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Stochastic Wave Equations with Constraints: Well-Posedness and Smoluchowski–Kramers Diffusion Approximation

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Abstract: We investigate the well-posedness of a class of stochastic second-order in time damped evolution equations in Hilbert spaces, subject to the constraint that the solution lies within the unitary sphere. Then, we focus on a specific example, the stochastic damped wave equation in a bounded domain of a d-dimensional Euclidean space, endowed with the Dirichlet boundary condition, with the added constraint that the L^2-norm of the solution is equal to one. We introduce a small mass $\mu > 0$ in front of the second-order derivative in time and examine the validity of a Smoluchowski–Kramers diffusion approximation. We demonstrate that, in the small mass limit, the solution converges to the solution of a stochastic parabolic equation subject to the same constraint. We further show that an extra noise-induced drift emerges, which in fact does not account for the Stratonovich-to-Itô correction term.

1. Introduction

The objective of this paper is twofold. Firstly, we aim to establish the existence and uniqueness of global solutions to stochastic second-order in time damped evolution equations in Hilbert spaces, while imposing the constraint that the norm of the solution is equal to one. Secondly, we focus on a specific case of such equations, namely the stochastic damped wave equation in a bounded domain of a d-dimensional Euclidean space, subject to the Dirichlet boundary condition, with the constraint that the L^2-norm of the solution is equal to one. In this case, we introduce an additional parameter μ, called mass, to such equation and aim to prove that the solution u_μ converges to the solution of a certain stochastic heat equation, with Dirichlet boundary conditions, satisfying the constraint that the L^2-norm of the solution is equal to one, as well. Unlike in all the examples studied in the existing literature, the limiting equation we obtain may not be

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of the Stratonovich form. However, one can give an independent proof of the existence and uniqueness of solutions to such limiting problem by employing methods similar to those used recently in [9].

The present paper is the first one to consider the problem of the well-posedness for the following class of evolution equations in a separable Hilbert space H

$$u_{tt}(t)+|u_t(t)|_H^2 u(t) = -A_0^2 u(t)+|A_0 u(t)|_H^2 - \gamma u_t(t)+\sigma(u(t), u_t(t)) \circ dW(t), \quad (1.1)$$

subject to the finite-codimension constraint of living on $M = S_H(0, 1)$, the unitary sphere of H, with the initial data (u_0, v_0) in TM, the tangent bundle of M. Here A_0 is a non-negative self-adjoint operator on H with domain $D(A_0)$, γ is a positive constant, $W(t)$ is a cylindrical Wiener process with reproducing kernel Hilbert space (RKHS) K having gamma-radonifying embedding in some Banach space E, and σ is a locally Lipschitz mapping defined on $\mathscr{H} := D(A_0) \times H$ with values in $\mathscr{L}(E, H)$, such that $\sigma(u, v)$ projects E onto $T_u M$. Due to its Stratonovich formulation, when written in Itô form equation (1.1) contains a non-trivial Stratonovich-to-Itô trace term.

It is worth noticing that Eq. (1.1) is a stochastic version of the constrained deterministic equation

$$u_{tt}(t) + |u_t(t)|_H^2 u(t) = -A_0^2 u(t) + |A_0 u(t)|_H^2 - \gamma u_t(t),$$

where the terms $|u_t(t)|_H^2 u(t)$ and $|A_0 u(t)|_H^2$ are added to $u_{tt}(t)$ and $-A_0^2 u(t)$, respectively, in order to ensure that the solution stays on the manifold M. Actually, if one adds to the deterministic equation above any stochastic perturbation such as

$$\sigma(u(t), u_t(t)) \circ dW(t),$$

under the assumption that for every $(u, v) \in \mathscr{H}$ the mapping $\sigma(u, v)$ projects E into the tangent space $T_u M$, due to the presence of the Stratonovich integral the invariance property holds also for the stochastic equation (1.1).

It is important to note that the model presented in Eq. (1.1) differs from the recent study of stochastic geometric wave equations by the first-named author and coauthors in [8]. In those works, the solution is constrained to a manifold within the Euclidean space (e.g., the sphere in $\mathbb{R}^3$), whereas here, the solution of (1.1) is restricted to a functional manifold - the set of all square-integrable functions with L^2-norm one. Although both classes of SPDEs exhibit nonlinearities with cubic growth, their nature differs significantly: in the former case, the nonlinearities are local, while in the latter, they are non-local. In particular, this fundamental distinction necessitates different mathematical techniques for their analysis.

However, the investigation of deterministic and stochastic constrained partial differential equations (PDEs) is not a new field of study. In this regard, we would like to mention the papers [13] by Caffarelli and Lin, and [49] by Rybka, where deterministic heat flows in Hilbert manifolds were explored. The motivation behind the former paper was to find a gradient flow approach to a specific minimization problem. A similar inquiry was undertaken for the stochastic 2-D Navier–Stokes equation by the first named author and Dhariwal in [4]. This work was preceded by the paper [6] by these two authors and Mariani, as well as the paper [14] by Caglioti, Pulvirenti, and Rousset, whose motivation was the occurrence of different dissipation timescales. Both in [6] and in [14], the study focused on the deterministic 2-D Navier–Stokes equations (NSEs). As well known, the L^2-norm of the solutions u^ν to such equations, with viscosity $\nu > 0$, converges to 0, as $t \to \infty$, while the L^2-norm of the (strong) solution u of the limiting

Euler equation, which corresponds to $\nu = 0$, remains constant. Thus, it was proposed in [14] to consider a modification of the Navier–Stokes equations in which the L^2-norm of the solution remains constant in time as well. Finally, we would like to mention the recent work of Hairer and Rosati [35], which examines the projected process of vector-valued linear SPDEs. This process corresponds to the angular component of the solution constrained to the unit sphere in L^2, and the study investigates its ergodic behavior.

A physical motivation for our model arises from the so-called relativistic limit of the Klein-Gordon equation; see, for example, [29,30,44] for mathematical treatments, and [55,57] for physical discussions. It follows from these and many related papers that the solutions u^c to the relativistic Klein-Gordon equation—with c denoting the speed of light—converge, in an appropriate sense, to solutions of the Schrödinger equation. Since the L^2-norm of Schrödinger solutions is conserved, it is natural to consider a modification of the Klein-Gordon equation that also preserves the L^2-norm of its solutions, in the hope that such modified equations will provide a better approximation to the limiting Schrödinger dynamics, at least over intermediate time scales.

The main result concerning the existence and uniqueness of solutions for Eq. (1.1) is formulated in Theorem 2.10, and an extension of that result to the case of more regular initial data is presented in Theorem 2.11. Both theorems, whose proofs are presented in Sect. 3, show that the stochastic constrained *wave* equation (1.1) admits a unique global solution living in the tangent bundle TM. Notice that the coefficients in Eq. (1.1) are only locally Lipschitz and have cubic growth. This means that the global well-posedness of the equation cannot be proven directly. Namely, we first consider the equation in its mild formulation and prove that there exists a local maximal solution that is defined up to a certain stopping time τ. Next, we prove that such solution $z = (u, v)$ stays on the tangent bundle TM. Finally we prove suitable a-priori bounds for the solution that allow to show that the solution is global and unique.

In the second part of the paper, the well posedness result for the abstract stochastic equation (1.1) is applied to the specific case of a stochastic damped wave equation in a bounded domain $D \subset \mathbb{R}^d$, endowed with the Dirichlet boundary condition and constrained to live in M, the unitary sphere in $H := L^2(D)$. Namely, we consider the equation

$$\begin{cases} \mu\, \partial_t^2 u_\mu(t,\xi) + \mu |\partial_t u_\mu(t)|_H^2 u_\mu(t,\xi) \\ \qquad\qquad = \Delta u_\mu(t,\xi) + |\nabla u_\mu(t)|_H^2 u_\mu(t,\xi) - \gamma\, \partial_t u_\mu(t,\xi) + \sigma(u_\mu(t))\, \partial_t w^Q(t,\xi) \\ u_\mu(0,\xi) = u_0(\xi), \quad \partial_t u_\mu(0,\xi) = v_0(\xi), \qquad u_\mu(t,\xi) = 0, \quad \xi \in \partial D, \end{cases} \tag{1.2}$$

depending on a positive parameter μ, where γ is a positive constant, w^Q is a Wiener process on H, with reproducing kernel Hilbert space K and covariance operator Q, and the mapping $\sigma : H_0^1(D) \to \mathscr{L}(H)$ is such that $\sigma(u)$ projects H onto the tangent space $T_u M$. Notice that, since here the diffusion coefficient depends on the unknown position u_μ and not on the velocity $\partial_t u_\mu$, the Stratonovich trace term is equal to zero and the Stratonovich and the Itô formulations coincide.

As we mentioned above, our aim is studying the limiting behavior of the solution u_μ of Eq. (1.2), when the mass μ goes to zero. Namely, we fix an arbitrary condition (u_0, v_0) that is sufficiently smooth and lives in the tangent bundle of M and we show that u_μ converges in probability, in a suitable functional space, to the unique solution of the equation

$$\begin{cases} \gamma\, \partial_t u(t,\xi) = \Delta u(t,\xi) + |\nabla u(t)|_H^2 u(t,\xi) - \dfrac{1}{2} \|\sigma(u(t))\|_{\mathscr{T}_2(K,H)}^2 u(t) + \sigma(u(t)) \partial_t w^Q(t,\xi), \\ u(0,\xi) = u_0(\xi), \quad u(t,\xi) = 0, \quad \xi \in \partial D. \end{cases} \tag{1.3}$$

In particular, this means that in the diffusion-approximation limit the term

$$\mu|\partial_t u_\mu(t)|_H^2 u_\mu(t)$$

converges, as $\mu \to 0$, to the non-trivial term

$$-\frac{1}{2}\|\sigma(u(t))\|^2_{\mathscr{T}_2(K,H)}u(t), \tag{1.4}$$

which depends on the diffusion coefficient σ through its $\mathscr{T}_2(K,H)$ norm, where $\mathscr{T}_2(K,H)$ denotes the space of Hilbert-Schmidt operators from K to H (see Sect. 2 for all details). It is important to stress that the coefficient (1.4) does not coincides with the Stratonovich-to-Itô correction term. Moreover, as we will show later with a concrete example, the solution to Eq. (1.3) does not coincide with the solution of the constrained parabolic equation perturbed by Stratonovich-type noise

$$\begin{cases} \gamma\,\partial_t u(t,\xi) = \Delta u(t,\xi) + |\nabla u(t)|_H^2 u(t,\xi) + \sigma(u(t)) \circ \partial_t w^Q(t,\xi), \\ u(0,\xi) = u_0(\xi), \quad u(t,\xi) = 0, \quad \xi \in \partial D, \end{cases} \tag{1.5}$$

which is the only example of constrained stochastic heat equation considered in the existing literature so far. In particular, our limiting result provides a new example of a constrained stochastic parabolic problem, which arises in a concrete situation such as the small mass limit for Eq. (1.2).

While this paper is the first one handling the case of SPDEs with constraints, a series of papers have investigated the validity of the so-called Smoluchowski–Kramers approximation, that describes the limiting behavior of the solution u_μ, as the mass μ vanishes. For the finite dimensional case, the existing literature is quite broad and we refer in particular to [32,33,36,37,40,53,54] (see also [18,25,42] for systems subject to a magnetic field and [19,38,45] for some related multiscaling problems). We should also mention here that a simple model of the Smoluchwski–Kramers phenomenon for stochastic SDEs has been investigated by Nelson in Chapters 9 and 10 of his famous book [43]. In recent years there has been an intense activity dealing with the Smoluchowski–Kramers diffusion approximation of infinite dimensional systems. To this purpose, we refer to [16,17,20,48,50] for the case of constant damping term (see also [24] where systems subject to a magnetic field are studied), and to [26,27] for the case of state-dependent damping. As a matter of fact, these two situations are quite different, as in the case of non-constant friction a noise-induced term emerges from the small mass limit.

The study of the Smoluchowski–Kramers approximation does not reduce only to the proof of the limit of the solutions u_μ. Actually, it is crucial to ascertain the stability of such an approximation in relation to other significant asymptotic characteristics exhibited by the two systems, such as their long-term behaviors, for instance. To this purpose, in [16,21] it is shown that the statistically invariant states of the stochastic damped wave equation (in case of constant friction) converge in a suitable sense to the invariant measure of the limiting equation. In the same spirit, the papers [22,23,27] are devoted to the analysis of the interplay between the small mass and the small noise limit. In particular, [27] studies the validity of a large deviation principle for the trajectories of the solution, while [22,23] deal with the study of the convergence of the quasi-potential, that describes, as known, the asymptotics of the exit times and the large deviation principle for the invariant measure.

The current paper is the first one addressing the small mass limit for constrained infinite-dimensional systems. To the best of our knowledge, the only other paper in the existing literature that investigates this particular problem is [1], which focuses on general manifolds in the finite-dimensional case. As mentioned in the introduction of [1], "Brownian motion of micro and nanoparticles occurring in complex environments can often be represented as two-dimensional or one-dimensional manifolds embedded within a three-dimensional space. For example, the motion of proteins on cellular membranes occurs effectively on two-dimensional manifolds". Thus, the physical motivation of [1] was to present Brownian motion on a manifold as the zero-mass limit of an inertial system. In the present paper, we will be able to address an analogous problem in the case of space-dependent systems.

The transition from a finite number of degrees of freedom to an infinite number presents considerable challenges and complexities. The strategy we follow in our proof is somehow standard: we first prove suitable uniform bounds with respect to $\mu \in (0,1)$ for the family of solutions $\{u_\mu\}_{\mu\in(0,1)}$, then, thanks to those bounds, we prove that the family $\{u_\mu\}_{\mu\in(0,1)}$ is tight in a suitable functional space and, finally, we identify any limiting point for the family $\{u_\mu\}_{\mu\in(0,1)}$ with the unique solution of the limiting problem (1.3). Nevertheless, the demonstration of these steps is quite challenging and necessitates the introduction of novel arguments and techniques.

Specifically, we must establish uniform bounds, with respect to μ, for the solutions of (1.2) within functional spaces possessing higher regularity than $\mathscr{H} = H^1_0(D) \times H$, and the presence of cubic terms in the equation adds an extraordinary level of complexity to proving such bounds. The need for delicate uniform bounds for the solution u_μ, with respect to $\mu \in (0,1)$, is not specific to this paper but is a key feature of all the existing literature on the Smoluchowski–Kramers approximation mentioned above. However, in this work, we require a priori bounds in spaces of higher regularity than those considered in previous papers, such as [16,17,21,26]. The first important reason for this necessity is that we need to take the limit of $|\nabla u_\mu(t)|^2_H u_\mu(t)$, and no integration by parts can be applied to handle the nonlocal term $|\nabla u_\mu(t)|^2_H$. To ensure the required tightness of $\{u_\mu\}_{\mu\in(0,1)}$ in $H^1(D)$, among other considerations, we require a priori bounds in $H^\alpha(D)$ for $\alpha > 1$. It is important to note that, in this context, obtaining bounds in $H^\alpha(D)$ for $\alpha \in (1,2)$ is not any easier than obtaining bounds in $H^2(D)$, due to the specific nature of Eq. (1.2). Another reason for requiring bounds in higher-regularity spaces is that, in the proof of the limit, we need uniform bounds for the solution in $L^2(0,T;H^2(D))$. Given the peculiar and highly nontrivial nature of Eq. (1.2), achieving this requires obtaining bounds for $(u_\mu, \sqrt{\mu}\,\partial_t u_\mu)$ in $H^3(D) \times H^2(D)$. Finally, we emphasize that the cubic nature of the nonlinearities introduces additional challenges, as we must establish uniform bounds for the fourth moments of the solution and its time derivative. For this purpose, see Lemmas 6.1 and 7.1.

Before concluding this introduction, we provide a brief overview of the contents of our paper. The first two sections are dedicated to the examination of the well-posedness of the abstract problem (1.1). In Sect. 2, we present the notation and assumptions, and describe how the abstract damped wave equation can be introduced in the deterministic setting. We then introduce the stochastically forced version of the equation and establish a series of preliminary results concerning the diffusion coefficient σ. Finally, we state the two main results concerning the existence and uniqueness of solutions: Theorem 2.10 and Theorem 2.11. Section 3 is dedicated to providing the proofs of these two theorems.

In the remaining seven sections of our paper, we delve into the examination of the validity of the Smoluchowski–Kramers diffusion approximation for the system (1.2). Section 4 is dedicated to introducing the necessary notation and assumptions. In Theorem 4.5, we present the main result of this study. In Sect. 5, we provide a concrete example to illustrate that our limiting equation (1.3) and equation (1.5) are, in fact, two distinct equations. The subsequent two sections focus on establishing the required uniform bounds for the solution $(u_\mu, \partial_t u_\mu)$ of Eq. (1.2). In Sect. 6, we prove bounds in $H^1_0(D) \times H$, while in Sect. 7, we establish bounds in $H^2(D) \times H^1_0(D)$. Section 8 addresses the proof of the tightness of $\{\mathcal{L}(u_\mu)\}_{\mu\in(0,1)}$ within the appropriate functional space. Finally, in Sect. 9, we conclude the proof of Theorem 2.11 by identifying any limit point of the family $\{\mathcal{L}(u_\mu)\}_{\mu\in(0,1)}$ as the unique solution of Eq. (1.3).

2. The Well-Posedness: Notations, Assumptions and Main Results

Let us briefly introduce the basic notations. We will denote by H a separable Hilbert space endowed with an inner product $\langle\cdot,\cdot\rangle_H$ and the corresponding norm $|\cdot|_H$. If E and F are Banach spaces, the class of all bounded linear operators from E to F will be denoted by $\mathscr{L}(E, F)$. We will use a shortcut notation $\mathscr{L}(E)$ for $\mathscr{L}(E, E)$. It is known that $\mathscr{L}(E, F)$ is also a Banach space. We will denote by $\mathscr{L}_2(E \times E; F)$ the Banach space of all bounded bilinear operators from $E \times E =: E^2$ to F. If K is another Hilbert space, we will denote by $\mathscr{T}_2(K, H)$, or $\gamma(K, H)$, the Hilbert space of all Hilbert-Schmidt operators from K to H, endowed with the natural inner product and norm. It is known that $\mathscr{T}_2(K, H) \hookrightarrow \mathscr{L}(K, H)$ continuously. If $\{e_j\}_{j\in\mathbb{N}}$ is an orthonormal basis of a separable Hilbert space K which is continuously embedded into a Banach space E and

$$\sum_{j=1}^{\infty} |e_j|_E^2 < \infty,$$

then for every $\Lambda \in \mathscr{L}_2(E \times E; H)$ we put

$$\mathrm{tr}_K(\Lambda) = \sum_{i=1}^{\infty} \Lambda(e_j, e_j). \tag{2.1}$$

If X is a normed vector space, $a \in X$ and $r > 0$, then we will denote by $B_X(a, r)$, respectively $S_X(a, r)$, the open ball, respectively the sphere, in X of radius r and center a.

In what follows, we shall assume that A_0 is a non-negative self-adjoint operator on H and we shall denote its domain by $D(A_0)$. If we put

$$A_1 := \sqrt{A_0^2 + \delta I}, \tag{2.2}$$

with $\delta = 0$ if A_0 is invertible (i.e. A_0 is injective, surjective and the inverse A_0^{-1} is bounded), and $\delta = 1$ otherwise, then A_1, with $D(A_1) = D(A_0)$, is a strictly positive self-adjoint operator on H and 0 belongs to $\rho(A_1)$, the resolvent set of the operator A_1. In particular, the inverse $A_1^{-1} : H \to H$ is bounded. Whenever we will use the space $D(A_1)$ we will always mean that it is endowed with the norm $|A_1 \cdot|_H$ and the corresponding inner product.

We will denote by $\mathscr{H}$ the Hilbert space

$$\mathscr{H} := D(A_1) \times H = D(A_0) \times H,$$

endowed with the following inner product

$$\langle z_1, z_2\rangle_{\mathscr{H}} = \langle A_1 u_1, A_1 u_2\rangle_H + \langle v_1, v_2\rangle_H,$$

The corresponding norm $|\cdot|_{\mathscr{H}}$ satisfies

$$|z|^2_{\mathscr{H}} = |A_1 u|^2_H + |v|^2_H, \quad z = (u, v) \in \mathscr{H}.$$

We will also use the following scale of Hilbert spaces

$$\mathscr{H}_\alpha := D(A_1^{1+\alpha}) \times D(A_1^\alpha), \quad \alpha \geq 0.$$

Each space $\mathscr{H}_\alpha$ is endowed with an inner product defined for every $z_i = (u_i, v_i) \in \mathscr{H}_\alpha$ by

$$\langle z_1, z_2\rangle_{\mathscr{H}_\alpha} := \langle A_1^{1+\alpha} u_1, A_1^{1+\alpha} u_2\rangle_H + \langle A_1^\alpha v_1, A_1^\alpha v_2\rangle_H.$$

Note that obviously $\mathscr{H}_0 = \mathscr{H}$, with equal norms and inner products.

Next, we introduce the linear operator $\mathscr{A}$ in the space $\mathscr{H}$ as follows,

$$\mathscr{A} z := (v, -A_0^2 u), \quad z = (u, v) \in D(\mathscr{A}) := \mathscr{H}_1. \tag{2.3}$$

It is well known that $\mathscr{A}$ generates a C_0 group (of exponential growth) $\mathscr{S} = (\mathscr{S}(t))_{t\in\mathbb{R}}$ on $\mathscr{H}$, see e.g. [10] and references therein. If A_0 is invertible (so that we take $\delta = 0$ in (2.2)) then $\mathscr{S}$ is a unitary group. The restriction $\mathscr{A}_\alpha$ of the operator $\mathscr{A}$ defined by

$$\mathscr{A}_\alpha z = \mathscr{A} z, \quad z \in D(\mathscr{A}_\alpha) := \mathscr{H}_{\alpha+1}$$

is the generator of a C_0 group $\mathscr{S}_\alpha = (\mathscr{S}_\alpha(t))_{t\in\mathbb{R}}$ on $\mathscr{H}_\alpha$ and $\mathscr{S}_\alpha(t)$ is the restriction of $\mathscr{S}(t)$ to the space $\mathscr{H}_\alpha$. In what follows, we will not make this distinction and denote all these objects without the subscript α, unless our approach could lead to ambiguity.

Let us also consider a separable Hilbert space K and a separable Banach space E such that $K \subset E$ continuously and the embedding

$$i : K \hookrightarrow E \text{ is gamma-radonifying.} \tag{2.4}$$

By the Kwapień-Szymański Theorem [41] there exists an orthonormal basis $\{e_j\}_{j\in\mathbb{N}}$ such that

$$\sum_{j\geq 1} |ie_j|^2_E < \infty.$$

We assume that $W_j = (W_j(t) : t \geq 0)$, $j \in \mathbb{N}$ is a sequence of iid real Wiener processes defined on some filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual assumptions. Let also

$$W(t) = \sum_{j=1}^{\infty} W_j(t) i e_j$$

be a E-valued Wiener process. The Reproducing Kernel Hilbert space of the law of $W(1)$ is equal to the space K and the process W can also be viewed as a canonical K-cylindrical Wiener process.

We assume now that γ is a positive constant (we call it the damping coefficient) and fix a mapping $\sigma_0 : \mathscr{H} \to \mathscr{L}(E, H)$. Our aim is to study a certain *constrained* version of the following abstract damped wave equation

$$u_{tt}(t) + A_0^2 u(t) = -\gamma u_t(t) + \sigma_0(u(t), u_t(t))\, dW(t), \tag{2.5}$$

with the following initial conditions

$$u(0) = u_0 \in D(A_0), \quad u_t(0) = v_0 \in H. \tag{2.6}$$

By *constrained* we mean that we want our solution u to stay on M, where M is the unit sphere in H, i.e.

$$M = S_H(0, 1) := \big\{x \in H : |x|_H = 1\big\}.$$

In particular, we need to assume that the initial data u_0 satisfy the same condition, i.e.

$$|u_0|_H = 1. \tag{2.7}$$

In what follows, it is convenient to use the tangent bundle of M which in the present framework can be defined as

$$\mathscr{M} := TM = \bigcup_{u \in M} \{(u, v) \in M \times T_u M\}, \qquad T_u M = \{v \in H : \langle u, v\rangle_H = 0\}.$$

Note that $\mathscr{M}$ is a Hilbert manifold modeled on the Hilbert space $H \times H$. Moreover, $\mathscr{M}$ is a closed subspace of $\mathscr{H}$. We will endow the former set with a metric inherited from the latter space.

One can heuristically see that if $u(t) \in M$, for all $t \geq 0$, then the following property is also verified

$$\langle u(t), u_t(t)\rangle_H = 0, \quad t \geq 0.$$

Hence, in addition to (2.7), we also need to assume the following condition on the initial data

$$\langle u_0, v_0\rangle_H = 0. \tag{2.8}$$

It is quite obvious that a solution to Eq. (2.5) with initial conditions (2.6) will not necessarily stay on the manifold M, even though the initial data (u_0, v_0) satisfy the compatibility conditions (2.7) and (2.8), see [5]. We will show below that it is possible to resolve this conundrum by modifying Eq. (2.5). In order to find the appropriate modification one can think of its deterministic part as the equation

$$u_{tt}(t) + \nabla_u \Phi_0(u(t)) = -\gamma\, u_t(t), \tag{2.9}$$

where the gradient ∇_u is understood in the H-sense and the *energy function* Φ_0 is defined by

$$\Phi_0(u) = \frac{1}{2}|A_0 u|_H^2, \quad u \in D(A_0).$$

Recalling that M is the unit sphere $S_H(0, 1)$, if Φ is the *restricted* energy functional

$$\Phi(u) = \Phi_0(u), \quad u \in D(A_0) \cap M,$$

we can replace the term $\nabla_u \Phi_0$ by $\widetilde{\nabla}_u \Phi_0$. Here, the gradient $\widetilde{\nabla}_u$ is understood as the gradient of M with respect to the *metric* on M inherited from H, i.e.

$$\widetilde{\nabla}_a \Phi = \Pi_a(\nabla_a \Phi_0) \in T_a M, \quad a \in M \cap D(A_0),$$

where, for every $a \in M$, we denote by $\Pi_a \in \mathscr{L}(H)$ the orthogonal projection onto $T_a M$, i.e.

$$\Pi_a : H \ni u \mapsto u - \langle u, a \rangle_H \, a \in H. \tag{2.10}$$

Since we have

$$\nabla_a \Phi_0 = A_0^2 a, \quad a \in D(A_0^2),$$

we infer that for $a \in D(A_0) \cap M$,

$$\widetilde{\nabla}_a \Phi = \Pi_a(A_0^2 a) = A_0^2 a - \langle A_0^2 a, a \rangle_H a = A_0^2 a - |A_0 a|_H^2 a.$$

Moreover, the *acceleration* term $u_{tt}(t)$ has also to be modified in a similar fashion so that $u(t)$ stays on M, i.e. we need to replace it by

$$\Pi_{u(t)}(u_{tt}(t)) = u_{tt}(t) - \langle u_{tt}(t), u(t) \rangle_H u(t).$$

Now, since we are assuming that $|u(t)|_H = 1$, it is immediate to check that

$$\langle u_{tt}(t), u(t) \rangle_H = -|u_t(t)|_H^2, \quad t \geq 0,$$

so that

$$\Pi_{u(t)}(u_{tt}(t)) = u_{tt}(t) + |u_t(t)|_H^2 u(t).$$

In this way we obtain the following constrained version of Eq. (2.9)

$$u_{tt}(t) + |u_t(t)|_H^2 u(t) = -A_0^2 u(t) + |A_0 u(t)|_H^2 u(t) - \gamma \, u_t(t). \tag{2.11}$$

The above heuristic argument can be made rigorous, as shown in the following two theorems, whose proofs are postponed to next section, where we will consider the more general stochastic case.

Theorem 2.1. *Assume that $f \in L^2_{\text{loc}}([0, \infty; H))$. If the initial data $(u_0, v_0) \in \mathscr{H}$ satisfy the compatibility conditions (2.7) and (2.8), then there exists a unique function $(u, v) \in C\big([0, \infty\big); \mathscr{H})$ such that*

$$u \in C^1\big([0, \infty\big); M), \quad u_t(t) = v(t), \ \ t \geq 0,$$

and u is a mild solution to the following equation

$$u_{tt}(t) + |u_t(t)|_H^2 u(t) = -A_0^2 u(t) + |A_0 \, u(t)|_H^2 u(t) - \gamma \, u_t(y) + \Pi_{u(t)} f(t). \tag{2.12}$$

Moreover, if we denote

$$\Psi(u, v) := \frac{1}{2}\left(|A_0 u|_H^2 + |v|_H^2\right), \quad (u, v) \in \mathscr{H}, \tag{2.13}$$

then for every $t \geq 0$ we have

$$\Psi(z(t)) = \Psi(z_0) - \gamma \int_0^t |v(s)|_H^2 \, ds + \int_0^t \langle v(s), f(s) \rangle_H \, ds.$$

The above result can be strengthened in the following way.

Theorem 2.2. *Assume that* $\alpha \geq 0$ *and fix* $f \in L^2_{\mathrm{loc}}([0,\infty; D(A_0^\alpha))$. *If the initial data* $(u_0, v_0) \in \mathscr{H}_\alpha$ *satisfy conditions* (2.7) *and* (2.8)*, then the unique solution to problem* (2.12)*, guaranteed by Theorem 2.1, satisfies*

$$(u, v) \in C\big([0,\infty); \mathscr{H}_\alpha\big), \quad u \in C^1\big([0,\infty); H_{\alpha+1}\big).$$

Notice that the proofs of the above two results require the following assertions about the non-linearities appearing in Eqs. (2.11) and (2.12).

Lemma 2.3. *The following function*

$$F : \mathscr{H} \ni z = (u, v) \mapsto |v|_H^2 u + |A_0 u|_H^2 u \in H,$$

is well defined and is a homogenous continuous polynomial of degree 3. *In particular, it is Lipschitz-continuous on balls. Moreover, the same result holds if* $\alpha \geq 0$ *and the spaces* H *and* $\mathscr{H}$ *are replaced respectively by* $D(A_0^\alpha)$ *and* $\mathscr{H}_\alpha$.

The proof of this result is obvious and we omit it. We just observe that the corresponding continuous trilinear function is given by

$$F : \mathscr{H}^3 \ni (z^1, z^2, z^3) \mapsto \langle v_1, v_2\rangle_H u_3 + \langle A_0\, u_1, A_0\, u_2\rangle_H u_3 \in H,$$

where $z^i = (u_i, v_i) \in \mathscr{H}$, for $i = 1, 2, 3$.

2.1. The stochastic constrained wave equation. First of all, we need to introduce the diffusion coefficient σ. We begin with a function $\sigma_0 : \mathscr{H} \to \mathscr{L}(E, H)$ which we assume to satisfy the following conditions.

Hypothesis 1. *The mapping*

$$\sigma_0 : \mathscr{H} \to \mathscr{L}(E, H)$$

is of linear growth and Lipschitz-continuous on balls, i.e. there exist $L \geq 0$ *and a sequence* $(L_n)_{n=1}^\infty$ *of nonnegative real numbers such that*

$$\begin{aligned} &\|\sigma_0(z)\|^2_{\mathscr{L}(E,H)} \leq L(1 + |z|^2_{\mathscr{H}}), \quad z \in \mathscr{H}, \\ &\|\sigma_0(z_2) - \sigma_0(z_2)\|^2_{\mathscr{L}(E,H)} \leq L_n |z_2 - z_1|^2_{\mathscr{H}}, \quad z_1, z_2 \in B_{\mathscr{H}}(0, n), \ n \in \mathbb{N}^*. \end{aligned} \tag{2.14}$$

Remark 2.4. Because of the Stratonovich integral we are going to use, we need to work with the space $\mathscr{L}(E, H)$ instead of the usual $\mathscr{T}_2(K, H)$. Note that in view of assumption (2.4), the former space is naturally emendable into the latter. To be precise, if $L \in \mathscr{L}(E, H)$ then $L \circ i \in \mathscr{T}_2(K, H)$ and the corresponding linear map is continuous. □

As in the deterministic case, we have to modify σ_0 by taking its tangential component. Thus, we define the function

$$\sigma : \mathscr{H} \to \mathscr{L}(E, H)$$

by setting for every $z = (u, v) \in \mathscr{H}$

$$\sigma(z) := \Pi_u \circ \big(\sigma_0(z)\cdot\big) = \sigma_0(z) - \langle \sigma_0(z)\cdot, u\rangle_H\, u \in \mathscr{L}(E, H), \tag{2.15}$$

where Π_u is the projection defined in (2.10). Note that in general

$$\sigma \neq \sigma_0.$$

In the above, we used the following notation for $B \in \mathscr{L}(E, H)$ and $u_1, u_2 \in H$,

$$\langle B\cdot, u_1\rangle_H u_2 = \{E \ni k \mapsto \langle Bk, u_1\rangle_H\, u_2 \in H\} \in \mathscr{L}(E, H).$$

Similarly, for $B \in \mathscr{T}_2(K, H)$ and $u_1, u_2 \in H$, we denote

$$\langle B\cdot, u_1\rangle_H u_2 = \{K \ni k \mapsto \langle Bk, u_1\rangle_H\, u_2 \in H\} \in \mathscr{L}(K, H).$$

In the latter case, it is obvious that for all $u_1, u_2 \in H$, $\langle B\cdot, u\rangle_H u$ is a bounded linear operator from K to H.

Moreover, if $B \in \mathscr{T}_2(K, H)$ and $u_1, u_2 \in H$, then

$$\langle B\cdot, u_1\rangle_H\, u_2 \in \mathscr{T}_2(K, H)$$

and

$$\|\langle B\cdot, u_1\rangle_H u_2\|_{\mathscr{T}_2(K,H)} \leq |u_1|_H\, |u_2|_H \|B\|_{\mathscr{T}_2(K,H)}, \quad u_1, u_2 \in H.$$

Actually, if $\{e_j\}_{j\in\mathbb{N}}$ is an orthonormal basis of K, then we have

$$\sum_{j=1}^{\infty} |\langle Be_j, u_1\rangle_H\, u_2|_H^2 \leq |u_1|_H^2 |u_2|_H^2 \sum_{j=1}^{\infty} |Be_j|_H^2 = |u_1|_H^2 |u_2|_H^2\, \|B\|_{\mathscr{T}_2(K,H)}^2.$$

We begin by noticing that, since Π_u is an orthogonal projection, as a consequence of the definition (2.15) for every $z = (u, v) \in \mathscr{H}$ and every $e \in E$ we have

$$|\sigma(z)e|_H \leq |\sigma_0(z)e|_H. \tag{2.16}$$

Moreover, σ satisfies the following properties.

Lemma 2.5. *The function σ is Lipschitz-continuous on balls and has cubic growth, as a function defined on $\mathscr{H}$ with values in $\mathscr{L}(E, H)$. More precisely, for every $z = (u, v) \in \mathscr{H}$*

$$\begin{aligned} &\|\sigma(z)\|_{\mathscr{L}(E,H)} \leq \|\sigma_0(z)\|_{\mathscr{L}(E,H)} \leq L(1 + |z|_{\mathscr{H}}),\\ &\|\sigma(z)\|_{\mathscr{L}(E,H)} \leq \sqrt{L}(1 + |z|_{\mathscr{H}})(1 + |u|_{D(A_0)}|u|_H), \quad . \end{aligned} \tag{2.17}$$

Moreover, for every $z \in M$ and $e \in E$

$$\sigma(z)\, e \in T_u M. \tag{2.18}$$

Proof. Let us begin by observing that property (2.18) is an immediate consequence of definition (2.15). Moreover, (2.17) follows from (2.14) and (2.16).

Next, by the definition of the function σ we have

$$\begin{aligned} |\sigma(z)(e)|_{D(A_0)} &\leq |\sigma_0(z)e|_{D(A_0)} + |\langle \sigma_0(z)e, u\rangle_H||u|_{D(A_0)}\\ &\leq |\sigma_0(z)e|_{D(A_0)} + |\sigma_0(z)e|_H|u|_H|u|_{D(A_0)}\\ &\leq \|\sigma_0(z)\|_{\mathscr{L}(E,D(A_0))}|e|_E + \|\sigma_0(z)\|_{\mathscr{L}(E,H)}|e|_E|u|_{D(A_0)}|u|_H\\ &\leq \sqrt{L}(1 + |z|_{\mathscr{H}})\big(1 + |u|_{D(A_0)}|u|_H\big)|e|_E, \end{aligned} \tag{2.19}$$

and hence (2.19) follows.

In order to prove that the map σ is Lipschitz-continuous on balls it is sufficient to prove that the second term on the RHS of (2.15), i.e. the map

$$(u,v)\in\mathscr{H}\mapsto\langle\sigma_0(u,v)\cdot,u\rangle_H u\in\mathscr{L}(E,H),$$

is Lipschitz-continuous on balls. For this purpose, if we fix $z_i=(u_i,v_i)\in\mathscr{H}$, $i=1,2$, we have

$$\begin{aligned}\langle\sigma_0(z_2)\cdot,u_2\rangle_H u_2-\langle\sigma_0(z_1)\cdot,u_1\rangle_H u_1&=\langle(\sigma_0(z_2)-\sigma_0(z_1))\cdot,u_2\rangle_H u_2\\&\quad+\langle\sigma(z_1)\cdot,u_2-u_1\rangle_H u_2+\langle\sigma_0(z_1)\cdot,u_1\rangle_H(u_2-u_1).\end{aligned}$$

This implies that

$$\begin{aligned}\|\langle\sigma_0(z_2)\cdot,u_2\rangle_H u_2-\langle\sigma_0(z_1)\cdot,u_1\rangle_H u_1\|_{\mathscr{L}(E,H)}&\leq|u_2|_H^2\|\sigma_0(z_2)-\sigma_0(z_1)\|_{\mathscr{L}(E,H)}\\&\quad+\|\sigma_0(z_1)\|_{\mathscr{L}(E,H)}|u_2-u_1|_H\left(|u_1|_H+|u_2|_H\right).\end{aligned}$$

Therefore, since the map $\sigma_0:\mathscr{H}\to\mathscr{L}(E,H)$ is Lipschitz-continuous on balls and of linear growth, the result follows. □

Once we have defined the diffusion operator σ, the stochastic version of Eq. (2.11) can be written as

$$u_{tt}(t)+|u_t(t)|_H^2u(t)=-A_0^2u(t)+|A_0u(t)|_H^2\,u(t)-\gamma u_t(t)+\sigma(u(t),u_t(t))\circ dW(t),\tag{2.20}$$

with the initial data satisfying $(u_0,v_0)\in\mathscr{M}$.

Remark 2.6. In this paper we have made the choice to consider the stochastic differential in the Stratonovich sense. For general constraints, this ensures that the constraint is preserved for all $t\geq 0$ In the specific case of our equation, where the solution is constrained to the unit sphere in L^2, we could have also used the Itô formulation while still maintaining the constraint. However, since the analysis in this case does not require different arguments or techniques, we focus solely on noise in the Stratonovich sense for the sake of brevity.

Remark 2.7. 1. The definition we gave in (2.15) for the function σ is not unique. We only require that condition (2.18) is satisfied. As a matter of fact, formula (2.15) is simply one of many that fulfills this assumption. Another one is the following

$$\widehat{\sigma}(z)=\phi(|u|_H)\,\sigma(z),\quad z\in\mathscr{H},$$

where $\phi:[0,\infty)\to[0,\infty)$ is an auxiliary C_0^∞ bump function such that

$$\phi(r)=\begin{cases}1,&\text{if}\quad|r-1|\leq\frac{1}{4},\\0,&\text{if}\quad|r-1|\geq\frac{1}{2}.\end{cases}$$

2. If we assume that the function σ_0 depends only on the first component u of $z=(u,v)$, i.e. there exists a function

$$g_0:H\to\mathscr{L}(E,H)\tag{2.21}$$

such that

$$\sigma_0(z)=g_0(u),\quad z=(u,v)\in\mathscr{H},\tag{2.22}$$

then we can modify g_0 by taking its tangent part. This means that with a slight abuse of notation, we can define

$$g(u) := \Pi_u(g_0(u)) = g_0(u) - \langle g_0(u)\cdot, u\rangle_H u \in \mathscr{L}(E, H), \quad u \in H.$$

Then, the function σ associated by formula (2.15) with the function σ_0 defined in formula (2.22) satisfies

$$\sigma(z) = g(u), \quad z = (u, v) \in \mathscr{L}(E, H).$$

In this case, it is possible to weaken assumptions on g_0 and to assume that g_0 is only defined on M and Lipschitz-continuous. Then by the classical Kirszbraun Theorem [39], see also [3, 12], we can find a (globally) Lipschitz-continuous and bounded extension of g_0 from M to the whole H.

3. We could have added a force f term to the above equation as in the deterministic equation (2.12), assuming only that f is an H-valued progressively measurable process such that $f \in L^2_{\mathrm{loc}}([0, \infty; H))$, $\mathbb{P}$-almost surely. But for the sake of simplicity of exposition we have not done so. Indeed, in this paper we concentrate on different issues. □

As we already mentioned in the present paper we decided to study the equation above in the Stratonovich sense. In fact, we can rewrite the Stratonovich term using the standard Itô differential, see e.g. [7]. To this purpose, with the notations we have introduced above, if we denote $z = (u, u_t)$, we rewrite the second order in time equation (2.20) as a system of two equations of first order in time

$$\begin{cases} dz(t) = \mathscr{A} z(t)\, dt + \big(-|u_t(t)|_H^2 u(t) + |A_0 u(t)|_H^2 u(t) - \gamma\, u_t(t)\big)\, dt \\ \qquad\qquad + (0, \sigma(z(t)))\, \circ\, dW(t), \\ z(0) = (u_0, v_0). \end{cases} \tag{2.23}$$

For a C^1-class function $G : \mathscr{H} \to \mathscr{L}(E, \mathscr{H})$, we define, see [7, Definition 3.1],

$$\int_0^t G(z(s))\, \circ\, dW(s) := \int_0^t G(z(s))\, dW(s) + \frac{1}{2}\int_0^t \mathrm{tr}_K\big[G'(z(s))G(z(s))\big]\, ds,$$

with tr_K defined as in (2.1). Note that (see comments after [7, Definition 3.1]) for all $z \in \mathscr{H}$ we have $d_z G = G'(z) \in \mathscr{L}(\mathscr{H}; \mathscr{L}(E, \mathscr{H}))$ so that

$$G'(z)\, G(z) := d_z G \cdot G(z) \in \mathscr{L}(E; \mathscr{L}(E, \mathscr{H})) \equiv \mathscr{L}_2(E \times E, \mathscr{H}),$$

i.e.

$$G'(z)G(z)(e_1, e_2) = \Big[d_z G\big(G(z)e_1\big)\Big]e_2, \quad (e_1, e_2) \in E \times E.$$

This means that $\mathrm{tr}_K\big[G'(z(s)G(z(s))\big]$ is a well defined element of $\mathscr{H}$ and satisfies

$$\mathrm{tr}_K\Big[G'(z)G(z)\Big] = \sum_{i=1}^{\infty}\Big[d_z G\big(G(z)e_j\big)\Big]e_j, \tag{2.24}$$

where $\{e_j\}_{j\in\mathbb{N}}$ is an orthonormal basis of K. In particular, if

$$G : \mathscr{H} \ni z = (u, v) \mapsto \{E \ni k \mapsto (0, \sigma(z)k) \in \mathscr{H}\} \in \mathscr{L}(E, \mathscr{H}),$$

where $\sigma : \mathscr{H} \to \mathscr{L}(E,H)$, then for every $z=(u,v)$ and $w=(x,y)$ in $\mathscr{H}$ we have the following expression for the Fréchet derivative of G,

$$[d_zG](w) = [d_{(u,v)}G](x,y) = \big(0, [\partial_u\sigma(z)](x) + [\partial_v\sigma(z)](y)\big),$$

where

$$\partial_u\sigma(z) \in \mathscr{L}\big(D(A_0), \mathscr{L}(E,H)\big), \quad \partial_v\sigma(z) \in \mathscr{L}\big(H, \mathscr{L}(E,H)\big),$$

are the directional Fréchet derivatives of function σ at z. Therefore, we deduce that for every $z \in \mathscr{H}$ we have

$$G'(z)G(z) = [d_zG](G(z)) = \big(0, [\partial_v\sigma(z)](\sigma(z))\big),$$

and

$$\mathrm{tr}_K\big[G'(z)G(z)\big] = \mathrm{tr}_K\big[[d_zG]G(z)\big] = \big(0, \mathrm{tr}_K\big[\partial_v\sigma(z)\,\sigma(z)\big]\big).$$

In view of formula (2.24) we have

$$\mathrm{tr}_K\Big[\partial_v\sigma(z)\,\sigma(z)\Big] = \sum_{i=1}^{\infty} \partial_v\sigma(z)\big(\sigma(z)e_j\big)e_j. \tag{2.25}$$

Thus, we have the following formula for the Stratonovich integral in Eq. (2.23)

$$\begin{aligned}\int_0^t \big(0, \sigma(z(s))\big) \circ dW(s) &= \int_0^t \big(0, \sigma(z(s))\big)\, dW(s) \\ &\quad + \frac{1}{2}\int_0^t \Big(0, \mathrm{tr}_K\big[\partial_v\sigma(z(s))\,\sigma(z(s))\big]\Big)\, ds.\end{aligned}$$

The above results explain why we need to make the following additional assumption.

Hypothesis 2. *The function $\sigma_0 : \mathscr{H} \to \mathscr{L}(E,H)$ is of C^1-class in the sense that the directional Fréchet derivative $\partial_v\sigma_0(z) \in \mathscr{L}\big(H, \mathscr{L}(E,H)\big)$ exists for every $z \in \mathscr{H}$ and the map*

$$\partial_v\sigma_0 : \mathscr{H} \ni z \mapsto \partial_v\sigma_0(z) \in \mathscr{L}\big(H, \mathscr{L}(E,H)\big),$$

is continuous. Moreover, the function

$$\mathscr{H} \ni z \mapsto \mathrm{tr}_K\big[\partial_v\sigma_0(z)\,\sigma(z)\big] \in H, \tag{2.26}$$

is Lipschitz-continuous on balls and has linear growth.

Remark 2.8. the framework of Remark 2.7-2, we have

$$\mathrm{tr}_K\big[\partial_v\sigma(z)\sigma(z)\big] = \mathrm{tr}_K\big[\partial_v g(z)\tilde{g}(z)\big] = 0, \quad z \in \mathscr{M}.$$

Thus, in this case we do not need Hypothesis 2 and, instead of (2.21), we can assume that $g_0 : H \to \mathscr{T}_2(K,H)$. Moreover, in Theorem 2.10 we will need to assume that g_0 is Lipschitz-continuous on balls and of linear growth, while in Theorem 2.11 we will need to assume that the following *restriction* map

$$g_0 : D(A_0) \to \mathscr{T}_2(K, D(A_0)),$$

is Lipschitz-continuous on balls and of linear growth. □

Proposition 2.9. *Assume that Hypothesis 2 holds. Then the map $\sigma : \mathscr{H} \to \mathscr{L}(E, H)$ defined in (2.15) is of C^1-class in the sense of Hypothesis 2 and*

$$\partial_v \sigma(z)(y) = \partial_v \sigma_0(z)(y) - \langle \partial_v \sigma_0(z)(y)\cdot, u\rangle_H \, u, \quad z, y \in \mathscr{H}.$$

In particular,

$$\big[\partial_v \sigma(z)(y)\big]e = \big[\partial_v \sigma_0(z)(y)\big]e - \big\langle\big[\partial_v \sigma_0(z)(y)\big]e, u, \big\rangle_H u, \quad z, y \in \mathscr{H}, \quad e \in E. \tag{2.27}$$

Moreover, the function

$$\mathscr{H} \ni z \mapsto \mathrm{tr}_K\big[\partial_v \sigma(z)\, \sigma(z)\big] \in D(A_0),$$

is Lipschitz-continuous on balls and of polynomial growth.

Proof. The first part of the result follows directly by applying classical results from Cartan's treatise [15]. □

We shall prove the following stochastic generalisation of Theorem 2.1.

Theorem 2.10. *Assume that Hypotheses 1 and 2 hold. Then for every $z_0 = (u_0, v_0) \in \mathscr{M}$ there exists a unique solution to the stochastic constrained wave equation (2.20). Namely, there exists a unique $\mathscr{M}$-valued continuous and adapted process $z(t) = (u(t), v(t))$ such that*

1. *the process u has M-valued C^1-class trajectories and*

$$v(t) = u_t(t), \quad t \geq 0,$$

2. *the process z is a mild solution of Eq. (2.20) with initial condition z_0, i.e. for every $t \geq 0$, $\mathbb{P}$-almost surely,*

$$\begin{aligned} z(t) = \mathscr{S}(t)z_0 &+ \int_0^t \mathscr{S}(t-s)\big(0, -|v(s)|_H^2 u(s) + |A_0 u(s)|_H^2 u(s) - \gamma v(s)\big)\, ds \\ &+ \int_0^t \mathscr{S}(t-s)\big(0, \sigma(z(s))\big)\, dW(s) \\ &+ \frac{1}{2}\int_0^t \mathscr{S}(t-s)\big(0, \mathrm{tr}_K\big[\partial_v \sigma(z(s))\, \sigma(z(s))\big]\, ds, \end{aligned} \tag{2.28}$$

where $\mathscr{S} = (\mathscr{S}(t))_{t\in\mathbb{R}}$ is the C_0 group in $\mathscr{H}$ generated by $\mathscr{A}$ defined in (2.3).

Moreover, if Ψ is the energy function defined in (2.13), then the following energy equality holds, for $t \geq 0$, $\mathbb{P}$-almost surely,

$$\begin{aligned} \Psi(z(t)) = \Psi(z_0) - \gamma \int_0^t |v(s)|_H^2\, ds + \int_0^t \langle v(s), \sigma(z(s))\, dW(s)\rangle_H \\ + \frac{1}{2}\int_0^t \langle v(s), \mathrm{tr}_K\big[\partial_v \sigma_0(z(s))\, \sigma(z(s))]\rangle_H\, ds + \frac{1}{2}\int_0^t \|\sigma(z(s)) \circ i\|_{\mathscr{T}_2(K,H)}^2\, ds. \end{aligned}$$

We will also prove the following strengthening of Theorem 2.10.

Theorem 2.11. *Suppose that* σ_0*, as well as the corresponding restriction map (denoted by the same symbol* σ_0*), map* $\mathscr{H}_1$ *into* $\mathscr{L}(E, D(A_0))$ *and are of linear growth and Lipschitz-continuous on balls. Then, if the initial condition* z_0 *belongs to* $\mathscr{H}_1 \cap \mathscr{M}$ *the unique solution to problem* (2.28) *guaranteed by Theorem* 2.10 *satisfies,* $\mathbb{P}$*-almost surely,*

$$(u, v) \in C\big([0, \infty); \mathscr{H}_1\big).$$

Theorems 2.10 and 2.11 will be proven in the next section. The strategy of our proof is as follows.

- Instead of Eq. (2.20) we will study its mild form (2.28), which admits a unique local maximal solution.
- We will prove that this solution stays on M. In other words we will prove that z stays on the tangent bundle $\mathscr{M}$ of M.
- We will prove that the solution is global by exploiting the energy functional Ψ and using the Hasminski criterion.

Remark 2.12. It is now quite obvious that our basic object should not be a function σ_0 defined on the whole space $\mathscr{H}$ and taking values in $\mathscr{L}(K, H)$, but a function

$$\widetilde{\sigma} : \mathscr{M} \to \mathscr{L}(K, H)$$

such that for every $(u, v) \in \mathscr{M}$, the range $R(\widetilde{\sigma}(u, v))$ is a subset of the tangent plane $T_u M$, i.e.

$$\widetilde{\sigma}(u, v)\, k \subset T_u M, \quad (u, v) \in \mathscr{M}, \quad k \in K.$$

Obviously, the last condition is equivalent to the following one

$$\langle \widetilde{\sigma}(u, v)k, u\rangle_H = 0, \quad (u, v) \in \mathscr{H}, \quad k \in K.$$

We would also need to assume that σ satisfies a natural modification of Hypothesis 1 and 2. In particular, we would need to assume that there exist $L \geq 0$ and a sequence $(L_n)_{n=1}^{\infty}$ of nonnegative real numbers such that for all $(u, v) \in \mathscr{M}$

$$\|\widetilde{\sigma}(u, v)\|^2_{\mathscr{L}(E,H)} \leq L(1 + |v|^2_H),$$

and for all $(u_1, v_1), (u_2, v_2) \in \mathscr{M}$, with $v_1, v_2 \in B_H(0, n)$, it holds

$$\|\widetilde{\sigma}(u_2, v_2) - \widetilde{\sigma}(u_1, v_1)\|^2_{\mathscr{L}(E,H)} \leq L_n |(u_2, v_2) - (u_1, v_1)|^2_{\mathscr{H}}.$$

Had we had decided to follow this path we would only need, purely for the purposes of the proof, to construct an extension σ of $\widetilde{\sigma}$ to the whole $\mathscr{H}$, with values in $\mathscr{L}(K, H)$, satisfying Hypothesis 1 and 2 and such function σ would automatically satisfy assertion (2.18) from Lemma 2.5. In the same vein, in the case the diffusion coefficient does not depend on the second variable, we could have started with a Lipschitz-continuous function

$$g : M \to \mathscr{L}(K, H)$$

such that

$$g(u)\, k \subset T_u M, \quad u \in M, \quad k \in K.$$

Notice that the last condition is equivalent to requiring that $\langle g(u)k, u\rangle_H = 0$, for all $u \in M$ and $k \in K$. □

We finish this section by noticing that the above stated results, and in particular Theorem 2.10, are also true for the following equation

$$\mu u_{tt}(t)+\mu|u_t(t)|_H^2 u(t) = -A_0^2 u(t)+|A_0 u(t)|_H^2 u(t)-\gamma u_t(t)+\sigma(u(t), u_t(t)) \circ dW(t), \tag{2.29}$$

where $\mu > 0$ is an arbitrary positive constant representing the mass of the object under consideration. As always, we consider the above with initial conditions $(u_0, v_0) \in \mathscr{H}$ that satisfy the constraints conditions (2.7) and (2.8). Obviously, Eq. (2.29) can be written in the following form

$$u_{tt}(t)+|u_t(t)|_H^2 u(t) = -\frac{1}{\mu} A_0^2 u(t)+\frac{1}{\mu}|A_0 u(t)|_H^2 u(t)-\frac{\gamma}{\mu} u_t(t)+\frac{1}{\mu}\sigma(u(t), v(t)) \circ dW(t), \tag{2.30}$$

which is of the form of Eq. (2.20).

An important difference is that we consider Eq. (2.30) in the spaces $\mathscr{H}$ or $\mathscr{H}_1$ which are independent of μ and the linear operator $\mathscr{A}$ and the C_0 group $\mathscr{S}$ on $\mathscr{H}$ introduced above are replaced by $\mathscr{A}_\mu$ and $\mathscr{S}_\mu$. Thus, in order to rigorously define the solution to Eq. (2.30) we introduce the linear operator $\mathscr{A}_\mu$ in the space $\mathscr{H}$ as follows,

$$\mathscr{A}_\mu z = (v, -\mu^{-1} A_0^2 u), \quad z = (u, v) \in D(\mathscr{A}_\mu) := D(A_0^2) \times D(A_0).$$

It is well known that $\mathscr{A}_\mu$ generates a C_0 group $\mathscr{S}_\mu = (\mathscr{S}_\mu(t))_{t\in\mathbb{R}}$ in $\mathscr{H}$, see e.g. [10] and references therein.

3. Proofs of Theorem 2.10 and Theorem 2.11

By using the C_0 group $\mathscr{S}(t)$ generated by the operator $\mathscr{A}$ in $\mathscr{H}$, we rewrite Eq. (2.20) in the mild form (2.28), on the whole space $\mathscr{H}$. We recall that, according to Lemma 2.5, the map σ is Lipschitz-continuous on balls and of cubic growth. Moreover, by Proposition 2.9, the map σ is of class C^1 in the sense of Hypothesis 2 and the function

$$\mathscr{H} \ni z \mapsto \mathrm{tr}_K\left[\partial_v \sigma(z)\, \sigma(z)\right] \in H,$$

is also Lipschitz-continuous on balls and of polynomial growth. Finally, by Proposition 2.3, the function

$$F : \mathscr{H} \ni z = (u, v) \mapsto |v|_H^2 u + |A_0 u|_H^2\, u \in H,$$

is Lipschitz-continuous on balls. Therefore, by proceeding in a standard way (compare, for example, [51, Theorem 1.5] or [2, Theorem 4.10]), we can find a unique maximal local mild solution $z(t) = \big(u(t), v(t)\big)$ for Eq. (2.28), defined up to a certain stopping time τ. In what follows, by following [10] with some important modifications as in [4,9], we will prove that

$$\mathbb{P}(\tau = \infty) = 1.$$

We first establish the following fundamental result, c.f. [9, Theorem 4.1].

Proposition 3.1. *The manifold $\mathscr{M}$ is invariant for the process $z(t)$, $t < \tau$. More precisely, we have*

$$\langle u(t), v(t)\rangle_H = 0, \quad t \in [0, \tau), \quad \mathbb{P}-a.s. \tag{3.1}$$

Before we embark on the proof, we state a few essential equalities.

Lemma 3.2. *For every $z=(u,v)\in\mathscr{H}$, with $u\in D(A_0^2)$, we have*

$$\begin{aligned}
&\langle u, -A_0^2 u + |A_0 u|_H^2 u\rangle_H = |A_0 u|_H^2\big(|u|_H^2-1\big),\\
&\langle u, \sigma(z)e\rangle_H = -\langle u, \sigma_0(z)e\rangle_H\big(|u|_H^2-1\big), \quad e\in E,\\
&\langle u, \mathrm{tr}_K\big[\partial_v\sigma(z)\sigma(z)\big]\rangle_H = -\langle u, \mathrm{tr}_K\big[\partial_v\sigma_0(z)\sigma(z)\big]\rangle_H\big(|u|_H^2-1\big).
\end{aligned} \tag{3.2}$$

Moreover, for every $z\in\mathscr{M}$, we have

$$\langle v, \sigma(z)\, e\rangle_H = \langle v, \sigma_0(z)\, e\rangle_H, \quad e\in E, \tag{3.3}$$

and

$$\langle v, \mathrm{tr}_K\big[\partial_v\sigma(z)\sigma(z)\big]\rangle_H = \langle v, \mathrm{tr}_K\big[\partial_v\sigma_0(z)(\sigma(z))\big]\rangle_H. \tag{3.4}$$

Proof. The first identity in (3.2) is obvious, because A_0 is a self-adjoint operator. The second one and (3.3) are straightforward consequences of the definition (2.15) of the function σ. Now, in order to prove the third identity in (3.2), we fix $z=(u,v)\in\mathscr{H}$, such that $u\in D(A_0^2)$, and we define

$$B := \sigma(z)\in\mathscr{L}(E,H),$$

and

$$T_0 := \partial_v\sigma_0(z)\in\mathscr{L}\big(H,\mathscr{L}(E,H)\big), \quad T := \partial_v\sigma(z)\in\mathscr{L}\big(H,\mathscr{L}(E,H)\big).$$

With these notations, we can rewrite formula (2.25) as follows

$$\mathrm{tr}_K\big[\partial_v\sigma(z)\,\sigma(z)\big] = \textstyle\sum_{j=1}^\infty \big[TBe_j\big]e_j.$$

Similarly we have

$$\mathrm{tr}_K\big[\partial_v\sigma_0(z)\,\sigma(z)\big] = \textstyle\sum_{j=1}^\infty \big[T_0\, Be_j\big]e_j.$$

Moreover, by (2.27) we have

$$\big[Ty\big]e = \big[T_0 y\big]e - \big\langle\big[T_0 y\big]e, u\big\rangle_H\, u, \quad y\in H, \quad e\in E,$$

and this implies that

$$\big\langle u, \big[Ty\big]e\big\rangle_H = \big\langle u, \big[T_0 y\big]e\big\rangle_H - \big\langle u, \big\langle\big[T_0 y\big]e, u\big\rangle_H\, u\big\rangle_H = \big\langle u, \big[T_0 y\big]e\big\rangle_H\,\big(1-|u|_H^2\big).$$

In particular,

$$\begin{aligned}
\big\langle u, \mathrm{tr}_K\big[\partial_v\sigma(z)\,\sigma(z)\big]\big\rangle_H &= \sum_{j=1}^\infty \big\langle u, \big[T\, Be_j\big]e_j\big\rangle_H\\
&= \big(1-|u|_H^2\big)\sum_{j=1}^\infty \big\langle u, \big[T_0\, B(e_j)\big]e_j\big\rangle_H\\
&= \big(1-|u|_H^2\big)\big\langle u, \mathrm{tr}_K\big[\partial_v\sigma_0(z)\,\sigma(z)\big]\big\rangle_H,
\end{aligned}$$

and the third identity in (3.2) follows.

Finally, assume that $z=(u,v)\in\mathscr{M}$. For every $y\in H$ and $e\in E$, we have

$$\big\langle v,[Ty]e\big\rangle_H=\big\langle v,[T_0y]e\big\rangle_H-\big\langle v,\langle[T_0y]e,u\rangle_H\,u\big\rangle_H=\big\langle u,[T_0y]e\big\rangle_H.$$

Therefore,

$$\begin{aligned}\big\langle v,\operatorname{tr}_K\big[\partial_v\sigma(z)\,\sigma(z)\big)\big]&=\sum_{j=1}^{\infty}\big\langle v,\big[TBe_j\big]e_j\big\rangle_H\big\rangle_H\\&=\sum_{j=1}^{\infty}\big\langle v,\big[T_0Be_j\big]e_j\big\rangle_H=\big\langle v,\operatorname{tr}_K\big[\partial_v\sigma_0(z)\,\sigma(z)\big]\big\rangle_H,\end{aligned}$$

and identity (3.4) follows. □

The proof of Proposition 3.1 will also use the following version of the Itô Lemma which is a special case of Lemma A.2.

Lemma 3.3. *Assume that a local process $z(t)=(x(t),y(t))$ is a solution to*

$$\begin{cases}dy(t)=\Big[-A_0^2x(t)+f(t)\Big]\,dt+g(t)\,dW(t),\\ dx(t)=y(t)dt,\end{cases}\tag{3.5}$$

where all processes are progressively measurable, f is H-valued, g is $\mathscr{T}_2(K,H)$-valued, and $x(t)$ is $D(A_0)$-valued such that for every $t\geq 0$,

$$\mathbb{E}\int_0^t\Big[\|g(s)\|^2_{\mathscr{T}_2(K,H)}+\|f(s)\|^2_H\Big]\,ds<\infty$$

In other words, we assume the above and

$$z(t)=\mathscr{S}(t)z_0+\int_0^t\mathscr{S}(t-s)\big(0,f(s)\big)\,ds+\int_0^t\mathscr{S}(t-s)\big(0,g(s)\big)\,dW(s),\quad t\geq 0.$$

Then, for every $t\geq 0$, $\mathbb{P}$-almost surely,

$$\begin{aligned}\langle x(t),y(t)\rangle_H=\langle x(0),y(0)\rangle_H&-\int_0^t|A_0x(s)|^2_H\,ds+\int_0^t\langle x(s),f(s)\rangle_H\,ds\\&+\int_0^t|y(s)|^2_H\,ds+\int_0^t\langle x(s),g(s)\,dW(s)\rangle_H.\end{aligned}\tag{3.6}$$

Moreover if Ψ is defined as in (2.13), then, for every $t\geq 0$, $\mathbb{P}$-almost surely,

$$\begin{aligned}\Psi(z(t))=\Psi(z(0))&+\int_0^t\langle y(s),f(s)\rangle_H\,ds\\&+\frac{1}{2}\int_0^t\|g(s)\|^2_{\mathscr{T}_2(K,H)}\,ds+\int_0^t\langle y(s),g(s)\,dW(s)\rangle_H.\end{aligned}$$

Proof of Proposition 3.1. Let us fix $k \in \mathbb{N}$. We will show that the processes φ and ψ defined by

$$\varphi(t) = \frac{1}{2}\big(|u(t)|_H^2 - 1\big), \quad t \in [0, \tau),$$

and

$$\psi(t) = \langle u(t), v(t)\rangle_H = \langle u(t), u_t(t)\rangle_H, \quad t \in [0, \tau),$$

satisfy the following system of linear stochastic differential equation

$$\begin{cases} d\varphi(t) = \psi(t)\,dt \\ d\psi(t) + \gamma\, d\varphi(t) = \alpha(t)\varphi(t)\,dt - 2\langle u(t), \sigma_0(z(s))dW(t)\rangle_H \varphi(t), \quad t \in [0, \tau), \end{cases} \tag{3.7}$$

for an appropriate process $\alpha(t)$ defined by

$$\alpha(t) = 2|A_0 u(t)|_H^2 - 2|v(t)|_H^2 - \langle u(t), \mathrm{tr}_K\big[\partial_v \sigma_0(z(t))\sigma(z(t))\big]\rangle_H.$$

Since $\varphi(0) = 0$ and $\psi(0) = \varphi_t(0) = 0$, this implies that $\varphi(t) = \psi(t) = 0$, and Proposition 3.1 follows. Thus, it is sufficient to prove (3.7).

Let us observe that the process $(u(t), v(t))$, $t \geq 0$, is a solution for system (3.5), with $x = u$, $y = v$, $g = \sigma(u, v)$ and with

$$\begin{aligned} f(s) &= -|v(s)|_H^2 u(s) + |A_0 u(s)|_H^2 - \gamma v(s) + \frac{1}{2}\mathrm{tr}_K\big[\partial_v \sigma(u(s), v(s))\sigma(u(s), v(s))\big], \\ &\quad s \geq 0. \end{aligned}$$

Thanks to identity (3.6) in Lemmas 3.3 and 3.2, we have

$$\begin{aligned} \psi(t) - \psi(0) &= \int_0^t \big(|v(s)|_H^2 - |A_0 u(s)|_H^2\big)\,ds \\ &\quad + \int_0^t \langle u(s), -|v(s)|_H^2 u(s) + |A_0 u(s)|_H^2 u(s) - \gamma v(s)\rangle_H\,ds \\ &\quad + \frac{1}{2}\int_0^t \langle u(s), \mathrm{tr}_K\big[\partial_v \sigma(z(s))\sigma(z(s))\big]\rangle_H\,ds + \int_0^t \langle u(s), \sigma(z(s))\,dW(s)\rangle_H \\ &= \int_0^t \Big(|A_0 u(s)|_H^2 - |v(s)|_H^2\Big)\Big(|u(s)|_H^2 - 1\Big)\,ds - \gamma\int_0^t \langle u(s), v(s)\rangle_H\,ds \\ &\quad - \int_0^t \big(|u(s)|_H^2 - 1\big)\langle u(s), \sigma_0(z(s))\,dW(s)\rangle_H \\ &\quad - \frac{1}{2}\int_0^t \langle u(s), \mathrm{tr}_K\big[\partial_v \sigma_0(z(s))\sigma(z(s))\big]\rangle_H \big(|u(s)|_H^2 - 1\big)\,ds \\ &= \int_0^t \Big(2|A_0 u(s)|_H^2 - 2|v(s)|_H^2 - \langle u(s), \mathrm{tr}_K\big[\partial_v \sigma_0(z(s))\,\sigma(z(s))\big]\rangle_H\Big)\varphi(s)\,ds \\ &\quad - \gamma\int_0^t \psi(s)\,ds - 2\int_0^t \langle u(s), \sigma_0(z(s))dW(s)\rangle_H \varphi(s). \end{aligned}$$

Hence equality (3.7) follows. □

Our next task is to show that the local maximal solution is in fact a global one.

Proposition 3.4. *We have*

$$\mathbb{P}(\tau = \infty) = 1. \tag{3.8}$$

Proof. We define the following stopping times

$$\tau_k := \inf\{t \in [0,\tau) : |z(t)|_{\mathscr{H}} \geq k\}, \quad k \in \mathbb{N}.$$

According to our definition, if $|z(t)|_{\mathscr{H}} < k$, for every $t \in [0,\tau)$, then $\tau_k = \tau$. The sequence of stopping times $\{\tau_k\}_{k\in\mathbb{N}}$ is non decreasing and

$$\tau^\star := \lim_{k\to\infty} \tau_k \leq \tau.$$

Thus, if we show that for every $t \geq 0$

$$\lim_{k\to\infty} \mathbb{P}(\tau_k \leq t) = 0,$$

we conclude that

$$\mathbb{P}(\tau < \infty) \leq \mathbb{P}\left(\tau^\star < \infty\right) = 0,$$

and (3.8) follows.

We apply the Itô Lemma A.2 to the function Ψ defined in (2.13), see Lemma 3.3. Since the process $z(t) = (u(t), v(t))$, $t \in [0,\tau)$ is a local solution to problem (2.20), we infer that

$$\begin{aligned} d\Psi(z(t)) &= \langle v(t), |A_0 u(t)|_H^2 u(t) - |v(t)|_H^2 u(t) - \gamma v(t)\rangle_H \, dt \\ &+ \langle v(t), \sigma(z(t))\, dW(t)\rangle_H + \frac{1}{2}\langle v(t), \mathrm{tr}_K\left[\partial_v \sigma(z(t))\, \sigma(z(t))\right]\rangle_H \, dt \\ &+ \frac{1}{2}\|\sigma(z(t))\|^2_{\mathscr{T}_2(K,H)}\, dt. \end{aligned}$$

Hence, thanks to (3.1) we have

$$\begin{aligned} d\Psi(z(t)) + \gamma|v(t)|_H^2\, dt &= \langle v(t), \sigma(z(t))\, dW(t)\rangle_H \\ &+ \frac{1}{2}\langle v(t), \mathrm{tr}_K\left[\partial_v \sigma(z(t))(\sigma(z))\right]\rangle_H \, dt + \frac{1}{2}\|\sigma(z)\|^2_{\mathscr{T}_2(K,H)}\, dt. \end{aligned}$$

Next note that since $z(t) \in \mathscr{M}$, in view of (3.3) and (3.4), we have

$$\langle v(t), \sigma(z(t))\, dW(t)\rangle_H = \langle v(t), \sigma_0(z(t))\, dW(t)\rangle_H,$$

and

$$\langle v(t), \mathrm{tr}_K\left[\partial_v \sigma v(z(t))\sigma(z(t))\right]\rangle_H = \langle v(t), \mathrm{tr}_K\left[\partial_v \sigma_0(z(t))\, \sigma(z(t))\right]\rangle_H,$$

so that

$$\begin{aligned} d\Psi(z(t)) + \gamma|v(t)|_H^2\, dt &= \langle v(t), \sigma_0(z(t))\, dW(t)\rangle_H \\ &+ \frac{1}{2}\langle v, \mathrm{tr}_K\left[\partial_v \sigma_0(z(t))\, \sigma(z(t))\right]\rangle_H \, dt + \frac{1}{2}\|\sigma(z(t))\|^2_{\mathscr{T}_2(K,H)}\, dt. \end{aligned}$$

Let us also observe that since $z(t) \in \mathscr{M}$, from inequality (2.17) in Lemma 2.5 we deduce that

$$\frac{1}{2}\|\sigma(z(t))\|^2_{\mathscr{T}_2(K,H)} \leq L(1 + |z(t)|^2_{\mathscr{H}}),$$

and, by assumption (2.26), there exists a constant $c > 0$ such that

$$\frac{1}{2}\langle v, \operatorname{tr}_K\big[\partial_v \sigma_0(z(t))\sigma(z(t))\big]\rangle_H \leq c(1+|z(t)|^2_{\mathscr{H}}).$$

Thus, if we put together all the estimates above, we deduce that for $t < \tau$

$$\Psi(z(t)) \leq \Psi(z(0)) + \int_0^t \langle v(s), \sigma_0(z(t))\, dW(s)\rangle_H + c\int_0^t \big(1+\Psi(z(s))\big)\, ds. \tag{3.9}$$

In particular, by taking the expectation of both sides of the stopped version of (3.9), we infer that for every $k \in \mathbb{N}$

$$\mathbb{E}\big(1+\Psi(z(t\wedge\tau_k))\big) \leq 1 + \psi(z(0)) + c\,\mathbb{E}\int_0^t \big(1+\Psi(z(s\wedge\tau_k)\big)\, ds.$$

As a consequence of Gronwall's Lemma, this gives

$$\mathbb{E}\left(1+\Psi(z(t\wedge\tau_k))\right) \leq \big(1+\Psi(z_0)\big)e^{ct}, \quad t \geq 0.$$

Now, since $|u(t\wedge\tau_k)|_H = 1$, this implies that

$$\mathbb{E}\,|z(t\wedge\tau_k)|^2_{\mathscr{H}} \leq c(t), \qquad t \geq 0,$$

so that

$$\lim_{k\to\infty} \mathbb{P}\,(\tau_k < t) = 0.$$

As we have explained above, this yields (3.8). □

The proof of Proposition 3.4 completes the proof of Theorem 2.10. Thus, we only need to prove Theorem 2.11. However, its proof is very similar to the one of Theorem 2.10 and for this reason we will only sketch it.

We fix $(u_0, v_0) \in \mathscr{M} \cap \mathscr{H}_1$. Since we are assuming that $\sigma_0(z) \in \mathscr{L}(E, D(A_0))$, if $z \in \mathscr{H}_1$, arguing as in the proof of Lemma 2.5 we can show the map $\sigma : \mathscr{H}_1 \to \mathscr{L}(E, D(A_0))$ is Lipschitz-continuous on balls and of cubic growth. Moreover, arguing as in the proof of Proposition 2.9, we can prove that the map σ is of C^1-class in the sense of Hypothesis 2 and the function

$$\mathscr{H}_1 \ni z \mapsto \operatorname{tr}_K\big(\partial_v\sigma(z)\sigma(z)\big) \in D(A_0),$$

is also Lipschitz-continuous on balls and of polynomial growth. Finally, it follows trivially from Proposition 2.3, that the function

$$F : \mathscr{H}_1 \ni z = (u, v) \mapsto |v|^2_H u + |A_0 u|^2_H u \in D(A_0),$$

is Lipschitz-continuous on balls and of polynomial growth. Therefore, by proceeding in a standard way (compare, for example, [51, Theorem 1.5] or [2, Theorem 4.10]), we can find a unique maximal local mild solution $z(t) = \big(u(t), v(t)\big)$, defined for $t < \xi$. In what follows we will prove that

$$\mathbb{P}\,(\xi = \infty) = 1. \tag{3.10}$$

For this aim, we define the following sequence of stopping times

$$\xi_k := \inf\{t \in [0, \tau) : |z(t)|_{\mathscr{H}_1} \geq k\}, \quad k \in \mathbb{N}.$$

To do this we could follow the proof of Theorem 2.10. But an easier way is available since by the uniqueness of solutions guaranteed by Theorem 2.10 we have

$$z(t) = \mathbf{z}(t), \quad t < \xi,$$

where $\mathbf{z}$ is the unique global solution from Theorem 2.10.

Thus we only need to prove a counterpart of Proposition 3.4, i.e. the local maximal solution $z(t) = (u(t), v(t))$, $t < \xi$, is a global one, i.e. $\xi = \infty$, $\mathbb{P}$-almost surely. The proof of this fact follows once we first apply the Itô Lemma A.2 to the following modification of the function Ψ defined in (2.13),

$$\Phi(z) := \frac{1}{2}\Big[|A_0^2 u|_H^2 + |A_0 v|_H^2\Big], \quad z = (u, v) \in \mathscr{H}_1,$$

and then apply the Gronwall Lemma. This allows to show that for every fixed $t \geq 0$

$$\lim_{k\to\infty} \mathbb{P}\,(\xi_k \leq t) = 0,$$

and this implies (3.10).

4. The Small Mass Limit: Notations, Assumptions and Main Results

Let D be a bounded and smooth domain in $\mathbb{R}^d$, with $d \geq 1$, and let H denote the Hilbert space $L^2(D)$, endowed with the usual scalar product $\langle\cdot,\cdot\rangle_H$ and the corresponding norm $|\cdot|_H$. It is well known that, if Δ is the Laplace operator on the domain D, endowed with the Dirichlet boundary conditions, then there exists a complete orthonormal system $\{e_j\}_{j\in\mathbb{N}} \subset H$ and a non-decreasing divergent sequence of positive real numbers $\{\alpha_j\}_{j\in\mathbb{N}}$, such that

$$\Delta e_j = -\alpha_j e_j, \quad k \in \mathbb{N}.$$

For every $\beta \in \mathbb{R}$, we denote by H^β the space $D((-\Delta)^\beta)$, endowed with the norm

$$|x|_{H^\beta}^2 := |(-\Delta)^\beta x|_H^2 = \sum_{j=1}^\infty \alpha_j^\beta |\langle x, e_j\rangle_H|^2,$$

and we set $\mathscr{H}_\beta := H^{\beta+1} \times H^\beta$. When $\beta = 0$, we simply denote $\mathscr{H}_0$ by $\mathscr{H}$. Moreover, we denote by M is the unit sphere in H

$$M = \{u \in H \,:\, |u|_H = 1\}\,.$$

Notice that by using interpolation for every $0 \leq \vartheta < \varrho$ and $u \in H^\varrho \cap M$ we have

$$|u|_{H^\vartheta} \leq |u|_{H^\varrho}^{\vartheta/\varrho}\, |u|_H^{1-\vartheta/\varrho} = |u|_{H^\varrho}^{\vartheta/\varrho}. \tag{4.1}$$

Throughout the rest of this paper, we will consider the following class of stochastic damped wave equations on D

$$\begin{cases} \mu\, \partial_t^2 u_\mu(t,\xi) + \mu|\partial_t u_\mu(t)|_H^2 u_\mu(t,\xi) \\ \qquad\qquad = \Delta u_\mu(t,\xi) + |\nabla u_\mu(t)|_H^2 u_\mu(t) - \gamma\, \partial_t u_\mu(t,\xi) + \sigma(u_\mu(t))\, \partial_t w^Q(t,\xi) \\ u_\mu(0,\xi) = u_0(\xi), \quad \partial_t u_\mu(0,\xi) = v_0(\xi), \quad u_\mu(t,\xi) = 0, \quad \xi \in \partial D, \end{cases} \tag{4.2}$$

depending on a positive parameter μ. Here, γ is a positive constant, w^Q is a Wiener process on H and the mapping $\sigma : H^1 \to \mathscr{L}(H)$ is such that $\sigma(u)$ projects H onto T_uM, for every $u \in H^1$. Namely, as in (2.15)

$$\sigma(u) = \sigma_0(u) - \langle \sigma_0(u), u \rangle_H u,$$

for some mapping $\sigma_0 : H^1 \to \mathcal{L}(H)$.

In this section, as well as in all following sections, we assume that σ_0 depends only on the first component, i.e. the domain of σ_0 is H^1 and not $H^1 \times H$ as in the previous sections. This stronger framework is precisely the one described in Remark 2.7 part (ii) (here we have decided to use the symbol σ_0 and not g_0). Moreover, this framework has the following consequence. The Itô-Stratonovich correction term $\mathrm{tr}_K\big[\partial_v\sigma(z)\,\sigma(z)\big]$, where σ is defined in (2.15), see also (4.4), is equal to 0. Hence, there is no need of introducing a Banach space E in which the Wiener process takes values. We may simply consider a cylindrical Wiener process w^Q on some separable Hilbert space K, called the reproducing kernel Hilbert space. If this Wiener process takes values in H, then its covariance operator Q belongs to $\mathscr{L}^+(H)$, the space of non-negative and symmetric operators of trace class. Note that in this case $K = Q(H)$, so that $w^Q(t,\xi)$ can be formally written as the sum

$$w^Q(t,\xi) = \sum_{j=1}^\infty \tilde{e}_j(\xi)\beta_j(t), \qquad t \geq 0, \qquad \xi \in D,$$

where $\{\tilde{e}_j\}_{j\in\mathbb{N}}$ is an orthonormal basis of K and $\{\beta_k\}_{j\in\mathbb{N}}$ is a sequence of mutually independent Brownian motions, all defined on the same stochastic basis $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$. Since $K = Q(H)$ and

$$\langle Qu, Q^{-1}v\rangle_K = \langle u, v\rangle_H, \quad u, v \in H,$$

we can assume that

$$\tilde{e}_j = Qe_j, \quad j \in \mathbb{N},$$

where $\{e_j\}_{j\in\mathbb{N}}$ is an orthonormal basis of H. We may assume, although this is not necessary, that $\{e_j\}_{j\in\mathbb{N}}$ diagonalizes the Laplacian Δ.

In what follows we assume a modified version of Hypothesis 1, namely we assume that the function σ_0 depends only on the first variable, see Remark 2.7(2), where we used an auxiliary notation g_0. Since σ_0 depends only on the first variable, we can relax the assumption by replacing the space $\mathscr{L}(E, H)$ by the space $\mathscr{T}_2(K, H)$.

Hypothesis 3. *The function $\sigma_0 : H^1 \to \mathscr{T}_2(K, H)$ is Lipschitz on balls and*

$$\sup_{u\in H^1} \|\sigma_0(u)\|_{\mathscr{T}_2(K,H)} < \infty. \tag{4.3}$$

If $u \in H^2$, then $\sigma_0(u) \in \mathscr{T}_2(K, H^1)$ and the corresponding function $\sigma_0 : H^2 \to \mathscr{T}_2(K, H^1)$ is Lipschitz on balls and

$$\sup_{u\in H^2} \|\sigma_0(u)\|_{\mathscr{T}_2(K,H^1)} < \infty.$$

We also assume the following strengthening of Hypothesis 3.

Hypothesis 4. *The function* $\sigma_0 : H^1 \to \mathscr{T}_2(K, H)$ *satisfies the following condition*

$$\sup_{u\in H^1} \|\sigma_0(u)\|_{\mathscr{T}_2(K,H)} \left(1 + |u|^2_{H^1}\right) < \infty.$$

Remark 4.1. In what follows, by $\sigma_0^*(u) \in \mathscr{T}_2(H, K) \subset \mathscr{L}(H, K)$ we will understand the Hilbert adjoint of the operator $\sigma_0(u) \in \mathscr{T}_2(K, H) \subset \mathscr{L}(K, H)$.

Note that in view of Hypothesis 3 the map

$$\sigma_0^* : H^1 \to \mathscr{T}_2(H, K) \subset \mathscr{L}(H, K)$$

is bounded. Moreover, for every $u \in H^1$,

$$\sup_{u\in H^1} \|\sigma_0(u)\|_{\mathscr{L}(K,H)} < \infty, \qquad \sup_{u\in H^1} \|\sigma_0^*(u)\|_{\mathscr{L}(H,K)} < \infty.$$

We have already seen that the diffusion coefficient σ is given by

$$\sigma(u)h = \sigma_0(u)h - \langle\sigma_0(u)h, u\rangle_H u, \quad u \in H^1, \quad h \in K. \tag{4.4}$$

In what follows we will use the following useful notation

$$\sigma_1(u)h := \langle\sigma_0(u)h, u\rangle_H u, \quad u \in H^1, \quad h \in K, \tag{4.5}$$

so that

$$\sigma(u) = \sigma_0(u) - \sigma_1(u), \quad u \in H^1.$$

We will also assume the following additional hypothesis.

Hypothesis 5. *If* $u \in H^3$, *then* $\sigma_0(u) \in \mathscr{T}_2(K, H^2)$ *and there exists* $c > 0$ *such that*

$$\|\sigma_0(u)\|_{\mathscr{T}_2(K,H^2)} \le c\left(1 + |u|_{H^2}\right), \quad u \in H^3.$$

Remark 4.2. Assume that for every $u \in H^1$

$$\sigma_0(u)\tilde{e}_k = \lambda_k(|u|_{H^1})e_k, \quad k \in \mathbb{N},$$

for some mappings $\lambda_k : [0, \infty) \to \mathbb{R}$ such that for every $R > 0$

$$r_1, r_2 \in [0, R] \Longrightarrow |\lambda_k(r_1) - \lambda_k(r_2)| \le c_{R,k}\, |r_1 - r_2|.$$

For every $\delta \ge 0$ and $u \in H^1$ we have

$$\|\sigma_0(u)\|^2_{\mathscr{T}_2(K,H^\delta)} = \sum_{k=1}^{\infty} |\sigma_0(u)\tilde{e}_k|^2_{H^\delta} = \sum_{k=1}^{\infty} \lambda_k^2(|u|_{H^1})\alpha_k^\delta,$$

and for every $u_1, u_2 \in B_R(H^1)$, we have

$$\begin{aligned}\|\sigma_0(u_1) - \sigma_0(u_2)\|^2_{\mathscr{T}_2(K,H^1)} &= \sum_{k=1}^{\infty} |\left[\sigma_0(u_1) - \sigma_0(u_2)\right] Qe_k|^2_{H^1} \\ &= \sum_{k=1}^{\infty} \left[\lambda_k(|u_1|_{H^1}) - \lambda_k(|u_2|_{H^1})\right]^2 \alpha_k \le \sum_{k=1}^{\infty} c^2_{R,k}\alpha_k.\end{aligned}$$

In particular, if

$$\Lambda_1 := \sup_{r\geq 0}\sum_{k=1}^{\infty}\lambda_k(r)\alpha_k < \infty, \quad c_R := \sum_{k=1}^{\infty} c_{R,k}^2\,\alpha_k < \infty,$$

we have that

$$\sup_{u\in H^1}\|\sigma_0(u)\|^2_{\mathcal{T}_2(K,H^1)} \leq \Lambda_1,$$

and

$$u_1, u_2 \in B_R(H^1) \Longrightarrow \|\sigma_0(u_1)-\sigma_0(u_2)\|^2_{\mathcal{T}_2(K,H^1)} \leq c_R,$$

so that Hypothesis 3 follows. Next, if we assume

$$\Lambda_2 := \sup_{r\geq 0}\frac{1}{1+r^4}\sum_{k=1}^{\infty}\lambda_k^2(r)\alpha_k^2 < \infty,$$

due to (4.1) we have

$$\|\sigma_0(u)\|^2_{\mathcal{T}_2(K,H^2)} = \sum_{k=1}^{\infty}\lambda_k^2(|u|_{H^1})\alpha_k^2 \leq \Lambda_2\left(1+|u|^4_{H^1}\right) \leq \Lambda_2\left(1+|u|^2_{H^2}\right).$$

Moreover, if we assume

$$\Lambda_3 := \sup_{r\geq 0}\sum_{k=1}^{\infty}\lambda_k^2(r)(1+r^4) < \infty,$$

we get

$$\sup_{u\in H^1}\|\sigma_0(u)\|^2_{\mathcal{T}_2(K,H)}\left(1+|u|^4_{H^1}\right) \leq \Lambda_3.$$

All this implies that Hypotheses 4 and 5 hold. □

The following result (as well as its proof) is similar to Lemma 2.5.

Lemma 4.3. *Assume that K is a separable Hilbert space. Assume that the functions σ and σ_1 are defined by formulae (4.4) and (4.5) respectively, where $\sigma_0 : H^1 \to \mathcal{T}_2(K, H)$.*

1. *If $u \in H^1$, then*

$$\|\sigma(u)\|^2_{\mathcal{T}_2(K,H)} \leq \|\sigma_0(u)\|^2_{\mathcal{T}_2(K,H)},$$

and, if the map $\sigma_0 : H^1 \to \mathcal{T}_2(K, H)$ satisfies (4.3), then

$$\sup_{u\in M\cap H^1}\|\sigma(u)\|_{\mathcal{T}_2(K,H)} < \infty. \tag{4.6}$$

Moreover, if Hypothesis 4 holds, then

$$\sup_{u\in M\cap H^1}\|\sigma(u)\|_{\mathcal{T}_2(K,H)}\left(1+|h|^2_{H^1}\right) < \infty, \tag{4.7}$$

2. *If $X \subset H^1$ is a Hilbert space, then σ_1 maps X into $\mathscr{T}_2(K, X)$ and for every $u \in X$,*
$$\|\sigma(u)\|_{\mathscr{T}_2(K,X)} \leq \|\sigma_0(u)\|_{\mathscr{T}_2(K,X)} + |u|_X |\sigma_0^*(u)u|_K.$$
In particular, for every $u \in X \cap M$,
$$\|\sigma(u)\|_{\mathscr{T}_2(K,X)} \leq \|\sigma_0(u)\|_{\mathscr{T}_2(K,X)} + c\,|u|_X. \tag{4.8}$$
3. *Under Hypothesis 3 there exists some $c > 0$ such that*
$$\|\sigma(u)\|_{\mathscr{T}_2(K,H^1)} \leq c\left(1 + |u|_{H^1}\right), \quad u \in H^2 \cap M. \tag{4.9}$$
4. *Under Hypothesis 5, there exists some $c > 0$ such that*
$$\|\sigma(u)\|_{\mathscr{T}_2(K,H^2)} \leq c\left(1 + |u|_{H^2}\right), \quad u \in H^3 \cap M. \tag{4.10}$$

Proof. For every $u \in H$, we have
$$|\sigma(u)\tilde{e}_k|_H^2 \leq |\sigma_0(u)\tilde{e}_k|_H^2, \quad k \in \mathbb{N}. \tag{4.11}$$
Hence, by summing the expression above over $k \in \mathbb{N}$, we obtain
$$\|\sigma(u)\|^2_{\mathscr{T}_2(K,H)} \leq \|\sigma_0(u)\|^2_{\mathscr{T}_2(K,H)}$$
and this implies (4.6) and, in case Hypothesis 4 holds, (4.7).

Next if $u \in H$, by the Parseval identity in K we get
$$\begin{aligned}\|\sigma_1(u)\|^2_{\mathscr{T}_2(K,X)} &= \sum_{k=1}^{\infty} |\langle \sigma_0(u)\tilde{e}_k, u\rangle_H u|_X^2 \\ &= |u|_X^2 \sum_{k=1}^{\infty} \langle \tilde{e}_k, \sigma_0^*(u)u\rangle_K^2 = |u|_X^2 |\sigma_0^*(u)u|_K^2,\end{aligned}$$
and, recalling that $\sigma(u) = \sigma_0(u) - \sigma_1(u)$, this proves (4.8).

Now, if we assume Hypothesis 3, for any $u \in H^1$ (4.8) gives
$$\|\sigma(u)\|_{\mathscr{T}_2(K,H^1)} \leq c\left(1 + |u|_H |u|_{H^1}\right),$$
and (4.9) follows.

Finally, under Hypothesis 5, we have
$$\|\sigma(u)\|^2_{\mathscr{T}_2(K,H^2)} \leq c\left(1 + |u|^2_{H^2}\right) + c\,|u|_H^2 |u|^2_{H^2},$$
and thus (4.10) follows. □

Equation (4.2) can be rewritten as the system
$$\begin{cases} du_\mu(t) = v_\mu(t)\,dt \\ \mu dv_\mu(t) = \left[\Delta u_\mu(t) + |u_\mu(t)|^2_{H^1} u_\mu(t) - \mu |v_\mu(t)|_H^2 u_\mu(t) - \gamma v_\mu(t)\right] dt \\ \qquad\qquad +\sigma(u_\mu(t))\,dw^Q(t), \\ u_\mu(0) = u_0, \quad v_\mu(0) = v_0, \quad u_\mu(t)_{|\partial D} = 0. \end{cases} \tag{4.12}$$

Thus, if we take

$$A_0^2 = -\Delta,\ H = L^2(D),\ D(A_0^2) = H^2(D),$$

we have

$$|A_0 u|_H = |\nabla u|_H,$$

and problem (4.2) is precisely problem (2.23). Moreover, if $\sigma_0 : H^1 \to \mathscr{L}_2(K, H)$ has linear growth and is Lipschitz continuous, then Hypothesis 1 is satisfied, and, since σ_0 is independent of v, Hypothesis 2 is satisfied as well. In particular, thanks to Theorem 2.10 we have the following result.

Theorem 4.4. *Assume that the function $\sigma_0 : H^1 \to \mathscr{T}_2(K, H)$ has linear growth and is Lipschitz-continuous on balls. Then, for every $z_0 = (u_0, v_0) \in \mathscr{M}$, there exists a unique solution to the stochastic constrained wave equation (4.12), i.e. an $\mathscr{M}$-valued continuous and an adapted process $z(t) = (u(t), v(t))$ such that*

1. *the process u has M-valued C^1 trajectories and*
$$v(t) = \partial_t u(t), \qquad t \geq 0;$$
2. *the process z is a mild solution of Eq. (2.20) with initial conditions (u_0, v_0), i.e. for every $t \geq 0$, $\mathbb{P}$-almost surely,*
$$z(t) = \mathscr{S}_\mu(t) z_0 + \frac{1}{\mu}\int_0^t \mathscr{S}_\mu(t-s)\big(0, -\mu\,|v(s)|^2 u(s) + |\nabla u(s)|^2 u(s) - \gamma v(s)\big)\,ds$$
$$+ \frac{1}{\mu}\int_0^t \mathscr{S}_\mu(t-s)\big(0, \sigma(u(s))\big)\,dw^Q(s),$$
where $\mathscr{S}_\mu = (\mathscr{S}_\mu(t))_{t\in\mathbb{R}}$ is the C_0 group in $\mathscr{H}$ generated by $\mathscr{A}_\mu$.

Moreover, the process z satisfies the following energy equality, for $t \geq 0$, $\mathbb{P}$-almost surely,

$$\frac{1}{2}\,|u(t)|_{H^1}^2 + \frac{\mu}{2}|v(t)|_H^2 = \frac{1}{2}\,|u_0|_{H^1}^2 + \frac{\mu}{2}|v_0|_H^2 - \gamma\int_0^t |v(s)|_H^2\,ds$$
$$+ \int_0^t \langle v(s), \sigma(u(s))\,dw^Q(s)\rangle_H + \frac{1}{2\mu}\int_0^t \|\sigma(u(s))\|_{\mathscr{T}_2(K,H)}^2\,ds.$$

Finally, if Hypothesis 3 is satisfied, and if $z_0 = (u_0, v_0) \in \mathscr{H}_1$, the above unique solution z belongs to $C\big([0, \infty); \mathscr{H}_1\big)$, $\mathbb{P}$-almost surely.

In what follows, we will study the asymptotic behavior of u_μ, when the parameter μ goes to zero and we will prove that the following diffusion approximation result holds.

Theorem 4.5. *Assume Hypotheses 3, 4 and 5 and fix $(u_0, v_0) \in \mathscr{H}_2 \cap \mathscr{M}$. Then, for every $\alpha \in [0, 2)$ and $q < 4/\alpha$ and every $T > 0$, we have*

$$\lim_{\mu\to 0}\mathbb{P}\left(|u_\mu - u|_{L^q(0,T;H^\alpha)} > \eta\right) = 0, \qquad \eta > 0, \tag{4.13}$$

where $u \in L^2(\Omega; L^4(0, T; H^1 \cap M) \cap L^2(0, T; H^2))$ is the unique solution of the equation

$$\left\{\begin{array}{l} \gamma\,\partial_t u(t,\xi) = \Delta u(t,\xi) + |u(t)|_{H^1}^2 u(t,\xi) - \frac{1}{2}\|\sigma(u(t))\|_{\mathscr{T}_2(K,H)}^2 u(t) + \sigma(u(t))\partial_t w^Q(t,\xi), \\ u(0,\xi) = u_0(\xi), \quad u(t,\xi) = 0, \quad \xi \in \partial D. \end{array}\right. \tag{4.14}$$

Remark 4.6. Although the limit of u_μ to u lies in $L^q(0, T, H^\alpha)$ for $\alpha < 2$, in Theorem 4.5, we require that (u_0, v_0) belongs to $\mathcal{H}_2 = H^3 \times H^2$. This requirement arises because, to establish the tightness of $\{u_\mu\}_{\mu \in (0,1)}$ and validate the limit (4.13), we need a priori bounds in the H^2-norm. Given the nature of the equation satisfied by u_μ, such bounds follow from additional estimates on $(u_\mu, \sqrt{\mu}\,\partial_t u_\mu)$ in $\mathcal{H}_2$.

However, in this context we note that (4.14) remains valid even if the initial conditions $(u_0^\mu, v_0^\mu) \in \mathcal{H}_2 \cap \mathcal{M}$ depend on μ, provided that the following conditions hold

$$\lim_{\mu \to 0} |(u_0^\mu, v_0^\mu) - (u_0, v_0)|_{\mathcal{H}_1} = 0,$$

for some $(u_0, v_0) \in \mathcal{H}_1$, and

$$\lim_{\mu \to 0} \sqrt{\mu}\, |(u_0^\mu, v_0^\mu)|_{\mathcal{H}_2} = 0.$$

5. A Few Comments About the Limiting Equation (4.14)

In [9] it is proven that for every $T > 0$ there exists a unique mild solution

$$u \in L^2(\Omega; C([0, T]; H^1 \cap M))$$

for the *constrained* parabolic equation

$$\begin{cases} \gamma\, \partial_t u(t, \xi) = \Delta u(t, \xi) + |u(t)|_{H^1}^2 u(t, \xi) + \sigma(u(t)) \circ \partial_t w^Q(t, \xi), \\ u(0, \xi) = u_0(\xi), \quad u(t, \xi) = 0, \quad \xi \in \partial D. \end{cases} \tag{5.1}$$

As we have seen above, Eq. (5.1) can be rewritten in terms of Itô's integral as

$$\begin{cases} \gamma\, \partial_t u(t, \xi) = \Delta u(t, \xi) + |u(t)|_{H^1}^2 u(t, \xi) + \dfrac{1}{2} \operatorname{tr}_K [\sigma'(u(t))\sigma(u(t))] + \sigma(u(t))\partial_t w^Q(t, \xi), \\ u(0, \xi) = u_0(\xi), \quad u(t, \xi) = 0, \quad \xi \in \partial D. \end{cases} \tag{5.2}$$

The same arguments used in [9] for Eq. (5.1) (or, equivalently, Eq. (5.2)) can be adapted to prove the well-posedness of the limiting Eq. (4.14) from Theorem 4.5. Namely, for every $u_0 \in H^1 \cap M$ and $T > 0$ there exists a unique adapted process $u \in L^2(\Omega; C([0, T]; H^1 \cap M))$ such that for every $\psi \in C_0^\infty(D)$ and $t \in [0, T]$

$$\begin{aligned} \gamma \langle u(t), \psi \rangle_H = \gamma \langle u_0, \psi \rangle_H &- \int_0^t \langle \nabla u(s)), \nabla \psi \rangle_H \, ds \\ &+ \int_0^t |\nabla u(s)|_H^2 \langle u(s), \psi \rangle_H \, ds + \int_0^t \langle \sigma(u(s)) dw^Q(s), \psi \rangle_H. \end{aligned}$$

However, as we will show in the example we are providing below, Eqs. (5.2) and (4.14) are different, as well as their respective solutions. This fact is somehow unexpected and shows how, as a consequence of the Smoluchowski–Kramers approximation of a damped stochastic wave equation, a new stochastic parabolic equation satisfying the same constraints as Eq. (5.2) is obtained. In particular, all this poses the intriguing question whether different stochastic parabolic equations can still describe a motion confined to the unitary sphere of L^2 (to this purpose see also [35]).

Let $K = \mathbb{R}$ and let

$$\sigma_0(u) := g(|u|_{H^1}^2) h, \quad u \in H^1,$$

where $g(t) = (1+t)^{-1}$, and $h \in H^2$, with $|h|_H = 1$. It is immediate to check that σ_0 satisfies Hypotheses 3, 4 and 5. If we define

$$\sigma(u) = \sigma_0(u) - \langle \sigma_0(u), u\rangle_H u, \quad u \in H^1,$$

we have the following identity.

Lemma 5.1. *For every $u \in H^1 \cap M$, we have*

$$\sigma'(u)\sigma(u) + |\sigma(u)|_H^2 u = \Lambda(u)\Big(1 + |u|_{H^1}^2\Big)^{-3}, \tag{5.3}$$

where

$$\Lambda(u) = \Big(\langle u, h\rangle_H\, |u|_{H^1}^2 - 2\langle u, h\rangle_{H^1} - \langle u, h\rangle_H\Big)\,(h - \langle u, h\rangle_H u)\,. \tag{5.4}$$

Proof. The mapping $\sigma_0 : H^1 \to H$ is differentiable, so that also the mapping $\sigma : H^1 \to H$ is differentiable and for every $u \in H^1 \cap M$ it holds

$$\begin{aligned}\sigma'(u)\sigma(u) = \sigma_0'(u)\sigma_0(u) &- \langle \sigma_0(u), u\rangle_H \sigma_0'(u)u - \langle \sigma_0'(u)\sigma_0(u), u\rangle_H u\\ &+ \langle \sigma_0(u), u\rangle_H \langle \sigma_0'(u)u, u\rangle_H u - |\sigma_0(u)|_H^2 u + 2\,|\langle \sigma_0(u), u\rangle_H|^2\, u\\ &- \langle \sigma_0(u), u\rangle_H \sigma_0(u).\end{aligned}$$

Since for every $u \in H^1 \cap M$ and $v \in H^1$ we have

$$\sigma_0'(u)v = 2\,g'(|u|_{H^1}^2)\langle v, u\rangle_{H^1} h = -2\,g^2(|u|_{H^1}^2)\langle v, u\rangle_{H^1} h,$$

this gives

$$\begin{aligned}\sigma'(u)\sigma(u) = &-2\,g^3(|u|_{H^1}^2)\,\langle h, u\rangle_{H^1} h + 2\,g^3(|u|_{H^1}^2)\,\langle h, u\rangle_H |u|_{H^1}^2 h\\ &+ 2\,g^3(|u|_{H^1}^2)\,\langle h, u\rangle_{H^1}\langle h, u\rangle_H u - 2\,g^3(|u|_{H^1}^2)\,\,|\langle h, u\rangle_H|^2\,|u|_{H^1}^2 u\\ &- g^2(|u|_{H^1}^2)u + g^2(|u|_{H^1}^2)\,|\langle h, u\rangle_H|^2\, u - g^2(|u|_{H^1}^2)\langle h, u\rangle_H h\\ &+ g^2(|u|_{H^1}^2)|\langle h, u\rangle_H|^2 u,\end{aligned}$$

so that

$$\begin{aligned}\sigma'(u)\sigma(u) = g^3(|u|_{H^1}^2)\,&\Big(\langle u, h\rangle_H\, |u|_{H^1}^2 - 2\langle u, h\rangle_{H^1} - \langle u, h\rangle_H\Big)\, h\\ &+ \Big(2|\langle u, h\rangle_H|^2 + 2\langle u, h\rangle_H\langle u, h\rangle_{H^1} - 1 - |u|_{H^1}^2\Big)\, u\end{aligned} \tag{5.5}$$

Now, since we are assuming that $|u|_H = 1$, we have

$$\begin{aligned}|\sigma(u)|_H^2 u &= |\sigma_0(u)|_H^2 - |\langle \sigma_0(u), h\rangle_H|^2\, u = g^2(|u|_{H^1}^2)\,\Big(1 - |\langle u, h\rangle_H|^2\Big)\, u\\ &= g^3(|u|_{H^1}^2)\,\Big(1 + |u|_{H^1}^2 - |\langle u, h\rangle_H|^2 - |u|_{H^1}^2\, |\langle u, h\rangle_H|^2\Big)\, u.\end{aligned}$$

Therefore, if we sum this expression with (5.5), we obtain (5.3), with Λ defined as in (5.4). □

The mapping $\Lambda : H^1 \to H^1$ we have introduced in Lemma 5.1 is continuous and the set

$$Z := \left\{ u \in H^1 \cap M \ : \ \Lambda(u) = 0 \right\},$$

is a closed subset of $H^1 \cap M$. It is immediate to check that, if $\bar{u} \in H^1 \cap M$ is such that

$$\langle \bar{u}, h \rangle_H = 0, \quad \langle \bar{u}, h \rangle_{H^1} \neq 0,$$

we have that $\Lambda(\bar{u}) = -2\langle \bar{u}, h \rangle_{H^1}(1 + |\bar{u}|^2_{H^1})^3 h \neq 0$, and this means that $Z^c := H^1 \cap M \setminus Z$ is a non-empty open set.

Now, we fix $u_0 \in Z^c$ and we denote by u the solution of the equation

$$\begin{cases} \gamma\, \partial_t u(t,\xi) = \Delta u(t,\xi) + |u(t)|^2_{H^1} u(t,\xi) - \dfrac{1}{2}|\sigma(u(t))|^2_H u(t) + \sigma(u(t)) d\beta_t, \\ u(0,\xi) = u_0(\xi), \quad u(t,\xi) = 0, \quad \xi \in \partial D, \end{cases}$$

where σ is the mapping introduced above and β_t is a standard Brownian motion. Moreover, we denote by $\tilde{u}$ the solution of the equation

$$\begin{cases} \gamma\, \partial_t \tilde{u}(t,\xi) = \Delta \tilde{u}(t,\xi) + |\tilde{u}(t)|^2_{H^1} \tilde{u}(t,\xi) + \dfrac{1}{2}\sigma'(\tilde{u}(t))\sigma(\tilde{u}(t)) + \sigma(\tilde{u}(t)) d\beta_t, \\ \tilde{u}(0,\xi) = u_0(\xi), \quad \tilde{u}(t,\xi) = 0, \quad \xi \in \partial D, \end{cases}$$

for the same mapping σ and the same Brownian motion β_t. Both equations admit a unique solution in $L^2(\Omega; C([0,T]; H^1 \cap M))$

Theorem 5.2. *The two solutions u and $\tilde{u}$ are different.*

Proof. We introduce the stopping time

$$\tau := \inf\{t \in [0,T] \ : \ u(t) \in Z\},$$

with the usual convention that $\inf \emptyset = T$. Since $u(0) = u_0 \in Z^c$ and Z is closed, we have that $\mathbb{P}(\tau > 0) = 1$. Now, if we assume that there exists some stopping time τ' such that $\mathbb{P}(\tau' > 0) = 1$ and

$$u(s) = \tilde{u}(s), \quad s \in [0, \tau'), \quad \mathbb{P} - \text{a.s.}$$

we have

$$-\int_0^t |\sigma(u(s))|^2_H u(s)\, ds = \int_0^t \sigma'(u(s))\sigma(u(s))\, ds, \quad t < \tau', \quad \mathbb{P} - \text{a.s.}$$

In particular

$$\int_0^t \Lambda(u(s))\, ds = 0, \quad t < \tau', \quad \mathbb{P} - \text{a.s.}$$

so that

$$\Lambda(u(t)) = 0, \quad \text{a.e. } t < \tau' \wedge \tau, \quad \mathbb{P} - \text{a.s.}$$

However, this is not possible, as $\mathbb{P}\left(u(t) \in Z^c,\ t < \tau\right) = 1$. □

6. A-priori Bounds. Part I

In what follows we prove a series of a priori-bounds for the solution of system (4.12).

Lemma 6.1. *Assume Hypothesis 3 and fix* $(u_0, v_0) \in \mathscr{H} \cap \mathscr{M}$. *Then, for every integer* $p \geq 1$ *and every* $T > 0$ *there exists a constant* $c_{T,p} > 0$ *such that for every* $\mu \in (0,1)$

$$\mu^p\,\mathbb{E}\sup_{s\in[0,t]}|\partial_t u_\mu(s)|_H^{2p} + \mathbb{E}\sup_{s\in[0,t]}|u_\mu(s)|_{H^1}^{2p} + \mu^{p-1}\,\mathbb{E}\int_0^t |\partial_t u_\mu(s)|_H^{2p}\,ds$$
$$\leq c_{T,p} + \frac{c}{\mu}\,\mathbb{E}\int_0^t |u_\mu(s)|_{H^1}^{2(p-1)}\,\|\sigma_0(u_\mu(s))\|^2_{\mathscr{T}_2(K,H)}\,ds. \tag{6.1}$$

Proof. Let us consider the function $K_p : H \ni u \mapsto |v|_H^{2p} \in \mathbb{R}$. Then the 2nd order Frechét derivative of K_p satisfies

$$D^2K_p(v) = 4p(p-1)\,|v|_H^{2(p-2)}\langle v,\cdot\rangle_H\langle v,\cdot\rangle_H + 2p\,|v|_H^{2(p-1)}\langle\cdot,\cdot\rangle_H. \tag{6.2}$$

Thus, if we set

$$I_p(u,v) := \mathrm{tr}\big[D^2K_p(\sigma(u)\cdot,\sigma(u)\cdot)\big] = \sum_{k=1}^\infty D^2K_p(\sigma(u)\tilde{e}_k,\sigma(u)\tilde{e}_k),$$

by (4.11) we have

$$I_p(u,v) = \sum_{k=1}^\infty\Big(4p(p-1)\,|v|_H^{2(p-2)}\langle v,\sigma(u)\tilde{e_k}\rangle_H^2 + 2p\,|v|_H^{2(p-1)}|\sigma(u)\tilde{e_k}|_H^2\Big)$$
$$\leq c_p|v|_H^{2(p-1)}\,\|\sigma_0(u)\|^2_{\mathscr{T}_2(K,H)}.$$

In particular, by the Itô Lemma A.2 applied to the function K_p and the process v_μ, we get

$$d\,|v_\mu(t)|_H^{2p} \leq -\frac{p}{\mu}|v_\mu(t)|_H^{2(p-1)}\,d|u_\mu(t)|^2_{H^1} - \frac{2\,p\,\gamma}{\mu}|v_\mu(t)|_H^{2p}\,dt$$
$$+\frac{c_p}{\mu^2}\,|v_\mu(t)|_H^{2(p-1)}\,\|\sigma_0(u_\mu(t))\|^2_{\mathscr{T}_2(K,H)}\,dt$$
$$+\frac{2p}{\mu}|v_\mu(t)|_H^{2(p-1)}\langle\sigma_0(u_\mu(t))\,dw^Q(t), v_\mu(t)\rangle_H, \tag{6.3}$$

so that

$$d\,|v_\mu(t)|_H^{2p} + \frac{p}{\mu}d\Big(|v_\mu(t)|_H^{2(p-1)}\,|u_\mu(t)|^2_{H^1}\Big) + \frac{2\,p\,\gamma}{\mu}|v_\mu(t)|_H^{2p}\,dt$$
$$\leq \frac{c_p}{\mu^2}\,|v_\mu(t)|_H^{2(p-1)}\,\|\sigma_0(u_\mu(t))\|^2_{\mathscr{T}_2(K,H)}\,dt$$
$$+\frac{2p}{\mu}|v_\mu(t)|_H^{2(p-1)}\langle\sigma_0(u_\mu(t))\,dw^Q(t), v_\mu(t)\rangle_H + \frac{p}{\mu}\,|u_\mu(t)|^2_{H^1}\,d\,|v_\mu(t)|_H^{2(p-1)}. \tag{6.4}$$

Next, if we use inequality (6.3) in (6.4), with p replaced by $p-1$, we get

$$\begin{aligned}
&d\,|v_\mu(t)|_H^{2p} + \frac{p}{\mu} d\left(|v_\mu(t)|_H^{2(p-1)}\,|u_\mu(t)|_{H^1}^2\right) + \frac{2\,p\,\gamma}{\mu}|v_\mu(t)|_H^{2p}\,dt\\
&\quad\leq \frac{c_p}{\mu^2}\,|v_\mu(t)|_H^{2(p-1)}\,\|\sigma_0(u_\mu(t))\|^2_{\mathscr{T}_2(K,H)}\,dt + \frac{2p}{\mu}|v_\mu(t)|_H^{2(p-1)}\langle\sigma_0(u_\mu(t))\,dw^Q(t), v_\mu(t)\rangle_H\\
&\quad\quad - \frac{p(p-1)}{\mu^2}|u_\mu(t)|^2_{H^1}|v_\mu(t)|_H^{2(p-2)}\,d|u_\mu(t)|^2_{H^1} - \frac{2\,p(p-1)\,\gamma}{\mu^2}|u_\mu(t)|^2_{H^1}|v_\mu(t)|_H^{2(p-1)}\,dt\\
&\quad\quad + \frac{p\,c_p}{\mu^3}\,|u_\mu(t)|^2_{H^1}|v_\mu(t)|_H^{2(p-2)}\,\|\sigma_0(u_\mu(t))\|^2_{\mathscr{T}_2(K,H)}\,dt\\
&\quad\quad + \frac{2p(p-1)}{\mu^2}\,|u_\mu(t)|^2_{H^1}|v_\mu(t)|_H^{2(p-2)}\langle\sigma_0(u_\mu(t))\,dw^Q(t), v_\mu(t)\rangle_H,
\end{aligned}$$

and this implies

$$\begin{aligned}
&d\,|v_\mu(t)|_H^{2p} + \frac{p}{\mu} d\left(|v_\mu(t)|_H^{2(p-1)}\,|u_\mu(t)|_{H^1}^2\right) + \frac{p(p-1)}{2\mu^2}\,d\left(|v_\mu(t)|_H^{2(p-2)}\,|u_\mu(t)|^4_{H^1}\right)\\
&\quad + \frac{2\,p\,\gamma}{\mu}|v_\mu(t)|_H^{2p}\,dt + \frac{2\,p(p-1)\,\gamma}{\mu^2}|u_\mu(t)|^2_{H^1}|v_\mu(t)|_H^{2(p-1)}\,dt\\
&\leq c_p\left(\frac{1}{\mu^2}\,|v_\mu(t)|_H^{2(p-1)} + \frac{1}{\mu^3}\,|u_\mu(t)|^2_{H^1}|v_\mu(t)|_H^{2(p-2)}\right)\|\sigma_0(u_\mu(t))\|^2_{\mathscr{T}_2(K,H)}\,dt\\
&\quad + \frac{2p}{\mu}|v_\mu(t)|_H^{2(p-1)}\langle\sigma_0(u_\mu(t))\,dw^Q(t), v_\mu(t)\rangle_H + \frac{p(p-1)}{2\mu^2}\,|u_\mu(t)|^4_{H^1}\,d|v_\mu(t)|_H^{2(p-2)}\\
&\quad + \frac{2p(p-1)}{\mu^2}\,|u_\mu(t)|^2_{H^1}|v_\mu(t)|_H^{2(p-2)}\langle\sigma_0(u_\mu(t))\,dw^Q(t), v_\mu(t)\rangle_H.
\end{aligned}$$

By proceeding in this way recursively, we obtain

$$\begin{aligned}
&\sum_{i=0}^{p}\frac{a_{i,p}}{\mu^i}\,d\left(|v_\mu(t)|_H^{2(p-i)}\,|u_\mu(t)|^{2i}_{H^1}\right) + \sum_{i=1}^{p}\frac{b_{i,p}}{\mu^i}\,|v_\mu(t)|_H^{2(p-i+1)}\,|u_\mu(t)|^{2(i-1)}_{H^1}\,dt\\
&\quad\leq \sum_{i=1}^{p}\frac{c_{i,p}}{\mu^{i+1}}|v_\mu(t)|_H^{2(p-i)}\,|u_\mu(t)|^{2(i-1)}_{H^1}\,\|\sigma_0(u_\mu(t))\|^2_{\mathscr{T}_2(K,H)}\,dt\\
&\quad\quad + \sum_{i=1}^{p}\frac{d_{i,p}}{\mu^i}|v_\mu(t)|_H^{2(p-i)}\,|u_\mu(t)|^{2(i-1)}_{H^1}\langle\sigma_0(u_\mu(t))\,dw^Q(t), v_\mu(t)\rangle_H.
\end{aligned}$$

Thus, if we integrate both sides with respect to time and then take the supremum, we get

$$\begin{aligned}
&\sum_{i=0}^{p}\frac{1}{\mu^i}\sup_{s\in[0,t]}|v_\mu(s)|_H^{2(p-i)}\,|u_\mu(s)|^{2i}_{H^1} + \sum_{i=0}^{p-1}\frac{1}{\mu^{i+1}}\int_0^t|v_\mu(s)|_H^{2(p-i)}\,|u_\mu(s)|^{2i}_{H^1}\,ds\\
&\quad\leq \frac{c_p}{\mu^p} + c\sum_{i=1}^{p}\frac{1}{\mu^{i+1}}\int_0^t|v_\mu(s)|_H^{2(p-i)}\,|u_\mu(s)|^{2(i-1)}_{H^1}\,\|\sigma_0(u_\mu(s))\|^2_{\mathscr{T}_2(K,H)}\,ds\\
&\quad\quad + c\sum_{i=1}^{p}\frac{1}{\mu^i}\sup_{s\in[0,t]}\left|\int_0^s|v_\mu(r)|_H^{2(p-i)}\,|u_\mu(r)|^{2(i-1)}_{H^1}\langle\sigma_0(u_\mu(r))\,dw^Q(r), v_\mu(r)\rangle_H\right|
\end{aligned}$$

$$=: \frac{c_p}{\mu^p} + \sum_{i=1}^p I_{i,p}(t) + \sum_{i=1}^p J_{i,p}(t). \tag{6.5}$$

Due to the boundedness of $\sigma_0 : H^1 \to \mathscr{T}_2(K, H)$, we have

$$\sum_{i=1}^p \mathbb{E}\, I_{i,p}(t) = \frac{1}{\mu^{p+1}} \mathbb{E} \int_0^t |u_\mu(s)|_{H^1}^{2(p-1)} \, \|\sigma_0(u_\mu(s))\|^2_{\mathscr{T}_2(K,H)} \, ds$$
$$+ \frac{1}{4} \sum_{i=2}^p \frac{1}{\mu^{i+1}} \mathbb{E} \int_0^t |v_\mu(s)|_H^{2(p-i)} \, |u_\mu(s)|_{H^1}^{2i} \, ds + c \sum_{i=2}^p \frac{1}{\mu^{i+1}} \mathbb{E} \int_0^t |v_\mu(s)|_H^{2(p-i)} \, ds. \tag{6.6}$$

Moreover,

$$\mathbb{E} \sum_{i=1}^p J_{i,p}(t) \le \frac{c}{\mu} \mathbb{E} \left(\int_0^t |v_\mu(s)|_H^{4(p-1)+2} \, \|\sigma_0(u_\mu(s))\|^2_{\mathscr{T}_2(K,H)} \, ds \right)^{\frac{1}{2}}$$
$$+ c \sum_{i=2}^p \frac{1}{\mu^i} \mathbb{E} \left(\int_0^t |v_\mu(s)|_H^{4(p-i)+2} |u_\mu(s)|_{H^1}^{4(i-1)} \, \|\sigma_0(u_\mu(s))\|^2_{\mathscr{T}_2(K,H)} \, ds \right)^{\frac{1}{2}}$$
$$\le \frac{1}{2\mu} \mathbb{E} \sup_{s \in [0,t]} |v_\mu(s)|_H^{2p} + \frac{c}{\mu^2} \mathbb{E} \int_0^t |v_\mu(s)|_H^{2(p-1)} \, ds$$
$$+ \sum_{i=2}^p \frac{1}{2\mu^i} \mathbb{E} \sup_{s \in [0,t]} |v_\mu(s)|_H^{2(p-i)} \, |u_\mu(s)|_{H^1}^{2i}$$
$$+ c \sum_{i=2}^p \frac{1}{\mu^i} \mathbb{E} \int_0^t |v_\mu(s)|_H^{2(p-i+1)} \, |u_\mu(s)|_{H^1}^{2(i-2)} \, ds.$$

Now, we have

$$c \sum_{i=2}^p \frac{1}{\mu^i} \mathbb{E} \int_0^t |v_\mu(s)|_H^{2(p-i+1)} \, |u_\mu(s)|_{H^1}^{2(i-2)} \, ds$$
$$= c \sum_{i=1}^{p-1} \frac{1}{\mu^{i+1}} \mathbb{E} \int_0^t |v_\mu(s)|_H^{2(p-i)} \, |u_\mu(s)|_{H^1}^{2(i-1)} \, ds$$
$$\le \sum_{i=1}^{p-1} \frac{1}{4\mu^{i+1}} \mathbb{E} \int_0^t |v_\mu(s)|_H^{2(p-i)} \, |u_\mu(s)|_{H^1}^{2i} \, ds + c \sum_{i=1}^{p-1} \frac{1}{\mu^{i+1}} \mathbb{E} \int_0^t |v_\mu(s)|_H^{2(p-i)} \, ds,$$

so that

$$\sum_{i=1}^p \mathbb{E}\, J_{i,p}(t) \le \frac{1}{2\mu} \mathbb{E} \sup_{s \in [0,t]} |v_\mu(s)|_H^{2p} + \sum_{i=2}^p \frac{1}{2\mu^i} \mathbb{E} \sup_{s \in [0,t]} |v_\mu(s)|_H^{2(p-i)} \, |u_\mu(s)|_{H^1}^{2i}$$
$$+ \sum_{i=1}^{p-1} \frac{1}{4\mu^{i+1}} \mathbb{E} \int_0^t |v_\mu(s)|_H^{2(p-i)} \, |u_\mu(s)|_{H^1}^{2i} \, ds + c \sum_{i=1}^{p-1} \frac{1}{\mu^{i+1}} \mathbb{E} \int_0^t |v_\mu(s)|_H^{2(p-i)} \, ds. \tag{6.7}$$

Therefore, if we take the expectation of both sides in (6.5) and replace (6.6) and (6.7) in it, we obtain

$$\begin{aligned}&\mu^p\,\mathbb{E}\sup_{s\in[0,t]}|v_\mu(s)|_H^{2p}+\mathbb{E}\sup_{s\in[0,t]}|u_\mu(s)|_{H^1}^{2p}+\mu^{p-1}\,\mathbb{E}\int_0^t|v_\mu(s)|_H^{2p}\,ds\\&\quad\leq c_{T,p}+\frac{1}{\mu}\mathbb{E}\int_0^t|u_\mu(s)|_{H^1}^{2(p-1)}\,\|\sigma_0(u_\mu(s))\|^2_{\mathscr{T}_2(K,H)}\,ds\\&\qquad+c\sum_{i=1}^p\mu^{(p-i)-1}\mathbb{E}\int_0^t|v_\mu(s)|_H^{2(p-i)}\,ds.\end{aligned}$$

In particular, by a recursive argument, this implies (6.1). □

Remark 6.2. In the case $p=2$, inequality (6.1) implies

$$\begin{aligned}&\mu^3\,\mathbb{E}\sup_{s\in[0,t]}|v_\mu(s)|_H^4+\mu\,\mathbb{E}\sup_{s\in[0,t]}|u_\mu(s)|_{H^1}^4+\mu^2\,\mathbb{E}\int_0^t|v_\mu(s)|_H^4\,ds\\&\quad\leq c_T+\mathbb{E}\int_0^t|u_\mu(s)|_{H^1}^2\,ds.\end{aligned}\tag{6.8}$$

□

Lemma 6.3. *Under Hypothesis 3, for every* $(u_0,v_0)\in\mathscr{H}_1\cap\mathscr{M}$ *and* $T>0$ *there exists a constant* $c_T>0$ *such that for every* $\mu\in(0,1)$ *and* $t\in[0,T]$

$$\begin{aligned}&\mathbb{E}\sup_{t\in[0,T]}|u_\mu(t)|_{H^2}^2+\mu\,\mathbb{E}\sup_{t\in[0,T]}|\partial_t u_\mu(t)|_{H^1}^2+\mathbb{E}\int_0^t|\partial_t u_\mu(s)|_{H^1}^2\,ds\\&\quad\leq\frac{c}{\mu}\mathbb{E}\int_0^t|u_\mu(s)|_{H^1}^2\,ds+\frac{c_T}{\mu}.\end{aligned}\tag{6.9}$$

Proof. The Itô Lemma A.2 gives

$$\begin{aligned}\frac{1}{2}(|u_\mu(t)|_{H^2}^2+\mu|v_\mu(t)|_{H^1}^2)&=\langle u_\mu(t),v_\mu(t)\rangle_{H^2}\,dt\\&\quad+\langle v_\mu(t),\Delta u_\mu(t)+|u_\mu(t)|_{H^1}^2u_\mu(t)-\mu|v_\mu(t)|_H^2u_\mu(t)-\gamma v_\mu(t)\rangle_{H^1}\,dt\\&\quad+\frac{1}{2\mu}\|\sigma(u_\mu(t))\|^2_{\mathscr{T}_2(K,H^1)}\,dt+\langle\sigma(u_\mu(t))dw^Q(t),v_\mu(t)\rangle_{H^1}\\&=\frac{1}{2}|u_\mu(t)|_{H^1}^2d|u_\mu(t)|_{H^1}^2-\frac{\mu}{2}\,|v_\mu(t)|_H^2\,d|u_\mu(t)|_{H^1}^2-\gamma\,|v_\mu(t)|_{H^1}^2\,dt\\&\quad+\frac{1}{2\mu}\|\sigma(u_\mu(t))\|^2_{\mathscr{T}_2(K,H^1)}\,dt+\langle\sigma(u_\mu(t))dw^Q(t),v_\mu(t)\rangle_H.\end{aligned}\tag{6.10}$$

Now,

$$\begin{aligned}\mu\,|v_\mu(t)|_H^2\,d|u_\mu(t)|_{H^1}^2&=\mu\,d\left(|v_\mu(t)|_H^2\,|u_\mu(t)|_{H^1}^2\right)-2|u_\mu(t)|_{H^1}^2\langle v_\mu(t),\mu\,dv_\mu(t)\rangle_H\\&\quad-\frac{1}{\mu}|u_\mu(t)|_{H^1}^2\|\sigma(u_\mu(t))\|^2_{\mathscr{T}_2(K,H)}\,dt\\&=\mu d\left(|v_\mu(t)|_H^2\,|u_\mu(t)|_{H^1}^2\right)+|u_\mu(t)|_{H^1}^2\,d|u_\mu(t)|_{H^1}^2\end{aligned}$$

$$
\begin{aligned}
&+ 2\gamma\, |u_\mu(t)|_{H^1}^2 |v_\mu(t)|_H^2\, dt \\
&- \frac{1}{\mu} |u_\mu(t)|_{H^1}^2 \|\sigma(u_\mu(t))\|_{\mathcal{T}_2(K,H)}^2 dt. \qquad (6.11)
\end{aligned}
$$

Thus, if we replace (6.11) into (6.10), we get

$$
\begin{aligned}
&\frac{1}{2} d\left(|u_\mu(t)|_{H^2}^2 + \mu |v_\mu(t)|_{H^1}^2 + \mu |v_\mu(t)|_H^2\, |u_\mu(t)|_{H^1}^2\right) \\
&\quad + \gamma\left(|u_\mu(t)|_{H^1}^2 |v_\mu(t)|_H^2 + |v_\mu(t)|_{H^1}^2\right) dt \\
&\quad = \frac{1}{2\mu}\left(|u_\mu(t)|_{H^1}^2 \|\sigma(u_\mu(t))\|_{\mathcal{T}_2(K,H)}^2 + \|\sigma(u_\mu(t))\|_{\mathcal{T}_2(K,H^1)}^2\right) dt \\
&\quad\ + \langle \sigma(u_\mu(t)) dw^Q(t), v_\mu(t)\rangle_{H^1}.
\end{aligned}
$$

In view of inequalities (4.6) and (4.9) in Lemma 4.3, this implies

$$
\begin{aligned}
&\frac{1}{2} d\left(|u_\mu(t)|_{H^2}^2 + \mu |v_\mu(t)|_{H^1}^2 + \mu |v_\mu(t)|_H^2\, |u_\mu(t)|_{H^1}^2\right) \\
&\quad + \gamma\left(|u_\mu(t)|_{H^1}^2 |v_\mu(t)|_H^2 + |v_\mu(t)|_{H^1}^2\right) dt \\
&\quad\ \le \frac{c}{2\mu}\left(|u_\mu(t)|_{H^1}^2 + 1\right) dt + \langle \sigma(u_\mu(t)) dw^Q(t), v_\mu(t)\rangle_{H^1}.
\end{aligned}
$$

Therefore, there exists a constant $c_T > 0$ such that after we integrate with respect to time and take the supremum, for every $\mu \in (0,1)$ we obtain

$$
\begin{aligned}
&\sup_{s\in[0,t]} |u_\mu(s)|_{H^2}^2 + \mu \sup_{s\in[0,t]} |v_\mu(s)|_{H^1}^2 + \int_0^t |v_\mu(s)|_{H^1}^2\, ds \\
&\quad \le \frac{c_T}{\mu} + \frac{c}{\mu}\int_0^t |u_\mu(s)|_{H^1}^2\, ds + c \sup_{s\in[0,t]} \left|\int_0^s \langle \sigma(u_\mu(r)) dw^Q(r), v_\mu(r)\rangle_{H^1}\right|. \qquad (6.12)
\end{aligned}
$$

By the Davis inequality, see [47], and Hypothesis 3, we have

$$
\begin{aligned}
&\mathbb{E} \sup_{s\in[0,t]} \left|\int_0^s \langle \sigma(u_\mu(r)) dw^Q(r), v_\mu(r)\rangle_{H^1}\right| \\
&\quad \le c\, \mathbb{E}\left(\int_0^t \|\sigma(u_\mu(r))\|_{\mathcal{T}_2(K,H^1)}^2 |v_\mu(r)|_{H^1}^2\, dr\right)^{\frac{1}{2}} \\
&\quad \le c\, \mathbb{E}\left(\int_0^t |v_\mu(r)|_{H^1}^2 \left(1 + |u_\mu(r)|_{H^1}^2\right) dr\right)^{\frac{1}{2}} \\
&\quad \le c\, \mathbb{E}\left(\int_0^t |v_\mu(r)|_{H^1}^2\, dr\right)^{\frac{1}{2}} + c\, \mathbb{E}\left(\int_0^t |v_\mu(r)|_{H^1}^2 |u_\mu(r)|_{H^1}^2\, dr\right)^{\frac{1}{2}} \\
&\quad \le c + \frac{1}{2}\mathbb{E}\int_0^t |v_\mu(r)|_{H^1}^2\, dr + c\, \mathbb{E}\left(\sup_{s\in[0,t]} |v_\mu(s)|_{H^1} \int_0^t |u_\mu(r)|_{H^1}^2\, dr\right)^{\frac{1}{2}} \\
&\quad \le c + \frac{1}{2}\mathbb{E}\int_0^t |v_\mu(r)|_{H^1}^2\, dr + \frac{\mu}{2}\mathbb{E} \sup_{s\in[0,t]} |v_\mu(s)|_{H^1}^2 + \frac{c}{\mu}\mathbb{E}\int_0^t |u_\mu(r)|_{H^1}^2\, dr. \qquad (6.13)
\end{aligned}
$$

Combining (6.13) and (6.12) we get

$$\begin{aligned}
&\mathbb{E}\sup_{s\in[0,t]}|u_\mu(s)|_{H^2}^2+\mu\,\mathbb{E}\sup_{s\in[0,t]}|v_\mu(s)|_{H^1}^2+\mathbb{E}\int_0^t|v_\mu(s)|_{H^1}^2\,ds\\
&\quad\leq\frac{c}{\mu}\mathbb{E}\int_0^t|u_\mu(s)|_{H^1}^2\,ds+\frac{c_T}{\mu},
\end{aligned}$$

and this implies (6.9). □

Lemma 6.4. *Under Hypothesis 3, for every $(u_0,v_0)\in\mathscr{H}\cap\mathscr{M}$, such that $v_0\in H^1$, and for every $T>0$ there exists a constant $c_T>0$ such that for every $\mu\in(0,1)$ and $t\in[0,T]$*

$$\mathbb{E}|u_\mu(t)|_{H^1}^2\leq c_T+c\,\mu\,\mathbb{E}\int_0^t|\partial_t u_\mu(s)|_{H^1}^2\,ds. \tag{6.14}$$

Proof. The Itô Lemma A.2 gives

$$\begin{aligned}
&\mu\,d\langle u_\mu(t),v_\mu(t)\rangle_{H^1}\\
&\quad=\mu\,\langle du_\mu(t),v_\mu(t)\rangle_{H^1}+\langle u_\mu(t),\mu\,dv_\mu(t)\rangle_{H^1}\\
&\quad=\Big(\mu\,|v_\mu(t)|_{H^1}^2+\langle u_\mu(t),\Delta u_\mu(t)\rangle_{H^1}+|u_\mu(t)|_{H^1}^2|u_\mu(t)|_{H^1}^2-\mu\,|v_\mu(t)|_H^2|u_\mu(t)|_{H^1}^2\\
&\qquad-\gamma\langle v_\mu(t),u_\mu(t)\rangle_{H^1}\Big)\,dt+\langle\sigma(u_\mu(t))dw^Q(t),u_\mu(t)\rangle_{H^1}\\
&\quad=\Big(\mu\,|v_\mu(t)|_{H^1}^2-|u_\mu(t)|_{H^2}^2+|u_\mu(t)|_{H^1}^4-\mu\,|v_\mu(t)|_H^2|u_\mu(t)|_{H^1}^2\Big)\,dt\\
&\qquad-\frac{\gamma}{2}d|u_\mu(t)|_{H^1}^2+\langle\sigma(u_\mu(t))dw^Q(t),u_\mu(t)\rangle_{H^1}.
\end{aligned}$$

Hence, if we integrate both sides above with respect to time, we get

$$\begin{aligned}
&\frac{\mu}{2}\frac{d}{dt}|u_\mu(t)|_{H^1}^2+\frac{\gamma}{2}|u_\mu(t)|_{H^1}^2+\int_0^t\Big(|u_\mu(s)|_{H^2}^2-|u_\mu(t)|_{H^1}^4\Big)\,ds\\
&\quad\leq\mu\int_0^t|v_\mu(s)|_{H^1}^2\,ds+\int_0^t\langle\sigma(u_\mu(s))dw^Q(s),u_\mu(s)\rangle_{H^1}+\frac{\mu}{2}\langle u_0,v_0\rangle_{H^1}+\frac{\gamma}{2}|u_0|_{H^1}^2.
\end{aligned}$$

Now, thanks to (4.1) for every $u\in H^2\cap M$ we have $|u|_{H^1}^4\leq|u|_{H^2}^2$. Therefore we can find $c>0$ independent of $\mu\in(0,1)$ such that

$$\begin{aligned}
\frac{d}{dt}|u_\mu(t)|_{H^1}^2+\frac{\gamma}{\mu}|u_\mu(t)|_{H^1}^2&\leq\frac{c}{\mu}+c\int_0^t|v_\mu(s)|_{H^1}^2\,ds\\
&\quad+\frac{2}{\mu}\int_0^t\langle\sigma(u_\mu(s))dw^Q(s),u_\mu(s)\rangle_{H^1}.
\end{aligned}\tag{6.15}$$

In particular, if we take the expectation of both sides in (6.15) we get

$$\frac{d}{dt}\mathbb{E}\,|u_\mu(t)|_{H^1}^2+\frac{1}{\mu}\,\mathbb{E}\,|u_\mu(t)|_{H^1}^2\leq\frac{c_T}{\mu}+c\int_0^t\mathbb{E}\,|v_\mu(s)|_{H^1}^2\,ds.$$

Finally, by a comparison argument this implies (6.14). □

Remark 6.5. By combining together (6.9) and (6.14), we have that for every $(u_0, v_0) \in \mathscr{H}_1 \cap \mathscr{M}$ and $T > 0$ there exists $c_T(u_0, v_0) > 0$ such that for every $\mu \in (0, 1)$

$$\mathbb{E}|u_\mu(t)|_{H^1}^2 \leq c_T(u_0, v_0) + c\,\mathbb{E}\int_0^t |u_\mu(s)|_{H^1}^2\,ds, \quad t \in [0, T].$$

Hence, from the Gronwall Lemma we conclude

$$\mathbb{E}|u_\mu(t)|_{H^1}^2 \leq c_T(u_0, v_0), \quad t \in [0, T]. \tag{6.16}$$

In view of (6.9), this also implies that

$$\int_0^T \mathbb{E}|\partial_t u_\mu(s)|_{H^1}^2\,ds \leq \frac{1}{\mu}\,c_T(u_0, v_0). \tag{6.17}$$

Moreover, thanks to (6.8) and (6.16), we get

$$\sup_{\mu\in(0,1)} \mu^{3/2}\,\mathbb{E}\sup_{t\in[0,T]} |\partial_t u_\mu(t)|_H^2 < \infty. \tag{6.18}$$

□

7. A priori Bounds. Part II

Now, we want to show that in fact, if the initial condition (u_0, v_0) belongs to $\mathscr{H}_2 \cap \mathscr{M}$, then the solution $(u_\mu, \partial u_\mu)$ belongs to $L^2(\Omega; C([0, T]; \mathscr{H}_2 \cap \mathscr{M}))$ and suitable uniform bounds with respect to $\mu \in (0, 1)$ are satisfied. In what follows, it will be fundamental to assume that Hypotheses 4 and 5 hold.

Lemma 7.1. *Under Hypotheses 3 and 4, for every $(u_0, v_0) \in \mathscr{H}_1 \cap \mathscr{M}$ and $T > 0$ there exists a constant $c_T > 0$ such that for every $\mu \in (0, 1)$ we have*

$$\begin{aligned}\mu^3\,\mathbb{E}\sup_{t\in[0,T]} |\partial_t u_\mu(t)|_{H^1}^4 + \mu\,\mathbb{E}\sup_{t\in[0,T]} |u_\mu(t)|_{H^2}^4 + \mu^2\int_0^T \mathbb{E}|\partial_t u_\mu(s)|_{H^1}^4\,ds\\ + \mu\int_0^T \mathbb{E}|u_\mu(s)|_{H^2}^2|\partial_t u_\mu(s)|_{H^1}^2\,ds \leq c_T + c_T\int_0^T \mathbb{E}\,|u_\mu(t)|_{H^2}^2\,dt.\end{aligned} \tag{7.1}$$

Proof. In order to prove (7.1) we apply the Itô Lemma A.2 to the function

$$K : H^1 \ni v \mapsto |v|_{H^1}^4 \in \mathbb{R}$$

and the H^1-valued process $v_\mu(t)$. Since $DK(v) = 4\,|v|_H^2 v$, we get

$$\begin{aligned}d\,|v_\mu(t)|_{H^1}^4 = {} & \frac{4}{\mu}\,|v_\mu(t)|_{H^1}^2\langle v_\mu(t), \Delta u_\mu(t) + |u_\mu(t)|_{H^1}^2 u_\mu(t)\\ & - \mu|v_\mu(t)|_H^2 u_\mu(t) - \gamma v_\mu(t)\rangle_{H^1}\,dt\\ & + \frac{1}{2\mu^2}\sum_{j=1}^\infty D^2K(v_\mu(t))\,(\sigma(u_\mu(t))\tilde{e_j}, \sigma(u_\mu(t))\tilde{e_j})\,dt\\ & + \frac{4}{\mu}|v_\mu(t)|_{H^1}^2\langle\sigma(u_\mu(t))dw^Q(t), v_\mu(t)\rangle_{H^1}.\end{aligned}$$

Thus, if we denote

$$I(u,v) := \sum_{j=1}^{\infty} D^2 K(v)\,(\sigma(u)\tilde{e_j}, \sigma(u)\tilde{e_j}),$$

we have

$$\begin{aligned}
&d\,|v_\mu(t)|_{H^1}^4 + \frac{4\gamma}{\mu}|v_\mu(t)|_{H^1}^4 \\
&\quad= -\frac{2}{\mu}|v_\mu(t)|_{H^1}^2\, d|u_\mu(t)|_{H^2}^2 + \frac{2}{\mu}|v_\mu(t)|_{H^1}^2 |u_\mu(t)|_{H^1}^2\, d|u_\mu(t)|_{H^1}^2 \\
&\qquad - 2\,|v_\mu(t)|_{H^1}^2 |v_\mu(t)|_H^2\, d|u_\mu(t)|_{H^1}^2 + \frac{1}{2\mu^2}\, I(v_\mu(t), u_\mu(t))\, dt \\
&\qquad + \frac{4}{\mu}|v_\mu(t)|_{H^1}^2 \langle \sigma(u_\mu(t)) dw^Q(t), v_\mu(t)\rangle_{H^1}.
\end{aligned} \tag{7.2}$$

The Itô A.2 Lemma gives

$$\begin{aligned}
d|v_\mu(t)|_{H^1}^2 &= \frac{2}{\mu}\langle v_\mu(t), \Delta u_\mu(t) + |u_\mu(t)|_{H^1}^2 u_\mu(t) - \mu|v_\mu(t)|_H^2 u_\mu(t) - \gamma v_\mu(t)\rangle_H\, dt \\
&\quad + \frac{1}{\mu^2}\|\sigma(u_\mu(t))\|_{\mathscr{T}_2(K,H^1)}^2\, dt + \frac{2}{\mu}\,\langle \sigma(u_\mu(t)) dw^Q(t), v_\mu(t)\rangle_{H^1} \\
&= -\frac{1}{\mu} d\,|u_\mu(t)|_{H^2}^2 + \frac{1}{2\mu} d\,|u_\mu(t)|_{H^1}^4 - |v_\mu(t)|_H^2 d|u_\mu(t)|_{H^1}^2 \\
&\quad - \frac{2\gamma}{\mu}|v_\mu(t)|_{H^1}^2\, dt \\
&\quad + \frac{1}{\mu^2}\|\sigma(u_\mu(t))\|_{\mathscr{T}_2(K,H^1)}^2\, dt + \frac{2}{\mu}\,\langle \sigma(u_\mu(t)) dw^Q(t), v_\mu(t)\rangle_{H^1}.
\end{aligned}$$

Hence, if we define

$$\Phi(u) := |u|_{H^2}^2 - \frac{1}{2}|u|_{H^1}^4,$$

it is not difficult to check that

$$\begin{aligned}
&-\frac{2}{\mu}|v_\mu(t)|_{H^1}^2\, d|u_\mu(t)|_{H^2}^2 + \frac{2}{\mu}|v_\mu(t)|_{H^1}^2 |u_\mu(t)|_{H^1}^2\, d|u_\mu(t)|_{H^1}^2 \\
&\quad= -\frac{2}{\mu} d\left(\Phi(u_\mu(t))|v_\mu(t)|_{H^1}^2\right) - \frac{2}{\mu^2}\,\Phi(u_\mu(t))\, d\Phi(u_\mu(t)) \\
&\quad - \frac{2}{\mu}\Phi(u_\mu(t))|v_\mu(t)|_H^2\, d|u_\mu(t)|_{H^1}^2 \\
&\qquad - \frac{4\gamma}{\mu^2}\Phi(u_\mu(t))|v_\mu(t)|_{H^1}^2 dt + \frac{2}{\mu^3}\,\Phi(u_\mu(t))\,\|\sigma(u_\mu(t))\|_{\mathscr{T}_2(K,H^1)}^2 dt \\
&\qquad + \frac{4}{\mu^2}\,\Phi(u_\mu(t))\langle \sigma(u_\mu(t)) dw^Q(t), v_\mu(t)\rangle_{H^1}.
\end{aligned} \tag{7.3}$$

In particular, if we replace (7.3) into (7.2), we get

$$\begin{aligned}
&d\left(|v_\mu(t)|^4_{H^1}+\frac{2}{\mu}\Phi(u_\mu(t))|v_\mu(t)|^2_{H^1}+\frac{1}{\mu^2}\,\Phi^2(u_\mu(t))\right)\\
&\quad+\left(\frac{4\gamma}{\mu}|v_\mu(t)|^4_{H^1}+\frac{4\gamma}{\mu^2}\Phi(u_\mu(t))|v_\mu(t)|^2_{H^1}\right)dt\\
&=-\frac{2}{\mu}\left(\Phi(u_\mu(t))+\mu\,|v_\mu(t)|^2_{H^1}\right)|v_\mu(t)|^2_H\,d|u_\mu(t)|^2_{H^1}\\
&\quad+\frac{2}{\mu^3}\,\Phi(u_\mu(t))\,\|\sigma(u_\mu(t))\|^2_{\mathcal{T}_2(K,H^1)}dt+\frac{1}{2\mu^2}\,I(v_\mu(t),u_\mu(t))\,dt\\
&\quad+\frac{4}{\mu^2}\,\Phi(u_\mu(t))\langle\sigma(u_\mu(t))dw^Q(t),v_\mu(t)\rangle_{H^1}\\
&+\frac{4}{\mu}|v_\mu(t)|^2_{H^1}\langle\sigma(u_\mu(t))dw^Q(t),v_\mu(t)\rangle_{H^1}.
\end{aligned}\tag{7.4}$$

Now, since similarly to (6.2) we have

$$D^2K(v)=8\langle v,\cdot\rangle_{H^1}\langle v,\cdot\rangle_{H^1}+4\,|v|^2_{H^1}\langle\cdot,\cdot\rangle_{H^1}.$$

remembering that $|u|_H=1$ from (2.19) we infer that

$$\begin{aligned}
D^2K\,(\sigma(u)\tilde{e}_j,\sigma(u)\tilde{e}_j)&=8\langle v,\sigma(u)\tilde{e}_j\rangle^2_{H^1}+4\,|v|^2_{H^1}|\sigma(u)\tilde{e}_j|^2_{H^1}\\
&\leq c\,|v|^2_{H^1}|\sigma(u)\tilde{e}_j|^2_{H^1}\\
&\leq c\,|v|^2_{H^1}\left(|\sigma_0(u)\tilde{e}_j|^2_{H^1}+|\sigma_0(u)\tilde{e}_j|^2_H|u|^2_{H^1}\right).
\end{aligned}\tag{7.5}$$

Therefore, if we sum both sides in (7.5) with respect to $j\in\mathbb{N}$, we get

$$\begin{aligned}
I(v,u)&=\sum_{j=1}^{\infty}D^2K\,(\sigma(u)\tilde{e}_j,\sigma(u)\tilde{e}_j)\\
&\leq c\,|v|^2_{H^1}\left(\|\sigma_0(u)\|^2_{\mathcal{T}_2(K,H^1)}+\|\sigma_0(u)\|^2_{\mathcal{T}_2(K,H)}|u|^2_{H^1}\right).
\end{aligned}$$

Thus, in view of Hypotheses 3 and 4 we infer that for some positive $c>0$,

$$\sup_{u\in H^1\cap M}I(v,u)\leq c\,|v|^2_{H^1},\quad v\in H^1.\tag{7.6}$$

According to (4.1), we have

$$\frac{1}{2}|u|^2_{H^2}\leq\Phi(u)\leq|u|^2_{H^2}.\tag{7.7}$$

Then, if we integrate both sides in (7.4) with respect to time, thanks to (7.6) we obtain

$$\begin{aligned}
&|v_\mu(t)|^4_{H^1}+\frac{1}{\mu^2}|u_\mu(t)|^4_{H^2}+\frac{1}{\mu}\int_0^t|v_\mu(s)|^4_{H^1}\,ds+\frac{1}{\mu^2}\int_0^t|u_\mu(s)|^2_{H^2}|v_\mu(s)|^2_{H^1}\,ds\\
&\quad\leq\frac{c}{\mu^2}+\frac{c}{\mu}\int_0^t|u_\mu(s)|^2_{H^2}|v_\mu(s)|^2_H|\langle u_\mu(s),v_\mu(s)\rangle_{H^1}|\,ds
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t |v_\mu(s)|_{H^1}^2 |v_\mu(s)|_H^2 \left|\langle u_\mu(s), v_\mu(s)\rangle_{H^1}\right| ds\\
&+\frac{c}{\mu^3}\int_0^t |u_\mu(s)|_{H^2}^2 \,\|\sigma(u_\mu(s))\|^2_{\mathscr{T}_2(K,H^1)}\,ds+\frac{c}{\mu^2}\int_0^t |v_\mu(s)|_{H^1}^2\,ds\\
&+\frac{4}{\mu^2}\int_0^t \Phi(u_\mu(s))\langle\sigma(u_\mu(s))\,dw^Q(s), v_\mu(s)\rangle_{H^1}\\
&+\frac{4}{\mu}\int_0^t |v_\mu(s)|_{H^1}^2\langle\sigma(u_\mu(s))\,dw^Q(s), v_\mu(s)\rangle_{H^1}=:\frac{c}{\mu^2}+\sum_{i=1}^6 J_i^\mu(t).
\end{aligned}
\tag{7.8}
$$

In what follows we will estimate each term $J_i^\mu(t)$. Since

$$
|\langle u, v\rangle_{H^1}|\leq |u|_{H^1}|v|_{H^1}\leq |u|_{H^2}^{1/2}|v|_{H^1},
$$

for $J_1^\mu(t)$ we have

$$
\begin{aligned}
J_1^\mu(t)&\leq\frac{c}{\mu}\int_0^t |u_\mu(s)|_{H^2}^{5/2}|v_\mu(s)|_H^2|v_\mu(s)|_{H^1}\,ds\\
&\leq\frac{1}{2\mu^2}\int_0^t |u_\mu(s)|_{H^2}^2|v_\mu(s)|_{H^1}^2\,ds+\frac{c}{\mu^2}\int_0^t |u_\mu(s)|_{H^2}^4\,ds+c\,\mu^6\int_0^t |v_\mu(s)|_H^{16}\,ds.
\end{aligned}
\tag{7.9}
$$

For $J_2^\mu(t)$, due to (4.1), we have

$$
\begin{aligned}
J_2^\mu(t)&\leq c\int_0^t |v_\mu(s)|_{H^1}^2|v_\mu(s)|_H^2|u_\mu(s)|_{H^1}|v_\mu(s)|_{H^1}\,ds\\
&\leq c\int_0^t |v_\mu(s)|_{H^1}^3|v_\mu(s)|_H^2|u_\mu(s)|_{H^2}^{1/2}\,ds\\
&\leq\frac{1}{4\mu}\int_0^t |v_\mu(s)|_{H^1}^4\,ds+\frac{c}{\mu^2}\int_0^t |u_\mu(s)|_{H^2}^4\,ds+c\,\mu^8\int_0^t |v_\mu(s)|_H^{16}\,ds.
\end{aligned}
\tag{7.10}
$$

As for $J_3^\mu(t)$, due to (4.7) we have

$$
J_3^\mu(t)\leq\frac{c}{\mu^3}\int_0^t |u_\mu(s)|_{H^2}^2\,ds.
\tag{7.11}
$$

For $J_4^\mu(t)$, we have

$$
J_4^\mu(t)\leq\frac{1}{4\mu}\int_0^t |v_\mu(s)|_{H^1}^4\,ds+\frac{c_T}{\mu^3}.
\tag{7.12}
$$

Therefore, if we replace (7.9), (7.10), (7.11) and (7.12) in (7.8), we get

$$
\begin{aligned}
&|v_\mu(t)|_{H^1}^4+\frac{1}{\mu^2}|u_\mu(t)|_{H^2}^4+\frac{1}{2\mu}\int_0^t |v_\mu(s)|_{H^1}^4\,ds+\frac{1}{2\mu^2}\int_0^t |u_\mu(s)|_{H^2}^2|v_\mu(s)|_{H^1}^2\,ds\\
&\leq\frac{c}{\mu^2}+\frac{c}{\mu^2}\int_0^t |u_\mu(s)|_{H^2}^4\,ds+\frac{c}{\mu^3}\int_0^t |u_\mu(s)|_{H^2}^2\,ds\\
&\quad+c\,\mu^6\int_0^t |v_\mu(s)|_H^{16}\,ds+\frac{c_T}{\mu^3}+J_5^\mu(t)+J_6^\mu(t).
\end{aligned}
$$

In particular, thanks to the Gronwall Lemma, for every $t \in [0, T]$ and $\mu \in (0,1)$ we get

$$\begin{aligned}
&\sup_{s\in[0,t]} |v_\mu(s)|_{H^1}^4 + \frac{1}{\mu^2} \sup_{s\in[0,t]} |u_\mu(s)|_{H^2}^4 + \frac{1}{2\mu} \int_0^t |v_\mu(s)|_{H^1}^4\, ds \\
&\quad + \frac{1}{2\mu^2} \int_0^t |u_\mu(s)|_{H^2}^2 |v_\mu(s)|_{H^1}^2\, ds \\
&\quad \leq \frac{c_T}{\mu^2} + \frac{c_T}{\mu^3} \int_0^t |u_\mu(s)|_{H^2}^2\, ds + \frac{c_T}{\mu^2} \int_0^t |u_\mu(s)|_{H^2}^4\, ds + c_T\, \mu^6 \int_0^t |v_\mu(s)|_H^{16}\, ds \\
&\quad + \frac{c_T}{\mu^3} + c_T \left(\sup_{s\in[0,t]} \left| J_5^\mu(s) \right| + \sup_{s\in[0,t]} \left| J_6^\mu(s) \right| \right).
\end{aligned} \tag{7.13}$$

Thanks to inequality (4.7) in Lemma 4.3 and inequality (7.7), we have

$$\begin{aligned}
\mathbb{E} \sup_{s\in[0,t]} \left| J_5^\mu(s) \right| &\leq \frac{c}{\mu^2} \mathbb{E} \left(\int_0^t |\Phi(u_\mu(s))|^2 |v_\mu(r)|_{H^1}^2\, ds \right)^{\frac{1}{2}} \\
&\leq \frac{1}{2\mu^2}\, \mathbb{E} \sup_{s\in[0,t]} |u_\mu(s)|_{H^2}^4 + \frac{c}{\mu^2} \mathbb{E} \int_0^t |v_\mu(s)|_{H^1}^2\, ds.
\end{aligned} \tag{7.14}$$

Similarly, we get

$$\begin{aligned}
\mathbb{E} \sup_{s\in[0,t]} \left| J_6^\mu(s) \right| &\leq \frac{c}{\mu}\, \mathbb{E} \left(\int_0^t |v_\mu(s))|_{H^1}^4 |v_\mu(r)|_{H^1}^2\, ds \right)^{\frac{1}{2}} \\
&\leq \frac{1}{2}\, \mathbb{E} \sup_{s\in[0,t]} |v_\mu(s)|_{H^1}^4 + \frac{c}{\mu^2} \mathbb{E} \int_0^t |v_\mu(s)|_{H^1}^2\, ds.
\end{aligned} \tag{7.15}$$

Therefore, if we take the expectation of both sides in (7.13), in view of (7.14) and (7.15), we obtain

$$\begin{aligned}
&\mathbb{E} \sup_{s\in[0,t]} |v_\mu(s)|_{H^1}^4 + \frac{1}{\mu^2} \mathbb{E} \sup_{s\in[0,t]} |u_\mu(s)|_{H^2}^4 \\
&\quad + \frac{1}{\mu} \mathbb{E} \int_0^t |v_\mu(s)|_{H^1}^4\, ds + \frac{1}{\mu^2} \mathbb{E} \int_0^t |u_\mu(s)|_{H^2}^2 |v_\mu(s)|_{H^1}^2\, ds \\
&\leq \frac{c_T}{\mu^2} + \frac{c_T}{\mu^3} \mathbb{E} \int_0^t |u_\mu(s)|_{H^2}^2\, ds + \frac{c_T}{\mu^2}\, \mathbb{E} \int_0^t |u_\mu(s)|_{H^2}^4\, ds + c_T\, \mu^6 \int_0^t \mathbb{E}\, |v_\mu(s)|_H^{16}\, ds \\
&\quad + \frac{c}{\mu^2} \mathbb{E} \int_0^t |v_\mu(s)|_{H^1}^2\, ds + \frac{c_T}{\mu^3}.
\end{aligned}$$

As a consequence of the Gronwall lemma, thanks to (6.17), after we multiply both sides by μ^3 we get

$$\begin{aligned}
&\mu^3\, \mathbb{E} \sup_{t\in[0,T]} |\partial_t u_\mu(t)|_{H^1}^4 + \mu\, \mathbb{E} \sup_{t\in[0,T]} |u_\mu(t)|_{H^2}^4 \\
&\quad + \mu^2 \int_0^t \mathbb{E} |\partial_t u_\mu(s)|_{H^1}^4\, ds + \mu\, \mathbb{E} \int_0^t |u_\mu(s)|_{H^2}^2 |v_\mu(s)|_{H^1}^2\, ds \\
&\quad \leq c_T + c_T \int_0^T \mathbb{E}\, |u_\mu(t)|_{H^2}^2\, dt + c_T\, \mu^9 \int_0^T \mathbb{E}\, |\partial_t u_\mu(s)|_H^{16}\, ds.
\end{aligned}$$

Now, according to (6.1) and Hypothesis 4, we have

$$\begin{aligned}\mu^9\,\mathbb{E}\int_0^T |\partial_t u_\mu(s)|_H^{16}\,ds &\leq c_T + c_T\,\mu\,\mathbb{E}\int_0^t |u_\mu(s)|_{H^1}^{14}\,\|\sigma_0(u_\mu(s))\|^2_{\mathscr{T}_2(K,H)}\,ds\\ &\leq c_T + c_T\,\mu\,\mathbb{E}\int_0^t |u_\mu(s)|_{H^1}^{10}\,ds,\end{aligned}$$

and if we use again (6.1) and Hypothesis 4 we have

$$\begin{aligned}\mu^9\,\mathbb{E}\int_0^T |\partial_t u_\mu(s)|_H^{16}\,ds &\leq c_T + c_T\,\mu\left(\frac{c_T}{\mu}+\frac{c}{\mu}\mathbb{E}\int_0^t |u_\mu(s)|_{H^1}^{8}\,\|\sigma_0(u_\mu(s))\|^2_{\mathscr{T}_2(K,H)}\,ds\right)\\ &\leq c_T + c_T\,\mathbb{E}\int_0^t |u_\mu(s)|_{H^1}^{4}\,ds.\end{aligned}$$

This allows to conclude that (7.1) holds. □

Lemma 7.2. *Assume Hypotheses 3, 4 and 5, and fix* $(u_0, v_0) \in \mathscr{H}_2 \cap \mathscr{M}$. *Then, for every* $T > 0$ *there exists* c_T *such that for every* $\mu \in (0, 1)$

$$\mu\,\mathbb{E}\sup_{s\in[0,t]}|v_\mu(t)|_{H^2}^2 + \mathbb{E}\int_0^t |v_\mu(s)|_{H^2}^2\,ds \leq \frac{c_T}{\mu}\mathbb{E}\int_0^t |u_\mu(s)|_{H^2}^2\,ds + \frac{c_T}{\mu}. \tag{7.16}$$

Proof. The Itô Lemma A.2 gives

$$\begin{aligned}&\frac{1}{2}d\left(|u_\mu(t)|_{H^3}^2 + \mu|v_\mu(t)|_{H^2}^2\right)\\ &= \langle u_\mu(t), du_\mu(t)\rangle_{H^3} + \langle v_\mu(t), \mu\,dv_\mu(t)\rangle_{H^2} + \frac{1}{2\mu}\|\sigma(u_\mu(t))\|^2_{\mathscr{T}_2(K,H^2)}\,dt\\ &= \frac{1}{2}|u_\mu(t)|_{H^1}^2\,d|u_\mu(t)|_{H^2}^2 - \frac{\mu}{2}|v_\mu(t)|_H^2\,d|u_\mu(t)|_{H^2}^2 - \gamma\,|v_\mu(t)|_{H^2}^2\,dt\\ &\quad+\frac{1}{2\mu}\|\sigma(u_\mu(t))\|^2_{\mathscr{T}_2(K,H^2)}\,dt + \langle\sigma(u_\mu(t))dw^Q(t), v_\mu(t)\rangle_{H^2}.\end{aligned} \tag{7.17}$$

Now, we have

$$|u_\mu(t)|_{H^1}^2\,d|u_\mu(t)|_{H^2}^2 = d\left(|u_\mu(t)|_{H^1}^2|u_\mu(t)|_{H^2}^2\right) - d|u_\mu(t)|_{H^1}^2|u_\mu(t)|_{H^2}^2, \tag{7.18}$$

and

$$\begin{aligned}\mu\,|v_\mu(t)|_H^2\,d|u_\mu(t)|_{H^2}^2 &\leq 2\mu\,|v_\mu(t)|_H^2\,|u_\mu(t)|_{H^2}|v_\mu(t)|_{H^2}\\ &\leq \gamma\,|v_\mu(t)|_{H^2}^2 + c\,|u_\mu(t)|_{H^2}^4 + c\,\mu^2\,|v_\mu(t)|_H^4.\end{aligned} \tag{7.19}$$

Thus, if we plug (7.18), and (7.19) into (7.17), we get

$$\begin{aligned}&\frac{1}{2}d\left(|u_\mu(t)|_{H^3}^2 - |u_\mu(t)|_{H^2}^2\,|u_\mu(t)|_{H^1}^2\right) + \frac{\mu}{2}\,d|v_\mu(t)|_{H^2}^2 + \frac{\gamma}{2}\,|v_\mu(t)|_{H^2}^2\,dt\\ &\quad\leq c\,|u_\mu(t)|_{H^2}^4\,dt + c\,\mu^2\,|v_\mu(t)|_H^4 - \frac{1}{2}\,|u_\mu(t)|_{H^2}^2\,d|u_\mu(t)|_{H^1}^2\,dt\\ &\quad\quad+\frac{1}{2\mu}\|\sigma(u_\mu(t))\|^2_{\mathscr{T}_2(K,H^2)}\,dt + \langle\sigma(u_\mu(t))dw^Q(t), v_\mu(t)\rangle_{H^2}.\end{aligned}$$

Due to (4.1), we have

$$\begin{aligned}|u_\mu(t)|^2_{H^2}\, d|u_\mu(t)|^2_{H^1} &\leq 2|u_\mu(t)|^2_{H^2}|u_\mu(t)|_{H^1}|v_\mu(t)|_{H^1} \leq 2|u_\mu(t)|^{5/2}_{H^2}|v_\mu(t)|^{1/2}_{H^2}|v_\mu(t)|^{1/2}_{H}\\ &\leq c\,|u_\mu(t)|^2_{H^2}|v_\mu(t)|^2_H + \frac{\gamma}{4}\,|v_\mu(t)|^2_{H^2} + c\,|u_\mu(t)|^4_{H^2},\end{aligned}$$

and thanks to inequality (4.10) from Lemma 4.3 this implies

$$\begin{aligned}&\frac{1}{2}d\left(|u_\mu(t)\ |^2_{H^3} - |u_\mu(t)|^2_{H^2}\,|u_\mu(t)|^2_{H^1}\right) + \frac{\mu}{2}\,d|v_\mu(t)|^2_{H^2} + \frac{\gamma}{4}|v_\mu(t)|^2_{H^2}\,dt\\ &\quad\leq \frac{c}{\mu}\,dt + \frac{c}{\mu}|u_\mu(t)|^2_{H^2}\,dt + c\,|u_\mu(t)|^4_{H^2}\,dt + c\,|u_\mu(t)|^2_{H^2}|v_\mu(t)|^2_H\,dt + c\,\mu^2\,|v_\mu(t)|^4_H\,dt\\ &\qquad+ \langle\sigma(u_\mu(t))dw^Q(t), v_\mu(t)\rangle_{H^2}.\end{aligned}$$

Since

$$|u_\mu(t)|^2_{H^3} - |u_\mu(t)|^2_{H^2}\,|u_\mu(t)|^2_{H^1} \geq 0,$$

after we integrate with respect to t, and take first the supremum in t and then the expectation, we get

$$\begin{aligned}&\mu\,\mathbb{E}\sup_{s\in[0,t]}|v_\mu(t)|^2_{H^2} + \frac{\gamma}{4}\mathbb{E}\int_0^t|v_\mu(s)|^2_{H^2}\,ds \leq \frac{c_T}{\mu} + \frac{c}{\mu}\mathbb{E}\int_0^t|u_\mu(s)|^2_{H^2}\,ds\\ &\quad+ c\int_0^t\mathbb{E}|u_\mu(s)|^4_{H^2}\,ds + c\int_0^t|u_\mu(s)|^2_{H^2}|v_\mu(s)|^2_H\,ds + c\,\mu^2\mathbb{E}\int_0^t|v_\mu(s)|^4_H\,ds\\ &\quad+ \mathbb{E}\sup_{s\in[0,t]}\left|\int_0^s\langle\sigma(u_\mu(r))dw^Q(r), v_\mu(r)\rangle_{H^2}\right|. \end{aligned}\tag{7.20}$$

According to inequality (4.10) from Lemma 4.3, and inequalities (6.9) and (6.16), we have

$$\begin{aligned}&\mathbb{E}\sup_{s\in[0,t]}\left|\int_0^s\langle\sigma(u_\mu(r))dw^Q(r), v_\mu(r)\rangle_{H^2}\right| \leq c\,\mathbb{E}\left(\int_0^t\left(1+|u_\mu(s)|^2_{H^2}\right)|v_\mu(s)|^2_{H^2}\,ds\right)^{1/2}\\ &\quad\leq \frac{\gamma}{8}\mathbb{E}\int_0^t|v_\mu(s)|^2_{H^2}\,ds + c\,\mathbb{E}\sup_{s\in[0,t]}|u_\mu(s)|^2_{H^2} + c_T \leq \frac{\gamma}{8}\mathbb{E}\int_0^t|v_\mu(s)|^2_{H^2}\,ds + \frac{c_T}{\mu}.\end{aligned}$$

and if we replace this into (7.20) we obtain

$$\begin{aligned}&\mu\,\mathbb{E}\sup_{s\in[0,t]}|v_\mu(t)|^2_{H^2} + \frac{\gamma}{8}\mathbb{E}\int_0^t|v_\mu(s)|^2_{H^2}\,ds \leq \frac{c}{\mu}\mathbb{E}\int_0^t|u_\mu(s)|^2_{H^2}\,ds\\ &\quad+ c\int_0^t\mathbb{E}|u_\mu(s)|^4_{H^2}\,ds\\ &\quad+ c\int_0^t|u_\mu(s)|^2_{H^2}|v_\mu(s)|^2_H\,ds + c\,\mu^2\mathbb{E}\int_0^t|v_\mu(s)|^4_H\,ds + \frac{c_T}{\mu}.\end{aligned}$$

Hence, (7.1) allows to conclude that (7.16) holds. □

Lemma 7.3. *Under Hypotheses 3, 4 and 5, for every* $(u_0, v_0) \in \mathscr{H}_2 \cap \mathscr{M}$ *and* $T > 0$ *there exists a constant* $c_T > 0$ *such that for every* $\mu \in (0,1)$ *and* $t \in [0,T]$

$$\mathbb{E}|u_\mu(t)|^2_{H^2} \leq c_T + c\,\mu\,\mathbb{E}\int_0^t |\partial_t u_\mu(s)|^2_{H^2}\,ds. \tag{7.21}$$

Proof. We apply the Itô Lemma A.2 to the function

$$K_\mu : H^2 \times H^2 \ni (u,v) \mapsto \mu\,\langle u, v\rangle_{H^2} \in \mathbb{R},$$

and by proceeding as in the the proof of Lemma 6.4, we get

$$\begin{aligned}\frac{\mu}{2}\frac{d}{dt}|u_\mu(t)|^2_{H^2} + \frac{\gamma}{2}|u_\mu(t)|^2_{H^2} + \int_0^t \left(|u_\mu(s)|^2_{H^3} - |u_\mu(t)|^2_{H^1}|u_\mu(t)|^2_{H^2}\right) ds\\ \leq \mu\int_0^t |v_\mu(s)|^2_{H^2}\,ds + \int_0^t \langle \sigma(u_\mu(s))dw^Q(s), u_\mu(s)\rangle_{H^2} + \frac{\mu}{2}\langle u_0, v_0\rangle_{H^2} + c.\end{aligned} \tag{7.22}$$

According to (4.1) we have

$$|u_\mu(s)|^2_{H^3} - |u_\mu(t)|^2_{H^1}|u_\mu(t)|^2_{H^2} \geq 0.$$

Moreover, by combining together (7.1), with (6.9) and (6.16), we have that for every $\mu \in (0,1)$

$$\mathbb{E}\int_0^t |u_\mu(s)|^4_{H^2}\,ds < \infty.$$

Due to inequality (4.10) from Lemma 4.3, this implies

$$\mathbb{E}\left|\int_0^t \langle \sigma(u_\mu(s))dw^Q(s), u_\mu(s)\rangle_{H^2}\right|^2 \leq c\,\mathbb{E}\int_0^t |u_\mu(s)|^4_{H^2}\,ds + c_T < \infty,$$

so that we can take the expectation of both sides in (7.22) and we get

$$\frac{d}{dt}\mathbb{E}|u_\mu(t)|^2_{H^2} + \frac{1}{\mu}\,\mathbb{E}|u_\mu(t)|^2_{H^2} \leq c\int_0^t \mathbb{E}|v_\mu(s)|^2_{H^2}\,ds + \frac{c_T}{\mu}.$$

By a comparison argument this gives (7.21). □

Remark 7.4. By combining together (7.16) and (7.21), we have that for every $(u_0, v_0) \in \mathscr{H}_2 \cap \mathscr{M}$ and $T > 0$ there exists a constant $c_T(u_0, v_0) > 0$ such that for every $\mu \in (0,1)$

$$\mathbb{E}|u_\mu(t)|^2_{H^2} \leq c_T(u_0, v_0) + c\,\mathbb{E}\int_0^t |u_\mu(s)|^2_{H^2}\,ds, \quad t \in [0,T].$$

Hence, the Gronwall Lemma allows to conclude

$$\mathbb{E}|u_\mu(t)|^2_{H^2} \leq c_T(u_0, v_0), \quad t \in [0,T]. \tag{7.23}$$

In particular, thanks again to (7.16),

$$\int_0^T \mathbb{E}|\partial_t u_\mu(s)|^2_{H^2}\,ds \leq \frac{1}{\mu}\,c_T(u_0, v_0). \tag{7.24}$$

□

Remark 7.5. Let us present an alternative proof of inequalities (7.7)

$$\frac{1}{2}\,|u|^2_{H^2} \le \Phi(u) := |u|^2_{H^2} - \frac{1}{2}\,|u|^4_{H^1} \le |u|^2_{H^2}.$$

The inequality on the right is obvious. The inequality on the left is a consequence of the following argument. Assume that $u \in M \cap H^2$. Then

$$\begin{aligned}\Phi(u) &= |\Delta u|^2_H - \frac{1}{2}|\nabla u|^4_H = \frac{1}{2}|\Delta u|^2_H + \frac{1}{2}\left(|\Delta u|^2_H - |\nabla u|^4_H\right)\\ &= \frac{1}{2}\,|\Delta u + |\nabla u|^2_H u|^2_H + \frac{1}{2}|\Delta u|^2_H \ge \frac{1}{2}|\Delta u|^2_H.\end{aligned}$$

Notice that here the crucial identity is

$$|\Delta u|^2_H - |\nabla u|^4_H = |\Delta u + |\nabla u|^2_H u|^2_H, \quad u \in M \cap H^2.$$

8. Tightness

We first need to introduce some notations and preliminary results. If E is a Banach space and $T > 0$, for every $0 < \sigma < 1$ and $1 \le p \le \infty$ we define

$$W^{\sigma,p}(0,T;E) := \left\{ f \in L^p(0,T;E) \;:\; [f]_{\dot W^{\sigma,p}(0,T;E)} < \infty \right\},$$

where

$$[f]_{\dot W^{\sigma,p}(0,T;E)} := \left(\int_0^T \int_0^T \frac{|f(t)-f(s)|^p_E}{|t-s|^{1+\sigma p}}\,dt\,ds \right)^{\frac{1}{p}}.$$

The space $W^{\sigma,p}(0,T;E)$, endowed with the norm

$$|\cdot|_{W^{\sigma,p}(0,T;E)} := |\cdot|_{L^p(0,T;E)} + [\cdot]_{\dot W^{\sigma,p}(0,T;E)},$$

is a Banach space. Moreover, for every $f \in L^p(0,T;E)$ and $h \in [0,T)$ we denote

$$\tau_h f(t) = f(t+h), \quad t \in [-h, T-h].$$

In [52, Lemma 5] it is proven that if $f \in W^{\sigma,r}(0,T;E)$, with $0 < \sigma < 1$ and $1 \le r \le \infty$, and if p is such that

$$p \le \infty, \text{ if } \sigma > \frac{1}{r}, \quad p < \infty, \text{ if } \sigma = \frac{1}{r}, \quad p \le r_\star := \frac{r}{1-\sigma r}, \text{ if } \sigma < \frac{1}{r},$$

then $f \in L^p(0,T;E)$ and there exists a constant c independent of f such that for every $h \ge 0$

$$|\tau_h f - f|_{L^p(0,T-h;E)} \le c \begin{cases} h^{\sigma+\frac{1}{p}-\frac{1}{r}}[f]_{\dot W^{\sigma,r}(0,T;E)}, & \text{if } r \le p \le \infty,\\ h^{\sigma} T^{\frac{1}{p}-\frac{1}{r}}[f]_{\dot W^{\sigma,r}(0,T;E)}, & \text{if } 1 \le r \le p.\end{cases} \tag{8.1}$$

According to (4.12), we have

$$\gamma\, du_\mu(t) + \mu\, dv_\mu(t) = \left(\Delta u_\mu(t) + |u_\mu(t)|^2_{H^1} u_\mu(t) - \mu |v_\mu(t)|^2_H u_\mu(t)\right) dt$$

$$+\,\sigma(u_\mu(t))\,dw^Q(t),$$

with $v_\mu(t) = \partial_t u_\mu(t)$. Then, if we integrate with respect to time, we get

$$\begin{aligned}\Phi_\mu(t) := \gamma u_\mu(t) + \mu\, v_\mu(t) = \gamma u_0 + \mu\, v_0 + \int_0^t \Delta u_\mu(s)\,ds + \int_0^t |u_\mu(s)|_{H^1}^2 u_\mu(s)\,ds\\ -\mu \int_0^t |v_\mu(s)|_H^2 u_\mu(s)\,ds + \int_0^t \sigma(u_\mu(s))\,dw^Q(s) =: I_\mu + \sum_{j=1}^4 J_{\mu,k}(t).\end{aligned} \tag{8.2}$$

Lemma 8.1. *Under Hypotheses 3, 4 and 5, for every $(u_0, v_0) \in \mathscr{H}_2 \cap \mathscr{M}$, and for every $T > 0$ and $\theta < 1/2$ we have*

$$\sup_{\mu\in(0,1)} \mathbb{E}\left[\Phi_\mu\right]_{\dot{W}^{\theta,2}(0,T;H)} < \infty. \tag{8.3}$$

Proof. We are going to estimate every term $J_{\mu,k}(t)$ in (8.2). Thanks to (7.23), we have

$$\mathbb{E}\,|J_{\mu,1}(t) - J_{\mu,1}(s)|_H^2 \leq \int_s^t \mathbb{E}\,|u_\mu(r)|_{H^2}^2\,dr\,|t-s| \leq c_T\,|t-s|,$$

and, due to (4.1),

$$\begin{aligned}\mathbb{E}\,|J_{\mu,2}(t) - J_{\mu,2}(s)|_H^2 &\leq \int_s^t \mathbb{E}\,|u_\mu(r)|_{H^1}^4\,dr|t-s| \leq \int_s^t \mathbb{E}\,|u_\mu(r)|_{H^2}^2\,dr\,|t-s|\\ &\leq c_T\,|t-s|.\end{aligned}$$

Next, due to (6.8) and (6.16), we have

$$\mathbb{E}\,|J_{\mu,3}(t) - J_{\mu,3}(s)|_H^2 \leq \mu^2 \int_s^t \mathbb{E}\,|v_\mu(r)|_H^4\,dr\,|t-s| \leq c_T\,|t-s|.$$

Finally, thanks to (4.6) we have

$$\mathbb{E}|J_{\mu,4}(t) - J_{\mu,4}(s)|_H^2 \leq \mathbb{E}\int_s^t \|\sigma(u_\mu(r))\|_{\mathscr{T}_2(K,H)}^2\,dr \leq c_T\,|t-s|.$$

Therefore, by combining together all these bounds, we conclude that (8.3) holds, for every $\theta < 1/2$. □

Lemma 8.2. *Under Hypotheses 3, 4 and 5, for every $(u_0, v_0) \in \mathscr{H}_2 \cap \mathscr{M}$ and $T > 0$ the family $\{\mathscr{L}(u_\mu)\}_{\mu\in(0,1)}$ is tight in $L^q(0, T; H^\alpha)$, for every $\alpha \in [0, 2)$ and $q < 4/\alpha$.*

Proof. Due to (8.1), for every $p < \infty$ there exists $\theta < 1/2$ such that the set

$$K_L := \left\{ f \in L^p(0,T;H) \,:\, [f]_{\dot{W}^{\theta,2}(0,T;H)} \leq L,\ \int_0^T |f(s)|_{H^2}^2\,ds \leq L \right\}$$

is relatively compact in $L^p(0, T; H)$, for every $L > 0$ (for a proof see e.g. [52, Theorem 3]).

Now, due to (7.23) and (7.24), we have that $u_\mu + \mu \partial_t u_\mu$ is bounded in $L^2(\Omega; L^2(0, T; H^2))$. Then, according to (8.3), for every $\epsilon > 0$ there exists $L_\epsilon > 0$ such that

$$\mathbb{P}\left(u_\mu + \mu\,\partial_t u_\mu \in K_{L_\epsilon}\right) \geq 1 - \epsilon, \qquad \mu \in (0, 1).$$

This means that the family $\{u_\mu + \mu \partial_t u_\mu\}_{\mu \in (0,1)}$ is tight in $L^p(0, T; H)$, for every $p < \infty$. Moreover, as a consequence of (6.18), the family $\{\mu \partial_t u_\mu\}_{\mu \in (0,1)}$ is tight in $L^\infty(0, T; H)$, and this allows to conclude that the family $\{u_\mu\}_{\mu \in (0,1)}$ is tight in $L^p(0, T; H)$, for every $p < \infty$.

In particular, for every $p < \infty$ and $\epsilon > 0$ there exists a relatively compact set $K_{1,\epsilon,p} \subset L^p(0, T; H)$ such that

$$\mathbb{P}\left(u_\mu \in K_{1,\epsilon,p}\right) \geq 1 - \frac{\epsilon}{2}, \quad \mu \in (0, 1).$$

Moreover, according to (7.23), for every $\epsilon > 0$ there exists $M_\epsilon > 0$ such that

$$\mathbb{P}\left(u_\mu \in K_{2,\epsilon}\right) \geq 1 - \frac{\epsilon}{2}, \quad \mu \in (0, 1),$$

where

$$K_{2,\epsilon} := \left\{ f \in L^2(0, T; H^2) \; : \; |f|_{L^2(0,T;H^2)} \leq M_\epsilon \right\}.$$

Now, for every $\alpha \in [0, 2)$ we have

$$|h|_{H^\alpha} \leq |u|_H^{1-\alpha/2}\,|u|_{H^2}^{\alpha/2}.$$

Therefore, if we set

$$q := \frac{4p}{\alpha p + 4 - 2\alpha}, \tag{8.4}$$

we get

$$|\tau_h f - f|_{L^q(0,T-h;H^\alpha)} \leq |\tau_h f - f|_{L^p(0,T-h;H)}^{1-\alpha/2} |\tau_h f - f|_{L^2(0,T-h;H^2)}^{\alpha/2}. \tag{8.5}$$

In view of the characterization of compact sets in $L^p(0, T; H)$ given in [52, Theorem 1], we have that

$$\lim_{h \to 0} \sup_{f \in K_{1,\epsilon,p}} |\tau_h f - f|_{L^p(0,T-h;H)} = 0.$$

Therefore, thanks to (8.5), we get

$$\lim_{h \to 0} \sup_{f \in K_{1,\epsilon,p}} |\tau_h f - f|_{L^q(0,T-h;H^\alpha)} = 0.$$

By applying again [52, Theorem 3], we conclude that the set $K_{1,\epsilon,p} \cap K_{2,\epsilon}$ is relatively compact in $L^q(0, T; H^\alpha)$ and this allows to conclude that the family $\{u_\mu\}_{\mu \in (0,1)}$ is tight in $L^q(0, T; H^\alpha)$, just by noticing that

$$\mathbb{P}\left(u_\mu \in K_{1,\epsilon} \cap K_{2,\epsilon,p}\right) \geq 1-\epsilon, \qquad \mu \in (0,1).$$

Finally, since we can take any arbitrary $p < \infty$, due to (8.4) we have that we can take any $q < 4/\alpha$ and our proof is concluded. □

9. Proof of Theorem 4.5

We start with the following fundamental identity.

Lemma 9.1. *Assume Hypotheses 3 and 4. Then, for every* $(u_0, v_0) \in \mathscr{H}_1 \cap \mathscr{M}$ *and every* $\mu > 0$ *and* $t \in [0,T]$*, we have*

$$\mu \int_0^t |\partial_t u_\mu(s)|_H^2 u_\mu(s)\,ds = \frac{1}{2\gamma}\int_0^t \|\sigma(u_\mu(s))\|^2_{\mathscr{T}_2(K,H)} u_\mu(s)\,ds + R_\mu(t),$$

where

$$\begin{aligned}
R_\mu(t) := {} & \frac{\mu^2}{2\gamma}|v_0|_H^2 u_0 - \frac{\mu^2}{2\gamma}|\partial_t u_\mu(t)|_H^2 u_\mu(t) + \frac{\mu}{\gamma}\int_0^t u_\mu(s)\langle \sigma_0(u_\mu(s))dw^Q(s), \partial_t u_\mu(s)\rangle_H \\
& + \frac{\mu^2}{2\gamma}\int_0^t |\partial_t u_\mu(s)|_H^2 \partial_t u_\mu(s)\,ds + \frac{\mu}{\gamma}\int_0^t u_\mu(s)\langle \partial_t u_\mu(s), \Delta u_\mu(s)\rangle_H\,ds \\
=: {} & \frac{\mu^2}{2\gamma}|v_0|_H^2 u_0 + \sum_{j=1}^4 I_{\mu,k}(t).
\end{aligned}$$

Proof. As a consequence of the Itô Lemma A.2, we have

$$\begin{aligned}
\frac{\mu^2}{2} d\,|v_\mu(t)|_H^2 = {} & \mu\langle v_\mu(t), \Delta u_\mu(t)\rangle_H\,dt - \mu\gamma|v_\mu(t)|_H^2\,dt + \frac{1}{2}\|\sigma(u_\mu(t))\|^2_{\mathscr{T}_2(K,H)}\,dt \\
& + \mu\langle \sigma(u_\mu(t))dw^Q(t), v_\mu(t)\rangle_H.
\end{aligned}$$

This implies that

$$\begin{aligned}
\frac{\mu^2}{2} d\left(|v_\mu(t)|_H^2 u_\mu(t)\right) = {} & \frac{\mu^2}{2} d|v_\mu(t)|_H^2 u_\mu(t) + \frac{\mu^2}{2}|v_\mu(t)|_H^2 v_\mu(t)\,dt \\
= {} & \mu\langle v_\mu(t), \Delta u_\mu(t)\rangle_H u_\mu(t)\,dt - \mu\gamma|v_\mu(t)|_H^2 u_\mu(t)\,dt \\
& + \frac{1}{2}\|\sigma(u_\mu(t))\|^2_{\mathscr{T}_2(K,H)} u_\mu(t)\,dt \\
& + \frac{\mu^2}{2}|v_\mu(t)|_H^2 v_\mu(t)\,dt + \mu\langle \sigma_0(u_\mu(t))dw^Q(t), v_\mu(t)\rangle_H u_\mu(t).
\end{aligned}$$

Rearranging all terms, we get

$$\begin{aligned}
\mu\gamma|v_\mu(t)|_H^2 u_\mu(t)\,dt = {} & \frac{1}{2}\|\sigma(u_\mu(t))\|^2_{\mathscr{T}_2(K,H)} u_\mu(t)\,dt - \frac{\mu^2}{2} d\left(|v_\mu(t)|_H^2 u_\mu(t)\right) \\
& + \mu\langle v_\mu(t), \Delta u_\mu(t)\rangle_H u_\mu(t)\,dt \\
& + \frac{\mu^2}{2}|v_\mu(t)|_H^2 v_\mu(t)\,dt + \mu\langle \sigma_0(u_\mu(t))dw^Q(t), v_\mu(t)\rangle_H u_\mu(t),
\end{aligned}$$

and the lemma follows once we divide both sides above by γ and integrate with respect to time. □

Lemma 9.2. *Under Hypotheses 3, 4 and 5, for every* $(u_0, v_0) \in \mathscr{H}_2 \cap \mathscr{M}$ *and* $T > 0$ *we have*

$$\lim_{\mu\to 0} \mathbb{E} \sup_{t\in[0,T]} |R_\mu(t)|_H = 0. \tag{9.1}$$

Proof. We use here the same notations as in Lemma 9.1 and we write

$$R_\mu(t) = \frac{\mu^2}{2\gamma}|v_0|_H^2 u_0 + \sum_{j=1}^{4} I_{\mu,k}(t), \quad t \in [0,T].$$

For $I_{\mu,1}(t)$ we have

$$|I_{\mu,1}(t)|_H = \frac{\mu^2}{2\gamma}\, |\partial_t u_\mu(t)|_H^2,$$

and, thanks to (6.18), we get

$$\lim_{\mu\to 0} \mathbb{E} \sup_{t\in[0,T]} |I_{\mu,1}(t)|_H = 0. \tag{9.2}$$

For $I_{\mu,2}(t)$, due to (4.6) we have

$$\begin{aligned}\mathbb{E} \sup_{t\in[0,T]} |I_{\mu,2}(t)|_H &\le c\,\mu \left(\int_0^T \mathbb{E}\|\sigma_0(u_\mu(s))\|^2_{\mathscr{T}_2(K,H)} |\partial_t u_\mu(s)|_H^2\, ds\right)^{\frac{1}{2}}\\ &= c\sqrt{\mu}\left(\mu \int_0^T \mathbb{E}\,|\partial_t u_\mu(s)|_H^2\, ds\right)^{\frac{1}{2}},\end{aligned}$$

and (6.17) allows to conclude that

$$\lim_{\mu\to 0} \mathbb{E} \sup_{t\in[0,T]} |I_{\mu,2}(t)|_H = 0. \tag{9.3}$$

For $I_{\mu,3}(t)$, we have

$$\mathbb{E} \sup_{t\in[0,T]} |I_{\mu,3}(t)|_H \le c\,\mu^2 \int_0^T \mathbb{E}\,|\partial_t u_\mu(s)|_H^3\, dt \le c_T\sqrt{\mu}\left(\mu^2 \int_0^T \mathbb{E}|\partial_t u_\mu(s)|_H^4\, ds\right)^{\frac{3}{4}},$$

so that, in view of (7.1) and (7.23), we have

$$\lim_{\mu\to 0} \mathbb{E} \sup_{t\in[0,T]} |I_{\mu,3}(t)|_H = 0. \tag{9.4}$$

Finally, since $|u_\mu|_H = 1$, for $I_{\mu,4}(t)$, we have

$$\begin{aligned}\mathbb{E} \sup_{t\in[0,T]} |I_{\mu,4}(t)|_H &\le c\,\mu \int_0^T \mathbb{E}|\partial_t u_\mu(s)|_{H^1} |u_\mu(s)|_{H^1}\, ds\\ &\le c\left(\mu \int_0^T \mathbb{E}|u_\mu(s)|_{H^1}^4\, ds\right)^{\frac{1}{4}} \left(\mu \int_0^T \mathbb{E}|\partial_t u_\mu(s)|_{H^1}^{\frac{4}{3}}\, ds\right)^{\frac{3}{4}}\\ &\le c_T\left(\mu \int_0^T \mathbb{E}|u_\mu(s)|_{H^1}^4\, ds\right)^{\frac{1}{4}} \left(\mu \int_0^T \mathbb{E}|\partial_t u_\mu(s)|_{H^1}^2\, ds\right)^{\frac{1}{2}} \mu^{\frac{1}{4}},\end{aligned}$$

and then, according to (6.9) and (6.16) and to (7.1) and (7.23), we have

$$\lim_{\mu\to 0}\ \mathbb{E}\ \sup_{t\in[0,T]} |I_{\mu,4}(t)|_H = 0. \tag{9.5}$$

As a result of (9.2), (9.3), (9.4) and (9.5) we obtain (9.1). □

9.1. Proof of Theorem 4.5. In Lemma 8.2 we have proven that the family $\{\mathscr{L}(u_\mu)\}_{\mu\in(0,1)}$ is tight in $L^q(0,T;H^\alpha)$, for every $\alpha\in[0,2)$ and $q<4/\alpha$. Here we take $\alpha\in[1,2)$. Thanks to (6.18), this implies that $\{\mathscr{L}(u_\mu,\mu\,\partial_t u_\mu)\}_{\mu\in(0,1)}$ is tight in $L^q(0,T;H^\alpha)\times L^\infty(0,T;H)$, so that, due to the Prohorov Theorem, there exists a weak limit point in the same space. Now, let us define

$$\mathscr{K}_T := \left[L^q(0,T;H^\alpha)\times L^\infty(0,T;H)\right]^2\times C([0,T];U),$$

where U is a Hilbert space containing the reproducing kernel K with Hilbert-Schmidt embedding. Thanks to the Skorokhod theorem for any two sequences $\{\mu_k^1\}_{j\in\mathbb{N}}$ and $\{\mu_k^2\}_{j\in\mathbb{N}}$, both converging to zero, there exist two subsequences, still denoted by $\{\mu_k^1\}_{j\in\mathbb{N}}$ and $\{\mu_k^2\}_{j\in\mathbb{N}}$, a sequence of random variables

$$\mathscr{Y}_k := ((\varrho_k^1,\vartheta_k^1),(\varrho_k^2,\vartheta_k^2),\hat{w}_k^Q),\quad k\in\mathbb{N},$$

in $\mathscr{K}_T$ and a random variable $\mathscr{Y}=(\varrho^1,\varrho^2,\hat{w}^Q))$ in $\mathscr{K}_T$, all defined on some probability space $(\hat{\Omega},\hat{\mathscr{F}},\hat{\mathbb{P}})$, such that

$$\mathscr{L}(\mathscr{Y}_k)=\mathscr{L}((u_{\mu_k^1},\ \mu_k^1\,\partial_t u_{\mu_k^1}),\ (u_{\mu_k^2},\ \mu_k^2\,\partial_t u_{\mu_k^2}),\ w^Q),\quad k\in\mathbb{N}, \tag{9.6}$$

and, for $i=1,2$,

$$\lim_{k\to\infty}|\varrho_k^i-\varrho^i|_{L^q(0,T;H^\alpha)}+|\vartheta_k^i|_{L^\infty(0,T;H)}+|\hat{w}_k^Q-\hat{w}^Q|_{C([0,T];U)}=0,\quad \mathbb{P}-\text{a.s.} \tag{9.7}$$

Notice that this implies that $\varrho^i(t)\in M$, $\hat{\mathbb{P}}$-a.s. and, due to (7.23), $\varrho^i\in L^2(\Omega,L^2(0,T;H^2))$, for $i=1,2$.

Next, a filtration $(\hat{\mathscr{F}}_t)_{t\geq 0}$ is introduced in $(\hat{\Omega},\hat{\mathscr{F}},\hat{\mathbb{P}})$, by taking the augmentation of the canonical filtration of $(\rho^1,\rho^2,\hat{w}^Q)$, generated by the restrictions of $(\rho^1,\rho^2,\hat{w}^Q)$ to every interval $[0,t]$. Due to this construction, $\hat{w}^Q$ is a $(\hat{\mathscr{F}}_t)_{t\geq 0}$ Wiener process with covariance Q^*Q (for a proof see [31, Lemma 4.8]).

Now, if we show that $\varrho^1=\varrho^2$, we have that u_μ converges in probability in $L^q(0,T;H^\alpha)$ to some $u\in L^2(\Omega;L^2(0,T;H^2))$. Actually, as observed by Gyöngy and Krylov in [34], if E is any Polish space equipped with the Borel σ-algebra, a sequence $(\xi_n)_{n\in\mathbb{N}}$ of E-valued random variables converges in probability if and only if for every pair of subsequences $(\xi_m)_{m\in\mathbb{N}}$ and $(\xi_l)_{l\in\mathbb{N}}$ there exists an E^2-valued subsequence $\eta_k:=(\xi_{m(k)},\xi_{l(k)})$ converging weakly to a random variable η supported on the diagonal $\{(h,k)\in E^2\ h=k\}$.

In order to show that $\varrho^1=\varrho^2$, we prove that they are both a solution of Eq. (4.14), which has pathwise uniqueness. Due to (9.6), we have that both $(\varrho_k^1,\vartheta_k^1)$ and (ρ_k^2,ϑ_k^2) satisfy Eq. (8.2), with w^Q replaced by $\hat{w}_k^Q$. Then, if we first take the scalar product in

H of each term in (8.2) with an arbitrary but fixed $\psi \in C_0^\infty(D)$ and then integrate by parts, we get

$$\begin{aligned}\langle \gamma \varrho_k^i(t) + \vartheta_k^i(t), \psi\rangle_H &= \langle \gamma\, u_0 + \mu_k v_0, \psi\rangle_H - \int_0^t \langle \nabla \varrho_k^i(s)), \nabla\psi\rangle_H\, ds\\ &+ \int_0^t |\varrho_k^i(s)|^2_{H^1} \langle \varrho_k(s), \psi\rangle_H\, ds - \mu_k \int_0^t |\vartheta_k^i(s)|^2_H \langle \varrho_k^i(s), \psi\rangle_H\, ds\\ &+ \int_0^t \langle \sigma(\rho_k^i(s)) d\hat{w}_k^Q(s), \psi\rangle_H =: \sum_{j=1}^4 I_{k,j}^i + \int_0^t \langle \sigma(\rho_k^i(s)) d\hat{w}_k^Q(s), \psi\rangle_H(t).\end{aligned}$$

Clearly

$$\lim_{k\to\infty} I_{k,1}^i = \langle \gamma u_0, \psi\rangle_H.$$

and, due to (9.7), since $\alpha \geq 1$, we have

$$\lim_{k\to\infty} I_{k,2}^i(t) = \int_0^t \langle \nabla \varrho^i(s), \nabla\psi\rangle_H\, ds, \quad \hat{\mathbb{P}} - \text{a.s.}$$

Moreover

$$\begin{aligned}&\left| I_{k,3}^i(t) - \int_0^t |\varrho^i(s)|^2_{H^1} \langle \varrho^i(s), \psi\rangle_H\, ds \right|\\ &\leq \int_0^t \left| |\varrho_k^i(s)|^2_{H^1} - |\varrho^i(s)|^2_{H^1} \right| \left| \langle \varrho_k^i(s), \psi\rangle_H \right| ds\\ &\quad + \int_0^t |\varrho^i(s)|^2_{H^1} \left| \langle \varrho_k^i(s) - \varrho^i(s), \psi\rangle_H \right| ds\\ &\leq |\psi|_H \int_0^t |\varrho_k^i(s) - \varrho^i(s)|_{H^1} \left(|\varrho_k^i(s)|_{H^1} + |\varrho^i(s)|_{H^1} \right) ds\\ &\quad + |\psi|_H \int_0^t |\varrho^i(s)|^2_{H^1}\, |\varrho_k^i(s) - \varrho^i(s)|_H\, ds\\ &\leq |\psi|_H\, |\varrho_k^i - \varrho^i|_{L^2(0,T;H^1)} \left(|\varrho_k^i|_{L^2(0,T;H^1)} + |\varrho^i|_{L^2(0,T;H^1)} \right)\\ &\quad + |\psi|_H |\varrho^i|_{L^2(0,T;H^2)} |\varrho_k^i - \varrho^i|_{L^2(0,T;H)}.\end{aligned}$$

Thanks again to (9.7) this allows to conclude that

$$\lim_{k\to\infty} I_{k,3}^i(t) = \int_0^t |\varrho^i(s)|^2_{H^1} \langle \varrho^i(s), \psi\rangle_H\, ds, \quad \hat{\mathbb{P}} - \text{a.s.}$$

Next, as a consequence of (9.6), due to Lemmas 9.1 and 9.2 we have

$$\lim_{k\to\infty} I_{k,4}^i = -\frac{1}{2} \int_0^\cdot \|\sigma(\varrho^i(s))\|^2_{\mathscr{T}_2(K,H)} \varrho^i(s)\, ds, \quad \text{in } L^2(\Omega; L^\infty(0,T;H)).$$

Now, for $i = 1, 2$ and $t \in [0, T]$, we define

$$
\begin{aligned}
M^i(t) := \langle \varrho^i(t), \psi\rangle_H - \langle \gamma\, u_0, \psi\rangle_H + \int_0^t \langle \nabla\varrho^i(s)), \nabla\psi\rangle_H \, ds \\
- \int_0^t |\varrho^i(s)|^2_{H^1} \langle \varrho(s), \psi\rangle_H \, ds \\
+ \frac{1}{2}\int_0^t \|\sigma(\varrho^i(s))\|^2_{\mathscr{T}_2(K,H)} \varrho^i(s)\, ds.
\end{aligned}
$$

By proceeding as in the proof of [31, Lemma 4.9], thanks to (9.7) and the limits above for $I^i_{k,j}$, $j = 1, 2, 3, 4$, we have that for every $t \in [0, T]$

$$
\left\langle M^i - \int_0^\cdot \langle \sigma(\varrho^i(s)) d\hat{w}^Q(s), \psi\rangle_H \right\rangle_t = 0, \qquad \hat{\mathbb{P}} - \text{a.s},
$$

where $\langle\cdot\rangle_t$ is the quadratic variation process. In particular, this implies that for $i = 1, 2$ the martingale M^i coincides with the stochastic integral

$$
\int_0^\cdot \langle \sigma(\varrho^i(s)) d\hat{w}^Q(s), \psi\rangle_H,
$$

and if we replace such stochastic integral in (9.1) we conclude that ρ^i is a solution of equation (4.14), with w^Q replaced by $\hat{w}^Q$. Hence, since both ρ^1 and ρ^2 satisfy the same equation (4.14), as we have explained above the pathwise uniqueness of Eq. (4.14) allows to conclude that (4.13) holds.

Data Availibility There are no data associated with this paper.

Appendix A. Ito Lemma

In this section we formulate and prove a version of the Itô Lemma that we use twice throughout the paper. Our approach follows the papers [11,56].

To make this section self-contained let us remind the framework. Assume that H is a real separable Hilbert space and that A_0 is a non-negative self-adjoint linear operator on H. We denote

$$
\mathscr{H} := D(A_0) \times H,
$$

and endow it with a norm (and the corresponding scalar product)

$$
|z|^2_{\mathscr{H}} := |A_0 x|^2_H + |x|^2_H + |y|^2_H, \quad z = (x, y) \in \mathscr{H}.
$$

Note that $D(A_0)$ is a Hilbert, and hence a Banach, space when endowed with the graph norm. Moreover, we define a linear operator $\mathscr{A}$ on $\mathscr{H}$ by (2.3), i.e.

$$D(\mathscr{A}) := D(A_0^2) \times D(A_0), \qquad \mathscr{A}z := (v, -A_0^2 u), \quad z = (u, v) \in D(\mathscr{A}).$$

It is well known that $\mathscr{A}$ generates a C_0 group (of exponential growth) $\mathscr{S} = (\mathscr{S}(t))_{t\in\mathbb{R}}$ on $\mathscr{H}$, see e.g. [10] and references therein. Let us point out that it does not matter which equivalent norm on $\mathscr{H}$ we choose.

Next, we introduce the function Φ about which we will formulate our Itô Lemma

$$\Phi(z) = \frac{1}{2}\left(|A_0 x|_H^2 + |y|_H^2 + \delta |x|_H^2\right) + \beta\langle x, y\rangle_H + F(x), \quad z = (x, y) \in \mathscr{H},$$

for some $F : D(A_0) \to \mathbb{R}$. The function Φ satisfies the following properties.

Lemma A.1. *Assume that the function $F : D(A_0) \to \mathbb{R}$ is of C^2-class in the Frechét sense. Then, for every $\delta \geq 0$ and $\beta \in \mathbb{R}$ the function $\Phi : \mathscr{H} \to \mathbb{R}$ is well defined and of C^2-class in the Frechét sense and its second derivative is bounded on balls. Moreover, for every $z = (x, y)$, $h = (h_1, h_2)$ and $k = (k_1, k_2) \in \mathscr{H}$, it holds*

$$D\Phi(z)\,h = \langle A_0 x, A_0 h_1\rangle_H + \langle y, h_2\rangle_H + \delta\langle x, h_1\rangle_H + \beta\left(\langle x, h_2\rangle_H + \langle y, h_1\rangle_H\right) + DF(x)\,h_1$$

and

$$\begin{aligned} D^2\Phi(z)(h, k) = {} & \langle A_0 h_1, A_0 k_1\rangle_H + \langle h_2, k_2\rangle_H + \delta\langle h_1, k_1\rangle_H \\ & + \beta\left(\langle h_2, k_1\rangle_H + \langle h_1, k_2\rangle_H\right) + D^2F(x)(h_1, k_1). \end{aligned}$$

Assume now that τ is an accessible stopping time with approximating sequence $(\tau_n)_{n=1}^{\infty}$. Moreover, assume that $f = (f(t) : t \in [0, \tau))$ is an H-valued process and $g = (g(t) : t \in [0, \tau))$ is a $\mathscr{T}_2(K, H)$-valued process, both progressively measurable and such that for every $k \in \mathbb{N}$ and every $t \geq 0$,

$$\mathbb{E}\int_0^{t\wedge\tau_k} \left(\|g(s)\|_{\mathscr{T}_2(K,H)}^2 + |f(s)|_H\right) ds < \infty.$$

Next we introduce the progressively measurable processes $\mathbb{f} = (\mathbb{f}(t) : t \in [0, \tau))$ and $\mathbb{g} = (\mathbb{g}(t) : t \in [0, \tau))$ which take values in $\mathscr{H}$ and $\mathscr{T}_2(K, \mathscr{H})$, respectively, defined by

$$\mathbb{f}(t) = (0, f(t)), \qquad \mathbb{g}(t) = (0, g(t)), \quad t \in [0, \tau).$$

Lemma A.2. *Assume that an $\mathscr{H}$-valued continuous local process $z(t) = (x(t), y(t))$, $t \in [0, \tau)$, is a mild solution to (3.5), i.e. for every $k \in \mathbb{N}$,*

$$z(t \wedge \tau_k) = z_0 + I_{\tau_k}(t \wedge \tau_k) + \int_0^t \mathbb{1}_{[0,\tau_k)}(r)\mathscr{S}(t - r)\,\mathbb{f}(r)dr, \quad t \geq 0, \tag{A.1}$$

where $z_0 = (x_0, y_0) \in \mathscr{H}$ and the process $I_{\tau_k} = \left(I_{\tau_k}(t) : t \geq 0\right)$ is defined by

$$I_{\tau_k}(t) := \int_0^t \mathbb{1}_{[0,\tau_k)}(r)\mathscr{S}(t - r)\mathbb{g}(r)dW(r).$$

Then

$$\begin{aligned}\Phi(z(t)) = \Phi(z_0) + \int_0^t \Big(\delta\langle x(s), y(s)\rangle_H + \beta\left(|y(s)|_H^2 - |A_0x(s)|_H^2\right) \\ + \langle \beta x(s) + y(s), f(s)\rangle_H + DF(x(s))\,y(s) + \frac{1}{2}\|g(t)e_i\|^2_{\mathscr{T}_2(K,H)}\Big)\,ds \\ + \int_0^t \langle \beta x(s) + y(s), g(t)\,dW(s)\rangle_H, \quad t \in [0,\tau).\end{aligned} \tag{A.2}$$

It is important to emphasize that Itô's formula (A.2) should be understood in the following stopped way. For every $k \in \mathbb{N}$, for every $t \geq 0$,

$$\begin{aligned}\Phi(z(t\wedge\tau_k)) = \Phi(z_0) + \int_0^{t\wedge\tau_k} \Big(\delta\langle x(s), y(s)\rangle_H + \beta\left(|y(s)|_H^2 - |A_0x(s)|_H^2\right) \\ + \langle \beta x(s) + y(s), f(s)\rangle_H + DF(x(s))\,y(s)x + \frac{1}{2}\|g(t)e_i\|^2_{\mathscr{T}_2(K,H)}\Big)\,ds \\ + \int_0^{t\wedge\tau_k} \langle \beta x(s) + y(s), g(t)\,dW(s)\rangle_H.\end{aligned} \tag{A.3}$$

Proof of Lemma A.2. Assume that $n \in \mathbb{N}$ is big enough so that $n \in \rho(\mathscr{A})$, i.e. $(nI + A_0^2)^{-1}$ exists and is bounded. Denote

$$\mathscr{R}_n = \begin{pmatrix} (nI + A_0^2)^{-1} & 0 \\ 0 & (nI + A_0^2)^{-1} \end{pmatrix}.$$

Since $(nI + A_0^2)^{-1}$ maps boundedly H into $D(A_0)$ and $D(A_0^2)$, the operator $\mathscr{R}_n$ is a bounded linear map in $\mathscr{H}$ and it maps boundedly $\mathscr{H}$ into $D(\mathscr{A})$. Moreover, by direct calculations one can verify that $\mathscr{R}_n$ commutes with $\mathscr{A}$ and therefore, it also commutes with the group $(\mathscr{S}(t) : t \in \mathbb{R})$.

From now on we assume that the processes z, f, g etc are as in the fomulation of the Lemma. We define a set of new processes as follows, for every $t \in [0,\tau)$,

$$\begin{aligned}&z_n(t) := \mathscr{R}_n\, z(t), \quad x_n(t) := (nI + A_0^2)^{-1}\, x(t), \quad y_n(t) := (nI + A_0^2)^{-1}\, y(t), \\ &f_n(t) := (nI + A_0^2)^{-1}\, f(t), \quad g_n(t) := (nI + A_0^2)^{-1}\, g(t), \\ &\mathbb{f}_n(t) := \mathscr{R}_n\, \mathbb{f}_n(t), \quad \mathbb{g}_n(t) := \mathscr{R}_n\, \mathbb{g}_n(t).\end{aligned}$$

We note then that for $t \in [0,\tau)$,

$$\begin{aligned}&z_n(0) := \mathscr{R}_n\, z_0 = (x_n(0), y_n(0)), \quad z_n(t) = (x_n(t), y_n(t)), \\ &\mathbb{f}_n(t) := (0, f_n(t)), \quad \mathbb{g}_n(t) := (0, g_n(t)).\end{aligned} \tag{A.4}$$

We also observe that z_n is an $D(A_0^3) \times D(A_0^2)$-valued continuous local process, f_n is an $D(A_0^2)$-valued local process and g_n is an is $\mathscr{T}_2(K, D(A_0^2))$-valued process. Both f_n and g_n are progressively measurable and for every $k \in \mathbb{N}$ and every $t \geq 0$,

$$\mathbb{E}\int_0^{t\wedge\tau_k} \left[\|g_n(s)\|^2_{\mathscr{T}_2(K,D(A_0^2))} + |f_n(s)|_{D(A_0^2)}\right] ds < \infty.$$

Moreover, by the commutativity property stated earlier, the process z_n satisfies identities (A.1) in $D(A_0^3)\times D(A_0^2)$ with appropriate and obvious modifications, i.e. for every $k\in\mathbb{N}$,

$$z_n(t\wedge\tau_k)=z_n(0)+I_{n,\tau_k}(t\wedge\tau_k)+\int_0^t \mathbb{1}_{[0,\tau_k)}(r)\mathscr{S}(t-r)\mathbb{f}_n(r)dr,\quad t\geq 0,$$

where $z_0=(x_0,y_0)\in\mathscr{H}$ and the process $I_{\tau_k}=\big(I_{\tau_k}(t):t\geq 0\big)$ is defined by

$$I_{n,\tau_k}(t):=\int_0^t \mathbb{1}_{[0,\tau_k)}(r)\mathscr{S}(t-r)\,\mathbb{g}_n(r)dW(r),\quad t\geq 0.$$

Moreover, by the Chojnowska-Michalik theorem, see [28] or [46, Theorem 12], see also the proof of [11, Proposition 6.1], for all $t\geq 0$,

$$z_n(t\wedge\tau_k)=z_n(0)+\int_0^{t\wedge\tau_k}[\mathscr{A}z_n(s)+\mathbb{f}_n(r)]\,dr+\int_0^{t\wedge\tau_k}\mathscr{A}\mathbb{g}_n(r)dW(r),\quad t\geq 0.$$

Next, by the classical strong version of the Itô Lemma we infer that

$$\begin{aligned}\Phi(z_n(t))=\Phi(z_0)&+\int_0^t D\Phi(z_n(s))(\mathscr{A}z_n(s)+\mathbb{f}_n(r))\,dr\\&+\frac{1}{2}\int_0^t \mathrm{tr}_K[D^2\Phi(z_n(s))(\circ\mathbb{g}(s),\circ\mathbb{g}(s))]\,ds\\&+\int_0^t D\Phi(z_n(s))\circ\mathbb{g}_n(s)\,dW(s).\end{aligned}$$

Applying Lemma A.1 as well as one of the identities in (A.4), we infer that for $s\in[0,\tau)$

$$\begin{aligned}D\Phi(z_n(s))(\mathscr{A}z_n(s)+\mathbb{f}_n(s))&=\delta\langle x_n(s),y_n(s)\rangle_H+\beta\left(|y_n(s)|_H^2-|A_0x_n(s)|_H^2\right)\\&\quad+DF(x_n(s))\,y_n(s)\langle y,f_n(s)\rangle_H+\beta\langle x_n(s),f_n(s)\rangle_H\end{aligned}$$

and, with $\{e_i\}_{i\in I}$ being an arbitrary orthonormal basis in K,

$$\begin{aligned}&\mathrm{tr}_K D^2\Phi(z_n(s))(\circ\mathbb{g}_n(s),\circ\mathbb{g}_n(s))=\sum_{i\in I}\langle g_n(s)e_i,g_n(s)e_i\rangle_H\\&=\sum_{i\in I}|g_n(s)e_i|_H^2=\|g_n(s)e_i\|^2_{\mathscr{T}_2(K,H)}.\end{aligned}$$

Therefore, we deduce that $\Phi(z_n(t))$ satisfies the desired identity (A.3), i.e. for every $t\geq 0$,

$$\begin{aligned}\Phi(z_n(t\wedge\tau_k))=\Phi(z_n(0))&+\int_0^{t\wedge\tau_k}\Big(\delta\langle x_n(s),y_n(s)\rangle_H+\beta\,(|y_n(s)|_H2-|A_0x_n(s)|_H2)\\&+\langle\beta x_n(s)+y_n(s),f_n(s)\rangle_H+DF(x_n(s))\,y_n(s)+\frac{1}{2}\|g_n(s)\|^2_{\mathscr{T}_2(K,H)}\Big)\,ds\\&+\int_0^{t\wedge\tau_k}\langle\beta x_n(s)+y_n(s),g_n(s)\,dW(s)\rangle_H.\end{aligned}\tag{A.5}$$

Observe that $\mathbb{P}$- a.s. for every compact interval $[0, T] \subset [0, \tau)$, the following convergences are satisfied uniformly on $[0, T]$

$$z_n(t) \to z(t) \text{ in } \mathscr{H}, \quad x_n(t) \to x(t) \text{ in } D(A_0), \quad y_n(t) \to y(t) \text{ in } H,$$

and, for every $t \geq 0$,

$$\mathbb{E} \int_0^{t \wedge \tau_k} \left[\|g_n(s) - g(s)\|^2_{\mathscr{T}_2(K, D(A_0^2))} + |f_n(s) - f(s)|_{D(A_0^2)} \right] ds < \infty.$$

Thus, we conclude the proof of (A.3) by taking the limit as $n \to \infty$ of equalities (A.5). Compare with the proofs of [56, Lemma 5.2] and/or [11, Proposition 6.1]. □

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