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A Quantitative Maximum Entropy Theorem for the Real Line

by

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0. Introduction.

We shall prove a deterministic and constructive maximum entropy theorem for the real line. The exact statement is given in Theorem 2.6. The essential feature of the theorem is an inequality of the form,

$$\int_{-\infty}^{\infty} \frac{\log G(\gamma)}{\pi(1+\gamma^2)}\, d\gamma \le \int_{-\infty}^{\infty} \frac{\log S(\gamma)}{\pi(1+\gamma^2)}\, d\gamma, \tag{0.1}$$

where S is a computable non-negative integrable function on the real line which extends given continuous data on an interval and where G is any one of a large class of functions extending the same data.

The classical maximum entropy theorem for discrete data is due to Burg, and Theorem 2.1 is the deterministic version of his result. In our view, and from a mathematical perspective, Theorem 2.6 is the continuous analogue of this deterministic theorem. There are also continuous analogues of Burg's theorem due to Dym and Gohberg [5] and Chover [2]. The former differs from Theorem 2.6 in using different logarithmic integrals than (0.1), thereby

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necessitating another presentation including both hypotheses and proof. The latter states a logarithmic inequality similar to (0.1), but is non-constructive and probabilistic, and has a different interpretation than Theorem 2.6.

Our notation is given in Section 1. Section 2 contains necessary definitions and discussion as well as statements of Burg's theorem (Theorem 2.1) and our result (Theorem 2.6). Theorem 2.6 is proved in several steps in Section 5; and we collect the little Fourier analysis required for the proof in Section 4. In Section 3 we focus on our chief hypothesis in Theorem 2.6 and pose a natural extension problem associated with a classical theorem due to Krein.

1. Notation.

$\mathbb{R}$ is the real line thought of as the time axis, and $\hat{\mathbb{R}}$ is the real line, the dual group of $\mathbb{R}$, thought of as the frequency axis. $\mathbb{Z}$ designates the integers and $\mathbb{T}_\Omega = \hat{\mathbb{R}}/2\Omega\mathbb{Z}$ is the compact group identified with the interval $[-\Omega,\Omega)$ for $\Omega > 0$ fixed; we write $\mathbb{T}$ instead of $\mathbb{T}_\pi$. $L^1(\hat{\mathbb{R}})$ is the space of complex ($\mathbb{C}$)-valued Lebesgue integrable functions G on $\hat{\mathbb{R}}$, normed by $\|G\|_{L^1(\hat{\mathbb{R}})} = \int |G(\gamma)|d\gamma$, where "$\int$" denotes integration over $\hat{\mathbb{R}}$. $L^1(\mathbb{T}_\Omega)$ is the space of $\mathbb{C}$-valued 2Ω-periodic locally Lebesgue integrable functions F on $\hat{\mathbb{R}}$, normed by $\|F\|_{L^1(\mathbb{T}_\Omega)} = \frac{1}{2\Omega}\int_{-\Omega}^{\Omega} |F(\gamma)|d\gamma$.

The *Fourier transform* of $G \in L^1(\hat{\mathbb{R}})$ is

$$\hat{G}(t) = \int G(\gamma) e^{-2\pi i t\gamma} d\gamma, \quad t \in \mathbb{R}.$$

The <u>Fourier series</u> of $F \in L^1(\mathbb{T}_\Omega)$ is $\sum \tilde{F}(j) e^{\pi i j\gamma/\Omega}$, summed over $\mathbb{Z}$, with <u>Fourier coefficients</u>,

$$\tilde{F}(j) = \frac{1}{2\Omega} \int_{-\Omega}^{\Omega} F(\gamma) e^{-i\pi j\gamma/\Omega} d\gamma.$$

"$\tilde{F}$" is not the customary notation for Fourier coefficients, but here we wish to distinguish it from "$\hat{F}$". $A(\mathbb{T}_\Omega)$ is the space of absolutely convergent Fourier series. For series of scalars we write $\sum' a_k$ instead of $\sum_{k \in \mathbb{Z}\setminus\{0\}} a_k$.

If $X \subseteq \mathbb{R}$ we write $PD(X)$ to denote the set of continuous functions $R : X-X \longrightarrow \mathbb{C}$ for which

$$\sum_{t,u \in F} c_t \bar{c}_u R(t-u) \geq 0$$

for all finite sets $F \subseteq X$ where $X-X = \{t-u : t,u \in X\}$. $PD(\mathbb{R})$ is the usual space of continuous <u>positive</u> <u>definite</u> functions on $\mathbb{R}$. Our setting in the sequel will be to take data $D \in PD([-T/2,T/2])$ for a fixed time $T > 0$.

An $(N+1)\times(N+1)$ matrix $R(N) = (r_{jk})$, $r_{jk} \in \mathbb{C}$ and $j,k = 0,\dots,N$, is <u>hermitian</u> if $r_{jk} = \bar{r}_{kj}$; and an hermitian matrix $R(N)$ is <u>positive</u> <u>definite</u> if

$$\sum c_j \bar{c}_k r_{jk} \geq 0 \tag{1.1}$$

for all $(N+1)$-tuples of complex numbers c_j and if equality in (1.1) implies each $c_j = 0$. An $(N+1)\times(N+1)$ matrix $R(N)$ is <u>Toeplitz</u> if it is constant on all diagonals of negative slope.

2. Maximum entropy theorems.

The following is Burg's theorem, which we use in Section 5. We refer to [4;9] for recent conceptually interesting proofs.

Theorem 2.1 (Burg, 1967). Given $N, \Omega > 0$ and $\{r_j : \bar{r}_j = r_{-j}$ where $|j| \le N$ and $r_0 > 0\} \subseteq \mathbb{C}$. Assume the $(N+1)\times(N+1)$ hermitian Toeplitz matrix,

$$R(N) = (r_{j-k}), \; j,k = 0,\dots,N,$$

is positive definite. There is a unique positive element $S_\Omega \in A(\mathbb{T}_\Omega)$ with Fourier coefficients $\{s_j : j \in \mathbb{Z}\}$ such that

$$\forall \; |j| \le N, \; s_j = r_j$$

and such that for every positive element $F \in A(\mathbb{T}_\Omega)$ satisfying the condition, $\tilde{F}(j) = r_j$ for $|j| \le N$, the inequality,

$$\text{(2.1)} \qquad \frac{1}{2\Omega}\int_0^{2\Omega} \log F(\gamma)\,d\gamma \le \frac{1}{2\Omega}\int_0^{2\Omega} \log S(\gamma)\,d\gamma,$$

is obtained.

Definition 2.2. Given N, Ω, and $\{r_j : \bar{r}_j = r_{-j}$ where $|j| \le N$ and $r_0 > 0\} \subseteq \mathbb{C}$. The Fourier series S_Ω of Theorem 2.1 is the Burg maximizer for the correlation data $\{r_j\}$.

Remark 2.3. S_Ω has the explicit form

$$S_\Omega(e^{\pi i\gamma/\Omega}) = p_n/|P_N(e^{\pi i\gamma/\Omega})|^2,$$

where

$$P_N(e^{\pi i\gamma/\Omega}) = \sum_{j=0}^{N} p_j e^{\pi i j\gamma/\Omega}$$

is the Szegö polynomial of degree N and p_N is a normalizing constant. The quadratic form $\sum a_j \bar{b}_k r_{j-k}$ defines an inner product of polynomials with coefficients $\{a_j\}$ and $\{b_j\}$. As such, $\sum p_j \bar{p}_k r_{j-k}$ is the norm $\|P_N\|$ of P_N; and the fact $(r_{j-k}) >> 0$ (especially the property $\sum a_j \bar{a}_k r_{j-k} = 0$ implies each $a_j = 0$), ensures that $\|\ldots\|$ is a genuine norm. Therefore, since P_N is not identically zero, we have $\|P_N\| > 0$. This positivity and the orthogonality of Szegö polynomials characterizes the positivity of $|P_N|$ on $\mathbb{T}_\Omega$, which, in turn, allows us to define S_Ω. The fact, $|P_N| > 0$, is due to Szegö; but the characterization in terms of orthogonality has been most simply and elegantly established in [9].

In order to formulate the analogue of Theorem 2.1 for the case of the real line, we introduce the following class of functions.

Definition 2.4. Let $L(\hat{\mathbb{R}})$ be the set of continuous, positive elements $G \in L^1(\hat{\mathbb{R}})$ for which

$$\int \frac{\log G(\gamma)}{\pi(1+\gamma^2)} \, d\gamma > -\infty \tag{2.2}$$

and

$$\sup_{\Omega \geq 1} \frac{1}{2\Omega} \sum_{j=-\infty}^{\infty} |\hat{G}(\frac{j}{2\Omega})| < \infty. \tag{2.3}$$

For positive $G \in L^1(\hat{\mathbb{R}})$ (or, more generally, for positive G satisfying $\int \frac{G(\gamma)}{\pi(1+\gamma^2)} d\gamma < \infty$), condition (2.2) is equivalent to the fact that G is in the Cartwright class, i.e., $(\log G(\gamma))/(\pi(1+\gamma^2)) \in L^1(\hat{\mathbb{R}})$.

Example 2.5. Let $G(\gamma) = 1/(\pi(1+\gamma^2))$. Then $\hat{G}(t) = \exp(-2\pi|t|)$ so that

$$\frac{1}{2\Omega} \sum |\hat{G}(\frac{j}{2\Omega})| \le \frac{1}{2\Omega} + \frac{1}{\pi}$$

by the integral test. Therefore, since

$$\int \frac{|\log g(\gamma)|}{\pi(1+\gamma^2)} d\gamma \le C + K \int_{\gamma \ge 10} \frac{\log \gamma}{\gamma^2} d\gamma < \infty,$$

we have $G \in L(\hat{\mathbb{F}})$.

We are now in a position to state our main result.

Theorem 2.6. Given $T > 0$ and $D \in PD[-T/2,T/2]$. For each $\Omega > 0$, let $S_\Omega \in A(\mathbb{T}_\Omega)$ be the Burg maximizer for the data $r_j = D(\frac{j}{2\Omega})$, $|j| \le 2T\Omega$. If $G \in L(\hat{\mathbb{F}})$ has the property that $\hat{G} = D$ on $[-T,T]$ then

$$\int \frac{\log G(\gamma)}{\pi(1+\gamma^2)} d\gamma \le \underline{\lim}_{\Omega\to\infty} \int \frac{\log[\frac{1}{2\Omega}S_\Omega(\gamma)]}{\pi(1+\gamma^2)} d\gamma < \infty. \tag{2.4}$$

As mentioned in the Introduction, the proof will be given in Section 5.

Remark 2.7. a. Theorem 2.6 can be extended to a larger class than $L(\hat{\mathbb{F}})$, e.g., [1, pp. 158-161] treats such an extension for $\mathbb{T}$. Of course, it may happen that there are no elements of $L(\hat{\mathbb{F}})$ whose transforms extend the given data; this issue is discussed in Section 3.

b. If property (2.2) of Definition 2.4 fails then Theorem 2.6 is immediate.

3. Extension and transition results.

The material in this section is not needed in the proof of Theorem 2.6; however, it illustrates the role of the $PD[-T/2,T/2]$

hypothesis and indicates some of the subtlety involved in choosing $G > 0$ for which $\hat{G} = D$ on $[-T,T]$.

We begin by stating Krein's theorem [8] on positive definite extensions, and then show that the transition from positive definite data on $\mathbb{R}$ to positive definite matrices can be effected for specific extensions by means of a Fourier uniqueness argument.

Theorem 3.1 (Krein, 1940). Let $X = [-T/2,T/2]$ for a fixed $T > 0$. For each $D \in PD(X)$ there is $P \in PD(\mathbb{R})$ for which $P = D$ on $X-X$.

Krein's theorem has had several significant lines of development, e.g., [3,10] give the flavor, plus further references, of two such developments. Besides providing a new proof of Theorem 3.1, Rudin [10] also demonstrates its failure in $\mathbb{R}^n$, $n > 1$.

Problem 3.2. In light of Theorem 2.6, where we suppose $G \in L(\hat{\mathbb{R}})$ has the property, $\hat{G} = D$ on $[-T,T]$, it is natural to ask for the following refinement of Krein's theorem: find further conditions on D (in Theorem 3.1) to ensure that the resulting $P = \hat{G}$ has the property that $G \in L^1(\hat{\mathbb{R}})$ and G is positive a.e. Of course, from Bochner's theorem we know that P (in Theorem 3.1) is $\hat{G}$ for some positive bounded measure G. In this regard, and in light of our assumption (and goal of weakening it) in Proposition 3.3, we point out that, because of the existence of totally disconnected sets having positive measure, there are non-negative functions $G \in L^1(\hat{\mathbb{R}})$ for which $\operatorname{supp} G = \hat{\mathbb{R}}$ but for which the argument in Proposition 3.3 fails.

Proposition 3.3. Given $T > 0$ and $D \in PD[-T/2,T/2]$. Assume there is a positive element $G \in L^1(\hat{\mathbb{R}})$ for which $\hat{G} = D$ on $[-T,T]$. For a fixed $N > 0$, define

$$r_j = D(\frac{jT}{N}), \quad |j| \le N$$

and the $(N+1)\times(N+1)$ Toeplitz matrix $R(N) = (r_{j-k})$, $j,k = 0,\dots,N$. $R(N)$ is hermitian and positive definite.

Proof. Since $D \in PD[-T/2,T/2]$ it is sufficient to prove that $c_0 = \dots = c_N = 0$ if $\sum c_j \bar{c}_k r_{j-k} = 0$.

By our assumption, $r_{j-k} = \hat{G}((j-k)T/N)$; and, so, if $\sum c_j \bar{c}_k r_{j-k} = 0$ we have

$$(3.1) \qquad 0 = \sum c_j \bar{c}_k \hat{G}(\frac{(j-k)T}{N}) = \int |\sum c_k e^{-2\pi ikT\gamma/N}|^2 G(\gamma)d\gamma ,$$

where the fact that $t_j - t_k = t_{j-k}$, for $t_j = jT/N$, is essential to the calculation. Since $G > 0$ a.e., (3.1) allows us to conclude that

$$\sum c_k e^{-2\pi ikT\gamma/N} = \sum c_k \hat{\delta}_{kT/N}(\gamma)$$

is identically zero. Consequently, by Fourier uniqueness or properties of polynomials, each $c_j = 0$. q.e.d.

4. A lemma from Fourier analysis.

Given $G \in L^1(\hat{\mathbb{R}})$. We set

$$G_\Omega(\gamma) = 2\Omega \sum_{k=-\infty}^{\infty} G(\gamma + 2k\Omega), \quad \gamma \in \hat{\mathbb{R}}.$$

G_Ω is 2Ω-periodic on $\hat{\mathbb{R}}$ and, in fact, $G_\Omega \in L^1(\mathbb{T}_\Omega)$ since it is obvious that $\|G_\Omega\|_{L^1(\mathbb{T}_\Omega)} \le \|G\|_{L^1(\hat{\mathbb{R}})}$. We also define the Fejér *kernel* $\{w_a : a > 0\}$,

$$w_a(\gamma) = \frac{a}{2\pi}\left[\frac{\sin(\frac{a\gamma}{2})}{\frac{a\gamma}{2}}\right]^2,$$

recalling that $\hat{w}_a(t) = \Delta_{a/(2\pi)}(t)$, where $\Delta_b(t) = \max\,(1-\frac{|\gamma|}{b},0)$.

The following is well known, e.g., [6;7].

Lemma 4.1. Given $F \in L^1(\mathbb{T}_\Omega)$, let F_e be the canonical extension of F as a 2Ω-periodic function on $\hat{\mathbb{R}}$. Then, for each $a > 0$, we have

$$\|F_e w_a\|_{L^1(\hat{\mathbb{R}})} \le 2\Omega\left[\sum_k \sup_{\gamma\in[2k\Omega,2(k+1)\Omega]} w_a(\gamma)\right]\|F\|_{L^1(\mathbb{T}_\Omega)}.$$

Remark 4.2. The proof of Lemma 4.1 is clear since

$$\|F_e w_a\|_{L^1(\hat{\mathbb{R}})} \le 2\Omega\sum_{k=-\infty}^{\infty}\frac{1}{2\Omega}\int_{2k\Omega}^{2(k+1)\Omega} |F_e(\gamma)w_a(\gamma)|\,d\gamma.$$

Also, $w_a(0) = a/(2\pi)$ and $0 \le w_a(\gamma) \le a/(2\pi^3k^2)$ for $\gamma \in [2\pi k/a, 2\pi(k+1)/a]$, $k \ge 1$. Consequently, we have the bound,

$$\sum_k \sup_{\gamma\in[2k\Omega,2(k+1)\Omega]} w_{\pi/\Omega}(\gamma) \le \frac{1}{\Omega}(1 + \frac{1}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}).$$

Remark 4.3. Given $G \in L^1(\hat{\mathbb{R}})$, we easily compute

(4.1) $$\forall\, j \in \mathbb{Z},\ \tilde{G}_\Omega(j) = \hat{G}(\frac{j}{2\Omega}).$$

In particular, if $\sum|\hat{G}(\frac{j}{2\Omega})| < \infty$ then $G_\Omega \in A(\mathbb{T}_\Omega)$, cf., Definition 2.4 and Remark 5.3.

5. Proof of the maximum entropy theorem for $\mathbb{R}$.

The following fact is intuitively clear but requires some verification. In particular, the dominated convergence theorem can not be directly applied.

Proposition 5.1. We have

$$\lim_{\Omega\to\infty} \int \frac{\log[2\Omega w_{\pi/\Omega}(\gamma)]}{\pi(1+\gamma^2)}\, d\gamma = 0. \tag{5.1}$$

In particular,

$$\forall\ \Omega \geq 1,\ -\infty < \int \frac{\log[w_{\pi/\Omega}(\gamma)]}{\pi(1+\gamma^2)}\, d\gamma < 0. \tag{5.2}$$

Proof. (i) Formally, we compute

$$\int \frac{\log[2\Omega w_{\pi/\Omega}(\gamma)]}{\pi(1+y^2)}\, d\gamma = \frac{2}{\Omega}\int_0^\infty \frac{\log|(\sin\lambda)/\lambda|}{(\frac{\pi}{2\Omega})^2+\lambda^2}\, d\lambda. \tag{5.3}$$

(ii) Let $b = b_\Omega = \pi/(2\Omega)$. Then

$$\left|\frac{2}{\Omega}\int_\pi^\infty \frac{|\log\lambda|}{b_\Omega^2+\lambda^2}\, d\lambda\right| \leq \frac{2}{\Omega}\int_\pi^\infty \frac{\log\lambda}{\lambda^2}\, d\lambda. \tag{5.4}$$

Also, we compute

$$\left|\frac{2}{\Omega}\int_\pi^\infty \frac{\log|\sin\lambda|}{b_\Omega^2+\lambda^2}\, d\lambda\right| \leq \frac{4}{\Omega}\sum_{k=1}^\infty \int_{k\pi}^{k\pi+\frac{\pi}{2}} \frac{|\log|\sin\lambda||}{b_\Omega^2+\lambda^2}\, d\lambda$$

$$= \frac{4}{\Omega}\sum_{k=1}^\infty \int_{k\pi}^{k\pi+\frac{\pi}{2}} \frac{|\log|\sin(\lambda-k\pi)||}{b_\Omega^2+\lambda^2}\, d\lambda$$

$$\leq \frac{4}{\Omega}\sum_{k=1}^\infty \int_0^{\pi/2} \frac{\left|\log\frac{\sin\gamma}{\gamma}\right|}{b_\Omega^2+(\gamma+k\pi)^2}\, d\gamma \ + \tag{5.5}$$

$$+ \frac{4}{\Omega} \sum_{k=1}^{\infty} \int_0^{\pi/2} \frac{|\log \gamma|}{b_\Omega^2 + (\gamma + k\pi)^2} \, d\gamma$$

$$\leq \frac{4}{\Omega} \log \frac{\pi}{2} \sum_{k=1}^{\infty} \int_0^{\pi/2} \frac{d\gamma}{b_\Omega^2 + (\gamma + k\pi)^2}$$

$$+ \frac{4}{\Omega} \sum_{k=1}^{\infty} \int_0^{\pi/2} \frac{|\log \gamma|}{b_\Omega^2 + (\gamma + k\pi)^2} \, d\gamma ,$$

where the first inequality is a consequence of integrating (computing Riemann sums) on $[k\pi, (k+1)\pi)$ "symmetrically about the point $k\pi + \pi/2$".

The two terms on the right hand side of (5.5) are estimated as follows:

$$\frac{c}{\Omega} \sum_{k=1}^{\infty} \int_0^{\pi/2} \frac{d\gamma}{b_\Omega^2 + (\gamma + k\pi)^2} \leq \frac{c}{\Omega} \sum_{k=1}^{\infty} \int_{2/\pi}^{\infty} \frac{d\lambda}{(k\lambda\pi)^2} = \frac{\pi c}{12\Omega}$$

and

$$\frac{4}{\Omega} \sum_{k=1}^{\infty} \int_0^{\pi/2} \frac{|\log \gamma|}{b_\Omega^2 + (\gamma + k\pi)^2} \, d\gamma \leq \frac{2}{3\Omega} \int_{2/\pi}^{\infty} \frac{|\log \lambda|}{\lambda^2} \, d\lambda .$$

Combining these estimates with (5.5), and incorporating this information with (5.4), we have proved

$$\lim_{\Omega \to \infty} \frac{2}{\Omega} \int_\pi^\infty \frac{\log|(\sin \lambda)/\lambda|}{(\frac{\pi}{2\Omega})^2 + \lambda^2} \, d\lambda = 0, \tag{5.6}$$

cf., (5.3).

(iii) For each $\alpha \in (0,\pi]$, the definition of the logarithm yields the estimate,

$$0 \leq \frac{2}{\Omega} \int_0^\alpha \frac{\log(\lambda/(\sin \lambda))}{b_\Omega^2 + \lambda^2} \, d\lambda \leq \frac{2}{\Omega} \int_0^\alpha \frac{\lambda - \sin \lambda}{\lambda^2 \sin \lambda} \, d\lambda . \tag{5.7}$$

Each of the above integrands is non-negative. For convenience, fix $\alpha \in [\pi/2,\pi)$. Using L'Hôpital's rule for the 0 endpoint, it is easy to see that there is $K_\alpha > 0$ such that

$$\forall\ \lambda \in (0,\alpha],\ \frac{\lambda - \sin \lambda}{\lambda^2 \sin \lambda} \le K_\alpha .$$

Substituting this bound in (5.7) we compute

$$\lim_{\Omega\to\infty} \frac{2}{\Omega} \int_0^\alpha \frac{\log|(\sin \lambda)/\lambda|}{(\frac{\pi}{2\Omega})^2+\lambda^2}\, d\lambda = 0. \tag{5.8}$$

(iv) Because of (5.3), (5.6), and (5.8) we shall have obtained (5.1) once we verify

$$\lim_{\Omega\to\infty} \frac{2}{\Omega} \int_\alpha^\pi \frac{\log(\lambda/(\sin \lambda))}{b_\Omega^2+\lambda^2}\, d\lambda = 0, \tag{5.9}$$

where $\alpha \in [\pi/2,\pi)$ is fixed. Since

$$\lim_{\Omega\to\infty} \frac{2}{\Omega} \int_\alpha^\pi \frac{\log \lambda}{\lambda^2}\, d\lambda = 0$$

and

$$\left|\int_\alpha^\pi \frac{\log(\lambda/(\sin \lambda))}{b_\Omega^2+\lambda^2}\, d\lambda\right| \le \left|\int_\alpha^\pi \frac{\log \lambda}{b_\Omega^2+\lambda^2}\, d\lambda\right| + \left|\int_\alpha^\pi \frac{\log \sin \lambda}{b_\Omega^2+\lambda^2}\, d\lambda\right|,$$

it is sufficient to prove

$$\lim_{\Omega\to\infty} \left|\frac{2}{\Omega} \int_\alpha^\pi \frac{\log \sin \lambda}{b_\Omega^2+\lambda^2}\, d\lambda\right| = 0. \tag{5.10}$$

To this end we make the following estimate:

$$\begin{aligned}\left|\frac{2}{\Omega} \int_\alpha^\pi \frac{\log \sin \lambda}{b_\Omega^2+\lambda^2}\, d\lambda\right| &\le \frac{2}{\Omega} \int_\alpha^\pi \frac{|\log|\sin(\lambda-\pi)||}{b_\Omega^2+\lambda^2}\, d\lambda \\ &\le \frac{2}{\Omega} \int_{\alpha-\pi}^0 \frac{|\log|\frac{\sin \gamma}{\gamma}||}{b_\Omega^2+(\gamma+\pi)^2}\, d\gamma + \frac{2}{\Omega} \int_{\alpha-\pi}^0 \frac{|\log|\gamma||}{b_\Omega^2+(\gamma+\pi)^2}\, d\gamma .\end{aligned} \tag{5.11}$$

Since $1 \geq (\sin \gamma)/\gamma \geq 2/\pi$ on $[-\pi/2, 0]$ and since $0 > \alpha - \pi \geq -\pi/2$, the first term on the right hand side of (5.11) is bounded by

$$\frac{2\log(\pi/2)}{\Omega}\int_{\alpha-\pi}^{0}\frac{d\gamma}{b_\Omega^2+(\gamma+\pi)^2} \leq \frac{2\log(\pi/2)}{\Omega}\int_{\alpha}^{\pi}\frac{d\lambda}{\lambda^2},$$

which tends to 0 as Ω tends to infinity.

Consequently, (5.10) is obtained once we control the second term on the right hand side of (5.11), viz., to show

$$\lim_{\Omega\to\infty}\frac{2}{\Omega}\int_{\alpha-\pi}^{0}\frac{|\log|\gamma||}{b_\Omega^2+(\gamma+\pi)^2}\,d\gamma = 0. \tag{5.12}$$

To this end we make the following estimate:

$$\frac{2}{\Omega}\int_{\alpha-\pi}^{0}\frac{|\log|\gamma||}{b_\Omega^2+(\gamma+\pi)^2}\,d\gamma \leq \frac{2}{\Omega}\int_{\alpha}^{\pi}\frac{|\log|\lambda-\pi||}{\lambda^2}\,d\lambda.$$

Also, we note that $|\log|\lambda-\pi|| \leq \log\pi + |\log(1-\frac{\lambda}{\pi})|$ on $[\alpha,\pi)$. Since

$$\lim_{\Omega\to\infty}\frac{2}{\Omega}\int_{\alpha}^{\pi}\frac{\log\pi}{\lambda^2}\,d\lambda = 0,$$

we shall have verified (5.12) when we prove

$$\lim_{\Omega\to\infty}\frac{2}{\Omega}\int_{\alpha}^{\pi}\frac{|\log(1-\frac{\lambda}{\pi})|}{\lambda^2}\,d\lambda = 0. \tag{5.13}$$

For $\lambda \in [\alpha,\pi)$ we obtain $|\log(1-\frac{\lambda}{\pi})| = \sum_{1}^{\infty}(\lambda/\pi)^n(1/n)$. Substituting this into the left hand side of (5.13) yields the estimate,

$$\frac{2}{\Omega}\int_{\alpha}^{\pi}\frac{|\log(1-\frac{\lambda}{\pi})|}{\lambda^2}\,d\lambda \leq \frac{2}{\pi\Omega}\log(\frac{\pi}{\alpha}) + \frac{2}{\pi\Omega}\sum_{n=2}^{\infty}\frac{1}{n(n-1)},$$

and the right hand side clearly tends to 0 as Ω tends to infinity. Thus, (5.13) is valid, which, by our chain of implications, gives (5.10); and, as we pointed out earlier in this exercise, (5.10) is sufficient to complete the result. q.e.d.

Proposition 5.2. Given $T > 0$ and $D \in PD[-T/2,T/2]$. There is a constant $C > 0$ such that for all $\Omega > 0$ and for all $G \in L(\hat{\mathbb{R}})$ for which $\hat{G} = D$ on $[-T,T]$, we have

$$(5.14) \qquad \int \frac{\log[G_\Omega(\gamma)w_{\pi/\Omega}(\gamma)]}{\pi(1+\gamma^2)}\,d\gamma \le \int \frac{\log[S_\Omega(\gamma)w_{\pi/\Omega}(\gamma)]}{\pi(1+\gamma^2)}\,d\gamma \le C,$$

where $G_\Omega(\gamma) = 2\Omega\sum G(\gamma+2k\Omega)$ is 2Ω-periodic on $\hat{\mathbb{R}}$, S_Ω is the Burg maximizer for the data $r_j = D(\frac{j}{2\Omega})$, $|j| \le 2T\Omega$, and S_Ω is considered as a 2Ω-periodic function on $\hat{\mathbb{R}}$.

Proof. (i) Because of Lemma 4.1 we have

$$(5.15) \qquad \|S_\Omega w_{\pi/\Omega}\|_{L^1(\hat{\mathbb{R}})} \le \frac{7}{3}D(0).$$

Then we invoke Jensen's inequality to obtain

$$\int \frac{\log[S_\Omega(\gamma)w_{\pi/\Omega}(\gamma)]}{\pi(1+\gamma^2)}\,d\gamma \le \log\frac{7D(0)}{3\pi}$$

from (5.15). This is the second inequality of (5.14).

(ii) For each $\Omega > 0$ choose $k_\Omega \ge 1$ such that $k_\Omega S_\Omega, k_\Omega G_\Omega \ge 1$ on $\hat{\mathbb{R}}$. This can be done since S_Ω and G_Ω are 2Ω-periodic on $\hat{\mathbb{R}}$ and positive on $\mathbb{T}_\Omega$.

We'll prove

$$(5.16) \qquad \int \frac{\log G_\Omega(\gamma)}{\pi(1+\gamma^2)}\,d\gamma \le \int \frac{\log S_\Omega(\gamma)}{\pi(1+\gamma^2)}\,d\gamma,$$

assuming, without loss of generality, that the left-hand side is not $-\infty$.

Note that $1+(2k\Omega)^2 \le 1+\gamma^2 \le 1+(2(k+1)\Omega)^2$ for all $k \ge 0$ and $\gamma \in [2k\Omega,\ 2(k+1)\Omega]$. Using the hypothesis, $k_\Omega S_\Omega \ge 1$, we make the estimate,

$$\sum_k \int_{2k\Omega}^{2(k+1)\Omega} \frac{\log(k_\Omega S_\Omega(\gamma))}{\pi(1+\gamma^2)}\,d\gamma \ge$$

$$\sum_{k=0}^{\infty} \frac{1}{\pi(1+(2(k+1)\Omega)^2)} \int_{2k\Omega}^{2(k+1)\Omega} \log(k_\Omega S_\Omega(\gamma))d\gamma$$

$$+ \sum_{k=-\infty}^{-1} \frac{1}{\pi(1+(2k\Omega)^2)} \int_{2k\Omega}^{2(k+1)\Omega} \log(k_\Omega S_\Omega(\gamma))d\gamma,$$

which by periodicity equals

$$\frac{2}{\pi} \sum_{k=1}^{\infty} \frac{1}{1+(2k\Omega)^2} \int_0^{2\Omega} \log(k_\Omega S_\Omega(\gamma))d\gamma.$$

By Burg's theorem (Theorem 2.1), this term dominates

$$\frac{2}{\pi} \sum_{k=1}^{\infty} \frac{1}{1+(2k\Omega)^2} \int_0^{2\Omega} \log(k_\Omega G_\Omega(\gamma))d\gamma \tag{5.17}$$

since $\sum(\log k_\Omega)/(1+(2k\Omega)^2)$ converges. Writing (5.17) as

$$\sum_{k=1}^{\infty} \frac{1}{\pi(1+(2k\Omega)^2)} \int_{2k\Omega}^{2(k+1)\Omega} \log(k_\Omega G_\Omega(\gamma))d\gamma \ +$$

$$\sum_{k=-\infty}^{0} \frac{1}{\pi(1+((2k+1)\Omega)^2)} \int_{2k\Omega}^{2(k+1)\Omega} \log(k_\Omega G_\Omega(\gamma))d\gamma$$

we see that, because $k_\Omega G_\Omega \geq 1$, this term dominates

$$\sum_{k=1}^{\infty} \int_{2k\Omega}^{2(k+1)\Omega} \frac{\log(k_\Omega G_\Omega(\gamma))}{\pi(1+\gamma^2)} d\gamma + \sum_{k=0}^{\infty} \int_{2k\Omega}^{2(k+1)\Omega} \frac{\log(k_\Omega G_\Omega(\gamma))}{\pi(1+\gamma^2)} d\gamma .$$

We obtain (5.16) by combining these inequalities and using the convergence of

$$\sum \log k_\Omega \int_{2k\Omega}^{2(k+1)\Omega} \frac{d\gamma}{\pi(1+\gamma^2)} = \log k_\Omega .$$

(iii) From Proposition 5.1 we know

$$|\int \frac{\log w_{\pi/\Omega}(\gamma)}{\pi(1+\gamma^2)} d\gamma| < \infty. \tag{5.18}$$

(5.14) is a consequence of (5.16) and (5.18). q.e.d.

Remark 5.3. The hypothesis, $G \in L(\hat{\mathbb{R}})$, can be weakened considerably in Proposition 5.2. However, because of the (lack of) generality we have chosen for the statement of Burg's theorem and because we have used Burg's theorem in Proposition 5.2, it is required that $G_\Omega \in A(\mathbb{T}_\Omega)$. This is a particular consequence of (2.3) in our definition of $L(\hat{\mathbb{R}})$.

Lemma 5.4. Given $G \in L(\hat{\mathbb{R}})$.

a. $\lim_{\Omega\to\infty} \int_{-\Omega}^{\Omega} |\sum{}' G(\gamma+2k\Omega)| d\gamma = 0.$

b. For all $\Omega \geq 1$,

$$G_\Omega(\gamma) = \sum \hat{G}(\frac{j}{2\Omega}) e^{\pi i j\gamma/\Omega} \in A(\mathbb{T}_\Omega).$$

c. For each $\epsilon > 0$, there is $\Gamma > 0$ such that

$$\forall\ \Omega \geq 1, \ \left| \int_{|\gamma|>\Gamma} \frac{\log \sum G(\gamma+2k\Omega)}{\pi(1+\gamma^2)}\, d\gamma \right| < \varepsilon .$$

Proof. a. We compute

$$\int_{-\Omega}^{\Omega} \left| \sum{}' G(\gamma+2k\Omega \right| d\gamma \leq \int_{|\lambda|\geq\Omega} |G(\lambda)|\, d\lambda$$

and the right hand side tends to 0 as Ω tends to infinity since $L(\hat{\mathbb{R}}) \subseteq L^1(\hat{\mathbb{R}})$.

b. Since $\tilde{G}_\Omega(j) = \hat{G}(\frac{j}{2\Omega})$ for $G \in L^1(\hat{\mathbb{R}})$, e.g., (4.1), we see that $G_\Omega \in A(\hat{\mathbb{T}}_\Omega)$ by property (2.3) of Definition 2.4.

c. Using part (b) as well as the positivity of G and property (2.3) from Definition 2.4, we have

$$0 < G(\gamma) \leq \sum_k G(\gamma+2k\Omega) \leq C, \tag{5.19}$$

where C is independent of $\gamma \in \hat{\mathbb{F}}$ and $\Omega \geq 1$. Consequently, we obtain the inequalities

$$\begin{aligned} \int_{|\gamma|>\Gamma} \frac{\log G(\gamma)}{\pi(1+\gamma^2)}\, d\gamma &\leq \int_{|\gamma|>\Gamma} \frac{\log \sum G(\gamma+2k\Omega)}{\pi(1+\gamma^2)}\, d\gamma \\ &\leq \int_{|\gamma|>\Gamma} \frac{\log C}{\pi(1+\gamma^2)}\, d\gamma \end{aligned} \tag{5.20}$$

for all $\Gamma \geq 0$ and all $\Omega \geq 1$.

For $\varepsilon > 0$ we choose Γ_ε such that

$$\left| \int_{|\gamma|>\Gamma_\varepsilon} \frac{\log G(\gamma)}{\pi(1+\gamma^2)}\, d\gamma \right| < \varepsilon \quad \text{and} \quad \left| \int_{|\gamma|>\Gamma_\varepsilon} \frac{\log C}{\pi(1+\gamma^2)}\, d\gamma \right| < \varepsilon ; \tag{5.21}$$

that this can be done is immediate for the "log C" term and follows from property (2.2) of Definition 2.4 for the "log $G(\gamma)$"

term. Ω is not involved in (5.21) and so part (c) follows from (5.20) and (5.21). q.e.d.

Proposition 5.5. Given $G \in L(\hat{\mathbb{R}})$. Then

$$\lim_{\Omega\to\infty} \int \frac{\log[G_\Omega(\gamma) w_{\pi/\Omega}(\gamma)]}{\pi(1+\gamma^2)} d\gamma = \int \frac{\log G(\gamma)}{\pi(1+\gamma^2)} d\gamma. \tag{5.22}$$

Proof. The right hand side of (5.22) exists by property (2.2) from Definition 2.4. Also, because of Proposition 5.1, it is sufficient to prove

$$\lim_{\Omega\to\infty} \int \frac{\log \sum G(\gamma+2k\Omega)}{\pi(1+\gamma^2)} d\gamma = \int \frac{\log G(\gamma)}{\pi(1+\gamma^2)} d\gamma. \tag{5.23}$$

Using (5.19), (5.20), and property (2.2) of Definition 2.4, we see that each integral on the left hand side of (5.23) converges and that

$$\begin{aligned} 0 &\le \left| \int \frac{\log \sum G(\gamma+2k\Omega) - \log G(\gamma)}{\pi(1+\gamma^2)} d\gamma \right| \\ &= \int \frac{\log \sum G(\gamma+2k\Omega)}{\pi(1+\gamma^2)} d\gamma - \int \frac{\log G(\gamma)}{\pi(1+\gamma^2)} d\gamma. \end{aligned} \tag{5.24}$$

Fix $\varepsilon > 0$. Choose $\Gamma = \Gamma_\varepsilon$ so large that

$$\left| \int_{|\gamma|>\Gamma} \frac{\log G(\gamma)}{\pi(1+\gamma^2)} d\gamma \right| < \frac{\varepsilon}{2} \quad \text{and} \quad \left| \int_{|\gamma|>\Gamma} \frac{\log \sum G(\gamma+2k\Omega)}{\pi(1+\gamma^2)} d\gamma \right| < \frac{\varepsilon}{2} \tag{5.25}$$

and

$$\int_{|\gamma|\le\Gamma} \frac{m}{\pi(1+\gamma^2)} d\gamma = 1, \quad m \in (1,\pi] \tag{5.26}$$

hold (noting that $\int \frac{d\gamma}{\pi(1+\gamma^2)} = 1$), where by Lemma 5.4c the second estimate of (5.25) is true independent of $\Omega \ge 1$.

Thus (5.24) leads to

$$0 \le \int \frac{\log \sum G(\gamma+2k\Omega)}{\pi(1+\gamma^2)}\, d\gamma - \int \frac{\log G(\gamma)}{\pi(1+\gamma^2)}\, d\gamma$$

$$= \int_{|\gamma|>\Gamma} \frac{\log \sum G(\gamma+2k\Omega)}{\pi(1+\gamma^2)}\, d\gamma - \int_{|\gamma|>\Gamma} \frac{\log G(\gamma)}{\pi(1+\gamma^2)}\, d\gamma$$

$$(5.27) \qquad + \frac{1}{m}\int_{|\gamma|\le\Gamma} \frac{m[\log \sum G(\gamma+2k\Omega) - \log G(\gamma)]}{\pi(1+\gamma^2)}\, d\gamma$$

$$= a_\varepsilon + b_\varepsilon + \frac{1}{m}\int_{|\gamma|\le\Gamma} \frac{m\ \log(1+(1/G(\gamma))\sum' G(\gamma+2k\Omega))}{\pi(1+\gamma^2)}\, d\gamma$$

$$\le \varepsilon + \frac{1}{m}\log \int_{|\gamma|\le\Gamma} \frac{m\ (1+(1/G(\gamma))\sum' G(\gamma+2k\Omega))}{\pi(1+\gamma^2)}\, d\gamma,$$

for all $\Omega \ge 1$, where, using (5.26), we have invoked Jensen's inequality and where $|a_\varepsilon|$, $|b_\varepsilon| < \varepsilon/2$. By definition of m, the integral on the right hand side of (5.27) is

$$1 + m\int_{|\gamma|\le\Gamma} \frac{\sum' G(\gamma+2k\Omega)}{\pi(1+\gamma^2)G(\gamma)}\, d\gamma \le$$

$$\le 1 + \frac{1}{G(\gamma_\varepsilon)}\int_{|\gamma|\le\Gamma} \left|\sum{}' G(\gamma+2k\Omega)\right| d\gamma,$$

$G(\gamma_\varepsilon) > 0$ being the maximum of G on $[-\Gamma,\Gamma]$. Substituting this information into the right hand side of (5.27) and taking "$\overline{\lim}_{\Omega\to\infty}$" we obtain

$$0 \le \overline{\lim_{\Omega\to\infty}} \left| \int \frac{\log \sum G(\gamma+2k\Omega)}{\pi(1+\gamma^2)}\, d\gamma - \int \frac{\log G(\gamma)}{(1+\gamma^2)}\, d\gamma \right|$$

$$\le \varepsilon + \overline{\lim_{\Omega\to\infty}} \frac{1}{m}\log\Big(1+\frac{1}{G(\gamma_\varepsilon)}\int_{|\gamma|\le\Gamma} \left|\sum{}' G(\gamma+2k\Omega)\right| d\gamma\Big) = \varepsilon,$$

where the equality is a consequence of Lemma 5.4a since Ω is eventually bigger than $\Gamma = \Gamma_{\varepsilon}$. (5.23) follows since this last estimate is true for all $\varepsilon > 0$. q.e.d.

Theorem 2.6 is now proved by combining Propositions 5.1, 5.2, and 5.5.

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