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THE PROBABILITY OF MULTIPLE CORRECT PACKET RECEPTIONS IN DIRECT-SEQUENCE SPREAD-SPECTRUM NETWORKS

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In this report, we provide an accurate analysis of the probabilities $P(l, m-l|K)$, $l = 0, 1, \cdots, m$ and $m \leq K$, of exactly l correct packet receptions in a group of m receivers, given that K packets are transmitted simultaneously from users employing direct-sequence spread-spectrum (DS/SS) signalling schemes. This quantity is essential for the design and performance evaluation of protocols for admission control, dynamic code allocation of multiple-access spread-spectrum packet radio networks; specific applications include networks of LEO satellites and multi-rate multi-media communications using CDMA (code-division multiple-access) techniques. The evaluations are carried out for DS/SS networks employing BPSK modulation with coherent demodulation and convolutional codes with Viterbi decoding. Systems with geographically dispersed receivers and systems with co-located receivers are considered.

First the exact multireception probabilities for synchronous **uncoded systems** are evaluated at the bit level; these results are essential for checking the accuracy of the other approximations used here. Our results establish that the Independent Receiver Operation Assumption (IROA) yields very good approximations whose accuracy increases as the number of chips per bit N increases. The IROA accuracy is not as satisfactory for co-located receivers when E_b/N_0 is small; for this case we develop an approximation based on the Gaussian multivariate distribution, which is more accurate than the IROA. Extensive comparisons of the exact expressions with the Gaussian and the IROA approximations are conducted. For **convolutional coded systems**, we derive the multireception packet probabilities following a new approach, the Joint First Error Event Approximation (JFEEA), which is based on the lower bound of the probabilities of all-correct packet receptions and the moments of random variables. We compare this approximation with the IROA and observe good agreement between the two.

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1. INTRODUCTION

Spread-spectrum multiple-access (SSMA) schemes are becoming increasingly important not only in military applications, but also in commercial communication networks. Mobile radio networks, satellite communications, air to air, ship to air, and cellular phone communications can all benefit from the multiple-access capability and the protection against multipath and other in-band narrowband interference that spread spectrum provides. An important quantity which is needed for the performance evaluation as well as the design and optimization of protocols in spread-spectrum radio networks is the probability that exactly l out of m packet transmissions are successful, given that K users transmit their packets simultaneously; this quantity is denoted by $P(l, m-l|K)$.

In the literature (see for example [1] and [2]) these multireception packet probabilities are typically approximated via the independent receiver operation assumption (IROA) method, which assumes that the m receivers operate independently of each other although the same K transmitted signals are present at all of them. This approximation is intuitively pleasing but its accuracy has not been verified (for direct-sequence systems) due to the lack of computationally efficient exact expressions for $P(l, m-l|K)$. The computation of these exact packet probabilities is a difficult combinatorial problem whose complexity is prohibitive even when the system parameters are small.

In our earlier work [??] we comprehensively analyzed exact expressions and approximations of $P(l, m-l|K)$ for frequency-hopped spread-spectrum (FH/SS) networks employing random frequency hopping patterns. In this report, we evaluate these quantities for direct-sequence spread-spectrum multiple-access (DS/SSMA) networks employing random signature sequences and examine thoroughly the accuracy of the IROA.

In order to provide a basis for reference against which the accuracy of all approximations will be checked, we first compute the exact expression for the joint packet probabilities at the bit level (i.e. packet length equals 1) for synchronous uncoded systems with dispersed or co-located receivers. The accuracy of the IROA at the bit level is investigated via comparisons with these exact probabilities. In addition, a new approximation based on the multivariate Gaussian distribution is obtained for both synchronous and asynchronous systems. For the system with co-located receivers and a certain range of parameters this approximation shows better agreement with the exact values than the IROA does. Based on these joint bit probabilities, the approximations for packet probabilities are derived with greater accuracy.

Typically DS/SSMA systems employ error-correcting codes for additional interference-rejection capability. Here, we analyze systems using binary convolutional codes with Viterbi decoding (trellis search) and hard-decisions at the receiver. A block diagram of the system is depicted in Figure ??. The evaluation of the multireception packet probabilities is complicated because of the fact that not only the errors at the decoder outputs are dependent, but also the errors at the decoder inputs of different receivers are dependent due to the common other-user interference present at all receivers.

Due to the difficulty in evaluating the packet probabilities of coded DS/SSMA systems, the assumption of independent errors at the decoder input and the union bound on the packet error probability (of a single receiver) are often used. Pursley and Taipale [??] have developed a tighter upper bound for the packet error probability for the single-receiver system. Here we consider the multireceiver case and take the dependence among users into consideration to derive a lower bound for the all-correct multireception packet probabilities. Then, the rest of the multireception probabilities are obtained from solving $m-1$ linear equations involving the moments of the random variable representing the number of correct receptions at each of the the m receivers (except for the all-correct event). We call this Joint First Error Event Approximation (JFEEA) method. The JFEEA is then compared with the IROA approximation to check the accuracy of the latter. Since the JFEEA does not provide upper or lower bounds (except the all-correct and the all-error probabilities) our comparisons focus on how close the two approximations are. Since the two approximations turn out to be reasonably close and they were obtained following completely independent (unrelated) methods we conclude that they help validate each other.

This report is organized as follows: In Section 2, exact expressions for $P(l, m-l|k)$ at the bit level are derived for synchronous uncoded systems with AWGN. In Section 3, the corresponding expressions based on the multivariate Gaussian approximation technique are derived for both synchronous and asynchronous systems. The approximation based on the independence assumption is cited in Section 4. Section 5 derives the JFEEA approximation for synchronous coded systems. The IROA for coded systems is cited in Section 6. Numerical results and comparisons of different approaches are presented in Section 7. In Section 8, conclusions are drawn.

2. DERIVATION OF EXACT EXPRESSIONS

For general binary DS/SSMA systems, the system model is that of [4]. The K transmitted signals are of the form

$$s_k(t) = \sqrt{2P} b_k(t) \Psi(t) a_k(t) cos(2\pi f_c t + \theta_k) \tag{1}$$

for $1 \leq k \leq K$. In (1), P is the signal power; f_c is the common carrier frequency; and θ_k, the phase introduced by the kth modulator, is modeled as a random variable uniformly distributed over $[0, 2\pi]$. The data waveform $b_k(t)$ consists of a sequence of mutually independent rectangular pulses $b_n^{(k)}$ of duration T and amplitude taking values $+1$ or -1 with equal probability. The shaping waveform $\Psi(t)$ is defined by $\Psi(t) = \psi(s)$, for $s = t \bmod T_c$, where $\psi(t)$ is an arbitrary time-limited function satisfying $\psi(t) = 0$, for $t \notin [0, T_c]$ (where T_c^{-1} is the chip rate), and is normalized to have energy equal to T_c (i.e., $\int_0^{T_c} \psi^2(t) dt = T_c$). The code waveform $a_k(t)$ consists of a sequence of rectangular pulse $a_l^{(k)}$ of duration T_c and has amplitude taking values $+1$ of -1. And $N = T/T_c$ is the number of chips per bit.

For K users transmitting simultaneously signals over an AWGN channel, the received signal at any receiver is

$$r(t) = \sum_{k=1}^{K} s_k(t - \tau_k) + n(t) \tag{2}$$

where $n(t)$ is a zero-mean white Gaussian noise process with two-sided spectral density $\frac{N_0}{2}$ and τ_k is the kth delay modeled as a uniformly distributed random variable in the interval $[0, T]$.

In a DS/SS packet radio network, packet errors are caused by a combination of noise at the receivers and interference between packet transmissions. The evaluation of joint packet probabilities is complicated, since the interference between packet transmissions produces dependent errors at the output of the demodulators and the bit errors among the successive bits in a packet are also dependent. Because these dependences prohibit the accurate analysis of joint packet probabilities even for uncoded systems, we make some necessary assumptions and present a worst-case analysis. It is well known that the packet error probability for a slotted, chip- and phase-synchronous network model is an upper bound on the packet error probability for an unslotted, chip- and phase-asynchronous model given that the maximum number of transmitters seen by both models are the same [6]. To make bit errors at the decoder input independent from bit to bit, we further assume that random signature sequences are used, as was the case in [5] and [6]. Recall that the random sequence model is convenient both for facilitating the performance evaluation of DS/SSMA systems and for modeling systems with large user populations for which we do not desire to distinguish in any way among the different users. Consequently, our assumptions are as follows :

- Random signature sequences are used.
- θ_ks and τ_ks in (1) are all zero.
- K in (2) remains constant over the duration of the packet.

The exact result of $P(l, m-l|K)$ is derived at bit level based on the perfectly synchronous system assumption stated above.

2.1 JOINT INTERFERENCE IN ONE CHIP

The key information for solving $P(l, m-l|K)$ is the mutually dependent interferences among the m receivers. We arrange these interferences at the chip level in order to reduce the number of combinations as much as possible so that the interferences at the bit level for small system parameters can be obtained quickly. Two simple examples of the derivation is presented in Figures 2 and 3.

Consider a single chip duration, the chips of the K users can be categorized into two groups: one is the set of chips on the m receivers we are interested in, the other is the set of the $K-m$ chips on all other receivers. We arrange the users so that the m users in which we are interested are *user* 0 to *user* $m-1$, while all other users are from *user* m to *user* $K-1$. Let $X_j(0 \leq j \leq K-1)$ be the "chip value" of the jth user, which means that X_j is the value of $b_j(t)a_j(t)$ during the chip considered; then the X_js are i.i.d. binary random variables taking values $+1$ or -1 with equal probability. Define the random vector $\mathbf{X} = (X_0, \ldots, X_{K-1})$, which can take any of the 2^K different vectors with equal probability 2^{-K}, and let $I_j(0 \leq j \leq m-1)$ denote the total interference that user j suffers from all the other $K-1$ users during the chip considered, then

$$I_j = X_j \sum_{\substack{i=0 \\ i \neq j}}^{K-1} X_i \tag{3}$$

Here interference is defined as the difference between the number of "agree with X_j" chips and that of "disagree with X_j" chips. Since the I_js are dependent random variables, they shouldn't be considered separately. We define the random vector $\mathbf{I} = (I_0, \ldots, I_{m-1})$ as the basic unit of interference in this interference-dependent system. Let S_I denote the sample space of $\mathbf{I}$ and N_I be the number of different vectors $\mathbf{I}$ can take (i.e., the number of elements in S_I), then

$$N_I = 1 + E(K) + \sum_{i=1}^{\alpha(K)} \sum_{j=b_i}^{e_i} \binom{m}{j} \tag{4}$$

where

$$E(K) = \begin{cases} 1, & \text{if } K = even \\ 0, & \text{if } K = odd \end{cases} \tag{5}$$

$$\alpha(K) = \begin{cases} \lfloor \frac{K}{2} \rfloor, & \text{if} K = odd \\ \frac{K}{2} - 1, & \text{if} K = even \end{cases} \tag{6}$$

$$b_i = \begin{cases} 0, & \text{if } K - i - m \geq 0 \\ m - (K - i), & \text{if } K - i - m < 0 \end{cases} \tag{7}$$

$$e_i = min(i, m) \tag{8}$$

Now we have N_I instead of 2^K interference vectors in one chip. Define three vectors

$$\begin{aligned} V_0 &= (\overbrace{K-1, K-1, \ldots, K-1}^{m \text{ terms}}) \\ V_1 &= (\overbrace{-1, -1, \ldots, -1}^{m \text{ terms}}) \\ V_{ij} &= (\overbrace{E_{i0}, \ldots, E_{i0}}^{j \text{ terms}}, \overbrace{E_{i1}, \ldots, E_{i1}}^{m-j \text{ terms}}) \end{aligned}$$

where

$$\begin{aligned} E_{i0} &= -K + 2i - 1 \\ E_{i1} &= K - 2i - 1 \end{aligned}$$

then the elements of S_I, denoted by $\mathbf{I}^l (0 \leq l \leq N_I - 1)$, and the probability mass function, $P(\mathbf{I} = \mathbf{I}^l)(0 \leq l \leq N_I - 1)$, are derived by the following algorithm

$$\mathbf{I}^0 = V_0; \tag{9}$$

$$P(\mathbf{I} = \mathbf{I}^0) = 2^{1-K}; \tag{10}$$

$$l = 0;$$

$$for(i = 1; i \leq \alpha(K); i++) for(j = b_i; j \leq e_i; j++)$$

$$for(k = 1; k \leq \binom{m}{j}; k++)\{ \quad l++;$$

$$\mathbf{I}^l = the\ kth\ permutation\ of\ V_{ij}; \tag{11}$$

$$P(\mathbf{I} = \mathbf{I}^l) = 2^{1-K}\binom{K-m}{i-j};\} \tag{12}$$

$$if(K == even)\{$$

$$\mathbf{I}^{N_I-1} = V_1; \tag{13}$$

$$P(\mathbf{I} = \mathbf{I}^{N_I-1}) = 2^{-K}\binom{K}{K/2};\} \tag{14}$$

The derivation of equations (4) to (14) is given in Appendix A.

2.2 JOINT INTERFERENCE IN ONE BIT

Let $\mathbf{I}^{<j>}(1 \leq j \leq N)$ be the interference vector during chip j in some data bit; then $\mathbf{I}^{<j>}$s are i.i.d. random vectors with distribution of that of $\mathbf{I}$ described in (10), (12) and(14). Now we are interested in the total interference in a data bit, denoted by $\mathbf{B} = (B_0, B_1, \cdots, B_{m-1})$, which is the sum of $\mathbf{I}^{<j>}$s,

$$\mathbf{B} = \sum_{j=1}^{N} \mathbf{I}^{<j>} \tag{15}$$

In a data bit, the random vector $\mathbf{I}^{<j>}$ can take the value of any of the N_I elements in S_I. Let $M_l(0 \leq l \leq N_I - 1)$ be the number of times that vector $\mathbf{I}^l$ is chosen during the N chips in the bit; then

$$\begin{cases} \sum_{l=0}^{N_I-1} M_l = N \\ 0 \leq M_l \leq N \end{cases} \tag{16}$$

The number of integral solutions for the above equation, denoted by N_B, is

$$N_B = \binom{N + N_I - 1}{N_I - 1} \tag{17}$$

Let vector $\mathbf{M}^w = (M_0^w, M_1^w, \cdots, M_{N_I-1}^w), 1 \leq w \leq N_B$, be the solutions of equation (16), and $\mathbf{B}^w = (B_0^w, B_1^w, \cdots, B_{m-1}^w)$ the total interference in a data bit, given the wth solution of equation (16); then

$$\mathbf{B}^w = \sum_{l=0}^{N_I-1} M_l^w \mathbf{I}^l \tag{18}$$

The probability that $\mathbf{B}$ equals $\mathbf{B}^w$ is

$$P(\mathbf{B} = \mathbf{B}^w) = \frac{N!}{\prod_{l=0}^{N_I-1} M_l^w!} \prod_{l=0}^{N_I-1} P(\mathbf{I} = \mathbf{I}^l)^{M_l^w} \tag{19}$$

2.3 MULTIRECEPTION PROBABILITIES AT BIT LEVEL

Consider a single bit l, for coherent reception and a correlation receiver matched to the jth user; the output [4] is

$$Z_j^l = \begin{cases} \sqrt{P/2}T(b_l^{(j)} - \frac{1}{N}B_j + \eta) & \text{if } b_l^{(j)} = -1 \\ \sqrt{P/2}T(b_l^{(j)} + \frac{1}{N}B_j + \eta) & \text{if } b_l^{(j)} = 1 \end{cases} \tag{20}$$

where η is a zero-mean Gaussian random variable with variance $(2E_b/N_0)^{-1}$, $E_b = PT$ is the energy per bit, $b_l^{(j)}$ the desired signal in the decision interval $[0, T]$, and B_j the jth component of the multiple-access interference vector $\mathbf{B}$. The probability that the lth bit of the jth user is incorrect is hence

$$\begin{aligned} Pe_j^l &= \frac{1}{2}Pr(Z_j^l > 0 | b_l^{(j)} = -1) + \frac{1}{2}Pr(Z_j^l < 0 | b_l^{(j)} = 1) \\ &= Pr(\eta > 1 + \frac{1}{N}B_j) \end{aligned} \tag{21}$$

The multireception probabilities at the bit level are considered for two kinds of systems. One is the system with the m receivers at independent sites; the other is the system with m co-located receivers. The packet probabilities for these two systems are denoted by $P_i(l, m-l|K)$ and $P_c(l, m-l|K)$, respectively. Due to the symmetry of the m receivers, we assume that only the first l packets are correctly received when deriving $P(l, m-l|K)$. Then

$$P_i(l, m-l|K) = \binom{m}{l} \prod_{j=0}^{l-1} Pr\{\eta_j \leq (1+\frac{1}{N}B_j)\} \prod_{j=l}^{m-1} Pr\{\eta_j > (1+\frac{1}{N}B_j)\} \tag{22}$$

where η_js are i.i.d. zero-mean Gaussian random variables with variance $(2E_b/N_0)^{-1}$. In terms of $P(\mathbf{B} = \mathbf{B}^w)$, (22) can be restated as

$$P_i(l, m-l|K) = \binom{m}{l} \sum_{w=1}^{N_B} \left\{ \prod_{j=0}^{l-1} \phi(\alpha_j^w) \prod_{j=l}^{m-1} \left[1 - \phi(\alpha_j^w)\right] \right\} P(\mathbf{B} = \mathbf{B}^w) \quad (23)$$

where

$$\alpha_j^w = \frac{1 + \frac{1}{N} B_j^w}{\sqrt{\frac{N_0}{2E_b}}} \quad (24)$$

$$\phi(\alpha) = \int_{-\infty}^{\alpha} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \quad (25)$$

For systems with co-located receivers,

$$\begin{aligned} P_c(l, m-l|K) = \binom{m}{l} Pr\{ \quad \eta &\le 1 + \frac{1}{N} B_0, \cdots, \eta \le 1 + \frac{1}{N} B_{l-1}, \\ \eta &> 1 + \frac{1}{N} B_l, \cdots, \eta > 1 + \frac{1}{N} B_{m-1}\} \end{aligned} \quad (26)$$

Define

$$L^w = \max_j \{1 + \frac{1}{N} B_j^w | j = l, \ldots, m-1\} \quad (27)$$

$$U^w = \min_j \{1 + \frac{1}{N} B_j^w | j = 0, \ldots, l-1\} \quad (28)$$

then (26) in terms of $P(\mathbf{B} = \mathbf{B}^w)$ becomes

$$P_c(l, m-l|K) = \binom{m}{l} \sum_{w=1}^{N_B} \left[\phi(\frac{U^w}{\sqrt{\frac{N_0}{2E_b}}}) - \phi(\frac{L^w}{\sqrt{\frac{N_0}{2E_b}}}) \right] 1(L^w, U^w) P(\mathbf{B} = \mathbf{B}^w) \quad (29)$$

where

$$1(L, U) = \begin{cases} 1, & \text{if } L < U \\ 0, & \text{if } L \ge U \end{cases} \quad (30)$$

3. THE MULTIVARIATE GAUSSIAN APPROXIMATION

In the previous section, we derived exact expressions for the probabilities $P(l, m-l|K)(l = 0, 1, \cdots, m, m \le K)$ at the bit level. Here we develop an approximation method based on the Gaussian multivariate distribution. The packet probabilities based on this approximation are denoted by $P_G(l, m-l|K)$.

Consider a single bit interval, let the random variables $x_{ij}(0 \le i \le K-1, 0 \le j \le m-1, i \ne j)$ denote the normalized interference term that user i has over user j, $x_{ij}^n(1 \le n \le N, 0 \le i \le K-1, 0 \le j \le m-1, i \ne j)$ denote the interference that user i causes to user j during the nth chip of the bit; then the total interference user j suffers from all the other users is

$$x_j = \sum_{\substack{i=0 \\ i \ne j}}^{K-1} x_{ij} = \sum_{\substack{i=0 \\ i \ne j}}^{K-1} \frac{1}{N} \sum_{n=1}^{N} x_{ij}^n \tag{31}$$

Since the x_{ij}s are i.i.d. random variables for a fixed j, x_j tends toward a Gaussian distribution for large K. Even K is not large, x_j approaches to a Gaussian distribution as long as the number of chips per bit N is large enough. This is because x_{ij}^ns are i.i.d. for fixed i and j (proved in Appendix B), and s_{ij}s tends to Gaussian for large N. Now, if we consider any linear combination of the x_js, say

$$\overline{z} = \sum_{j=0}^{m-1} a_j x_j \tag{32}$$

then

$$\overline{z} = \sum_{j=0}^{m-1} a_j \Big(\sum_{\substack{i=0 \\ i \ne j}}^{K-1} \frac{1}{N} \sum_{n=1}^{N} x_{ij}^n \Big) = \frac{1}{N} \sum_{i=0}^{K-1} \sum_{n=1}^{N} \sum_{\substack{j=0 \\ j \ne i}}^{m-1} a_j x_{ij}^n = \frac{1}{N} \sum_{i=0}^{K-1} \sum_{n=1}^{N} z_{in} \tag{33}$$

Consider a fixed n, since $x_{ij}^n = x_{ji}^n$ when $0 \le i, j \le m-1$ and $i \ne j$, random variables $z_{0n}, \cdots, z_{m-1\ n}$ are $m-1$ dependent; and the number of $m-1$ dependent z_{in}s in (33) are Nm. But all other $N(K-m)$ z_{in}s $(m \le i \le K-1, 1 \le n \le N)$ are i.i.d. random variables. It turns out that $\overline{z}$ is a sum of i.i.d. and $m-1$ dependent RVs and, as $NK \to \infty$, tends to have a Gaussian distribution. Consequently, all x_js are jointly Gaussian if the number of NK is sufficiently large.

Define the m-dimensional column vectors

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{m-1} \end{bmatrix} \quad and \quad \underline{\mu} = \begin{bmatrix} \mu \\ \mu \\ \vdots \\ \mu \end{bmatrix} \tag{34}$$

where μ is the mean of x_js.
Then we have the multivariate Gaussian probability density function (pdf)

$$p_{\underline{x}}(\underline{x}) = \frac{1}{\sqrt{(2\pi)^m |\Sigma|}} e^{-\frac{1}{2}(\underline{x}-\underline{\mu})^T \Sigma^{-1} (\underline{x}-\underline{\mu})} \tag{35}$$

where Σ is the $m \times m$ covariance matrix with diagonal elements

$$a = E\{(x_j - \mu)^2\} \tag{36}$$

and off-diagonal elements (all of which are equal due to the symmetry of our problem)

$$b = E\{(x_j - \mu)(x_{j'} - \mu)\} \tag{37}$$

a and b are calculated in Appendix C on the basis of the signaling scheme (DS synchronous or asynchronous) and the presence of AWGN.
Let b_{l1}, b_{u1}, b_{l2}, and b_{u2} be system parameters such that user j is successful when $x_j \in [b_{l1}, b_{u1}]$ and unsuccessful when $x_j \in [b_{l2}, b_{u2}]$; then the multivariate Gaussian approximation of $P(l, m-l|K)$ is

$$P_G(l, m-l|K) = \overbrace{\int_{b_{l1}}^{b_{u1}} \cdots \int_{b_{l1}}^{b_{u1}}}^{l} \overbrace{\int_{b_{l2}}^{b_{u2}} \cdots \int_{b_{l2}}^{b_{u2}}}^{m-l} p_{\underline{x}}(\underline{x}) dx_0 \cdots dx_{m-1} \tag{38}$$

We define

$$\underline{x}_l = \left[\overbrace{b_{l1}, \cdots, b_{l1}}^{l}, \overbrace{b_{l2}, \cdots, b_{l2}}^{m-l} \right]^T \tag{39}$$

$$\underline{x}_u = \left[\overbrace{b_{u1}, \cdots, b_{u1}}^{l}, \overbrace{b_{u2}, \cdots, b_{u2}}^{m-l} \right]^T \tag{40}$$

and have

$$P_G(l, m-l|K) = \int_{\underline{x}_l}^{\underline{x}_u} \frac{1}{\sqrt{(2\pi)^m |\Sigma|}} e^{-\frac{1}{2}(\underline{x}-\underline{\mu})^T \Sigma^{-1} (\underline{x}-\underline{\mu})} d\underline{x} \tag{41}$$

The integral in (41) can be simplified, if the exponent is converted from a quadratic to a sum-of-squares form. The result of this conversion is

$$-\frac{1}{2}\underline{x}^T \Sigma^{-1} \underline{x} = -\frac{1}{2(a-b)} \left[\sum_{j=0}^{m-1} x_j{}^2 - \frac{b}{a+(m-1)b} \left(\sum_{j=0}^{m-1} x_j \right)^2 \right] \tag{42}$$

and the determinant of Σ is found to be

$$det(\Sigma) = (a-b)^{m-1}[a+(m-1)b] \tag{43}$$

(42) and (43) are derived in Appendix D.
Using $u_j = \frac{x_j}{\sqrt{a-b}}$ makes (42)

$$-\frac{1}{2} \left[\sum_{j=0}^{m-1} u_j{}^2 - \frac{b}{a+(m-1)b} \left(\sum_{j=0}^{m-1} u_j \right)^2 \right] \tag{44}$$

But the square in the above equation can be eliminated with the help of the following integral transform [7]

$$e^{\frac{1}{2}\sigma^2\phi^2} = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} dy \; e^{\phi y} e^{-\frac{y^2}{2\sigma^2}} \tag{45}$$

with

$$\sigma^2 = \frac{b}{a+(m-1)b} \tag{46}$$

and

$$\phi = \sum_{j=0}^{m-1} u_j \tag{47}$$

After some manipulations in which we use (43),(45),(46) and (47), we obtain the basic result

$$P_G(l, m-l|K) =$$
$$\int_{-\infty}^{+\infty} dy \; \frac{e^{-\frac{1}{2}y^2}}{\sqrt{2\pi}} [Q(Z(b_{l1})) - Q(Z(b_{u1}))]^l [Q(Z(b_{l2})) - Q(Z(b_{u2}))]^{m-l} \tag{48}$$

where

$$Z(x) = \frac{x-\mu-\sqrt{b}y}{\sqrt{a-b}} \tag{49}$$

$$Q(x) = \frac{1}{\sqrt{2\pi}}\int_x^\infty e^{-\frac{u^2}{2}}du \tag{50}$$

Equation (48) gives a method for calculating $P_G(l, m-l|K)$ with linear computational complexity in m.

Consider the system with AWGN, random variable η is added to x_j; from equation (21) we know that, for user j to be successful, $x_j+\eta$ has to be greater than -1. Hence, $b_{l1}=-1$, $b_{l2}=-\infty$, $b_{u1}=\infty$ and $b_{u2}=-1$. The parameters a and b also need to be changed to account for the AWGN : for the system with independent receivers,

$$\begin{cases} a' = a + \frac{N_0}{2E_b} \\ b' = b \end{cases} \tag{51}$$

for the system with co-located receivers,

$$\begin{cases} a' = a + \frac{N_0}{2E_b} \\ b' = b + \frac{N_0}{2E_b} \end{cases} \tag{52}$$

Finally, (48) becomes

$$P_G(l, m-l|K) = \int_{-\infty}^{+\infty} dy \, \frac{e^{-\frac{1}{2}y^2}}{\sqrt{2\pi}}[Q(z_1)]^l[1-Q(z_1)]^{m-l} \tag{53}$$

where

$$z_1 = -\frac{1+\mu+\sqrt{b'}y}{\sqrt{a'-b'}} \tag{54}$$

4. THE INDEPENDENCE ASSUMPTION (IROA)

The assumption of independence between the packet errors of the users is commonly made for simplifying the evaluation of $P(l, m-l|K)$. The relevant expressions for synchronous systems are given in this section. The numerical results of these expressions will be compared with the exact results derived in Section 2 to determine the validity of IROA assumption.

The exact bit error probability ρ for a binary DS/SSMA system is [9]

$$\rho = Q(\sqrt{\frac{2E_b}{N_0}}) + \frac{1}{\pi}\int_0^{\infty} \frac{sin(u)}{u}\Phi_2(u)(1-\Phi_1(u))du \tag{55}$$

where

$$\begin{aligned} \Phi_1(u) &= \left[cos\left(\frac{u}{N}\right)\right]^{N(K-1)} && (56) \\ \Phi_2(u) &= exp\left[-\frac{N_0}{4E_b}u^2\right] && (57) \end{aligned}$$

and $Q(x)$ is defined as (50).

Based on the IROA, the packet probabilities $P_{iid}(l, m-l|K)$ is

$$P_{iid}(l, m-l|K) = \binom{m}{l}\rho^l(1-\rho)^{m-l} \tag{58}$$

5. CODED SYSTEMS

Pursley and Taipale [6] have developed a tight upper bound of the packet error probability for the hard-decision single-receiver case. In this section, we consider the multireceiver system using binary convolutional coding and hard-decision Viterbi decoding. Approximations (JFEEA) for the packet probabilities $P(l, m-l|K)(l = 0, 1, \ldots, m$ and $m \leq K)$ are obtained by using first-event error [8], Pursley and Taipale's bound [6], multi-dimensional extension, and moments of random variables.

The difficulties in evaluating the joint packet probabilities in a multi-receiver system are twofold: first, the stream of input to the decoders is dependent from bit to bit (horizontal dependence); second, the other-user interference terms present at the m receives in the same time slot are correlated (vertical dependence). To facilitate the analysis, the assumptions of zero delays and zero phases among the users, as well as the constant number of equal power transmitters are again made. These assumptions render among the successive bits of input stream to the decoder independent and yield a worst-case evaluation of the system [6]. For a convolutional coded system using Viterbi decoding, another horizontal dependence exists since the errors out of the decoder are not independent. The first-event error is used to circumvent the difficulty caused by this dependence.

There is a considerable difference between the single-receiver case and the multireceiver case when it comes to the evaluation of packet probabilities. In the single-receiver case, there are only two events, namely the error event and the correct event whereas in the multireceiver case there are 2^m different events. One of the events is that all m receives have correct packet receptions. In this case, a union bound on $P(m, 0|K)$ can be obtained based on Pursley and Taipale's work [6]. But all other non-symmetric joint packet probabilities (i.e. the probabilities that l packets are successful, while $m - l$ packets are unsuccessful, where $l \neq m$) can not be obtained from the union bound. A technique is thus introduced for deriving these non-symmetric joint packet probabilities based on solving a system of $m - 1$ linear equations involving the lower bound on the all-correct packet probabilities and the moments of random variable representing the number of correct receptions at each of the m receivers. These linear equations give the result of the packet probabilities as long as the algorithm which solves them can reach the required accuracy.

5.1 LOWER BOUND ON $P(l,0|K)$

Denote a convolutional code with a $k-$input, $n-$output linear sequential circuit with input memory m by (n,k,m). Then, for an information sequence of length kM, the trellis diagram contains $M+m+1$ time units (or nodes) and 2^{kM} distinct paths through the trellis corresponding to the 2^{kM} codewords of length $n(M+m)$. Since convolutional codes are group codes, for purposes of analysis we can assume, without loss of generality, that the transmitted message of each transmitter is the all-zeros message. The all-correct event happens if no path ever causes an error event for all m receivers. Consider $P(m,0|K)$, an error occurs when any of the m Viterbi decoders selects any of the non-all-zeros paths as it merges with the all-zeros path. So, $P(m,0|K)$ is the probability that none of the non-all-zeros paths of all m trellises are selected by the Viterbi algorithm. Let us classify the paths in the trellis into M classes, from class 0 to class $M-1$. Class i consists all the paths that first deviate from node i. Denote the number of paths in class i by N_i, and define the following events :

$K^{i,*}$: the joint event that in all m trellises, all paths with class number greater than or equal to i do not cause error events.

K_i^*: the joint event that in all m trellises, all paths in class i do not cause error events.

$K_i^{l,*}$: the joint event that in all m trellises, all paths in class i with path indexes greater than or equal to l do not cause error events.

$K_{i,l}^*$: the joint event that in all m trellises, all m lth paths in class i do not cause error events.

Then according to [6], the lower bound of $P(m,0|K)$ can be derived as follows:

$$
\begin{aligned}
P(m,0|K) &= P(K^{0,*}) = P(K^{1,*}|K_0^*)P(K_0^*) \\
&\geq P(K^{1,*})P(K_0^*) \geq \cdots \geq \prod_{i=0}^{M-1} P(K_i^*)
\end{aligned}
\tag{59}
$$

and

$$
\begin{aligned}
P(K_i^*) &= P(K_i^{1,*}) = P(K_i^{2,*}|K_{i,1}^*)P(K_{i,1}^*) \\
&\geq P(K_i^{2,*})P(K_{i,1}^*) \geq \cdots \geq \prod_{l=1}^{N_i} P(K_{i,l}^*)
\end{aligned}
\tag{60}
$$

If in each class we include paths of all lengths originating from the node, then all M classes are the same and (60) becomes

$$P(K_0^*) \geq \prod_{l=1}^{\infty} P(K_{0,l}^*) \tag{61}$$

Consequently,

$$P(m, 0|K) \geq \left[\prod_{l=1}^{\infty} P(K_{0,l}^*)\right]^M \tag{62}$$

Let $P(l)$ denote the probability that in all m trellises, any of the m paths with index l causes an error event, then by (62),

$$\begin{aligned} P(m, 0|K) &\geq \left[\prod_{l=1}^{\infty} P(K_{0,l}^*)\right]^M = \left[\prod_{l=1}^{\infty}(1 - P(l))\right]^M \\ &\geq \left(1 - \sum_{l=1}^{\infty} P(l)\right)^M \triangleq (1 - P_u)^M \end{aligned} \tag{63}$$

The sum P_u on the right-hand side of (63) is a vector version of the union bound of first-event error probability discussed by Viterbi in [8]. The evaluation of P_u is based on the generating function $T(D)$ [8]. Suppose

$$T(D) = \sum_{d=d_{free}}^{\infty} a_d D^d \tag{64}$$

Let P_d denote the probability that in all m trellises, one or more distance d paths cause error events, then

$$P_u = \sum_{d=d_{free}}^{\infty} a_d P_d \tag{65}$$

The evaluation of P_d is not easy because of the dependence among m users. Let us define the random variables $x_{ij}(0 \leq i \leq m-1, 1 \leq j \leq d)$, which are the bits of interest for some distance d path in all m trellises, so that

$x_{ij} = 1,$ if the jth interested bit of trellis i is correct with probability $(1-\rho)$.

$x_{ij} = 0,$ if the jth interested bit of trellis i is incorrect with probability ρ.

where ρ is given in (55).
Define random variables $x_i (0 \leq i \leq m-1)$ so that

$$x_i = \sum_{j=1}^{d} x_{ij} \tag{66}$$

Then in terms of x_is, $\overline{P_d} = 1 - P_d$ is

$$\overline{P_d} = \begin{cases} P(x_0 \geq \frac{d+1}{2}, x_1 \geq \frac{d+1}{2}, \cdots, x_{m-1} \geq \frac{d+1}{2}), & d \text{ odd} \\ \frac{1}{2}P(x_0 = \frac{d}{2}, x_1 = \frac{d}{2}, \cdots, x_{m-1} = \frac{d}{2}) & \\ +P(x_0 \geq \frac{d}{2}+1, x_1 \geq \frac{d}{2}+1, \cdots, x_{m-1} \geq \frac{d}{2}+1), & d \text{ even} \end{cases} \tag{67}$$

Since the input stream to the decoder is assumed to be independent from bit to bit, x_{ij} is independent of $x_{ij'}$ when $j \neq j'$. Consequently, x_i is the sum of d i.i.d. random variables and has mean $d(1-\rho)$ and variance $d\rho(1-\rho)$. In order to determine how much x_i and $x_{i'}$ are dependent, we derive the covariance, b, of them

$$\begin{aligned} b &= E[(x_i - d(1-\rho))(x_{i'} - d(1-\rho))] \\ &= E\left\{\left[\sum_{j=1}^{d}(x_{ij} - (1-\rho))\right]\left[\sum_{j=1}^{d}(x_{i'j} - (1-\rho))\right]\right\} \\ &= \sum_{j=1}^{d}(E[x_{ij}x_{i'j}] - (1-\rho)^2) = d\left[P(x_{ij} = 1, x_{i'j} = 1) - (1-\rho)^2\right] \end{aligned} \tag{68}$$

The term $P(x_{ij} = 1, x_{i'j} = 1)$ is in fact $P(2, 0|K)$ at the bit level we have derived in section 2, and $(1-\rho)^2$ is the IROA approximation of $P(x_{ij} = 1, x_{i'j} = 1)$ in section 4. According to our numerical results, $P(x_{ij} = 1, x_{i'j} = 1) - (1-\rho)^2$ is negligible for general system parameters. Moreover, since the distance d of interest can not be very large, b is also negligible. Consequently, P_d can be derived with enough accuracy based on the independence assumption. Assume that x_{ij} and $x_{i'j}$ are independent, P_d becomes

$$P_d = 1 - \overline{P_d} = 1 - [P_c(d)]^m \tag{69}$$

where

$$P_c(d) = \begin{cases} \sum_{k=(d+1)/2}^{d} \binom{d}{k}(1-\rho)^k \rho^{d-k}, & \text{if } d = odd \\ \frac{1}{2}\binom{d}{d/2}(1-\rho)^{d/2}\rho^{d/2} + \sum_{k=d/2+1}^{d}\binom{d}{k}(1-\rho)^k\rho^{d-k} & \text{if } d = even \end{cases} \tag{70}$$

By applying (69) to (65), then to (63), the lower bound of $P(l,0|K)$ is obtained.

5.2 THE JFEEA APPROXIMATION

Given $P(l,0|K)(1 \leq l \leq m)$, the probabilities $P(l,m-l|K)(0 \leq l \leq m-1)$ are derived in the following way : Define the random variable $V = V_1+V_2+\cdots+V_m$, where the V_is are binary random variables which take value 1, if the ith receiver decodes correctly, and 0, if the ith receiver's decoder fails. Then the nth moment of V is

$$\begin{aligned}
\mu_n &= \sum_{l=1}^{m} l^n P(l,m-l|K) && (71)\\
&= E[(V_1+V_2+\cdots+V_m)^n] && (72)\\
&= \sum_{v_1=0}^{1}\cdots\sum_{v_m=0}^{1}(v_1+v_2+\cdots+v_m)^n P(v_1,v_2,\cdots,v_m) && (73)
\end{aligned}$$

By substituting μ_n for $n = 1,2,\cdots,m-1$ into (71), $m-1$ linear equations can be obtained to solve $P(l,m-l|K)$, for $l = 1,2,\cdots,m-1$, i.e.

$$\begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_{m-1} \end{bmatrix} = \begin{bmatrix} 1^1 & 2^1 & \cdots & m^1 \\ 1^2 & 2^2 & & m^2 \\ \vdots & & \ddots & \vdots \\ 1^{m-1} & 2^{m-1} & \cdots & m^{m-1} \end{bmatrix} \begin{bmatrix} P(1,m-1|K) \\ P(2,m-2|K) \\ \vdots \\ P(m,0|K) \end{bmatrix} \qquad (74)$$

Then, the probability $P(0,m|K)$ is obtained by

$$P(0,m|K) = 1 - \sum_{l=1}^{m} P(l,m-l|K) \qquad (75)$$

The nth moment of V can be obtained by expanding (73)

$$\begin{aligned}
\mu_n = \sum_{n_1}\cdots\sum_{n_m}\frac{n!}{n_1!n_2!\cdots n_m!}\sum_{v_1=0}^{1}\cdots\sum_{v_m=0}^{1}\\
{v_1}^{n_1}{v_2}^{n_2}\cdots{v_m}^{n_m}P(v_1,v_2,\cdots,v_m) \qquad (76)
\end{aligned}$$

where

$$\begin{cases} n_1+n_2+\cdots+n_m = n \\ 0 \leq n_i \leq n \end{cases} \qquad (77)$$

Consider the expansion of (76), if $n_i = 0$, $v_i^{n_i}$ is always 1 and the sum over v_i is eliminated; moreover, for a nonzero n_i, v_i has to be one to avoid the whole term becoming zero. Consequently, The sum in (76) can be divided into n groups, Group 1 to Group n; in Group j, only j of the n_is are nonzero; i.e.

$$\mu_n = \sum_{j=1}^{n} M_{nj} P(v_1 = v_2 = \cdots = v_j = 1) \tag{78}$$

where M_{nj} is the number of terms in Group j.
Because of the symmetry of all m receivers, M_{nj} equals $\binom{m}{j} N_{nj}$, where N_{nj} is the number of terms in the expansion of $(v_1 + v_2 + \cdots + v_j)^n$ that contain all j variables. N_{nj} can be computed iteratively

$$N_{nj} = j^n - \sum_{i=1}^{j-1} \binom{j}{i} N_{ni} \tag{79}$$

where $N_{n1} = 1$.
Finally, the nth moment of V becomes

$$\mu_n = \sum_{j=1}^{n} \binom{m}{j} N_{nj} P(j, 0|K) \tag{80}$$

After obtaining $\mu_1, \mu_2, \cdots, \mu_{m-1}$, we can derive the packet probabilities by applying (74).

5.3 PACKET PROBABILITIES FOR UNCODED SYSTEMS

For the purpose of comparison, we also cite the packet probabilities for uncoded systems. Both exact expressions and approximation can be obtained. Since the system is assumed to be synchronous, and the covariance between two bits is negligible, $P(l, 0|K)$ can be very well approximated by

$$P(l, 0|K) = (1 - \rho)^{lM} \tag{81}$$

where M is the packet length. The packet probabilities are derived by Applying $P(l, 0|K)(1 \le l \le m)$ to the moments method.

If we use the packet probabilities at the bit level derived in section 2, exact packet probabilities can be derived by changing (81) to

$$P(l, 0|K) = [P_{bit}(l, 0|K)]^M \tag{82}$$

6. APPROXIMATIONS BASED ON IROA

If we assume that the packet errors are independent among users, and apply the upper bound of single-receiver packet error probability $P(E)$ in [6], the approximation of multireceiver packet probabilities are

$$P_{iid}(l, m-l|K) = \binom{m}{l}(1 - P(E))^l P(E)^{m-l} \tag{83}$$

where

$$1 - P(E) = [1 - P_u(\rho)]^M \tag{84}$$

In (84), $P_u(\rho)$ is the union bound discussed by Viterbi [8],

$$P_u(\rho) = \sum_{d=d_{free}}^{\infty} a_d P_d(\rho) \tag{85}$$

where

$$P_d(\rho) = \begin{cases} \sum_{k=0}^{(d-1)/2} \binom{d}{k} \rho^k (1-\rho)^{d-k}, & \text{if } d = odd \\ \frac{1}{2}\binom{d}{d/2}\rho^{d/2}(1-\rho)^{d/2} + \sum_{k=0}^{d/2-1}\binom{d}{k}\rho^k(1-\rho)^{d-k} & \text{if } d = even \end{cases} \tag{86}$$

a_d is determined by the generating function (64), and ρ is obtained by (55).

7. NUMERICAL RESULTS

In this section we present extensive comparisons between the exact expressions (whenever available) and the approximations.

The first group of results (Tables 1 to 5) is concerned with synchronous uncoded DS/SSMA systems. Multireception probabilities at the bit level are compared since for those the exact expressions have been obtained. Packet probabilities of systems with geographically dispersed receivers are shown in Tables 1 to 4. Then Table 5 shows results for systems with co-located receivers. The exact results for $P(l, m-l|K)$ are compared to those obtained via the multivariate Gaussian and the IROA approximations.

In Tables 1 and 3 E_b/N_0 varies while all other parameters (K, m, N) are held constant. Tables 2 and 4 illustrate the comparisons when (m, K) vary, for fixed values of N and E_b/N_0. From the results it becomes clear that the IROA approximation is superior to the Gaussian approximation. This is because the covariance of the interference present at two receivers (during the same bit interval, is $1/N$, usually a small number. According to our results, even when $N = 31$ the IROA approximation is very accurate, and the covariance is negligible. Consequently, the IROA approximation, which requires the least computational effort, is a sufficiently accurate approximation for any range of parameters. Our results also show that the accuracy of the multivariate Gaussian approximation improves as N and K become larger.

In Table 5 we present the comparisons for systems with co-located receivers. The Gaussian approximation is better than the IROA approximation when E_b/N_0 is small. However, as E_b/N_0 increases, the IROA approximation yields more accurate results than the Gaussian approximation. Consider the covariance b', which is the sum of $\frac{1}{N}$ and $\frac{N_0}{2E_b}$. If E_b/N_0 is small such that $\frac{N_0}{2E_b} >> \frac{1}{N}$, then $\frac{N_0}{2E_b}$ dominates, the receivers are highly correlated and the IROA is not appropriate. In the opposite case where b is dominant, IROA is the better approximation.

From Tables 1 to 5, we also notice that, the larger l is, the smaller the relative error of the $P(l, m-l|K)$ is. Since we are usually interested in the multireception packet probabilities for large l, the approximations (in particular the IROA) are appropriate.

The second group of results (Tables 6 to 8) is concerned with synchronous coded DS/SSMA systems; convolutional codes with Viterbi decoding and hard-decisions are employed by the DS/SSMA system.

In Tables 6 and 7 we present the JFEEA of the multireception packet probabilities for coded systems in which rate 1/2 convolutional codes are used. The relative errors in these tables are computed by

$$Error = \frac{|IROA - JFEEA|}{(JFEEA + IROA)/2} * 100$$

The results obtained by JFEEA and those computed based on the IROA are almost identical. Since the two approximation techniques were derived following completely different methodologies and independent assumptions, we interpret the observed closeness to each other as a sign of common accuracy. Of course true verification/validation can only be achieved via comparisons with the exact expressions, unfortunately these are completely out of reach for the coded systems. In this context we feel sufficiently confident to claim that the IROA has satisfactory accuracy for approximating the multireception probabilities of the coded DS/SSMA systems as well.

In Table 8 we compare the packet probabilities of coded and uncoded systems. It is evident that the performance level of the uncoded system is not acceptable and convolutional coding is absolutely necessary. For the uncoded system, we compared the exact result and that obtained from the IROA. It shows that the difference between them are negligible. Therefore, the errors caused by the correlation between users in the coded system are also negligible.

8. CONCLUSIONS AND DISCUSSION

For DS/SSMA systems we derived the exact expressions for the multireception probabilities $P(l, m-l|K)$ at the bit level (given that packet length equals one); this is the only case that these expressions can be computed and provides a basis for comparisons of all approximations considered. The effects of AWGN were taken into account. For systems with geographically dispersed receivers, we conclude that the IROA approximation is sufficiently accurate. For systems with co-located receivers, the multivariate Gaussian approximation gives better results when E_b/N_0 is small. But as the E_b/N_0 increases, the IROA approximation becomes superior. We also observed that the closer l is to m, the smaller the relative error is. So in most cases we have interest, the approximations are close enough to the exact values. For the convolutional coded systems with Viterbi decoding, the packet probabilities obtained by JFEEA and that computed from IROA are very close. We conclude that the IROA approximation is sufficiently accurate for coded systems as well.

The results of this report find applications in all our current work on CDMA networks. They are used for comparison of the performance of admission policies for voice and data traffic in CDMA networks when threshold and graceful degradation models are used. (refer to [10]). Also in our work on networks of LEO satellites using CDMA (refer to [11]-[12]) $p(m|k)$ plays an important role. The same is true for the derivation of dynamic CDMA code allocation schemes in [13]. Finally, in [14] we extend some of the work in this report to CDMA systems with multi-rate traffic (this finds applications in PCS communications and wireless video).

Moreover, the work in this report opens the way for using the graceful degradation model, which results in more realistic assessment of the CDMA multiple-access capability, in CDMA networks. The work of other researchers in the CDMA area will also benefit.

Finally, the effects of channel fading, signal shadowing, and power control must be incorporated into our analysis to enable the use of our findings to real-life CDMA networks. This can be done at the expense of a moderately large analytical and computational effort. Actually, we expect that the IROA will be valid within some range of CDMA system (network) and channel (fading, shadowing) parameters but it may be difficult to determine this range without further work.

APPENDIX A

Figures 2 and 3 are two examples for the derivation of equations (4) to (14), given K is odd and even, respectively. We list all possible K-dimensional vectors that $\mathbf{X}$ can take, where the value of X_0 is assumed to be always -1 since the symmetry of the problem; hence the probability that $\mathbf{X}$ takes any of these vectors is 2^{1-K}. The corresponding interference vector $\mathbf{I}$ are given and rearranged below $\mathbf{X}$ in the two figures. We found that they can be categorized into three groups, denoted by Group 1, Group 2, and Group 3. Group 1 consists a single vector $\mathbf{I}^0 = V_0$; (9), (10) and the term "1" in (4) correspond to this vector. Group 2, which exists only when K is even, contains $\frac{1}{2}\binom{K}{K/2}$ identical vectors with all components -1; (13), (14) and the term "$E(K)$" in (4) correspond to this vector.

All the remaining vectors are belong to Group 3. These interference vectors are again categorized into $\sum_{i=1}^{\alpha(K)} \sum_{j=b_i}^{e_i}$ subgroups (see (6)-(8)). In subgroup ij, each vector is a permutation of V_{ij} (see(11)), where i determines the values of E_{i0} and E_{i1}, and j determines the number of E_{i0}s in V_{ij}. Consider the vector $\mathbf{I}^* = (I_0, I_1, \cdots, I_{K-1})$ in subgroup ij (for all j); it consists i entries taking value E_{i0} and $K-i$ entries taking value E_{i1}. If $K-i \geq m$, vector $\mathbf{I} = (I_0, I_1, \cdots, I_{m-1})$ can have all its entries being E_{i1}. If $K-i < m$, at least $m-(K-i)$ entries of $\mathbf{I}$ have to be E_{i0}. This explains (7). Since the number of available E_{i0}s in $\mathbf{I}^*$ is i and the dimension of $\mathbf{I}$ is m, we can not have any vector with more than $min(i,m)$ entries being E_{i0}. This explains (8). The probability that a vector in subgroup ij happens is determined by the number of permutations of the set with $i-j$ E_{i0}s and $K-m-i+j$ E_{i1}s and is given in (12).

APPENDIX B

For a synchronous system, the independence of x_{ij}^ns for fixed i, j is clear; they are i.i.d. distributed and each of them takes values $+1$ or -1 with equal probability.

For the asynchronous case (Figure 4), let τ_{ij} be the time delay between user i and user j, and ϕ_{ij} be the phase difference between user i and user j. τ_{ij} is assumed to be uniformly distributed over $[0, T_c)$. ϕ_{ij} is in general a function of the phase of the interfering transmission and the delay, but it has been shown that ϕ_{ij} *mod* 2π is statistically independent of τ_{ij} and assumed to be uniformly distributed over $[0, 2\pi)$.

Define $r_{ij} = \frac{\tau_{ij}}{T_c}$; then r_{ij} is uniformly distributed over $[0, 1)$ and the distribution of x_{ij}^n is

$$\begin{cases} Pr(x_{ij}^n = 1) = \frac{1}{4} \\ Pr(x_{ij}^n = -1) = \frac{1}{4} \\ Pr(x_{ij}^n = 1 - 2r_{ij}) = \frac{1}{4} \\ Pr(x_{ij}^n = 2r_{ij} - 1) = \frac{1}{4} \end{cases}$$

The derivation of the above equation is explained in Figure 4, in which $c_l^n (0 \leq l \leq K-1, 1 \leq n \leq N)$ denotes the chip value of user l in the nth chip interval. Since τ_{ij} and ϕ_{ij} are fixed during the bit interval, $x_{i,j}^n$s are independent. In Figure 4 note that

$$\begin{cases} P(x_{ij}^2 = x_0 | c_i^2 = 1) = P(x_{ij}^2 = x_0 | c_i^2 = -1) \\ P(x_{ij}^3 = x_1 | c_i^2 = 1) = P(x_{ij}^3 = x_1 | c_i^2 = -1) \end{cases}$$

and

$$\begin{cases} P(x_{ij}^2 = x_0) = \frac{1}{2} P(x_{ij}^2 = x_0 | c_i^2 = 1) + \frac{1}{2} P(x_{ij}^2 = x_0 | c_i^2 = -1) \\ P(x_{ij}^3 = x_1) = \frac{1}{2} P(x_{ij}^3 = x_1 | c_i^2 = 1) + \frac{1}{2} P(x_{ij}^3 = x_1 | c_i^2 = -1) \end{cases}$$

so

$$\begin{cases} P(x_{ij}^2 = x_0) = P(x_{ij}^2 = x_0 | c_i^2 = 1) = P(x_{ij}^2 = x_0 | c_i^2 = -1) \\ P(x_{ij}^3 = x_1) = P(x_{ij}^3 = x_1 | c_i^2 = 1) = P(x_{ij}^3 = x_1 | c_i^2 = -1) \end{cases}$$

and

$$\begin{aligned} & P(x_{ij}^2 = x_0, x_{ij}^3 = x_1) \\ = \; & \frac{1}{2} P(x_{ij}^2 = x_0, x_{ij}^3 = x_1 | c_i^2 = 1) + \frac{1}{2} P(x_{ij}^2 = x_0, x_{ij}^3 = x_1 | c_i^2 = -1) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}P(x_{ij}^2 = x_0 | c_i^2 = 1)P(x_{ij}^3 = x_1 | c_i^2 = 1) + \\
&\quad \frac{1}{2}P(x_{ij}^2 = x_0 | c_i^2 = -1)P(x_{ij}^3 = x_1 | c_i^2 = -1) \\
&= P(x_{ij}^2 = x_0)P(x_{ij}^3 = x_1)
\end{aligned}
$$

This proves the independence of x_{ij}^ns for fixed i, j.

APPENDIX C

The method developed in Section 3 for approximating the probability $P(l, m-l|K)$ requires only knowledge of the three quantities μ, a and b. These quantities are the mean of the total interference a user suffers from all the other $K-1$ users in a bit duration and the diagonal and off-diagonal terms of the covariance matrix Σ [see (34) - (37)]. Here we evaluate these quantities for both synchronous and asynchronous systems.

For a synchronous system, $x_{ij}^n (1 \leq n \leq N, 0 \leq i \leq K-1, 0 \leq j \leq m-1, i \neq j)$ (the interference term that user i has over user j during the nth chip of the bit) takes values $+1$ or -1 with equal probability. The normalized interference that user j suffers from all the other $K-1$ users in a bit duration, x_j, is given in (31). And μ becomes

$$\mu = E(x_j) = \frac{1}{N} \sum_{\substack{i=0 \\ i \neq j}}^{K-1} \sum_{n=1}^{N} E[x_{ij}^n] = 0 \tag{87}$$

Since $x_{ij} = \sum_{n=1}^{N} x_{ij}^n$ and $x_{kl} = \sum_{n=1}^{N} x_{kl}^n$ are independent as long as $(i \neq l \vee j \neq k)$ and $(i \neq k \vee j \neq l)$, and for fixed i, j, x_{ij}^ns are independent, the variance a is

$$\begin{aligned} a &= \frac{1}{N^2} E\left[\left(\sum_{\substack{i=0 \\ i \neq j}}^{K-1} \sum_{n=1}^{N} x_{ij}^n\right)^2\right] = \frac{1}{N^2} \sum_{\substack{i=0 \\ i \neq j}}^{K-1} E\left[\left(\sum_{n=1}^{N} x_{ij}^n\right)^2\right] \\ &= \frac{1}{N^2} \sum_{\substack{i=0 \\ i \neq j}}^{K-1} \sum_{n=1}^{N} E\left[\left(x_{ij}^n\right)^2\right] = \frac{K-1}{N} \end{aligned} \tag{88}$$

and the covariance b is

$$\begin{aligned} b &= \frac{1}{N^2} E\left[\left(\sum_{\substack{i=0 \\ i \neq j}}^{K-1} \sum_{n=1}^{N} x_{ij}^n\right)\left(\sum_{\substack{i=0 \\ i \neq j'}}^{K-1} \sum_{n=1}^{N} x_{ij'}^n\right)\right] \\ &= \frac{1}{N^2} E\left[\left(\sum_{n=1}^{N} x_{j'j}^n\right)\left(\sum_{n=1}^{N} x_{jj'}^n\right)\right] = \frac{1}{N^2} \sum_{n=1}^{N} E\left[x_{j'j}^n x_{jj'}^n\right] = \frac{1}{N} \end{aligned} \tag{89}$$

For the asynchronous case, the normalized interference that user j suffers from all the other $K-1$ users in a single bit duration is

$$x_j = \frac{1}{N} \sum_{\substack{i=0 \\ i \neq j}}^{K-1} \left(\sum_{n=1}^{N} x_{ij}^n \right) cos\phi_{i,j} \tag{90}$$

where ϕ_{ij} is the phase difference between user i and user j and is defined in Appendix B. In Appendix B, we also proved that x_{ij}^ns are independent for fixed i, j. So the mean value of x_j is

$$\mu = E\left[\frac{1}{N} \sum_{\substack{i=0 \\ i \neq j}}^{K-1} \left(\sum_{n=1}^{N} x_{ij}^n \right) cos\phi_{ij} \right] = \frac{1}{N} \sum_{\substack{i=0 \\ i \neq j}}^{K-1} \sum_{n=1}^{N} E(x_{ij}^n) E(cos\phi_{ij}) = 0 \tag{91}$$

Note that for fixed j, $i \neq i'$, and any n, n', the random variables x_{ij}^n and $x_{i'j}^{n'}$ are independent; and for fixed i, j and $n \neq n'$, the random variables x_{ij}^n and $x_{ij}^{n'}$ are independent. The variance a is

$$\begin{aligned} a &= \frac{1}{N^2} E\left[\left(\sum_{\substack{i=0 \\ i \neq j}}^{K-1} \sum_{n=1}^{N} x_{ij}^n cos\phi_{ij} \right)^2 \right] = \frac{1}{N^2} \sum_{\substack{i=0 \\ i \neq j}}^{K-1} E\left[\left(\sum_{n=1}^{N} x_{ij}^n cos\phi_{ij} \right)^2 \right] \\ &= \frac{1}{N^2} \sum_{\substack{i=0 \\ i \neq j}}^{K-1} \sum_{n=1}^{N} E\left[\left(x_{i,}^n \right)^2 \right] E\left[cos^2\phi_{ij} \right] = \frac{K-1}{3N} \end{aligned} \tag{92}$$

and covariance b is

$$\begin{aligned} b &= \frac{1}{N^2} E\left[\left(\sum_{\substack{i=0 \\ i \neq j}}^{K-1} \sum_{n=1}^{N} x_{ij}^n cos\phi_{ij} \right) \left(\sum_{\substack{i=0 \\ i \neq j'}}^{K-1} \sum_{n=1}^{N} x_{ij'}^n cos\phi_{ij'} \right) \right] \\ &= \frac{1}{N^2} E\left[\left(cos\phi_{j'j} \sum_{n=1}^{N} x_{j'j}^n \right) \left(cos\phi_{jj'} \sum_{n=1}^{N} x_{jj'}^n \right) \right] \\ &= \frac{1}{N^2} E\left[cos\phi_{j'j} cos\phi_{jj'} \right] \sum_{n=1}^{N} E\left[x_{j'j}^n x_{jj'}^n \right] = \frac{1}{3N} \end{aligned} \tag{93}$$

APPENDIX D

In (41), Σ takes the form

$$\Sigma = \begin{bmatrix} a & b & b & \cdots & b \\ b & a & & \vdots & \vdots \\ b & \vdots & a & \vdots & \vdots \\ \vdots & \vdots & & \ddots & b \\ b & b & \cdots & b & a \end{bmatrix}_{m\times m} = \begin{bmatrix} a-b & & 0 \\ & \ddots & \\ 0 & & a-b \end{bmatrix}_{m\times m} + \begin{bmatrix} b & \cdots & b \\ \vdots & \ddots & \vdots \\ b & \cdots & b \end{bmatrix}_{m\times m}$$

and, consequently,

$$\Sigma = A + \underline{u}\underline{u}^T$$

where $\underline{u}^T = \sqrt{b}[11...1]_{1\times m}$ and $A = (a-b)I$. Moreover,

$$\Sigma^{-1} = A^{-1} - \frac{(A^{-1}\underline{u})(\underline{u}^T A^{-1})}{1 + \underline{u}^T A^{-1}\underline{u}} = \frac{1}{a-b}I - \frac{\underline{u}\,\underline{u}^T}{(a-b)(a+(m-1)b)}$$

so that, if we define

$$\overline{a} = \frac{1}{a-b} \qquad \overline{b} = \frac{b}{(a-b)(a+(m-1)b)}$$

then

$$\Sigma^{-1} = \overline{a}I - \begin{bmatrix} \overline{b} & \cdots & \overline{b} \\ \vdots & \ddots & \vdots \\ \overline{b} & \cdots & \overline{b} \end{bmatrix}_{m\times m} \triangleq \overline{a}I - \overline{B}$$

Consider the exponent of the integrand in (41). After a shift of variables to account for the mean, it can be simplified to

$$-\frac{1}{2}\underline{x}^T\Sigma^{-1}\underline{x} = -\frac{1}{2(a-b)}\left[\sum_{j=0}^{m-1} x_j{}^2 - \frac{b}{a+(m-1)b}\left(\sum_{j=0}^{m-1} x_j\right)^2\right]$$

One way to diagonalize Σ^{-1} is to find its eigenvalues and the corresponding eigenvectors and then create a transformation matrix with the eigenvectors as its columns. The eigenvalues of Σ^{-1} are given by the equation

$$det(\lambda I - \Sigma^{-1}) = 0$$

But $\lambda I - \Sigma^{-1} = (\lambda - \overline{a})I + \overline{B}$, where B is of rank 1, and

$$\begin{aligned} det(\lambda I - \Sigma^{-1}) &= (\lambda - \overline{a})^{m-1} det(I + \frac{1}{(\lambda - \overline{a})}\overline{B}) \\ &= (\lambda - \overline{a})^{m-1}[1 + \frac{m\overline{b}}{\lambda - \overline{a}}] = (\lambda - \overline{a})^{m-1}(\lambda - (\overline{a} - m\overline{b})) \end{aligned}$$

Hence

$$\lambda_1 = \lambda_2 = \cdots = \lambda_{m-1} = \overline{a} \triangleq \lambda_a$$

and

$$\lambda_m = \overline{a} - m\overline{b} \triangleq \lambda_b$$

where

$$\lambda_a = \frac{1}{a-b} \qquad \lambda_b = \frac{1}{a + (m-1)b}$$

Finally, notice that

$$det(\Sigma) = (a-b)^{m-1}[a + (m-1)b]$$

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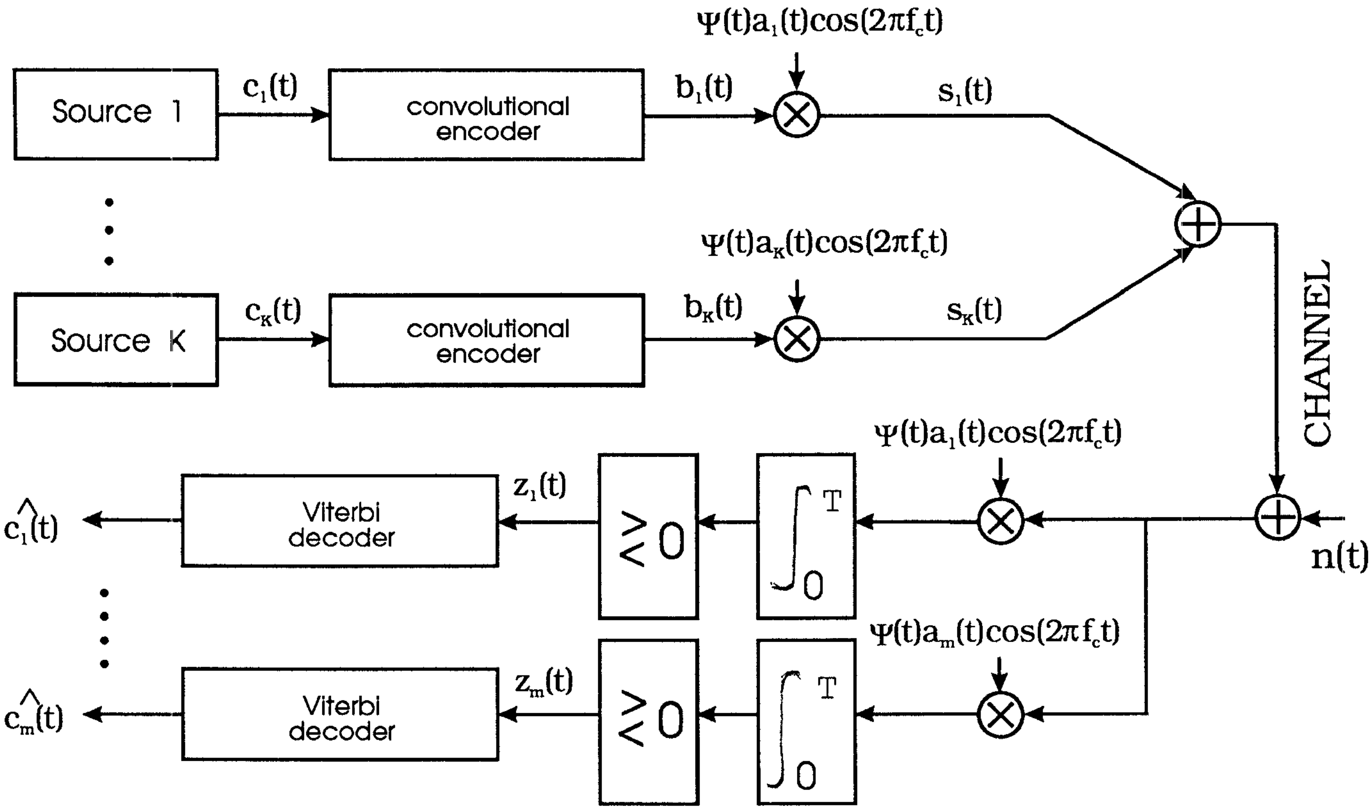


Figure 1: A multiple access system using direct-sequence spread spectrum and convolutional code with hard-decision Viterbi decoding

$$
\begin{array}{lrrrrrrrrrrrrrrrr}
X_0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
X_1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
X_2 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\
X_3 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
X_4 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1
\end{array}
$$

$$
\begin{array}{lrrrrrrrrrrrrrrrr}
I_0 & 4 & 2 & 2 & 0 & 2 & 0 & 0 & -2 & 2 & 0 & 0 & -2 & 0 & -2 & -2 & -4 \\
I_1 & 4 & 2 & 2 & 0 & 2 & 0 & 0 & -2 & -4 & -2 & -2 & 0 & -2 & 0 & 0 & 2 \\
I_2 & 4 & 2 & 2 & 0 & -4 & -2 & -2 & 0 & 2 & 0 & 0 & -2 & -2 & 0 & 0 & 2 \\
I_3 & 4 & 2 & -4 & -2 & 2 & 0 & -2 & 0 & 2 & 0 & -2 & 0 & 0 & -2 & 0 & 2 \\
I_4 & 4 & -4 & 2 & -2 & 2 & -2 & 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 & -2 & 2
\end{array}
$$

group	3					1
i	$i = 1$		$i = 2 = \alpha(K)$			
$\binom{m}{j}$	$\binom{3}{0}$	$\binom{3}{1}$	$\binom{3}{0}$	$\binom{3}{1}$	$\binom{3}{2}$	1
I_0	2	-4 2 2	0	-2 0 0	-2 -2 0	4
I_1	2	2 -4 2	0	0 -2 0	-2 0 -2	4
I_2	2	2 2 -4	0	0 0 -2	0 -2 -2	4
I_3	2 -4	2	-2	-2 0	0	4
I_4	-4 2	2	-2	0 -2	0	4
$\binom{K-m}{i-j}$	$\binom{2}{1}$	$\binom{2}{0}$	$\binom{2}{2}$	$\binom{2}{1}$	$\binom{2}{0}$	1

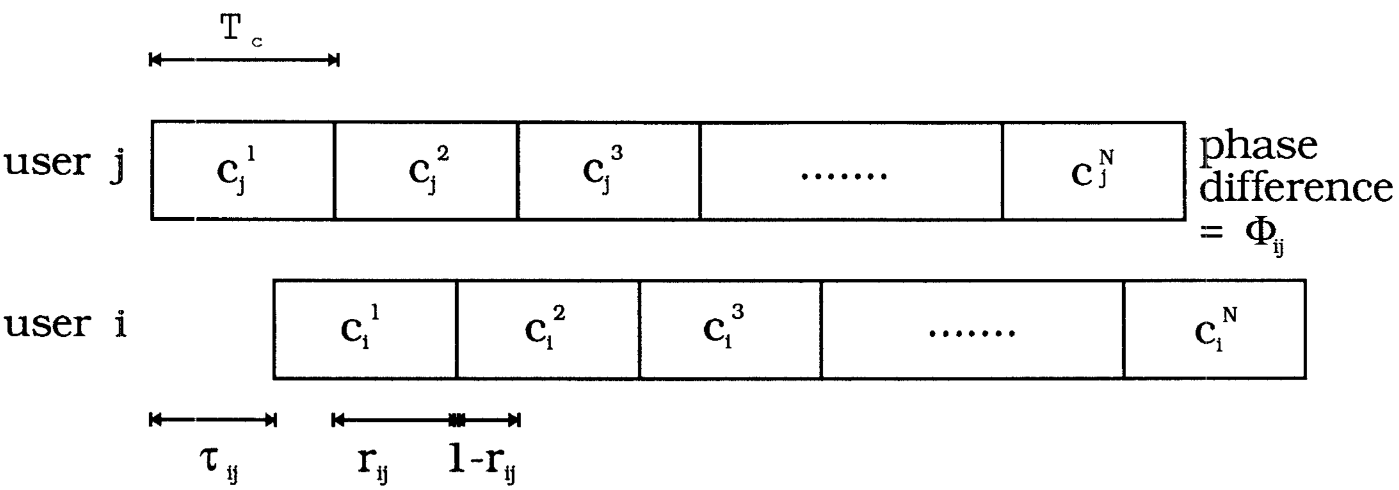
Figure 2: An example of deriving $P(l, m-l|K)$ in a perfectly synchronous system with K odd; $K = 5$. $m = 3$.

$$
\begin{array}{lrrrrrrrr}
X_0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
X_1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\
X_2 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 \\
X_3 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1
\end{array}
$$

$$
\begin{array}{lrrrrrrrr}
I_0 & 3 & 1 & 1 & -1 & 1 & -1 & -1 & -3 \\
I_1 & 3 & 1 & 1 & -1 & -3 & -1 & -1 & 1 \\
I_2 & 3 & 1 & -3 & -1 & 1 & -1 & -1 & 1 \\
I_3 & 3 & -3 & 1 & -1 & 1 & -1 & -1 & 1
\end{array}
$$

group	3		2	1
i	$i = 1 = \alpha(K)$			
$\binom{m}{j}$	$\binom{2}{0}$	$\binom{2}{1}$	$E(K) = 1$	1
I_0	1	1 -3	-1	3
I_1	1	-3 1	-1	3
I_2	1 -3	1	-1	3
I_3	-3 1	1	-1	3
$\binom{K-m}{i-j}$	$\binom{2}{1}$	$\binom{2}{0}$	$\frac{\binom{K}{K/2}}{2}$	1

Figure 3: An example of deriving $P(l, m - l|K)$ in a perfectly synchronous system with K even; $K = 4$. $m = 2$.



prob	c_j^2	c_i^1	c_i^2	x_{ij}^2
1/8	-1	-1	-1	1
1/8	-1	-1	1	$2r_{ij} - 1$
1/8	-1	1	-1	$1 - 2r_{ij}$
1/8	-1	1	1	-1
1/8	1	-1	-1	-1
1/8	1	-1	1	$1 - 2r_{ij}$
1/8	1	1	-1	$2r_{ij} - 1$
1/8	1	1	1	1

Figure 4: The interference in an asynchronous DS/SS system

Table 1

Packet probabilities of uncoded systems computed using exact, Gaussian and IROA models with packet length = 1, $N = 31$, $(m, K) = (2, 6)$, and $E_b/N_0 = 10, 12, 14, 16$

(a) $E_b/N_0 = 10$

$P(l, m-l\|K)$	Exact Value	Gaussian		IROA	
		Approx.	Error (%)	Approx.	Error (%)
$P(0,2\|6)$	0.00019027	0.00052210	1.7E+02	0.00021761	14.
$P(1,1\|6)$	0.029123	0.028548	-2.0	0.029068	-0.19
$P(2,0\|6)$	0.97069	0.97093	0.025	0.97071	0.0028

(b) $E_b/N_0 = 12$

$P(l, m-l\|K)$	Exact Value	Gaussian		IROA	
		Approx.	Error (%)	Approx.	Error (%)
$P(0,2\|6)$	9.4084E-05	0.00035618	2.8E+02	0.00012838	36.
$P(1,1\|6)$	0.022473	0.022061	-1.8	0.022404	-0.31
$P(2,0\|6)$	0.97743	0.97758	0.015	0.97747	0.0035

(c) $E_b/N_0 = 14$

$P(l, m-l\|K)$	Exact Value	Gaussian		IROA	
		Approx.	Error (%)	Approx.	Error (%)
$P(0,2\|6)$	5.3540E-05	0.00027113	4.1E+02	8.7273E-05	63.
$P(1,1\|6)$	0.018577	0.018270	-1.7	0.018509	-0.36
$P(2,0\|6)$	0.98137	0.98146	0.0091	0.98140	0.0034

(d) $E_b/N_0 = 16$

$P(l, m-l\|K)$	Exact Value	Gaussian		IROA	
		Approx.	Error (%)	Approx.	Error (%)
$P(0,2\|6)$	3.5013E-05	0.00022492	5.4E+02	6.6674E-05	90.
$P(1,1\|6)$	0.016261	0.016019	-1.5	0.016198	-0.39
$P(2,0\|6)$	0.98370	0.98376	0.0053	0.98374	0.0032

Table 2

Packet probabilities of uncoded systems computed using exact, Gaussian and IROA models with packet length = 1, $N = 31$, $E_b/N_0 = 10$, and different (m, K)s

(a) $(m, K) = (2, 3)$

$P(l, m-l \mid K)$	Exact Value	Gaussian		IROA	
		Approx.	Error (%)	Approx.	Error (%)
$P(0,2\|3)$	8.0756E-06	2.6503E-05	2.3E+02	2.3296E-06	-71.
$P(1,1\|3)$	0.0030365	0.0030731	1.2	0.0030480	0.38
$P(2,0\|3)$	0.99696	0.99690	-0.0055	0.99695	-0.00057

(b) $(m, K) = (2, 4)$

$P(l, m-l \mid K)$	Exact Value	Gaussian		IROA	
		Approx.	Error (%)	Approx.	Error (%)
$P(0,2\|4)$	2.6958E-05	9.9920E-05	2.7E+02	2.0006E-05	-26.
$P(1,1\|4)$	0.0088917	0.0088489	-0.48	0.0089055	0.16
$P(2,0\|4)$	0.99108	0.99105	-0.0031	0.99107	-0.00070

(c) $(m, K) = (2, 5)$

$P(l, m-l \mid K)$	Exact Value	Gaussian		IROA	
		Approx.	Error (%)	Approx.	Error (%)
$P(0,2\|5)$	7.8821E-05	0.00025694	2.3E+02	8.1036E-05	2.8
$P(1,1\|5)$	0.017846	0.017595	-1.4	0.017842	-0.025
$P(2,0\|5)$	0.98207	0.98215	0.0075	0.98208	0.00023

(d) $(m, K) = (3, 4)$

$P(l, m-l \mid K)$	Exact Value	Gaussian		IROA	
		Approx.	Error (%)	Approx.	Error (%)
$P(0,3\|4)$	2.3677E-07	5.5100E-06	2.2E+03	8.9481E-08	-62.
$P(1,2\|4)$	8.0165E-05	0.00028323	2.5E+02	5.9749E-05	-25.
$P(2,1\|4)$	0.013257	0.012990	-2.0	0.013299	0.31
$P(3,0\|4)$	0.98666	0.98672	0.0060	0.98664	-0.0021

(e) $(m, K) = (3, 5)$

$P(l, m-l \mid K)$	Exact Value	Gaussian		IROA	
			Error (%)		Error (%)
$P(0,3\|5)$	8.3708E-07	1.5100E-05	1.7E+03	7.2949E-07	-13.
$P(1,2\|5)$	0.00023395	0.00072551	2.1E+02	0.00024092	3.0
$P(2,1\|5)$	0.026536	0.025667	-3.3	0.026522	-0.051
$P(3,0\|5)$	0.97323	0.97359	0.037	0.97324	0.00070

Table 3

Packet probabilities of uncoded systems computed using exact, Gaussian and IROA models with packet length = 1, $N = 63$, $(m, K) = (2, 12)$, and $E_b/N_0 = 10, 12, 14, 16$

(a) $E_b/N_0 = 10$

$P(l, m-l\|K)$	Exact Value	Gaussian		IROA	
		Approx.	Error (%)	Approx.	Error (%)
$P(0,2\|12)$	0.00027542	0.00045670	66.	0.00030337	10.
$P(1,1\|12)$	0.034284	0.033941	-1.0	0.034228	-0.16
$P(2,0\|12)$	0.96544	0.96560	0.017	0.96547	0.0029

(b) $E_b/N_0 = 12$

$P(l, m-l\|K)$	Exact Value	Gaussian		IROA	
		Approx.	Error (%)	Approx.	Error (%)
$P(0,2\|12)$	0.00016172	0.00030609	89.	0.00019055	18.
$P(1,1\|12)$	0.027285	0.027021	-0.97	0.027227	-0.21
$P(2,0\|12)$	0.97255	0.97267	0.012	0.97258	0.0030

(c) $E_b/N_0 = 14$

$P(l, m-l\|K)$	Exact Value	Gaussian		IROA	
		Approx.	Error (%)	Approx.	Error (%)
$P(0,2\|12)$	0.00010884	0.00022990	1.1E+02	0.00013615	25.
$P(1,1\|12)$	0.023119	0.022905	-0.92	0.023064	-0.24
$P(2,0\|12)$	0.97677	0.97686	0.0095	0.97680	0.0028

(d) $E_b/N_0 = 16$

$P(l, m-l\|K)$	Exact Value	Gaussian		IROA	
		Approx.	Error (%)	Approx.	Error (%)
$P(0,2\|12)$	8.2269E-05	0.00018890	1.3E+02	0.00010790	31.
$P(1,1\|12)$	0.020610	0.020428	-0.88	0.020559	-0.25
$P(2,0\|12)$	0.97931	0.97938	0.0077	0.97933	0.0026

Table 4

Packet probabilities of uncoded systems computed using exact, Gaussian and IROA models with packet length = 1, $N = 63$, $E_b/N_0 = 12$, and different (m, K)s

(a) $(m, K) = (2, 8)$

$P(l, m-l\|K)$	Exact Value	Gaussian		IROA	
		Approx.	Error (%)	Approx.	Error (%)
$P(0,2\|8)$	1.3411E-05	3.9956E-05	2.0E+02	1.6312E-05	22.
$P(1,1\|8)$	0.0080508	0.0080270	-0.30	0.0080450	-0.072
$P(2,0\|8)$	0.99194	0.99193	-0.00028	0.99194	0.00029

(b) $(m, K) = (2, 9)$

$P(l, m-l\|K)$	Exact Value	Gaussian		IROA	
		Approx.	Error (%)	Approx.	Error (%)
$P(0,2\|9)$	2.9506E-05	7.5774E-05	1.6E+02	3.5941E-05	22.
$P(1,1\|9)$	0.011931	0.011869	-0.52	0.011918	-0.11
$P(2,0\|9)$	0.98804	0.98806	0.0016	0.98805	0.00065

(c) $(m, K) = (2, 10)$

$P(l, m-l\|K)$	Exact Value	Gaussian		IROA	
		Approx.	Error (%)	Approx.	Error (%)
$P(0,2\|10)$	5.7113E-05	0.00012983	1.3E+02	6.8992E-05	21.
$P(1,1\|10)$	0.016498	0.016382	-0.70	0.016474	-0.14
$P(2,0\|10)$	0.98344	0.98349	0.0044	0.98346	0.0012

(d) $(m, K) = (2, 11)$

$P(l, m-l\|K)$	Exact Value	Gaussian		IROA	
		Approx.	Error (%)	Approx.	Error (%)
$P(0,2\|11)$	0.00010000	0.00020564	1.1E+02	0.00011936	19.
$P(1,1\|11)$	0.021650	0.021467	-0.85	0.021612	-0.18
$P(2,0\|11)$	0.97825	0.97833	0.0080	0.97827	0.0020

(e) $(m, K) = (3, 4)$

$P(l, m-l\|K)$	Exact Value	Gaussian		IROA	
		Approx.	Error (%)	Approx.	Error (%)
$P(0,3\|4)$	5.0049E-11	3.9929E-09	7.9E+03	6.4060E-12	-87.
$P(1,2\|4)$	2.0532E-07	1.2475E-06	5.1E+02	1.0346E-07	-50.
$P(2,1\|4)$	0.00055677	0.00056650	1.7	0.00055696	0.034
$P(3,0\|4)$	0.99944	0.99943	-0.0011	0.99944	-9.0E-06

Table 5

Packet probabilities of systems withed co-located receivers computed using exact, Gaussian and IROA models with $N = 63$, different E_b/N_0s and (m, K)s

(a) Packet probabilities for $(m, K) = (2, 3)$ and $E_b/N_0 = 10, 14$

$E_b/N_0 = 10$					
$P(l, m-l \mid K)$	Exact Value	Gaussian		IROA	
		Approx.	Error (%)	Approx.	Error (%)
$P(0,2\|3)$	3.9966E-05	5.1719E-05	29.	5.3793E-08	-1.0E+02
$P(1,1\|3)$	0.00038395	0.00036608	-4.7	0.00046376	21.
$P(2,0\|3)$	0.99958	0.99958	0.00061	0.99954	-0.0040

$E_b/N_0 = 14$					
$P(l, m-l \mid K)$	Exact Value	Gaussian		IROA	
		Approx.	Error (%)	Approx.	Error (%)
$P(0,2\|3)$	8.3406E-08	2.7247E-07	2.3E+02	2.4740E-11	-1.0E+02
$P(1,1\|3)$	9.7818E-06	1.0277E-05	5.1	9.9478E-06	1.7
$P(2,0\|3)$	0.99999	0.99999	-6.8E-05	0.99999	-8.3E-06

(b) Packet probabilities for $(m, K) = (2, 12)$ and $E_b/N_0 = 10, 14$

$E_b/N_0 = 10$					
$P(l, m-l \mid K)$	Exact Value	Gaussian		IROA	
		Approx.	Error (%)	Approx.	Error (%)
$P(0,2\|12)$	0.00096942	0.0013022	34.	0.00030337	-69.
$P(1,1\|12)$	0.032896	0.032250	-2.0	0.034228	4.0
$P(2,0\|12)$	0.96613	0.96645	0.032	0.96547	-0.069

$E_b/N_0 = 14$					
$P(l, m-l \mid K)$	Exact Value	Gaussian		IROA	
		Approx.	Error (%)	Approx.	Error (%)
$P(0,2\|12)$	0.00023073	0.00040640	76.	0.00013615	-41.
$P(1,1\|12)$	0.022875	0.022552	-1.4	0.023064	0.83
$P(2,0\|12)$	0.97689	0.97704	0.015	0.97680	-0.0097

Table 6

Packet probabilities of coded systems computed using JFEEA and IROA models with $(m, K) = (3, 8)$, $N = 63$, $M = 1000$, $E_b/N_0 = 10$ and different constraint lengths

(a) Constraint Length = 6

$P(l, m-l\|K)$	JFEEA	IROA	Error(%)
$P(0,3\|8)$	2.2315E-14	2.2329E-14	0.060
$P(1,2\|8)$	2.3764E-09	2.3788E-09	0.10
$P(2,1\|8)$	8.4473E-05	8.4473E-05	5.6E-06
$P(3,0\|8)$	0.99992	0.99992	2.4E-10

(b) Constraint Length = 7

$P(l, m-l\|K)$	JFEEA	IROA	Error(%)
$P(0,3\|8)$	2.7756E-15	2.7863E-15	0.39
$P(1,2\|8)$	5.9342E-10	5.9402E-10	0.10
$P(2,1\|8)$	4.2213E-05	4.2213E-05	2.8E-06
$P(3,0\|8)$	0.99996	0.99996	5.9E-11

(c) Constraint Length = 8

$P(l, m-l\|K)$	JFEEA	IROA	Error(%)
$P(0,3\|8)$	7.1054E-15	6.9234E-15	2.6
$P(1,2\|8)$	1.0887E-09	1.0898E-09	0.10
$P(2,1\|8)$	5.7176E-05	5.7176E-05	3.8E-06
$P(3,0\|8)$	0.99994	0.99994	1.1E-10

(d) Constraint Length = 9

$P(l, m-l\|K)$	JFEEA	IROA	Error(%)
$P(0,3\|8)$	5.9952E-15	6.1764E-15	3.0
$P(1,2\|8)$	1.0089E-09	1.0099E-09	0.10
$P(2,1\|8)$	5.5041E-05	5.5041E-05	3.7E-06
$P(3,0\|8)$	0.99994	0.99994	1.0E-10

Table 7

Packet probabilities of coded systems computed using JFEEA and IROA models with $K = 12$, $N = 63$, $E_b/N_0 = 10$, $M = 1000$, constraint length = 6 and different ms

(a) $m = 3$

$P(l, m-l\|K)$	JFEEA	IROA	Error(%)
$P(0,3\|12)$	2.1906E-08	2.1968E-08	0.28
$P(1,2\|12)$	2.3444E-05	2.3466E-05	0.094
$P(2,1\|12)$	0.0083551	0.0083551	0.00053
$P(3,0\|12)$	0.99162	0.99162	2.2E-06

(b) $m = 4$

$P(l, m-l\|K)$	JFEEA	IROA	Error(%)
$P(0,4\|12)$	6.1289E-11	6.1527E-11	0.39
$P(1,3\|12)$	8.7380E-08	8.7628E-08	0.28
$P(2,2\|12)$	4.6757E-05	4.6800E-05	0.094
$P(3,1\|12)$	0.011109	0.011109	0.00079
$P(4,0\|12)$	0.98884	0.98884	4.5E-06

(c) $m = 5$

$P(l, m-l\|K)$	JFEEA	IROA	Error(%)
$P(0,5\|12)$	-	-	-
$P(1,4\|12)$	3.0339E-10	3.0677E-10	1.1
$P(2,3\|12)$	2.1784E-07	2.1846E-07	0.28
$P(3,2\|12)$	7.7710E-05	7.7782E-05	0.093
$P(4,1\|12)$	0.013847	0.013847	0.0011
$P(5,0\|12)$	0.98607	0.98607	7.5E-06

(d) $m = 6$

$P(l, m-l\|K)$	JFEEA	IROA	Error(%)
$P(0,6\|12)$	-	-	-
$P(1,5\|12)$	-	-	-
$P(2,4\|12)$	9.1920E-10	9.1775E-10	0.16
$P(3,3\|12)$	4.3446E-07	4.3569E-07	0.28
$P(4,2\|12)$	0.00011624	0.00011635	0.093
$P(5,1\|12)$	0.016570	0.016570	0.0013
$P(6,0\|12)$	0.98331	0.98331	1.1E-05

Table 8
Packet probabilities of coded and uncoded systems computed using JFEEA and IROA models with $N = 31$, $E_b/N_0 = 10$, $M = 1000$, constraint length = 6 and different (K, m)s

(a) $(K, m) = (4, 2)$

$P(l, m-l\|K)$	CODED(IROA)	UNCODED(EXACT)	UNCODED(IROA)
$P(0,2\|4)$	3.720769446E-13	0.9775254410	0.9775245442
$P(1,1\|4)$	1.219961460E-06	0.02234593619	0.02234772989
$P(2,0\|4)$	0.9999987800	0.0001286228078	0.0001277259569

(b) $(K, m) = (4, 3)$

$P(l, m-l\|K)$	CODED(IROA)	UNCODED(EXACT)	UNCODED(IROA)
$P(0,3\|4)$	2.269599047E-19	0.9664796216	0.9664769617
$P(1,2\|4)$	1.116230153E-12	0.03313745812	0.03314274748
$P(2,1\|4)$	1.829941073E-06	0.0003814461631	0.0003788473512
$P(3,0\|4)$	0.9999981701	1.474086790E-06	1.443506513E-06

(c) $(K, m) = (5, 2)$

$P(l, m-l\|K)$	CODED(IROA)	UNCODED(EXACT)	UNCODED(IROA)
$P(0,2\|5)$	6.823868745E-09	0.9997635304	0.9997635304
$P(1,1\|5)$	0.0001651997748	0.0002364556484	0.0002364555849
$P(2,0\|5)$	0.9998347934	1.394938883E-08	1.398111702E-08

(d) $(K, m) = (5, 3)$

$P(l, m-l\|K)$	CODED(IROA)	UNCODED(EXACT)	UNCODED(IROA)
$P(0,3\|5)$	5.636973550E-13	0.9996453165	0.9996453166
$P(1,2\|5)$	2.046991514E-08	0.0003546416293	0.0003546414390
$P(2,1\|5)$	0.0002477791922	4.184324145E-08	4.193839160E-08
$P(3,0\|5)$	0.9997522003	1.641679688E-12	1.653152073E-12