not* ω*-categorical, unless* $E = \emptyset$*.*

Remark 2.34. If $E = \emptyset$ then $(\mathfrak{A}, \mathbf{0})$ is indeed ω-categorical. For any $n \geq 1$, a complete n-type $(x_1, \ldots, x_n)$ only needs to make a decision for each x_i as to whether $x_i \in U$, and whether $x_i = \mathbf{0}$. Thus there are finitely many complete n-types for any n. Likewise $(\mathfrak{A}^{\downarrow}, \mathbf{0})$ is ω-categorical for having even fewer n-types.

Proof. [Proof of Proposition 2.33]

1. Assume for contradiction that $\mathrm{End}_{E,\mathbf{0}}({\mathfrak{A}^{\downarrow}}_0) \neq \{\mathbf{1}\}$, so there exists $\sigma \in \mathrm{End}_{E,\mathbf{0}}({\mathfrak{A}^{\downarrow}}_0)$ and $a \neq b \in {\mathfrak{A}^{\downarrow}}_0$ s.t. $\sigma(a) = b$. Note that $a \neq \mathbf{0}$ for σ preserves $\mathbf{0}$. By connectedness, there exists a finite *signed path* $\mathbf{0} \ldots a \subseteq {\mathfrak{A}^{\downarrow}}_0$, i.e. for each $(i, i+1)$ pair on $[0, a]$, either $(i, i+1) \in E^{{\mathfrak{A}^{\downarrow}}_0}$ or $(i+1, i) \in E^{{\mathfrak{A}^{\downarrow}}_0}$. Take $\mathfrak{D} := {\mathfrak{A}^{\downarrow}}_0[\{\mathbf{0}, \ldots, a\}]$, which is again connected thanks to that signed path. Let $h_1 : \mathfrak{D} \hookrightarrow {\mathfrak{A}^{\downarrow}}_0$ the embedding. Now, $h_2 := \sigma \circ h_1 \in \mathrm{Hom}_{E,\mathbf{0}}(\mathfrak{D}, {\mathfrak{A}^{\downarrow}}_0)$ with $b = h_2(a) \neq h_1(a) = a$, contradicting PHR. We show that conversely, even $\big(\mathrm{End}_{E,\mathbf{0}}({\mathfrak{A}^{\downarrow}}_0) = \{\mathbf{1}\}\big) +$ FT $\not\Rightarrow$ PHR.[25] In fact, Figure 2.4 below has $\big(\mathrm{End}_{E,\mathbf{0}}({\mathfrak{A}^{\downarrow}}_0) = \{\mathbf{1}\}\big) +$ FT but no

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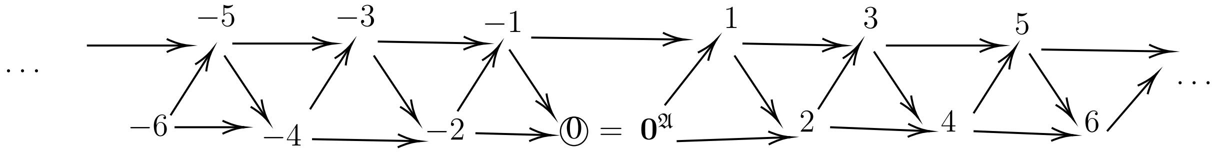
[25] If the implication were true, for Theorem 2.31 we would have algebraic conditions without reference of "all finite connected τ-structures", which would be considerably cleaner.

PHR: $\mathfrak{D}$ embeds in $(\mathfrak{A}^{\downarrow}, \mathbf{0})$ in 2 ways. Verification is routine .[26]



$\big(\mathfrak{A}^{\downarrow}, 0\big) := (\mathbb{Z},\ E,\ \mathbf{0})$ as above, $\mathfrak{D} :\ \textcircled{0} = \mathbf{0}^{\mathfrak{D}} \longrightarrow 1$

Figure 2.4: Counter Example $\mathfrak{A}^{\downarrow}$

2. Assume $E \neq \emptyset$ and assume for the sake of contradiction that $|A^{\downarrow}{}_0| < \aleph_0$.

    (a) If $|A^{\downarrow}{}_0| = 1$, take $(a_1, a_2) \in E$. Note that $a_1, a_2 \neq \mathbf{0}$ since $\mathfrak{A}^{\downarrow}$ is isolated. The translation $\sigma : x \mapsto x - a_1$ fails to be an $\{E\}$-endomorphism, as $(\sigma(a_1), \sigma(a_2)) \notin E^{\mathfrak{A}^{\downarrow}}$. This contradicts FT.

    (b) Otherwise $\aleph_0 > |A^{\downarrow}{}_0| \geq 2$. Let $b \in A^{\downarrow}{}_0 \setminus \{\mathbf{0}\}$ be of maximal absolute value in $A^{\downarrow}{}_0$. As $\mathfrak{A}^{\downarrow}$ has FT and endomorphisms preserve connectedness, by repeatedly applying the translation $x \mapsto x + b$ we obtain that $\{kb\}_{k<\omega} \subset \mathfrak{A}^{\downarrow}{}_0$, contradicting $\aleph_0 > |A^{\downarrow}{}_0|$.

Remark 2.35. One might also want to note that a claim stronger than Prop. 2.33(1), that PHR $\implies$ $\mathrm{End}_{E,\mathbf{0}}((\mathfrak{A}^{\downarrow}, \mathbf{0})) = \{1\}$, is also false. Consider $\mathfrak{A}^{\downarrow} := (\mathbb{Z}; E, \mathbf{0} := 0)$ with edges defined as

$$\cdots - 8 \to -7 \to -6 \to -5 \qquad -4 \leftarrow -3 \qquad -2 \leftarrow -1 \qquad 0 \to 1 \to 2 \to 3 \to 4 \to \dots$$

Any connected finite $\{E, \mathbf{0}\}$-structure must homomorphically map to the non-negative, so a $\{E, \mathbf{0}\}$-homomorphism exists iff $\mathfrak{D}$ is loopless, directed acyclic, and each vertex has a *non-negative* signed distance from $\mathbf{0}^{\mathfrak{D}}$ (i.e. never "$a \to \mathbf{0}^{\mathfrak{D}}$"). Each $\{E, \mathbf{0}\}$-homomorphism from such $\mathfrak{D}$s should it exist, must be unique, mapping each vertex to its distance from $\mathbf{0}^{\mathfrak{D}}$. However $\mathrm{End}_{\{E,\mathbf{0}\}}((\mathfrak{A}^{\downarrow}, \mathbf{0}))$ contains swapping $-4 \leftarrow -3$ with $-2 \leftarrow -1$.

---

[26] Without naming the constant, each vertex has in-degree 2 and out-degree 2, so any translation is an $\{E\}$-endomorphism — in fact even an automorphism, yielding FT. Furthermore, after naming $\mathbf{0}^{\mathfrak{A}} := 0$, its predecessors and successors are fixed: let $\sigma \in \mathrm{End}_{E,\mathbf{0}}(\mathfrak{A}^{\downarrow}, \mathbf{0})$. $\sigma(2), \sigma(1) \in \{2, 1\}$ yet if $\sigma(2) = 1$, neither $\sigma(1) = 1$ nor $\sigma(1) = 2$ would give $E^{\mathfrak{A}}(\sigma(1), \sigma(2))$. It follows that $\sigma(2) = 2$. Likewise, $\sigma(\pm 1) = \pm 1$ and $\sigma(-2) = -2$. For $|n| \geq 3$, $\sigma(n) = n$ follows from induction. The fact that vertices further away are fixed follow from induction, so $\mathrm{End}_{E,\mathbf{0}}(\mathfrak{A}^{\downarrow}{}_0) = \mathrm{End}_{E,\mathbf{0}}(\mathfrak{A}^{\downarrow}, 0) = \{1\}$. On the other hand, $\mathfrak{D} := \mathbf{0}^{\mathfrak{D}} = 0 \to 1$ embeds into both $\mathbf{0} \to 1 \subset \mathfrak{A}$ and $\mathbf{0} \to 2 \subset \mathfrak{A}$.

3. We have $\mathrm{End}_{E,\mathbf{0}}(\mathfrak{A}^{\downarrow}{}_0) = \{\mathbf{1}\}$ and $|A^{\downarrow}{}_0| = \aleph_0$. Now $\mathrm{Aut}_{E,\mathbf{0}}(\mathfrak{A}^{\downarrow}, \mathbf{0}) \subseteq \mathrm{Aut}_{E,\mathbf{0}}(\mathfrak{A}^{\downarrow}{}_0) \subseteq \mathrm{End}_{E,\mathbf{0}}(\mathfrak{A}^{\downarrow}{}_0) = \{\mathbf{1}\}$, so $(\mathfrak{A}^{\downarrow}, \mathbf{0}) = (\mathbb{Z}, E, \mathbf{0})$ is rigid, and $\mathrm{Aut}_{E,\mathbf{0}}(\mathfrak{A}^{\downarrow}, \mathbf{0})$ on $\mathbb{Z}$ has infinitely many orbits. $(\mathfrak{A}, \mathbf{0})$ being an expansion of $(\mathfrak{A}^{\downarrow}, \mathbf{0})$ is also rigid.

□

Proposition 2.33 (2) implies that, when $|A^{\downarrow}{}_0| < \aleph_0$, we need a different condition to guarantee a generalized lowerbound simialr to Section 2.3, which we address next.

Definition 2.36. Let $\mathfrak{A} = (\mathbb{Z}, E, U)$ so $(\mathfrak{A}^{\downarrow}, \mathbf{0}) = (\mathbb{Z}, E, \mathbf{0})$ with $\mathfrak{A}^{\downarrow}{}_0$ its the central component. Say $\mathfrak{A}^{\downarrow} = (\mathbb{Z}, E)$ has *Transitive Endomorphism-action on Hom sets (TEH)* if for any finite, connected E-structure $\mathfrak{D}$, $\mathrm{End}_E(\mathfrak{A}^{\downarrow}[A^{\downarrow}{}_0])$ acts *transitively* on $\mathrm{Hom}_E(\mathfrak{D}, \mathfrak{A}^{\downarrow}[A^{\downarrow}{}_0])$ by post-composition.

Note that $\mathfrak{A}^{\downarrow}[A^{\downarrow}{}_0]$ is the substructure of $\mathfrak{A}^{\downarrow}$ induced on $\mathfrak{A}^{\downarrow}{}_0$'s central component.

Proposition 2.37. *If* $|A^{\downarrow}{}_0| < \aleph_0$ *and* $\mathfrak{A}^{\downarrow}$ *has* TEH*, then:*

$$\mathrm{CSP}(\mathfrak{A}, \mathbf{0}) \leq_T^p \varinjlim \{U, \mathrm{CSP}(\mathfrak{A}), \mathrm{CSP}(\mathfrak{A}^{\downarrow}[A^{\downarrow}{}_0])\} \tag{2.5}$$

Changing $\mathrm{CSP}(\mathfrak{A}^{\downarrow}[A^{\downarrow}{}_0])$ to $\text{SEARCH-CSP}(\mathfrak{A}^{\downarrow}[A^{\downarrow}{}_0])$ would give us a stronger statement that remains correct, but we use the following fact to get the current more succinct form:

Proposition 2.38 (e.g. Exercise in [38]without proof)**.** *Let* $\mathfrak{A}$ *be a* finite *relational structure. Then* $\text{SEARCH-CSP}(\mathfrak{A}) \leq_T^p \mathrm{CSP}(\mathfrak{A})$.

Proof of Proposition 2.38. A fun exercise. Conceptually mimics the Search to Decision reduction for SAT. As in graphs, a *core* $\mathfrak{B} \subset_{\text{ind}} \mathfrak{A}$ is a minimal image of endomorphisms, i.e. there exists no $\sigma \in \text{End}(\mathfrak{A})$ s.t. $\sigma(\mathfrak{A}) \subsetneq \mathfrak{B}$. For finite $\mathfrak{A}$s, cores always exist and are unique up to isomorphism [13]. are isomorphic,[27] so hereinafter we talk about "*the*" core of $\mathfrak{A}$.

Let $\mathfrak{B}$ be the core of $\mathfrak{A}$, then inclusion $\mathfrak{B} \hookrightarrow \mathfrak{A}$ and the definitional endomorphism $\mathfrak{A} \to \mathfrak{B}$ shows homomorphic equivalence of $\mathfrak{A}$ and $\mathfrak{B}$, so suffices to show SEARCH-CSP$(\mathfrak{A}) \leq_T^p$ CSP$(\mathfrak{B})$.[28]

To make the description less cumbersome let's define POINTED-CSP: fix arbitrary $b \in B$. POINTED-CSP$(\mathfrak{B}, b)$ takes a pair $(\mathfrak{D}, d)$ with $\mathfrak{D}$ a finite τ-structure, $d \in D$, and returns 1 iff there exists $h \in \text{Hom}_\tau(\mathfrak{D}, \mathfrak{B})$ s.t. $h(d) = b$.

Claim: POINTED-CSP$(\mathfrak{B}, b) \leq_m^p$ (hence $\leq_T^p$) CSP$(\mathfrak{B})$.

Proof of Claim. Upon a POINTED-CSP$(\mathfrak{B}, b)$ input $(\mathfrak{D}, d)$, let $\widetilde{\mathfrak{D}} := \dfrac{\mathfrak{D} \cup \mathfrak{B}}{b \sim d}$.[29]. This is a polynomial time construction since $|B| \sim O(1)$. I claim $(\mathfrak{D}, d) \in$ POINTED-CSP$(\mathfrak{B}, b) \iff \widetilde{\mathfrak{D}} \in$ CSP$(\mathfrak{B})$.

1. $\implies$: let $h : (\mathfrak{D}, d) \to (\mathfrak{B}, b)$ be a basepoint-preserving homomorphism. Then $\widetilde{h} : \widetilde{\mathfrak{D}} \to \mathfrak{B}, h(x) := \begin{cases} h(x), & \text{if } x \in D \\ x, & \text{o.w.} \end{cases}$ is a homomorphism.
2. $\impliedby$: let $h : \widetilde{\mathfrak{D}} \to \mathfrak{B}$ be a homomorphism. Then $h|_{\mathfrak{B}} : \mathfrak{B} \to \mathfrak{B}$ is an endomorphism of the finite core $\mathfrak{B}$. Since an endomorphism of a core is an automorphism,[30]

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[27] (e.g. [39] for proof of isomorphism of cores in the special case of graphs) Let $\mathfrak{B}_1, \mathfrak{B}_2 \subset_{\text{ind}} \mathfrak{A}$ be cores, witnessed by $e_1 : \mathfrak{A} \to \mathfrak{B}_1, e_2 : \mathfrak{A} \to \mathfrak{B}_2$, two surjective homomorphisms (otherwise $\mathfrak{B}_*$ is not a core). Then $e_1|_{\mathfrak{B}_2}$ is also surjective, for otherwise $(e_1|_{\mathfrak{B}_2}) \circ e_2$ gives an induced substructure of $\mathfrak{B}_1$ which is an endomorphic image. Dually $e_2|_{\mathfrak{B}_1}$ is surjective. Therefore $|B_1| = |B_2|$, $e_1|_{\mathfrak{B}_2}$ and $e_2|_{\mathfrak{B}_1}$ are bijective homomorphisms, and we have the isomorphism.

[28] Note: $\mathfrak{A}$ is input-independent, hence core$(\mathfrak{A}) = \mathfrak{B}$.

[29] That is, take a copy of $\mathfrak{D}$ and a copy of $\mathfrak{B}$, glue b and d.

[30] $e : \mathfrak{B} \to \mathfrak{B}$ must be surjective, so bijective. As Im$e \subseteq_{\text{ind}} \mathfrak{B}$, the bijectivity says Im$e = \mathfrak{B}$. In particular this says e preserves the number of solutions for each R_i, i.e. there cannot be $(a_1, \ldots, a_{r_i}) \notin R_i^{\mathfrak{B}}$ s.t. $(e(a_1), \ldots, e(a_{r_i})) \in R_i^{\mathfrak{B}}$. It follows that e preserves the relations bidirectionally, hence an automorphism.

there exists $\sigma \in \mathrm{Aut}(\mathfrak{B}) = \mathrm{End}(\mathfrak{B})$ s.t. $\sigma \circ (h|_{\mathfrak{B}}) = 1_{\mathfrak{B}}$. Now $\sigma \circ (h|_{\mathfrak{D}})$ is a homomorphism $\mathfrak{D} \to \mathfrak{B}$ sending d to $\sigma(h([d])) = \sigma(h([b])) = b$, which is the desired witness.

$\square$

Analogously if we define a m-POINTED-CSP($\mathfrak{B}, b_1, \ldots, b_m$) it also Karp-reduces to CSP($\mathfrak{B}$), the reduction being $(\mathfrak{D}, d_1, \ldots, d_m) \rightsquigarrow \widetilde{\mathfrak{D}} := \dfrac{\mathfrak{D} \cup \mathfrak{B}}{\{d_i \sim b_i\}_{i \in [1,m]}}$.[31]

An algorithm that calls the CSP($\mathfrak{B}$)-oracle to solve SEARCH-CSP($\mathfrak{A}$) therefore goes as:

```
if $\mathfrak{D} \notin$ CSP($\mathfrak{B}$) then
    Reject
for $i \in [1, |D|]$ do
    for $j \in [1, |B|]$ do
        if $(\mathfrak{D}, d_1, \ldots, d_i) \in$ POINTED-CSP($\mathfrak{B}, h(d_1), \ldots, h(d_{i-1}), b_j$) then
            Set $h(d_i) := b_j$
            break //break inner iteration, go to next $i$
    end
    reject
end
```

Algorithm 2: Querying CSP($\mathfrak{B}$)-oracle to solve SEARCH-CSP($\mathfrak{A}$)

Algorithm 2 finds a full homomorphism $\mathfrak{D} \to \mathfrak{B} \iff \mathfrak{D} \in \mathrm{CSP}(\mathfrak{B}) = \mathrm{CSP}(\mathfrak{A})$ so it solves SEARCH-CSP($\mathfrak{A}$); furthermore it calls the CSP($\mathfrak{B}$)-oracle $O(nk)$ times. $\Longrightarrow$

---

[31] Note e.g. when $m = 2$ in the $\Longleftarrow$ direction has $\sigma \circ h|_{\mathfrak{D}}$ sending $[d_i]$ to $[b_i]$ for *both* i as each $b_i \in B$.

is by definition, and to see $\Longleftarrow$, note that the existence of an $h : \mathfrak{D} \to \mathfrak{B}$ means for each $i \in [1, |D|]$ there exists some $j \in [1, |B|]$ s.t. $h(d_i) = b_j$, i.e. the existence of an accepting path.[32] ☐

Proof of Proposition 2.37. As before, decompose $\mathfrak{D} = (D, E^{\mathfrak{D}}, U^{\mathfrak{D}}, \mathbf{0}^{\mathfrak{D}})$ into connected components and let the CSP($\mathfrak{A}$) handle the non central components.

For the central component, call SEARCH-CSP($\mathfrak{A}^{\downarrow}[A^{\downarrow}{}_0]$) oracle, which thanks to $|A^{\downarrow}{}_0| < \aleph_0$ and Fact 2.38 Cook reduces to CSP($\mathfrak{A}^{\downarrow}[A^{\downarrow}{}_0]$). If we found an $h \in \text{Hom}_E((D, E^{\mathfrak{D}}), \mathfrak{A}^{\downarrow}[A^{\downarrow}{}_0])$, by the assumption, all $h' \in \text{Hom}_E((D, E^{\mathfrak{D}}), \mathfrak{A}^{\downarrow}[A^{\downarrow}{}_0])$ is of the form σh for different σs in $\text{End}_E(\mathfrak{A}^{\downarrow}[A^{\downarrow}{}_0])$.

Note that $|A^{\downarrow}{}_0| < \aleph_0 \implies |\text{End}_E(\mathfrak{A}^{\downarrow}[A^{\downarrow}{}_0])| \leq |A^{\downarrow}{}_0|^{|A^{\downarrow}{}_0|} < \aleph_0$ and $|\text{End}_E(\mathfrak{A}^{\downarrow}[A^{\downarrow}{}_0])|$ is input-independent. Each $\sigma_i \in |\text{End}_E(\mathfrak{A}^{\downarrow}[A^{\downarrow}{}_0])|$ is a finite domain function, whose encoding is also input-independent.For each of the $O(1)$-many $\sigma_i \in |\text{End}_E(\mathfrak{A}^{\downarrow}[A^{\downarrow}{}_0])|$, compute $\sigma_i h$ (Since σ_i is input-independent and $A^{\downarrow}{}_0$ is constant-sized, time and space cost is dominated by computing h, which is again dominated by the oracle call to CSP($\mathfrak{A}^{\downarrow}[A^{\downarrow}{}_0]$)); then check if $\sigma_i h$ preserves $\mathbf{0}^{\mathfrak{D}}$ in $O(1)$ time and if $\sigma_i h$ preserves the unary by calling the U-oracle. ☐

Corollary 2.39. *Let* $U \in \mathsf{P}$.

1. *If* $\mathfrak{A}^{\downarrow} = (\mathbb{Z}, E)$ *is* PHR-FT *and if* SEARCH-CSP($\mathfrak{A}^{\downarrow}$) $\leq_T^p$ CSP($\mathfrak{A}^{\downarrow}$), *OR*

2. *If* $|A^{\downarrow}{}_0| < \aleph_0$, $\mathfrak{A}^{\downarrow}$ *has* TEH, *and if* CSP($\mathfrak{A}^{\downarrow}[A^{\downarrow}{}_0]$) $\leq_T^p$ CSP($\mathfrak{A}^{\downarrow}$),

---

[32] The algorithm also has the benefit that whenever a homomorphism is found, the solution tuple $(h(d_1), \ldots, h(d_{|D|})) = (b_{s,1}, \ldots, b_{s,|D|})$ is the minimal element in the dictionary order by indices in the target. That is, for any other solution $(b'_{s,1}, \ldots, b'_{s,|D|}) \in B^{|D|} = (b_1 \prec \cdots \prec b_{|B|})^{|D|}$, $(b_{s,1}, \ldots, b_{s,|D|}) \preceq (b'_{s,1}, \ldots, b'_{s,|D|})$.

then CSP$(\mathfrak{A}, \mathbf{0}) \equiv_T^p$ CSP$(\mathfrak{A})$.

Proof. With assumption set 1, by $U \in \mathtt{P}$ and CSP$(\mathfrak{A}^{\downarrow}) \leq_m^p$ CSP$(\mathfrak{A})$, all other terms in the direct limit are reduced to CSP$(\mathfrak{A})$. Likewise for assumption set 2. □

We summarize our main results of this section in Figure 2.5.

$$\mathfrak{A} := (\mathbb{Z},\ \mathrm{E},\ U)$$

$$\begin{tikzcd}
\text{CSP}\ (\mathfrak{A}, \mathbf{0}) \arrow[dr, "{``\leq_T^p\text{''}}", "(**)"'] & & \text{CSP}\ (\mathfrak{A}, \mathbf{0}) \\
& \text{CSP}\ (\mathfrak{A}) \arrow[ur, "\leq_m^p"] &
\end{tikzcd}$$

(**): up to $\lim_{\rightarrow}\{U,$SEARCH-CSP$(\mathfrak{A}^{\downarrow}), \bullet\}$
with conditions (e.g. (PHR + FT))

OR

(**): up to $\lim_{\rightarrow}\{U,$ CSP$(\mathfrak{A}^{\downarrow}[A_0^{\downarrow}]), \bullet\}$
with conditions (e.g. TEH)

Figure 2.5: Results for $\mathfrak{A} := (\mathbb{Z}, E, U)$,$\mathfrak{A}^{\downarrow} := (\mathbb{Z}, E)$.

2.6 Future Directions

In Sections 2.3 and 2.4, we studied how properties of U impacts the complexity of CSP$(\mathbb{Z}, \mathrm{succ}, U)$. In particular, Section 2.3 identified some properties for U to guarantee a relatively tame behavior of CSP$(\mathbb{Z}, \mathrm{succ}, U)$, while Section 2.4 characterized CSP$(\mathbb{Z}, \mathrm{succ}, U)$ by gaps in U. Some questions that initially motivated our research in that direction remains open:

Problem 2.40. What are the *equivalent* conditions for CSP$(\mathbb{Z}, \mathrm{succ}, U)$ to be tractable? Are there Us such that CSP$(\mathbb{Z}, \mathrm{succ}, U)$ is NP-intermediate or coNP-intermediate, assuming P≠NP? In particular, is the answer to Question **2.26** yes and can $\leq_T$ there be refined to $\equiv_m^p$?

In Section 2.5 we generalized our quasi-lowerbound results to arbitrary $(\mathbb{Z}, E, U)$ where the underlying digraph has nice properties. Some next steps may include:

1. Study if with reasonable properties on E, the $\varinjlim()$ in the quasi-lowerbound could be removed.
2. Refine the characterization of complexity of CSP$(\mathbb{Z}, E, U)$ by properties on E *and* U.

As mentioned in Section 4.1, we believe these are solid first steps towards characterizing the CSP of non-ω-categorical binary structures. In fact, one meaningful subclass of non-ω-categorical structures where $(\mathbb{Z}, \text{succ}, U)$ lives is those that are *mutually algebraic* (e.g. [40],[41]) ones, which also include all $\mathfrak{A} = (\mathbb{Z}, E, U)$s where the underlying $\mathfrak{A}^{\downarrow}$ has *(universally) bounded degree*. Hence one may ask:

Problem 2.41. Let $\mathfrak{A}^{\downarrow} = (\mathbb{Z}, E)$ be of degree $d \geq 2$, i.e. $\deg_{\mathfrak{A}^{\downarrow}}(v) \leq d$ for any $v \in \mathbb{Z}$.

1. Is CSP$(\mathfrak{A}^{\downarrow})$ either in `P` or in `NPC`? If not, are there meaningful dividing lines for tractability at least?
2. Can we fully characterize the complexity of CSP$(\mathbb{Z}, E, U)$ $(U \subseteq \mathbb{Z})$ now that we further restricted the underlying E by degree?

Some facts to bear in mind: that in general it is not true that the CSP of infinite digraphs have a P-NPC dichotomy [13, 37].[33] On the other hand, for any $d \geq 2$ the disjoint union of all (isomorphic types of) finite *undirected* graph of degree ≤ 2 has the

---

[33] See also, statuses of some other open questions in [13] or here. Also [37] showed that there are coNP-intermediate ω-categorical structures.

same CSP as K_{d+1}, by e.g. Brook's theorem. It also has the same CSP with $\mathfrak{M}_\alpha$ where $d-1=k_\alpha \in (\frac{2}{\alpha}, \frac{2}{\alpha}+1]$, per Proposition 4.8 from last Chapter.

Some final comments on CSP of (Z, succ, U) : in fact the research here was originally motivated by dichotomizing mutually algebraic structures, which can be characterized as models of weakly minimal and trivial theories [40], and as noted above can be viewed as a generalization of a hereditary graph property of bounded degree graphs [42]. It was recently proved that mutual algebraicity affords a dividing line for speed of growth of hereditary properties [42]. Although per a construction in [13], the computational complexity of mutually algebraic structures can be all over the places, it may still be interesting to study the dividing line between intractible and tractible CSPs of mutually algebraic structures.

Chapter 3: $\mathfrak{M}_{\alpha,\mathrm{TF}}$ and Shelah-Spencer Triangle-Free Random Graphs

3.1 Introduction

For general introduction on Hrushovski-Fraïssé classes, see Chapter 1. Here we focus on the following classes of Hrushovski-Fraïssé classes of relational structures, and, as advertised, aim at axiomatizing the first order theory of their generic structures.

Let $L := \{\overline{R_1}, \cdots, \overline{R_s}\}$ be a finite collection of finitary, *irreflexive* and *symmetric* relational symbols: that is, respectively, we assert for each $\overline{R_*}$ to not contain tuples with identical coordinates and we identify two tuples of solutions up to permutation of entries. For example, when $s = 1$ and R_1 is binary, L is just the language of simple undirected graphs. Denote the arities $\mathrm{ar}(\overline{R_t}) =: \overline{r_t} \geq 2$ for each $t \in [1, s]$. For $(\alpha_t)_t \in (0, 1]^s$, define the $(\vec{\alpha})$-*predimension* on each L-structure $\mathfrak{A} = (A, R_1^{\mathfrak{A}}, \cdots, R_s^{\mathfrak{A}})$:

$$\delta_{\vec{\alpha}}(\mathfrak{A}) := v(\mathfrak{A}) - \sum_{t\in[1,s]} \alpha_t \cdot e_t(\mathfrak{A}) \tag{3.1}$$

where, just as for graphs, $v(\mathfrak{A}) = |A|$ the "number of vertices", $e_t(\mathfrak{A}) = |\frac{R_t^{\mathfrak{A}}}{\sim}|$ with $\sim$ identifying all permutation of entries of a hyper-edge,[1] so e.g. $(x, y) \sim (y, x)$ are considered the same binary edge. Now define the class of hereditarily $\vec{\alpha}$-linearly sparse

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[1] That is, more cumbersomely, $(v_1, \cdots, v_{r_t}) \sim (\sigma(v_1), \cdots, \sigma(v_{r_t}))$ for each $\sigma \in S_{r_t}$ the symmetric group. Also by irreflexivity, no $i \neq j \in [1, r_t]$ exists s.t. $v_i = v_j$.

L-structures[2]

$$\mathcal{K}_{\vec{\alpha}} := \{\text{finite } L\text{-structures } \mathfrak{A} : \forall \mathfrak{B} \subseteq_{\text{ind}} \mathfrak{A}, \delta_{\vec{\alpha}}(\mathfrak{B}) \geq 0\}. \tag{3.2}$$

Next we fix some notations for "clique" in the r-ary setting. Recall the smallest binary clique is $K_3 = K_{2+1}$; likewise, let $(m_t)_t \in (\mathbb{N}^{>0})^s$. For each t, denote $K_{\overline{R_t},m_t} :=$ the complete $\overline{R_t}$-hypergraph on vertices $\{1, 2, \cdots, \overline{r_t}+m_t\}$. Note that $K_{\overline{R_t},m_t}$ has $\binom{\overline{r_t}+m_t}{\overline{r_t}}$ hyperedges — recall symmetricityof the relation. Lastly, consider the subclass of $\mathcal{K}_{\vec{\alpha}}$ that *avoids the cliques* $(K_{\overline{R_t},m_t})_t$:

$$\mathcal{K}'_{\alpha,(m_t)_t} := \mathcal{K}_\alpha \cap \bigcap_{t\in[1,s]} (K_{\overline{R_t},m_t}\text{-free}) \tag{3.3}$$

When the forbidden configurations $(K_{\overline{R_t},m_t})_t$ are clear from context, we shorthand $\mathcal{K}'_{\alpha,(m_t)_t}$ as $\mathcal{K}'_\alpha$.

Example 3.1. *Let $s := 2$, $\overline{R_1}, \overline{R_2}$ be binary and ternary respectively, $\alpha_1 = \alpha_2 = \frac{1}{2}$. For $\{m_1 := 2, m_2 := 1\}$, $K_{\overline{R_1},m_1} = K_{\overline{R_1},2}$ is the (labelled) 4-clique ${}^1_4\boxtimes^2_3$, while $K_{\overline{R_2},1}$ is the "ternary 4-hyperclique", that is, the $\{\overline{R_2}\}$-structure on $\{1, 2, 3, 4\}$ with hyperedges $\binom{4}{3}$.*

Then $\mathcal{K}_{\vec{\alpha}}$ is the collection of L-structures whose number of vertices is at least half the total number of edges and hyperedges, while $\mathcal{K}'_{\vec{\alpha}}$ is the subclass of $\mathcal{K}_{\vec{\alpha}}$ avoiding both ${}^1_4\boxtimes^2_3$ and the ternary 4-hyperclique. It's a proper subclass because e.g. the $\delta_{\frac{1}{2}}(H) \geq 0$ for each $H \subseteq^1_4 \boxtimes^2_3$.

---

[2] Note that when $\vec{\alpha}, \vec{r}$ are both diagonal (i.e. α_ts are identical, and so are the $\overline{r_t}$s), it can be verified using computation similar to what's done in Chapter 4 that the class $\mathcal{K}_{\vec{\alpha}}$ has a "mad" characterization, i.e. $\{\text{finite } L - \text{structures } \mathfrak{A} : \max_{\mathfrak{B}\subseteq\mathfrak{A}} \frac{\sum_{v\in B} \deg_{\mathfrak{B}}(v)}{v(\mathfrak{B})} \geq \frac{s\overline{r}}{\alpha}\}$.

Since the argument for the key lemmas in Section 3.2 are a bit different when the relation is binary (as opposed to tertiaries and above), we keep the discussion there for general relational structures over finite signatures. For practicality concerns it is recommended that one imagines (finite simple undirected) graphs as running examples.

We define *strong structures* ($\leq$) and *generic structures* as follows, similar to Chapter 4 Preliminaries (Section 4.2) below: $G \leq H \iff \Big(G \subseteq H \bigwedge \delta_\alpha(G) \leq \delta_\alpha(G')$ for all $G' : G \subseteq_{\text{ind}} G' \subseteq_{\text{ind}} H\Big)$, where $\subseteq_{ind}$ denote the substructure relation. For $\mathcal{K} := \mathcal{K}_{\vec{\alpha}}$ or $\mathcal{K}'_{\vec{\alpha}}$, $(\mathcal{K}, \preceq)$ has a *generic structure* $\mathfrak{M}_{\alpha,*}$, which is a union of $\{G_i\}_{i<\omega} \subseteq \mathcal{K}$ (therefore countably infinite) with $G_i \leq G_{i+1}$, s.t. $\forall\Big(G \leq H \in \mathcal{K}_\alpha$ and $f : G \hookrightarrow \mathfrak{M}_\alpha$ as induced subgraphs where the image $f(G) \leq \mathfrak{M}_\alpha\Big)$, there exists an extension embedding $\widetilde{f} : H \hookrightarrow \mathfrak{M}_{\alpha,*}$ s.t. $\widetilde{f}(H) \leq \mathfrak{M}_{\alpha,*}$. It is routine to verify that for any L, any $\vec{\alpha}$ and any $(m_t)_t$, the classes $(\mathcal{K}_{\vec{\alpha}}, \leq)$ and $(\mathcal{K}'_{\vec{\alpha},(m_t)_t}, \leq)$ both satisfies $A_1 \sim A_5+$ amalgamation property in [43], therefore they each have an up-to-isomorphism unique generic structure. We denote these generic structures as $\mathfrak{M}_{\vec{\alpha}}, \mathfrak{M}'_{\vec{\alpha},(m_t)_t}$ respectively for $\mathcal{K}_{\vec{\alpha}}, \mathcal{K}'_{\vec{\alpha},(m_t)_t}$. As above, we denote the latter generic structure as $\mathfrak{M}'_{\vec{\alpha}}$ when the forbidden configurations are clear.

Laskowski showed in [1] that for $\vec{\alpha} \in (0,1)^s$ s.t. the components are $\mathbb{Q}$-linearly independent with 1, *Th*($\mathfrak{M}_{\vec{\alpha}}$), the first order theory of $\mathfrak{M}_{\vec{\alpha}}$, admits a Π_2 set of axiomatizations. Such axiomatizations were later generalized to arbitrary $\vec{\alpha} \in (0,1)^s$ (e.g. [44][3]). In Section 3.2 we show that Th($\mathfrak{M}'_{\vec{\alpha},(m_t)_t}$) admits a similar set of axiomatizations; the

---

[3] In [44] they also have some simple axiomatization for subclasses of $\mathcal{K}_{\vec{\alpha}}$ forbidding *reduced* substructures, which requires there to be no non-trivial substructure with smaller δ_α. As for example a triangle is *not reduced* for $\alpha \leq \frac{2}{3}$, witnessed by a singleton substructure, our approach in this section is still interesting at least for covering different ranges of αs.

An alternative approach of our job here would be to verify that $\mathfrak{M}'_{\vec{\alpha},(m_t)_t}$ as a subclass of $\mathfrak{M}_\alpha$ satisfies the "Approximate Extension Property" in [44]. Since the Approximate Extension Property is essentially a generalization of Proposition 4.2 of [1] to arbitrary $\alpha \in (0,1)$, we believe the tricks used to forbid triangle in the AEP approach for arbitrary α will be similar.

approach is highly reminiscent of [1]'s, but involves some new technicality, especially in the binary case.

It is verified that, in $L = \{E\}$ the language of graphs and when $\vec{\alpha} \in ((0,1) \setminus \mathbb{Q})^s$, the aforementioned set of Π_2 axiomatizations are *contained in* the almost sure theory of Shelah-spencer graphs. Using the *completeness* of Th$(\mathfrak{M}_\alpha)$, one may see that (e.g. [1, 22, 23]) the same set actually *axiomatizes* the almost sure theory of Shelah-spencer graphs. In Section 3.3 we show some efforts in making similar connections between the Π_2 axioms of Th$(\mathfrak{M}'_{\vec{\alpha}})$ (with $L = \{E\}$ and the only banned configuration K_3) and the almost sure theory of "Shelah-Spencer *triangle-free* graphs".

3.2 Simple Axiomatization of Th$(\mathfrak{M}'_{\vec{\alpha},(m_t)_t})$

Let the finite signature $L = (R_t)_{t \in [1,s]}$, density parameters $\vec{\alpha} \in (0,1]^s$, arities $(r_t)_t \in (\mathbb{N}^{\geq 2})^s$, and forbidden clique sizes $(m_t)_t \in (\mathbb{N}^{\geq 1})^s$ be arbitrary, but fixed throughout this section. With the configurations chosen, description of the forbidden cliques $(K_{\overline{R_t},m_t})_t$ are clear, so hereinafter we abbreviate $(\mathcal{K}'_{\alpha,(m_t)_t}, \mathfrak{M}'_{\vec{\alpha},(m_t)_t})$ as $(\mathcal{K}'_{\vec{\alpha}}, \mathfrak{M}'_{\vec{\alpha}})$.

Consider the following set of $\forall\exists$ sentences in L, which we denote by $\mathcal{S}'_{\vec{\alpha}}$:

1. "Any finite substructure is a member of $\mathcal{K}'_{\vec{\alpha}}$".

Remark 3.2. This is a collection of sentences parameterized by $n < \omega$, size of vertex set of the substructures. For example if $L = \{E\}$, $\alpha = \frac{1}{2}, m = 1$ (so banning triangles),

$$\mathcal{S}'_\alpha \ni \psi_3 \quad :\equiv \quad \forall xyz\Big(\big(\neg E(x,y) \wedge E(y,z) \wedge E(x,z)\big) \bigvee \big(E(x,y) \wedge \neg E(y,z) \wedge E(x,z)\big)$$

$$\bigvee \underbrace{\cdots\cdots}_{\text{description of other finitely many proper substructures of } K_3 \text{ on } \{x,y,z\}} \Big)$$

2. “For all $\mathfrak{A} \leq \mathfrak{B}$ from $\mathcal{K}'_{\vec{\alpha}}$, every copy of $\mathfrak{A}$ as induced substructure extends to a copy of $\mathfrak{B}$ as induced substructure.

   Remark 3.3. This is of course parameterized by $(\mathfrak{A} \leq \mathfrak{B}) \in {\mathcal{K}'_{\vec{\alpha}}}^2$. For example in the language of graphs banning triangles,

$$\mathcal{S}'_\alpha \ni \phi_{P_1,P_2} :\equiv \forall x \exists y (E(x,y))$$

   Therefore any model $\mathcal{M} \models \mathcal{S}'_\alpha$ contains no globally isolated vertices. Note: this does not contradict the fact that $P_1 \hookrightarrow \mathcal{M}$.

   The main result of this section is that $\mathcal{S}'_{\vec{\alpha}}$ axiomatizes Th$(\mathfrak{M}'_{\vec{\alpha}})$.

Theorem 3.4. *Th($\mathfrak{M}'_{\vec{\alpha}}$) $=$ Cn($\mathcal{S}'_{\vec{\alpha}}$) (Cn$=$ set of logical consequences).*

Our two core technical lemmas towards Theorem 3.4 are modifications of Lemma 4.1, Proposition 4.2 of [1]. Admittedly my arguments benefited largely from Proofs in [1], but I carefully and substantially adapted the approach to avoid generation of the forbidden configurations. Before diving into that, let us state without proof the following classical results [1, 45] which we directly use in the proof:

Fact 3.5 ([1, 45])**.** *Fix $\alpha \in (0,1) \smallsetminus \mathbb{Q}$.*

1. *There are infinitely many pairs $(a,b) \in (\mathbb{N}^{\geq 1})^2$ s.t. $|\frac{a}{b} - \alpha| < \frac{1}{b}$, and $G_\alpha := \{b - a\alpha : a, b \in \mathbb{N}^{\geq 1}\}$ is dense in $\mathbb{R}$.*

2. *For each $n \in \mathbb{N}^{\geq 1}$, let $q_n :=$ the unique integer satisfying $0 < nq_n\alpha < \alpha$. Denote $q_n^+ := q_n + 1$.*

   Since $\alpha < 1$, $q_n < q_m$ whenever $n < m < \omega$.

3. *Let $p \in \mathbb{N}^{\geq 1}$. Call p* locally optimal *with respect to α if*

$$|p - q_p^+\alpha| < |n - q_n^+\alpha|$$

   for all $1 \leq n \leq p$. Since G_α is dense in $\mathbb{R}$, infinitely many positive integers p are locally optimal with respect to α.

   Now fix $p > 1$ locally optimal. For each $1 \leq n < p$, define

$$d_n := n - q_n\alpha,$$

   then define

$$d_p := p - q_p^+\alpha$$

   Note that:

   (a) For each $n \in [1, p)$, $d_n \in (0, \alpha)$;

   (b) $d_p < 0$;

   (c) $d_n - d_m < \alpha$ whenever $1 \leq n < m \leq p$. Note the verification of this when $m = p$ uses the local optimality of p.

   Then define $\langle s_n : 1 \leq n \leq p\rangle$ by $s_1 := q_1$; $s_n := q_n - q_{n-1} - 1$ for $1 < n < p$;

and $s_p := q_p^+ - q_{p-1} - 1$. *Then*

(d) *for* $n \in [1,p)$, $\sum_{i\in[1,n]} s_i$ *telescopes and equals* $q_n - (n-1)$*; thus*

(e) $\sum_{i\in[1,p]} s_i = q_p^+ - (p-1)$.

By $q_n < \frac{n}{\alpha} < q_n + 1$ $(\forall n \in \mathbb{N}^{\geq 1})$ *one obtains that*

(f) $0 \leq s_n < \frac{1}{\alpha}$ *for all* $n \in [1,p)$ *and* $0 \leq s_p < 1 + \frac{1}{\alpha}$.

Our first core lemma: any sufficiently large $\mathfrak{B} \in \mathcal{K}'_{\vec{\alpha}}$ has a "minimal" extension $\mathfrak{D} \in \mathcal{K}'_{\vec{\alpha}}$ that's "infinitesimally" denser than $\mathfrak{B}$. Colors in the statements and proofs below highlight some differences between my approach and [1]; note that as mentioned, we have new technicality especially in the forbidden of configurations (in particular for arity 2).

Lemma 3.6. *Denote* $\underline{\alpha} := \min_{t\in[1,s]}\{\alpha_t\}$, *and* $\overline{r} := \max_{r\in[1,s]}\{\overline{r_t}\}$.

Let $\emptyset \neq \mathfrak{B} \in \mathcal{K}'_{\vec{\alpha}}$, $X := \{e_1, e_2, \cdots, e_k\}$ *disjoint from* B *with* $k > \overline{r}(1/\underline{\alpha} + \overline{r})$ *a set of new, isolated vertices*[4] . *Define* $\mathfrak{B}^* := \mathfrak{B} \oplus_\emptyset \mathfrak{X}$, *and fix* $\epsilon > 0$. *Then there exists* $\mathfrak{D} \in \mathcal{K}'_{\vec{\alpha}}$ *extending* $\mathfrak{B}^*$ *s.t.*

1. $-\epsilon < \delta_{\vec{\alpha}}(\mathfrak{D}/\mathfrak{B}^*) < 0$, *and*

2. *for any proper* $\mathfrak{D}' \subsetneq \mathfrak{D}$, $\delta_{\vec{\alpha}}(\mathfrak{D}'/(\mathfrak{D}' \cap \mathfrak{B}^*)) \geq 0$.

Moreover $\mathfrak{D}$ *could be chosen so that* $R^{\mathfrak{D}} = R^{\mathfrak{B}^*} = R^{\mathfrak{B}}$ *for all but one* $R \in L$.

Remark 3.7. The $\mathfrak{D}$ obtained here achieves "maximal density" (of hyperedges) among all its induced substructures.

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[4] i.e., let $R^{\mathfrak{X}} = \emptyset$ for any $R \in L$ in the L-structure $\mathfrak{X}$ on X.

Proof. For each $t \in [1, s]$ denote $\alpha_{\overline{R_t}}$ by α_t. Fix $t^* \in [1, s]$ an index at which $\alpha_{t^*} = \underline{\alpha}$ is realized. Since $\mathfrak{B} \in K'_\alpha$ and $\mathfrak{B} \neq \emptyset$, $\delta(\mathfrak{B}^*) > \delta(\mathfrak{B}) > 0$ and by construction $\mathfrak{B}^* \in K'_\alpha$.[5] One may assume $\epsilon < \min\{\delta(\mathfrak{B}), \underline{\alpha}\}$.[6] Since $G_{\underline{\alpha}}$ is dense in $\mathbb{R}$, one may find infinitely many locally optimal integers p with $-\epsilon < p - q_p^+ \underline{\alpha} < 0$; we fix one such $p > 2$ so large that $p(1/\underline{\alpha} - 1) > |\mathfrak{B}^*|$. Define the sequences $\langle d_n, s_n : 1 \leq n \leq p \rangle$ as in Facts 3.5, with respect to $\underline{\alpha}$. Our bound on $k = |X|$ and $f)$ in Facts 3.5 ensures that $|X| > \overline{r}(\overline{r} + 1/\underline{\alpha}) > \overline{r}(1 + 1/\underline{\alpha}) > \overline{r} \cdot \max\{s_i : 1 \leq i \leq p\}$; it also follows from $e)$ that $\sum_{i=1}^{p} s_i = q_p^+ - (p-1) > q_p - (p-1) \overset{*}{>} p/\underline{\alpha} - p > |\mathfrak{B}^*| > |X| = k$. [7]

Let $C := \{c_1, \cdots, c_p\}$ be disjoint from B^*; let $I(B)$:= the set of isolated vertices (w.r.t. $\overline{R_{t^*}}$) in B. We build a finite structure $\mathfrak{D}$ built from the following conditions:

1. The universe of $\mathfrak{D}$ is $B^* \cup C$;
2. $\mathfrak{B}^* \subseteq \mathfrak{D}$;
3. For each relation $R \neq \overline{R_{t^*}}$, $R^{\mathfrak{D}} = R^{\mathfrak{B}^*}$.
4. For each $1 \leq i \leq p$, there are exactly s_i subsets $Q_{i,1}, \cdots, Q_{i,s_i}$ drawn from B^* each of size $\overline{r_{t^*}} - 1$ s.t.
    (a) $Q_{i,j} \cup \{c_i\} \in \overline{R_{t^*}}^{\mathfrak{D}}$ for each $i \in [1, p], j \in [1, s_i]$ and
    (b) in each $Q_{i,j}$ there is at least 1 element from $X \cup I(B)$.

This is possible since for each i:

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[5] Any $\mathfrak{C} \subset \mathfrak{B}^*$, $\mathfrak{C} \cap \mathfrak{B}$ is a substructure of $\mathfrak{B}$ lying in K'_α, and so is $\mathfrak{C} \cap \mathfrak{X}$. Since $\mathfrak{B} \cap \mathfrak{X} = \emptyset$, $\mathfrak{C} = (\mathfrak{C} \cap \mathfrak{B}) \oplus (\mathfrak{C} \cap \mathfrak{X}) \in K'_\alpha$.

[6] Otherwise, take $0 < \epsilon' < \min\{\delta(\mathfrak{B}), \underline{\alpha}\}$ and prove the two inequalities for ϵ'; note that in this case $0 < \epsilon' < \epsilon$ so the inequalities, in particular (1), still hold for ϵ.

[7] (*): we have that $q_p > p/\underline{\alpha} - 1$ by the definition of q_p. Therefore $q_p - (p-1) > p/\underline{\alpha} - p$, as desired.

   (a) if $\overline{r_{t^*}} \geq 3$:

      i. If $I(B) = B$ then $B^* = X \cup B = X \cup I(B)$. Note that $|X \cup I(B)| > |X| > \overline{r_{t^*}} \geq 3$ so number of choices for $Q_{i,j}$s are $\binom{|X \cup I(B)|}{\overline{r_{t^*}}-1} \geq |X| > s_i$ meaning we have sufficiently many choices.

      ii. Otherwise $|B \smallsetminus I(B)| \geq 2$ (note that $B \smallsetminus I(B)$ is the set of *non-isolated* vertices). So at least one has $\binom{|B \smallsetminus I(B)|}{1} \cdot \binom{|X \cup I(B)|}{\overline{r_{t^*}}-2} \geq |X| > s_i$ choices for the $Q_{i,j}$s.

   (b) if $\overline{r_{t^*}} = 2$, all $Q_{i,j}$s are singletons and they all come from $X \cup I(B)$. So number of choices simply amounts to $\binom{|X \cup I(B)|}{1} \geq |X| > s_i$.

   In fact we shall soon give a *canonical way* of choosing these $Q_{i,j}$s later for each scenario, which we adopt as our algorithm creating $Q_{i,j}$s. Also, though seemingly obvious, we shall argue later that no $K_{\overline{R_{t^*}},m_{t^*}}$ is created here.

5. Each $b \in X \cup I(B)$ is in at least one of the $Q_{i,j}$s: this is possible since:

   (a) If $\overline{r_{t^*}} \geq 3$:

      i. If $I(B) = B$, recall that $X \cup I(B) = B^*$. A *canonical way* to do Step 4 is simply to give a listing $\{\eta_1, \eta_2, \cdots, \eta_{|B^*|}\}$ of B^*, and walk through the list setting $\{\eta_1, \cdots, \eta_{\overline{r_{t^*}}-1}\} =: Q_{1,1}$, $\{\eta_{\overline{r_{t^*}}}, \cdots, \eta_{2\overline{r_{t^*}}-2}\} =: Q_{1,2}$, etc, only coming back to the beginning and keeping walking when necessary. Since total size of the $Q_{i,j}$s are $\sum_{i=1}^{p} s_i \cdot (\overline{r_{t^*}} - 1) > \sum_{i=1}^{p} s_i > |B^*|$, it is guaranteed that the list will eventually be exhausted.

      For each $i^{**} \in [1, q]$, this process does not generate repeated $Q_{i^{**},j}$s, as

$s_{i^{**}}(\overline{r_{t^*}}-1) < \max_i(s_i)\cdot\overline{r} < k < |B^*|$.[8]

    ii. If $I(B) \subsetneq B$ then similarly, in Step 4 one may list $\{\eta_1, \cdots, \eta_{|X\cup I(B)|}\}$, and let $\{\eta_1, \cdots, \eta_{\overline{r_{t^*}}-2}\} \subset Q_{1,1}$, $\{\eta_{\overline{r_{t^*}}-1}, \cdots, \eta_{2\overline{r_{t^*}}-4}\} \subset Q_{1,2}$, etc, with the remaining elements drawn from $B \setminus I(B)$ arbitrarily. We still have $\sum_{i=1}^{p} s_i\cdot(\overline{r_{t^*}}-2) \geq \sum_{i=1}^{p} s_i > |B^*| > |X \cup I(B)|$ and $s_{i^{**}}(\overline{r_{t^*}}-2) < \max_i(s_i)\cdot\overline{r} < k \leq |X\cup I(B)|$, so exhaustion is guaranteed and repetition within the same i^{**} is avoided.

  (b) If $\overline{r_{t^*}} = 2$: in Step 4 list $\{\eta_1, \cdots, \eta_{|X\cup I(B)|}\}$ and choose $Q_{i,j}$s as above. Now we have total size of all $Q_{i,j}$s: $\sum_{i\leq p} s_i > |B^*| \geq |X\cup I(B)|$ and fixing any $i^{**} \in [1,p]$, total size of $Q_{i^{**},j}$s: $s_{i^{**}}\cdot 1 < \max_i(s_i)\cdot\overline{r} < k \leq |X \cup I(B)|$, yielding again the exhaustion and non-repetition.

6. there is exactly one subset $Z_i \subset B^*$ of size $\overline{r^*}-2$ such that $Z_i \cup \{c_i, c_{i+1}\} \in \overline{R_{t^*}}^{\mathfrak{D}}$ for each $1 \leq i < p$,[9] which does not create $K_{\overline{R_{t^*}},m_{t^*}}$ either (See later argument) and

7. $\overline{R_{t^*}}^{\mathfrak{D}}$ contains no other subsets of D.

We argue that no forbidden structure is created in $\mathfrak{D}$. Since the only modified relation is $\overline{R_{t^*}}$, it suffices to show that no $K_{\overline{R_{t^*}},m_{t^*}}$ is created; by construction a $K_{\overline{R_{t^*}},m_{t^*}}$ must involve either 1 or 2 vertices from C, so it suffices to show that neither Step 4 nor Step 6 creates any $K_{\overline{R_{t^*}},m_{t^*}}$:

1. If some $K_{\overline{R_{t^*}},m_{t^*}}$ were created in clause (4), there must exist $i \in [1,p]$, $x_1, \cdots, x_{\overline{r_{t^*}}+m_{t^*}-1} \in \bigcup_{j\in[1,s_i]} Q_{i,j}$ where:

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[8] So the exhaustion will happen at some $i^{**} > 1$.

[9] Note that in arity $\overline{r_{t^*}} = 2$ this simply means $\{c_i, c_{i+1}\} \in \overline{R_{t^*}}^{\mathfrak{D}}$ for each $i \in [1, p-1]$.

   (a) for each size $\overline{r_{t^*}}$ subset $S \subset \{x_1, x_{\overline{r_{t^*}}+m_{t^*}-1}\}$, $S \in \overline{R_{t^*}}^{\mathfrak{D}}$; and

   (b) for each size $\overline{r_{t^*}} - 1$ subset $S' \subset \{x_1, x_{\overline{r_{t^*}}+m_{t^*}-1}\}$,$\{c_i\} \cup S' \in \overline{R_{t^*}}^{\mathfrak{D}}$.

   However, let $\bigcup_{j \leq n} S_j$ be a subset cover of $\{x_1, \cdots, x_{\overline{r_{t^*}}+m_{t^*}-1}\}$ with each $|S_j| = \overline{r_{t^*}}$, then $S_1, \cdots, S_n \in \overline{R_{t^*}}^{\mathfrak{D}} \bigwedge \bigcup_j S_j \subset \bigcup_{j \in [1,s_i]} Q_{i,j} \subset B^*$ implies each $S_j \in \overline{R_{t^*}}^{\mathfrak{D}}|_{\mathfrak{B}^*} = \overline{R_{t^*}}^{\mathfrak{B}^*} = \overline{R_{t^*}}^{\mathfrak{B} \smallsetminus I(\mathfrak{B})}$, which gives $\bigcup_j S_j = \{x_1, \cdots, x_{\overline{r_{t^*}}+m_{t^*}-1}\} \subset B \smallsetminus I(B)$. On the other hand by Clauses 4b,6,7 , in $\mathfrak{D}$ no c_i is $\overline{R_{t^*}}$-related with any size-($\overline{r_{t^*}} - 1$) subset *purely* drawn from $B \smallsetminus I(B)$, contradicting Clause 1b.

2. For Clause (6) to create an $K_{\overline{R_{t^*}}, m_{t^*}}$ one would need some $i \in [1, p-1]$, $x_1, \cdots, x_{\overline{r_{t^*}}+m_{t^*}-2} \in \bigcup_j Z_j \subset B^*$ s.t.

   (a) $S \in \overline{R_{t^*}}^{\mathfrak{D}}$ for each $S \subset \{x_1, \cdots, x_{\overline{r_{t^*}}+m_{t^*}-2}\}$ with size $\overline{r_{t^*}}$;

   (b) $\{c_i\} \cup S', \{c_{i+1}\} \cup S' \in \overline{R_{t^*}}^{\mathfrak{D}}$ for each $S' \subset \{x_1, \cdots, x_{\overline{r_{t^*}}+m_{t^*}-2}\}$ with size $\overline{r_{t^*}} - 1$;

   (c) $\{c_i, c_{i+1}\} \cup S'' \in \overline{R_{t^*}}^{\mathfrak{D}}$ for each $S'' \subset \{x_1, \cdots, x_{\overline{r_{t^*}}+m_{t^*}-2}\}$ with size $\overline{r_{t^*}} - 2$; $(\star^{\star}\star)$

   Note that since $\overline{r_{t^*}} + m_{t^*} \geq 3$, Clause ($2c$) (Clause $(\star^{\star}\star)$) is not a trivial condition. We split the cases due to $\overline{r_{t^*}}$:

   (a) If $\overline{r_{t^*}} \geq 3$: Observe $\overline{r_{t^*}} + m_{t^*} - 2 \geq \overline{r_{t^*}} - 1 > \overline{r_{t^*}} - 2 \geq 1$ so $\binom{\overline{r_t}+m_t-2}{\overline{r_{t^*}}-2} > 1$, meaning Clause (2c) above would describe more than 1 elements in $\overline{R_{t^*}}^{\mathfrak{D}}$, yet we have exactly one relation $\{c_i, c_{i+1}, \underbrace{\cdots}_{=Z_i \subset \mathfrak{B}^*}\} \in \overline{R_{t^*}}^{\mathfrak{D}}$, contradiction.

   (b) If $\overline{r_{t^*}} = 2$, Clause (2c) above says $\{c_i, c_{i+1}\} \in \overline{R_{t^*}}^{\mathfrak{D}}$. Since a clique containing c_i, c_{i+1} must in particular contain a triangle ${}_{c_i}\overset{*}{\blacktriangle}_{c_{i+1}}$, this says there exists some $b \in X \cup I(B)$ [10] s.t. ${}_{c_i}\overset{b}{\blacktriangle}_{c_{i+1}} \hookrightarrow \overline{R_{t^*}}^{\mathfrak{D}}$; by construction, we have $b \in Q_{i,j} \cap Q_{i+1,j'}$ for some $j \in [1, s_i], j' \in [1, s_{i+1}]$. This is impossible, because the total size of $Q_{i,*}, Q_{i+1,*}$ is $s_i + s_{i+1} \leq 2 \cdot \max_i s_i < k < |X \cup I(B)|$; pursuant to our algorithm creating $Q_{i,j}$s, this means we cannot exhaust $X \cup I(B)$ just by making $Q_{i,*}, Q_{i+1,*}$s, so they don't touch each other, i.e. $Q_{i,j} \cap Q_{i+1,k} = \emptyset$ for any $j \in [1, s_i], k \in [1, s_{i+1}]$.

We have established that $K_{\overline{R_{t^*}},m_{t^*}} \not\rightarrow \mathfrak{D}$; since no other relations are changed, this means once we have $\mathfrak{D} \in K_\alpha$ we would have $\mathfrak{D} \in K'_\alpha$. Moreover, observe that once we establish the inequalities in Lemma 3.6, by $\delta(\mathfrak{D}) = \delta(\mathfrak{D}/\mathfrak{B}^*) + \delta(\mathfrak{B}^*) \underbrace{>}_{\text{Inequality (1)}} -\epsilon + \epsilon = 0$ and $\delta(\mathfrak{D}') = \delta(\mathfrak{D}'/(\mathfrak{D}' \cap \mathfrak{B}^*)) + \delta(\mathfrak{D}' \cap \mathfrak{B}^*) \underbrace{\geq}_{\text{Inequality (2)}} 0 + \delta(\underbrace{\mathfrak{D}' \cap \mathfrak{B}^*}_{\subseteq \mathfrak{B}^* \in K'_\alpha}) \geq 0$ we would have $\mathfrak{D} \in K_\alpha$. Thus the only remaining job is to verify the two inequalities in the statement of Lemma 3.6.

First, $D - B^*$ has p elements; Clause (4) contributes to $\sum_{i=1}^p s_i$ more subsets in $\overline{R_{t^*}}^{\mathfrak{D}}$ while Clause (6) adds $p - 1$ more. Then $\delta(\mathfrak{D}/\mathfrak{B}^*) = \Big(p - \underline{\alpha}(\sum_{i=1}^p s_i + (p-1))\Big) = \Big(p - \underline{\alpha} q_p^+\Big) \in (-\epsilon, 0)$ by our choice of locally optimal integer p at the beginning. The first inequality is proved.

To show the second inequality, let us begin with the proper substructures fully containing $\mathfrak{B}^*$. For each $1 \leq n < p$, denote $\mathfrak{D}_n \subseteq \mathfrak{D}$ with universe $B^* \cup \{c_1, \cdots, c_n\}$ and for $1 \leq n < m \leq p$ denote $\mathfrak{D}_{n,m} \subseteq \mathfrak{D}$ with universe $B^* \cup \{c_{n+1}, \cdots, c_m\}$. We have

$$\delta(\mathfrak{D}_n/\mathfrak{B}^*) = n - \underline{\alpha}(\sum_{i=1}^{n} s_i + (n-1)) = n - q_n\underline{\alpha} = d_n > 0 \tag{3.4}$$

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[10] note that in arity 2 we only drew elements from $X \cup I(B)$ in Step 4 and drew no element from B^* in Step 6; that is Z_is are empty.

and

$$\begin{aligned}
\delta(\mathfrak{D}_{n,m}/\mathfrak{B}^*) &= m-n-\underline{\alpha}(\underbrace{(m-n-1)}_{\text{Clause } 6}+\underbrace{\sum_{i=m+1}^{n} s_i}_{\text{Clause } 4}) \\
&= m-\underline{\alpha}(\sum_{i=1}^{m} s_i+m-1)-(n-\underline{\alpha}(\sum_{i=1}^{n} s_i+n-1))+\underline{\alpha} \\
&= \begin{cases} m-q_m\underline{\alpha}-(n-q_n\underline{\alpha})+\underline{\alpha}, & \text{if } m \neq p \\ p-q_p^+\underline{\alpha}-(n-q_n\underline{\alpha})+\underline{\alpha} & , \text{if } m = p \end{cases} \\
&= d_m-d_n+\underline{\alpha} > 0
\end{aligned}$$

since for any $\emptyset \neq A \subsetneq C$ letting $\mathfrak{D}_A :=$ the substructure of $\mathfrak{D}$ with the universe $B^* \cup A$, $\mathfrak{D}_A$ is a free join over $\mathfrak{B}^*$ of substructures of the form $\mathfrak{D}_n$ and $\mathfrak{D}_{n,m}$s[11], by closure of K'_α under free join one has $\delta(\mathfrak{D}_A/\mathfrak{B}^*) \geq 0$. Since $\delta(\mathfrak{B}^*/\mathfrak{B}^*) = 0$, we conclude that for any proper $\mathfrak{B}^* \subseteq \mathfrak{D}' \subsetneq \mathfrak{D}$, the second inequality holds.

For general $\mathfrak{D}' \subsetneq \mathfrak{D}$, denote by $\mathfrak{B}^*_0, \mathfrak{D}^*$ resp. the substructure of $\mathfrak{D}$ with universe $D' \cap B^*$, $D' \cup B^*$. Then $\mathfrak{D}^*$ is a join of $\mathfrak{D}'$ and $\mathfrak{B}^*$ over $\mathfrak{B}^*_0$, and it suffices to show $\delta(\mathfrak{D}'/\mathfrak{B}^*_0) \geq 0$.

1. If $\mathfrak{D}^* \neq \mathfrak{D}$, then from the predimension lemma (Lemma 2.3) in [1], $\delta(\mathfrak{D}^*/\mathfrak{B}^*) \leq \delta(\mathfrak{D}'/\mathfrak{B}^*_0)$. Note that $\mathfrak{D}^*$ is a proper substructure of $\mathfrak{D}$ containing $\mathfrak{B}^*$, so by the above $\delta(\mathfrak{D}^*/\mathfrak{B}^*) \geq 0$. We obtain $\delta(\mathfrak{D}'/\mathfrak{B}^*_0) = \delta(\mathfrak{D}'/(\mathfrak{D}' \cap \mathfrak{B}^*)) \geq \delta(\mathfrak{D}^*/\mathfrak{B}^*) \geq 0$ as desired.

2. If $\mathfrak{D}^* = \mathfrak{D}$ then $\mathfrak{B}^*_0 \neq \mathfrak{B}^*$ since $\mathfrak{D}'$ is proper. The universe D' must contain all of

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[11] e.g. for $p = 31$, $A = \{c_i \in C : i = 2^g$ for some g$\}$, then $\mathfrak{D}_A = \bigoplus_{\mathfrak{B}^*}\{\mathfrak{D}_2, \mathfrak{D}_{3,4}, \mathfrak{D}_{7,8}, \mathfrak{D}_{15,16}\}$. Also there is no partition of $\mathfrak{D}$ into the *free* join of any $\mathfrak{D}_n, \mathfrak{D}_{n',m'}$s *over* $\mathfrak{B}^*$: if $n' \geq n$, we miss the relation $(c_n, c_{n+1}, \cdots)$, while if $n' < n$, $\mathfrak{D}_n \cap \mathfrak{D}_{n',m'} \neq \mathfrak{B}^*$.

C (for $B^* \cap C = \emptyset$) and avoid some $b \in \mathfrak{B}^* \setminus \mathfrak{B}^*_0$;

(a) if $b \in I(B) \cup X$ then by Clause (5), $(b, c_i, \cdots) \in \overline{R_{t^*}}^{\mathfrak{D}}$ for some i; but then $\overline{R_{t^*}}^{\mathfrak{D}^*} = \overline{R_{t^*}}^{\mathfrak{D}} \neq \overline{R_{t^*}}^{\mathfrak{B}^*_0} \cup \overline{R_{t^*}}^{\mathfrak{D}'}$, as both $\mathfrak{B}^*_0, \mathfrak{D}'$ avoids b;

(b) Otherwise $b \in B \setminus I(B)$, then some $(b, \cdots) \in \overline{R_{t^*}}^{\mathfrak{B}} \subseteq \overline{R_{t^*}}^{\mathfrak{D}}$ but $\mathfrak{B}^*_0, \mathfrak{D}'$ avoids b so again $\overline{R_{t^*}}^{\mathfrak{D}^*} = \overline{R_{t^*}}^{\mathfrak{D}} \neq \overline{R_{t^*}}^{\mathfrak{B}^*_0} \cup \overline{R_{t^*}}^{\mathfrak{D}'}$.

By lemma 2.3 [1], $\delta(\mathfrak{D}'/\mathfrak{B}^*_0) \geq \delta(\mathfrak{D}^*/\mathfrak{B}^*) + \underline{\alpha} = \delta(\mathfrak{D}/\mathfrak{B}^*) + \underline{\alpha} \underbrace{>}_{\text{first inequality}} -\epsilon + \underline{\alpha} > 0$, as desired.

$\square$

Another core result: For any strong pair $(\mathfrak{A} \leq \mathfrak{B}) \in {\mathcal{K}'_{\vec{\alpha}}}^2$ and a finite collection of forbidden structures $\Phi \subset_{\text{fin}} \mathcal{K}'_{\vec{\alpha}}$, there exists an extension $\mathcal{K}'_{\alpha} \ni \mathfrak{D} \supseteq \mathfrak{B}$ that strongly extends $\mathfrak{A}$, is "infinitesimally sparser" than $\mathfrak{A}$, and avoids all forbidden structures in Φ.

Lemma 3.8. *Let $\mathfrak{A} \leq \mathfrak{B} \in \mathcal{K}'_{\vec{\alpha}}$, $\mu > 0$, and a finite set $\Phi \subset K'_{\vec{\alpha}}$ where $\mathfrak{B} \subset \mathfrak{C}$ yet $\mathfrak{B} \not\leq \mathfrak{C}$ for each $\mathfrak{C} \in \Phi$. Then there exists $\mathcal{K}'_{\vec{\alpha}} \ni \mathfrak{D}^* \supseteq \mathfrak{B}$, satisfying:*

1. *$\delta(\mathfrak{D}^*/\mathfrak{A}) < \mu$;*
2. *$\mathfrak{A} \leq \mathfrak{D}^*$; and*
3. *no $\mathfrak{C} \in \Phi$ isomorphically embeds into $\mathfrak{D}^*$ over $\mathfrak{B}$ (note: not "over $\mathfrak{B}^*$", which would be weaker).*

Proof. Note that if $\mathfrak{B} = \emptyset$ then $\mathfrak{A} \leq \mathfrak{B}$ implies $\mathfrak{A} = \emptyset$ and Φ has to be empty as we require $\mathfrak{B} \not\leq \mathfrak{C}$ for any $\mathfrak{C} \in \Phi$. The conclusion then follows by taking any $\mathfrak{D}^*$ with $\delta(\mathfrak{D}^*) < \mu$, such as $\mathfrak{B}$ itself. Also when $\mathfrak{A} = \mathfrak{B}$, taking $\mathfrak{D}^* := \mathfrak{B}$ does the job. So we assume $\mathfrak{B} \neq \emptyset$, $\mathfrak{A} \neq \mathfrak{B}$ (so by the assumption that α_Rs are irrational, $\delta(\mathfrak{B}/\mathfrak{A}) > 0$). Let $X = \{e_1, \cdots, e_k\}$ disjoint from B with $k > \overline{r}(1/\underline{\alpha} + \overline{r})$, $R^{\mathfrak{X}} = \emptyset$ for all $R \in L$ and $\mathfrak{B}^* := \mathfrak{B} \oplus \mathfrak{X}$. Note that $\mathfrak{A} \leq \mathfrak{B} \implies \mathfrak{A} \leq \mathfrak{B}^*$ by construction. It is also without loss of generality to assume $\delta(\mathfrak{C}/\mathfrak{B}) < 0$ for all $\mathfrak{C} \in \Phi$, as for the cases where $\delta(\mathfrak{B}/\mathfrak{C}) \geq 0$ one may replace $\mathfrak{C}$ with a minimal $\mathfrak{C}'$ s.t. $\mathfrak{B} \subset \mathfrak{C}' \subset \mathfrak{C}$ and $\delta(\mathfrak{C}') < \delta(\mathfrak{B})$.[12]

Now fix $\mathbb{Z} \ni s' > \max_{\mathfrak{C} \in \Phi}\{|\mathfrak{C}|\}$ and then choose $\epsilon > 0$ so small that

1. $\epsilon < \min\{\mu, \delta(\mathfrak{B}/A), \underline{\alpha}\}$: note that this guarantees $\epsilon < \delta(\mathfrak{B}^*/A)$ as well; and
2. $s'\epsilon < -\delta(\mathfrak{C}/\mathfrak{B}) \leq -\delta(\mathfrak{C}/\mathfrak{B}^* \cap \mathfrak{C})$ (recall $\delta(\mathfrak{C} \cap \mathfrak{B}^*) \geq \delta(\mathfrak{C} \cap \mathfrak{B}) = \delta(\mathfrak{B})$ by construction of $\mathfrak{B}^*$ so indeed $\delta(\mathfrak{C}/(\mathfrak{C} \cap \mathfrak{B}^*)) \leq \delta(\mathfrak{C}/\mathfrak{B}) < 0$.

We first work with $\mathfrak{B}^*$ instead of $\mathfrak{B}$ to find $\mathfrak{D}^*$ s.t. $\delta(\mathfrak{D}^*/\mathfrak{A}) < \mu$ and $\mathfrak{A} \leq \mathfrak{D}^*$. Use Lemma (3.6) to find $\mathfrak{D} \in K'_\alpha$ for $(\mathfrak{B}^*, \epsilon)$: note that $\mathfrak{D}$ is also an extension of $\mathfrak{B}$. Let $\gamma = -\delta(\mathfrak{D}/\mathfrak{B}^*)$ so $0 < \gamma < \epsilon < \delta(\mathfrak{B}^*/\mathfrak{A})$ by Inequality (1) in Lemma (3.6). Choose an integer h s.t. $h\gamma \leq \delta(\mathfrak{B}^*/\mathfrak{A}) \leq (h+1)\gamma$, which exists since $1 \cdot \gamma < \delta(\mathfrak{B}^*/\mathfrak{A})$ and $h\gamma \overset{h\to\infty}{\to} \infty$. Let $\{\mathfrak{D}_i : i < h\}$ be h copies of $\mathfrak{D}$ with $D_i \cap D_j = B^*$ for all $i \neq j$ and $\mathfrak{D}^* := \bigoplus_{i<h,/\mathfrak{B}^*} \mathfrak{D}_i$ be the free join of $\{\mathfrak{D}_i : i < h\}$ over $\mathfrak{B}^*$.

The predimension equation for free join gives $\delta(\mathfrak{D}^*/\mathfrak{B}^*) = h \cdot \delta(\mathfrak{D}/\mathfrak{B}^*) = -h\gamma$, hence $\delta(\mathfrak{D}^*/\mathfrak{A}) = \delta(\mathfrak{D}^*/\mathfrak{B}^*) + \delta(\mathfrak{B}^*/\mathfrak{A}) = \delta(\mathfrak{B}^*/\mathfrak{A}) - h\gamma \leq \gamma < \epsilon < \mu$, as desired.

To see $\mathfrak{A} \leq \mathfrak{D}^*$, choose any $\mathfrak{C}$ s.t. $\mathfrak{A} \subseteq \mathfrak{C} \subseteq \mathfrak{D}^*$. Write $\mathfrak{B}^*_0 := \mathfrak{C} \cap \mathfrak{B}^*$ and $\mathfrak{C}_i := \mathfrak{C} \cap \mathfrak{D}_i$ for each $i < h$. As substructure of $\mathfrak{B}^*$,

1. if $\mathfrak{B}^*_0 = \mathfrak{B}^*$, then $\mathfrak{B}^* = \mathfrak{B}^* \cap \mathfrak{C}$ implies $\mathfrak{B}^* \subseteq \mathfrak{C}$ and $\mathfrak{C}_i = \mathfrak{C} \cap \mathfrak{D}_i \supseteq \mathfrak{B}^* \cap \mathfrak{D}_i = \mathfrak{B}^*$, and $\mathfrak{C}$ is the free join of $\mathfrak{C}_i$ over $\mathfrak{B}^*$. By Lemma 3.6 we have $\delta(\mathfrak{D}'/\mathfrak{B}^*) \geq -\gamma$

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[12] Having found a $\mathfrak{D}^*$ w.r.t. such $\mathfrak{C}'$, if $\mathfrak{C}$ were to isomorphically embed into $\mathfrak{D}^*$ over $\mathfrak{B}$ so would $\mathfrak{C}'$ by restricting the embedding, contradiction. Also such $\mathfrak{C}'$ must exist, for $\mathfrak{B} \not\leq \mathfrak{C}$.

(in fact it's either $-r$ or non-negative depending whether $\mathfrak{D}' = \mathfrak{D}$) for all $\mathfrak{D}'$ s.t. $\mathfrak{B}^* \subseteq \mathfrak{D}' \subseteq \mathfrak{D}_i \cong \mathfrak{D}$. Then in particular, $\delta(\mathfrak{C}_i/\mathfrak{B}^*) \geq -\gamma$ for each $i < h$. This gives $\delta(\mathfrak{C}) = \delta(\mathfrak{B}^*) + \sum_{i<h} \delta(\mathfrak{C}_i/\mathfrak{B}^*) \geq \delta(\mathfrak{B}^*) - h \cdot \gamma \geq \delta(\mathfrak{A})$.

2. if $\mathfrak{B}^*_0 \neq \mathfrak{B}^*$, then we still have $\mathfrak{A} \leq \mathfrak{B} \leq \mathfrak{B}^* \implies \delta(\mathfrak{B}^*_0/\mathfrak{A}) \geq 0$ by $\mathfrak{A} \subseteq \mathfrak{B}^*_0 \subsetneq \mathfrak{B}^*$, and $\mathfrak{C}_i \subsetneq \mathfrak{D}_i \cong \mathfrak{D}$ for each $i < h$[13] so $\delta(\mathfrak{C}_i/(\mathfrak{C}_i \cap \mathfrak{B}^*)) = \delta(\mathfrak{C}_i/\mathfrak{B}^*_0) \geq 0$ by $\mathfrak{D}_i \cap \mathfrak{B}^* = \mathfrak{B}^*$[14] and Lemma 3.6. This gives $\delta(\mathfrak{C}/\mathfrak{A}) = \delta(\mathfrak{C}/\mathfrak{B}^*_0) + \delta(\mathfrak{B}^*_0/\mathfrak{A}) \geq 0$ since $\mathfrak{C}$ is a free join of $\{\mathfrak{C}_i : i < h\}$ over $\mathfrak{B}^*_0$.

The second clause in the statement of the proposition is therefore proved, as $\delta(\mathfrak{C}/\mathfrak{A}) \geq 0$ in either case above. For the last clause, Let $\mathfrak{C}$ be s.t. $|\mathfrak{C}| < s'$, $\mathfrak{B} \subseteq \mathfrak{C} \subseteq \mathfrak{D}^*$, i.e. there exists an isomorphic embedding of $\mathfrak{C}$ into $\mathfrak{D}^*$ over $\mathfrak{B}$. It suffices to show $\delta(\mathfrak{C}/\mathfrak{B}) \geq -s'\epsilon$, which by condition (2) above for s', ϵ means for any $\mathfrak{C}' \in \Phi$, $\mathfrak{C}' \ncong \mathfrak{C}$.

Write $\mathfrak{C}_i := \mathfrak{C} \bigcap \mathfrak{D}_i (\supseteq \mathfrak{C} \cap \mathfrak{B}^*)$. Since $|\mathfrak{C}| < s'$ and $\mathfrak{D}_i$, $\mathfrak{D}_j$ $(j \neq i)$ are disjoint over $\mathfrak{B}^*$, there are fewer than s' indices is for which $\mathfrak{C}_i \supsetneq \mathfrak{B}^* \cap \mathfrak{C}$.[15] For these is, $\delta(\mathfrak{C}_i/(\underbrace{\mathfrak{B}^* \cap \mathfrak{C}}_{=\mathfrak{B}^* \cap \mathfrak{C}_i}))$ is either ≥ 0 (when $\mathfrak{C}_i \subsetneq \mathfrak{D}_i$) or in $(-\epsilon, 0)$ (when $\mathfrak{C}_i = \mathfrak{D}_i$) by Lemma (3.6) for $(\mathfrak{D} \cong \mathfrak{D}_i, \mathfrak{B}^*, \epsilon)$. For the other indices $j \in \{0, \cdots, h-1\}$, we have $\mathfrak{C}_j = \mathfrak{B}^* \cap \mathfrak{C}$ so $\delta(\mathfrak{C}_j/(\mathfrak{B}^* \cap \mathfrak{C})) = 0$. This says $\sum_{i<h} \delta(\mathfrak{C}_i/(\mathfrak{C} \cap \mathfrak{B}^*)) \geq 0 + s' \cdot (-\epsilon)$. Also note that $0 \leq \delta(\mathfrak{C} \cap \mathfrak{B}^*) - \delta(\mathfrak{C} \cap \mathfrak{B}) \leq k = |X|$, i.e. $\delta(\mathfrak{C} \cap \mathfrak{B}^*) \geq \delta(\mathfrak{C} \cap \mathfrak{B}) = \delta(\mathfrak{B})$.

Finally note that C the free join of $\mathfrak{C}_i$ over $\mathfrak{B}^* \cap \mathfrak{C}$, so

$$\delta(\mathfrak{C}) = \delta(\mathfrak{C} \cap \mathfrak{B}^*) + \sum_{i<h} \delta(\mathfrak{C}_i/(\mathfrak{C} \cap \mathfrak{B}^*)) \geq \delta(\mathfrak{B}) + s' \cdot (-\epsilon) \implies \delta(\mathfrak{C}/\mathfrak{B}) \geq -s'\epsilon, \quad (3.5)$$

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[13] Otherwise $\mathfrak{B}^* \subseteq \mathfrak{D}_i \subseteq \mathfrak{C}$ for some i, which gives $\mathfrak{B}^*_0 = \mathfrak{B}^*$.

[14] Hence $\mathfrak{C} \cap \mathfrak{B}^* = \mathfrak{C} \cap (\mathfrak{D}_i \cap \mathfrak{B}^*) = \mathfrak{C}_i \cap \mathfrak{B}^*$.

[15] Indeed, $\mathfrak{C}_i, \mathfrak{C}_j$ are disjoint over $\mathfrak{B}^* \cap \mathfrak{C}$. So if there at at least s' indices where $\mathfrak{C}_i \supsetneq \mathfrak{B}^* \cap \mathfrak{C}$, one would have $|C| \geq |C \smallsetminus (\mathfrak{B}^* \cap \mathfrak{C})| \geq s$ by picking one element from each $\mathfrak{C}_i \smallsetminus (\mathfrak{B}^* \cap \mathfrak{C}) \neq \emptyset$.

as desired. $\square$

Remark 3.9. If we take $\mathfrak{D}^* = \bigoplus_{j \leq h} \mathfrak{D}$ instead (that is, one more copy of $\mathfrak{D}$), then one may prove that $(\mathfrak{A}, \mathfrak{D}^*)$ is a *mimimal pair*, that is, $\delta(\mathfrak{D}^*/\mathfrak{A}) < 0$ yet for any $\mathfrak{D}'$ s.t. $\mathfrak{A} \subseteq \mathfrak{D}' \subsetneq \mathfrak{D}$, $\delta(\mathfrak{D}'/\mathfrak{A}) \geq 0$.

Indeed, by the analysis in the proof one may take h to be s.t. $h\gamma \leq \delta(\mathfrak{B}^*/\mathfrak{A}) < (h+1)\gamma$ instead; now we have $\delta(\mathfrak{D}^*/\mathfrak{A}) = \delta(\mathfrak{D}^*/\mathfrak{B}^*) + \delta(\mathfrak{B}^*/\mathfrak{A}) = -(h+1)\gamma + \delta(\mathfrak{B}^*/\mathfrak{A}) < 0$. On the other hand, to see $\mathfrak{A} \leq \mathfrak{D}'$ for any $\mathfrak{A} \subseteq \mathfrak{D}' \subsetneq \mathfrak{D}^*$, write $\mathfrak{B}_0^* := \mathfrak{D}' \cap \mathfrak{B}^*$ and $\mathfrak{C}_i := \mathfrak{D}' \cap \mathfrak{D}_i$ for each $0 \leq i \leq h$. As substructure of $\mathfrak{B}^*$,

1. if $\mathfrak{B}_0^* = \mathfrak{B}^*$, then $\mathfrak{B}^* = \mathfrak{B}^* \cap \mathfrak{D}'$ implies $\mathfrak{B}^* \subseteq \mathfrak{D}'$ and $\mathfrak{C}_i = \mathfrak{D}' \cap \mathfrak{D}_i \supseteq \mathfrak{B}^* \cap \mathfrak{D}_i = \mathfrak{B}^*$, and $\mathfrak{D}'$ is the free join of $\mathfrak{C}_i$s over $\mathfrak{B}_0^* = \mathfrak{B}^*$. By Lemma 3.6 we have $\delta(\mathfrak{D}'/\mathfrak{B}^*) \geq -\gamma$ (in fact it's either $-r$ or non-negative depending whether $\mathfrak{D}' = \mathfrak{D}$) for all $\mathfrak{D}'$ s.t. $\mathfrak{B}^* \subseteq \mathfrak{D}' \subseteq \mathfrak{D}_i \cong \mathfrak{D}$; furthermore there are at most h out of the $h+1$ indices within $\{0, 1, \cdots, h\}$ s.t. $\mathfrak{C}_i = \mathfrak{D}_i$, for otherwise we would have $\mathfrak{D}' = \mathfrak{D}^*$. Then in particular, $\delta(\mathfrak{C}_i/\mathfrak{B}^*) \geq -\gamma$ for each $i \leq h$, and there are at most h out of $h+1$ indices for which $\delta(\mathfrak{C}_i/\mathfrak{B}^*)$ is possibly negative. This gives $\delta(\mathfrak{C}) = \delta(\mathfrak{B}^*) + \sum_{i \leq h} \delta(\mathfrak{C}_i/\mathfrak{B}^*) \geq \delta(\mathfrak{B}^*) - h \cdot \gamma \geq \delta(\mathfrak{A})$.

2. if $\mathfrak{B}_0^* \neq \mathfrak{B}^*$, then we still have $\mathfrak{A} \leq \mathfrak{B} \leq \mathfrak{B}^* \implies \delta(\mathfrak{B}_0^*/\mathfrak{A}) \geq 0$ by $\mathfrak{A} \subseteq \mathfrak{B}_0^* \subsetneq \mathfrak{B}^*$, and $\mathfrak{C}_i \subsetneq \mathfrak{D}_i \cong \mathfrak{D}$ for each $i \leq h$[16] so $\delta(\mathfrak{C}_i/(\mathfrak{C}_i \cap \mathfrak{B}^*)) = \delta(\mathfrak{C}_i/\mathfrak{B}_0^*) \geq 0$ by $\mathfrak{D}_i \cap \mathfrak{B}^* = \mathfrak{B}^*$[17] and Lemma 3.6. This gives $\delta(\mathfrak{C}/\mathfrak{A}) = \delta(\mathfrak{C}/\mathfrak{B}_0^*) + \delta(\mathfrak{B}_0^*/\mathfrak{A}) \geq 0$ since $\mathfrak{C}$ is a free join of $\{\mathfrak{C}_i : i \leq h\}$ over $\mathfrak{B}_0^*$.

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[16] Otherwise $\mathfrak{B}^* \subseteq \mathfrak{D}_i \subseteq \mathfrak{D}'$ which gives $\mathfrak{B}_0^* = \mathfrak{B}^*$.

[17] Hence $\mathfrak{D}' \cap \mathfrak{B}^* = \mathfrak{D}' \cap (\mathfrak{D}_i \cap \mathfrak{B}^*) = \mathfrak{C}_i \cap \mathfrak{B}^*$.

This finishes our proof of the remark that $(\mathfrak{A}, \mathfrak{D}^*)$ is minimal. This also serves as a detailed verification of the result on $(\mathcal{K}'_{\vec{\alpha}}, \mathfrak{M}'_{\vec{\alpha}})$ analogous to Lemma 6.7 in [1], a step towards more important conclusions such as Theorem 3.10 below.

Let $\mathfrak{M} \models \mathcal{S}'_{\vec{\alpha}}$. For each $(\mathfrak{A} \leq \mathfrak{B}, \Phi)$ triple as described in Lemma 3.8, using a disjoint union of arbitrarily many copies of $\mathfrak{D}$ one can obtain infinitely many embeddings of $\mathfrak{B}$ to $\mathfrak{M}$ avoiding Φ. This in particular says that for a sufficiently saturated ("realizing sufficiently many types") model $\mathfrak{M}^* \models \mathcal{S}'_{\vec{\alpha}}$, every $\mathfrak{B} \in \mathcal{K}'_{\alpha}$ *strongly embeds* in $\mathfrak{M}^*$ (paramount to Propositions $4.4 \sim 4.6$, [1]). These further allows us to characterize the existence of an embedding extension to $\mathfrak{M}^*$ by a "maximally embeddable" member of $\mathcal{K}'_{\vec{\alpha}}$ formed from a finite chain of strong pairs (cf. Theorem 5.4, [1]). This is then used to reduce *each L-formula*, up to $\mathcal{S}'_{\vec{\alpha}}$-equivalence, to a boolean combination of (1-layer, or "near-quantifier-free") existential formulae (cf. Theorem 5.6 [1]), which leads to the completeness of $\mathrm{Cn}(\mathcal{S}'_{\vec{\alpha}})$ (cf. Corollary 5.7 [1]). Now one may verify that $\mathfrak{M}'_{\vec{\alpha}} \models \mathcal{S}'_{\vec{\alpha}}$ (cf. Corollary 5.10 [1]), so by completeness, $\mathrm{Th}(\mathfrak{M}'_{\vec{\alpha}}) = \mathrm{Cn}(\mathcal{S}'_{\vec{\alpha}})$ as desired. All the results named in this paragraph are proved using the same lines of argument as in [1], changing $(\mathcal{K}_\alpha, \mathfrak{M}_\alpha, S_\alpha)$ to $(\mathcal{K}'_{\vec{\alpha}}, \mathfrak{M}'_{\vec{\alpha}}, \mathcal{S}'_{\vec{\alpha}})$ wherever applicable. As such, Theorem 3.4 is established. Note, though, that Corollary 5.8 [1] establishing connection with Shelah-Spencer graphs does not go through for our new configuration-forbidden classes, as the "easy lower bound of Theorem 3 in [23]" is nowhere close to being easy, at least for the author, in our new configuration-forbidden setting. We will exhibit our work in that direction in the next subsection, but let us highlight some model-theoretical implications of the results from this subsection:

Theorem 3.10. *Let $\vec{\alpha}$ have each entry in $(0,1)$. Then $\mathcal{S}'_{\vec{\alpha}}$ axiomatizes Th($\mathfrak{M}'_{\vec{\alpha}}$), so Th($\mathfrak{M}'_{\vec{\alpha}}$) admits a Π_2, in fact near-model-complete, axiomatization.*

Moreover, Th($\mathfrak{M}'_{\vec{\alpha}}$) is stable but unsuperstable if $\vec{\alpha}$ is $\mathbb{Q}$-linearly independent from 1 (for example if all α_is are irrational), and ω-stable otherwise.

Proof. The first paragraph follows from the main result of this subsection. The first subcase of the second paragraph follows from modifying Section 7 of [1], changing $(\mathcal{K}_\alpha, \mathfrak{M}_\alpha, S_\alpha)$ to $(\mathcal{K}'_{\vec{\alpha}}, \mathfrak{M}'_{\vec{\alpha}}, \mathcal{S}'_{\vec{\alpha}})$ wherever applicable. The ω-stability result comes from ([43]); note that as $\mathcal{S}'_{\vec{\alpha}}$ axiomatizes Th($\mathfrak{M}'_{\vec{\alpha}}$), $\mathcal{S}'_{\vec{\alpha}}$ is also ω-stable for $\vec{a}$ linearly dependent with 1 (e.g. when $\vec{a}$ is length 1 and rational). See also, e.g. [24]. $\square$

3.3 Connections with $(0,1)$-law of the Shelah-Spencer Triangle-Free Random Graphs

From now on we fix $L = \{E\}$, where E is again irreflexive and symmetric binary. So L-structures are just (simple, undirected) graphs. Fix $\alpha \in (0,1) \setminus \mathbb{Q}$.[18] As usual let $(v(G), e(G))$ denote resp. the number of vertices and edges. First we make rigorous the definitions of the almost sure theory of Shelah-Spencer graphs.

Definition 3.11. Define for each $n \in \omega$:

$$\begin{aligned} L[n] &:= \{\text{finite graphs with vertex set } [1,n] := \{1,\cdots,n\}\} \\ p(n) &:= n^{-\alpha} \end{aligned}$$

---

[18] It is known that for $\alpha = 1$, there is no 0-1 law in $G(n, n^{-\alpha})$ — see e.g. [46] where it was argued that the sentence "the graph is triangle-free" has asymptotic probability $e^{\frac{-1}{6}}$. We do not know if there is a 0-1 law in $G(n, n^{-\alpha}, \text{TF})$ with our probability measure (*infra*) for general $\alpha \in (0,1)$.

Define the measure $\mu_n : \mathcal{P}(L[n]) \to [0,1]$ by

$$\mu_n(\{G\}) = p(n)^{e(G)}(1-p(n))^{\binom{n}{2}-e(G)} \tag{3.6}$$

then extending by finite additivity. This is a probability measure:

$$\begin{aligned} \mu_n(L[n]) &= \sum_{G\in L[n]} p(n)^{e(G)}(1-p(n))^{\binom{n}{2}-e(G)} = \sum_{0\leq j\leq \binom{n}{2}} \binom{\binom{n}{2}}{j} p(n)^j(1-p(n))^{\binom{n}{2}-j} \\ &= (p(n)+(1-p(n)))^{\binom{n}{2}} = 1 \end{aligned}$$

Conceptually the probability measure describes a graph on $[1,n]$ where each edge appears with probability $p(n)$. The space $(L[n], \mathcal{P}(L[n]), \mu_n)$ is the probability space of the *Shelah-Spencer* graph (on n vertices) when we set the edge-probability $p(n) := n^{-\alpha}$. We shorthand the space by $G(n,p(n))$. Lastly, letting $\mathrm{Sn}_L :=$ the collection of all first order L-sentences, then the *almost sure theory* of $G(n,p(n))$ is

$$a.s.(G(n,p(n))) := \Big\{\varphi \in Sn_L : \lim_{n\to\infty} \mu_n(\{G \in L[n] : G \vDash \varphi\}) = 1\Big\} \tag{3.7}$$

That is, the set of first order sentences that are asymptotically satisfied by "almost every graph".

Definition 3.12. Define:

$$\mathbb{K}_\alpha[n] \quad := \quad \{\text{finite graphs } G \text{ on } [1,n] \text{ s.t. } \forall H \subset_{\text{ind}} G, \delta_\alpha(H) \geq 0\} \tag{3.8}$$

$$\mathbb{K}_\alpha \quad := \quad \bigcup_{n\in\omega} \mathbb{K}_\alpha[n] \tag{3.9}$$

Note that $\mathbb{K}_\alpha$ also satisfies A1$\sim$ A5 + AP in [43] so has a generic structure $\mathcal{M}_\alpha$; furthermore $\mathbb{K}_\alpha$ differs with $\mathcal{K}_\alpha$ from last section (Eqn 3.2) (with generic structure $\mathfrak{M}_\alpha$) only at the "labelledness" of vertices. Unrolling the definition of generic structures and using their respective uniqueness, $\mathcal{M}_\alpha \cong_L \mathfrak{M}_\alpha$; in particular they share the same first order theory.

We have, impressively, that for $\alpha \in (0,1] \setminus \mathbb{Q}$, the model theory of the generic structure $\mathfrak{M}_\alpha$ matches perfectly with the almost-sure theory of Shelah-Spencer:

Fact 3.13 (e.g. [1, 22, 23])**.**

$$Th(\mathfrak{M}_\alpha) = a.s.(G(n,p(n)) \tag{3.10}$$

And as a result, $G(n,p(n)) = G(n,n^{-\alpha})$ has a 0-1 law when $\alpha \in (0,1) \setminus \mathbb{Q}$ and $p(n) = n^{-\alpha}$: indeed, let a first order sentence $\sigma \notin$ a.s.$G(n,p(n))$. Then $\sigma \notin$ Th$(\mathfrak{M}_\alpha)$, so $\neg\sigma \in$ Th$(\mathfrak{M}_\alpha) =$ a.s.$G(n,p(n))$, using completeness of the left hand side. It follows from definition that $\lim_{n\to\infty} \mu_n(\{G \in L[n] : G \models \varphi\}) = 0$, as desired.

It would be great if the connection stands with the triangle-free constraint placed on both sides. More precisely, letting TF:= "triangle-free",

Definition 3.14. Let

$$L'[n] := L[n] \cap TF \tag{3.11}$$

Then $\mathbb{K}'_\alpha = \mathbb{K}_\alpha \cap (\bigcup_{n\in\omega} L'[n]) = \mathbb{K}_\alpha\cap$ TF once again has a generic structure isomorphic to $\mathfrak{M}'_\alpha$ (Section 3.1), the generic of $\mathcal{K}'_\alpha$. Note also that in $L = \{E\}$ and with the forbidden configuration K_3, $\mathfrak{M}'_\alpha = \mathfrak{M}_{\alpha,\text{TF}}$ in Chapter 4. We can denote the axiomatization $\mathcal{S}'_{\vec{\alpha}}$ in Theorem 3.10 as $\mathcal{S}_{\alpha,TF}$.

On the other side, define $\nu_n : \mathcal{P}(L'[n]) \to [0,1] : \nu_n(\{G\}) = \frac{\mu_n(\{G\})}{\mu_n(L'[n])}$.[19] Denote the new probability space $(L'[n], \mathcal{P}(L'[n]), \nu_n)$ by $G(n, p(n); TF)$, then define its *almost sure theory* analogously:

$$a.s.(G(n,p(n);TF)) := \Big\{\varphi \in Sn_L : \lim_{n\to\infty} \nu_n\big(\{G \in L'[n] : G \vDash \varphi\}\big) = 1\Big\} \tag{3.12}$$

Again as the name suggests, a.s.$G(n,p(n);TF)$ is the set of sentences that are almost surely satisfied by sampling from $G(n,p(n);TF)$ with as $n \to \infty$. Note the almost-sure theory exists no matter whether a 0-1 law exists; it is its completeness or non-completeness that is hard:

Problem 3.15. Does Th$(\mathfrak{M}'_\alpha) = a.s.G(n, n^{-\alpha}; TF)$?

The proposition below justifies our approach of studying $\mathcal{S}_{\alpha,\text{TF}}$ for establishing the equation. Indeed, now instead of verifying Th$(\mathfrak{M}_{\alpha,\text{TF}}) \subseteq$ a.s.G$(n, n^{-\alpha};$ TF), we "only" need to verify $\mathcal{S}_{\alpha,\text{TF}} \subseteq$ a.s.G$(n, n^{-\alpha};$ TF) (Theorem 3.10), which is *a priori* simpler as we know precisely and concretely what sentences are in $\mathcal{S}_{\alpha,\text{TF}}$.

Proposition 3.16. *If $\mathcal{S}_{\alpha,TF} \models$ a.s. $G(n, n^{-\alpha}; TF)$, then Th($\mathfrak{M}_{\alpha,TF}$) = a.s. $G(n, n^{-\alpha}; TF)$; in particular, the RHS is complete and therefore $G(n, n^{-\alpha}; TF)$ would have a* 0-1 *law.*

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[19] In the languages of probability: for each $\varphi \in Sn_L$, $\nu_n(\{G \in L'[n] : G \vDash \varphi\}) = \frac{\sum \mu_n(\{G \in L[n]: G \vDash TF \wedge G \vDash \varphi\})}{\mu_n\{G \in L[n]: G \vDash TF\}} = \frac{Pr[G(n,p(n)) \vDash \varphi \wedge TF]}{Pr[G(n,p(n)) \vDash TF]} = Pr[G(n,p(n)) \vDash \varphi \mid G(n,p(n)) \vDash TF]$. Intuitively, think of the random graph as still obtained by flipping a coin at each of the $\binom{n}{2}$ edges with head probability $n^{-\alpha}$, but throwing a way the sample once a triangle appears.

Proof. Indeed, since Th($\mathfrak{M}_{\alpha,\mathrm{TF}}$) = Cn($\mathcal{S}_{\alpha,\mathrm{TF}}$),[20] the assumption that $\mathcal{S}_\alpha \models$ a.s. $G(n, n^{-\alpha}; \mathrm{TF})$ implies Th($\mathfrak{M}_{\alpha,\mathrm{TF}}$) $\subseteq$ a.s. $G(n, n^{-\alpha}; \mathrm{TF})$. But then if the inclusion is proper, let $\sigma \in$ RHS $\setminus$ LHS; by completeness of LHS, $\neg\sigma \in$ LHS $\subseteq$ RHS, so $\sigma, \neg\sigma \in$ RHS, contradicting the definition of RHS. ☐

Some concluding remarks:

Remark 3.17.

1. For $G(n, p(n))$ when $p(n) := c \in [0, 1]$, the famous Fagin-GKLT theorem (e.g. [23, 47, 48, 49*] says that $G(n, p(n))$ does admit a 0-1 law.

2. Another interesting open problem[21]: by e.g. [23, 47*], $G(n, n^{-\alpha})$ does not admit a 0-1 law when α is rational, and it is (relatively) well-known exactly what sentences witness the failure of 0-1 law for rational αs. [24] studied the model-theoretic properties of limits of G($n, n^{-\alpha_n}$) = Th($\mathfrak{M}_{\alpha_n}$) where $\{\alpha_n\}_n$ is a sequence of irrationals approaching a rational α. One may look into similar directions for $G(n, n^{-\alpha}; \mathrm{TF})$ when $\alpha \in \mathbb{Q} \cap (0, 1)$.

3. Literature (e.g. [30]) says when the measure on $L'[n]$ is uniform (i.e. $\nu'_n(\{G\}) = \frac{1}{|L'[n]|}$), there exists a an almost sure theory w.r.t. ν'_n but it does *not* agree with Th($\mathfrak{M}'_\alpha$) (nor does it agree with theory of the generic structure of the class of all triangle-free finite graphs w.r.t. the partial order $\subseteq_{ind}$). In particular, the sentence $\sigma_{C_5} :\equiv$"there exists a C_5 in the graph" is not in the almost sure theory w.r.t. ν'_n. On the other hand, $\sigma_{C_5} \in \mathcal{S}_{\alpha,\mathrm{TF}}$ for all $\alpha \in (0, 1]$, by $C_5 \in \mathcal{K}_{\alpha,\mathrm{TF}} = \mathrm{Age}(\mathfrak{M}_{\alpha,\mathrm{TF}})$. A reasonable first step towards answering Question 3.15 would therefore be deciding

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[20] $Cn(T)$ for a first order theory T is the logcial consequences of T. In particular if $T_1 = Cn(T_2)$ with $T_2 \subseteq T_1$ then $T_1 \equiv T_2$.

[21] As was kindly pointed out by an audience member when I was giving a practice talk.

if $\sigma_{C_5} \in$ a.s.$G(n, p(n); TF)$.

A potentially relevant fact is that there are at most $(\frac{n}{5})^5$ pentagons in an n-vertex TF graph [50], which seems to weigh in favor of answering the Question 3.15 negatively. One may imagine proving something like "for any $G = G(n, p(n))$ ($n >> 0$), in any triangle-removal process of G, immediately after removing all triangles one almost surely have removed all pentagons as well".

4. Another way to approach this is to modify [23]'s original proof of the 0-1 law of G$(n, n^{-\alpha})$ into our G$(n, n^{-\alpha}$; TF$)$ setting. We believe this approach would work if we can prove a similar key lemma as [23]'s Lemma 1, roughly stating that for sufficiently large n, the probability for a fixed pair $(f : H_0 \to G(n, n^{-\alpha}; \text{TF}), B \subset_{\text{fin}} G(n, n^{-\alpha}; \text{TF}))$ to be "bad" (there existing no suitable $\widetilde{f}$ extending f to a particular $H_1 \supseteq H_0$) is less than a half.

   To show that lemma, one needs a probabilistic argument that does not depend on the assertion that "probability that an arbitrary injective map preserves e chosen edges is $(n^{-\alpha})^e$", which holds for $G(n, n^{-\alpha})$ but not necessarily for $G(n, n^{-\alpha}$; TF$)$. We are missing some counting mechanisms in the triangle-free probability space.

   Lastly, in Chapter 4 e.g. Proposition 4.11, one may see that forbidding *cliques other than* K_3 in some sense makes the constraint satisfaction problems "easier" to characterize. Therefore one may also wonder if a similar "complete first order theory of the generic" — "almost sure theory" correspondence holds or fails for subclasses of $\mathcal{K}_\alpha$ forbidding *other cliques*, of which we are unsure.

Chapter 4: Homomorphic Presentation of Hrushovski-Fraïssé classes and CSP of their generic structures

4.1 Introduction

For general discussion of Hrushovski-Fraïssé classes, see Chapter 1. In this chapter we inspect into selected Hrushovski-Fraïssé classes of *graphs*[1] parameterized by an $\alpha > 0$ through different lenses, namely homomorphisms. We aim at finding a small subclass that homomorphically "presents" the original class. For $\alpha > \frac{5}{6}$ such "presents" can be made a singleton by application of known results; my paper proves that:

1. For $\alpha = \frac{5}{6}$, there exists a subclass homomorphically presenting the original class, such that each element in the subclass has, among other structural properties:
   (a) exactly 1 *copy of* a special substructure E_1 and E_2 respectively (Theorem 4.29), which are vertex-disjoint and "far" apart (Lemma 4.19),
   (b) forests as subgraphs near the copies of E_1, E_2 (Theorem 4.42),
   (c) no "very dense" subgraphs outside of the copies of E_1, E_2 (Theorem 4.42), and
   (d) no long paths of consecutive vertices with small degrees (Theorem 4.42).

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[1] i.e. (symmetric, simple) structures in the language $\{E\}$ of binary relation.

The class has a finite homomorphic presentation *if and only if* the aforementioned subclass is finite (Theorem 4.29), which we conjecture to be the case (e.g. Subsection 4.5).

2. These structural properties generalize to different ranges of *smaller* α provided reasonable conditions (Theorems 4.29, 4.42; Porism 4.40, Porism 4.26 etc).

Precise statements and proofs of these results are exhibited in §3. Most of these structural properties are obtained by showing certain induced quotient homomorphisms exist, some of which ensured in part by the large distances between special substructures (Lemma 4.19), which is in turn partly guaranteed by the local density and absence of certain configurations. Some attempts in theoretically developing similar "density – existence of quotient homomorphism" interplay are recorded as one of the future directions (Subsection 4.6.1).

Our study was originally motivated by study of the *constraint satisfaction problems* (CSP) of infinite graphs, a vividly active field of research (e.g. [13, 14, 51, 52, 53*]). Our infinite graphs of interest record the information of Hrushovski-Fraïssé classes in a nice way and its CSP is naturally related to homomorphic presentations. I address such connections in Section (4.3). In §5 we collect interesting combinatorial findings relevant to our study, and point to potential alternative directions to finitizing the homomorphic presentation for relatively large αs.

4.2 Preliminaries

Simple graphs are those where $(\forall x \in V(G))((x,x) \notin E(G))$, and *undirected* graphs have $\forall (x,y) \in E(G)\big((x,y) = (y,x)\big)$. In this chapter we shall omit the adjective "undirected, simple" for graphs, and assume graphs are finite unless otherwise noted. by $(v(G), e(G))$ we denote $(|V(G)|, |E(G)|)$. For $n \geq 2$, P_n denotes the path 1–$\cdots$–n. For $n \geq 3$, C_n denotes the cycle 1–2–$\cdots$–n–1, K_n the clique on $\{1, \cdots, n\}$.

Let H, G be graphs; $H \subseteq G$ denotes the subgraph relation, i.e. $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We use $\subseteq_{ind}$ to denote the induced subgraph relation, i.e. $G \subseteq H \iff \big(V(G) \subseteq V(H) \bigwedge E(G) = \{(x,y) \in E(H) : x, y \in V(G)\}\big)$, or equivalently, G is the substructure of H induced on $V(G)$. For $U \subseteq V(G)$, the *substructure of* G *induced by* U is denoted by $G[U]$. For example, $K_4[\{1,2,3\}]$ is the triangle C_3. Let A, B be graphs; $A \coprod B$ denotes the disjoint union of A, B. If $\mathcal{A}$ is a (finite) set of graphs, $\coprod \mathcal{A} := \coprod_{G \in \mathcal{A}} G$.

For $a \in V(G)$, let $N_G(a)$ denote the set of neighbors of a in G. We omit G when it's clear from context. For $H \subseteq G$ and $k \geq 1$, $N_{G,\leq k}(H) := \big\{x \in G : \mathrm{dist}_G(\{x\}, H) \leq k\big\}$. That is, all vertices in G of distance at most k from H. By slight abuse of notation, $N_{G,\leq k}(H)$ also denotes $G[N_{G,\leq k}(H)]$, the subgraph induced thereby. Note that $N_{G,\leq 1}(H) = G[H \cup N_G(H)]$ and $N_{G,\leq k}(H) \supseteq N_{G,\leq k'}(H)$ whenever $k \geq k'$. Lastly, for H a graph and $E \subseteq E(H)$, $H \smallsetminus E$ denotes the subgraph on $V(H)$ with edges in E deleted.

When a function $f : A \to B$ is surjective, we write $f : A \twoheadrightarrow B$; when injective, we write $f : A \hookrightarrow B$. For a set S, we use the same notation for "the decision problem of S-membership" . For example, $\mathcal{K}_\alpha$ also refers to the decision problem "is an input graph $G \in \mathcal{K}_\alpha$?". We denote by $x \in! Y \iff Y = \{x\}$. For $H_1, H_2 \subseteq G$, denote by $E(H_1, H_2)$ the set of edges of the form $(x, y) \in E(G) : x \in V(H_1), y \in V(H_2)$. Denote by $E(H_1) := E(H_1, H_1)$.

Definition 4.1 (subgraph distance)**.** Let $H_1, H_2 \subseteq_{\text{ind}} G$. Define $\mathit{dist}_G(H_1, H_2) :=$

$$\begin{cases} 0 & \text{if } H_1 \cap H_2 \neq \emptyset \\ \min\{k-1 : x_1 - \cdots - x_k \subseteq_{\text{ind}} G, x_1 \in H_1, x_k \in H_2\} & \text{if } H_1 \cap H_2 = \emptyset \bigwedge H_1, H_2 \subseteq \text{the same} \\ & \text{component of } G \\ \infty & \text{otherwise} \end{cases}$$

For any graph G, possibly countably infinite, denote:

1. the complement graph $\widetilde{G}$ on the same vertex set $V(G)$ with edges exactly those missing from G.
2. for any $k \geq 1$, $kG :=$ the disjoint union of k copies of G.

Recall that given graphs G, H, the set of *graph homomorphisms* $\mathrm{Hom}_{\{E\}}(G, H) := \{f : V(G) \to V(H) : E^G(x, y) \implies E^H(f(x), f(y))\}$. We omit the signature $\{E\}$ when no confusion may arise. Let $\mathcal{S}, \mathcal{T}$ be classes of finite graphs. We write $\mathcal{S} \to \mathcal{T}$ if for each $s \in \mathcal{S}$ there exists $t \in \mathcal{T}$ s.t. s homomorphically maps to t. We say $\mathcal{T}$ *homomorphically presents* $\mathcal{S}$ if $\mathcal{S} \to \mathcal{T}$ and $\mathcal{T} \subseteq \mathcal{S}$.

4.2.1 The Hrushovski-Fraïssé Classes (in the language $\{E\}$)

We introduce the Hrushovski-Fraïssé Classes that this paper focuses on. Fix any $\alpha > 0$, denote by $\delta_\alpha(G){:=}v(G) - \alpha e(G)$ the "*α-predimension*" of G, measuring the sparsity of G. The *hereditarily α-sparse* class of graphs $\mathcal{K}_\alpha := \big\{$(finite simple undirected) graphs $G : \delta_\alpha(G) \geq 0$, and $(\forall G' \subseteq_{\text{ind}} G)(\delta_\alpha(G') \geq 0)\big\}$. Intuitively, it is the collection of finite graphs which are both "globally" and "locally" linearly sparse with respect to α.

A natural partial order on $\mathcal{K}_\alpha$ is: $G \preceq H \iff \Big(G \subseteq H \bigwedge \delta_\alpha(G) \leq \delta_\alpha(G')$ for all $G' : G \subseteq_{\text{ind}} G' \subseteq_{\text{ind}} H\Big)$, which we call the *strong substructure* relation. The strong substructure relation naturally extends to "finite, infinite" pairs (G, H) where $G \in \mathcal{K}_\alpha$ and H is s.t. all its finite substructures are in $\mathcal{K}_\alpha$.

$(\mathcal{K}_\alpha, \preceq)$ has a *generic structure* $\mathfrak{M}_\alpha$, which is a union of $\{G_i\}_{i<\omega} \subseteq \mathcal{K}_\alpha$ (therefore countably infinite) with $G_i \preceq G_{i+1}$, s.t. $\forall\Big(G \preceq H \in \mathcal{K}_\alpha$ and $f : G \hookrightarrow \mathfrak{M}_\alpha$ as induced subgraphs, where the image $f(G) \preceq \mathfrak{M}_\alpha\Big)$, there exists an extension embedding $\widetilde{f} : H \hookrightarrow \mathfrak{M}_\alpha$ s.t. $\widetilde{f}(H) \preceq \mathfrak{M}_\alpha$.[2] For existence of a unique-up-to-isomorphism generic structure in $\mathcal{K}_\alpha$, see e.g. [1, 43].

Let $\mathcal{K}_{\alpha,\mathit{TF}}$ and $\mathfrak{M}_{\alpha,\mathit{TF}}$ be the triangle-free subclass of $\mathcal{K}_\alpha$ and its generic structure, resp. Denote by *TF* the collection of (finite undirected simple) triangle-free graphs. $\mathcal{K}_{\alpha,K_n-\mathit{free}}$, $\mathfrak{M}_{\alpha,K_n-\mathit{free}}$ are defined analogously to the triangle-free case. The existence of such generic structures can be obtained from e.g. [1, 43] as well. For the purpose of the paper, by "$(\alpha, \preceq)$-Hrushovski-Fraïssé classes" we refer to $\mathcal{K}_\alpha$, $\mathcal{K}_{\alpha,\text{TF}}$ or $\mathcal{K}_{\alpha,K_n-\text{free}}$.

For possibly infinite graph M, *Age*$(M) := \{$(isomorphic types of finite) G embeds

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[2] By extension we mean $\widetilde{f}\restriction_G = f$. Also when the right hand side is an infinite structure, $f(G) \preceq \mathfrak{M}_\alpha$ simply means $f(G) \leq \mathfrak{B}$ for all $\mathfrak{B} \subseteq_{\text{fin, ind}} \mathfrak{M}_\alpha$.

into M as induced subgraph $\}$. Any Hrushovski-Fraïssé class $\mathcal{K}_{\alpha,*}$ has the following properties:

1. (hereditary property, "HP") If $A \in \mathcal{K}_{\alpha,*}$ and $B \subseteq A$, $B \in \mathcal{K}_{\alpha,*}$;
2. (joint embedding property, "JEP") If $A, B \in \mathcal{K}_{\alpha,*}$ then there exists $C \in \mathcal{K}_{\alpha,*}$ s.t. A, B both embed to C. Indeed, just take $C := A \coprod B$.
3. (amagamation with respect to $\preceq$, "$\preceq$-AP") If $A, B, C \in \mathcal{K}_{\alpha,*}$ and $h_1 : A \hookrightarrow B, h_2 : A \hookrightarrow C$ both embeddings, *and* $A \cong h_1(A) \preceq C$, then $\exists D \in \mathcal{K}_{\alpha,*}$, $g_1 : B \hookrightarrow D, g_2 : C \hookrightarrow D$ both embeddings s.t. $g_1 h_1 = g_2 h_2$. Indeed, one may check that $D := \frac{B \coprod C}{\{h_1(a)=h_2(a)\}_{a \in V(A)}}$, i.e. the glue of a copy of B and a copy of C along the isomorphic images of h_is, are in $\mathcal{K}_{\alpha,*}$. Furthermore if $A \cong h_1 A \preceq C$ then $g_1 B \preceq D$, and vice versa.[3]

For a finite graph G, Define *mad*$(G) := \max_{H \subseteq G} \frac{2e(H)}{v(H)}$, the <u>m</u>ax <u>a</u>verage <u>d</u>egree of subgraphs; dropping the hereditarity, $\mathrm{ad}(G) := \frac{2e(G)}{v(G)}$ is its average degree. Observe that for $\alpha > 0$,[4]

$$\begin{aligned} G \in \mathcal{K}_\alpha &\iff \min_{H \subseteq G} \delta_\alpha(H) \geq 0 \\ &\iff \max_{H \subseteq G} \frac{e(H)}{v(H)} \leq \frac{1}{\alpha} \\ &\iff \mathrm{mad}(G) \leq \frac{2}{\alpha} \end{aligned}$$

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[3] Indeed, for any H^* where $B \cong g_1 B \subseteq H^* \subseteq D$, observe that $\delta_\alpha(H^*) - \delta_\alpha(g_1 B) = \delta_\alpha(H^* \cap g_2(C)) - \delta_\alpha(g_2 h_2 A) \geq 0$ as $g_2(C) \supseteq H^* \cap g_2(C) \supseteq g_2 h_2 A$ and $g_2(C) \cong C \succeq h_2(A) \cong g_2 h_2(A)$.

[4] Note, however, this does not extend to a full "order reversing" relation, i.e. it is not true that $\delta_\alpha(H) \geq \delta_\alpha(G) \iff \mathrm{mad}(H) \leq \mathrm{mad}(G)$. For example, let $\alpha = \frac{5}{6}$; then $\delta_\alpha(C_7) > \delta_\alpha(P_2)$, yet $\mathrm{mad}(C_7) = 2 > \mathrm{mad}(P_2) = 1$.

Observe that the membership decision for $\mathcal{K}_\alpha$ when $\alpha > 1$ is uninteresting: since $\delta_\alpha(P_2) = 2 - \alpha$, if $\alpha > 2$, then $G \in \mathcal{K}_{\alpha,\text{TF}} \iff G$ is edgeless; since $\frac{e(C_n)}{v(C_n)} = 1$ and for any tree T we have $\delta_\alpha(T) = v(T)(1-\alpha) - \alpha$ which drops as $v(T)$ increases, if $\alpha \in (1, 2]$ then $G \in \mathcal{K}_{\alpha,\text{TF}} \iff \Big(G$ is a forest, and each component T has $v(1 - \alpha) - \alpha \geq 0\Big)$. These membership decisions are all in $\mathtt{P}$. Also note that $\mathcal{K}_\alpha = \mathcal{K}_{\alpha,\text{TF}} = \mathcal{K}_{\alpha,K_n-\text{free}}(n \geq 4)$ whenever $\alpha > 1$.

4.2.2 Constraint Satisfaction Problems (in the langauge $\{E\}$)

Definition of *constraint satisfaction problems* are introduced in Chapter 1, which we briefly recall here with languages pertaining the context of graphs. Given a graph G, possibly infinite, the *constraint satisfaction problem* is the membership decision of the set: *CSP*$(M) := \{H : \text{Hom}_{\{E\}}(H, M) \neq \emptyset\}$. We are interested in cases where M arises as the generic structure of a Hrushovski-Fraïssé class. For finite or infinite graphs M_1, M_2, we say they are *pp-equivalent* if CSP(M_1) = CSP(M_2); this is justified by the well-known fact that up to Karp-equivalence, CSP$(M) = \{\sigma : \sigma$ is a pp-$\{E\}$-sentence and $M \models \sigma\}$ where a pp-$\{E\}$-sentence is a first order sentence in the language of graphs built from atomics, conjunction and existential quantification.

When $\alpha > 1$, the CSP for $\mathfrak{M}_\alpha$ ($= \mathcal{K}_{\alpha,\text{TF}} = \mathcal{K}_{\alpha,K_n-\text{free}}$) is uninteresting as well. Indeed, when $\alpha > 2$, $\mathcal{K}_\alpha$ is homomorphically presented by the singleton $\{(\{\bullet\}, \emptyset)\}$, so CSP$(\mathfrak{M}_\alpha)$=CSP$((\{\bullet\}, \emptyset)) = \mathcal{K}_\alpha \in \mathtt{P}$. For $\alpha \in (1, 2]$, $\mathcal{K}_\alpha$ is homomorphically presented by $\{P_2\} \subseteq \mathcal{K}_\alpha$ so CSP$(\mathfrak{M}_\alpha)$ = CSP$(P_2) \in \mathtt{P}$.

In fact, we can even address the homomorphic presentation for $\alpha \in (\frac{5}{6}, 1]$ here. Per

Propositions 4.8 and 4.11 below, in such cases $\mathcal{K}_\alpha = \mathcal{K}_{\alpha, K_n-\text{free}}(n \neq 4)$ are presented by $\{K_3\}$; corollary to [54], for $\alpha \in (\frac{5}{6}, 1]$ we have $\mathcal{K}_{\alpha,\text{TF}} \to \{C_5\}$, the RHS presenting the LHS. Therefore in the discussions to follow, we assume $\alpha \in (0, 1]$, and mostly focus on $\alpha \in (0, \frac{5}{6}]$.

We close this section with a definition, which serves as the starting point of our study of homomorphic presentation for $\mathcal{K}_{\alpha,\text{TF}}$.

Definition 4.2 (E_1,E_2 (e.g. [55])). Define E_1, E_2 as in Figure 4.1.

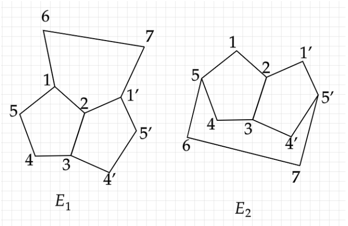

Figure 4.1: E_1 and E_2

4.3 Motivation: Constraint Satisfaction Problems of Hrushovski-Fraïssé Classes

By well-known classical results (e.g. [34]), if $\mathcal{K}$ is a class with HP and JEP then there exists a countable generic structure $\mathfrak{M}_\mathcal{K}$ with $\text{Age}(\mathfrak{M}_\mathcal{K}) = \mathcal{K}$. Recall that $\text{CSP}(\mathfrak{M}_\mathcal{K}) = \{G : \text{Hom}_{\{E\}}(G, \mathfrak{M}_\mathcal{K}) \neq \emptyset\}$. Since the (setwise) homomorphic image induces a finite substructure, this gives

$$\text{CSP}(\mathfrak{M}_\mathcal{K}) = \{G : \text{Hom}_{\{E\}}(G, B) \neq \emptyset \text{ for some } B \in \mathcal{K}\} \tag{4.1}$$

For any classes of finite graphs $\mathcal{K}$ and countable graphs $\mathfrak{M}$, if $(\mathcal{K}, \mathfrak{M})$ satisfies (4.1), we also say $\mathfrak{M}$ *homomorphically presents* $\mathcal{K}$. Note that $\mathfrak{M}$ homomorphically presents $\mathcal{K}$ if and only if Age($\mathfrak{M}$) $\leftrightarrow \mathcal{K}$.[5] Let $\mathcal{K}$ have HP and JEP; if in addition $\mathcal{K}$ has AP for *all*, not just *strong*, embeddings, then with our $\tau = \{E\}$[6] one may obtain an *ultrahomogenous* generic $\mathfrak{M}_{\mathcal{K}}$ (see e.g. [34]), which certainly homomorphically presents $\mathcal{K}$. Countable ultrahomogenous (simple, undirected) graphs have been fully characterized in [31, 56*], whose CSPs are all well-understood. Indeed,

Remark 4.3. For finite ultrahomogenous graphs one resorts to the dichotomy in [5]; for countably infinite ultrahomogenous graphs,

1. if it is a mK_n or its complement, then it admits the same CSP with its largest clique; in that case the CSP is either a finite colorability problem which is NP-complete or the collection of all finite graphs, which is decidable in $O(1)$-time.

2. CSP of Rado graph, which is isomorphic to its complement, is $O(1)$-time decidable.

3. CSP of the K_t-free Hensen graph is the same as K_t-freeness decision, which is of time class $O(N^t)$ with t a constant; the complement of a K_t-free Hensen graph has a copy of K_x for any $x \geq 3$ so its CSP is once again $O(1)$-time decidable.

In fact, [51] has characterized the complexity of CSP of any countable relational structure S (with finite signature) where each relation is first order definable from the same countable ultrahomogenous graph.[7]

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[5] In particular if Age($\mathfrak{M}$) presents $\mathcal{K}$, so does $\mathfrak{M}$. This is the case when e.g. $\mathfrak{M}$ is the generic structure of a class $\mathcal{K}$.

[6] Or e.g. any countable relational signature.

[7]

Remark 4.4. However it is not to be inferred that the complexity of CSP of any ultrahomogenous relational structure behaves similarly. In fact there are ultrahomogenous *directed* graphs with co-NPComplete CSP (Proposition 13.3.1 [13]), and in general the CSP of (first order reducts of) ultrahomogenous, finitely-

Now let $\alpha \in (0,1]$, $\mathcal{K}_{\alpha,*}$ be a Hrushovski-Fraïssé class ($\mathcal{K}_\alpha, \mathcal{K}_{\alpha,\text{TF}}$ or $\mathcal{K}_{\alpha,K_n-\text{free}}$), and $\mathfrak{M}_{\alpha,*}$ be its generic structure and therefore a homomorphic presentation. Unfortunately $\mathfrak{M}_{\alpha,*}$ is not even ω-categorical (see e.g. [1]; for example, $\mathfrak{M}_\alpha \not\cong \mathfrak{M}_{\alpha,\text{ e.c.}}$, an existentially closed structure adopting the same first order axiomatization), let alone ultrahomogenous. It would be great if one could obtain some countable ultrahomogenous or finite $\mathfrak{M}'_{\alpha,*}$ with the same CSP, or equivalently the same pp-theory, so we could get clearer understanding of the original CSP($\mathfrak{M}_{\alpha,*}$). Note that the attempt is not *a priori* void because pp-equivalence is even weaker than elementary equivalence. However, the observation below indicates that ultrahomogeneity and pp-equivalence do not go well in our context.

Proposition 4.5.

1. *For any* $\alpha \in (0, \frac{5}{6}]$*, no countable ultrahomogenous graph* G *satisfies* CSP($\mathfrak{M}_{\alpha,\mathit{TF}}$) $=$ CSP(G).
2. *For* $n \geq 4$*,* $\alpha \in (0, \frac{2(n+2)}{n^2+3n-8}]$*, no countable ultrahomogenous* G *satisfies* CSP($\mathfrak{M}_{\alpha,K_n-\mathit{free}}$) $=$ CSP(G).

Proof. For $\alpha \in (0, \frac{5}{6}]$, $E_1 \in$ CSP($\mathcal{K}_{\alpha,\text{TF}}$) $\neq$ CSP(C_5); for $n \geq 4, \alpha \in (0, \frac{2(n+2)}{n^2+3n-8}]$, $E_1 \in$ CSP($\mathfrak{M}_{\alpha,K_n-\text{free}}$) $\neq$ CSP(C_5). So let $G \not\cong C_5$ be a countable ultrahomogenous graph. We first address $\mathfrak{M}_{\alpha,\text{TF}}$ for $\alpha \in (0, \frac{5}{6}]$:

1. If G is infinite: since $K_3 \notin$ CSP($\mathcal{K}_{\alpha,\text{TF}}$), it cannot be the case that CSP($\mathcal{K}_{\alpha,\text{TF}}$) $=$ CSP(G) if G is an infinite Turán graph, or a disjoint union of cliques, or the Rado

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bounded structures is a famous open problem.

graph, or the complement of a Hensen graph, or the K_n-free Hensen graph for $n \geq 4$. If G is the TF-Hensen graph, CSP(G) contains all the Andrásfai graphs, which are infinitely many per Proposition 4.59(1), while for any fixed α, CSP$(\mathcal{K}_{\alpha,\text{TF}})$ only contains finitely many Andrásfai graphs.

2. If $G \not\cong C_5$ is finite, per [56], G is pp-equivalent to a clique, which is not in CSP$(\mathcal{K}_{\alpha,\text{TF}})$.

Now let $n \geq 4, \alpha \in (0, \frac{2(n+2)}{n^2+3n-8}]$. Consider the graph $C_5 \odot K_{n-3}$ obtained by taking a disjoint copy of C_5 and K_{n-3}, then connecting every pair $(v, v') : v \in C_5, v' \in K_{n-3}$. [8] For example when $n = 4$ this is just a 5-wheel. We have ad$(C_5 \odot K_{n-3}) = \frac{5(n-1)+(n-3)(n+1)}{n+2} = \frac{n^2+3n-8}{n+2} \leq \frac{2}{\alpha}$, and it is routine to verify $C_5 \odot K_{n-3}$ is its own densest subgraph.[9]. Lastly $C_5 \odot K_{n-3}$ is K_n-free, and has $\chi = n$.[10] It follows that $C_5 \odot K_{n-3} \in$ CSP$(\mathfrak{M}_{\alpha,K_n-\text{free}}) \setminus$ CSP(K_{n-1}). On the other hand, $K_n \in$ CSP$(K_n) \setminus$ CSP$(\mathfrak{M}_{\alpha,K_n-\text{free}})$. So $\mathfrak{M}_{\alpha,K_n-\text{free}}$ is not pp-equivalent to any finite or infinite cliques, nor the Rado graph nor

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[8] Such coonstructions were mentioned elsewhere for other purposes, e.g. here.

[9] It suffices to verify the following:

1. For any $0 \leq m$, ad$(C_5 \odot K_m) \leq$ ad$(C_5 \odot K_{m+1})$. This can be seen from the following proposition:

   Proposition 4.6. *Let G be a graph with $v(G) \geq 2$. Then ad$(G - x) <$ ad(G) iff $d_G(x) > \frac{ad(G)}{2}$, and ad$(G - x) >$ ad(G) iff $d_G(x) < \frac{ad(G)}{2}$.*

   Proof of Proposition 4.6. Observe that ad$(G - x) = \frac{2e(G-x)}{v(G-x)} = \frac{2\left(e(G)-d_G(x)\right)}{v(G)-1}$. Assuming $v(G) - 1 \geq 1$, we have $\frac{(e(G)-d_G(x))}{v(G)-1} > \frac{e(G)}{v(G)} \iff d_G(x) < \frac{e(G)}{v(G)}$ and $\frac{(e(G)-d_G(x))}{v(G)-1} < \frac{e(G)}{v(G)} \iff d_G(x) > \frac{e(G)}{v(G)}$. Lastly, $\frac{e(G)}{v(G)} = \frac{\text{ad}(G)}{2}$. □

2. For any $m \geq 0$, $m' \in [0, 4]$, let $H_{m,m'}$ be the induced subgraph of $C_5 \odot K_m$ containing the copy of K_m and an size-m' part of C_5, with $v(H_{m,m'}) = m + m'$ and largest edge density. Then ad$(H_{m,m'}) \leq$ ad$(H_{m,m'+1})$ (where $H_{m,5}$ is $C_5 \odot K_m$). When $m' = (0, 1, 2)$, ad$(H_{m,m'}) =$ ad$(K_{(m,m+1,m+2)}) = (m - 1, m, m + 1)$; when $m' = 3$, it can be computed that ad$(H_{m,m'}) \in (m + 2, m + 3)$. Finally for $m' = 4, 5$ one can obtain the increase by Proposition 4.6.

[10] It needs 3 vertices from the copy of C_5 to form an n-clique, which is impossible; on the other hand, we need 3 colors for C_5 and $n - 3$ other colors for K_{n-3}.

a complement of Hensen.

By Erdős e.g. [46], there exists K_n-free graphs of arbitrary chromatic number. Therefore for any $\alpha \in (0, \frac{2(n+2)}{n^2+3n-8}]$ there exists $H \in K_n\text{-free} \setminus \text{CSP}(\mathfrak{M}_\alpha) = \text{CSP}(K_{k_\alpha})$. Since $\text{CSP}(\mathfrak{M}_\alpha)$ contains $\text{CSP}(\mathfrak{M}_{\alpha,K_n-\text{free}})$, $H \in K_n\text{-free} \setminus \text{CSP}(\mathfrak{M}_{\alpha,K_n-\text{free}})$; also $C_5 \odot K_{n-3} \in \text{CSP}(\mathfrak{M}_{\alpha,K_n-\text{free}}) \setminus K_{n-1}\text{-free}$. Therefore $\mathfrak{M}_{\alpha,K_n-\text{free}}$ is not pp-equivalent to a Hensen graph. $\square$

So how about reducing $\mathfrak{M}_{\alpha,*}$ to a finite, pp-equivalent graph? We have results in this direction, most of which connect with our study of finite presentation.

4.3.1 $\mathfrak{M}_\alpha$

We know the full picture here. Recall that $G \in \mathcal{K}_\alpha \iff \text{mad}(G) \leq \frac{2}{\alpha}$. Moreover, Golderg [57] e.g.introduced a polynomial time algorithm to compute $\text{mad}(G)$: the idea was to construct a digraph $\text{mad}(\widetilde{G})$ parameterized by a guess of $\frac{\text{mad}(G)}{2}$ (maximum density), find its min-cut, and adjust the guess accordingly. It takes $\log(v(G))$ iterations. [11] This puts $\mathcal{K}_\alpha \in \texttt{P}$, and therefore $\text{CSP}(\mathfrak{M}_\alpha) \in \texttt{NP}$, where each $G \in \text{CSP}(\mathfrak{M}_\alpha)$ is equipped with a short witness $\big(B \in \mathcal{K}_\alpha, h \in \text{Hom}(G, B)\big)$, $v(B) \leq v(G)$. In fact more can be said, thanks to the following fact on chromatic number:

Fact 4.7 (folklore, stated without proof in e.g. [59]). *For any finite G, $\chi(G) \leq \lfloor mad(G) \rfloor + 1$.*

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[11] The algorithm has been improved by multiple subsequent works. For more discussions on this, see, e.g., [58].

Also, to refrain from using $\log(n)$ iterations, note that there are $\leq \binom{n}{2} + \binom{n-1}{2} + \cdots \binom{2}{2} \sim O(n^3)$ possible $\text{mad}(G)$ values for G with $v(G) =: n$; therefore it takes $O(n^3)$ time to find the two possible $\text{mad}(G)$ values closest to $\frac{2}{\alpha}$, i.e. poly-time to find a $\widetilde{\alpha}$ s.t. $\text{mad}(G) > \frac{2}{\widetilde{\alpha}} \iff \text{mad}(G) > \frac{2}{\alpha}$ and $\text{mad}(G) < \frac{2}{\widetilde{\alpha}} \iff \text{mad}(G) \leq \frac{2}{\alpha}$. Then we run 1 iteration of Goldberg's algorithm with the guess $g := \frac{1}{\widetilde{\alpha}}$.

Proof. We prove by induction on $|G|$, much like the Brook's theorem. When $|G| \leq 2$ the result is immediate.

Assume the inequality holds for n. When $|G| = n+1$, take $v \in G$ with maximal degree, and consider $G - v := G[V(G) \setminus \{v\}]$. By inductive hypothesis, $\chi(G-v) \leq \lfloor \text{mad}(G-v) \rfloor + 1$.

1. If v is connected to every vertex in $G-v$, then $\chi(G) = \chi(G-v)+1$; on the other hand, if H is any induced subgraph of $G-v$, with density $\frac{e}{b}$, then $H \cup \{v\}$ has density $\frac{e+b}{b+1}$. This means the density of densest subgraph jumps by at least $\frac{e+b}{b+1} - \frac{e}{b}$, and using $e \leq \frac{b(b-1)}{2}$ we get $\frac{e+b}{b+1} - \frac{e}{b} \geq 1/2$. It follows that the mad of G, being twice the density of densest subgraph jumps by at least $2 \cdot \frac{1}{2} = 1$. Therefore $\lfloor \text{mad}(G) \rfloor \geq \lfloor \text{mad}(G-v) \rfloor + 1 \geq \chi(G-v) = \chi(G) - 1$, as desired .
2. otherwise, v is not connected to every vertex in $G-v$, so there is at least one vertex $v' \in G-v$ which could share a color with v. It follows that $\chi(G) = \chi(G-v) \leq \lfloor \text{mad}(G-v) \rfloor + 1 \leq \lfloor \text{mad}(G) \rfloor + 1$, since any induced subgraph of $G-v$ is also an induced subgraph of G.

□

Proposition 4.8. *For* $\alpha \in (0,1]$*, let* k_α *be the unique integer in* $(\frac{2}{\alpha}, \frac{2}{\alpha}+1]$*. Then* $\text{CSP}(\mathfrak{M}_\alpha) = \text{CSP}(K_{k_\alpha})$*. In particular,* $\text{CSP}(\mathfrak{M}_\alpha)$ *is NP-complete, noting that* $\chi(K_{k_\alpha}) = k_\alpha \geq 3$.

Proof. $\supseteq$: When $\frac{1}{\alpha} \geq \frac{k_\alpha - 1}{2} = \frac{\frac{k_\alpha(k_\alpha-1)}{2}}{k_\alpha} = \frac{e(K_{k_\alpha})}{v(K_{k_\alpha})} = \frac{\text{mad}(K_{k_\alpha})}{2}$, $K_{k_\alpha} \in \mathcal{K}_\alpha \subset \text{CSP}(\mathfrak{M}_\alpha)$.

$\subseteq$: Let $G \in \text{CSP}(\mathfrak{M}_\alpha)$ witnessed by $(B \in \mathcal{K}_\alpha, h : G \to B)$. Then $\max_{H \subseteq B} \frac{e(H)}{v(H)} \leq \frac{1}{\alpha}$, so $\text{mad}(B) \leq \frac{2}{\alpha} < k_\alpha$. It follows from Fact 4.7 that $\chi(B) \leq \lfloor \text{mad}(B) \rfloor + 1 \leq$

$\lfloor \frac{2}{\alpha} \rfloor + 1 = k_\alpha$, so $G \xrightarrow{h} B \xrightarrow{c} K_{k_\alpha} \in \text{Hom}(G, K_{k_\alpha})$ where c is any K_{k_α}-coloring of B, as desired. $\square$

Note this also says $\mathcal{K}_\alpha$ is finitely homomorphically presented by $\{K_{k_\alpha}\}$, for each $\alpha \in (0, 1]$.

4.3.2 $\mathfrak{M}_{\alpha,\text{TF}}$

We have that $\mathcal{K}_{\alpha,\text{TF}} = \mathcal{K}_\alpha \cap \text{TF} \in \mathtt{P}$ since $\mathcal{K}_\alpha \in \mathtt{P}$ by discussions above and $\text{TF} \in \mathtt{P}$ by an algorithm no worse than enumerating all triples of vertices. Similarly witnessed by $\big(B \in \mathcal{K}_{\alpha,\text{TF}}, h \in \text{Hom}(G, B)\big)$, $v(B) \leq v(G)$ whenever $G \in \text{CSP}(\mathcal{K}_{\alpha,\text{TF}})$ we have $\text{CSP}(\mathcal{K}_{\alpha,\text{TF}}) \in \mathtt{NP}$.

Per prior art [54], for $\alpha \in (\frac{5}{6}, 1]$, $\mathcal{K}_{\alpha,\text{TF}}$ is presented by $\{C_5\}$. It follows that for such α, $\text{CSP}(\mathcal{K}_{\alpha,\text{TF}}) = \text{CSP}(C_5)$ is NP-complete. Note that:

Proposition 4.9. *Let* $\alpha \in (0, \frac{5}{6}]$, $\mathcal{B} \subseteq \mathcal{K}_{\alpha,TF}$ *be a homomorphic presentation such that* $\Big((\mathcal{B}$ *is finite*$) \iff (\mathcal{K}_{\alpha,TF}$ *has a finite presentation*$)\Big)$.

1. *If* $|\mathcal{B}| < \aleph_0$, *then* $\text{CSP}(\mathfrak{M}_{\alpha,TF})$ *is* NP-complete.
2. *If* $|\mathcal{B}| = \aleph_0$ *then* $\text{CSP}(\mathfrak{M}_{\alpha,TF}) \neq$CSP*$(H)$ for any finite H.*

Indeed, with the assumptions of Proposition 4.9:

1. If $\mathcal{B}$ is finite: if $\max_{B \in \mathcal{B}} \chi(B) \leq 2$ then all graphs in $\text{CSP}(\mathfrak{M}_{\alpha,\text{TF}})$ are bipartite, absurd; therefore $\chi(\coprod \mathcal{B}) \geq 3$, yielding the NP-completeness by [5].
2. Otherwise, per the assumptions $\mathcal{K}_{\alpha,\text{TF}}$ cannot be finitely presented. Assume for the sake of contradiction that $\text{CSP}(\mathfrak{M}_{\alpha,\text{TF}}) = \text{CSP}(H)$ for some finite H, and take $(h \in$

Hom$(H, \mathfrak{M}_{\alpha,\text{TF}})$, $B := \mathfrak{M}_{\alpha,\text{TF}}[\text{Im}(h)] \in \mathcal{K}_{\alpha,\text{TF}}$. We have CSP$(\mathfrak{M}_{\alpha,\text{TF}})$=CSP$(H) \subseteq$ CSP$(B) \subseteq$ CSP$(\mathfrak{M}_{\alpha,\text{TF}})$, so $\mathcal{K}_{\alpha,\text{TF}} \to \{B\}$, contradicting non-existence of finite $\mathcal{K}_{\alpha,\text{TF}}$-presentation.

The **main contribution** of Section 4.4 is to find such an "indicator" $\mathcal{B}$ as in the assumption of Proposition 4.9, and give some interesting structural properties of members in $\mathcal{B}$.

We close the subsection with another notable fact. By [60] every planar triangle-free graph homomorphically maps to H_5, the Greenwood-Gleason graph, which is regular of degree 5 and triangle-free. So

Corollary 4.10. *For any $\alpha \in (0, \frac{2}{5}]$, any triangle-free planer graph G is in* CSP$(\mathcal{K}_{\alpha,TF})$.[12]

Proof. H_5 is regular of degree 5 so ad$(H) \leq 5$ for every $H \subseteq H_5$, meaning $H_5 \in \mathcal{K}_{\alpha,\text{TF}}$. ☐

4.3.3 $\mathfrak{M}_{\alpha,K_n-\text{free}}$

For αs sufficiently close to 1 with respect to n, the K_n-free constraint makes no difference. Note that as $n \to \infty$, the lower bound of α below approaches 0.

Proposition 4.11. *Let $n \geq 4$. Then for $\alpha \in (\frac{2}{n-1}, 1]$,* CSP$(\mathfrak{M}_{\alpha,K_n-free})$ = CSP$(\mathfrak{M}_\alpha)$.

Proof. LHS $\subseteq$ RHS by definition; conversely, observe that mad$(K_n) = \frac{\frac{n(n-1)}{2}}{n} = \frac{n-1}{2}$, so when $\frac{1}{\alpha} < \frac{n-1}{2}$, $\mathcal{K}_\alpha \subseteq K_n$-free, i.e. $\mathcal{K}_\alpha = \mathcal{K}_\alpha \cap K_n$-free = $\mathcal{K}_{\alpha,K_n\text{-free}}$. ☐

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[12] So the planar pieces of *such* CSP$(\mathcal{K}_{\alpha,\text{TF}})$ and CSP(H_5) coincide, and are both the collection of all planar triangle-frees.

Therefore for the correct (n, α_n), $\mathfrak{M}_{\alpha,K_n-\text{free}}$ admits the same homomorphic presentation as $\mathcal{K}_{\alpha_n}$. We collect the consequences on $\mathfrak{M}_{\alpha,K_n-\text{free}}$ in Table 4.1.

$n\backslash$CSP($\mathfrak{M}_{\alpha,K_n-\text{free}}$)/$\alpha$	$\alpha > 1$	$\alpha \in (\frac{5}{6}$,1]	($\frac{2}{3}$,$\frac{5}{6}$]	($\frac{1}{2}$, $\frac{2}{3}$]	... ($\frac{2}{n-1}$, $\frac{2}{n-2}$]	(0, $\frac{2}{n-1}$]
$n = 3$	CSP(P_2/P_1)	CSP(C_5)	NP	NP	CSP(P_2)	NP
4	CSP(P_2/P_1)	CSP(K_3)	CSP(K_3)	NP	CSP(K_3)	NP
5	CSP(P_2/P_1)	CSP(K_3)	CSP(K_3)	CSP(K_4)	CSP(K_4)	NP
... $n \geq 5$...	CSP(P_2/P_1)	CSP(K_3)	CSP(K_3)	CSP(K_4)	... CSP(K_{n-1})	NP

Table 4.1: Results on CSP($\mathfrak{M}_{\alpha,K_n-\text{free}}$). the CSP($P_2/P_1$) column has CSP($P_2$) for $\alpha \in (1, 2]$ and CSP(P_1) for $\alpha > 2$, as discussed in Section 4.2. All "NP"s above are conjectured NPC.

4.4 Homomorphic Presentation of $\mathcal{K}_{\alpha,\text{TF}}$

As forecast in Section 4.3, the main task of this section is, for suitable $\alpha \in (0, \frac{5}{6}]$, to find a $\mathcal{B} \subseteq \mathcal{K}_{\alpha,\text{TF}}$ presenting $\mathcal{K}_{\alpha,\text{TF}}$ where $\mathcal{B}$ is finite if and only if $\mathcal{K}_{\alpha,\text{TF}}$ has any finite presentation, and then characterize $\mathcal{B}$ using structural information of its members. First we show that, for α reasonably close to $\frac{5}{6}$, any graph in $\mathcal{K}_{\alpha,\text{TF}}$ which does not homomorphically map to C_5 must contain an isomorphic copy of E_1 or E_2.

Proposition 4.12. *Let* $\alpha \in (\frac{32}{39}, 1]$. $\big(B \in \mathcal{K}_{\alpha,TF} \bigwedge B \not\to C_5\big) \implies E_1$ *or* E_2 *embeds into* B *as induced subgraph.*

Proof. When $\alpha \in (\frac{5}{6}, 1]$ the statement is vacuously true, so fix $\alpha \in (\frac{32}{39}, \frac{5}{6}]$ and let $B \in \mathcal{K}_{\alpha,\text{TF}}$, $B \not\to C_5$. First, note that by finiteness of B, B contains a *minimal* subgraph G' s.t.

(*) $G' \not\to C_5$, yet each proper subgraph of G' admits a homomorphism to C_5.

Indeed, we start from $G_0 := B$, and one may obtain $G_{i+1} \subsetneq G_i$ if there exists such a proper subgraph of G_i that does not homomorphically map to C_5.[13] Such $G' \models$ (*) are

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[13] This descending chain must terminate before reaching null graph as when $n \leq 6$ every triangle-free

called the C_5-*critical* graphs (e.g. [55]). It is routine to verify that E_1, E_2 are C_5-criticals.

Assume for contradiction that $G' \notin \{E_1, E_2\}$. By [55], if $\big(G' \notin \{K_3, E_1, E_2\}\big) \bigwedge \big(G' \models (*)\big)$, then $5v(G') - 4e(G') \leq 1$. Moreover $G' \subseteq B \in \mathcal{K}_{\alpha,\text{TF}}$, so $v(G') - \alpha e(G') \geq 0$. We obtain $v(G') \leq \frac{1}{5-\frac{4}{\alpha}} < 8$. On the other hand, [61] says the smallest C_5-critical graph has $v(G') = 8$, contradiction.

To see that G' must be an *induced* subgraph, assume $G^* := B[V(G')] \supsetneq G'$. Then $e(G^*) \geq e(G')+1 = 13$, so $\delta_\alpha(G^*) = v(G^*) - \alpha e(G^*) < 10 - 13 \times \frac{32}{39} < 0$, contradicting $G^* \subseteq B \in \mathcal{K}_{\alpha,\text{TF}}$. $\square$

We show that E_1, E_2 enjoys some "rigidity" and distance properties allowing us to further simplify the presentation of $\mathcal{K}_{\frac{5}{6},\text{TF}}$.

Definition 4.13 ((α, TF)-homomorphic rigidity)**.** Fix $\alpha \in (0, 1]$. A graph $H \in \mathcal{K}_{\alpha,\text{TF}}$ is (α, TF)-*homomorphically rigid* if for any $B \in \mathcal{K}_{\alpha,\text{TF}}$, any homomorphism $h : H \to B$ is injective.

Corollary 4.14. *E_1, E_2 are (α, TF)-homomorphically rigid for any $\alpha \in (\frac{32}{39}, \frac{5}{6}]$. In fact, we also have for any $B \in \mathcal{K}_{\alpha,\mathit{TF}}$, any injective homomorphism $h : E_i \to B$ must be an embedding. In addition, E_1, E_2 are* the only *C_5-criticals inside $\mathcal{K}_{\alpha,\mathit{TF}}$ for $\alpha \in (\frac{32}{39}, \frac{5}{6}]$.*

Proof. For $(\alpha$, TF)-homomorphic rigidity: assume otherwise, then there exists $a \in (\frac{32}{39}, \frac{5}{6}], i \in [2], B \in \mathcal{K}_{\alpha,\text{TF}}$, and $h \in \text{Hom}(E_i, B) \setminus \text{Hom}_{\text{inj}}(E_i, B)$. Consider $B^* := B[\text{Im}(h(E_i))] \in \mathcal{K}_{\alpha,\text{TF}}$; we must have $B^* \not\to C_5$, for otherwise $h \circ -$ would give a homomorphism $E_i \to C_5$. By Proposition 4.12, $E_1 \subseteq B^*$ or $E_2 \subseteq B^*$, so $v(B^*) \geq 10$. But $v(B^*) \leq 9$ by non-injectivity of h, contradiction.

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graph admits a homomorphism to C_5, and for any n, connected graphs on n vertices with $(n-1)$ edges homs to C_5. Recall subgraphs are obtained by either deleting vertices or deleting edges.

That any $\mathcal{K}_{\alpha,\mathrm{TF}}$-homomorphism $h : E_i \to B$ must also be an embedding was shown while proving Proposition 4.12. That E_1, E_2 are the only C_5-criticals inside $\mathcal{K}_{(\frac{32}{39},\frac{5}{6}],\mathrm{TF}}$ follows from my proof of Proposition 4.12. $\square$

Remark 4.15. Note that $E_1 \not\to E_2$ and $E_2 \not\to E_1$: indeed, otherwise since e.g. a homomorphism $h : E_1 \to E_2$ must be injective, we would have a copy of E_1 as subgraph of E_2; since $v(E_1) = v(E_2)$, $e(E_1) = e(E_2)$ and $E_1 \ncong E_2$[14] this is impossible.

We have a more general observation along the lines of Corollary 4.14.

Definition 4.16. Fix $\alpha \in (0, 1]$. We say $A \in \mathcal{K}_{\alpha,\mathrm{TF}}$ is *maximally $\mathcal{K}_{\alpha,TF}$* if adding any missing edge e to A excludes the new graph from $\mathcal{K}_{\alpha,\mathrm{TF}}$.

Example 4.17. *$\forall\alpha \in (0, 1], C_5$ is maximally $\mathcal{K}_{\alpha,TF}$ because it is maximally triangle-free; $\forall\alpha \in (\frac{10}{13}, 1]$, E_1, E_2 are both maximally $\mathcal{K}_{\alpha,TF}$ because $\delta_\alpha(E_i + e) = 10 - 13\alpha < 0$ for any missing edge e.* [15]

Proposition 4.18. *Let $\alpha \in (0, 1]$ and let $A \in \mathcal{K}_{\alpha,TF}$. Then the following are equivalent:*

1. *For any $B \in \mathcal{K}_{\alpha,TF}$, any homomorphism $h : A \to B$ is an embedding.*

2. *A is (α, TF)-homomorphically rigid and maximally $\mathcal{K}_{\alpha,TF}$.*

In particular, if A is (α, TF)-homomorphically rigid and $\delta_\alpha(A) = 0$, then $A \models$ (1).

Proof. 1. $\Longrightarrow$: Let $B \in \mathcal{K}_{\alpha,\mathrm{TF}}, h : A \to B$ a homomorphism. Since an embedding is injective, by arbitrariness of (B, h), A is (α, TF)-homomorphically rigid. For the other clause of RHS, assume A is not maximally $\mathcal{K}_{\alpha,\mathrm{TF}}$ and let e be a missing edge

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[14] It is routine to verify that $\mathrm{Aut}(E_1) = S_3$ and $\mathrm{Aut}(E_2) = \mathbb{Z}/2 \times \mathbb{Z}/2$.

[15] Note that E_2 is not maximally triangle-free: e.g. $E_2 + (6, 2) \in \mathrm{TF}$.

that witnesses it. Then $\iota_e : A \hookrightarrow A + e$ induced by the identity on vertices is an injective homomorphism and not an embedding, contradicting LHS.

2. $\Longleftarrow$: assume $\neg$ LHS; then it suffices to show that $(A$ is (α, TF)-homomorphically rigid$) \implies \neg(A$ is maximally $\mathcal{K}_{\alpha,\text{TF}})$. Now $(\neg$ LHS$)\wedge((\alpha, \text{TF})$-homomorphic rigidity) implies there is an injective, non-embedding homomorphism. . Let $(B \in \mathcal{K}_{\alpha,\text{TF}}, \iota : A \to B)$ be one such homomorphism.

   By injectivity of ι, $V(A) \overset{\iota}{\cong}_{\textbf{Set}} \text{Im}(\iota)$ and $\iota(E(A)) \subseteq E(B[\text{Im}(\iota)])$; by ι being a non-embedding, $B[\text{Im}(\iota)] \ncong A$, so $\iota(E(A)) \subsetneq E(B[\text{Im}(\iota)])$. Pick $e \in E(B[\text{Im}(\iota)]) \setminus \iota(E(A))$ and let $V(B^*) := \text{Im}(\iota), E(B^*) := \iota(E(A)) \cup \{e\}$. Then $\mathcal{K}_{\alpha,\text{TF}} \ni B[\text{Im}(\iota)] \supseteq B^* \cong A + e'$ for some e' missing from $E(A)$, as desired.

   The "In particular" part follows from the fact that $\delta_\alpha(A) = 0 \implies$ for any missing edge e, $\delta_\alpha(A + e) < 0$.[16] □

The property below stipulates that two induced copies of E_1, E_2 in the same graph cannot be too close.

Lemma 4.19. *Let* $\alpha \in (\frac{24}{29}, \frac{5}{6}]$, *let* $B \in \mathcal{K}_{\alpha,TF}$, $H_1, H_2 \subseteq B$ *s.t.* $H_i \cong E_i$. *Then* $dist_B(H_1, H_2) \geq 6$.

Likewise if H_1, H_2 *are both isomorphic to* E_1 *(resp.* E_2*) and* $V(H_1) \neq V(H_2)$ *then* $dist_B(H_1, H_2) \geq 6$.

Proof. Note that $\frac{32}{39} < \frac{24}{29} < \frac{5}{6}$, so per Corollary 4.14, each $\mathcal{K}_{\alpha,\text{TF}}$-homomorphism from H_i is an embedding. We first prove that $H_1 \cap H_2 = \emptyset$, whenever $(H_1, H_2) \cong (E_1, E_1), (E_1, E_2)$,

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[16] Note that in general $\neg\big((\delta_\alpha(A) = 0) \implies (A$ is (α, TF)-homomorphically rigid$)\big)$. Indeed, consider $A \in (E_1)_a\text{–}5\text{–}_b(E_2)_b\text{–}5\text{–}_a(E_1) \in \mathcal{K}_{\alpha,\text{TF}}$ (see comments above Corollary 4.20 for notations) with $\delta_\alpha(A) = 0$, but $A \to (E_1)_a\text{–}5\text{–}_b(E_2)$. Note, also, that for $\delta_\alpha(A)$ to be 0 it is necessary that $\alpha \in \mathbb{Q}$.

or (E_2, E_2). Assume for the sake of contradiction that $V(H_1) \cap V(H_2) \neq \emptyset$.

1. If $|V(H_1) \cap V(H_2)| = 10$: then $V(H_2) = V(H_1)$ so $H_i \cong E_i$. $V(H_1) = V(H_2)$ induces both a copy of E_2 and a copy of E_1. As $E_1 \ncong E_2$ and $\delta_\alpha(E_1) - \alpha = \delta_\alpha(E_2) - \alpha = 10 - 13\alpha < 0$, this is impossible.

2. If $|V(H_1) \cap V(H_2)| \in [6, 9]$, then $|V(H_2) \smallsetminus V(H_1)| = 10 - |V(H_1) \cap V(H_2)| \in [1, 4]$ respectively.

    (a) If $|V(H_2) \smallsetminus V(H_1)| = 1$: let $u \in! V(H_2) \smallsetminus V(H_1)$. Since $\deg_{H_2}(u) \geq 2$, we have $e(B[H_1 \cup H_2]) \geq e(H_1) + 2 = 14$. It follows that $\delta_\alpha(B[H_1 \cup H_2]) \leq 11 - 14\alpha < 0$.

    (b) If $|V(H_2) \smallsetminus V(H_1)| = 2$: let $u_1 \neq u_2 \in V(H_2) \smallsetminus V(H_1)$. Since $\deg_{H_2}(u_1), \deg_{H_2}(u_2) \geq 2$ and there is at most one edge $(u_1, u_2) \in E(H_2)$, we have that $e(B[H_1 \cup H_2]) \geq e(H_1) + 2 \times 2 - 1 = 15$ and so $\delta_\alpha(B[H_1 \cup H_2]) \leq (10 + 2) - 15\alpha < 0$.

    (c) If $|V(H_2) \smallsetminus V(H_1)| = 3$: let u_1, u_2, u_3 enumerate $V(H_2) \smallsetminus V(H_1)$. By triangle-freeness, $e(B[\{u_i\}_i]) \leq 2$. Once again $\deg_{H_2}(u_i) \geq 2$, so $e(B[H_1 \cup H_2]) \geq e(H_1) + 2 \times 3 - 2 = 16$, and $\delta_\alpha(B[H_1 \cup H_2]) \leq (10 + 3) - 16\alpha < 0$.

    (d) If $|V(H_2) \smallsetminus V(H_1)| = 4$: let u_1, u_2, u_3, u_4 enumerate $V(H_2) \smallsetminus V(H_1)$. By triangle-freeness, $e(B[\{u_i\}_i]) \leq 4$. But if $e(B[\{u_i\}_i]) = 4$, then (after renaming if need be) $u_1u_2u_3u_4u_1$ must form a copy of C_4. On the other hand, each E_i is C_4-free, contradiction.

        It follows that $e(B[\{u_i\}_i]) \leq 3$, and $e(B[H_1 \cup H_2]) \geq e(H_1) + 2 \times 4 - 3 = 17$, and $\delta_\alpha(B[H_1 \cup H_2]) \leq 10 + 4 - 17\alpha < 0$.

3. If $|V(H_1) \cap V(H_2)| = 5$: by triangle-freeness, $e(B[V(H_1) \cap V(H_2)]) \leq 6$. But if $e(B[V(H_1) \cap V(H_2)]) = 6$, $B[V(H_1) \cap V(H_2)] \cong K_{2,3}$, while each E_i is $K_{2,3}$-free. It follows that $e(B[H_1 \cap H_2]) \leq 5$. Therefore $e(B[H_1 \cup H_2]) = e(H_1) + e(H_2) - e(B[H_1 \cap H_2]) \geq 24 - 5 = 19$, and $\delta_\alpha(B[H_1 \cup H_2]) \leq 15 - 19\alpha < 14 - 17\alpha < 0$.

4. If $|V(H_1) \cap V(H_2)| \in [1, 4]$: these are dual to Cases in (2a)$\sim$(2d).
   (a) If $|V(H_1) \cap V(H_2)| = 4$, then by analysis in (2d) $e(B[H_1 \cap H_2]) \leq 3$, so $e(B[H_1 \cup H_2]) \geq 24 - 3$ and $\delta_\alpha(B[H_1 \cup H_2]) \leq 16 - 21\alpha < 0$.
   (b) If $|V(H_1) \cap V(H_2)| = 3$, then $\delta_\alpha(B[H_1 \cup H_2]) \leq 17 - (24 - 2)\alpha < 0$.
   (c) If $|V(H_1) \cap V(H_2)| = 2$, then $\delta_\alpha(B[H_1 \cup H_2]) \leq 18 - (24 - 1)\alpha < 0$.
   (d) If $|V(H_1) \cap V(H_2)| = 1$, then $\delta_\alpha(B[H_1 \cup H_2]) \leq 19 - (24 - 0)\alpha < 0$.

We have proved that $H_1 \cap H_2 = \emptyset$ as desired. Now assume for the sake of contradiction that $\mathrm{dist}_B(H_1, H_2) \leq 5$; then there exists $1 \leq k = \mathrm{dist}_B(H_1, H_2) + 1 \leq 6$, $x_1 \in H_1, x_k \in H_2$, $x_2, \cdots, x_{k-1} \in V(B) \smallsetminus (V(H_1) \cup V(H_2))$,[17] s.t. $x_1 – \cdots – x_k \subseteq B$. But then $\delta_\alpha\big(B[H_1 \cup H_2 \cup \{x_2, \cdots, x_{k-1}\}]\big) \leq (2\times 10 - 2\times 12\alpha) + (4 - 5\alpha) = 24 - 29\alpha < 0$, contradiction. $\square$

Let us denote by $(E_1)_a – k –_b (E_2) := \dfrac{E_1 \coprod P_{k+2} \coprod E_2}{a_{E_1} \sim 1_{P_{k+2}}, b_{E_2} \sim (k+2)_{P_{k+2}}}$, that is a copy of E_1 and a copy of E_2 joined by a path with k vertices (of degree 2) in the middle and "anchor points" $a \in V(E_1), b \in V(E_2)$. Define $(E_1) – k – (E_2) := \big\{(E_1)_a – k –_b (E_2) : a \in V(E_1), b \in V(E_2)\big\}$. For any finite $H_1, \cdots, H_m$, define $(H_1)_{(a_1)} – k_1 –_{(b_1)} (H_2)_{(a_2)} – \cdots – k_{m-1} –_{(b_{m-1})} (H_m)$ analogously.

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[17] Otherwise there exists shorter paths linking H_1, H_2 contradicting our choice of k.

Corollary 4.20. *For $\alpha \in (\frac{24}{29}, \frac{5}{6}]$, every graph in (E_1)–5–(E_2) is (α, TF)-homomorphically rigid.*

Proof. To see that (E_1)–5–$(E_2) \subseteq \mathcal{K}_{\alpha,\text{TF}}$, let $G \in (E_1)$–5–(E_2) and $H \subseteq_{\text{ind}} G$ be connected. If $H \cap E_2 = \emptyset$, $\delta_\alpha(H) = t - t\alpha + \delta_\alpha(H \cap E_1) \geq 0$ for some $t \in [0, 5]$; dually if $H \cap E_1 = \emptyset, \delta_\alpha(H) \geq 0$. Finally if H intersects both E_1 and E_2, $\delta_\alpha(H) = \delta_\alpha(H \cap E_1) + \delta_\alpha(H \cap E_2) + 5 - 6\alpha \geq 0$.

Let $G \in (E_1)$–5–(E_2), $B \in \mathcal{K}_{\alpha,\text{TF}}$ and $h \in \text{Hom}(G, B)$. Per Lemma 4.19, $B[h(E_1)] \cap B[h(E_2)] = \emptyset$ with distance at least 6, so $v(B[\text{Im}(h)]) \geq v(E_1) + v(E_2) + 6 - 1 = 25 = v(G)$, yielding injectivity. □

Remark 4.21.

1. The range of α guaranteeing $H_1 \cap H_2 = \emptyset$ can be slightly wider: observe that that part of the proof only requires $\alpha > \frac{14}{17}$.

2. Proof of the second part generalizes below.

Proposition 4.22 (More general density-distance trade off). *Fix $\alpha \in (0, 1]$, and let $r_1, r_2 \geq 0$. If there exists $F_1, F_2 \subseteq G \in \mathcal{K}_{\alpha,TF}$ s.t.*

1. *$\delta_\alpha(F_1) = r_1, \delta_\alpha(F_2) = r_2$, and*

2. *$F_1 \cap F_2 = \emptyset$*

   Then $dist_G(F_1, F_2) \geq \frac{1-(r_1+r_2)}{1-\alpha}$.

Proof. Without loss of generality one may assume F_1, F_2 belong to the same connected component of G, for otherwise by definition $\text{dist}_G(F_1, F_2) = \infty > \frac{1-(r_1+r_2)}{1-\alpha}$. Further one may assume r_1, r_2 is so small that $\frac{1-(r_1+r_2)}{1-\alpha} > 1$ for $F_1 \cap F_2 = \emptyset \implies \text{dist}_G(F_1, F_2) \geq 1$ and if $\frac{1-(r_1+r_2)}{1-\alpha} \leq 1$ there is nothing left to be shown.

By way of contradiction, assume $\text{dist}_G(F_1, F_2) < \frac{1-(r_1+r_2)}{1-\alpha}$ then because $1 \leq \text{dist}_G(F_1, F_2) < \infty$ there exists a shortest path $x_0 \in F_1, x_1, \cdots, x_{k-1} \notin F_1 \cup F_2, x_k \in F_2$ in G with $1 \leq k < \frac{1-(r_1+r_2)}{1-\alpha}$. Consider the subgraph $G' := G[F_1 \cup \{x_1, \cdots, x_{k-1}\} \cup F_2]$. Observe that $\delta_\alpha(G') \leq v(F_1)+k-1+v(F_2)-\alpha(e(F_1)+k+e(F_2)) = \delta_\alpha(F_1)+\delta_\alpha(F_2)+k(1-\alpha)-1 < 0$, contradicting $G \in \mathcal{K}_{\alpha,\text{TF}}$. $\square$

4.4.1 $\mathcal{K}_{\frac{5}{6},\text{TF}}$ Reduces to $(1,1)$-Types

Definition 4.23 ((m,n)-type)**.** Let G be a finite graph, $m, n \in \mathbb{N}$. G is *of (m,n)-type* if there are exactly m copies of E_1 and n copies of E_2 in G as induced subgraphs.

Our main result for this subsection is that we can simplify a $G \in \mathcal{K}_{\frac{5}{6},\text{TF}}$ by reducing them to $(1,1)$-types. Before showing that, a very helpful lemma:

Lemma 4.24. *Let G be any finite graph, $\alpha \in (0,1]$. Let $V(G) := V_1 \coprod V_2$ $\big($So $E(G) = E(V_1) \coprod E(V_2) \coprod E(V_1, V_2)\big)$. Let $h : G \twoheadrightarrow G'$ be s.t. $h\restriction_{V_2} = id$ and h is h is a quotient induced by $h\restriction_{V_1}$: that is, $(x,y) \in E(hV_1, hV_2 = V_2) \iff \Big(\exists x_0 \in h^{-1}(x)$ s.t. $(x_0, y) \in E(V_1, V_2)\Big)$. Let $H' \subseteq_{ind} G'$. Then:*

1. *(upstairs) $\delta_\alpha(G[h^{-1}(H') \cup V_1]) - \delta_\alpha(G[V_1]) \leq \delta_\alpha(G[h^{-1}(H')]) - \delta_\alpha(G[h^{-1}(H') \cap V_1])$.*
2. *(downstairs) likewise, $\delta_\alpha(G'[V(H') \cup h(V_1)]) - \delta_\alpha(G'[h(V_1)]) \leq \delta_\alpha(H') - \delta_\alpha(G'[V(H') \cap h(V_1)])$.*
3. *(upstairs - downstairs) $\delta_\alpha(G[h^{-1}(H')]) - \delta_\alpha(G[h^{-1}(H') \cap V_1]) \leq \delta_\alpha(H') - \delta_\alpha(G'[V(H') \cap h(V_1)])$.*

Proof. (upstairs) and (downstairs) essentially follow from the $\preceq$-AP property of $\mathcal{K}_{\alpha,\mathrm{TF}}$, but we list them for ease of application (especially in combination with (upstairs-downstairs)) in later proofs.

1. For (upstairs),

$$\begin{aligned}
\text{LHS} &= \delta_\alpha(G[h^{-1}(H') \smallsetminus V_1]) - \alpha e(V_1 \cap h^{-1}(H'), h^{-1}(H') \smallsetminus V_1) \\
&\quad -\alpha e(V_1 \smallsetminus h^{-1}(H'), h^{-1}(H') \smallsetminus V_1) \\
&\leq \delta_\alpha(G[h^{-1}(H') \smallsetminus V_1]) - \alpha e(V_1 \cap h^{-1}(H'), h^{-1}(H') \smallsetminus V_1) \\
&= \text{RHS}.
\end{aligned}$$

2. Likewise for (downstairs),

$$\begin{aligned}
\text{LHS} &= \delta_\alpha(G'[V(H') \smallsetminus h(V_1)]) - \alpha e(h(V_1) \cap V(H'), V(H') \smallsetminus h(V_1)) \\
&\quad -\alpha e(h(V_1) \smallsetminus V(H'), V(H') \smallsetminus h(V_1)) \\
&\leq \delta_\alpha(G'[V(H') \smallsetminus h(V_1)]) - \alpha e(h(V_1) \cap V(H'), V(H') \smallsetminus h(V_1)) \\
&= \text{RHS}.
\end{aligned}$$

3. For (upstairs - downstairs), first consider the function: $\pi_h : E_G\big(V_1 \cap h^{-1}(H'), h^{-1}(H') \smallsetminus V_1\big) \to E_{G'}\big(h(V_1) \cap V(H'), V(H') \smallsetminus h(V_1)\big)$, $\pi_h(x,y) = (h(x), y)$. By the assumption that h is an induced quotient, π_h is surjective. It follows that

$$e_G\big(V_1 \cap h^{-1}(H'), h^{-1}(H') \smallsetminus V_1\big) \geq e_{G'}\big(h(V_1) \cap V(H'), V(H') \smallsetminus h(V_1)\big) \quad (4.2)$$

Therefore

$$\begin{aligned}
\text{LHS} &= \delta_\alpha(G[h^{-1}(H') \setminus V_1]) - \alpha e_G(h^{-1}(H') \setminus V_1, h^{-1}(H') \cap V_1) \\
&\overset{h\restriction_{V_2}=\text{id}}{=} \delta_\alpha(G'[V(H') \setminus h(V_1)] - \alpha e_G(h^{-1}(H') \setminus V_1, h^{-1}(H') \cap V_1) \\
&\overset{\text{Inequality (4.2)}}{\leq} \delta_\alpha(G'[V(H') \setminus h(V_1)] - \alpha e_{G'}\big(h(V_1) \cap V(H'), V(H') \setminus h(V_1)\big) \\
&= \text{RHS},
\end{aligned}$$

as desired.

□

We present a conceptual illustration of Lemma 4.24 in Figure 4.2.

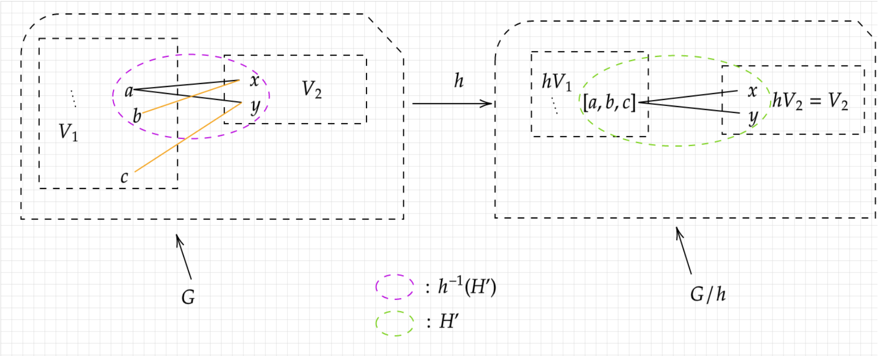

Figure 4.2: With $H', h^{-1}(H')$ as circled by the dashed ovals, the edges $((b, x), (c, y))$ contribute to -2α in LHS(upstairs), $-\alpha$ in RHS(upstairs), and 0 in RHS(upstairs-downstairs).

Lemma 4.25. *For $G \in \mathcal{K}_{\frac{5}{6},TF}$, let $m \geq 2$ be the number of copies of E_1 in G, enumerated as $o_1, \cdots, o_m$. Let π be the quotient map identifying $o_i(x) \sim o_j(x)$ whereever $i \neq j \in [1, m]$ and $x \in V(E_1)$, π being identity on $G \setminus \coprod_{j \leq m} Im(o_j)$.* [18] *Then the quotient graph $G/\pi \in \mathcal{K}_{\frac{5}{6},TF}$, and the same claim holds for E_2.*

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[18] Note that $([a], [b]) \in E((G/\pi)[\pi o_1]) \iff \bigvee_{1 \leq t \leq m} \Big((\pi^{-1}(a) \cap o_t, \pi^{-1}(b) \cap o_t) \in E(G)\Big)$.

Proof. We focus on E_1 as the same lines of arguments go through for E_2. Fix $m \geq 2$. Let G^* be a minimal counter example to the claim with respect to m; that means:

1. $G^* \in \mathcal{K}_{\frac{5}{6},\text{TF}}$, G^* has m copies of E_1, yet $G^*/\pi \notin \mathcal{K}_{\alpha,\text{TF}}$ (G^* is a counter example), and
2. $\forall H \subsetneq G^*$, $\Big(\big(H \text{ has } m \text{ copies of } E_1\big) \implies \big(H/(\pi \restriction_H) \in \mathcal{K}_{\alpha,\text{TF}}\big)\Big)$ (G^* is minimal, in that no proper subgraph is a counter example with respect to m).[19]

Such a minimal G^* always exists as long as there exists any counter example $(G^*)_0$: if a subgraph $H \subseteq (G^*)_0$ containing all m orbits also satisfies $H/(\pi \restriction_H) \notin \mathcal{K}_{\alpha,\text{TF}}$, just take $(G^*)_1 := H$ and reiterate. Since $(G^*)'$ is finite, this iteration terminates.

As $G^*/\pi \notin \mathcal{K}_{\alpha,\text{TF}}$, one of the following 3 cases happens:

1. $G^*/\pi \notin$ TF;
2. $G^*/\pi \in$ TF, yet for some $H \subseteq G^*/\pi$ s.t. $H \supseteq \pi o_1$, $\delta_\alpha(H) < 0$ (note that πo_1 is the unique copy of E_1 in G^*/π);
3. $G^*/\pi \in$ TF, and for every $H \subseteq G^*/\pi$ s.t. $H \supseteq \pi o_1$, $\delta_\alpha(H) \geq 0$; however there exists an $H \subseteq G^*/\pi$, $H \not\supseteq \pi o_1$, with $\delta_\alpha(H) < 0$.

To derive the contradiction it suffices to show that none of the three cases can happen. Indeed,

1. G^*/π must be triangle-free. Suppose not, and let $a, b, c \in G^*/\pi$ induce a triangle. Either one or two out of the three vertices are in $[\pi o_1] \cong E_1$.

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[19] This asserts weaker inductive hypothesis than the one with out assuming $\#O_{\text{inj}}(E_1, H) = m$, so provides an *a priori* stronger conclusion.

   (a) If (wlog) only a is in πo_1, then we must have an $x \in V(E_1)$ with $\{o_1(x), o_2(x), \cdots, o_m(x)\} = \iota^{-1}(a)$ s.t. $o_i(x)$–b–c–$o_j(x) \subseteq G^*$ for some $i \neq j \in [1, m]$ (if $i = j$ then the subgraph $G^*[o_i \cup \{b, c\}]$ would not have been triangle-free). This gives $\text{dist}_{G^*}(o_i, o_j) \leq 3$, contradicting Lemma 4.19.

   (b) Otherwise, (wlog) only $a, b \in \pi o_1$. For ease of notation, denote by $a_i^{-1} :=$ the unique vertex in $\pi^{-1}(a) \cap o_i$, Likewise for b_j^{-1} $(i, j \in [1, k])$. Then upstairs in $G^* \in \text{TF}$, c must connect to at least one a_i^{-1} and at least one b_j^{-1}, by the definition of π; furthermore we must have $(a_i^{-1}, b_i^{-1}), (a_j^{-1}, b_j^{-1}) \in E(G^*)$ because all the $G[o_i]$s are isomorphic to E_1, meaning $i \neq j$ by triangle-freeness. But then $\text{dist}_{G^*}(o_i, o_j) \leq 2$, contradiction.

2. Next we argue that for any $H \subseteq G^*/\pi$ s.t. $H \supseteq \pi o_1$, $\delta_\alpha(H) \geq 0$, falsifying case (2). Since $\delta_\alpha(E_1) - \alpha < 0$ and $\forall i \neq j \in [1, m]$ $\text{dist}_{G^*}(o_i, o_j) \geq 6 > 1$, $\coprod_{j \in [1,m]} o_j \subseteq_{\text{ind}} G$. Also, $\pi o_1 \subseteq_{\text{ind}} G/\pi$ and $\pi o_1 = \left(\coprod_{j \in [1,m]} o_j\right)/\pi$.

   (a) If $\pi o_1 \subseteq H \subsetneq G^*/\pi$: consider $\widetilde{H} := G^*[\pi^{-1}(H)]$. As $\pi o_1 \subseteq H$, $\coprod_{j \leq m} o_j \subseteq \widetilde{H} \subsetneq G^*$. By minimality of G^* and the fact that there are m copies of E_1 in $\widetilde{H}$, $H = \widetilde{H}/(\pi \restriction_{\widetilde{H}}) \in \mathcal{K}_{\alpha,\text{TF}}$, so in particular $\delta_\alpha(H) \geq 0$.

   (b) Otherwise, $H = G^*/\pi$. But then by my Lemma 4.24(3) applied to $H' := G^*/\pi, V_1 := \coprod_{j \leq m} o_j$, we have

$$\delta_\alpha(G^*) \leq \delta_\alpha(G^*/\pi) - \delta_\alpha(\pi o_1) + \delta_\alpha(\coprod_{j \leq m} o_j) \tag{4.3}$$

   When $\alpha = \frac{5}{6}$, $\delta_\alpha(o_1) = \delta_\alpha(\coprod_{j \leq m} o_j) = 0$ so $\delta_\alpha(G^*/\pi) \geq \delta_\alpha(G^*) \geq 0$ as desired.[20] **It should be noted that** when $\alpha < \frac{5}{6}$, we only get $\delta_\alpha(G^*/\pi) \geq \delta_\alpha(G^*) - \sum_{2\leq j\leq m} \delta_\alpha(o_j) < \delta_\alpha(G^*)$ as $\delta_\alpha(o_j) > 0$, so no similar conclusion about $\delta_\alpha(G^*/\pi)$ can be drawn.[21]

3. We are left with the third case, which we will essentially reduce to the second case by (mainly, in the non-trivial subcase) looking downstairs. Let $H \subseteq G^*/\pi$ be s.t. $H \not\supseteq \pi o_1$.

   (a) If $H \subsetneq \pi o_1$ or $H \subseteq G^*/\pi \smallsetminus \pi o_1$, then $G^*[\pi^{-1}(H)] \subsetneq \coprod_j o_j$ or $G^*[\pi^{-1}(H)] \subseteq G^* \smallsetminus \coprod_j o_j$, respectively. with $\delta_\alpha(H) = \frac{\delta_\alpha(G^*[\pi^{-1}(H)])}{m} \geq 0$ in the first case and $\delta_\alpha(H) = \delta_\alpha(G^*[\pi^{-1}(H)]) \geq 0$ in the second case.

   (b) Otherwise, H must intersect both πo_1 and $G^*/\pi - \pi o_1$. We want to show that $\delta_\alpha(H) \geq 0$ as well. Since $\delta_\alpha((G^*/\pi)[V(H)]) \leq \delta_\alpha(H)$, it is without loss of generality to assume $H = (G^*/\pi)[V(H)]$. Then by Lemma 4.24(2) applied to $H' := H$,$V_1 := \coprod_{j\leq m} o_j$, we get

$$\delta_\alpha((G^*/\pi)[V(H)]) - \delta_\alpha(\pi o_1 \cap H) \quad \geq \quad \delta_\alpha((G^*/\pi)[V(H) \cup \pi o_1]) - \delta_\alpha(\pi o_1),$$

which implies $\delta_\alpha((G^*/\pi)[V(H)]) \geq \delta_\alpha(\underbrace{(G^*/\pi)[V(H) \cup \pi o_1]}_{\supseteq \pi o_1, \text{reduces to Case (2)}}) - \underbrace{\delta_\alpha(\pi o_1)}_{0} + \delta_\alpha(\underbrace{\pi o_1 \cap H}_{\subseteq \pi o_1}) \geq$

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[20] In fact Inequality 4.3 can be made into equality using $\mathrm{dist}_{G^*}(o_i, o_j) > 2 \implies$ there exists no common neighbor of $o_i, o_j \implies$ The $\geq$ in Inequality (4.2) is in fact $=$. We keep the current presentation for ease of generalizing to Porism 4.26(2).

[21] Such obstruction is not vacuous: per (a slightly modification of arguments in) [1], for any $\alpha \in (0, 1] \smallsetminus \mathbb{Q}$, as long as $\delta_\alpha(H) > 0$ for sufficiently large $H \in \mathcal{K}_{\alpha,\mathrm{TF}}$, for any $\epsilon \in (0, \alpha)$ there exists $H' \in \mathcal{K}_{\alpha,\mathrm{TF}}$ s.t. $\delta_\alpha(H) - \epsilon < \delta_\alpha(H') < \delta_\alpha(H)$.

Note: Case (2) can also be argued without using the language of "minimal counter example" by directly using Lemma 4.24 to both subcases (2a)(2b).

0.

In conclusion, none of the three cases happen, so G^* does not exist. By arbitrariness of m, we are done.[22] $\square$

The lines of argument generalize when appropriate assumptions are imposed:

Porism 4.26. *Let* $\alpha \in (0, 1]$.

1. *If* $H \in \mathcal{K}_{\alpha,TF}$ *is s.t.*
   - (a) $\delta_\alpha(H) = 0$[23] *and*
   - (b) $\forall B \in \mathcal{K}_{\alpha,TF}$, *any two different copies of* H *as subgraphs have distance at least* 4 *in* B,[24]

   Then for any $B \in \mathcal{K}_{\alpha,TF}$, *whenever there are at least* 2 *copies of* H *in* B *as subgraphs, the quotient induced by gluing all copies of* H *remains in* $\mathcal{K}_{\alpha,TF}$. *In particular, there exists* $B' \in \mathcal{K}_{\alpha,TF}$ *of* $(1,1)$*-type s.t.* $B \to B'$.

2. *If* $H \in \mathcal{K}_\alpha$ *is s.t.*
   - (a) $\delta_\alpha(H) = 0$ *and*
   - (b) $\forall B \in \mathcal{K}_\alpha$, *any two different copies of* H *as subgraphs have distance at least* 2 *in* B,

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[22] Remark: one may be tempted to rephrase the proof of Lemma 4.25 using the language of "join of graphs" over the common isomorphic subgraph o_is (e.g. [1, 24*]), but there are obstacles: first, *a priori* there is no canonical way to decide what the "host graphs" $G_{1\,\text{ind}} \supseteq o_1, \cdots, G_{m\,\text{ind}} \supseteq o_m$ are; moreover there is no immediately easy way of arguing that the *non-free* join of G_ms over the o_ms remain in $\mathcal{K}_{\alpha,\text{TF}}$, noting there are edges across different G_i, G_js which may be impacted by performing the join. One therefore believes that resorting to Lemma 4.24 remains the most convenient, and probably necessary, way.

[23] This implies all *injective* $\mathcal{K}_{\alpha,\text{TF}}$-homomorphisms out of H are embeddings, and kills the "inconclusive inequality issue" noted in the proof of Lemma 4.25.

[24] This can be enforced by e.g. conditions in Proposition 4.22, but we keep the weaker assumptions.

Then for any $B \in \mathcal{K}_\alpha$, whenever there are at least 2 copies of H in B as subgraphs, the quotient induced by gluing all copies of H remains in $\mathcal{K}_\alpha$. In particular, there exists $B' \in \mathcal{K}_\alpha$ of $(1,1)$-type s.t. $B \to B'$.

Remark 4.27. Note that for any $\alpha \in (0,1]$ and any $B \in \mathcal{K}_\alpha$, if $\delta_\alpha(B) = 0$ and $\delta_\alpha(B') > 0$ $(\forall B' \subsetneq B)$, then any two distinct copies of B in a host graph $G \in \mathcal{K}_\alpha$ must be vertex-disjoint. [25] In fact,

Proposition 4.28. *Let $k \geq 3$ and $\alpha := \frac{2}{k}$. Then there is an infinite collection $\{H_i\}_{i\in\omega} \subseteq \mathcal{K}_{\alpha,TF}$ s.t. for each i, $\delta_\alpha(H_i) = 0$ and $\big(\delta_\alpha(H_i') > 0$ whenever $H_i' \subsetneq H_i\big)$.*

Proof. We first prove the case for $k = 3$. Consider the subcollection $\mathcal{P}$ of generalized Petersen graphs (See discussions above Definition 4.47) with girth 5. Each $H \in \mathcal{P}$ is regular of degree 3 so $\delta_\alpha(H) = v(H) - \frac{2}{3}e(H) = v(H) - \frac{2}{3} \cdot \frac{3v(H)}{2} = 0$, and for any $H' \subsetneq H$, we have some vertex of degree less than 3, which means $\mathrm{ad}(H') < 3$, i.e. $v(H') > \frac{2}{3}e(H') = \alpha e(H')$.

The crux of the observation above is that $\mathcal{K}_{\alpha,\mathrm{TF}}$ contains infinitely many graphs regular of degree $\frac{2}{\alpha}$. Fixing $k \geq 4, \alpha = \frac{2}{k}$ this is always true. For each k let $n > 2k$ be arbitrary, then there is a k-regular, triangle-free subgraph of $K_{n,n}$ with vertex-size $2n$, namely by connecting 1_L with $1_R, 2_R, \cdots, k_R$, 2_L with $2_R, 3_R, \cdots, (k+1)_R$, and so on, with the vertices labelled modulo n.[26] Since n can be arbitrarily large, there are infinitely many k-regular, triangle-free graphs in $\mathcal{K}_{\alpha,\mathrm{TF}}$, as desired. ☐

We now reveal how Lemma 4.25 and Porism 4.26 can be used to simplify the pre-

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[25] Indeed, otherwise assume in $G \in \mathcal{K}_\alpha$ two copies o_1, o_2 of B intersect at C; then $\delta_\alpha(\frac{(o_1 \coprod o_2)}{C}) = \delta_\alpha(o_1) + \delta_\alpha(o_2) - \delta_\alpha(C) = -\delta_\alpha(C) < 0$.

[26] See e.g. the comments in here.

sentation of $\mathcal{K}_{\alpha,\mathrm{TF}}$. Let $\mathcal{S}, \mathcal{T}$ be sets of finite graphs. Recall that $\mathcal{S} \to \mathcal{T} \iff$ for each $s \in \mathcal{S}$ there exists $t \in \mathcal{T}$ s.t. s homomorphically maps to t. Note that this relation is transitive. Also recall that $\mathcal{T}$ is a *homomorphic presentation* of $\mathcal{S}$ if $\left(\mathcal{S} \to \mathcal{T} \bigwedge \mathcal{T} \subseteq \mathcal{S}\right)$. Just like in Definition 4.23, given $\vec{H} := \{H_1, \cdots, H_g\}$ $(g < \omega)$ and $(c_1, \cdots, c_g) \in \mathbb{N}^g$, we say a finite graph G is *of* $(c_1, \cdots, c_g)_{\vec{H}}$*-type* if there are exactly c_1 copies of H_1 as subgraphs in $G, \cdots$, exactly c_g copies of H_g as subgraphs in G.

Theorem 4.29.

1. *Let $\alpha := \frac{5}{6}$, and let*

   $$\mathcal{S}_0 := \left\{G \in \mathcal{K}_{\alpha,\mathit{TF}} : G \textit{ is connected, of } (1,1) \textit{ -type, and } (\frac{5}{6}, \mathit{TF})\textit{-homomorphically rigid} \right\}$$

   Then $\mathcal{K}_{\alpha,\mathit{TF}} \to \mathcal{S}_0$.

2. *Fix $\alpha \in (0,1]$. If $\vec{H}$ is a tuple of* (α, TF)-homomorphically rigid *graphs each satisfying the assumptions(1a)(1b) in Porism 4.26(1), then $\mathcal{K}_{\alpha,\mathit{TF}} \to \mathcal{S}_{0,\vec{H}}$, where*

   $$\mathcal{S}_{0,\vec{H}} := \left\{G \in \mathcal{K}_{\alpha,\mathit{TF}} : G \textit{ is connected, of } (1, \cdots, 1)_{\vec{H}}\textit{-type, and } (\alpha, \mathit{TF})\textit{-homomorphically rigid} \right\}$$

3. *In either case above, $|\mathcal{S}_{0,*}| < \aleph_0 \iff \Big($there exists a finite homomorphic presentation of $\mathcal{K}_{\alpha,\mathit{TF}}\Big)$.*

Proof. 1. Denote $\mathcal{S}_2 := \left\{G \in \mathcal{K}_{\alpha,\mathrm{TF}} : G \text{ is connected}\right\}$, $\mathcal{S}_1 := \left\{G \in \mathcal{S}_2 : G \text{ is of } (1,1)\text{-type}\right\}$. We prove that $\mathcal{K}_{\alpha,\mathrm{TF}} \to \mathcal{S}_2 \to \mathcal{S}_1 \to \mathcal{S}_0$.

(a) $\mathcal{K}_{\alpha,\mathrm{TF}} \to \mathcal{S}_2$: take $G \in \mathcal{K}_{\alpha,\mathrm{TF}}$ and let $C_1, \cdots, C_t$ be its connected components. Then any $G^* \in (C_1)_*\text{–}5\text{–}_*(C_2)\text{–}\cdots{}_*\text{–}5\text{–}_*(C_t)$(Notations above Corollary 4.20) is triangle-free, connected and in fact lies in $\mathcal{K}_\alpha$.[27] Embedding each component to G^* gives a homomorphism $h : G \to G^* \in \mathcal{K}_{\alpha,\text{TF}}$.

(b) $\mathcal{S}_2 \to \mathcal{S}_1$: take $G \in \mathcal{S}_2$. If G is of $(0,*)$-type, consider $\iota : G \hookrightarrow G^* \in (E_1)_*\text{–}5\text{–}_*(G) \subseteq \mathcal{K}_{\alpha,\text{TF}}$; likewise when G is of $(*,0)$-type. Therefore, it is without loss of generality to consider G of (m,n)-types with $m,n \geq 1$. But per Lemma 4.25, such Gs homomorphically map to a $(1,1)$-type by composing two quotient maps. Note that homomorphisms preserve connectedness.

(c) $\mathcal{S}_1 \to \mathcal{S}_0$: take $G \in \mathcal{S}_1$, and let $G_0 := G$. If G is not (α,TF)-homomorphically rigid, there exists $G_1 \in \mathcal{K}_{\alpha,\text{TF}}$, $h_0 : G_0 \twoheadrightarrow G_1$ non-injective homomorphism. Since $v(G_1) < v(G_0) < \infty$, such a chain must terminate with an (α,TF)-homomorphically rigid, connected graph G_t. Since E_1, E_2 are (α,TF)-homomorphically rigid, letting o_1 be the unique copy of E_1 in G_0 and o_2 for E_2, then we must have $h_{t-1} \circ \cdots h_0 \restriction_{o_i}$ injecting E_i as a subgraph of G_t. On the other hand, if there are more than one copy of E_i in G_t then per Lemma 4.25, G_t would not be (α,TF)-homomorphically rigid. It follows that $G_t \in \mathcal{S}_0$ as desired.

Note that similar ideas can be generalized to the following claim, whose formal proof just uses the finiteness of elements in $\mathcal{K}_{\alpha,\text{TF}}$s:

Claim 4.30. *Let $\alpha \in (0,1]$ and let $\mathcal{S}_\alpha := \{G \in \mathcal{K}_{\alpha,TF} : G$ is (α, TF)-homomorphically rigid$\}$. Then $\mathcal{S}_\alpha$ is homomorphically universal in that for*

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[27] Similar to proof of Corollary 4.20, let H be a connected induced subgraph of G^*. If H intersects with exactly one of the C_js, there exists $t \leq 10$ s.t. $\delta_\alpha(H) = \delta_\alpha(C_j \cap H) + t - t\alpha > 0$; if e.g. H intersects with C_j, C_{j+1}, then $\delta_\alpha(H) = \delta_\alpha(C_j \cap H) + \delta_\alpha(C_{j+1} \cap H) + 5 - 6\alpha + t' - t'\alpha > 0$ for some $t' \leq 10$.

any $G \in \mathcal{K}_{\alpha,TF}$ there exists $G' \in \mathcal{S}_\alpha$ s.t. $G \to G'$.

2. Very similar to Part (1). Define $\mathcal{S}_{2,\vec{H}}$, $\mathcal{S}_{1,\vec{H}}$ analogously.
    (a) $\mathcal{K}_{\alpha,\text{TF}} \to \mathcal{S}_{2,\vec{H}}$: for each $\alpha \in (0,1]$ there exists $k >> 0$ s.t. $k-(k+1)\alpha \geq 0$. We follow the same lines of argument, changing 5 to k.

    (b) $\mathcal{S}_{2,\vec{H}} \to \mathcal{S}_{1,\vec{H}}$: we reduce to types $(c_1, \cdots, c_g)_{\vec{H}}$ with each $c_j \geq 1$ by the same "gluing" trick, changing 5 to k; such types reduce to $(1, \cdots, 1)_{\vec{H}}$ types per Porism 4.26(1).

    (c) $\mathcal{S}_{1,\vec{H}} \to \mathcal{S}_{0,\vec{H}}$: we obtain $h_0, \cdots, h_{t-1}$ analogously, resulting in G_t connected, (α, TF)-homomorphically rigid. For the type $(c_1, \cdots, c_g)_{\vec{H}}$ of G_t, each $c_j \geq 1$ by the homomorphic rigidity of H_j, and each $c_j \leq 1$ by (α, TF)-homomorphic rigidity of G_t and Porism 4.26(1).

3. In either case $\mathcal{S}_{0,*}$ is a homomorphic presentation of $\mathcal{K}_{\alpha,\text{TF}}$, so we prove the converse. Assume $|\mathcal{S}_{0,*}| = \aleph_0$; then for any $n \in \omega$ there exists $G \in \mathcal{S}_{0,*}$ s.t. $v(G) > n$ and each $\mathcal{K}_{\alpha,\text{TF}}$-homomorphism from G is injective. Assume for the sake of contradiction that $\{M_1, \cdots, M_k\} \subseteq \mathcal{K}_{\alpha,\text{TF}}$ presents $\mathcal{K}_{\alpha,\text{TF}}$; take $G \in \mathcal{S}_{0,*}$, $v(G) > \max_{i\in[k]}\{v(M_i)\}$, we have a contradiction, as some M_j has to host G as a subgraph.

□

Therefore to find a finite homomorphic presentation of $\mathcal{K}_{\frac{5}{6},\text{TF}}$, it suffices to study subclass $\mathcal{S}_0$; for $\alpha \in (0, \frac{5}{6})$, a potential approach is to find $\vec{H}$ playing the roles of $\{E_1, E_2\}$.

4.4.2 More Structural Properties of Homomorphic Presentation of $\mathcal{K}_{\alpha,\mathrm{TF}}$

We continue to refine the homomorphic presentation $\mathcal{S}_0 \subseteq \mathcal{K}_{\frac{5}{6},\mathrm{TF}}$ (and resp. $\mathcal{S}_{0,\vec{H}} \subseteq \mathcal{K}_{\alpha,\mathrm{TF}}$ in general), seeking structural descriptions of the members in $\mathcal{S}_{0,*}$. First, except for trivial cases, (α, TF)-homomorphically rigid graphs cannot have pendant vertices.

Proposition 4.31. *Let $\alpha \in (0,1], G \in \mathcal{K}_{\alpha,TF}$ with $e(G), v(G) \geq 2$. If there exists $x \in V(G)$ with $\deg_G(x) \leq 1$, then G is not (α, TF)-homomorphically rigid.*

Proof. If $\deg_G(x) = 0$ then $G \twoheadrightarrow G/(x \sim y)$ for any $y \neq x$, and $G/(x \sim y) \cong G \smallsetminus \{x\} \subseteq G \in \mathcal{K}_{\alpha,\mathrm{TF}}$. Assume $\deg_G(x) = 1$. Let $y \in! N_G(x)$. If $\deg_G(y) = 1$ as well, then x–y is an isolated edge in G, so $G \twoheadrightarrow G/(x \sim x, y \sim y') \cong G \smallsetminus \{x\text{–}y\} \in \mathcal{K}_{\alpha,\mathrm{TF}}$ for any $(x', y') \in E(G) \smallsetminus \{x\text{–}y\}$.

Next assume $\deg_G(y) \geq 2$ then there exists $x' \neq x \in N_G(y)$. I claim that the canonical quotient $h : G \twoheadrightarrow G/(x \sim x')$ remains in $\mathcal{K}_{\alpha,\mathrm{TF}}$. Indeed, for any $H' \subseteq_{\mathrm{ind}} G/h$:

- if $[x'] = [x] \notin H'$ then $H \cong G[h^{-1}(H)] \in \mathcal{K}_{\alpha,\mathrm{TF}}$;
- if $[x'] = [x] \in H'$ then $H' \cong G[h^{-1}(H') \smallsetminus x] \in \mathcal{K}_{\alpha,\mathrm{TF}}$.

$\square$

Proposition 4.32. *Let $G \in \mathcal{K}_{\frac{5}{6},TF}$ be of $(1,1)$-type, with o_i the unique copy of E_i in G as (induced) subgraph. If $\chi\big(G \smallsetminus (o_1 \coprod o_2)\big) = 1$, then G is not $(\frac{5}{6}, TF)$-homomorphically rigid.*

Proof. Indeed, the only monochromatic graphs are edgeless. If there exists $x \in V(G \smallsetminus o_1 \coprod o_2)$ s.t. $x \in N_G(o_1) \cap N_G(o_2)$ then $\delta_\alpha(G[o_1 \cup o_2 \cup \{x\}]) \leq 1 - 2\alpha < 0$. Therefore it must be the case that each $x \in V(G \smallsetminus o_1 \coprod o_2)$ is in at most one of $N_G(o_1)$ and $N_G(o_2)$. Pick any $x \in V(G \smallsetminus o_1 \coprod o_2)$.

- If $x \in N_G(o_1)$, then because $e\big(x, (G \smallsetminus o_1 \coprod o_2) \smallsetminus \{x\}\big) = 0$, $e(x, o_2) = 0$ and $\delta_\alpha(o_1) + 1 - 2\alpha = -\frac{2}{3} < 0$, we have $\deg_G(x) = 1$. Per Proposition 4.31, G is not (α, TF)-homomorphically rigid.
- Dually if $x \in N_G(o_2)$, we get $\deg_G(x) = 1$ and G is not (α, TF)-homomorphically rigid.
- Otherwise $\deg_G(x) = 0$; again G is non-(α, TF)-homomorphically rigid by Proposition 4.31.

□

In fact, for any $\alpha \in (0, \frac{5}{6}]$, any $(\alpha, \text{TF}]$-homomorphically rigid graph does not contain six consecutive vertices globally of degree 2:

Lemma 4.33. *Let* $\alpha \in (0, \frac{5}{6}], G \in \mathcal{K}_{\alpha,TF}$ *with* $e(G), v(G) \geq 2$. *If*

$$(*) \text{ There exists } x_1 - x_2 - \cdots - x_6 \subseteq G, \text{ with each } \deg_G(x_i) = 2, i \in [1, 6]$$

Then

$$(**)\ G \text{ is not } (\alpha, \text{TF}) - \text{homomorphically rigid}$$

Proof. Let G be as described and assume $(*)$. Let $y \in! \big(N_G(x_6) \smallsetminus \{x_5\}\big)$.

1. If $y \in \{x_1, \cdots, x_4\}$: if $y = x_2, x_3$ or x_4 then $N_G(y) \supseteq \{x_{i-1}, x_{i+1<6}, x_6\} \implies \deg_G(y) \geq 3$, contradicting the assumption; hence $y = x_1$, and x_1-x_2-$\cdots$-x_6-x_1 is

a copy of C_6 disconnected from $G \smallsetminus \{x_1, \cdots, x_6\}$, by the degree assumptions. It follows that $G \twoheadrightarrow G \smallsetminus \{x_1, \cdots, x_6\} \in \mathcal{K}_{\alpha,\text{TF}}$, for $C_6 \to P_2 \hookrightarrow G \smallsetminus \{x_1, \cdots, x_6\}$. Therefore G is not (α, TF)-homomorphically rigid, as desired.

2. If $\deg_G(y) = 1$, per Proposition 4.31, G is not (α, TF)-homomorphically rigid.

3. If $\deg_G(y) \geq 2$, $y \notin \{x_1, \cdots, x_5, x_6\}$ and $N_G(y) \smallsetminus \{x_6\} \subseteq \{x_1, \cdots, x_5\}$, the only possible case is $N_G(y) = \{x_1, x_6\}$. Indeed, if $x_i := x_2, x_3, x_4$, or resp. $x_5 \in N_G(y) \smallsetminus \{x_6\}$ then that x_i has degree ≥ 3, its neighbors containing x_{i-1}, x_{i+1} and y.

   Now when $N_G(y) = \{x_1, x_6\}$, $G[\{x_1, \cdots, x_6, y\}] \cong C_7$. Furthermore each vertex of this copy of C_7 has degree 2 in G, so $G[\{x_1, \cdots, x_6, y\}]$ is isolated. It follows that $G \twoheadrightarrow \big((G \smallsetminus \{x_1, \cdots, x_6, y\}) \coprod C_5\big) \in \mathcal{K}_{\alpha,\text{TF}}$ via $C_7 \twoheadrightarrow C_5$.

4. We are left with the case where $\deg_G(y) \geq 2, y \notin \{x_1, \cdots, x_6\}$, and $\exists x \in N_G(y) \smallsetminus \{x_1, \cdots, x_6\}$. The idea is two do a "one-step" collapse that reduces the vertex and edge count both by 1. Consider the quotient map $h : G \twoheadrightarrow G/(x_6 \sim x)$.

   (a) G/h is triangle-free: assume not. Then x_6–$*$–$*$–$x \subseteq G$. Since $N_G(x_6) = \{y, x_5\}$, that copy of P_4 is either x_6–x_5–$*$–x or x_6–y–$*$–x. The latter is impossible, since $((y, x) \in E(G)) \wedge G \in \text{TF}$; the former forces $* = x_4$ and $x = x_3$. But then $N_G(x_3) \supseteq \{x_4, x_2, y\}$, contradicting $\deg_G(x_3) = 2$.

   (b) For any $H \subseteq_{\text{ind}} G/h$:

      - if $V(H) \cap \{x_1, \cdots, x_5\} \neq \emptyset$: then $\delta_\alpha(H) = \delta_\alpha(H \cap \{x_1, \cdots, x_5\}) + \delta_\alpha(\underbrace{H \smallsetminus \{x_1, \cdots, x_5\}}_{\subseteq G}) - \alpha e(H \cap \{x_1, \cdots, x_5\}, H \smallsetminus \{x_1, \cdots, x_5\}) \geq \delta_\alpha(H \cap$

$\{x_1, \cdots, x_5\}) - \alpha e(H \cap \{x_1, \cdots, x_5\}, H \smallsetminus \{x_1, \cdots, x_5\})$.

  - If $x_1, x_5 \in V(H)$: then $H \cap \{x_1, \cdots, x_5\}$ induces a subgraph of P_5 containing both ends. Such subgraphs have $\delta_\alpha \geq \delta_\alpha(P_5) = 5 - 4\alpha$. It follows that $\delta_\alpha(H \cap \{x_1, \cdots, x_5\}) - \alpha e(H \cap \{x_1, \cdots, x_5\}, H \smallsetminus \{x_1, \cdots, x_5\}) \geq 5 - 4\alpha - 2\alpha \geq 0$.
  - If exactly one of $x_1, x_5 \in V(H)$: then $H \cap \{x_1, \cdots, x_5\}$ induces a subgraph of P_4 containing at least one end, whose δ_α is at least 1; on the other hand, $e(H \cap \{x_1, \cdots, x_5\}, H \smallsetminus \{x_1, \cdots, x_5\}) \leq 1$. Thus $\delta_\alpha(H \cap \{x_1, \cdots, x_5\}) - \alpha e(H \cap \{x_1, \cdots, x_5\}, H \smallsetminus \{x_1, \cdots, x_5\}) \geq 1 - 1\alpha > 0$.
  - If $x_1, x_5 \notin V(H)$ then $e(H \cap \{x_1, \cdots, x_5\}, H \smallsetminus \{x_1, \cdots, x_5\}) = 0$; since $H \cap \{x_1, \cdots, x_5\} \subseteq P_3$, its $\delta_\alpha \geq 0$, so $\delta_\alpha(H \cap \{x_1, \cdots, x_5\}) - \alpha e(H \cap \{x_1, \cdots, x_5\}, H \smallsetminus \{x_1, \cdots, x_5\}) \geq 0$.

- If $V(H) \cap \{x_1, \cdots, x_5\} = \emptyset$ then $H \cong G[h^{-1}(V(H)) \smallsetminus \{x_6\}] \in \mathcal{K}_{\alpha,\text{TF}}$.

Having examined every possible case we conclude that $G \models (**)$, as desired. [28] $\square$

Note also that the number "6" in Lemma 4.33 is tight at least for $\alpha \in (\frac{24}{29}, \frac{5}{6}]$, witnessed by the (α, TF)-homomorphic rigidity of (E_1)–5–(E_2) (Corollary 4.20). Noting that $(5+1)(\lceil \frac{t}{6} \rceil - 1) < t$ when $t > 6$, the following is also immediate:

Corollary 4.34. *Let $\alpha \in (0, \frac{5}{6}]$, $G \in \mathcal{K}_{\alpha,TF}$ be (α, TF)-homomorphically rigid and o*

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[28] It is worthy of noting that any finite graph $G \models (*) \implies C_6, C_7, C_8$ or P_8 embeds into G but not necessarily the converse. Assume LHS and let $x_0 := N_G(x_1) \smallsetminus \{x_2\}, x_7 := N_G(x_6) \smallsetminus \{x_5\}$; the first two subcases were discussed: (a) $x_0 = x_6 \iff x_7 = x_1$, in which case $G[\{x_0, \cdots, x_7\}] \cong C_6$; (b) $x_0 \neq x_6$ but $x_0 = x_7$, whence $G[x_0, \cdots, x_7] \cong C_7$; otherwise depending on whether $(x_0, x_7) \in E(G)$, we have $G[\{x_0, x_1, \cdots, x_7\}] \cong C_8 \vee G[\{x_0, x_1, \cdots, x_7\}] \cong P_8$. For counter example of the converse, take two copies of induced C_8 then add edges $\{(x_{i,\text{inner}}, x_{i,\text{outer}})\}_{i \in [8]}$. This graph is regular of degree 3.

a copy of cycle C_t as subgraph in G with $t > 6$. Then there are at least $\lceil \frac{t}{6} \rceil$ vertices $y_1, \cdots, y_{\lceil \frac{t}{6} \rceil} \in Im(o)$ such that each $\deg_G(y_j) \geq 3$ ($j \in [1, \lceil \frac{t}{6} \rceil]$).[29]

In addition,

Corollary 4.35. *Let $\mathcal{S}_0$ be defined as in Theorem 4.29. Define*

$$\mathcal{T} := \left\{ G \in \mathcal{S}_0 : \forall x \in V(G - o_1 \coprod o_2), \deg_G(x) = 2 \right\}.$$

Then $|\mathcal{T} \cap (\frac{5}{6}, TF) - homomorphically\ rigid| \leq 2^{100} - 1$, and $G \setminus o_1 \coprod o_2 \cong \coprod_{[k]} P_5$ for some $k \leq 100$ whenever $G \in \mathcal{T}$, o_i the unique copy of E_i in G.

Proof. If $G \setminus (o_1 \coprod o_2) = \emptyset$ then G is disconnected and so $G \notin \mathcal{T}$. Take $G \in \mathcal{T} \cap (\frac{5}{6}, \text{TF})$-homomorphically-rigid; let $x \in V(G - o_1 \coprod o_2)$ and $C_x \subseteq_{\text{ind}} G \setminus o_1 \coprod o_2$ be the component of $G \setminus o_1 \coprod o_2$ containing x.

Claim 4.36. $|V(C_x) \cap N_G(o_1)| = |V(C_x) \cap N_G(o_2)| = 1$.

Proof of claim. $|V(C_x) \cap N_G(o_1)|, |V(C_x) \cap N_G(o_2)| \geq 1$ because G is connected.

Assume for the sake of contradiction that $|V(C_x) \cap N_G(o_1)| \geq 2$ then let $a \neq b \in V(C_x) \cap N_G(o_1)$, $c \in V(C_x) \cap N_G(o_2)$. Note $c \notin \{a, b\}$ for $N_G(o_1) \cap N_G(o_2) = \emptyset$. Then there exists simple paths $bp_1c, ap_2c \subseteq C_x \subseteq G \setminus o_1 \coprod o_2$. If p_1, p_2 share some vertex other than c, then there must be $v' \in (p_1 \cap p_2) \setminus \{c\}$ with $3 \leq \deg_{G \setminus o_1 \coprod o_2}(v') \leq \deg_G(v')$; therefore $p_1 \cap p_2 = c$ but then $\deg_G(c) \geq 3$ because $c \in N_G(o_2)$ while $p_1, p_2 \cap o_2 = \emptyset$. Likewise we can disprove $|N_G(o_2) \cap V(C_x)| \geq 2$. □

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[29] let the number of $\deg_G \geq 3$ vertices in o be t'; these vertices may dissect o into less than t' "($\deg_G = 2$)-arcs" if two of them are next to each other, but there are no more than t' such arcs. So the total number of vertices in o is, by Lemma 4.33, at most $5 \times t' + t'$.

On the other hand, let $x_0 \in! V(C_x) \cap N_G(o_1)$ then $\deg_G(x_0) = 2$, for otherwise $\delta_\alpha(G[o_1 \cup \{x_0\}]) \leq 0 + 1 - 2\alpha < 0$. So C_x has degree sequence $1, 2, \cdots, 2, 1$ in $G \setminus o_1 \coprod o_2$ and is a path: indeed, send the two deg-1 vertices to $1, n \in P_n$ for $n := v(C_x)$ and observe that this extends uniquely to an isomorphism. Since $G[o_1 \coprod C_x \coprod o_2] \in \mathcal{K}_{\alpha,\text{TF}}$, $n - (n+1)\alpha + \delta_\alpha(o_1 \coprod o_2) \geq 0 \implies n \geq 5$; but (α, TF)-homorigidity requires $n < 6$, by Lemma 4.33. It follows that $C_x \cong P_5$.

Lastly, if for some $(x_0, y_0) \in V(o_1) \times V(o_2)$ there are components $C_1 \neq C_2 \subseteq G \setminus o_1 \coprod o_2$ both connected to (x_0, y_0), G will not be $(\frac{5}{6}, \text{TF})$-homomorphically rigid, for $G/(C_1 \sim C_2) \cong G \setminus C_1 \in \mathcal{K}_{\alpha,\text{TF}}$. Hence components of $G \setminus o_1 \coprod o_2$ can be identified by $(x, y) \in V(o_1) \times V(o_2)$ the unique pair of connection points in (o_1, o_2); there are at most $\sum_{1 \leq p \leq |V(o_1) \times V(o_2)|} \binom{V(o_1 \times V(o_2))}{p} = 2^{100} - 1$ such graphs up to isomorphism. $\square$

Recall that $N_{G,\leq k}(H) := \left\{x \in G : \text{dist}_G(\{x\}, H) \leq k\right\}$ the collection of vertices in G of distance at most k from H and it also denotes the induced subgraph $G[N_{G,\leq k}(H)]$. We formulate the intuition that (for large α) the neighborhood of a marginally dense subgraph must be very sparse. The observation below in particular applies to any $B \in \mathcal{S}_0$ with o_i the unique copy of E_i in B, as $\delta_\alpha(o_i) = 0$ for $\alpha := \frac{5}{6}$.

Lemma 4.37. *Let $\alpha \in (\frac{4}{5}, \frac{5}{6}]$, $H \subseteq G \in \mathcal{K}_{\alpha,\textit{TF}}$, $\delta_\alpha(H) = 0$. Then $N_{G,\leq 2}(H) \setminus E(H)$ is a forest consisting of $v(H)$ trees.*

Proof. Let $G' := N_{G,\leq 2}(H) \setminus E(H)$. Note that $v(H) \geq 5$ and $e(H) \geq 6$ by $\delta_\alpha(H) = 0$.

4.4.2.0.1 (1) For $x \neq y \in V(H)$, $N_{G',\leq 2}(\{x\})$ and $N_{G',\leq 2}(\{y\})$ are disconnected Assume otherwise, then $N_{G',\leq 2}(\{x\}) \cap N_{G',\leq 2}(\{y\}) \neq \emptyset$ for some $x \neq y \in V(H)$, or

$\Big(\big((N_{G',\leq 2}(\{x\}) \cap N_{G',\leq 2}(\{y\}) = \emptyset\big) \wedge E_{G'}((N_{G',\leq 2}(\{x\}), N_{G',\leq 2}(\{y\}) \neq \emptyset\Big)$ for some $x \neq y \in V(H)$.

1. Case (a): let $z \in N_{G',\leq 2}(\{x\}) \cap N_{G',\leq 2}(\{y\})$. There is a path $p_1 : x\text{--}\cdots\text{--}z$ and $p_2 : y\text{--}\cdots\text{--}z \subseteq G' \subseteq G$, each consisting of at most 2 edges. Therefore $\delta_\alpha(G[H \cup p_1 \cup p_2]) \leq \delta_\alpha(H) + 3 - 4\alpha < 0$.

2. Case (b): similar to the above. Let $z \in N_{G',\leq 2}(\{x\}), z' \in N_{G',\leq 2}(\{y\})$ be s.t. $(z, z') \in E(G') \subseteq E(G)$. There is a path $p_1 : x\text{--}\cdots\text{--}z$ and $p_2 : y\text{--}\cdots\text{--}z' \subseteq G' \subseteq G$, each consisting of at most 2 edges. Therefore $\delta_\alpha(G[H \cup p_1 \cup p_2]) \leq \delta_\alpha(H) + 4 - (2 \times 2 + 1)\alpha < 0$.

   Either case derives contradiction, as desired.

4.4.2.0.2 (2) For $x \in V(H)$, $N_{G',\leq 2}(\{x\})$ is connected This is by definition of $N_{G',\leq 2}(\{x\})$: indeed, for any $y \neq z \in N_{G',\leq 2}(\{x\})$, there is a path from y to z by joining the path $x\text{--}\cdots\text{--}y$ and $x\text{--}\cdots\text{--}z$.

Therefore $G' = \coprod_{x \in V(H)} N_{G',\leq 2}(\{x\})$, each summand being its own connected component.

4.4.2.0.3 (3) Each $N_{G',\leq 2}(\{x\})$ for $x \in V(H)$ is acyclic Assume otherwise, then $N_{G',\leq 2}(\{x\})$ contains a cycle. Since $N_{G',\leq 2}(\{x\}) \subseteq G' \subseteq G \in \text{TF}$, the cycle has size ≥ 4. By definition of $N_{G',\leq 2}(\{x\})$, this means that, regardless of whether x is in the cycle, there exists a path in G' of length at least 3 starting from x: that is, $x\text{--}a\text{--}b\text{--}c \subseteq G'$, $|\{x, a, b, c\}| = 4$. Note that $c \in N_{G',\leq 2}(\{x\})$, so:

1. if $\mathrm{dist}_{G'}(c, x) = 1$, $\delta_\alpha(G[H \cup \{a, b, c\}]) \leq 3 - 4\alpha < 0$.

2. if $\mathrm{dist}_{G'}(c, x) = 2$ witnessed by a path p, $\delta_\alpha(G[H \cup \{a, b, p\}]) \leq 4 - 5\alpha < 0$.

   Each case induces a contradiction, as desired.

$\square$

Next we show that for some α, (α, TF)-homomorphically rigid graphs have tighter bounds on local density of substructures. Per Fact 4.7 and the fact that $H \in \mathcal{K}_\alpha \iff \mathrm{mad}(H) \leq \frac{2}{\alpha}$, we have whenever $\alpha \in (\frac{2}{3}, 1]$, any $G \in \mathcal{K}_{\alpha,\mathrm{TF}}$ has $\chi(G) \leq 3$. Now,

Lemma 4.38. *Fix $\alpha := \frac{5}{6}$. Let $G \in \mathcal{K}_{\alpha,TF}$ be of type $(1, 1)$ and let o_1, o_2 denote the unique copy of E_1, E_2, respectively.*

If:

$$(*) \text{ there exists } H \subseteq \big(G \smallsetminus (o_1 \coprod o_2)\big) \text{ with } \delta_\alpha(H) < \frac{1}{2},$$

then

$$(**)\ G \text{ is } \textit{not } (\alpha, \textit{TF}) - \text{homomorphically rigid.}$$

Proof. Let G be as described and assume $(*)$. Since $4-4\alpha > 3-3\alpha \geq \frac{1}{2}$ and $H \in \mathcal{K}_{\alpha,\mathrm{TF}}$, one may manually check that either $v(H) \geq 6$ or $H \cong K_{3,2}$. Furthermore,

Claim 4.39. *there does not exist $x \neq y \in V(H)$, $z, t \in V(G) \smallsetminus V(H)$, s.t. ($x$–$z$–$t$–$y \subseteq G$ or x–z–$y \subseteq G$).*

Proof of Claim 4.39. Otherwise consider the induced subgraph $H' := G[V(H) \cup \{z, t\}]$. It has $\delta_\alpha(H') = \delta_\alpha(H) + 2 - \alpha - \alpha e(\{z, t\}, H) \leq \delta_\alpha(H) + 2 - 3\alpha \leq \delta_\alpha(H) - \frac{1}{2} < 0$, contradiction.

Similarly if some $x \neq y \in V(H)$ with common neighbor $z \notin V(H)$, then $\delta_\alpha(G[V(H) \cup \{z\}]) \leq \delta_\alpha(H) + 1 - 2\alpha < \delta_\alpha(H) + 2 - 3\alpha < 0$. $\square$

Now we show that $H \cong K_{3,2} \implies$ there exists a $\mathcal{K}_{\frac{5}{6},\text{TF}}$-homomorphism h from H that induces a quotient h' on G, s.t. $G/h' \in \mathcal{K}_{\frac{5}{6},\text{TF}}$, and $v(H) \geq 6 \implies$ a similar (h, h') as described above exists. Together this will imply $G \models (**)$.

4.4.2.0.4 **Case 1:** $H \cong K_{3,2}$ Any homomorphism $h : H \to P_2$ is non-injective, and in fact surjective for chromatic number concerns. Take one such h and let h' be the quotient map on G induced by h,[30]

Assume not. Then one of three subcases would happen:

1. $G/h' \notin$ TF;
2. $G/h' \in$ TF, but there exists $H' \subseteq G/h'$ s.t. $H' \supseteq \text{Im}(h' \restriction_H)$ (i.e. H' contains that particular copy of P_2 as the image of H), and $\delta_\alpha(H') < 0$;
3. $G/h' \in$ TF, and for every $H'' \subseteq G/h'$ s.t. $H'' \supseteq \text{Im}(h' \restriction_H)$ we have $\delta_\alpha(H'') \geq 0$, yet there exists an $H' \subseteq G/h'$, $H' \not\supseteq \text{Im}(h' \restriction_H)$, with $\delta_\alpha(H') < 0$.

However none of the subcases could happen:

4.4.2.0.4.1 Subcase 1(1) Assume this, and let a, b, c induce a triangle in G/h'. Then either one or two out of the three vertices are in $\text{Im}(h' \restriction_H)$.

1. If (without loss of generality) only a is in $\text{Im}(h' \restriction_H)$, then there must be $x \neq y \in (h')^{-1}(a)$, s.t. x–b–c–$y \subseteq G \in$ TF. This violates Claim 4.39.

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[30] That is, $x \mapsto h(x)$ if $x \in V(H)$, id elsewhere. Note that for $x \in V(G), y \in V(G) \setminus V(H)$, $E_{G/h}(h(x), y) \iff \big(\exists x_0 \in h^{-1}(x), E_G(x_0, y)\big)$.

2. Otherwise, without loss of generality $\{a, b\} = V(\text{Im}(h' \restriction_H))$. But then there exists $a^* \in (h')^{-1}(a), b^* \in (h')^{-1}(b)$ s.t. a^*–c–$b^* \subseteq G$. This also violates Claim 4.39.

4.4.2.0.4.2 Subcase 1(2) Assume this. Then $H' \supseteq \text{Im}(h' \restriction_H) = P_2 \in \mathcal{K}_{\alpha,\text{TF}}$. Per Lemma 4.24,

$$\delta_\alpha(G[(h')^{-1}(H')]) - \delta_\alpha(\underbrace{(h')^{-1}(P_2)}_{H}) \leq \delta_\alpha(H') - \delta_\alpha(\underbrace{(G/h')[V(H') \cap P_2]}_{P_2})$$

Hence $\delta_\alpha(\delta_\alpha(G[(h')^{-1}(H')])) \leq \delta_\alpha(H') - (2 - \alpha) + \delta_\alpha(H) < \delta_\alpha(H') - 1 + \frac{1}{2} < -\frac{1}{2}$ while $G \in \mathcal{K}_{\alpha,\text{TF}}$, contradiction.

4.4.2.0.4.3 Subcase 1(3) Assume this. It is without loss of generality to further assume H' is induced. Also, $H' \nsubseteq \text{Im}(h' \restriction_H)$ and $H' \nsubseteq G[V(G) \smallsetminus \text{Im}(h' \restriction_H)]$, for both RHSs are in $\mathcal{K}_{\frac{5}{6},\text{TF}}$. Combined with the assumption that $\text{Im}(h' \restriction_H) \nsubseteq H'$ and $v(\text{Im}(h' \restriction_H)) = 2$, we have $|V(\text{Im}(h' \restriction_H)) \cap H'| = 1$. We denote this vertex by a. An application of Lemma 4.24(1) and (3) gives

$$\delta_\alpha(G[(h')^{-1}(H') \cup H]) - \delta_\alpha(H) \leq \delta_\alpha(H') - \delta_\alpha(\{a\})$$

Therefore $\delta_\alpha(G[(h')^{-1}(H') \cup H]) \leq \delta_\alpha(H') - 1 + \delta_\alpha(H) < \delta_\alpha(H') - \frac{1}{2} < 0$, contradiction.

We conclude, as desired, that $H \cong K_{3,2} \implies$ there exists a $\mathcal{K}_{\frac{5}{6},\text{TF}}$-homomorphism h from H that induces a quotient h' on G, s.t. $G/h' \in \mathcal{K}_{\frac{5}{6},\text{TF}}$.

Now $\chi(H) > 1$ because otherwise H contains no edges and $\delta_\alpha(H) \geq 1 > \frac{1}{2}$. On the other hand, $H \in \mathcal{K}_{\alpha,\text{TF}}$ so by the discussion above the statement of this Proposition,

$\chi(H) \leq 3$.

4.4.2.0.5 **Case 2**: $v(H) \geq 6$, $\chi(H) = 2$. Once again H non-injectively surjects to P_2 and it is routine to verify that similar argument to the above carries.

4.4.2.0.6 **Case 3**: $v(H) \geq 6, \chi(H) = 3$. Any homomorphism $h : H \to C_5$ is non-injective. Such homomorphisms must exist, for $H \in \mathcal{K}_{\frac{5}{6},\text{TF}}$ has type $(0, 0)$ (Proposition 4.12). h is once again surjective for chromatic number concerns. Take one such h and let h' be the quotient map on G induced by h. which again is non-injective. I claim that $G/h' \in \mathcal{K}_{\alpha,\text{TF}}$.

Assume not; then again we have three subcases whose statements are identical to Subcases 1(1)∼ 1(3).

4.4.2.0.6.1 Subcase 2(1) The refutation of this subcase is almost identical to Subcase 1(1), with "without loss of generality $\{a, b\} = V(\text{Im}(h' \restriction_H))$" changed to "without loss of generality $\{a, b\} \subseteq V(\text{Im}(h' \restriction_H))$ and $c \notin V(H)$".

4.4.2.0.6.2 Subcase 2(2) Almost identical to Subcase 1(2) too, with P_2 changed to C_5 and $2 - \alpha$ changed to $5 - 5\alpha \geq \frac{5}{6} > \frac{1}{2} > \delta_\alpha(H)$.

4.4.2.0.6.3 Subcase 2(3) Substituting H' with $(G/h')[V(H')]$ if necessary, which has $\delta_\alpha < 0$ once the original H' is so, one may assume H' is induced. By Lemma 4.24(1)(3) again,

$$\delta_\alpha(H') - \delta_\alpha(H' \cap \mathrm{Im}(h' \restriction_H)) \geq \delta_\alpha((G/h')[H' \cup \mathrm{Im}(h' \restriction_H)]) - \delta_\alpha(\underbrace{\mathrm{Im}(h' \restriction_H)}_{\cong C_5})$$

But then $\delta_\alpha(H') \geq \delta_\alpha((G/h')[H' \cup \mathrm{Im}(h' \restriction_H)]) + \delta_\alpha(H' \cap \mathrm{Im}(h' \restriction_H)) - \delta_\alpha(\mathrm{Im}(h' \restriction_H))$. Observe that whenever $\alpha \in [\frac{4}{5}, \frac{5}{6}]$, $\delta_\alpha(C_5) = \delta_\alpha(\mathrm{Im}(h' \restriction_H)) = 5 - 5\alpha \leq 1 \leq \delta_\alpha(S)$ for any $S \subsetneq C_5$, so $\delta_\alpha(H' \cap \mathrm{Im}(h' \restriction_H)) - \delta_\alpha(\mathrm{Im}(h' \restriction_H)) \geq 0$; the first term $\delta_\alpha((G/h')[H' \cup \mathrm{Im}(h' \restriction_H)]) \geq 0$ by $G \in \mathcal{K}_{\alpha,\mathrm{TF}}$. Therefore $\delta_\alpha(H') \geq 0$, contradiction.

With all cases covered, we conclude that (*) $\implies$ (**), as desired. ☐

Our derivation above shall work for a wider range of α when we impose stricter conditions: this witnesses a trade-off between α-range and structural freedoms. Let *"odd-K_4", "odd-K_3^2"*, two types of subdivisions of K_4 and C_3, be defined as in Figure 4.3 from [2]; note that E_1, E_2 are instances of odd-K_4.

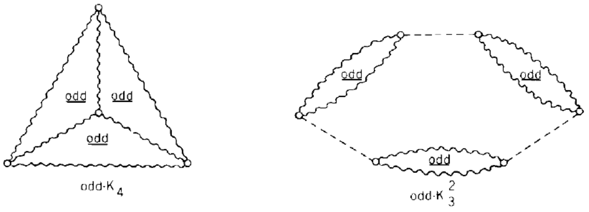

Figure 4.3: "Odd-K_4" and "Odd-K_3^2"; figure credit: [2]. Wriggled and dotted lines stand for (pairwise only disjoint) paths. Dotted lines may have length zero. Wriggled lines have positive length. The word "odd" in a face indicates that the surrounding cycle is odd.

Porism 4.40. *Fix $\alpha \in [\frac{4}{5}, \frac{5}{6}]$. Let $G \in \mathcal{K}_{\alpha,TF}$. If:*

$$(*) \text{ there exists } H \subseteq G \text{ with } \Big(\delta_\alpha(H) < \frac{2}{5} \bigwedge H \textit{ is free of odd-}K_4,\ \textit{odd-}K_3^2\Big),$$

then

$$(**)\ G \text{ is } \textit{not } (\alpha, \textit{TF}) - \text{homomorphically rigid.}$$

Proof. Assume G is as described and satisfies $(*)$. Since $3-3\alpha \geq \frac{1}{2}$ and $H \in \mathcal{K}_{\alpha,\text{TF}}$, $\delta_\alpha(H) < \frac{2}{5} < 1$, once again $\chi(H) \in \{2,3\}$ and either $v(H) \geq 6$ or $H \cong K_{3,2}$. Claim 4.39 holds, because $\delta_\alpha(H) + 2 - 3\alpha < \frac{2}{5} + 2 - 3\alpha \leq 0$. My proofs of **Cases 1,2** therefore then go through.

For **Case 3** ($\chi(H) = 3, v(H) \geq 6$), then as H is non-bipartite and free of odd-K_4 and odd-K_3^2, $H \twoheadrightarrow C_{2k+1} \twoheadrightarrow C_5$ where $C_{2k+1 \geq 5}$ is the shortest odd cycle in $H \in$ TF [2]. The rest of my proof of Lemma 4.38 goes through for $\alpha \in [\frac{4}{5}, \frac{5}{6}]$ (whence $\delta_\alpha(C_5) \leq \delta_\alpha(S)$ for any $S \subsetneq C_5$). □

Remark 4.41. Recall [55] proves that if H is C_5-critical, $4v(H) - 5e(H) \leq 2$. Our Porism 4.40 provides a conditional "converse": for $\alpha \in (\frac{4}{5}, \frac{5}{6}]$, if G is (α, TF)-homomorphically rigid, then $\forall H \subseteq G$, $\big((H \text{ is odd-}K_4, \text{odd-}K_3^2\text{-free}) \text{ implies } (4v(H) - 5e(H) = 5(v(H) - \frac{4}{5}e(H)) > 5\delta_\alpha(H) \geq 2)\big)$.

We collect all description about homomorphic presentation of $\mathcal{K}_{\alpha,\text{TF}}$ which we just proved using properties of (α, TF)-homomorphic rigid graphs.

Theorem 4.42.

1. *Fix* $\alpha := \frac{5}{6}$. *There exists a homomorphic presentation of* $\mathcal{B} \subseteq \mathcal{K}_{\alpha,\textit{TF}}$ *such that*
   - *(a)* $\mathcal{B} \subseteq \mathcal{S}_0$ *(my Theorem 4.29), and maintains the property that* $\Big(\mathcal{B}$ *is finite* $\iff \mathcal{K}_{\alpha,\textit{TF}}$ *admits a finite presentation*$\Big)$*; furthermore,*
   - *(b)* $\forall B \in \mathcal{B}$, *with* o_1, o_2 *respectively the unique copy of* E_1, E_2 *in* B*:*

        i. $\min_{x \in V(B)} \deg_G(x) = 2$, [31]

        ii. *B does not contain 6 consecutive vertices globally of degree 2,*

        iii. *$N_{B,\leq 2}(o_i) \smallsetminus E(o_i)$ is a forest of 10 trees $(\forall i \in [2])$; moreover for $x \in V(o_1), y \in V(o_2)$, the two trees rooted at x, y are disjoint.*[32] *and*

        iv. *$\chi(B \smallsetminus o_1 \coprod o_2) \in \{2, 3\}$, and for each $H \subseteq (B \smallsetminus o_1 \coprod o_2)$, $\delta_\alpha(H) \geq \frac{1}{2}$;*

        v. *$(B \smallsetminus o_1 \coprod o_2) \to C_5$;*[33]

    (c) *Moreover, $\exists B \in \mathcal{B}$ s.t.*

        i. *Emb$\big((E_1)_*\text{–}5\text{–}_*(E_2), B)\big) \neq \emptyset$, and*

        ii. *for every $(m, n) \in \{(1, 4), (2, 3), (3, 2), (4, 1)\}$, Emb$\big((E_1)_*\text{–}m\text{–}_*(C_5)_*\text{–}n\text{–}_*(E_2), B\big) \neq \emptyset$.*

2. *Let $\alpha \in (0, \frac{5}{6}]$. Then there exists a homomorphic presentation $\mathcal{B} \subseteq \mathcal{K}_{\alpha,TF}$ s.t.*

    (a) *$\forall B \in \mathcal{B}$ with $v(B), e(B) \geq 2$:*

        i. *$\min_{x \in V(B)} \deg_G(x) \geq 2$ $\big(\min_{x \in V(B)} \deg_G(x) = 2$ if $\alpha > \frac{2}{3}\big)$, and $\chi(B) \in [2, k_\alpha]$ with $k_\alpha \in! (\frac{2}{\alpha}, \frac{2}{\alpha} + 1] \cap \mathbb{N}$;*

        ii. *B does not contain 6 consecutive vertices globally of degree 2,*

    (b) *If $\alpha \in [\frac{4}{5}, \frac{5}{6}]$, then* in addition, *$\forall B \in \mathcal{B}$,*

        i. *if $H \subseteq B$ is free of odd-K_4 and free of odd-K_3^2, then $\delta_\alpha(H) \geq \frac{2}{5}$,*

        ii. *if $\alpha \neq \frac{4}{5}$, then for $H \subseteq B$, $\delta_\alpha(H) = 0 \implies N_{B,\leq 2}(H) \smallsetminus E(H)$ is a forest of $v(H)$ trees.*

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[31] Note that $\min_{x \in B} \deg_B(x) \leq \mathrm{ad}(B) \leq \mathrm{mad}(B) \leq \frac{2}{\alpha}$.

[32] This follows from $\mathrm{dist}_B(o_1, o_2) \geq 6$. In fact denote those two trees by $T_x := N_{B\leq 2}(\{x\}) \smallsetminus E(o_1), T_y := N_{B\leq 2}(\{y\}) \smallsetminus E(o_2)$, we even have $E_B(T_x, T_y) = \emptyset$ for otherwise there will be a path of length 5 connecting E_1, E_2.

[33] However note that e.g. $(E_1)\text{–}0\text{–}(C_5)\text{–}0\text{–}(E_2) \cap \mathcal{K}_{\frac{5}{6},\mathrm{TF}} = \emptyset$.

(c) *If* $\exists\vec{H}$ *as described in Theorem* 4.29(2), *then* in addition *one may take* $\mathcal{B} \subseteq \mathcal{S}_{0,\vec{H}}$.

Proof. Only Part (1c) requires further arguments. Let $\mathcal{A} := (E_1)–5–(E_2) \cup \big((E_1)–m–C_5–n–(E_2)\big)_{m+n=5,m,n\geq 0}$, and let $G_0 := \coprod \mathcal{A}$. Each component of G_0 is either a member of $(E_1)–5–(E_2)$, which is in $\mathcal{K}_{\alpha,\mathrm{TF}}$ (Corollary 4.20), or of $(E_1)–m–C_5–n–(E_2)$, whose $\mathcal{K}_{\alpha,\mathrm{TF}}$-membership is routine to check.[34] Thus $G_0 \in \mathcal{K}_{\alpha,\mathrm{TF}}$. Let $G' := \frac{G}{\utilde{\mathrm{Hom}}_{\mathrm{inj}}(E_1,G),\utilde{\mathrm{Hom}}_{\mathrm{inj}}(E_2,G)}$; per Lemma 4.25, $G' \in \mathcal{K}_{\alpha,\mathrm{TF}}$. Each component of G_0 embeds into G' naturally; G' is connected, of $(1,1)$-type, and one may take $B :=$ any (α,TF)-homomorphically rigid graph s.t. $\mathrm{Hom}(G',B) \neq \emptyset$, which always exists by finiteness of G'.[35]

It remains to show that the homomorphisms from $(E_1)_*–5–_*(E_2)$ and $(E_1)_*–m–{}_*(C_5)_*–n–_*(E_2)$ to B via G' remain embeddings. As $\delta_\alpha((E_1)_*–5–_*(E_2)) = 0$ and $(E_1)_*–5–_*(E_2)$ is $(\frac{5}{6},\mathrm{TF})$-homomorphically rigid (Corollary 4.20), the claim for $(E_1)_*–5–_*(E_2)$ follows by Proposition 4.18; as for each $G \in (E_1)\text{-}m\text{-}(C_5)\text{-}n\text{-}(E_2)$, per Proposition 4.18 again and the fact that $\delta_{\frac{5}{6}}(G) = 0$ it suffices to show such G is $(\frac{5}{6},\mathrm{TF})$-homomorphically rigid. A more general statement is proved below, from which our theorem follows. ☐

Claim 4.43. $(E_1)–m–(C_5)–n–(E_2)$ *are* (α, TF)*-homomorphically rigid for* $\alpha \in (\frac{29}{35},\frac{5}{6}]$, $(m,n) \in \{(1,4),(2,3),(3,2),(4,1)\}$.

Proof. Let $G \in (E_1)–m–(C_5)–n–(E_2)$, o_1, o_2 denote the unique copy of E_1, E_2 in G respectively, and let o_3 denote the unique copy of C_5 in G that is disjoint from o_1, o_2.

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[34] With slight abuse let us use E_1 and E_2 to denote respectively the unique copy of E_i in G respectively, and let C_5 denote the unique copy of C_5 in G that is disjoint from E_1, E_2. G contains no triangle as it merely adds paths across (E_1, C_5) or (C_5, E_2). Let connected $H \subseteq G$; if $H \cap E_1 = \emptyset$ then $\delta_\alpha(H) \geq \delta_\alpha(H \cap E_1) + \delta_\alpha(–0–C_5) = \delta_\alpha(H \cap E_1) + 5 - (1+5)\alpha \geq 0$; dually when $H \cap E_2 = \emptyset$. Otherwise, $H \cap E_1, H \cap E_2 \neq \emptyset$, whence $\delta_\alpha(H) = \delta_\alpha(H \cap E_1) + \delta_\alpha(H \cap E_2) + \delta_\alpha(–m–C_5–n–) = \delta_\alpha(H \cap E_1) + \delta_\alpha(H \cap E_2) + m + n + 5 - (m+n+2+5)\alpha \geq 0$.

[35] And by arguments in proof of Theorem 4.29, homomorphisms preserve connectedness and $(1,1)$-type.

Take arbitrary $B \in \mathcal{K}_{\frac{5}{6},\mathrm{TF}}$ and $h : G \to B$ homomorphism.

First, by Lemma 4.19, $B[ho_1] \cap B[ho_2] = \emptyset$. If $ho_3 \cap ho_1 \neq \emptyset$ and $ho_3 \not\subseteq ho_1$, we'd have $\delta_\alpha(B[ho_1 \cup ho_3]) < 0$ [36] violating $B \in \mathcal{K}_{\frac{5}{6},\mathrm{TF}}$; if $ho_3 \subseteq ho_1$ then $\mathrm{dist}_B(ho_1, ho_2) \leq 4+1 < 6$ violating Lemma 4.19. It follows that $ho_1 \cap ho_3 = \emptyset$, and dually $ho_1 \cap ho_2 = \emptyset$, so $h \restriction_{\mathrm{Im}o_1 \coprod \mathrm{Im}o_2 \coprod \mathrm{Im}o_3}$ is injective.

Now let p be the shortest path in $B[\mathrm{Im}h]$ connecting o_1, o_3, and let q be the shortest path in $B[\mathrm{Im}h]$ connecting o_2, o_3. If $(p \cap q) \smallsetminus ho_3 \neq \emptyset$, then once again $\mathrm{dist}_B(ho_1, ho_2) \leq 5 < 6$, contradicting Lemma 4.19. Therefore $(p \cap q) \smallsetminus ho_3 = \emptyset$. We also have $p \cap ho_2 = q \cap ho_1 = \emptyset$ by distances concern.

Lastly observe that $|p \smallsetminus (ho_1 \coprod ho_3)| \leq m$ and $|q \smallsetminus (ho_2 \coprod ho_3)| \leq n$; by the fact that $\mathrm{dist}_B(ho_i, ho_3) \leq \mathrm{dist}_G(o_i, o_3)$ (per definition of homomorphisms and subgraph distance). if either of the inequalities are strict then $B[ho_1 \cup ho_2 \cup ho_3 \cup p \cup q] \leq 5 + 2 \times 10 + (m+n) - 1 - \alpha(5 + 2 \times 12 + (m+n-1) + 2) = 29 - 35\alpha < 0$, thanks to $(p \cap q) \smallsetminus ho_3 = \emptyset$; so $|p \smallsetminus (ho_1 \coprod ho_3)| = m$ and $|q \smallsetminus (ho_1 \coprod ho_3)| = n$. These two sets are disjoint, so it follows $v(\mathrm{Im}(h)) \geq 2 \times 10 + 5 + m + n = 30 = v(G)$ so that h is injective, as desired.

Note that the argument for their (α, TF)-rigidity in fact goes through for all $\alpha \in (\frac{29}{35}, \frac{5}{6}]$. □

Remark 4.44. We discuss the particular B in Theorem 4.42(1c) because, the E_1-5-E_2 and E_1-m-C_5-n-E_2 are building blocks for our *candidates* towards a finite homomorphic presentation of $\mathcal{K}_{\frac{5}{6},\mathrm{TF}}$. For example, Corollary 4.35 says that the only $(\frac{5}{6}, \mathrm{TF})$-homomorphic

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[36] $\mathrm{Im}ho_1$ would have at least one "Boomerang arc", i.e. a path $xp'y$ with $x, y \in ho_1 \wedge p' \cap ho_1 = \emptyset$, of length ≤ 5 attached to it, formed by members of $ho_3 \smallsetminus (ho_1)$, and $\delta_\alpha(B[\mathrm{Im}ho_1 \cup p']) \leq 0 + 4 - 5\alpha < 0$.

rigids G where the $\deg_G(x) = 2$ for every $x \in V(G \setminus o_1 \coprod o_2)$ comes from gluing members of (E_1)–5–(E_2) along their respective copies of E_1, E_2.

It should also be noted that techniques above do not easily generalize to $\mathcal{K}_{\alpha,K_n-\text{free}}$. Indeed, each graph homomorphism h can be decomposed into a sequence of pairwise quotients followed by an injective homomorphism[37]; In $\mathcal{K}_{\alpha,\text{TF}}$ it is necessary that in any such decomposition, any pairwise quotient is not of the form $G_i/a \sim b$ where a–$*$–$*$–$b \subseteq_{\text{ind}} G_i$. This imotivated some proofs on quotient graphs above, and it is unclear for $n \geq 4$ in terms of finding similar conditions on pairwise-quotient decomposition of homomorphisms.

4.5 Open Problems and Future Directions

It is unfortunate that we are unable to close the case for any $\alpha \in (0, \frac{5}{6}]$ in terms of finitizability of homomorphic presentation of $\mathcal{K}_{\alpha,\text{TF}}$, but I believe my results have brought us closer thereto. In particular:

4.5.0.0.1 ***For*** $\alpha := \frac{5}{6}$: we know more structural information about homomorphic presentation of $\mathcal{K}_{\alpha,\text{TF}}$ than smaller αs. One may even ask:

Problem 4.45. Fix $\alpha := \frac{5}{6}$. Is it true that $\{B\}$ where B is as addressed in Theorem (4.42)(1c) forms a $\mathcal{K}_{\frac{5}{6},\text{TF}}$-presentation?

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[37] For each non-singleton h-fibre, glue the preimage one by one. The quotient graph $(G/(\underbrace{\pi_n \circ \pi_{n-1} \cdots \circ \pi_1}_{\pi})$ has a well-defined natural bijection to hG on their vertices, namely $\iota : x \mapsto h(\pi^{-1}(x))$; edgewise, one may prove by induction on n using definition of induced quotients that $E_{G/\pi}(x_1, x_2) \iff \Big((\exists x_1^* \in \pi^{-1}(x_1), x_2^* \in \pi^{-1}(x_2))E_G(x_1^*, x_2^*)\Big)$, which implies $E_{h_G}(hx_1, hx_2)$. This says $\iota \in \text{Hom}_{\text{inj}}(G/\pi, hG)$.

Evidence supporting a positive answer to Problem 4.45 is *not* limited to our knowledge that (E_1)–5–(E_2), (E_1)–m–(C_5)–n–(E_2) are $(\frac{5}{6}, \text{TF})$-homomorphically rigid. In fact, Let us define a *homomorphic basis* $\mathcal{B}$ as a reduced homomorphic presentation, i.e. a homomorphic presentation $\mathcal{B}$ s.t. $\forall B_1 \neq B_2 \in \mathcal{B}, (B_1 \not\to B_2) \bigwedge (B_2 \not\to B_1)$. Existence of finite presentation implies existence of finite basis, by removing any homomorphically dependent element.

Proposition 4.46. *Let $\alpha \in (0,1]$. If $\mathcal{K}_{\alpha,TF}$ admits a finite homomorphic presentation, then it admits a singleton homomorphic basis, where the unique element is connected.*

Proof. For each α there exists $t_\alpha >> 0$ s.t. $t_\alpha - \alpha(t_\alpha + 1) > 0$. Let $\mathcal{B} = \{B_1, \cdots, B_m\}$ be a finite homomorphic *basis*. Assume for the sake of contradiction that $m \geq 2$, let $B' := (B_1)_*$–t_α–$_*(B_2)_*$–$\cdots$–t_α–$_*(B_m) \in \mathcal{K}_{\alpha,\text{TF}}$ (proof of Theorem 4.29). We must have $\text{Hom}(B', B_i) = \emptyset$ for any $i \in [m]$, for otherwise letting B_{i^*} be the counter example and $j \in [m] \smallsetminus \{i^*\}$ we would have $h \restriction_{B_j} \in \text{Hom}(B_j, B_{i^*})$. But then B' witnesses the failure of $\mathcal{B}$ as a homomorphic presentation, contradiction. □

Therefore, if $\mathcal{K}_{\frac{5}{6},\text{TF}}$ does admit finite presentation and hence singleton basis, then either B explicitly constructed from Theorem 4.42 or some $B' \models \big($Theorem 4.42(1c) and B homomorphically injects to $B'\big)$[38] would be the unique basis element.[39] I also suspect that such $\{B'\}$ is the common $\mathcal{K}_{\alpha,\text{TF}}$-homomorphic basis for every $\alpha \in (\frac{4}{5}, \frac{5}{6}]$. [40]

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[38] There could be other substructures in B'.

[39] A next step could be to check whether the following claims are true: (a) For $\alpha = \frac{5}{6}$ or reasonably below, there are no (α, TF)-homomorphically rigid graphs in (E_1)–x–(C_7)–y–(E_2). (b) $G \in \mathcal{S}_0$ (Theorem (4.42)) implies $\forall x \in V(G \smallsetminus o_1 \coprod o_2)$, $\deg_{G \smallsetminus o_1 \coprod o_2}(x) \leq 3$. Another possible intermediary goal: let G^* be a minimal $(1,1)$-type counter example to the claim that "$\forall G \in \mathcal{K}_{\frac{5}{6},\text{TF}}, \text{Hom}(G, B) \neq \emptyset$ where B is as described in Theorem (4.42)(1c)". Then an induced C_5 in $G \smallsetminus o_1 \coprod o_2$ has at least 3 G-strings attached, where a string is a path of global degrees (≥ 3)–2–$\cdots$–2–(≥ 3). The claim mimics Lemma 15 in [55].

[40] Another possibly interesting question which might give some insight towards finding the homomor-

4.5.0.0.2 ***For*** $\alpha \in (0, \frac{2}{3}]$ All triangle-free graphs regular of degree 3 are in such $\mathcal{K}_{\alpha,\mathrm{TF}}$, which in particular include all *generalized Petersen graphs* $G(n,k)$ where $k < \frac{n}{2}$, $V(G(n,k)) = \{u_i\}_{i\in[n]} \coprod \{v_i\}_{i\in[n]}$, $E(G) = \{(u_i, u_{i+1}),\ (u_i, v_i),\ (v_i, v_{i+k})\}_i$.I doubt if there is a finite homomorphic presentation, and if not, its proof may require a machinery more relaxed and general than (α, TF)-homomorphic rigidity. Here is a possible approach:

Definition 4.47. Fix $\alpha \in (0, \frac{2}{3}]$. Define the partial function $f_\alpha : \omega \times \omega \to \omega$,[41]

$$(n,k) \mapsto \min\big\{v(B) : B \in \mathcal{K}_{\alpha,\mathrm{TF}},\ \text{and } G(n,k) \to B\big\}$$

Problem 4.48. Fix $\alpha \in (0, \frac{2}{3}]$. Let $\omega \times \omega$ admit the dictionary order. Does there exist $\{(a_i, b_i)\}_{i\in\omega}$ an increasing sequence in $\omega \times \omega$, on which $f_\alpha \downarrow$ and is strictly monotone increasing?

Note: a positive answer to Problem 4.48 would disprove the existence of finite presentation for that α.[42] Also, in any range of α, it would be helpful to characterize the sufficient conditions for (α, TF)-homomorphic rigidity, not just necessary conditions.

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phic basis of $\mathcal{K}_{\alpha,\mathrm{TF}}$ for $\alpha \in (\frac{4}{5}, \frac{5}{6}]$ is whether there are any absolutely TF-homomorphically rigids in $\mathrm{CSP}(\mathfrak{M}_{\alpha,\mathrm{TF}})$ for some $\alpha \in (\frac{4}{5}, \frac{5}{6})$ other than C_5 and P_2. Note that SCE_1 is a TF-homomorphically rigid graph in $\mathcal{K}_{\alpha,\mathrm{TF}}$ for $\alpha \in (0, \frac{4}{5}]$, $C_5 \cong A_2, P_2$ are TF-homomorphically rigid graphs in $\mathcal{K}_{\alpha,\mathrm{TF}}$ for $\alpha \in (0,1]$. None of the Andrásfai graphs, except for C_5 and P_2, lie in $\mathrm{CSP}(\mathfrak{M}_{\alpha,\mathrm{TF}})$ for any $\alpha \in (\frac{4}{5}, \frac{5}{6})$. Per [3] and Proposition 4.59(2), one may know the answer to this question by answering if any subgraph of the Petersen's graph $G(5,2)$ is homomorphically rigid, but without further insights this would have to be brute-forced.

Also note: if $B' \in \mathcal{K}_{\frac{5}{6},\mathrm{TF}}$ homomorphically bounds all $\mathcal{K}_{(\frac{4}{5},\frac{5}{6}],\mathrm{TF}} := \bigcup_{\alpha\in(\frac{4}{5},\frac{5}{6}]} \mathcal{K}_{\alpha,\mathrm{TF}}$ then there should be no absolutely TF-homomorphically rigid graphs in $\mathcal{K}_{(\frac{4}{5},\frac{5}{6}),\mathrm{TF}} \setminus \mathcal{K}_{\frac{5}{6},\mathrm{TF}}$ for otherwise B' has to contain an overly-dense subgraph.

[41] The domain is wherever $G(n,k)$ makes sense.

[42] If such $\{(a_i, b_i)\}_i$ exists, it is also necessary that for each $i, 2 \nmid a_i \vee 2 \mid b_i$, because only then will we have $\chi(G(a_i, b_i)) = 3$.

4.6 Appendix

Remark 4.49. We point out that “copies of subgraphs” can be stated in an more algebraically accurate way and we choose our current presentation in the main text for succinctness. Let A, B be (labelled) finite graphs; observe that Aut(A) acts on $\text{Hom}_{\text{inj}}(A, B)$ inducing a natural equivalent relation: $h_1 \sim h_2 \iff \big((\exists \sigma \in \text{Aut}(A))(h_1 = h_2 \circ \sigma)\big)$. Note that $B[\text{Im}(h_1)] = B[\text{Im}(h_2)]$. Denote by $\underset{\sim}{\boldsymbol{Hom}}_{\boldsymbol{inj}}(A, B) := \frac{\text{Hom}_{\text{inj}}(A,B)}{\sim}$ and the “number of Aut(A)-orbits in $\text{Hom}_{\text{inj}}(A, B)$” by $\#\boldsymbol{O}_{\boldsymbol{inj}}(A, B) := |\underset{\sim}{\text{Hom}}_{\text{inj}}(A, B)|$.

Example 4.50. *We compute that $\#O_{inj}(C_5, E_1) = 3$, confirming our intuition that “E_1 contains 3 copies of C_5. Indeed, note that the Aut(A)-action on Hom$_{inj}(A, B)$ is free, so $\#O_{inj}(A, B) = \frac{|Hom_{inj}(A,B)|}{|Aut(A)|}$. Now Aut($C_5$) $= D_5$ so $|Aut(C_5)| = 10$, while one may count that $|Hom_{inj}(C_5, E_1)| = 30$ by elementary methods.*[43]

Remark 4.51. We are aware of a relevant notation in the combinatorics community: let τ be a finite relational language and let $\mathfrak{A}, \mathfrak{B}$ be finite τ-structures s.t. $\text{Emb}(\mathfrak{B}, \mathfrak{A}) \neq \emptyset$. Then $\binom{\mathfrak{A}}{\mathfrak{B}} := \{\mathfrak{C} \subseteq \mathfrak{A}$ (as substructures): $\mathfrak{C} \cong \mathfrak{B}\}$. We still adopt $\underset{\sim}{\text{Hom}}_{\text{inj}}(-,-)$ and $\#O_{\text{inj}}(-,-)$ mainly because $\underset{\sim}{\text{Hom}}_{\text{inj}}(A,-)$ collects the number of *injective homomorphisms* up to Aut(A)-action, while $\binom{-}{A}$ collects such *embeddings*. In other words, when

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[43] Since $E_1 \in$ TF, $\text{Hom}(C_5, E_1) = \text{Hom}_{\text{inj}}(C_5, E_1)$. For any $h \in \text{Hom}_{\text{inj}}(C_5, E_1) = \text{Hom}(C_5, E_1)$,

1. If $h(1_{C_5})$ has degree 2 in E_1, then $h(2_{C_5})$ uniquely determines the homomorphism. There are 6×2 such homomorphisms.
2. If $h(1_{C_5}) \in \{1_{E_1}, 1'_{E_1}, 3_{E_1}\}$, then $h(2_{C_5})$ can be either of degree 2 (two choices) or of degree 3. When $h(2_{C_5})$ has degree 2, h is determined; when $h(2_{C_5}$ has degree 3, $h(2_{C_5})$ must be 2_{E_1}, and $h(3_{C_5})$ has two choices, and then no further freedom for $h(4_{E_1}), h(5_{E_1})$. Therefore there are $3 \times (2 \times 1 + 1 \times 2) = 12$ such homomorphisms.
3. lastly, if $h(1_{C_5}) = 2_{E_1}$ then we have 3 choices for $h(2_{C_5})$; for each choice of $h(2_{C_5})$ we have 2 choices for $h(3_{C_5})$. We obtain the last $3 \times 2 = 6$ homomorphisms.

The total amounts to $2 \times 12 + 6 = 30$.

we say "copies of subgraphs A in B" we mean $\underset{\sim}{\mathrm{Hom}}_{\text{inj}}(A, B)$ while "copies of *induced* subgraphs A in B" we mean $\binom{B}{A}$.

In cases such as when A is maximally triangle-free and B is triangle-free or situations as desribed in Proposition 4.18, $\underset{\sim}{\mathrm{Hom}}_{\text{inj}}(A, B) = \binom{B}{A}$, but in general $\underset{\sim}{\mathrm{Hom}}_{\text{inj}}$ fits more closely in our framework. For that, and also due to time constraint, in the appendix section we refrain from changing notations such as $\underset{\sim}{\mathrm{Hom}}_{\text{inj}}(A, B)$ or $\#O_{\text{inj}}(A, B)$ into words.

In the appendix we also occasionally use the notations below.Once again, due to time constraints we do not change them into words in the appendix.

Notation 4.52. Let H be any finite graph. $\mathrm{Hom}(H, -)$, $\mathrm{Hom}_{\text{inj}}(H, -)$, $\mathrm{Emb}(H, -)$ are resp. the functors **Graph** $\to$ **Set** that assigns $B \mapsto \mathrm{Hom}_{\{E\}}(H, B), B \mapsto \{h \in \mathrm{Hom}(H, B) : h$ is injective$\}$, and $B \mapsto \{h \in \mathrm{Hom}_{\text{inj}}(H, B) : h$ is an embedding as substructure$\}$. In our context, substructures are just induced subgraphs.

Fix $\alpha \in (0, 1]$ and $H \in \mathcal{K}_{\alpha,\text{TF}}$. $\mathrm{Hom}_{\mathcal{K}_{\alpha,\text{TF}}}(H, -)$, $\mathrm{Hom}_{\text{inj},\mathcal{K}_{\alpha,\text{TF}}}(H, -)$, $\mathrm{Emb}_{\mathcal{K}_{\alpha,\text{TF}}}(H, -)$ are resp. the functors $\mathcal{K}_{\alpha,\text{TF}} \to$ **Set** that assigns $B \mapsto \mathrm{Hom}_{\mathcal{K}_{\alpha,\text{TF}}}(H, B), B \mapsto \{h \in \mathrm{Hom}_{\mathcal{K}_{\alpha,\text{TF}}}(H, B) : h$ is injective$\}$, and $B \mapsto \{h \in \mathrm{Hom}_{\text{inj},\mathcal{K}_{\alpha,\text{TF}}}(H, B) : h$ is an embedding as substructure$\}$.

Now comes the material contents of the appendix section. Here we collect some more partial results generated while attempting to finitize the presentation of $\mathcal{K}_{\alpha,\text{TF}}$s. These frameworks could potentially develop to something more interesting presuming the reader's interest, insight, and technical ingenuity. A few further comments are also included.

4.6.1 Towards Homomorphism-Gluability Study of Hrushovski-Fraïssé Classes

Straight from the definition of ($\preceq$-AP)or by simple vertex-edge counting we know for an $(\alpha, \preceq)$-Hrushovski-Fraïssé Class $\mathcal{K}$ ($\mathcal{K}_\alpha, \mathcal{K}_{\alpha,\text{TF}}, \mathcal{K}_{\alpha,K_n-\text{free}}$), $\iota_1 \in \text{Emb}_{\mathcal{K}}(A, B)$, $\iota_2 \in \text{Emb}_{\mathcal{K}}(A, C)$ such that ι_1 is *strong*, i.e. $\delta_\alpha(A) = \delta_\alpha(\iota_1 A) \leq \delta_\alpha(B')$ for any $\iota_1 A \subseteq B' \subseteq B$, ι_1, ι_2 glue to make a new embedding $\iota_1 \sim \iota_2 \in \text{Emb}_{\mathcal{K}}(A, \frac{B \coprod C}{\{\iota_1(a) \sim \iota_2(a)\}_{a \in V(A)}})$. We would like to study how this generalizes to *$\mathcal{K}$-homomorphisms*. First, a naïve generalization where we assert a $\preceq$-condition on the source and the target:

Proposition 4.53. *Fix arbitrary $\alpha \in (0, 1]$, and let $\mathcal{K}$ be an $(\alpha, \preceq)$-Hrushovski-Fraïssé class. Let $h_1 \in \mathit{Hom}_{\mathcal{K}}(G_1, H_1), h_2 \in \mathit{Hom}_{\mathcal{K}}(G_2, H_2)$, $\iota_i \in \mathit{Emb}_{\mathcal{K}}(G_0, G_i)$ ($i \in [2]$) be s.t.:*

1. *$h_1\iota_1(G_0) \cong h_2\iota_2(G_0)$;*
2. *$h_1\iota_1(G_0) \preceq H_1$ or $h_2\iota_2(G_0) \preceq H_2$; and*
3. *$\iota_1(G_0) \preceq G_1$ or $\iota_2(G_0) \preceq G_2$.*

then $\exists \frac{h_1 \coprod h_2}{h_1\iota_1 \sim h_2\iota_2} \in \mathit{Hom}_{\mathcal{K}}\left(\frac{G_1 \coprod G_2}{\iota_1 \sim \iota_2}, \frac{H_1 \coprod H_2}{h_1\iota_1 \sim h_2\iota_2}\right)$. Note the choices of $i \in [1, 2]$ in (2) and (3) are independent.

Proof. Let $Q_s := \frac{G_1 \coprod G_2}{\iota_1 \sim \iota_2}$, $Q_t := \frac{H_1 \coprod H_2}{h_1\iota_1 \sim h_2\iota_2}$. Let h send each $v \in Q_s$ to

$$\begin{cases} h_1(v), & \text{if } v \in G_1 \setminus \iota_1 G_0, \\ h_2(v), & \text{if } v \in G_2 \setminus \iota_2 G_0, \\ [h_i\iota_i v_0], & \text{if } v = [\iota_i v_0] \end{cases}$$

This is well-defined because for any $v = [\iota_1 v_0] = [\iota_2 v_0]$, $hv = [h_1\iota_1 v_0] = [h_2\iota_2 v_0]$ remains the same vertex in Q_t. h' is also a graph homomorphism, which follows from the facts that $E(Q_s) = E(G_1) \cup E(G_2)$ and $E(Q_t) = E(H_1) \cup E(H_2)$, and each h_i is a graph homomorphism.

It therefore suffices to show that Q_s, Q_t are respectively in $\mathcal{K}$. For Q_s, it follows from the $\preceq$-AP property; for Q_t, apply the $\preceq$-AP property on the target to obtain $Q_t \in \mathcal{K}_\alpha$; [44] and lastly, if $\mathcal{K}$ is $\mathcal{K}_{\alpha,\text{TF}}$ or $\mathcal{K}_{\alpha,K_n-\text{free}}$, Q_t forbids the corresponding clique as well. This is really just the definition of $\frac{H_1 \coprod H_2}{\{h_1\iota_1(a) \sim h_2\iota_2(a)\}_{a\in G_0}}$, because $E(Q_t) = E(H_1) \cup E(H_2)$, while if a K_n were to emerge from the gluing there must be $e \in E_{Q_t}(H_1 \setminus h_1\iota_1(G_0), H_2 \setminus h_2\iota_2(G_0))$, avoiding $E(H_1) \cup E(H_2)$. $\square$

Note that when $h_2 = \text{id}_{G_2}$, the conditions (2) and (3) can be combined into $\iota_2(G_0) \preceq G_2$. One might want to explore the connection of this simple observation with, for example, the upstairs-downstairs inequality in Lemma 4.24. However, the observation below shows that the $\preceq$ conditions are usually too strong when it comes to gluing homomorphisms in $\mathcal{K}_{\alpha,\text{TF}}$, even when restricted to cases where we are interested.

Proposition 4.54. *Let* $\alpha \in (0,1]$, $G_1 \in \mathcal{K}_{\alpha,TF}$, $H \subseteq G_1$ *be s.t.* $\delta_\alpha(H) = 0$ *and* H *is* (α, TF)*-homomorphically rigid.*[45] *Let* $\iota \in Emb(\emptyset \neq G_0, G_1)$ *be s.t.* $\iota(G_0) \cap H = \emptyset$ *and* $E_{G_1}(H, \iota(G_0)) \neq \emptyset$. *Then* $\big(\neg(\iota(G_0) \preceq G_1)\big)$, *and* $\forall h \in Hom_{\mathcal{K}_{\alpha,TF}}(G_1, -)$, $\big((h\iota G_0 \cap$

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[44] Without loss of generality assume $h_1\iota_1 G_0 \preceq G_1$. Let $B \subseteq_{\text{ind}} Q_t$. Then

$$\begin{aligned}
\delta_\alpha(B) &= \delta_\alpha(Q_t[B \cap H_2]) + \delta_\alpha(Q_t[B \cap H_1]) - \delta_\alpha(Q_t[B \cap [h_1\iota_1 G_0]]) \\
&\geq \delta_\alpha(Q_t[B \cap H_1]) - \delta_\alpha(Q_t[B \cap [h_1\iota_1 G_0]]) \\
&\geq \delta_\alpha(Q_t[B \cap H_1 \cup h_1\iota_1 G_0]) - \delta_\alpha(Q_t[[h_1\iota_1 G_0]]) \\
&\geq 0,
\end{aligned}$$

where $[H] := \{[v] : [v] \in V(H)\}$.

[45] For example, $\alpha = \frac{5}{6}$, $G_1 \in \mathcal{S}_0$, $H \in! \underset{\sim}{\text{Hom}}_{\text{inj}}(E_1, G)$.

$hH = \emptyset\big) \implies \neg\big(h\iota(G_0) \preceq hG_1\big)$. *See Figure 4.4 for illustration.*

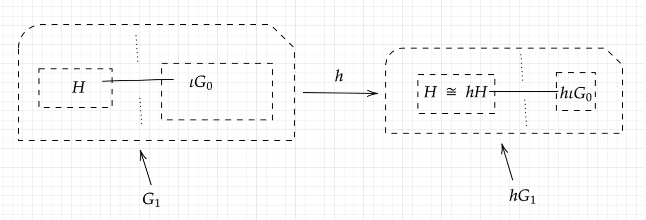

Figure 4.4: Example where the G_0 embeddings and disjoint homomorphisms can't be strong.

Proof. Indeed, consider $G^* := G_1[V(H) \cup V(\iota G_0)]$; note that $\iota(G_0) \subseteq G^* \subseteq G_1$, and $e_{G_1}(H, \iota G_0) > 0$ per assumption. However $\delta_\alpha(G^*) = \delta_\alpha(H) + \delta_\alpha(\iota G_0) - \alpha e_{G_1}(H, \iota G_0) = \delta_\alpha(\iota G_0) - \alpha e_{G_1}(H, \iota G_0) < \delta_\alpha(\iota G_0)$.

Applying Proposition 4.18 to H, we have $h(H) \cong H$ for any $h \in \mathrm{Hom}_{\mathcal{K}_{\alpha,\mathrm{TF}}}(G_1, -)$. Moreover by definition of homomorphisms and the assumption that $h\iota G_0 \cap hH = \emptyset$, we have $e_{hG_1}(hH, h\iota(G_0)) \neq \emptyset$. Then again set $G^* = (hG_1)[V(hH) \cup V(h\iota G_0)]$; we have $h\iota G_0 \subseteq G^* \subseteq hG_1$ yet $\delta_\alpha(G^*) = \delta_\alpha(hH) + \delta_\alpha(h\iota G_0) - \alpha e_{hG_1}(hH, h\iota G_0) < \delta_\alpha(h\iota G_0)$. $\square$

I exhibit another set of conditions sufficient for gluability of homomorphisms, which does not seek $\preceq$-relations. What might be interesting is that instead of requiring "relative density" like in the $\preceq$-criterion, we require the "relative sparsity" of the common image compared to the number of "out-going" edges. [46]

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[46] The goal was: by showing if graphs $G_0 \in \mathcal{K}_{\alpha,\mathrm{TF}}$ are either "relatively very dense" (Proposition 4.53) or "relatively very sparse" (Proposition 4.55 then homomorphisms may glue over it, somehow deduce that there are finitely many "mildly dense" graphs in $\mathcal{K}_{\alpha,\mathrm{TF}}$ which either 1-1 correspond to elements in $\mathcal{S}_0$ (Theorem 4.29) which would close the case, or correspond to "maximal ep-types" triggering Bodirsky's characterization for pp-equivalence to a finite or ω-categorical structure [13, 62*] to achieve nicer presentation. As the location of this subsection indicates, I have not been able to achieve this.

Proposition 4.55. *Fix arbitrary* $\alpha \in (0,1]$, *and let* $\mathcal{K}$ *be an* $(\alpha, \preceq)$*-Hrushovski-Fraïssé class. Let* $h_1 \in \mathit{Hom}_{\mathcal{K}}(G_1, H_1), h_2 \in \mathit{Hom}_{\mathcal{K}}(G_2, H_2)$, $\iota_i \in \mathit{Emb}_{\mathcal{K}}(G_0, G_i)$ $(i \in [2])$ *be s.t.* $\frac{G_1 \coprod G_2}{\iota_1 \sim \iota_2} \in \mathcal{K}_{\alpha,\mathit{TF}}$ *and:*

1. $h_1\iota_1(G_0) \cong h_2\iota_2(G_0)$;
2. *For each connected* $H \subseteq_{\mathit{ind}} h_1\iota_1 G_0$, $\delta_\alpha(H) \geq \alpha\big(e_{h_1G_1}(H,\, h_1G_1 \smallsetminus h_1\iota_1G_0) + e_{h_2G_2}(H,\, h_2G_2 \smallsetminus h_2\iota_2G_0)\big)$.

Then $\exists \frac{h_1 \coprod h_2}{h_1\iota_1 \sim h_2\iota_2} \in \mathit{Hom}_{\mathcal{K}_{\alpha,\mathit{TF}}}\left(\frac{G_1 \coprod G_2}{\iota_1 \sim \iota_2}, \frac{H_1 \coprod H_2}{h_1\iota_1 \sim h_2\iota_2}\right)$.

Proof. Once again we let $Q_s := \frac{G_1 \coprod G_2}{\iota_1 \sim \iota_2}, Q_t := \frac{H_1 \coprod H_2}{h_1\iota_1 \sim h_2\iota_2}$. The construction and proof of all but $Q_t \in \mathcal{K}_\alpha$ is identical to that of Proposition 4.53, and they use merely Condition (1). Also note that Condition (2) is equivalent to the same assertion without the adjective "connected", by summing over all connected components of $H \subseteq_{\text{ind}} h_1\iota_1G_0$ on the LHS and RHS. To see that $Q_t \in \mathcal{K}_\alpha$, let $H \subseteq_{\text{ind}} Q_t$. We have

$$\begin{aligned}
\delta_\alpha(H) \quad &= \quad \delta_\alpha(\underbrace{Q_t[H \cap (h_1G_1 \smallsetminus h_1\iota_1G_0)]}_{\subseteq h_1G_1}) \\
&\qquad + \delta_\alpha(\underbrace{Q_t[H \cap (h_2G_2 \smallsetminus h_2\iota_2G_0)]}_{\subseteq h_2G_2}) + \delta_\alpha(H \cap [h_1\iota_1G_0]) \\
&\qquad - \alpha e_{Q_t}(H \cap (h_1G_1 \smallsetminus h_1\iota_1G_0), H \cap [h_1\iota_1G_0]) \\
&\qquad - \alpha e_{Q_t}(H \cap (h_2G_2 \smallsetminus h_2\iota_2G_0), H \cap [h_2\iota_2G_0]) \\
&\geq \quad \delta_\alpha(H \cap [h_1\iota G_0]) \\
&\qquad - \alpha e_{Q_t}(H \cap (h_1G_1 \smallsetminus h_1\iota_1G_0), H \cap [h_1\iota_1G_0]) \\
&\qquad - \alpha e_{Q_t}(H \cap (h_2G_2 \smallsetminus h_2\iota_2G_0), H \cap [h_2\iota_2G_0])
\end{aligned}$$

$$\begin{aligned}
&\overset{H':=H\cap[h_1\iota_1G_0]\subseteq[h_1\iota_1G_0]}{=} && \delta_\alpha(H') \\
& && -\alpha e_{Q_t}(H\cap(h_1G_1\setminus h_1\iota_1G_0),H') \\
& && -\alpha e_{Q_t}(H\cap(h_2G_2\setminus h_2\iota_2G_0),H') \\
&\geq && \delta_\alpha(H')-\alpha e_{Q_t}(h_1G_1\setminus h_1\iota_1G_0,H')-\alpha e_{Q_t}(h_2G_2\setminus h_2\iota_2G_0,H') \\
&= && \delta_\alpha(H')-\alpha e_{h_1G_1}(h_1G_1\setminus h_1\iota_1G_0,H')-\alpha e_{h_2G_2}(h_2G_2\setminus h_2\iota_2G_0,H') \\
&\geq && 0
\end{aligned}$$

as desired. [47] □

We close this subsection with an example applying Proposition 4.55 to find a gluing.

Example 4.56. *Let G_1, G_2 be as defined in Figure 4.5. Let $G_0 := P_5$. $\iota_i : G_0 \hookrightarrow G_i$ forming the pink paths, i.e. $i_{P_5} \mapsto \gamma_i$ or γ_i'. It is routine to verify that $Q_s := \frac{G_1\coprod G_2}{\gamma_1\sim\gamma_1',\cdots,\gamma_5\sim\gamma_5'}$ is in $\mathcal{K}_{\frac{5}{6},TF}$.*

Let $H_2 := \frac{E_1\coprod P_6}{*\sim 1_{P_6}}$ where $dist_G(*,a) = 2$, then $\exists h_2 : G_2 \twoheadrightarrow H_2$. Indeed, Observe that $a\alpha c_1\beta_3\beta_2\beta_1 ba \cong C_7$ and each edge in E_2 lies in either an induced copy of C_7 or an induced copy of C_5. In the former case, isomorphically map $a\alpha c_1\beta_3\beta_2\beta_1 ba$ to the induced C_7, fixing (a,b); in the latter case, choose a canonical homomorphisim $C_7 \twoheadrightarrow C_5$ preserving $dist_*(\{a\},\{c\})$: for example $\beta_1\sim\beta_3, \beta_2\sim b$. Let $H_1 := G_1$ and $h_1 := id$. We want to show that h_1 and h_2 glues via $h_1\iota_1 \sim h_2\iota_2$ to form a homomorphism in $\mathcal{K}_{\frac{5}{6},TF}$, which boils down to checking Condition 2, as obviously $h_1\iota_1G_0 \cong h_2\iota_2G_0$. The setup is illustrated in Figure 4.5.

To see this, note that $\delta_\alpha(P_5) - 2\alpha = 5 - 6\alpha = 0$ and for any connected proper

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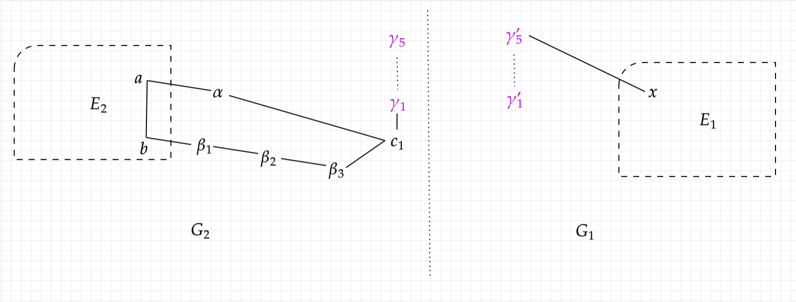
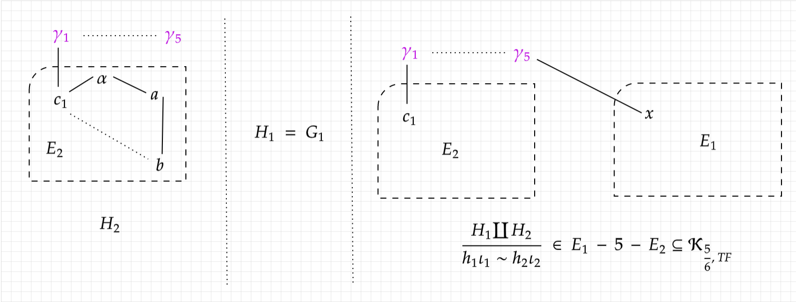
[47] Note that Condition 2 is weaker than the condition obtained by changing H on the RHS to $h_1\iota_1G_0$.



(a) Example 4.56: the source.



(b) Example 4.56: the target and the gluing.

Figure 4.5: Example 4.56 illustrated

subgraph $H \subset \gamma_1 - \cdots - \gamma_5$*, it must miss either* γ_1 *or* γ_5*, or miss both. In either case,* $e_{h_1G_1}(H,\ h_1G_1 \setminus h_1\iota_1G_0) + e_{h_2G_2}(H,\ h_2G_2 \setminus h_2\iota_2G_0) \leq 1$*, so* $\delta_\alpha(H) - \alpha\Big(e_{h_1G_1}(H,\ h_1G_1 \setminus h_1\iota_1G_0) + e_{h_2G_2}(H,\ h_2G_2 \setminus h_2\iota_2G_0)\Big) \geq m - (m-1)\alpha - 1 \times \alpha = m(1-\alpha) > 0$ *for* *some* $m \in [1, 4]$.[48]

4.6.2 On (Absolute) TF-homomorphic rigidity

One may wonder if the concepts of (α, TF)-homomorphic rigidity could generalize to other (Hrushovski-)Fraïssé classes. For example, one may define α-homomorphic rigidity in a similar way: $H \in \mathcal{K}_\alpha$ is *α-homomorphically rigid* if $\mathrm{Hom}_{\mathcal{K}_\alpha}(H, -) = \mathrm{Hom}_{\mathrm{inj}, \mathcal{K}_\alpha}(H, -)$. We have yet to find a clean combinatorial description or application scenario of α-homomorphic rigidity, given that we already know the finite presentation of $\mathcal{K}_\alpha$. On the other hand, the (absolute) *TF-homomorphic rigidity*, which we naturally define by $\big(H \in \mathrm{TF}$ and $\mathrm{Hom}_{TF}(H, -) = \mathrm{Hom}_{\mathrm{inj,TF}}(H, -)\big)$, captures some fairly interesting combinatorial phenomena.

Proposition 4.58. *For a finite graph $\emptyset \neq H \in TF$, the following are equivalent:*

1. *H is TF-homomorphically rigid.*
2. *$\forall \alpha \in (0, \frac{2}{mad(H)}]$, H is (α, TF)-homomorphically rigid.*

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[48] *Remark* 4.57. We say $G \in \mathcal{K}_{\frac{5}{6}, \mathrm{TF}}$ with $o_i \in! \utilde{\mathrm{Hom}}_{\mathrm{inj}}(E_i, G)$ is $(a, b; c)$-*shaped* if $G \smallsetminus (o_1 \coprod o_2)$ has exactly 1 vertex x^* of global degree 3, and $G = \frac{E_1 \coprod P_{a+2} \coprod P_{b+2} \coprod P_{c+2} \coprod E_2}{(a+2) \sim (b+2) \sim 1_{P_{c+2}}(\sim x^*), 1_{P_{a+2}} \sim q, 1_{P_{b+2}} \sim r, (c+2) \sim t}$ for some $q, r \in o_1, t \in o_2$; one may define $(a; b, c)$-*shaped* analogously. For example $\frac{G_1 \coprod G_2}{\iota_1 \sim \iota_2}$ in Example 4.56 is $(5; 1, 3)$ or $(5; 3, 1)$-shaped. It can be derived that we need

$$\begin{cases} a, b, c & \geq \ 0; \\ a+b, b+c, c+a & \geq \ 4 \\ a+b+c & \geq \ 9 \end{cases}$$

in order for $G \in \mathcal{K}_{\frac{5}{6}, \mathrm{TF}}$. We *conjecture* that all $(a; b, c)$ and $(a, b; c)$-shaped $G \in \mathcal{K}_{\frac{5}{6}, \mathrm{TF}}$ are not $(\frac{5}{6}, \mathrm{TF})$-homomorphically rigid and map to (E_1)–5–(E_2),[49] and it is our hope that tricks similar to Example 4.56 using Proposition 4.55 might help prove it.

Note: at least for $(0, 5; 4)$-shaped we already need some gluing technique slightly different from Example 4.56, yet still explainable by Proposition 4.55, namely by leaving x^* *outside* the image ho_1. We believe this small class of examples could help obtain better usage or improvement of the gluing techniques, although **we are aware** that whether $(a; b, c)$-shaped and $(a, b; c)$-shaped homomorphically map to (E_1)–m–(C_5)–n–(E_2) do not directly impact the existence of finite $\mathcal{K}_{\frac{5}{6}, \mathrm{TF}}$-presentation.

3. *H has enough P_4s; that is, for any $(a,b) \in \binom{V(H)}{2} \setminus E(H)$, $a{-}*{-}*{-}b \subseteq_{ind} H$.*

Proof. Unpacking definitions.

1. (2) $\implies$ (1): Note for each $\alpha \in (0, \frac{2}{\text{mad}(H)}]$ we have $H \in \mathcal{K}_{\alpha,\text{TF}}$. Let $h \in \text{Hom}_{\text{TF}}(H,B)$ and take $0 < \alpha < \min(\frac{2}{\text{mad}(H)}, \frac{2}{\text{mad}(B)})$; then $H, B \in \mathcal{K}_{\alpha,\text{TF}}$ and $h \in \text{Hom}_{\mathcal{K}_{\alpha,\text{TF}}}(H,B)$ is injective per (2). By arbitrariness of B, h, H is TF-homomorphically rigid.

2. (1) $\implies$ (2): each $h \in \text{Hom}_{\mathcal{K}_{\alpha,\text{TF}}}(H,B) \subseteq \text{Hom}_{\text{TF}}(H,B)$ and is therefore injective.

3. $\neg$(1) $\implies$ $\neg$(3): Let $(\mathfrak{B}, h)$ witness $\neg$(1). Since h is non-injective, there exists $(a,b) \in \binom{v(H)}{2} \setminus E(H)$ s.t. $h(a) = h(b)$. But then it cannot be the case that $a{-}*{-}*{-}b \subseteq_{\text{ind}} H$.

4. $\neg$(3) $\implies$ $\neg$(1): let $(a,b) \in \binom{V(H)}{2} \setminus E(H)$ be s.t. $a{-}*{-}*{-}b \not\subseteq_{\text{ind}} H$. Since $(a,b) \notin E(H)$, the map $H \twoheadrightarrow \frac{H}{a\sim b}$ is a homomorphism; also, $\frac{H}{a\sim b} \in \text{TF}$.[50] So $h \in \text{Hom}_{\text{TF}}(H,-) \setminus \text{Hom}_{\text{inj,TF}}(H,-)$.

$\square$

Examples of graphs that has enough P_4s include C_5, Petersen's graph, and the following graph that I call SCE$_1$ ("skew" and "contracted" E_1). We finish our discussion on TF-homomorphic rigidity or equivalently P_4-enoughness with the following results.

Proposition 4.59. *1. There are infinitely many TF-homomorphically rigid graphs, so TF does not admit a finite presentation.*

---

[50] Assume not; then one of the three vertices in any triangle in $H/a \sim b$ must be $[a,b]$. So there exists $c \in N_H(a), d \in N_H(b)$ s.t. $(c,d) \in E(H)$. Note that $(a,d),(c,b) \notin E(H)$ for o.w. $H[a,d,c]$ or resp. $H[b,d,c]$ would be a triangle. But then $a{-}c{-}d{-}b \subseteq_{\text{ind}} H$.

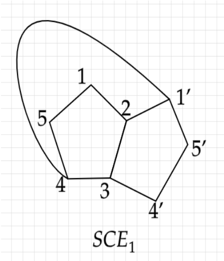

Figure 4.6: To my knowledge SCE$_1$ was first mentioned without naming in [3] as an example with mad $= \frac{10}{4}$ which does not admit a homomorphism to the Petersen's graph. This also says CSP($\mathfrak{M}_{\frac{4}{5}+\epsilon,\text{TF}}$) $\subsetneq$ CSP($\mathfrak{M}_{\frac{4}{5},\text{TF}}$). To see that SCE$_1$ has enough P_4s, note that the only non-adjacent pair (a, b) not contained in an induced copy of C_5 ($123451, 1'234'5'1', 1'21541', 1'434'5'1'$) are $(1, 5')$, $(1, 4')$, $(5, 5')$, $(5, 4')$. They are respectively connected by the following induced copies of P_4: $121'5', 1234', 541'5', 5434'$.

2. *When $\alpha \in (\frac{4}{5}, 1]$, there are finitely many TF-homomorphically rigid graphs in* CSP($\mathfrak{M}_{\alpha,TF}$).

3. *for any $d \in \omega \setminus 0$, there are finitely many graphs with max vertex degree d that have enough P_4s.*

Proof. 1. Consider the Andrásfai graphs $\{A_n\}_n$ where A_n has $[1, 3n-1]$ as vertices and each k is connected with $k+j$ for $j \cong 1 \mod 3$. It can be proved that $(k, k+1)$ and $(k, k-1) \in E(A_n)$ for each n by $(k-1) + v(A_n) = k - 1 + (3n-1) = k + 3(n-1) + 1$, and each $A_n \in$ TF. For each (k, p) pair, if $p = k + 3m$ for some m then $[k]$ – $[k+3(m-1)+1]$–$[k+3m-1]$–$[p] \subseteq_{\text{ind}} A_n$; if $p = k + 3m + 2$ then $[k]$–$[k+3(m+1)+1]$–$[k+3m+3]$–$[p] \subseteq_{\text{ind}} A_n$. It follows that each A_n has enough P_4s. A finite presentation would contain each A_n as subgraph by homomorphic rigidity, which is impossible.

2. Per [3], every $G \in$ CSP($\mathfrak{M}_{\frac{4}{5}+\epsilon,\text{TF}}$) maps to the Petersen's graph $G(5, 2)$. It must follow that $G \cong G^*$ for some $G^* \subseteq G(5, 2) \in$ TF, G being TF-homomorphically rigid.

3. Let $N_d := d + 2 + d^3$. We argue that for any finite G with $v(G) > N_d$, G cannot have enough P_4s. Indeed, pick any $v_0 \in V(G)$. It has at most d neighbors, so $|V(G) \setminus (x_0 \cup N_G(x_0))| \geq n - (d+1) > d^3$. On one hand we need $n - (d+1) > d^3$ different copies of induced P_4s of the form v_0–$*$–$*$–$*$, as the *rightmost* $*$ runs through $V(G) \setminus (x_0 \cup N(x_0))$; on the other hand there are at most d^3 copies of induced P_4s of the form v_0–$*$–$*$–$*$, making a choice of neighbors at each $*$.

□

4.6.3 Each $\mathcal{K}_{\alpha,\text{TF}}$ has Unbounded Treewidth and Treedepth; Independent Proof for $\alpha \in (\frac{8}{9}, 1]$

There were efforts in exploring results in e.g. [59] for bounded tree-width or bounded tree-depth classes of sparse-graphs. We show in this subsection that one should not expect to apply those results in naïve ways; in particular, the classes $\mathcal{K}_{\alpha,K_n-\text{free}}$ in some sense have little control over how much their members "look like trees". We then record a proof that for $\alpha \in (\frac{8}{9}, 1]$, CSP($\mathfrak{M}_{\alpha,\text{TF}}$) = CSP($C_5$) which we independently did before finding out existing literatures on $(\frac{5}{6}, 1]$.

Definition 4.60. For a graph G, a *tree decomposition* is a pair $(\{V_i\}_{i\leq n}, T)$, where $\{V_i\}_{i\leq n}$ is a collection of subsets of $V(G)$ and T is a tree with $V(T) = \{V_1, \cdots, V_n\}$, satisfying

1. $V(G) \subseteq \bigcup_i V_i$;
2. $E(G) \subseteq \bigcup_i \binom{V_i}{2}$;
3. For each $v \in V(G)$, the induced subgraph $T[\{V_i : v \in V_i\}]$ is connected.

The *tree width tw*(G) is $\min_{(\{V_i\}_i,T)\text{ tree-decomposes }G}\ \max_i(|V_i|-1)$. For example, for any m, tw$(K_m) = m-1$, with only the trivial tree decomposition; for any tree T, td(T)=1 with the tree decomposition $(\{V_i\}_{i\in[1,e(T)]}, T')$ where each V_j contains exactly the two ends of the j-th edge and $(V_i, V_j) \in E(T') \iff V_i \cap V_j \neq \emptyset$.

A *rooted forest* is a disjoint union of rooted trees. For a rooted forest $(\vec{r}, \coprod_i F_i)$ with $\vec{r}$ the roots, the *height of vertex* $v \in F_i$ is 1 plus the length of the path r_iF_iv. *Height of* $(\vec{r}, \coprod_i F_i)$ is $\max_{i,v\in F_i}$ height(v). *Closure* of F, denoted as *clos*(F), is the graph defined on vertex set $V(F)$ and edge set $\{(x,y) : x \neq y$ and x is an ancestor of y in the tree order$\}$.[51] Finally, for any graph G, its *tree-depth td(G)* is the minimum height of rooted forest F s.t. $G \subseteq$ clos(F).

If $\mathcal{K}_{\alpha,K_n-\text{free}}$ has *bounded* treewidth or *bounded* treedepth, then there is some hope in characterizing its sparsity and possibly tying such characterizations to CSP($\mathfrak{M}_{\alpha,K_n-\text{free}}$), per e.g. [59, 63][52]. Unfortunately $\mathcal{K}_{\alpha,K_n-\text{free}}$ has unbounded tree-width and unbounded tree-depth.

Proposition 4.61. *For $n \geq 3$, $\alpha \in (0,1)$, $\mathcal{K}_{\alpha,K_n-\mathit{free}}$ has unbounded tree-width and unbounded tree-depth.*

Proof. For unboundedness of tree width in $\mathcal{K}_{\alpha,K_n-\text{free}}$, first observe that for finite graphs G, H, if G is a minor of H, then tw$(G) \leq$ td(H). Indeed, for each minor operation there is a corresponding operation on the tree-decomposition maintaining the upper bound of tw (see e.g. [64]).

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[51] For a rooted tree (r,T), the natural *Tree order* is as follows: $x \leq y \iff$ there exists a path rTy passing through x.

[52] See e.g. p20 of [63], "the Full Picture".

Now observe that for any $m, n \geq 3, \alpha \in (0,1)$, there exists $G_m \in \mathcal{K}_{\alpha, K_n-\text{free}}$ s.t. K_m is a minor of G_m. Indeed, sufficiently subdividing each edge of K_m will yield one such G_m. Therefore tw$(G_m) \geq$ tw$(K_m) = m - 1$, as desired.

For unboundedness of tree-depth in $\mathcal{K}_{\alpha, K_n-\text{free}}$, note that the following recursive formula computes td(G) for any finite G [59]:

$$td(G) = \begin{cases} 1, & \text{if } |G| = 1; \\ 1 + \min_{v \in V(G)} td(G - v), & \text{if } G \text{ is connected and } |G| > 1; \\ \max_{i=1}^{p} td(G_i) & \text{o.w., i.e. } G \text{ has } p \text{ components.} \end{cases}$$

Then we have by e.g. induction on the length that for a path P_n of length m, td$(P_m) \geq \lfloor \log(m) \rfloor$. As any $\mathcal{K}_{\alpha, K_n-\text{free}}$ contains all P_ms, unboundedness follows. □

We exhibit the following proof despite that the result is superseded by [54], as my approach was independently discovered and is mildly technically interesting. [53]

Proposition 4.62. *Let $\alpha \in (\frac{8}{9}, 1]$; then CSP$(\mathfrak{M}_{\alpha,TF}) \subseteq$ CSP(C_5).*

Proof. Let $G \in$ CSP$(\mathfrak{M}_{\alpha,\text{TF}})$ via h, and denote $B := \mathfrak{M}_{\alpha,\text{TF}}[h(V(G))]$. Then $B \in \mathcal{K}_{\alpha,\text{TF}}$, so in particular ad$(B) \leq$ mad$(B) < \frac{9}{4}$. Also $\min_{x \in V(B)} \deg_B(x) \leq 2$ as mad$(B) < 3$. We prove $B \in$ CSP(C_5) by induction on $v(B)$. When $v(B) \leq 4$ and triangle-free, B is bipartite, so homomorphic to C_5 via P_2.

Assume that $\Big(B \in \mathcal{K}_{\alpha,\text{TF}}$ implies Hom$(B, C_5) \neq \emptyset\Big)$ holds whenever $v(B) \leq n$. Now consider $B \in \mathcal{K}_{\alpha,\text{TF}}$ with $v(B) = n + 1$.

---

[53] Note, though, that one has a hard time generalizing to smaller αs using similar techniques, as even for $\alpha = \frac{8}{9}$ there are graphs without 2-2-2 paths. For $\alpha \in (\frac{5}{6}, \frac{8}{9}]$ it is guaranteed that 2-2-$\leq$ 3 paths appear [65], but approaches similar to the below fails for one subcase.

1. If $\min_x \deg_B(x) \le 1$: let $x \in V(B)$ attain $\min_x \deg_B(x)$, and consider $B - x := B[V(B) \smallsetminus \{x\}]$. Now $v(B-x) \le n$ and $B - x \in \mathcal{K}_{\alpha,\text{TF}}$ by hereditariness, so by inductive hypothesis there exists $h \in \text{Hom}(B-x, C_5)$. If $\deg(x) = 0$, extend h arbitrarily to x; if $\deg(x) = 1$, extend h by $x \mapsto h(a) + 1 \mod 5$ where a is xs unique neighbor.

2. Otherwise $\min_x \deg_B(x) = 2$. By [65], we have a 2-2-2 path $avb \subseteq B$; since $B \in \text{TF}$, $avb \subseteq_{\text{ind}} B$. In particular, $N_B(v) = \{a, b\}$, $N_B(a) = \{v, c\}$, $N_B(b) = \{v, d\}$ for some $c \notin \{b, v\}$, $d \notin \{a, v\}$. Now consider the induced subgraph $B - v := B[V(B) \smallsetminus \{v\}]$. Hereditarity gives $B - v \in \mathcal{K}_{\alpha,\text{TF}}$, so the inductive hypothesis says there is a $g \in \text{Hom}(B - v, C_5)$, where the min distance between $g(a), g(b)$ is at most 2. Note that the inductive hypothesis itself does not rely on the existence of a 2-2-2 path.

   If $g(a) = g(b)$, map v to $g(a)+1 \mod 5$; if $g(a), g(b)$ has distance 2, map v to their unique common neighbor on C_5. We are left with the case where $|(g(a) - g(b)) \mod 5| = 1$; without loss of generality, assume $g(a) = 1, g(b) = 2$.

   (a) If $g(c) \neq g(b)$, then $g(c) = 5$. Construct $\tilde{g}(x) := \begin{cases} g(x), & \text{if } x \in B - \{v, a\}, \\ 4, & \text{if } x = a \\ 3, & \text{if } x = v \end{cases}$;
   we have $\widetilde{g} \in \text{Hom}(B, C_5)$. Dually if $g(d) \neq g(a)$, then $g(d) = 3$ and one can remap b to 4, v to 5.

(b) Otherwise $g(c) = g(b), g(d) = g(a)$. Construct $\tilde{g}(x) := \begin{cases} g(x), & \text{if } x \in B - \{v, a, b\}, \\ 3, & \text{if } x = a \\ 5, & \text{if } x = b \\ 4, & \text{if } x = v \end{cases}$

In all cases we have established $G \xrightarrow{h} B \xrightarrow{\tilde{g}} C_5$, as desired. A pictorial illustration is attached in Figure 4.7.

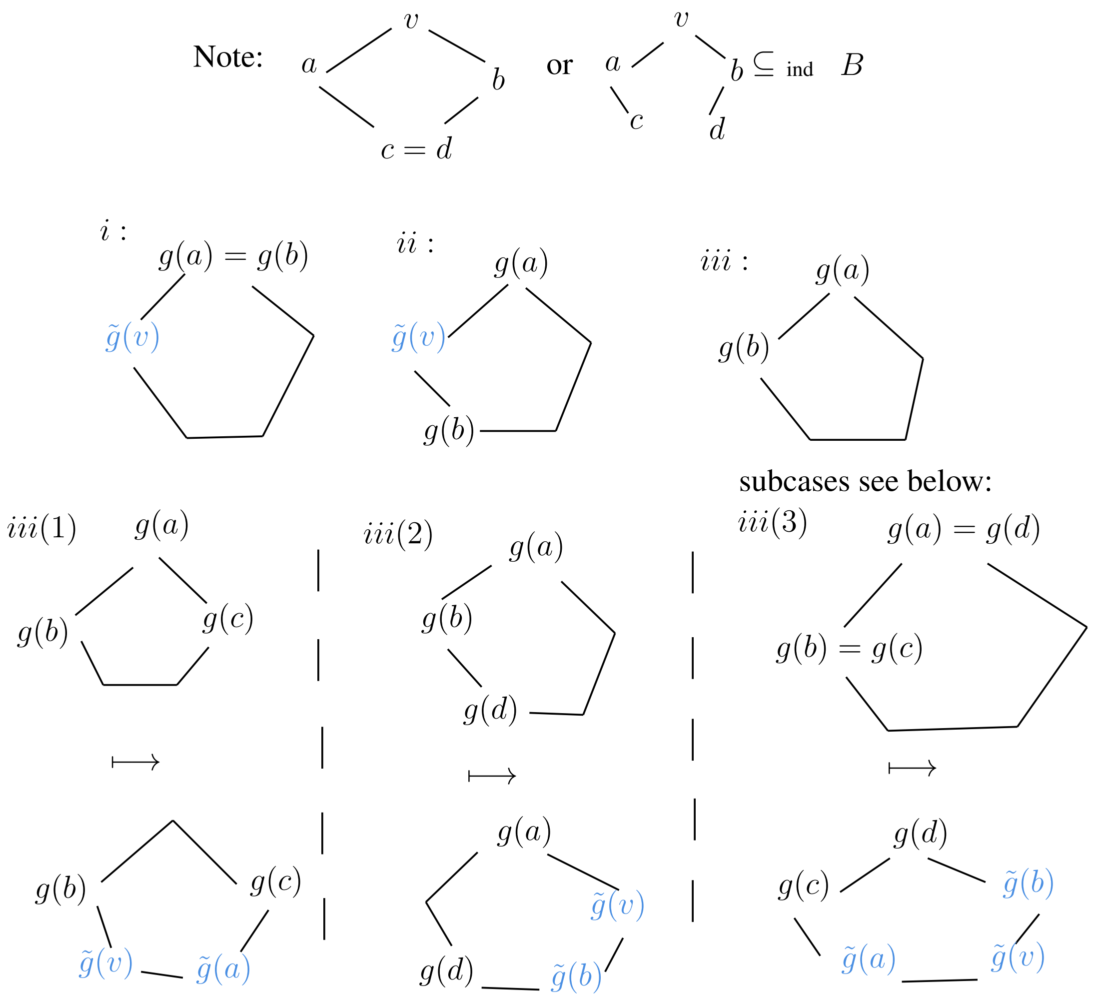

Figure 4.7: Modifying $g : B - v \to C_5$ to $\tilde{g} : B \to C_5$

$\square$

4.6.4 Closing Remarks: Potential Approaches to ω-categorical Homomorphic Presentation of $\mathcal{K}_{\alpha,\mathrm{TF}}$

ω-categorical structures are those whose first order theory has exactly 1 countably infinite model up to isomorphism. Conceptually they are the "second best thing" one could hope for as a homomorphic presentation of non finitely-presentable or ultrahomogenously presentable classes. We exhibit two such potential approaches for $\mathcal{K}_{\alpha,\mathrm{TF}}$.

A class $\mathcal{K}$ has *Homomorphic Amalgamation Property (HAP)* ([4]) if $\forall A \in \mathcal{K}$, $h_1 \in \mathrm{Hom}_{\mathcal{K}}(A, B)$, $g_1 \in \mathrm{Emb}_{\mathcal{K}}(A, C)$, $\big(\exists (D \in \mathcal{K}$ and $h_2 \in \mathrm{Emb}_{\mathcal{K}}(B, D)$, $g_2 \in \mathrm{Hom}_{\mathcal{K}}(A, D))$ s.t. $h_2 h_1 = g_2 g_1\big)$. See Figure 4.8.

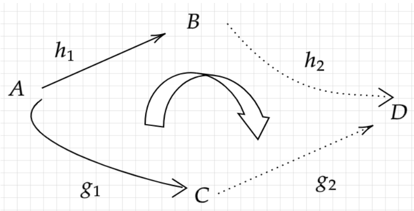

Figure 4.8: Homomorphic Amalgamation Property (HAP), [4]

Applying Theorem 3.5, Lemma 4.4 and Theorem 6.12 of [4], we have the following:

Proposition 4.63. *Let* $\alpha \in (0, \frac{5}{6}]$*. If there exists an* $\mathcal{S} \subseteq \mathcal{K}_{\alpha,TF}$ *has HP, JEP and HAP, then there exists a countable,* ω-categorical $\mathfrak{M}'_{\alpha,\mathrm{TF}}$ *s.t.* $\mathrm{CSP}(\mathfrak{M}'_{\alpha,TF}) = \mathrm{CSP}(\mathfrak{M}_{\alpha,TF})$*; in fact* $\mathfrak{M}'_{\alpha,TF}$ *will be a generic structure of* $\mathcal{S}$*, i.e. Age*$(\mathfrak{M}'_{\alpha,TF}) = \mathcal{S}$*.*

It appears tempting to seek such an $\mathcal{S}$ to simplify our homomorphic presentation; however, below I proved that the first potential approach fails for many αs and my proof *uses my results on* (α, TF)*-homomorphic rigidity*. So this may serve as another interesting application of my findings in Section 4.4.

Proposition 4.64.

1. *Let* $\alpha \in (\frac{32}{39}, \frac{5}{6}]$*. Any* $\mathcal{S} \subseteq \mathcal{K}_{\alpha,TF}$ *with* $C_5, E_1, E_1 - \{6\} \in \mathcal{S}$ *does not admit* HAP.

2. *Let* $\alpha \in (0, \frac{4}{5}]$*. Any* $\mathcal{S} \subseteq \mathcal{K}_{\alpha,TF}$ *with* C_5*, SCE*$_1$ *(Figure 4.6), SCE*$_1 - \{1\} \in \mathcal{S}$ *does not admit* HAP.

3. *In either case above, any* $\mathcal{S} \subseteq \mathcal{K}_{\alpha,TF}$ as homomorphic presentation *with* HP *must contain* E_1 *(Case(1)) or SCE*$_1$ *(Case(2)).*

Proof.

1. let $A := E_1 - \{6\}$, $B := C_5$, $C := E_1$. Let $g_1 \in \text{Emb}_{\mathcal{S}}(A, C)$ be the natural embedding. Per Proposition 4.12, there exists $h_1 \in \text{Hom}_{\mathcal{S}}(A, B)$. Now for any $D \in \mathcal{S}$ and any $h_2 \in \text{Emb}_{\mathcal{S}}(B, D) \subseteq \text{Emb}_{\mathcal{K}_{\alpha,\text{TF}}}(B, D)$, $|\text{Im}(h_2 h_1)| \leq |C_5| = 5$; on the other hand Per (α, TF)-homomorphic rigidity of E_1, any $g_2 \in \text{Hom}_{\mathcal{S}}(C, D) \subseteq \text{Hom}_{\mathcal{K}_{\alpha,\text{TF}}}(C, D)$ must be injective, so $|\text{Im} g_2 g_1| = 9$.

2. The same idea as above. Let $A :=$ SCE$_1$-$\{1\}$, $B := C_5$, $C :=$ SCE$_1$, g_1 be the natural embedding. $h_1 := \frac{A}{5 \sim 3, 4 \sim 2} \in \text{Hom}(A, B)$. By TF-homomorphic rigidity of C, $|\text{Im} g_2 g_1| = 7$ while $|\text{Im} h_2 h_1| \leq 5$ for any $h_2 \in \text{Emb}_{\mathcal{S}}(B, -), g_2 \in \text{Hom}_{\mathcal{S}}(C, -)$.

3. By (α, TF)-homomorphic rigidity, any $\mathcal{S} \subseteq \mathcal{K}_{\alpha,\text{TF}}$ must contain a B hosting E_1 (Case (1)) or SCE$_1$ (Case (2)) as subgraph, then $\mathcal{S}$ must contain E_1 or SCE$_1$ respectively per hereditary property.

□

The second potential approach is based on [66], and to the best of the author's knowledge remains viable. A generalized second order τ-formua generalizes the *guarded first order formula*, which are defined recursively as follows:

1. atomic τ-formulas are generalized first order formulas;
2. if φ and ψ are generalized (first order) formulas, so are $\varphi \bigwedge \psi$, $\varphi \bigvee \psi$ and $\neg\varphi$.
3. ("guarded") if $\psi(\overline{x}, \overline{y})$ is a guarded formula and $\alpha(\overline{x}, \overline{y})$ is an atomic formula whose free variables contain all free variables of $\psi(\overline{x}, \overline{y})$, then $\exists \overline{y}(\alpha(\overline{x}, \overline{y}) \bigwedge \psi(\overline{x}, \overline{y}))$ is a generalized formula. Note that by closure under negation, so is $\forall \overline{y}(\alpha(\overline{x}, \overline{y}) \implies \psi(\overline{x}, \overline{y}))$.

A *guarded second order τ-formula* is defined using the same recursive rules as guarded first order formula, but in addition allows second order quantification, i.e. $\exists S_1 \varphi$, $\forall S_1 \varphi$ are guarded second order formulas wherever φ is guarded second order formula and $S_1 \notin \tau$ is a new relational symbol. As usual a *guarded second order sentence* is a guarded second order formula with no free variables. If a class $\mathcal{C}$ of finite τ-structures is *axiomatized* by a guaraded second order sentence (GSO), i.e. if there exists $\Phi_{\mathcal{C}} \in$ GSO s.t. $\mathfrak{A} \in \mathcal{C} \iff \mathfrak{A} \models \Phi_{\mathcal{C}}$, then by abuse we say $\mathcal{C} \in$ *GSO*. It is known that

Fact 4.65 (Reducing to ω-categorical, [66])**.** *Let* $\mathfrak{M}$ *be* any τ*-structure. If* CSP$(\mathfrak{M}) \in$ GSO*, then there exists a* countable, ω-categorical τ*-structure* $\mathfrak{B}$ *s.t.* CSP$(\mathfrak{M}) =$ CSP$(\mathfrak{B})$.

Example 4.66 (Based on Example 11 in [66], with more clarification)**.** *Consider the GSO* *sentence in $\tau = \{E\}$ the language of digraphs*

$$\Phi :\equiv \forall X \forall t\Bigg(\bigg(X(t)\bigg) \implies \bigg(\exists s,y\Big(X(s) \wedge X(y) \wedge \forall z\big(X(z) \implies (\neg E(s,z) \vee \neg E(y,z))\big)\Big)\bigg)\Bigg) \tag{4.4}$$

Observe that:

1. *$\neg\Phi$ asserts the existence of a subset $\emptyset \neq X \subseteq V(G)$, s.t. $\forall s,y \in X$, there exists $z \in X$ connecting to both s and y. ($(s,z) \in E(G)$ and $(y,z) \in E(G)$).*

2. *$\neg\Phi$ is "closed under τ-homomorphisms" for finite τ-structures: for any finite digraph $\mathfrak{A}$, $\mathfrak{B}$, if $\Big(\mathfrak{A} \models \neg\Phi$ and $\mathfrak{A} \to \mathfrak{B}\Big)$ then by preservation of E we have $\mathfrak{B} \models \neg\Phi$.*

3. *Consider $[\![\Phi]\!] := \{$finite τ-structures $\mathfrak{A} : \mathfrak{A} \models \Phi\}$. Then $[\![\Phi]\!]$ is closed under disjoint unions: indeed for subsets X spanning two components, just take s,y from different components. Hence $[\![\Phi]\!]$ has Joint Embedding property. It is also closed under taking substructures, for $\mathfrak{A} \subseteq_{ind} \mathfrak{B} \implies \mathfrak{A} \to \mathfrak{B}$. It follows that there exists some countable τ-structure $\mathfrak{M}$ with Age$(\mathfrak{M}) = [\![\Phi]\!]$. Furthermore Age$(\mathfrak{M}) =$ CSP$(\mathfrak{M})$, also by the fact that $\neg\Phi$ is closed under homomorphisms.*

4. *it is not obvious whether $\mathfrak{M}$ is ω-categorical. However, as $\Phi \in$ GSO, we conclude using Fact 4.65 that there exists some ω-categorical, countable τ-structure $\mathfrak{M}'$ satisfying CSP$(\mathfrak{M}') =$ CSP$(\mathfrak{M})$(= Age($\mathfrak{M}$) = $[\![\Phi]\!]$).*

Recall that for any $\alpha \in (0,1]$, $\mathfrak{M}_{\alpha,\text{TF}}$ is Π_2-axiomatized, i.e. axiomatized by a *set* of first order $\forall\exists$-sentences: see Chapter 3 of this dissertation. If one further finds a guarded second order *singleton* axiomatization of $\mathcal{K}_{\alpha,\mathrm{TF}}$, then per Fact 4.65 there would be an ω-categorcial $\mathfrak{M}'_{\alpha,\mathrm{TF}}$ homomorphically presenting $\mathcal{K}_{\alpha,\mathrm{TF}}$, via $\mathrm{CSP}(\mathfrak{M}'_{\alpha,\mathrm{TF}}) = \mathrm{CSP}(\mathfrak{M}_{\alpha,\mathrm{TF}})$. The hardness of this approach lies, among other things, in capturing the interplay between triangle-freeness and sparsity within one *single* GSO sentence.

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