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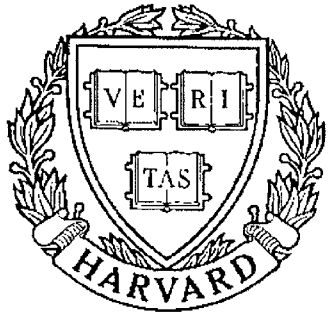

Supported by the National Science Foundation Engineering Research Center Program (NSFD CD 8803012), Industry and the University

Maximal Range for Generalized Stability-Application to Physically Motivated Examples

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TR 89-103r1

To appear in International Journal of Control

Maximal Range for Generalized Stability - Application to Physically Motivated Examples

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Abstract

Some recent results on guardian maps and their application to generalized robust stability are reviewed and a characterization of the maximum stability range is obtained. This framework is then applied to the analysis of robust stability in several physically motivated examples.

Key Words: Robust stability, Stability, Linear systems, Matrices, Polynomials

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1. Introduction

Recently, the authors developed a new approach for the study of generalized stability of families of real (or complex) matrices and polynomials using so-called guardian and semiguardian maps (Saydy *et al* (1990, 1988a, 1989) and Saydy (1988b)).

In this paper, after recalling some of the main results, we present a closed-form expression for the maximal range of stability of one-parameter families of matrices or polynomials relative to many domains of practical interest of the complex plane. The determination of the maximal range requires finding the zeros of a polynomial which depends on the family and domain of stability under consideration. This framework is then applied to the study of three physically motivated examples: (1) crane stabilization with damping and stability margin specifications, (2) satellite attitude control and (3) control of a digital tape transport.

2. Guardian Maps and Robust Stability

The guardian map approach was introduced in Saydy *et al* (1988a, 1988b, 1990) as a unifying tool for the study of generalized stability of parametrized families of matrices or polynomials. Some of the basic concepts used in this approach are reviewed below. We use the notation $\overline{\mathcal{D}}$ and $\partial\mathcal{D}$ to denote the closure and the boundary of a given set $\mathcal{D}$.

2.1. Guardian Maps

Basically, guardian maps are scalar valued maps on the set of real $n \times n$ matrices[1] that take nonzero values on the set of "stable" matrices and vanish on the boundary of that set. As the concept of guardian maps is normally applied to to generalized stability problems (wherein eigenvalues are confined within open subsets of the complex plane other than the open left-half plane), it is useful to allow the set of stable matrices to be any given open subset of $\mathbb{R}^{n\times n}$. Thus we have the following definition.

[1] While in this paper we restrict ourselves to families of matrices, all results extend readily to families of polynomials as well (Saydy *et al.*, 1990).

Definition 1. (Saydy *et al*, 1990) Let $\mathcal{S}$ be an open subset of $\mathbb{R}^{n\times n}$ and let ν map $\mathbb{R}^{n\times n}$ into $\mathbb{C}$. We say that ν *guards* $\mathcal{S}$ if for all $A \in \overline{\mathcal{S}}$, the equivalence

$$\nu(A) = 0 \quad \Longleftrightarrow \quad A \in \partial\mathcal{S} \tag{1}$$

holds. The map ν is said to be *polynomic* if it is a polynomial function of the entries of its argument.

Of special interest are sets of the form $\mathcal{S}(\Omega)$, where

$$\mathcal{S}(\Omega) = \{A \in \overset{n\times n}{\mathbb{R}} : \sigma(A) \subset \Omega\}, \tag{2}$$

Ω is an open subset of the complex plane and $\sigma(A)$ denotes the spectrum of A. Such sets $\mathcal{S}(\Omega)$ will be referred to as (*generalized*) *stability sets.*

For example, the map $\nu : A \mapsto \det(A \oplus A)$, where $\oplus$ denotes Kronecker sum, guards the set of $n \times n$ Hurwitz stable real matrices $\mathcal{S}(\overset{\circ}{\mathbb{C}}_-)$. This follows from the property that the spectrum of the Kronecker sum of two square matrices A and B consists of all pairwise sums of eigenvalues of A and B (Lancaster and Tismenetsky, 1985).

2.2. Robust stability

The robust stability problem for parametrized families of matrices or polynomials may be stated as follows. Let $r = (r_1, \ldots, r_k) \in U$, where U is a pathwise connected subset of $\mathbb{R}^k$, and let $A(r)$ be an element of $\mathbb{R}^{n\times n}$ which depends continuously on the parameter vector r. Given an open subset $\mathcal{S}$ of $\mathbb{R}^{n\times n}$, we seek basic conditions for $A(r)$ to lie within $\mathcal{S}$ for all values of r in U. The next theorem gives a basic necessary and sufficient condition for this problem both for guarded sets $\mathcal{S}$. Typically, $\mathcal{S}$ is a stability set of the form $\mathcal{S}(\Omega)$ where Ω is a given subset of the complex plane.

Theorem 1. (Saydy *et al*, 1990) Let $\mathcal{S}(\Omega)$ be guarded by the map ν. The family $\{A(r) : r \in U\}$ is stable relative to Ω if and only if

(i) it is nominally stable, i.e., $A(r^0) \in \mathcal{S}(\Omega)$ for some $r^0 \in U$, and,

(ii) $\nu(A(r)) \neq 0, \quad \forall r \in U.$

In the case of polynomic guardian maps, the theorem above yields computable conditions for robust stability of polynomially parametrized families of matrices. In Saydy *et al* (1990), polynomic guardian maps are constructed for many stability sets of practical interest.

2.3. Maximal interval of generalized stability

Let Ω be an open subset of the complex plane such that $\mathcal{S}(\Omega)$ is guarded by a polynomic map ν and consider the one-parameter family of matrices

$$A(r) = A_0 + rA_1 + \ldots + r^m A_m, \tag{3}$$

where $A_0, \ldots, A_m$ are given real square matrices. Let r^0 be a nominal value of interest and assume that $A(r^0)$ is stable relative to Ω.

Denote

$$\nu(r) := \nu\left(A(r)\right). \tag{4}$$

We seek to find the largest open interval of parameter values $(r_{\min}, r_{\max})$ containing r^0 for which

$$A(r) \in \mathcal{S}(\Omega) \qquad \forall\, r \in (r_{\min}, r_{\max}). \tag{5}$$

Note that since $\mathcal{S}(\Omega)$ is open and $A(r)$ is continuous in r, the fact that $A(r^0) \in \mathcal{S}(\Omega)$ guarantees the existence of an open interval containing r^0 in which the family above is stable.

The theorem below easily follows from Theorem 1.

Theorem 2. Let $\mathcal{S}(\Omega)$ be guarded by a polynomic map ν and let $A(r^0)$ be stable relative to Ω. Let[2]

$$\mathcal{Z}^- = \{r < r^0 :\ \nu(r) = 0\} \cup \{-\infty\} \tag{6}$$

$$\mathcal{Z}^+ = \{r > r^0 :\ \nu(r) = 0\} \cup \{+\infty\}. \tag{7}$$

[2] Note that $\nu(r^0) \neq 0$ since $A(r^0) \in \mathcal{S}(\Omega)$ and $\mathcal{S}(\Omega)$ is guarded by ν.

Then

$$r_{\min} = \max \{z : \ z \in \mathcal{Z}^-\} \tag{8}$$

$$r_{\max} = \min \{z : \ z \in \mathcal{Z}^+\} \tag{9}$$

□

Note that in the case of Hurwitz stability of the convex hull of two matrices, it was shown in Fu and Barmish (1988), based on a result of Bialas (1985), that the maximum range can be obtained from the eigenvalues of a suitably constructed matrix.

3. Examples of Application

We apply the results of the previous section to three examples. The last two from Franklin *et al* (1986) were studied in Bhattacharyya (1986) (see also Biernacki *et al.* (1987)) where controllers achieving robust stabilization within a prescribed set in the the parameter space were synthesized.

Example 1: Crane

In this first example we treat the case of a crane, a simplified model of which consists of two masses connected by a light inextensible rod (Fig. 1). This system may be described in state space form by (Hwang and Schmitendorf (1984))

$$\dot{x}(t) \ = \ A(r)x(t) + b(r)u$$

where

$$A(r) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -(1+r) & 0 \end{pmatrix}, \qquad b(r) = \begin{pmatrix} 0 \\ 1+r \\ 0 \\ -(1+r) \end{pmatrix}$$

and $r := \frac{m}{M}$ is an uncertain parameter.

Our goal is to synthetize a feedback controller $u = kx$ which stabilizes the above uncertain system around the nominal value $r^0 = 1$ with full information on the maximal range of stability as well as guaranteed damping ratio and stability margin. To this end,

let us assume that it is desired to place the eigenvalues of the above uncertain system in the domain Ω of Fig. 2 specified by a slope $\alpha = 2$ and a margin $\sigma = 0.5$ with complete knowdelge of the maximal parameter range for which these eigenvalues remain in Ω.[3]

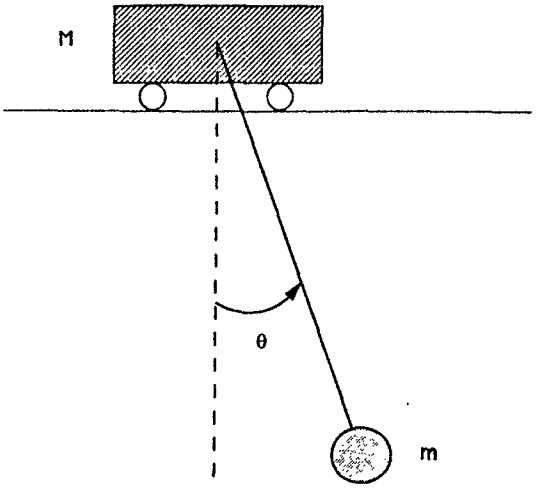

Fig. 1. Crane of Example 1

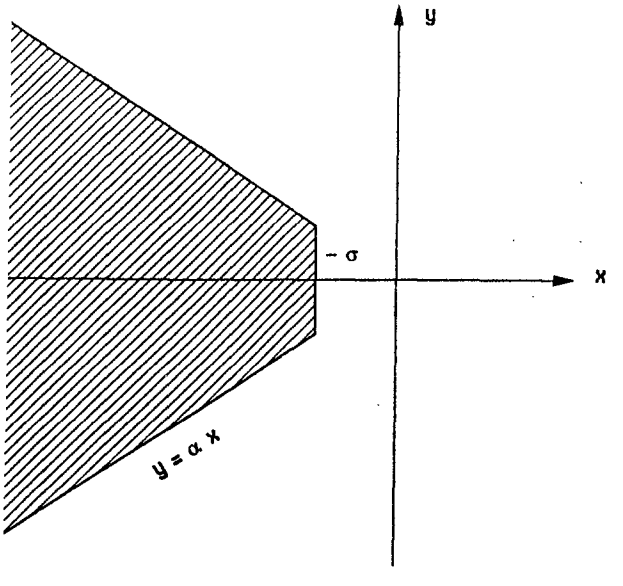

Fig. 2. Stability domain for Example 1

[3] Note that the controllability of the nominal system makes it possible to prespecify Ω.

Accordingly, let $\Delta = \{-1 \pm 0.5j,\ -1 \pm j\} \subset \Omega$ be desired closed-loop eigenvalues for the nominal system. Then one easily finds that the required feedback is [1.25 3.25 $-$ 1.375 1.25] $:= k_0$. To determine $(r_{\min},\ r_{\max})$, the largest range of stability relative to Ω, we may apply Theorem 2 to $A_{\rm cl}(r) := A(r) - bk_0$, provided we can endow $\mathcal{S}(\Omega)$ with a polynomic guardian map. It turns out that the mapping given by $\nu(A) = \nu_1(A)\nu_2(A)$, where

$$\nu_1(A) = \det\left\{\frac{1-\alpha^2}{2}A \cdot A - \frac{1+\alpha^2}{2}A^2 \cdot I\right\}$$

and

$$\nu_2(A) = \det\left((A+\sigma I)\cdot I\right)\ \det\left(A+\sigma I\right),$$

is one such map (Saydy *et al*, (1990)). We obtain the polynomials

$$\begin{aligned}\nu_1(A_{\rm cl}(r)) = 1.933\,10^4(&r^6 + 4.92757\,r^5 + 9.04694\,r^4 + 6.94471\,r^3 + 0.828279\,r^2\\ &- 1.62776\,r - 0.630638)\end{aligned}$$

$$\nu_2(A_{\rm cl}(r)) = \frac{1}{4096}(18r^3 + 187r^2 + 80r - 25)\ \ \frac{1}{32}\,(9r+11)$$

which vanish at the real values

$$\{-1.0,\ -1.4399487,\ -0.9490848,\ 0.4614538252\},$$

$$\{-9.927086,\ -0.670474,\ 0.20867,\ -1.22222\},$$

respectively. Recalling that $r^0 = 1$, we conclude by virtue of Theorem 2 that $r_{\min} = 0.4614538252$, $r_{\max} = +\infty$, i.e. that the closed loop uncertain system is asymptotically stable for all values $r \in (r_{\min},\ \infty)$ and unstable at $r = r_{\min}$. (At this value of r, the closed loop eigenvalues are $-0.76734499 \pm j\,0.4117427 \in \Omega$, and $-0.694108944 \pm j\,1.388217872 \in \partial\Omega$.)

Example 2: Satellite attitude control

A satellite system is modeled as two masses connected by a spring with torque constant k and viscous damping constant d (Fig. 3). The equations of motion are given by

$$J_1\ddot{\theta}_1 + d(\dot{\theta}_1 - \dot{\theta}_2) + k(\theta_1 - \theta_2) = T_c$$

$$J_2\ddot{\theta}_2 + d(\dot{\theta}_2 - \dot{\theta}_1) + k(\theta_2 - \theta_1) = 0$$

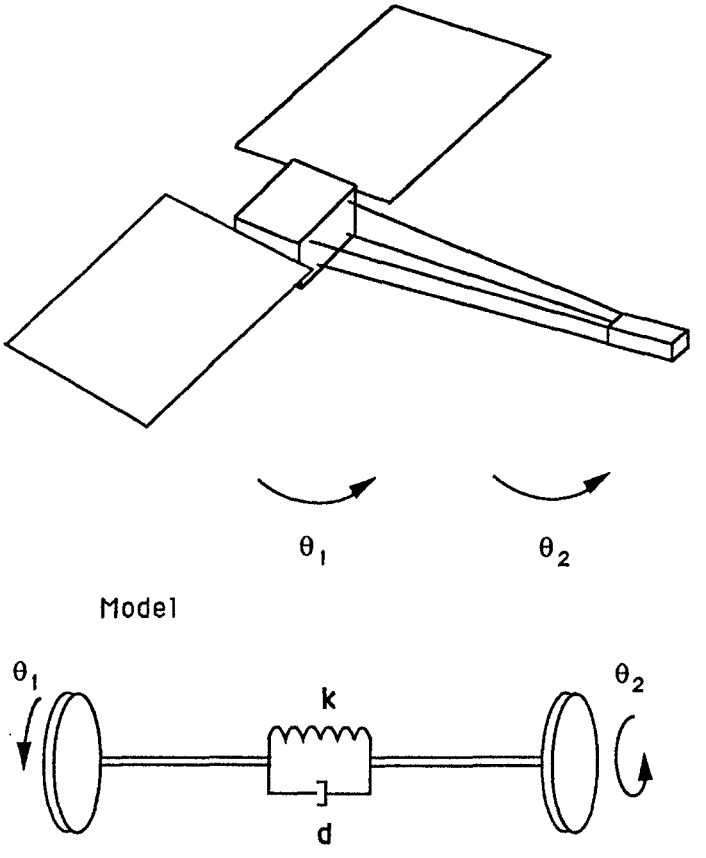

Fig. 3. Sketch of a satellite

where J_1 and J_2 are inertias and T_c is the control torque. Choosing the state vector to be $x = (\theta_2, \dot{\theta}_2, \theta_1, \dot{\theta}_1)$ and letting u, r_1, r_2 denote denote T_c, k and d respectively, we obtain the state equations

$$\dot{x} = A(r_1, r_2)x + bu$$

$$y = cx$$

with

$$A(r_1, r_2) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{r_1}{J_2} & -\frac{r_2}{J_2} & \frac{r_1}{J_2} & \frac{r_2}{J_2} \\ 0 & 0 & 0 & 1 \\ -\frac{r_1}{J_1} & -\frac{r_2}{J_1} & \frac{r_1}{J_1} & \frac{r_2}{J_1} \end{pmatrix}, \qquad b = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{J_1} \end{pmatrix}$$

$$c = (0, 0, 1, 0).$$

Physical analysis leads to the conclusion that the parameters r_1 and r_2 vary as a result of temperature fluctuations within the range (Franklin *et al.*, 1986):

$$0.09 \leq r_1 \leq 0.4,$$

$$0.04\sqrt{\frac{r_1}{10}} \leq r_2 \leq 0.2\sqrt{\frac{r_1}{10}}.$$

The matrix $A(r_1, r_2)$ is singular for every value of the parameters r_1 and r_2 and is hence Hurwitz unstable. Letting $u = -ky$ for some scalar k, we obtain a closed loop matrix $A_k(r_1, r_2) := A(r_1, r_2) - k\ b\ c$.

As we have seen, the set of Hurwitz stable matrices $\mathcal{S}(\mathring{\mathbb{C}}_-)$ is guarded by the polynomic map $\nu : \ A \mapsto \det(A \oplus A)$. Rather than using this map (it involves in our case the computation of a 16×16 determinant), we use the alternate (polynomic) guardian map (see Propositions 7 and 8 in Saydy *et al* (1990))

$$\nu : \ A \mapsto \nu_1(A)\nu_2(A)$$

with

$$\nu_1(A) = \det(A \cdot I)$$

$$\nu_2(A) = \det(A)$$

where the matrix $A \cdot I$ denotes the bialternate product of A and I and has dimension 6.[4] With $J_1 = J_2 = 1$ we obtain[5]

$$\begin{aligned} \nu\left(A_k(r_1, r_2)\right) &= \left(\frac{1}{64}k^3 r_2^2\right)(r_1) \\ &= \frac{1}{64}k^3 r_1 r_2^2. \end{aligned}$$

We can therefore conclude using Theorem 1 that if k is any controller which stabilizes the system for some nominal values $r_1^0 > 0$, $r_2^0 > 0$, then it also stabilizes the satellite

[4] For an $n \times n$ matrix A, the matrices $A \oplus A$ and $A \cdot I_n$ are $n^2 \times n^2$ and $\frac{n(n-1)}{2} \times \frac{n(n-1)}{2}$ respectively.

[5] Computer algebra codes were used to carry out the computations symbolically in both examples.

system globally in the open first quadrant, i.e. for all strictly positive r_1 and r_2, but stability is lost if r_1 or r_2 vanishes.

Example 3: Control of a digital tape transport system

The objective in this last example is to control the speed and the tension of the tape at the read/write head. A model of such a system is (Bhattacharyya, 1987)

$$\dot{x} = A(r_1, r_2)x + b\ u$$

$$y = c\ (r_1, r_2)x$$

where $A(r_1, r_2)$ is a 5×5 companion matrix with last row given by

$$A_{51}(r_1, r_2) = 0$$

$$A_{52}(r_1, r_2) = -0.1045\ 10^{-4} r_1$$

$$A_{53}(r_1, r_2) = -(0.35\ 10^{-4} r_1 + 0.01045\ r_2 + 0.2725)$$

$$A_{54}(r_1, r_2) = -(0.25\ 10^{-4} r_1 + 0.035\ r_2 + 1.5225)$$

$$A_{55}(r_1, r_2) = -(0.025\ r_2 + 2.25)$$

and where

$$b = (0, 0, 0, 0, 1)^T,\ c(r_1, r_2) = (0.3\ 10^{-4} r_1,\ 0.03\ r_2,\ 0,\ 0,\ 0).$$

It is assumed that the parameters r_1 and r_2 are subject to perturbations around their nominal values of $r_1^0 = 4.0\ 10^4$ and $r_2^0 = 20$. Let $A_k(r_1, r_2)$ again denote the closed loop matrix $A(r_1, r_2) - bkc(r_1, r_2)$. It is shown in (Bhattacharyya (1987), p. 71) that the gain $k = 0.00827$ stabilizes the system robustly with stability margin 9.99. We propose to analyze the performance of this controller using the approach presented here. Using the guardian map of the previous example we obtain for $k = 0.0827$ that

$$\begin{aligned}\nu_1\left(A_k(r_1, r_2)\right) = 1.5258\ 10^{-23}[&1.2056\ 10^{12}\ r_2^4 + (5.0782\ 10^9\ r_1 + 1.7002\ 10^{14})r_2^3 \\ &+ (1.7008\ 10^7\ r_1^2 + 3.9995\ 10^{11}\ r_1 + 7.8397\ 10^{15})r_2^2 \\ &+ (1.214\ 10^4\ r_1^3 + 1.4755\ 10^9\ r_1^2 + 1.6002\ 10^{12} r_1 + 1.3643\ 10^{17})r_2 \\ &+ (4.1306\ 10^5\ r_1^3 + 3.1718\ 10^4\ r_1^2 - 1.8761\ 10^{14}\ r_1)]\end{aligned}$$

$$\nu_2(A_k(r_1, r_2)) = -2.48\ 10^{-6}\ r_1.$$

Let us freeze one of the parameter at its nominal value, say, $r_1 = r_1^0 = 4.0\ 10^4$. Then one can show using Sturm sequences for example that the univariate polynomial $\nu_1(A_k(r_1^0, r_2))\nu_2(A_k(r_1^0, r_2))$ has no zeros in $[0, +\infty)$, implying by virtue of Theorem 1 that the family $A_k(r_1^0, r_2)$ is Hurwitz stable for all values $r_2 \in [0, \infty)$. Repeating the same computation in the direction $r_2 = r_2^0 = 20$ yields the similar conclusion that the family $A_k(r_1, r_2^0)$ is Hurwitz stable for all $r_1 \in (0, +\infty)$. This suggests that the controller $k = 0.0827$ possibly achieves stability for all positive values of the parameters r_1, r_2.

To investigate this question we use the following result:

Fact. (Saydy *et al.*, 1990) Let Ω be a subset of the complex plane such that $\mathcal{S}(\Omega)$ is guarded by a given real polynomic map ν. Let $\mathcal{A} := \{A(r_1, r_2) : (r_1, r_2) \in [\alpha_1, \beta_1] \times [\alpha_2, \beta_2]\}$ be a nominally stable family of real matrices; e.g., $A(\alpha_1, \alpha_2) \in \mathcal{S}(\Omega)$. Then the family $\mathcal{A}$ is stable relative to Ω if and only if $U_{\rm cr}^2 = \emptyset$ and the univariate polynomials p_{α_1}, and p_{r_1}, $r_1 \in U_{\rm cr}^1$, have no zeros in $[\alpha_2, \beta_2]$. Here, for each r_1, p_{r_1} denotes the univariate polynomial $\nu(A(r_1, \cdot))$,

$$U_{\rm cr}^1 := \left\{ r_1 \in [\alpha_1, \beta_1] : \ \det B(p_{r_1}, p'_{r_1}) = 0 \right\},$$

$$U_{\rm cr}^1 := \left\{ r_1 \in [\alpha_1, \beta_1] : \ p_{r_1}(\alpha_2) p_{r_1}(\beta_2) = 0 \right\}.$$

where $B(p, q)$ denotes the Bezoutian of the polynomials p and q and prime denotes derivative.

□

Note: the following remarks regarding the application of the fact above are in order:

(i) the interval within which r_1 lies may be any interval (i.e. not necessarily closed);

(ii) the factor $p_{r_1}(\beta_2)$ may be omitted in $U_{\rm cr}^2$ if $\beta_2 = +\infty$;

(iii) it is necessary to verify that the polynomial p_{α_1} has no zeros in $[\alpha_2, \beta_2]$ only if $U_{\rm cr}^1$ turns out to be empty.

Clearly $\nu_2(A_k(r_1, r_2))$ does not vanish in the first open quadrant and we can thus redefine p_{r_1} as $\nu_1(A_k(r_1, r_2))$. It can be verified using Sturm sequences that the polynomial

$\det B(p_{r_1}, p'_{r_1})$, given (modulo a constant factor) by

$$1.7383\, r_1^{12} - 2.5952\, 10^4\, r_1^{11} - 4.3427\, 10^9\, r_1^{10} + 2.0134\, 10^{14}\, r_1^9$$
$$- 1.3604\, 10^{18}\, r_1^8 - 1.403\, 10^{23}\, r_1^7 + 5.2263\, 10^{27}\, r_1^6 - 8.5599\, 10^{31}\, r_1^5$$
$$+\ 6.5237\, 10^{35}\, r_1^4 - 2.9793\, 10^{38}\, r_1^3 - 2.4777\, 10^{43}\, r_1^2 + 4.896\, 10^{46}\, r_1 + 5.1342\, 10^{50}$$

has only one zero in $(0, +\infty)$, namely 14005.4616. With the notation of the fact above and $(r_1, r_2) \in (0, +\infty) \times [0, +\infty)$ we thus have

$$U_{\rm cr}^1 = \{r_1 \in (0, \infty) :\ \det\, B(p_{r_1}, p'_{r_1}) = 0\} = \{14005.4615\}$$
$$U_{\rm cr}^2 = \{r_1 \in (0, \infty) :\ p_{r_1}(0) = 0\} = \{5518.349\}.$$

It follows that the closed-loop matrix $A_k(r_1, r_2)$ is not Hurwitz stable for all (r_1, r_2) in $(0, \infty) \times [0, \infty)$ since $U_{\rm cr}^2 \neq \emptyset$. In fact one can easily check that for $r_1 = 14005.4615$, the polynomial p_{r_1} has no zeros in $[0, \infty)$. We therefore conclude by virtue of the fact above that the Hurwitz stability of the closed-loop system is guaranteed in the smaller rectangle

$$(r_1, r_2) \in (5518.349, +\infty) \times [0, +\infty).$$

Acknowledgments

This work was supported in part by the National Science Foundation's Engineering Research Centers Program: NSFD CDR 8803012, as well as by the NSF under Grants ECS-86-57561, DMC-84-51515 and by the Air Force Office of Scientific Research under Grant AFOSR-87-0073.

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